Particle Accelerator Simulation - A look at bounded regions of electric field

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In this report we aim to simulate the basic motion of charged particles in magnetic field, experiencing an accelerating electric force. This set-up lies at the heart of particle accelerators. We analyse the program against physical predictions(which?) and discover that for a bunch of 10 protons doing 100 revolutions, the time steps of 10^{-5} and 10^{-9} inside and outside the accelerating region respectively and an electric field phase of $\pi/2$, provide the best parameters for efficient acceleration.

CONTENTS

I. INTRODUCTION AND BACKGROUND THEORY

Functioning particle accelerators, in their simplest representation, can be thought of as a system of charged particles moving in an uniform magnetic field and accelerated at points by one (or more) electric field(s). This sketches the starting point in their simulation, goal on which we focus our efforts in this report.

Ignoring the electric field for the moment, the initial conditions we use are: $B = (0, 0, 1.0 \times 10^{-7})$T and $velocity = (0, 0.1, 0)$m/s. In this case, we expect orbital motion on the x-y plane for the charged particle, with the *radius* of orbit found by equating the centripetal and magnetic (Lorentz) forces:

$$qvB = \frac{mv^2}{r} \Rightarrow r = \frac{mv}{qB}. \tag{1}$$

Interestingly, the angular frequency of the rotation is then independent on velocity (and by extension, energy):

$$\omega = 2\pi f = \frac{2\pi}{\tau} = 2\pi * \frac{v}{2\pi r} = \frac{qB}{m}. \tag{2}$$

It is worth pointing out that this equation ignores relativistic effects, which could be discussed in an extension of this project.

Moving one step forward, we can consider an electric field accelerating the charged particle. The simplest way of representing this is as an infinite strip of width W lying on the x-axis. Since the particle (in our case, a proton), should go around in a circle, we want this electric field to be time-varying, so to push the particle in the right direction as it moves around the orbit. We will also see more physically realistic alternatives to the infinite strip: one or two bounded regions of coherently varying electric field, located at the origin or at the opposite side of the accelerator respectively. In any case, the formula describing the strength of the field is:

$$E = E_0 \sin(\omega t), \tag{3}$$

where E_0 is $1.0 \times 10^{-7} V/m$. We will soon see that introducing a phase term can generate a more efficient acceleration of the particle.

In the next section we will describe the methods that we used for our simulation. Further, in section III we will present and comment different tests that we used to test the accuracy and limitations of the programs. Finally, in section IV we will provide a conclusion for our efforts.

II. METHODS

In order to simulate a bounded region of electric field (either a strip, one or two boxes) we created an additional class, named *LocalisedEField*, which inherits from the *EMField* class. This class has two methods named *getElectric* which override the one in the parent file. These two return the electric field at a given position of a particle, or at a given point in space respectively. The difference between them and the *EMField* methods is that they yield a zero-magnitude electric field for points outside the region defined by the user. These domains are constructed using a so-called *Rectangle* object, that we defined. It is implicitly assumed that the field is contained in a rectangular shape. This is more physically accurate than an infinite strip. In any case, this object can simulate both situations named above (infinite strip and bounded regions). Let us give a brief description of it.

The *Rectangle* object has the following parameters: a length, a width, a position of its centre and a orientation (given by the length unit vector). The main feature of this class is a method that allows the user to determine if a point is within the rectangle. The application of this method is immediately obvious in our case: we can check whether a particle is passing a region of electric field. Thus, the *LocalisedEField* object is constructed using a magnitude, a direction and a bounding region defined by the *Rectangle* object. This allows for an easy extension to more complicated set-ups which require localised magnetic fields for refocusing the beam. We do not consider these here but they could be looked at as a future direction.

As hinted at previously, in this report we explore three types of region configurations. The first one is a strip infinite on the x-axis and bounded by $W = 0.05 * radius$ on the y-axis, where $radius$ is defined in Eq. 1. The second case consists of one rectangle of $length = radius/4$ and $width = W$, centred at the origin, and the third one considers an additional rectangle of the same size, centred at $2.95 * 10^{-2}$. This is the empirical value (obtained in the conditions defined so far) for the radius of the first accelerated orbit (which is different than the non-accelerated version in Eq. 1).

The value for the *length* of the rectangles ($radius/4$) was chosen so the two rectangles do not overlap. It is otherwise a rather random value, and so it could obscure the analysis of how a spread in position of a bunch of protons affects the efficiency of the accelerator. Unfortunately, tests were done using the two-rectangle configuration when a strip was instead intended. Nevertheless, interesting results still came out of this analysis: for example, we managed to simulate a realistic situation where a particle leaves the first accelerating region for ever and get 'stuck' instead to the second region (see Fig. 4).

Before we jump to looking at results we need to talk about how we programmed the equations of motion. We have looked at three algorithms: Euler, Euler Mid-point and Euler-Cromer. These assume the time is broken down in steps, as such:

$$t_n = nD, \tag{4}$$

where t_n is the time at the nth iteration, and D is the time step. We can then represent the position, velocity and acceleration at t_n as x_n, v_n and a_n respectively. In this framework, the algorithms are given as such:

- Euler

$$\begin{aligned} v_{n+1} &= v_n + a_n D \\ x_{n+1} &= x_n + v_n D \end{aligned} \tag{5}$$

- Euler Mid-point

$$\begin{aligned} v_{n+1} &= v_n + a_n D \\ x_{n+1} &= x_n + 1/2(v_n + v_{n+1})D \end{aligned} \tag{6}$$

- Euler-Cromer

$$\begin{aligned} x_{n+1} &= x_n + v_n D \\ v_{n+1} &= v_{n+1} + a_n D \end{aligned} \tag{7}$$

Notice how Euler-Cromer differs from Euler just by the exchange of the two lines. For a more detailed description see Ref. [1].

A main theme of this report is looking at how different time steps affect the simulation. This is especially important for regions of accelerating field, as they result in a smaller density of measurements per unit of space (since the velocity increases rapidly), and thus, physical information about the accelerating field can be effectively 'lost' by a particle who skips the region in just a few steps. To try to go around this issue, in the later pages of this report we looked at decreasing the time step when a particle is inside an accelerating region. To implement this, at each time step, we checked if the Rectangle object (defining the accelerating region) contains a particle. Moreover, we added an extra layer of safety in case the proton does a big first leap inside the accelerating domain. To account for this problem, the particle is effectively taken back in time as soon as it reaches the area of electric field, we then decrease the time step, and re-start the operation.

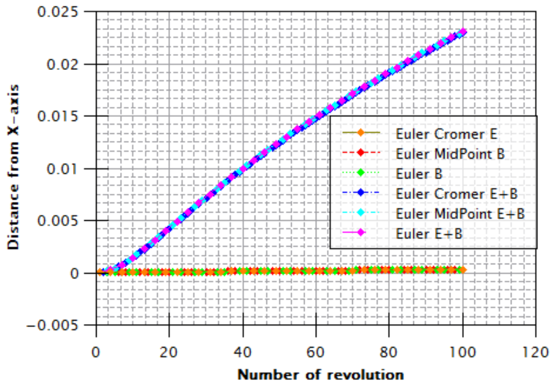
In the next section we will test and comment on the accuracy of the methods presented here.

The values defined so far in the report will be used throughout, unless explicitly stated so.

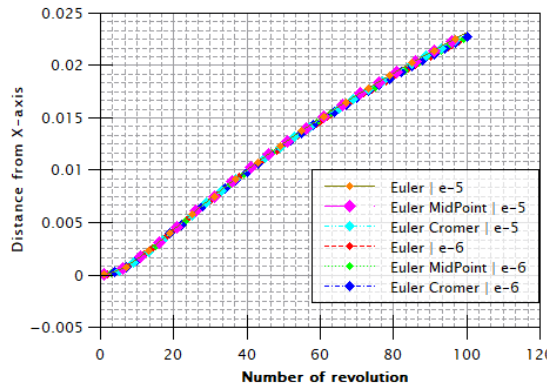
III. RESULTS & DISCUSSION

Let us now look at what happens when we introduce a strip of electric field in the simulation. We took a proton, used a time step of 10^{-5} and performed 100 revolutions. The results are shown in Fig. 9a. This nicely portrays the expanding orbits in the presence of an accelerating field. However, further tests are needed as this is not a particularity of this case, but occurs in the case of magnetic field only as well, for all algorithms, as shown in Fig. 9b. In order to bring out this difference, Fig. 1a shows the three algorithms in two different cases: with and without accelerating electric field. The plot depicts the x-coordinate of the particle when a new orbit is recorded. Thus we have 100 orbits and 100 points. It is evident from this picture that there is a clear distinction between the case of magnetic field only and that including an accelerating electric field. Moreover, from 1b it is apparent that the difference it is not only due to time steps, but to the fact that we have an electric field. Later in the report we look at what happens when we decrease the time step inside the accelerating region relative to that outside of it.

Interestingly, in the case of magnetic field only, the different algorithms agree to machine precision (17 significant figures) on the difference in energy between the final and initial steps of 500 revolutions. Although, it is possible that



(a) The upper curve represents the motion of the particle in the case of both magnetic and electric fields, while the lower curve represents only the magnetic field case. In both scenarios, the time step used is of 10^{-5} seconds.



(b) The main curve is a superposition of six curves representing the motion of a charged particle in magnetic field only, computed for the three algorithms, with two different time steps: 10^{-5} and 10^{-6} seconds.

FIG. 1. Plots of distance from origin on the x-axis for a particle experiencing a uniform magnetic and an accelerating electric fields. The measurements are registered at each completed orbit. The program is run until 100 revolutions are achieved. The equations of motions are implemented in parallel using three different algorithms: Euler, Euler-MidPoint and Euler-Cromer. Details of the parameters and methods are given in Secs. I and II.

a distinction between the three algorithms arises when electric field is introduced. But in this case, conservation of energy is not a good measure of accuracy, since the electric field does work on the particles, and measuring this work proves tricky. Therefore, instead of providing more in depth arguments, we invoke the following plot (Fig. 10), to motivate why we choose to use only Euler-Cromer later in the report. It simply performs better than the other two, when a radial electric field is present.

Henceforth, only the Euler-Cromer algorithm will be used, unless stated otherwise.

A. Addition of Phase

In general, the sinusoidal dependence of the electric field (see Eq. 3) can include a phase term:

$$E = E_0 \sin(\omega t + \phi). \tag{8}$$

This turns out to affect how well the particle is accelerated. It also plays a role in refocusing a bunch of particles in the direction of travel, and raises problems of phase stability. We do not tackle these issues here.

To get an insight into how the phase affects the simulation, we look at the energy after 100 revolutions of a proton, initially at the origin, at nine different values of the phase: in steps of $p/4$ between 0 and 2π. Since the origin is included in the accelerating region, we expect that the most efficient acceleration will be given by an electric field that reaches the maximum value at the start of the simulation. This corresponds to setting $phi = \pi/2$ in Eq. 8. The results are shown in table I. It is worth pointing out that at this point we implemented different time steps for when the particle is inside or outside the accelerating region. This is important as an accelerated motion results in a smaller density of measurements, which can cause issues, as discussed in Sec. II. We explore these in the future pages. At the moment we do not need to worry about them, as the aim of table I is only to provide a comparison between different phases, and the results should still hold for varying the time steps.

From table I it is evident that odd multiples of $\pi/2$ give the highest final energies, and thus electric field varying with this phase give the most efficient acceleration. As discussed previously, this result is physical and expected, since if $\phi = (2n + 1)\pi/2$, the electric field peaks exactly when the particle passes through the accelerating region, thus giving the most efficient acceleration. Future work could be done on looking at a finer spectrum of phase. Although, we do not see any physical or computational reason for why any other value than $\pi/2$ should be the most efficient, so we stop here for now.

Henceforth we will only be using the value of the phase of $\pi/2$, unless stated otherwise.

Phase	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
Total energy	8.57E-29	2.38E-27	3.34E-27	2.37E-27	8.78E-29	2.18E-27	3.28E-27	2.55E-27	8.57E-29

TABLE I. Total energy of an accelerating proton after 100 revolutions as a function of the electric field phase (see Eq. 8). The initial conditions are presented in section I. We use varying time steps: 1.0×10^{-5} outside and 1.0×10^{-6} inside the domains of electric field respectively. The accelerating region is a box centred at origin, of width 0.05×radius in Eq. 1. The algorithm used it Euler-Cromer. The accelerating field varies according to equation 8. It takes the maximum value of 1.0×10^{-7} and is oriented along the y direction.

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B. Electric Field Configurations

As hinted in the Methods section (see II), a more physical way of implementing an electric field is to restrict it to a region of space.

1. One Box

As a first course of action we confined the electric field in a box centred at origin. Details of the parameters are given in Sec. II. The resulting plot is shown in Fig. 2.

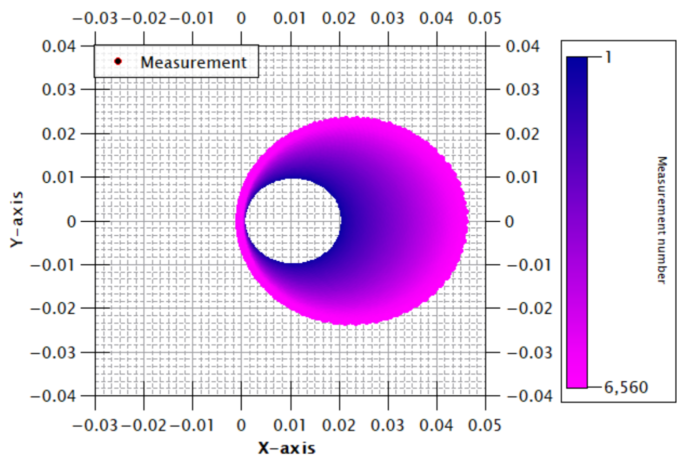


FIG. 2. Real space plot of 100 revolutions of a charged particle experiencing a uniform magnetic field and an electric field centred in a region around the origin, as defined in Sec. II. The colour map indicates the position in time of the particle: dark blue for early in time and magenta for later in time. The time steps used were: 10^{-5} s for outside and 10^{-6} s for inside the region of electric field. The phase of the electric field is 0. Compare with Fig. 3.

From Fig. 2, it is interesting to see how the particle effectively gets 'trapped' by the electric field region, while increasing its radius of orbit in the opposite direction. The apparent wide band of magenta on the left hand side is an artifact of plotting: the later measurements are brought to front while the old ones are left in the back. This is still interesting nevertheless, as in a perfect simulation we would expect the position of the particle when it reaches the left side to not change with time, since there is no accelerating field on the right hand side to increase the radius of the orbit. This suggests that the slight shift of the position of the orbit on the left is a result of computation, and can be attributed to the same phenomenon that increases the radius of the orbit even in the absence of electric field (see Fig. 9b).

In Fig. 3 we plotted the same situation, but for the case where the phase of the electric field is $\pi/2$. We can see that with the same number of revolutions and measurements, the proton reached much longer distances ($2.85 * 10^{-1}$ for $\phi/2$ versus $4.48 * 10^{-2}$ for $\phi = 0$). It serves as a confirmation to our result that the situation with $\pi/2$ provides a more efficient acceleration than $\phi = 0$.

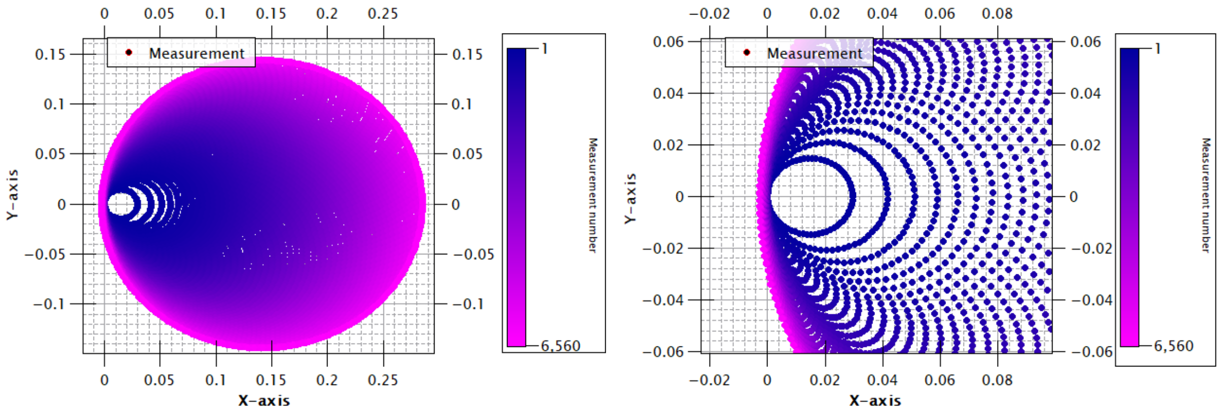


FIG. 3. Real space plot of 100 revolutions of a charged particle experiencing a uniform magnetic field and an electric field centred in a region around the origin, as defined in Sec. II. The colour map indicates the position in time of the particle: dark blue for early in time and magenta for later in time. The time steps used were: 10^{-5} s for outside and 10^{-6} s for inside the region of electric field. The phase of the electric field is $\pi/2$.

2. Two Boxes

As discussed above, we decided to investigate the case of boxes containing mutually coherent electric fields of the same strength. This set-up could be used to simulate a particle accelerator of constant radius with two accelerating regions, or it could also simulate the initial stage of acceleration, where the radius increases dramatically before it stops growing when reaching the outer ring.

Details about the boxes configuration and parameters are given in Sec. II.

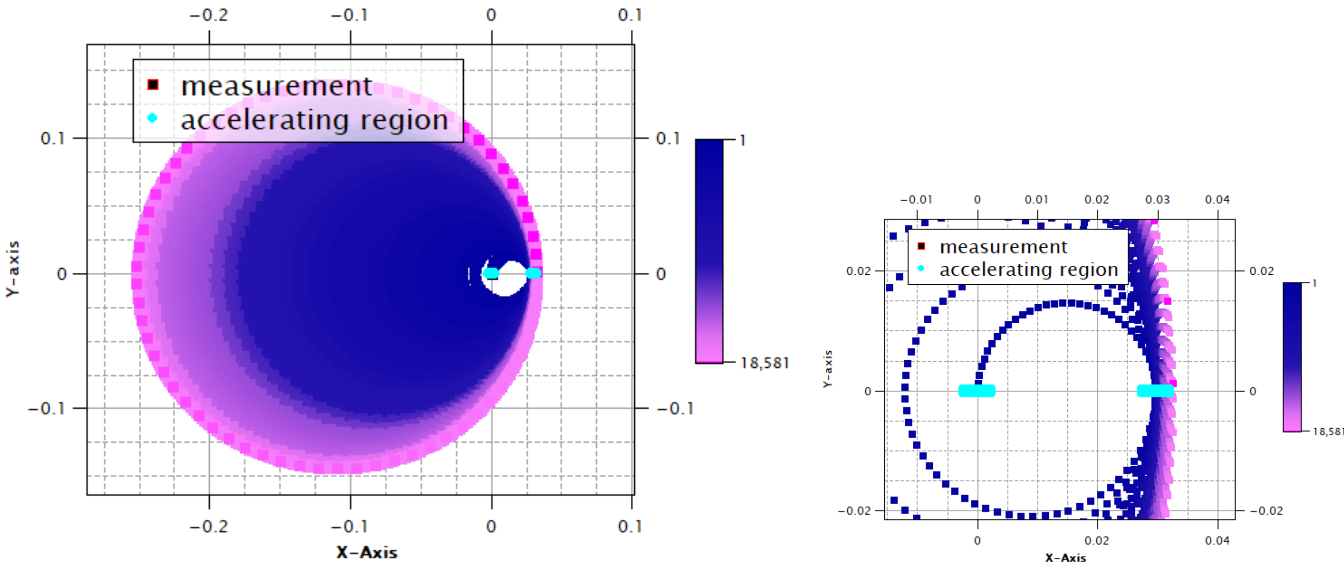
As a first test for this case, we plotted the points inside a box centred on the x-axis, away from the origin. Figure 11 shows such a plot, confirming that the method works as it should.

Next, we centre the box at position $(0, 2.95 * 10^{-2})$ m, as described in Sec. II, and add an additional one centred at the origin. We then proceed to accelerate our proton for 100 revolutions using time steps of 10^{-5} s outside and 10^{-8} s inside the regions of electric field (we used 10^{-8} just to be sure, but it turns out that 10^{-6} gives the same results). The measurements are plotted in Fig. 4. We can see that they are as expected from the discussion of the one box before: the orbit detaches from the first box and gets 'trapped' by the second box. From here, the situation is almost identical to that presented in Fig. 3, but with the orbit increasing in the left rather than right side.

As a further test of the efficiency of the acceleration in this configuration, we plotted the average energy of a bunch of 10 protons performing 100 revolutions. The image is shown in Fig. 5a. We can see that the overall trend is of increasing energy. This is a physical result as particles increase in kinetic energy at each revolution due to the electric field. Nevertheless, we would expect a roughly linear curve as the energy should increase by the same amount each time. The fact that this does not happen seems to be a computational artifact which remains an open question. In Fig. 5b we can see the small flat bands in energy corresponding to when no particle is within the electric field region, and thus the bunch does not change in energy.

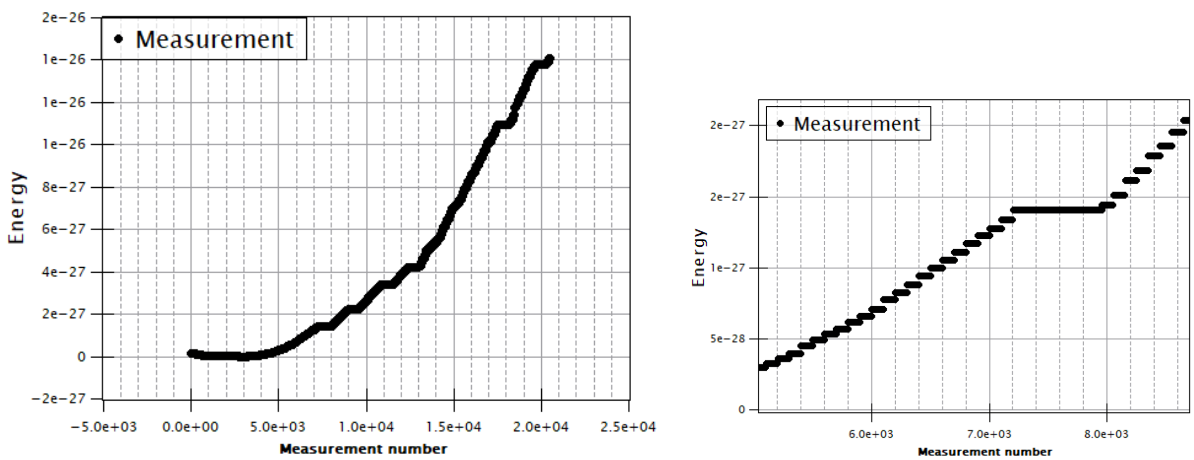
Other interesting features of this plot are: the longer-than-expected flat areas in the energy and the initial decrease. The former is strange since it seems to imply that particles sometimes revolve without gaining energy. The latter suggests that the bunch initially loses energy before gaining it. We have not investigated this further but we think this might be due to having two boxes instead of just one (compare with Fig. 7). Another explanation for these interesting features could be that 10^{-8} s is too big of a time step. Indeed, in the next subsection we will see how decreasing it further, makes some of these strange effects disappear in the case of an infinite strip of electric field.

On another train of thought, a curious phenomenon happens when we try to decelerate the particle by reversing the direction of the electric field in the second region. Fig. 6 shows the result, which is rather surprising. We expected that the orbit would detach from the first box and attach to the second box, where it would continue to grow in the direction right, as the particle reverses velocity of travel. However, by analysing the close up figure on the right, it is apparent that the second region slows down the particle but does not stop it completely. The particle then goes



(b) Close up of the box on the left.

FIG. 4. Real space plot of a particle experiencing a uniform magnetic field and two regions of electric field illustrated by the cyan boxes. The details of parameters are presented in Sec. II. The time step used were 10^{-5} s outside and 10^{-8} s inside the accelerating region. The colour code represents the measurement number which tracks the orbit of the proton: from dark blue to magenta in the order of time. A measurement was taken each 1000 time steps.



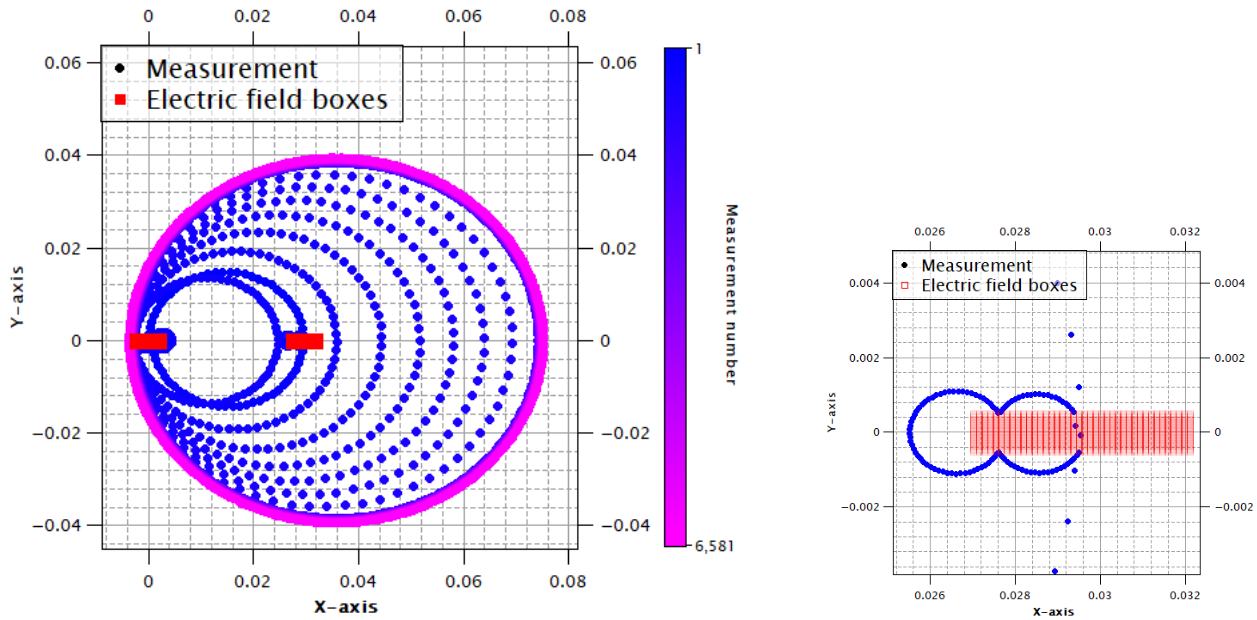
(b) Close up of the box on the left.

FIG. 5. Plot of energy as a function of measurement number (i.e. time) in the case of 10 protons doing 100 revolutions in the configuration of two boxes. The details of parameters are presented in Sec. II. The time step used were 10^{-5} s outside and 10^{-8} s inside the accelerating region. A measurement was taken each 1000 time steps.

ahead and does skipping orbits along the region until the varying electric field matches the direction of the velocity. After this point the particle leaves the domain and does not come back in future iterations.

We are unsure yet what causes this behaviour. We tried decreasing the time step to 10^{-8} inside the area of electric field, but the graph looks the same. A further decrease to 10^{-10} breaks the program, as the usual method of recognition for a completed orbit fails. This constitutes a topic of further research.

In addition, the particle also leaves the first accelerating region at some point and does not accelerate anymore. This can be seen in the fact that the final orbit is stable (sudden change from blue to pink in the outer part of the plot, and no further gradient). This is most likely caused by fact that the box on the right increases the orbit of the proton, which gets 'trapped' at the edge of the box on the left. Further computational errors (slow expansion of orbit



(b) A close up look of the box on the right.

FIG. 6. Real space plot of a particle experiencing a uniform magnetic field and two regions of electric field illustrated by the red boxes. In contrast to the case of Fig. 4, the electric fields are pointing in opposite directions to each other. The details of parameters are presented in Sec. II. The time step used were 10^{-5} s outside and 10^{-6} s inside the accelerating region (but the graph looks the same even for reduced time steps of 10^{-8}). The colour code represents the measurement number which tracks the orbit of the proton: from dark blue to magenta in the order of time. A measurement was taken each 1000 time steps.

due to magnetic field), cause the proton to get completely detached from the boxes, point at which it just does usual orbits in magnetic field.

3. Infinitely Long Strip

We now briefly look at the case of an infinitely long strip of accelerating field with the aim of testing variations in the time step. For this purpose we choose to look at the energy gained as a function of time (i.e. increasing measurement number (not number of measurements!)). The results are shown in Fig. 7.

Fig. 7a depicts energy steps which indicate the points where the particle finds itself in the accelerating region. Outside of these regions the particle does not gain energy and we have flat areas. It is also apparent that steps fall in two main categories: small and large, which indicate that the particle gains more energy during some revolutions than during others. We speculate that step size is the reason for this: if the particle's first step in the accelerating region is closer to the boundary, it is going to experience the electric field for longer, thus gaining more energy.

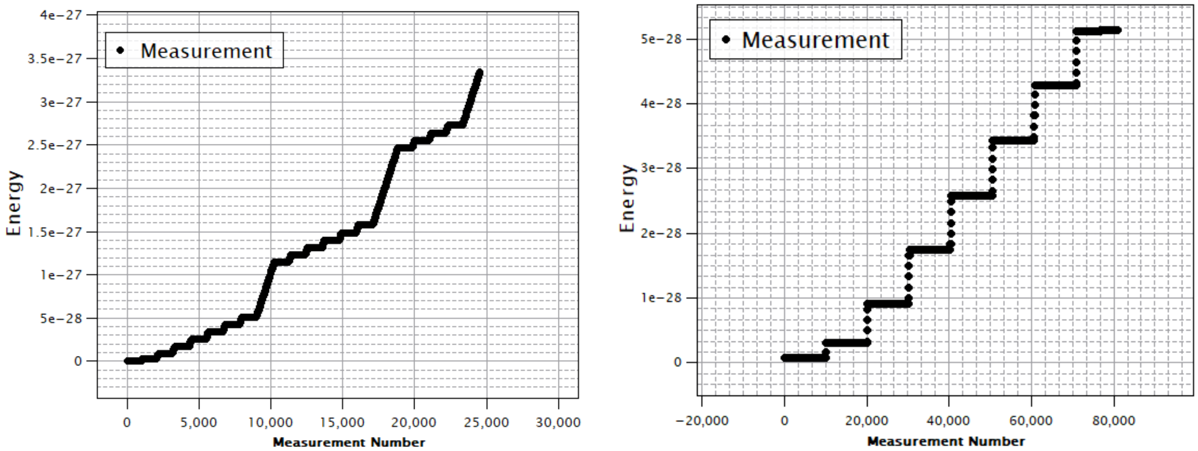
Indeed, in Fig. 7b we see that reducing the time step regularises the steps in Fig. 7a. It is now clear that the particle gains the same amount of energy in each passing through the accelerating region and that the two categories mentioned previously were just computational artifacts. This result is in line with physical intuition.

In the next subsection we will look at introducing errors in the initial distribution of a bunch of electrons, for the case of two accelerating regions, presented in Fig. 4.

C. Introducing Spread

To start this section, we need to mention that *we consider a bunch of 10 protons throughout, unless explicitly stated so.*

The aim here is to present the consideration of spread in the initial position of particles, in the case shown in Fig. 4. We would like to underline now that the following discussion is not very useful generally, as the case of 2 boxes depends on the rather arbitrary value for the length of the box, of $radius/4$. However, we spent a lot of time unknowingly collecting data for this case (when we thought we were using the infinite strip instead). Nevertheless, we



(a) Reduced time step of 10^{-8}.

(b) Reduced time step of 10^{-9}.

FIG. 7. Energy plot as a function of measurement number for a proton performing 100 revolutions in uniform magnetic field and experiencing an electric field over an infinite strip on the x-axis, as described in section II. A measurement was taken each 1000 time steps.

will investigate this situation and try to provide comments which could apply to the case of an infinite strip as well (which actually raises concerns about its physicality, so one may say it's just as bad as having to choose an arbitrary value for the length of the boxes).

The first observation that we make is that, in the non-relativistic case, introducing spreads should not mess the period of oscillation, as this is independent on speed and radius.

Given the boxes configuration, the obvious maximum limit for the full spread of the bunch on the x-axis should be given by half of the length of the box, $radius/8$. That is because, if a particle finds itself outside the box, it would not experience an electric field as its orbit would be given by a classical magnetic case, while the other particles would accelerate. The same considerations apply to the spread on the y-axis which is larger than the width, with the mention that a particle that initially finds itself outside the first box can nevertheless access the second box and get accelerated. However, this would introduce a lag with respect to the other particles.

Moreover, since the radius of the orbit depends on velocity, and the velocity depends on how much a particle is accelerated, we expect the radius of the first completed orbit to range from maximal for particles which start off at the lower bound of the accelerating region, to minimal for particles starting off outside this region.

Another point of consideration for the spread on the x-axis is that there should be a central region which no particle crosses. This would correspond to the realistic scenario of a particle accelerator, which is 'empty' in the middle.

We tested the spread in x under the assumptions that the spread in y is 0. Investigating the spread in y could constitute a future project. Now, we had to choose what we should measure in order to get a quantitative result for the maximum spread which still gives an efficient orbit. Some ways that we thought of were: plot the average position of the bunch and check that it does an orbit, plot the energy as a function of time and check that it increases, and lastly, check for consistency in the number of measurements between multiple re-runs of the algorithm (as the algorithm should give the same result if nothing changes). Out of these, we first focus on the last one and later on the second one.

Let us first look for consistency in the number of measurements for 10 revolutions for 10 protons. We tested spreads in X ranging from 0 to $radius$, in steps of $radius/10$. Interestingly, the number of step per 10 revolutions changes dramatically for each of the 10 cases, as shown in table II.

Initial spread in X in units of radius/10	0	1	2	3	4	5	6	7	8	9	10
Number of steps for 10 rev	7100	7100	7100	7100	531	1171	1314	1143	46	598	14064

TABLE II. Number of measurements as a function of initial spread of a bunch of 10 protons in uniform magnetic and two localised electric fields. Details of parameters are given in Sec. II. We used varying time steps: 1.0×10^{-5} outside and 1.0×10^{-8} inside the accelerating region.

Moreover, when we re-run the code for the spread $= 10/10{*}radius$, we do not get the same number of measurements, but 5300. This signifies that the program is not stable for this value of the spread and we thus discard it.

Next, we can see that the first four have the same number of measurements. We thus decide to investigate this further. Let us re-run the cases between 3/10 and 4/10 spread, in steps of 0.2/10 to check for stability. The results are shown in table III. We can see here that spread=$4/10 * radius$ returns a different value than that in table II. Thus we discard spreads larger than $3.5 * radius$.

Initial spread in X in units of radius/10	3	3.2	3.4	3.6	3.8	4
Number of steps for 10 rev	7100	541	7100	8053	7852	1165

TABLE III. Number of measurements as a function of initial spread of a bunch of 10 protons in uniform magnetic and two localised electric fields. Details of parameters are given in Sec. II. We used varying time steps: 1.0×10^{-5} outside and 1.0×10^{-8} inside the accelerating region.

Now let us look in more detail at the range $3-3.5$ in steps of 0.1.

Initial spread in X in units of radius/10	3	3.1	3.2	3.3	3.4	3.5
Number of steps for 10 rev	7100	1281	6900	6900	9252	7100

TABLE IV. Number of measurements as a function of initial spread of a bunch of 10 protons in uniform magnetic and two localised electric fields. Details of parameters are given in Sec. II. We used varying time steps: 1.0×10^{-5} outside and 1.0×10^{-8} inside the accelerating region.

What is interesting in table IV is that 3.5 gives the same number of measurements as the case of 0 spread. Nevertheless, on a second trial, it gives 8643. This is in contrast with the value 3/10∗radius, which consistently gives 7100. Thus when the spread in position in the y-direction is 0 along with the spread in velocity and energy, the maximum spread in X that we can trust is 3/10∗radius.

To solidify the confidence level, we created a plot of the energy as a function of the measurement number, for different sizes of the spread. The results are shown in Fig. 8. We can see that, in accordance to our previous result, the simulation works consistently until the spread in X of 3/10*Radius, but gives different results for when the spread is 4/10∗Radius. What is interesting to note is the initial decrease in energy, before it starts going up. When comparing with Fig. 5a, we can see that such behaviour is unimportant when considering a 100 revolutions instead of just 5. This behaviour signals a initial decrease in the average velocity of the bunch.

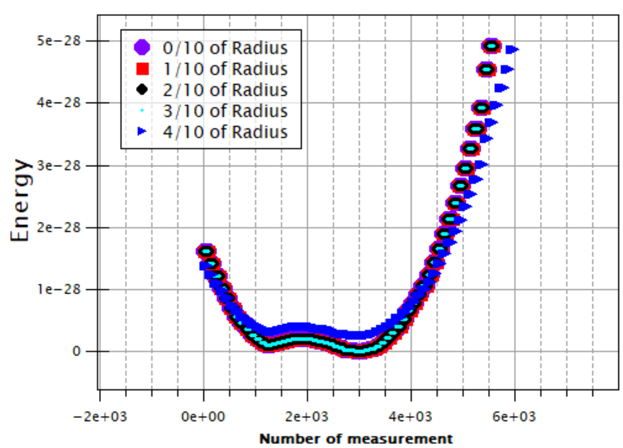


FIG. 8. A plot of energy versus measurement number (i.e. time) for a bunch of 10 protons doing 5 revolutions in uniform magnetic field and localised electric field defined by the two-boxes configurations. Details of the parameters are given in section II. The time step used was 10^{-5} outside and 10^{-8} inside the accelerating region. A measurement was taken each 1000 time steps.

In spite of this apparent agreement, we ought to be careful before we can say for sure that $0.3 * radius$ is a spread that still gives efficient acceleration. This is because it runs against the intuition described at the beginning of the section: that the spread should be smaller than half of the box length $0.125 * radius$. The possible reason for the agreement is that the frequency of orbit does not depends on the velocity, and thus a particle that does not experience electric field should complete the orbit in the same amount of time as one that does. Thus really, the results obtained in this section might be generally valid and similar to those obtained in the case of an infinite strip or indeed no accelerating field at all!

IV. CONCLUSION

In this report we have analysed the workings of three basic models of a particle accelerator. All included a uniform magnetic field. The first one included a bounded region of varying electric field (see Eq. 3), centred at origin. The second one contained two bounded regions centred at the origin and at $2.95 * 10^{-2}$ on the x-axis respectively. The last model consisted of an infinite strip of electric field stretching along the x-axis and bounded on the y-axis.

We explored three numerical algorithms: Euler, Euler-MidPoint and EulerCromer. They all cause diverging orbits in magnetic field, however, this divergence is much smaller than in the case where we have an accelerating electric field (see Figs. 1a and 1b). On this scale, the difference between the algorithms becomes unnoticeable. In addition, in the case of magnetic field only, the three agree to 17 significant figures on the energy change of the system after 500 revolutions. Nevertheless, Euler-Cromer is the only one which does not diverge in the case of radial electric field. Thus, we decided to do the rest of the calculations using this algorithm.

We have shown that all three models simulate an increasing orbit, but, whereas the infinite strip causes an expansion of the orbit in all directions (see Fig. 9a), the one and two-box models effectively 'trap' the orbit on one box and expand the orbit in the opposite direction (see Figs. 3 and 4).

Next, we discovered that, as expected, odd multiples of $\phi = \pi/2$ for the phase of the electric field (see Eq. 8)) result in the most efficient acceleration. There were 2 orders of magnitude difference in the final energy of a proton accelerated for 100 revolutions by an electric field of phase 0 and $\pi/2$ respectively. This difference was pictorially shown in the comparison between Figs. 9a and 11.

We implemented varying time steps: the program switches to small time steps once inside the accelerating region. We saw that we get an increase in energy (which means efficient acceleration) for bunch of 10 protons with no spread (see Fig. 5) using time steps of 10^{-5} and 10^{-8} respectively. In addition, in Fig. 7 we showed that decreasing the reduced time step to 10^{-9} explains the uneven energy steps. As a reminder, energy steps should appear as the particles gain energy only when inside the region of electric field. Between these periods the energy should stay the same.

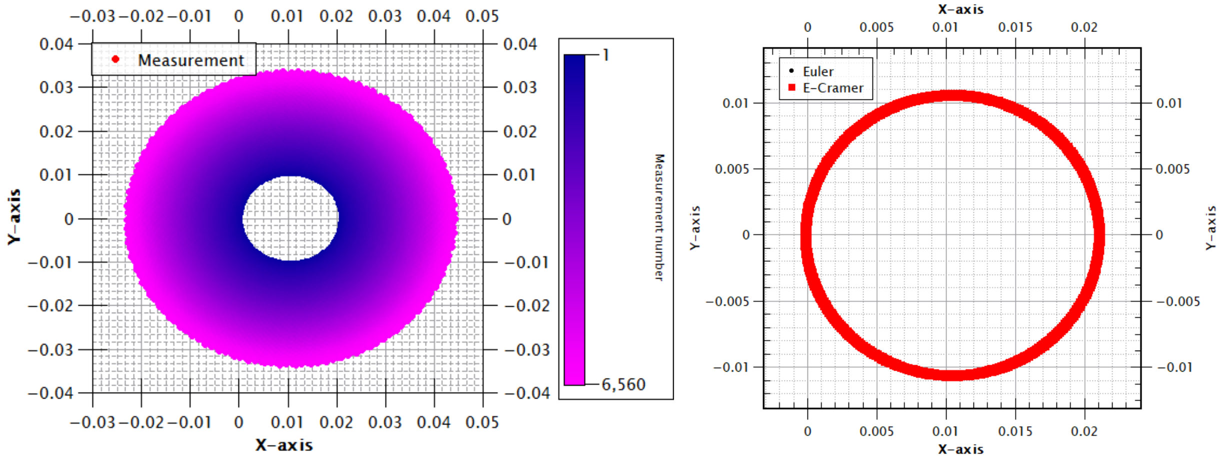
Next, for the two-box case, we saw that the maximum spread on X that gives consistent results on different re-runs is $0.3 * radius$ although this is bigger than the size of the accelerating box itself, which is $0.25 * radius$.

We conclude that our best simulation is provided by the case of uniform magnetic field with an infinite accelerating strip laying on the x-axis, with time steps of 10^{-5} that reduce to 10^{-9} inside the accelerating regions.

FUTURE DIRECTIONS?

V. APPENDIX

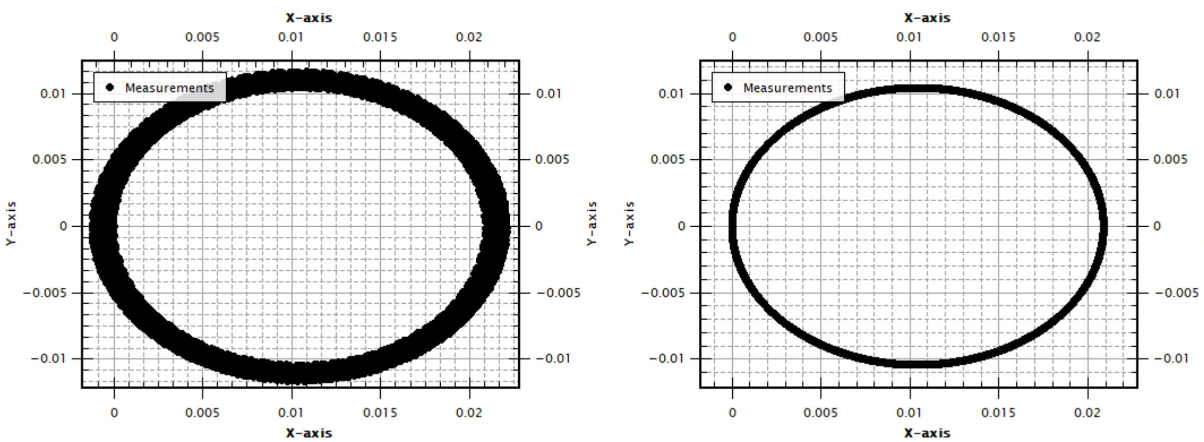
[1] A. Cromer, American Journal of Physics 49, 455 (1981).



(a) Movement of a particle in a magnetic field. A strip of time-varying accelerating field runs horizontally through. It is infinite on the x-axis and bounded by W on the y-axis. The measurements are performed every 1000 time steps. The algorithms used are Euler, Euler-Midpoint and Euler-Cromer, they overlap. The particle ran for 100 revolutions with a time step of 10^{-5}, giving 6560 measurements. The colour map is a representation of time-evolution of the orbit.

(b) This figure shows a plot of the 6560 measurements (100 revolutions) of a proton in a uniform magnetic field of 10^{-7} T with initial velocity in the +y direction of 0.1 ms-1. The figure includes both Euler and Euler-Cromer methods, although they are superimposed on each other.

FIG. 9.



(a) Euler method - 7152 measurements

(b) Euler-Cromer method - 6560 measurements

FIG. 10. This figure shows two plots of a proton experiencing a radial electric field. The electric field is generated by a central particle whose charged is computed to give the same theoretical motion as that of the proton moving in uniform magnetic field of 1e-7 T with initial velocity in the +y direction of 0.1 ms-1.

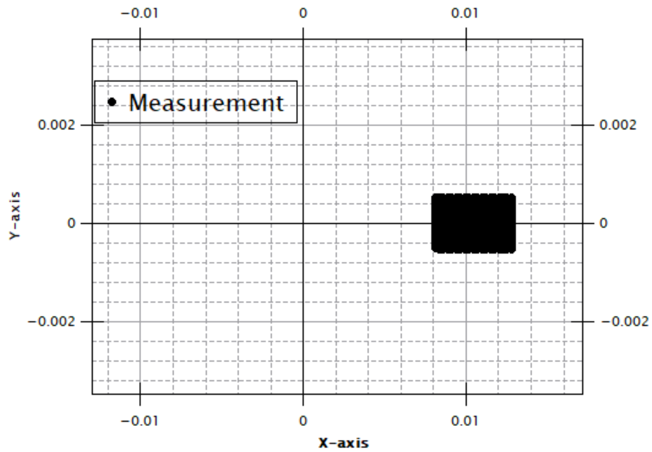


FIG. 11. Real space plot of a function that prints a measurement for each point inside the box defined by the Rectangle object presented in Sec. II.