# Detective quantum efficiency of imaging systems with amplifying and scattering mechanisms

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We have analyzed the influence of stochastic amplifying and scattering mechanisms on the transfer of signal and noise through multistage imaging systems in terms of multivariate moment-generating functions. Stochastic amplification of photon noise by one stage of an imaging system is shown to constitute an effective signal to the next, while the underlying photon-noise component is unaffected by a subsequent scattering process. In the case of stationary, photon-limited inputs, these considerations then lead to useful expressions for the noise power spectrum and detective quantum efficiency for multistage image systems. The application of these results to the analysis of radiographic screen–film imaging systems is discussed.

## INTRODUCTION

A complete description of an imaging system must include the relationships that govern the propagation of signal and noise throughout the system. From these relationships many other useful quantities are derived, such as detective quantum efficiency (DQE), that further characterize the system. Signal-transfer characteristics have received widespread attention, for which Fourier-based linear-system methods are well established in the literature, and they have been applied to diverse imaging systems and technologies. The direct application of these methods to noise propagation may often be appropriate and successful but can also lead to apparent paradoxes and confusion if the essential differences between signal and noise is not delineated.

The topic of noise transfer through stochastic amplification processes has a long history. It was discussed in the context of electron multiplier tubes by Shockley and Pierce.[1] Subsequent authors include Mandel[2] and Zwieg.[3] Zwieg's work is especially noteworthy here, since his results are reported in terms of DQE. However, these investigations are from the statistics viewpoint of a point process (e.g., cascading of variances), not from that of imaging systems with spatial resolution. In this latter respect, Doerner[4] reported on the Wiener spectrum analysis of photographic granularity and presented the theoretical framework for the linear transfer of granularity through a photographic printing process. That analysis included the use of a known photographic granularity pattern as a test object for the determination of the transfer function of a lens, recognizing the fact that a similar noise-transfer theory might be applied to screen–film systems in Wiener spectrum terms. Rossmann[5] consolidated this approach for radiographic screen–film systems; later, Kemperman and Trabka[6] presented a rigorous analysis of the exposure fluctuations arriving from the screen and providing the exposure to the photographic process. Our own radiographic modeling[7,8] was based on this previous work. In this context, the analysis of noise spectra of cascaded linear systems has been reported, where input is uncorrelated Poisson noise.[9,10]

The purpose of this paper is to provide a general theoretical framework for analyzing the influence of stochastic amplifying and scattering mechanisms on the transfer of image statistics through multistage imaging systems. Our results, which are cast in terms of multivariate moment-generating functions, are general in that they are applicable to stationary as well as nonstationary processes. In the special case of stationary processes, relationships between input and output noise power spectra (NPS) are described. For photon-limited inputs, these considerations lead to useful expressions for the DQE. In this case stochastic amplification of photon noise by one stage of an imaging system is shown to constitute an effective signal to the next, while the underlying photon-noise component is unaffected by a subsequent scattering process.

## THEORY

The general noise-propagation characteristics for systems containing arbitrary numbers of scattering and amplifying processes can be obtained in terms of a proper description of elementary amplification and scattering processes. The method used in this paper to describe these elementary processes is to express the multivariate moment-generating function (MMGF) of the output signal in terms of the MMGF of the input signal and the parameters that identify the amplification or the scattering process. Processes of arbitrary complexity can be constructed by suitably cascading the results of the elementary processes. The resultant MMGF is used to determine the statistical moments of the output signal. The expression for the output moments can become quite involved if the number of cascaded processes is large. However, for a wide-sense-stationary input, where the definition of NPS is meaningful, the relationship be-

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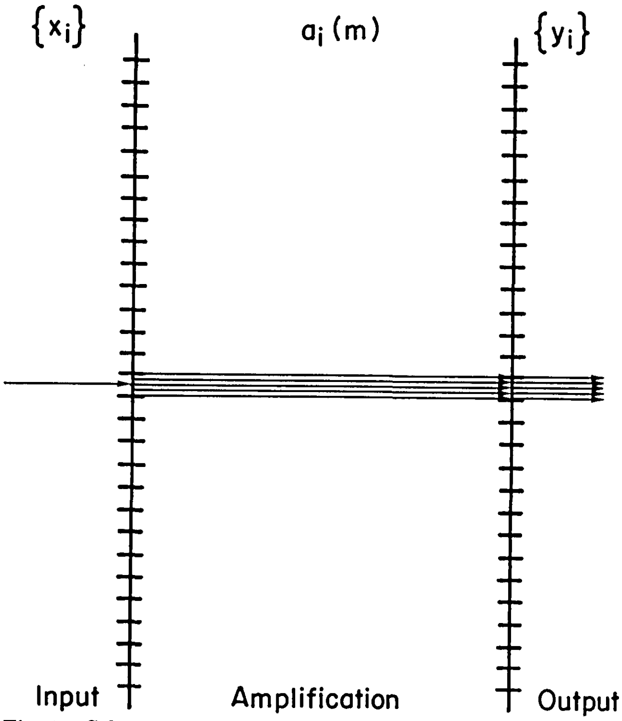


Fig. 1. Schematic diagram of the elementary amplification process. $\{x_i\}$ is the number of input quanta in each pixel, $\{y_i\}$ is the number of output quanta in each pixel after amplification, and $a_i(m)$ describes the amplification process at the $i$th pixel.

tween the input and output NPS takes a simple form. The NPS at the output of each cascaded level is expressed uniquely in terms of the input NPS and the process parameters of that level. Thus knowledge of the input NPS and the system parameters is all that is required to determine the NPS at the output of any number of cascaded processes. For notational simplicity, our analysis is presented for the one-dimensional case; however, the extension to two dimensions is straightforward.

### Amplification Process

We consider an elementary amplification process, as illustrated in Fig. 1. For instance, this may be an intermediate stage of a more complex process. The input and output image planes are divided into elementary cells or pixels. The random variable $\{x_i\}$ represents the number of input quanta in each pixel, and the random variable $\{y_i\}$ represents the number of output quanta in each pixel after amplification. The amplification process is described by the set of probabilities

$$a_i(m) = \Pr\{y_i = m | x_i = 1\}, \tag{1}$$

which denotes the probability of $m$ ($m \geq 0$) output quanta resulting from a single input at the $i$th pixel.

We define $X(\mathbf{s})$ and $Y(\mathbf{s})$ as the MMGF's associated with the joint probability distribution of the random variables $\{x_i\}$ and $\{y_i\}$, respectively.[11] Thus

$$X(\mathbf{s}) = X(s_1, s_2, \ldots, s_n) = \sum_{i_1} \cdots \sum_{i_n} s_1^{i_1} s_2^{i_2} \ldots s_n^{i_n} \times \Pr\{x_1 = i_1, x_2 = i_2, \ldots, x_n = i_n\}, \tag{2}$$

with a similar definition for $Y(\mathbf{s})$. The moment-generating function (MGF) associated with the amplification process at the $i$th pixel may likewise be expressed as

$$A_i(s) = \sum_j s^j \Pr\{y_i = j | x_i = 1\} = \sum_j s^j a_i(j). \tag{3}$$

Assuming that the amplification process is independent of location (i.e., space invariant), the subscript $i$ in $A_i(s)$ can be removed. Our aim is first to express $Y(\mathbf{s})$ in terms of $X(\mathbf{s})$ and $A(s)$ and then to find the statistics of $\{y_i\}$. By definition,

$$Y(\mathbf{s}) = \sum_{i_1} \cdots \sum_{i_n} s_1^{i_1} \ldots s_n^{i_n} \Pr\{y_1 = i_1, \ldots, y_n = i_n\}. \tag{4}$$

This can be expressed in terms of the joint probability distribution of $\{x_i\}$ as

$$Y(\mathbf{s}) = \sum_{i_1} \cdots \sum_{i_n} s_1^{i_1} \ldots s_n^{i_n} \sum_{j_1} \cdots \sum_{j_n} \times \Pr\{y_1 = i_1, \ldots, y_n = i_n | x_1 = j_1, \ldots, x_n = j_n\} \times \Pr\{x_1 = j_1, \ldots, x_n = j_n\}. \tag{5}$$

Since each amplification event is independent of all others, we may reorder the summation to obtain

$$Y(\mathbf{s}) = \sum_{j_1} \cdots \sum_{j_n} [A(s_1)]^{j_1} \ldots [A(s_n)]^{j_n} \times \Pr\{x_1 = j_1, \ldots, x_n = j_n\}, \tag{6}$$

where

$$[A(s_i)]^{j_i} = \sum_{k_i} s_i^{k_i} \Pr\{y_i = k_i | x_i = j_i\}. \tag{7}$$

This follows from the well-known property that the MGF of a random variable, which is a sum of independent random variables, is the product of their MGF's. By inspection, we get

$$Y(\mathbf{s}) = X[A(s_1), A(s_2), \ldots, A(s_n)]. \tag{8}$$

Equation (8) is a direct generalization of the MGF for the univariate compound process.[11] This input–output relationship is shown in the block diagram of Fig. 2(a).

The statistics of $\{y_i\}$ follow directly from the MMGF given in Eq. (8). In general, the $n$th moment of the output is a function of the first $n$ moments of the input quanta and

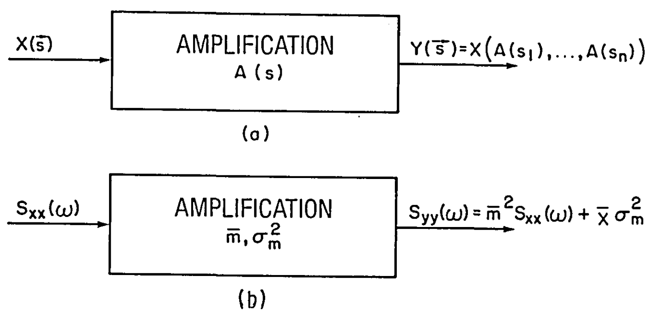


Fig. 2. Block diagram showing the input–output relationships for the amplification process: (a) input–output MMGF; (b) input–output NPS.

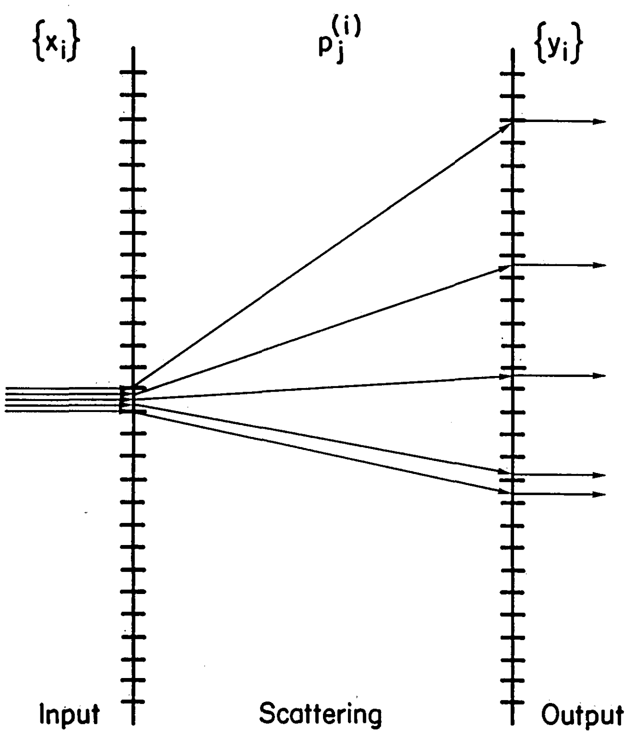


Fig. 3. Schematic diagram of the elementary scattering process. $\{x_i\}$ is the number of input quanta in each pixel, $\{y_i\}$ is the number of output quanta in each pixel after scattering, and $p_j^{(i)}$ describes the scattering process.

those of the amplification process. We now calculate the mean, the variance, and the covariance of $\{y_i\}$. The following relationships are known:

$$\left.\frac{\partial X(\mathbf{s})}{\partial s_i}\right|_{\mathbf{s}=1} = \bar{x}_i, \tag{9}$$

$$\left.\frac{\partial^2 X(\mathbf{s})}{\partial s_i \partial s_j}\right|_{\mathbf{s}=1} = \begin{cases} \sigma_{x_i}^2 - \bar{x}_i + \bar{x}_i^2 & \text{if } i = j \\ \operatorname{cov}(x_i, x_j) + \bar{x}_i\bar{x}_j & \text{if } i \neq j \end{cases}, \tag{10}$$

where $\bar{x}_i$ and $\sigma_{x_i}^2$ denote, respectively, the mean and the variance of the random variable $x_i$, and $\operatorname{cov}(x_i, x_j) = E\{(x_i - \bar{x}_i)(x_j - \bar{x}_j)\}$. Similarly,

$$\left.\frac{\partial A(s)}{\partial s}\right|_{s=1} = \bar{m}, \tag{11}$$

$$\left.\frac{\partial^2 A(s)}{\partial s^2}\right|_{s=1} = \sigma_m^2 - \bar{m} + \bar{m}^2. \tag{12}$$

As a result of the chain rule of differentiation, the mean value of $y_i$ is given by

$$\bar{y}_i = \left.\frac{\partial Y(\mathbf{s})}{\partial s_i}\right|_{\mathbf{s}=1} = \sum_l \left.\frac{\partial X[A(s_1), \ldots, A(s_n)]}{\partial A(s_l)} \frac{\partial A(s_l)}{\partial s_i}\right|_{\mathbf{s}=1} = \bar{m}\bar{x}_i. \tag{13}$$

The covariance and the variance of $y_i$ are calculated based on a similar approach:

$$\operatorname{cov}(y_i, y_j) = \left.\frac{\partial^2 Y(\mathbf{s})}{\partial s_i \partial s_j}\right|_{\mathbf{s}=1} - \bar{y}_i\bar{y}_j = \bar{m}^2 \operatorname{cov}(x_i, x_j) + \sigma_m^2 \bar{x}_i \delta_{ij}, \tag{14}$$

$$\sigma_{y_i}^2 = \operatorname{cov}(y_i, y_i) = \sigma_{x_i}^2 \bar{m}^2 + \sigma_m^2 \bar{x}_i, \tag{15}$$

where $\delta_{ij}$ is the Kronecker delta (i.e., $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$). Although Eq. (14) is new, Eq. (15) is a familiar expression within the general theory of compound processes.[11] It has appeared frequently to describe the noise of detectors not involved with spatial resolution, such as electron and photomultiplier tubes. The works of Shockley and Pierce,[1] Mandel,[2] Jones,[12] and Zweig[3] provide several examples of this result, and the implications in DQE terms for multistage processes have been discussed by Shaw.[13]

The NPS, which, for a two-dimensional spatial process, we denote as the Wiener spectrum, is meaningful when the process is stationary in the wide sense, that is, when $\bar{x}_i = \bar{x}$, for all $i$ and $\operatorname{cov}(x_i, x_j) = R_x^{i-j}$. Equation (14) may then be written as

$$R_y^{i-j} = \bar{m}^2 R_x^{i-j} + \sigma_m^2 \bar{x} \delta_{ij}. \tag{16}$$

The Fourier transform of $R_y^{i-j}$, taken in the limit of arbitrary small pixel size, is shown in Appendix A to yield

$$S_{yy}(\omega) = \bar{m}^2 S_{xx}(\omega) + \sigma_m^2 \bar{x}, \tag{17}$$

where $\bar{x}$ is now the number of input quanta per unit area, and $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are the input and output NPS, respectively. A block diagram representation of Eq. (17) is shown in Fig. 2(b).

### Scattering Process

We now consider an elementary scattering process, as illustrated in Fig. 3. The random variables $\{x_i\}$ denote the number of input quanta in each pixel, and $\{y_i\}$ denotes the number of output quanta in each pixel after scattering. The stochastic scattering process is defined in terms of the sets of transition probabilities $\{p_j^{(i)}\}$, describing the probability that a quantum in pixel $i$ will be scattered to pixel $j$. Conservation of quanta demands that

$$\sum_j p_j^{(i)} = 1 \text{ for all } i.$$

As before, we define $X(\mathbf{s})$ and $Y(\mathbf{s})$ as the MMGF associated with the random variables $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, respectively. Our aim is first to express $Y(\mathbf{s})$ in terms of $X(\mathbf{s})$ and $p_j^{(i)}$ and then to find the statistics of $\{y_i\}$.

The distribution of $\{y_i\}$, given one quantum at pixel $i$ as the only input, is multinomial with a MMGF given by $\sum_j p_j^{(i)} s_j$, for which we use the conventional notation $(\mathbf{p}^{(i)} \cdot \mathbf{s})$. Assuming that all the scattering events are independent, the output MMGF, given $k_i$ input quanta at pixel $i$, is $(\mathbf{p}^{(i)} \cdot \mathbf{s})^{k_i}$. Similarly, the output MMGF, when there are $k_1$ input quanta incident at pixel 1, $k_2$ quanta incident at pixel 2, ... etc., is

$$Y(\mathbf{s}|x_1 = k_1, \ldots, x_n = k_n) = (\mathbf{p}^{(1)} \cdot \mathbf{s})^{k_1} \ldots (\mathbf{p}^{(n)} \cdot \mathbf{s})^{k_n}. \tag{18}$$

Since

$$Y(\mathbf{s}) = \sum_{k_1} \ldots \sum_{k_n} Y(\mathbf{s}|x_1 = k_1, \ldots, x_n = k_n) \times \Pr(x_1 = k_1, \ldots, x_n = k_n), \tag{19}$$

we obtain

$$Y(\mathbf{s}) = X[(\mathbf{p}^{(1)} \cdot \mathbf{s}), (\mathbf{p}^{(2)} \cdot \mathbf{s}), \ldots, (\mathbf{p}^{(n)} \cdot \mathbf{s})], \tag{20}$$

which is the general expression that we are seeking. Figure 4(a) is the block-diagram representation of Eq. (20).

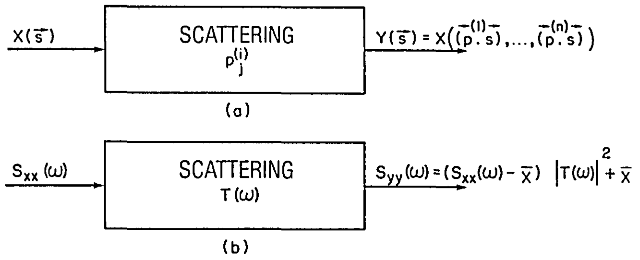


Fig. 4. Block diagram showing the input–output relationships for the scattering process; (a) input–output MMGF and (b) input–output NPS.

We now find the mean, the variance, and the covariance of $\{y_i\}$. We assume that the scattering process is space invariant, i.e., $p_j^{(i)} = p_{i-j}$, and we shall refer to the set of transition probabilities $\{p_k\}$ as the point-spread function (PSF) of the stochastic scattering process. It is easily shown that

$$\left.\frac{\partial(\mathbf{p}^{(j)}\cdot \mathbf{s})}{\partial s_i}\right|_{s=1} = p_{i-j}. \tag{21}$$

Using Eqs. (9), (10), and (21) and the chain rule for differentiation, we get

$$\bar{y}_i = \sum_j \bar{x}_j p_{i-j}, \tag{22}$$

$$\mathrm{cov}(y_i, y_j) = \sum_l \sum_k \mathrm{cov}(x_l, x_k) p_{i-l} p_{j-k} - \sum_k \bar{x}_k p_{i-k} p_{j-k} + \delta_{ij} \sum_k \bar{x}_k p_{i-k}, \tag{23}$$

$$\sigma_{y_i}{}^2 = \mathrm{cov}(y_i, y_i) = \sum_l \sum_k \mathrm{cov}(x_l, x_k) p_{i-l} p_{j-k} + \sum_k \bar{x}_k p_{i-k}(1 - p_{i-k}). \tag{24}$$

Note that the output statistics would be different if the scattering process were modeled as deterministic (i.e., if $p_{i-j}$ denoted the portion of input quanta at pixel $i$ that scattered to pixel $j$). Equation (22) would remain the same, but Eq. (24) would then be

$$\sigma_{y_i}{}^{2'} = \sum_l \sum_k \mathrm{cov}(x_l, x_k) p_{i-l} p_{i-k}, \tag{25}$$

which lacks the second term, $\sum_k \bar{x}_k p_{i-k}(1 - p_{i-k})$. Since $0 \leq p_k < 1$ for all $k$, this term is always positive, which implies that the stochastic scattering will always result in a larger variance.

For a wide-sense-stationary input, Eq. (23) may be rewritten as

$$R_y{}^{i-j} = \sum_l \sum_k R_x{}^{l-k} p_{i-l} p_{j-k} - x \sum_k p_{i-k} p_{j-k} + \bar{x}\delta_{ij}. \tag{26}$$

The output NPS is found as the Fourier transform of $R_y{}^{i-j}$, taken in the limit of arbitrary small pixel size, and is shown in Appendix A to be

$$S_{yy}(\omega) = (S_{xx}(\omega) - \bar{x})|T(\omega)|^2 + \bar{x}, \tag{27}$$

where $T(\omega)$ is the modulation transfer function (MTF) of the scattering process, given by the Fourier transform of the scattering PSF,

$$T(\omega) = \int p(\xi)\exp(-2\pi i\omega\xi)\mathrm{d}\xi. \tag{28}$$

Figure 4(b) is the block-diagram representation of Eq. (27). Equation (27) demonstrates that, in effect, the correlated component $[S_{xx}(\omega) - \bar{x}]$ of the input noise passes through the transfer function but the uncorrelated component does not. This has important implications for the use of the NPS as input (target spectra) for the measurement of MTF, since it shows that the output noise of a stochastic scattering process is not, in general, simply the product of the input noise and the MTF (as would be the case with deterministic scattering). Only in the limit $S_{xx}(\omega) \gg \bar{x}$ is there a simple input/output noise ratio that determines the MTF. In the limit when the input is purely uncorrelated Poisson noise, $S_{xx}(\omega) = \bar{x}$, the output noise becomes independent of the transfer function

$$S_{yy}(\omega) = \bar{x}.$$

This is expected, since the stochastic scattering of uncorrelated quanta does not affect their statistics.

### Multistage Processes

Equations (8) and (20) form the basis of the influence of stochastic amplifying and scattering mechanisms analysis on the transfer of signal statistics. They may be applied to systems of arbitrary complexity by cascading them any number of times, with the output of each stage forming the input to the next. For wide-sense-stationary input, Eqs. (17) and (27) describe the propagation of the Wiener spectrum through various stages of amplification and scattering. Each stage need only be described in terms of its scattering process $[T(\omega)]$ and the mean and variance of its amplification process ($\bar{m}$ and $\sigma_m{}^2$) to specify the ouput NPS completely.

We now demonstrate the general utility of the above relationships by considering an example. An important case for radiographic imaging is one in which primary quanta are first amplified and then scattered, as shown in Fig. 5. This is a reasonable description of events in a screen–film system,[7,8] which is the problem that precipitated the present analysis. To find the output NPS $S_{zz}(\omega)$, we cascade Eqs. (17) and (27):

$$S_{zz}(\omega) = [\bar{m}^2 S_{xx}(\omega) + \bar{x}\sigma_m{}^2 - \bar{m}\bar{x}]|T(\omega)|^2 + \bar{m}\bar{x}. \tag{29}$$

This expression is equivalent, for the case of uncorrelated Poisson input noise [i.e., $S_{xx}(\omega) = \bar{x}$], to the results obtained by Fu,[9] Fu and Roehrig,[10] and Kemperman and Trabka.[6] We note that when amplification and scattering both occur in an imaging system, the order in which they act is important. For instance, in our example, if the scattering preced-

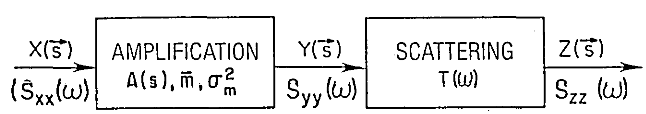


Fig. 5. Block diagram representing amplification followed by scattering.

ed the amplification, the expression for the output NPS [Eq. (29)] would be given by

$$S_{zz}(\omega) = \bar{m}^2[S_{xx}(\omega) - \bar{x}]|T(\omega)|^2 + \bar{x}(\bar{m}^2 + \sigma_m^{\ 2}). \quad (30)$$

The spectra defined in Eqs. (29) and (30) have approximately the same value at low spatial frequencies but generally approach different limits at higher frequencies. The quantity $(\bar{m}^2 + \sigma_m^{\ 2})\bar{x}$ need not necessarily be larger than $\bar{m}\bar{x}$, but it will be larger whenever $\bar{m} > 1$. When the input is uncorrelated Poisson noise, and when scattering takes place before amplification, the output NPS is independent of spatial frequency, as discussed before. To find the output MMGF $Z(\mathbf{s})$, which is a complete statistical description of $\{z_i\}$, we cascade Eqs. (8) and (20):

$$Z(\mathbf{s}) = X[A(\mathbf{p}^{(1)} \cdot \mathbf{s}), A(\mathbf{p}^{(2)} \cdot \mathbf{s}), \ldots, A(\mathbf{p}^{(n)} \cdot \mathbf{s})]. \quad (31)$$

From Eq. (31), the mean, the covariance, and the variance of $z_i$, after some algebraic manipulations, are given by

$$\bar{z}_i = \bar{m} \sum_k \bar{x}_k p_{i-k}, \quad (32)$$

$$\operatorname{cov}(z_i, z_j) = \bar{m}^2 \sum_k \sum_l \operatorname{cov}(x_k, x_l) p_{i-k} p_{j-l} + \sum_k \bar{x}_k p_{i-k}[(\sigma_m^{\ 2} - \bar{m})p_{j-k} + \bar{m}\delta_{ij}], \quad (33)$$

and

$$\sigma_{z_i}^{\ 2} = \bar{m}^2 \sum_k \sum_l \operatorname{cov}(x_k, x_l) p_{i-k} p_{i-l} + \sum_k \bar{x}_k p_{i-k}[(\sigma_m^{\ 2} - \bar{m})p_{i-k} + \bar{m}]. \quad (34)$$

Higher-order moments also follow from Eq. (31).

## DETECTIVE QUANTUM EFFICIENCY

We now examine the effects of stochastic amplification and scattering on the DQE of imaging systems. The effect of stochastic amplification alone, in single-channel detectors, has been discussed by Zweig,[3] and the implications for the conversion of x-ray quanta to light were discussed by Swank[14] and later by Dick and Motz.[15]

DQE can be calculated from the ratio of input noise (which is taken to be uncorrelated Poisson noise associated with input quantum fluctuations) to output noise when these have been cast in common units by means of the system transfer function.[16] This definition of DQE was generalized by Shaw[17] to include the exposure and spatial-frequency dependence needed for photographic imaging systems. The working equation may be expressed as

$$\mathrm{DQE}(q, \omega) = g^2(q, \omega) \frac{q}{S(q, \omega)}, \quad (35)$$

where $q$ is the incident number of quanta per unit area, $g(q, \omega)$ is the transfer function of the imaging system, and $S(q, \omega)$ is the NPS of the output.

### Amplification

In Eq. (17) we have shown that stochastic amplification increases the noise associated with the quanta detected by an imaging system in two ways. First, the quantum noise is itself amplified, and second, noise associated with the stochastic nature of the amplifying mechanism is introduced into the output. The resulting DQE is given by

$$\mathrm{DQE} = \frac{1}{1 + \dfrac{\sigma_m^{\ 2}}{\bar{m}^2}}, \quad (36)$$

where $\sigma_m^{\ 2}$ is the variance in the amplification process and $\bar{m}$ is the mean amplification. This result, which is independent of exposure level or spatial frequency, is in agreement with previous work on simple detectors as reported by Zweig.[3] A noiseless amplification process, where $\sigma_m^{\ 2} = 0$, has unit DQE. The magnitude of the effect of added noise on DQE varies with its coefficient of variation $\sigma_m/\bar{m}$.

### Scattering

Stochastic scattering reduces the transfer function (and hence the output signal) and the noise (as a function of spatial frequency) in an imaging system. However, the reduction of noise is always less than proportional to the reduction in signal. Therefore, scattering processes will, in general, decrease the DQE of an imaging system. The DQE, which follows from the noise analysis for stochastic scattering, is given by

$$\mathrm{DQE}(\omega) = |T(\omega)|^2, \quad (37)$$

where $T(\omega)$ is the MTF of the scattering process. This result follows from the observation that scattering of the primary quanta in an imaging system reduces the system-transfer function but does not decrease the output NPS.

### Amplification and Scattering of Incident Quanta

Many imaging systems involve amplification and scattering of the incident quanta. When amplification is followed by scattering, the DQE, which follows from our NPS analysis in Eq. (29), can be expressed as

$$\mathrm{DQE}(\omega) = \frac{1}{1 + \dfrac{\epsilon}{\bar{m}} + \dfrac{1}{\bar{m}|T(\omega)|^2}}, \quad (38)$$

where

$$\epsilon = \frac{\sigma_m^{\ 2}}{\bar{m}} - 1 \quad (39)$$

is the Poisson excess.

Comparison of this expression for DQE with the preceding DQE results shows that they are obtained in the appropriate limits; that is, when $|T(\omega)|$ is unity for all $\omega$, the result for amplification alone is recovered, and when $\bar{m} = 1$ and $\sigma_m^{\ 2} = 0$, the result for scattering is obtained. Note, however, that the DQE for amplification followed by scattering cannot be readily deduced from the separate results, nor do the effects of scattering and amplification commute. It should be clear that scattering before amplification will, in general, have a more adverse effect on system DQE than scattering that follows amplification, when amplification is by a factor in excess of unity.

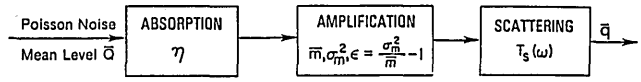


Fig. 6. Schematic diagram of a model for imaging by a radiographic screen–film system. X-ray quanta are absorbed with efficiency $\eta$; these are stochastically amplified within the phosphor screen to produce an average of $\bar{m}$ light quanta for each absorbed x-ray quantum. These light quanta are scattered within the screen–film system before exposing the film.

### Radiographic Imaging

A particular imaging system of practical interest to which this analysis has been applied is screen–film radiography.[7,8,18–20] In these systems, shown schematically in Fig. 6, x-ray quanta are incident upon a fluorescent phosphor. A fraction $\eta$ of the incident x-ray quanta are absorbed. This can be treated within our theoretical framework as a stochastic amplification process in which

$$\begin{aligned} a(0) &= 1 - \eta, \\ a(1) &= \eta, \\ a(i) &= 0 \qquad \text{for } i \neq 0 \text{ or } 1. \end{aligned}$$

The energy carried by each absorbed x ray is partially converted into visible light quanta, which are emitted by the phosphor. This may be described by a stochastic amplification process of mean $\bar{m}$ and variance $\sigma_m^2$ that is governed by the intrinsic efficiency of the phosphor particles as well as the optical characteristics of the phosphor coating. Finally, during emission, the light quanta are scattered within the screen–film system. This scattering process is described by $T_s(\omega)$. The NPS of the light image, which exposes the photographic film in these systems, can be expressed as

$$S(\bar{Q}, \omega) = \left[\eta \bar{m}^2 \bar{Q}\left(1 + \frac{\epsilon}{\bar{m}}\right)\right] |T_s(\omega)|^2 + \eta \bar{m} \bar{Q}, \tag{40}$$

where $\bar{Q}$ is the number of incident x-ray quanta per unit area. The DQE is therefore given by

$$\mathrm{DQE}(\bar{Q}, \omega) = \frac{\eta}{1 + \dfrac{\epsilon}{\bar{m}} + \dfrac{1}{\bar{m}|T_s(\omega)|^2}}. \tag{41}$$

## CONCLUSIONS

A general theoretical framework has been given for analyzing the influence of stochastic amplifying and scattering mechanisms on the transfer of image statistics through multistage imaging systems. MMGF's have been used to describe the output statistics at each stage. This theoretical development is general in that it can be applied to stationary as well as nonstationary processes. In the special case of stationary processes, relationships between input and output NPS have been described. For photon-limited inputs, these considerations lead to useful expressions for the DQE. In this case, stochastic amplification of photon noise by one stage of an imaging system is shown to constitute an effective signal to the next, while the underlying photon-noise component is unaffected by a subsequent scattering process.

## APPENDIX A

As the pixel size becomes infinitesimally small, Eq. (16) becomes

$$R_y(u) = \bar{m}^2 R_x(u) + \sigma_m^2 \bar{x} \delta(u), \tag{A1}$$

where $R_y$ is a function of only one argument, $u$, which denotes the location difference, and $\delta(u)$ is the impulse function.

The power spectrum $S_{yy}(\omega)$ is defined as

$$S_{yy}(\omega) = \int R_y(\xi) \exp(-i2\pi\omega\xi) \mathrm{d}\xi. \tag{A2}$$

Taking the Fourier transform of both sides of Eq. (A1) results in the desired expression.

Applying a similar approach to Eq. (26), and noting that summations will be replaced by integrals, after a change of variables we get

$$\begin{aligned} R_y(u) = &\int\int R_x(u_1 - u_2) p(u_2) p(u_1 - u) \mathrm{d}u_1 \mathrm{d}u_2 \\ &- \bar{x} \int p(u_1) p(u_1 - u) \mathrm{d}u_1 + \bar{x}\delta(u). \end{aligned} \tag{A3}$$

Taking the Fourier transform of both sides of Eq. (A3) results in

$$\begin{aligned} S_{yy}(\omega) &= \int\int\int R_x(u_1 - u_2) p(u_2) p(u_1 - u) \exp(-i2\pi\omega u) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u \\ &\quad - \bar{x} \int p(u_1) p(u_1 - u) \exp(-i2\pi\omega u) \mathrm{d}u_1 \mathrm{d}u + \bar{x}, \end{aligned} \tag{A4}$$

and it follows that (* denotes a complex conjugate)

$$\begin{aligned} S_{yy}(\omega) = T^*(\omega) &\int\int R_x(u_1 - u_2) p(u_2) \exp(-i2\pi\omega u_1) \mathrm{d}u_1 \mathrm{d}u_2 \\ &- \bar{x} T^*(\omega) \int p(u_1) \exp(-i2\pi\omega u_1) \mathrm{d}u_1 + \bar{x} \end{aligned} \tag{A5}$$

and

$$S_{yy}(\omega) = [S_{xx}(\omega) - \bar{x}] |T(\omega)|^2 + \bar{x}, \tag{A6}$$

which is the desired result.

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