3. Justificati egalitatea

$$\frac{1}{(1+x^2)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n}, \quad \forall x \in I \subseteq \mathbb{R}$$

si determinati multimea de convergenta I a seriei din membrul drept.

$\sum_{n=0}^{\infty} \underbrace{(-1)^n (n+1)}_{a_n} x^{2n}$ → coeficient x^{2n}

Seria de puteri $\sum_{n=0}^{\infty} \boxed{P_n}(x-x_0)^n$ funcție de la x^n

1) calculăm raza de convergență

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

2) Mulțimea de convergență

$(x_0 - r ; x_0 + r)$

$\frac{1}{(x^2+1)^2}$ = Seria constată

$\pm\infty$

$x^2+1=0$

$x^2=-1$

$x=\pm i$

pentru $x=\pm i$ seria nu e convergentă

dar cum $x \in I \subseteq \mathbb{R} \to \forall x \in \mathbb{R}$. Seria e convergentă $\forall x \in \forall I \subseteq \mathbb{R} \ldots$

$\boxed{I=\mathbb{R}}$

3. Determinati punctele critice si punctele de extrem local (specificand tipul acestora) pentru functia

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(\underline{x}, \underline{y}) = (x + xy + y^2)\sqrt{e^x}$$

$\nabla f = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\}$

$\left(e^{u(x)}\right)' = u(x)' e^{u(x)}$

$\left(e^{\frac{x}{2}}\right)' = \frac{1}{2} e^{\frac{x}{2}}$

$\frac{\partial f}{\partial x} = (1+y)\sqrt{e^x} + \frac{1}{2}(x + xy + y^2) e^{\frac{x}{2}}$

$= \sqrt{e^x}\left(1 + y + \frac{x}{2} + \frac{xy}{2} + \frac{y^2}{2}\right)$

$\sqrt{e^x}\left(\underbrace{1+y}_{} + \frac{x}{2}\underbrace{(1+y)}_{} + \frac{y^2}{2}\right)$

$\sqrt{e^x}\left((1+y)(1+\frac{x}{2}) + \frac{y^2}{2}\right)$

$\frac{\partial f}{\partial y} = (x+2y)\sqrt{e^x}$

$\nabla f = \{0\}$

$\sqrt{e^x}\left((1+y)(1+\frac{x}{2}) + \frac{y^2}{2}\right) = 0$

$\sqrt{e^x}(x+2y) = 0$

$\begin{cases} (1+y)(1+\frac{x}{2}) + \frac{y^2}{2} = 0 \\ x+2y = 0 \end{cases} \Rightarrow x = -2y$

$(1+y)(1-y) + \frac{y^2}{2} = 0$

$1 - y^2 + \frac{y^2}{2} = 0$

$1 - \frac{y^2}{2} = 0 \Rightarrow y^2 = 2$

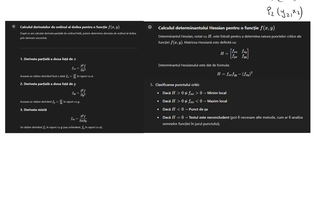
$y = \pm\sqrt{2}$

$y_1 = \sqrt{2}$ $x_1 = -2\sqrt{2}$

$y_2 = -\sqrt{2}$ $x_2 = +2\sqrt{2}$

$P_1(y_1, x_1)$

$P_2(y_2, x_2)$



$\frac{\partial^2 f}{\partial x^2} = \left(\sqrt{e^x}\left((1+y)(1+\frac{x}{2}) + \frac{y^2}{2}\right)\right)'$

$\frac{1}{2}\sqrt{e^x}\left((1+y)(1+\frac{x}{2}) + \frac{y^2}{2}\right) + \sqrt{e^x}\left(\frac{1}{2} + \frac{y}{2}\right)$

$\frac{\partial^2 f}{\partial y^2} = \left((x+2y)\sqrt{e^x}\right)'$

$(x\sqrt{e^x} + 2y\sqrt{e^x})' = 2\sqrt{e^x}$

$= \sqrt{e^x}\left(\frac{1}{2}(1+y)(1+\frac{x}{2}) + \frac{y^2}{2} + \frac{1}{2}(1+y) + \frac{y}{2}\right)$

$= \sqrt{e^x}\left(\frac{1}{2}(1+y)\left(1+\frac{x}{2}+1\right) + \frac{y^2}{2}\right)$

$= \frac{1}{2}\sqrt{e^x}\left((1+y)(2+\frac{x}{2}) + \frac{y^2}{2}\right)$

$= \frac{1}{2}\sqrt{e^x}(x+2y) + \sqrt{e^x}$

$= \sqrt{e^x}\left(\frac{x}{2} + 2y + 1\right)$

$= \frac{1}{2} e^{\sqrt{2}}\left((1-\sqrt{2})(2+\sqrt{2}) + 1\right) = \frac{1}{2} e^{\sqrt{2}}\left(\sqrt{2}(1-\sqrt{2})(1+\sqrt{2}) + 1\right)$

$\frac{1}{2} e^{\sqrt{2}}\left(1 - \sqrt{2}\right)$

$\sqrt{e^{-\sqrt{2}}} = e^{-\sqrt{2}}$

$P_1(-2\sqrt{2}, \sqrt{2})$

$P_2(2\sqrt{2}, -\sqrt{2})$

$$H(f) = \begin{pmatrix} \frac{1}{2}\sqrt{e^{-\sqrt{2}}}\left((1+\sqrt{2})(2-\sqrt{2}) + 1\right) & \sqrt{e^{-2\sqrt{2}}}\left(-2\sqrt{2} + \sqrt{2} + 1\right) \\ \sqrt{e^{-2\sqrt{2}}}\left(-2\sqrt{2} + \sqrt{2} + 1\right) & 2\sqrt{e^{-2\sqrt{2}}} \end{pmatrix}$$

$(\sqrt{2}+1)(\sqrt{2}-1) = 2-1$

$$H(f) = \begin{pmatrix} \frac{1}{2} e^{-\sqrt{2}}\left(\sqrt{2}(1+\sqrt{2})(\sqrt{2}-1) + 1\right) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2e^{-\sqrt{2}} \end{pmatrix}$$

$$H(f) = \begin{pmatrix} \frac{1}{2} e^{-\sqrt{2}}(\sqrt{2}+1) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2e^{-\sqrt{2}} \end{pmatrix}$$

$|H(f)| = \frac{1}{2} e^{-\sqrt{2}}(\sqrt{2}+1) \cdot 2e^{-\sqrt{2}} - \left(e^{-\sqrt{2}}\right)^2$

$\Delta_2 = |H(f)| = e^{-2\sqrt{2}}(\sqrt{2}+1) - e^{-2\sqrt{2}}$

$e^{-2\sqrt{2}}\sqrt{2}$

$\Delta_1 > 0$, $\Delta_2 < 0$ } definita e negativ definită
este un punct de maxim

$\Delta_1 > 0$, $\Delta_2 > 0$ } def e pozitiv definită
este un punct de minim

$y_2 = -\sqrt{2}$ $x_2 = +2\sqrt{2}$

$d^2 f(x,y)(u_1, u_2) = \frac{\partial^2 f}{\partial x^2}(x,y) \cdot \boxed{u_1^2} + \frac{\partial^2 f}{\partial y^2}(x,y) u_2^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x,y) u_1 u_2$

$(2\sqrt{2}, -\sqrt{2})$ $d^2 f(x,y)(u_1, u_2) = \frac{1}{2} e^{\sqrt{2}}(1-\sqrt{2}) u_1^2 + 2e^{\sqrt{2}} u_2^2 + 2e^{\sqrt{2}}(\sqrt{2}(-\sqrt{2}+1)) u_1 u_2$

$\frac{1}{2} e^{\sqrt{2}}(1-\sqrt{2}) u_1^2 + 2e^{\sqrt{2}} u_2^2 + 2e^{\sqrt{2}} u_1 u_2$

$2e^{\sqrt{2}}\left(u_1^2 \frac{(1-\sqrt{2})}{4} + u_2^2 + u_1 u_2\right)$

$u = (u_1^1, u_2^1)$ a.î. $d^2 f(u) > 0$

$v = (u_1^2, u_2^2)$ a.î. $d^2 f(v) < 0$ } diff e indefinită

$(u_1 = 0 \quad u_2 = 1)$

$d^2 f(0,1) = 2e^{\sqrt{2}}(1) > 0$

$(u_1 = 1 \quad u_2 = 0)$ $1-\sqrt{2} < 0$ $\sqrt{2} = 1{,}73$, $\sqrt{2} = 1{,}41$

$d^2 f(1,0) = 2e^{\sqrt{2}}\left(\frac{1-\sqrt{2}}{4}\right) < 0$

diff e indefinită

$\Rightarrow P_2(2\sqrt{2}, -\sqrt{2}) \to$ punct șa

1) gradientul funcției

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ → vector, nu multime

2) egalăm gradientul cu 0

3) Matricea Hessiană

4) Δ_1 Δ_2

$\Delta_1 > 0$, $\Delta_2 < 0$ neg definită maxim

$\Delta_1 > 0$, $\Delta_2 > 0$ poz definită minim

$\Delta_1 < 0$, $\Delta_2 < 0$ indef (treci la forci $d^2 f(x,y)(u_1, u_2)$)

5) $d^2 f(x,y)(u_1, u_2)$

$u = (u_1^1, u_2^1)$ – alege autor

$v = (u_1^2, u_2^2)$

dacă $\exists\, u, v$ a.î. $d^2 f(u) \leq 0$, $d^2 f(v) > 0$ (sau invers) $d^2 f$ indefinită

$\Rightarrow P(x,y)$ punct șa

$d^2 f(u) > 0$ $P(x,y)$ punct de minim $\forall u \in \mathbb{R}^2$

$d^2 (f(u) < 0$ $P(x,y)$ pct maxim $\forall u \in \mathbb{R}^2$

f:[-1,1] -> R, f(x,y) = (x+y)/rad(x^2 + y^2) daca (x,y) diferit de O2 sau f(x,y) = 0 daca (x,y) egal O2.
a) studiati convergenta in (0,0) justificati.
b) calculati derivatele partiale in (0,0).
c) care este inf f(A) si sup f(A)

$f(0,0) -$

$$f(x,y) = \begin{cases} \frac{x+y}{\sqrt{x^2+y^2}} & (x,y) \neq O_2 \\ 0 & (x,y) = O_2 \end{cases}$$

b) Calculul derivatelor parțiale în $(0, 0)$

Derivata parțială în raport cu x:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h}{|h|} - 0}{h} = \lim_{h\to 0} \frac{h}{|h|h}.$$

Pentru $h > 0$, fracția este $1/h$ (tinde la $+\infty$), iar pentru $h < 0$, fracția este $-1/h$ (tinde la $-\infty$).

Deci derivata **nu există**.

Similar, pentru derivata parțială în raport cu y:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h}{|h|} - 0}{h} = \lim_{h\to 0} \frac{h}{|h|h}.$$

Aceasta are acelaşi comportament, deci şi derivata parţială în raport cu y **nu există**.

$\frac{x^2+y^2+2xy}{x^2+y^2} = 1 + \frac{2xy}{x^2+y^2}$

$\sqrt{\frac{(x+y)^2}{x^2+y^2}} = \sqrt{1 + \frac{2xy}{x^2+y^2}} = \sqrt{1 + \frac{2}{\frac{y}{x} + \frac{x}{y}}} > \sqrt{2}$

$\frac{x}{y} + \frac{y}{x} \geq 2$ $\forall x,y \in [0,1]$

$\frac{x}{y} + \frac{y}{x} \leq -2$ $\forall x,y \in [-1,0)$

$a_n = (n, n)$ $a = \frac{x}{y}$ $\frac{x}{y} + \frac{y}{x} \geq 2$

$b_n = (n, n)$ $a + \frac{1}{a} \geq 2$ $\forall a \in \mathbb{R}_+$

$\lim a_n = \lim b_n$

$\lim f(a_n) \neq \lim f(b_n)$ } $\Rightarrow \nexists \lim f$

$\sqrt{1 + \frac{2}{\frac{x}{y} + \frac{y}{x}}} \leq \sqrt{2}$

$\sqrt{1 + \frac{2}{\frac{x}{y} + \frac{y}{x}}} \geq 0$

$0 \leq f(x,y) \leq \sqrt{2}$ $x,y \in (0,1]$

$f(x,y) \geq 0$ $x,y \in [-1,0)$

$f(0,0) - 0$

$\lim_{(x,y)\to O_2} \frac{x+y}{\sqrt{x^2+y^2}}$

$x = x$, $y = 0$: $\lim_{(x,0)\to O_2} \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \pm 1$

$\Rightarrow$ nu există limita pe $y = 0$

$x = 0$, $y = y$: $\frac{y}{|y|} = \pm 1$

$\Rightarrow$ nu există

c) Calculul $\inf f(A)$ și $\sup f(A)$

Observăm că:

$$f(x, y) = \frac{x + y}{\sqrt{x^2 + y^2}}$$

Rescriem în coordonate polare, unde $x = r\cos\theta$ și $y = r\sin\theta$:

$$f(r, \theta) = \frac{r\cos\theta + r\sin\theta}{\sqrt{r^2}} = \frac{r(\cos\theta + \sin\theta)}{r} = \cos\theta + \sin\theta.$$

Astfel, valorile lui $f(x, y)$ depind doar de θ:

$$g(\theta) = \cos\theta + \sin\theta.$$

Determinăm valorile extreme ale lui $g(\theta)$. Derivăm:

$$g'(\theta) = -\sin\theta + \cos\theta.$$

Egalăm cu zero:

$$\cos\theta - \sin\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} + k\pi.$$

Calculăm valorile funcției:

$$g\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$g\left(\frac{5\pi}{4}\right) = \cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

Astfel:

$$\inf f(A) = -\sqrt{2}, \quad \sup f(A) = \sqrt{2}.$$

$\lim_{h\to 0} \frac{f(\theta + h) - f(\theta)}{h}$

$\inf f(A) = \min$

$\sup f(A) = \max$

$f(-1,-1) = \frac{-1 + (-1)}{\sqrt{1+1}} = \frac{-2}{\sqrt{2}}$

$a_n = \left(\frac{1}{n}, 0\right)$

$b_n = \left(0, \frac{1}{n}\right)$

$\lim_{n\to\infty} a_n = (0,0)$

$\lim_{n\to\infty} b_n = (0,0)$

$\lim_{n\to\infty} \frac{\frac{1}{n} + 0}{\sqrt{\frac{1}{n^2} + 0}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left|\frac{1}{n}\right|} = \pm 1$

$\lim_{n\to\infty} \frac{0 + \frac{1}{n}}{\sqrt{0 + \frac{1}{n^2}}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left|\frac{1}{n}\right|} = \pm 1$

$\frac{2xy}{x^2+y^2}$

$\sqrt{1 + \frac{2xy}{x^2+y^2}} \leq \frac{x+y}{x+y}\sqrt{2}$

$\leq \frac{x+y}{\sqrt{x^2+y^2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$