iulian.simion@ubbcluj.ro

1. Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

2. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

5. Use the classification of quadrics to determine what surfaces are described by the following equations

1. $xz + xy + yz = 1$
2. $x^2 - 2xz - y^2 - z^2 = 1$
3. $xz + xy + yz = -1$
4. $5x^2 + 3y^2 + xz = 1$

6. Determine the points P of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

7. Show that the line

$$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \rangle \quad \text{is tangent to the quadric} \quad \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the tangency point.

8. Prove that the intersection of a quadric in $\mathbb{E}^3$ with a plane is either the empty set or a point or a line or two lines or an ellipse or a hyperbola or a parabola.

9. Prove that the intersection of an ellipsoid with a plane is either the empty set or a point or an ellipse.

10. Show that the ellipsoid $\mathcal{E}_{a,b,b}$ is the locus of points for which the sum of the distances to two given points is constant. Such a surface is called *ellipsoid of revolution*.

11. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution.

12. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

- Give an equation of $\mathcal{S}$.
- Find the parameters of the point $P(3, \sqrt{3}, 1)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give an equation of $T_P\mathcal{S}$.

13. Prove that the intersection of an elliptic cone with a plane is either a point or a line or an ellipse or a hyperbola or a parabola.

1. Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \right\rangle.$$

$\mathcal{E} \cap \ell$ = points on ℓ satisfying the eq. of the ellipsoid

$$\frac{(4+2t)^2}{16} + \frac{(-6-3t)^2}{12} + \frac{(-2-2t)^2}{4} - 1 = 0$$

$$\Leftrightarrow \quad (2+t)^2 + (1+t)^2 - 1 = 0$$

$$\Leftrightarrow \quad (t+1)(t+2) = 0$$

the parameters $t=-1$ and $t=-2$ correspond to points on ℓ which satisfy the eq. of $\mathcal{E}$

$t=-1$ corresponds to $P_1(2,-3,0)$

$t=-2$ — " — $P_2(0,0,2)$

The tangent planes in these points are

$$T_{P_1}\mathcal{E} : \frac{2x}{16} - \frac{3y}{12} + \frac{0 \cdot z}{4} - 1 = 0 \quad \Leftrightarrow \quad x - 2y - 8 = 0$$

$$T_{P_2}\mathcal{E} : \frac{0x}{16} + \frac{0 \cdot y}{12} + \frac{2 \cdot z}{4} - 1 = 0 \quad \Leftrightarrow \quad z = 2$$

2. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

Intersection with planes $\parallel Oxy$: $\quad \mathcal{E} \cap z=h$ for some $h \in \mathbb{R}$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 - \frac{h^2}{16}$$

$$\Leftrightarrow \quad \frac{x^2}{4\left(1-\frac{h^2}{16}\right)} + \frac{y^2}{9\left(1-\frac{h^2}{16}\right)} = 1 \qquad \text{(in the plane } z=h\text{)}$$

which gives

- the empty set if $1 < \frac{h^2}{16} \Leftrightarrow h \in \left]-\infty, -4\right[\cup \left]4, \infty\right[$
- the point $(0,0,4)$ if $h=4$
- the point $(0,0,-4)$ if $h=-4$
- an ellipse if $h \in (-4,4)$
  with semi-major axis $\sqrt{9\left(1-\frac{h^2}{16}\right)}$
  and semi-minor axis $\sqrt{4\left(1-\frac{h^2}{16}\right)}$

The other cases are treated similarly.

3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

$$\mathcal{E} \cap \ell : \begin{cases} \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \\ y = x \\ z = x \end{cases}$$

$$\Rightarrow \frac{x^2}{4} + \frac{x^2}{3} + \frac{x^2}{9} = 1$$

$$\Rightarrow 25x^2 = 4 \cdot 9$$

$$\Rightarrow x = \pm \frac{6}{5}$$

$\Rightarrow$ we obtain two points : $P_1\left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}\right)$ and $P_2\left(-\frac{6}{5}, -\frac{6}{5}, -\frac{6}{5}\right)$

- The tangent planes in these points are

$$T_{P_1}\mathcal{E} : \frac{\frac{6}{5}x}{4} + \frac{\frac{6}{5}y}{3} + \frac{\frac{6}{5}z}{9} = 1$$

$$\Leftrightarrow \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = \frac{5}{6}$$

and $T_{P_2}\mathcal{E} : \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = -\frac{5}{6}$

4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}}: \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

- for $P(x_0, y_0, z_0) \in \mathcal{E}$ we have

$$T_P \mathcal{E} : \frac{x_0 x}{4} + \frac{y_0 y}{9} + \frac{z_0 z}{8} = 1$$

- $T_P \mathcal{E} \parallel \pi \Rightarrow$ the normal vectors $(3, -2, 5)$ and $\left(\frac{x_0}{4}, \frac{y_0}{9}, \frac{z_0}{8}\right)$

are proportional

$$\frac{3}{\frac{x_0}{4}} = \frac{-2}{\frac{y_0}{9}} = \frac{5}{\frac{z_0}{8}}$$

$$\Leftrightarrow \frac{x_0}{12} = \frac{y_0}{-18} = \frac{z_0}{40}$$

$$\Leftrightarrow \begin{cases} y_0 = -\frac{3}{2} x_0 \\ z_0 = \frac{10}{3} x_0 \end{cases}$$

- with the condition that $P \in \mathcal{E}$ we have

$$\frac{x_0^2}{4} + \frac{\left(-\frac{3}{2} x_0\right)^2}{9} + \frac{\left(\frac{10}{3} x_0\right)^2}{8} = 1 \quad \dots \Leftrightarrow x_0^2 = \frac{9}{17}$$

so we obtain two points $P_1\left(\frac{3}{\sqrt{17}}, -\frac{9}{\sqrt{17}}, \frac{10}{\sqrt{17}}\right)$ and $P_1\left(-\frac{3}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{10}{\sqrt{17}}\right)$

$$\Rightarrow T_{P_1} \mathcal{E} : \frac{x}{4} - \frac{y}{6} - \frac{5z}{12} = \frac{\sqrt{17}}{3} \quad \text{and} \quad T_{P_2} \mathcal{E} : \frac{x}{4} - \frac{y}{6} + \frac{5z}{12} = -\frac{\sqrt{17}}{3}$$

5. Use the classification of quadrics to determine what surfaces are described by the following equations

1. $xz + xy + yz = 1$

The matrix associated to this equation is $Q = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

the characteristic polynomial is $\begin{vmatrix} \lambda & 1/2 & 1/2 \\ 1/2 & \lambda & 1/2 \\ 1/2 & 1/2 & \lambda \end{vmatrix} = \lambda^3 + \frac{1}{8} + \frac{1}{8} - \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{\lambda}{4}$

$$-\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} = 0$$

We notice that $\lambda = 1$ is a root

$$\begin{array}{r|l} -\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} & \lambda - 1 \\ \cline{2-2} -\lambda^3 + \lambda^2 \phantom{+\frac{3}{4}\lambda+\frac{1}{4}} & -\lambda^2 - \lambda - \frac{1}{4} \\ \cline{1-1} -\lambda^2 + \frac{3}{4}\lambda + \frac{1}{4} & \\ -\lambda^2 + \lambda \phantom{+\frac{1}{4}} & \\ \cline{1-1} -\frac{1}{4}\lambda + \frac{1}{4} & \end{array}$$

$$(\lambda - 1)\left(-\lambda^2 - \lambda - \frac{1}{4}\right) = 0$$

$$\Leftrightarrow -(\lambda+1)\left(\lambda - \frac{1}{2}\right)^2 = 0$$

so the roots are -1 and $\frac{1}{2}$ with $\frac{1}{2}$ having multiplicity 2

$\Rightarrow$ rank $Q = 3$ and the signature of Q is $(2, 1)$

↳ one negative eigenvalue

↑ two positive eigenvalues

$\Rightarrow$ the possibilities for this surface are

either a hyperboloid (of one sheet or of two sheets)

or an elliptic cone

Method 2 we can make linear changes of variables which correspond to affine changes of coordinates in order to see the canonical form.

$$xy + yz + zx = 1$$

replace y by $x+y$ to obtain

$$x^2 + xy + xz + yz + zx = 1$$

$$\Leftrightarrow \quad x^2 + xy + yz + 2zx = 1$$

$$\Leftrightarrow \quad \left(x^2 + xy + \frac{y^2}{4}\right) - \frac{y^2}{4} + yz + 2zx = 1$$

$$\Leftrightarrow \quad \left(x + \frac{y}{2}\right)^2 - \left(\frac{y^2}{4} - yz + z^2\right) + z^2 + 2zx = 1$$

$$\Leftrightarrow \quad \left(x + \frac{y}{2}\right)^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2zx = 1$$

Replace $x + \frac{y}{2}$ by x

$$\Leftrightarrow \quad x^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2z\left(x - \frac{y}{2}\right) = 1$$

$$\Leftrightarrow \quad x^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2zx - zy = 1$$

Replace $\frac{y}{2} - z$ by y

$$\Leftrightarrow \quad x^2 - y^2 + z^2 + 2zx - z(2y + 2z) = 1$$

$$\Leftrightarrow \quad x^2 - y^2 + z^2 + 2zx - 2yz - 2z^2 = 1$$

$$\Leftrightarrow \quad x^2 - y^2 - z^2 + 2zx - 2yz = \underline{1}$$

$$\Leftrightarrow \quad (x^2 + 2zx + z^2) - z^2 - (y^2 + 2yz + z^2) + z^2 - z^2 = 1$$

$$\Leftrightarrow \quad (x+z)^2 - (y+z)^2 - z^2 = 1$$

Replace $x+z$ by x and $y+z$ by y to obtain

$$x^2 - y^2 - z^2 = 1 \qquad \text{or} \qquad y^2 + z^2 - x^2 = \underline{-1}$$

the eq. of a hyperboloid of two sheets.

6. Determine the points P of the ellipsoid

$$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

Let $P(x_0, y_0, z_0)$ be a point on $\mathcal{E}$

Then $T_p\mathcal{E}: \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$

and $T_p\mathcal{E} \cap Ox = \begin{bmatrix} a^2/x_0 \\ 0 \\ 0 \end{bmatrix}$

$T_p\mathcal{E} \cap Oy = \begin{bmatrix} 0 \\ b^2/y_0 \\ 0 \end{bmatrix}$

$T_p\mathcal{E} \cap Oz = \begin{bmatrix} 0 \\ 0 \\ c^2/z_0 \end{bmatrix}$

"$T_p\mathcal{E}$ intersects the coordinate axis in congruent sements" means that the distances prom the intersection points to the origin are equal

so $\left|\frac{x_0}{a^2}\right| = \left|\frac{y_0}{b^2}\right| = \left|\frac{z_0}{c^2}\right| \Rightarrow x_0 = \pm \frac{a^2}{c^2} z_0$ and $y_0 = \pm \frac{b^2}{c^2} z_0$

since $P \in \mathcal{E}$ we have $\frac{\left(\pm \frac{a^2}{c^2} z_0\right)^2}{a^2} + \frac{\left(\pm \frac{b^2}{c^2} z_0\right)^2}{b^2} + \frac{z_0^2}{c^2} = 1$

$$\Leftrightarrow z_0^2 \left(\frac{a^4}{c^4} + \frac{b^4}{c^4} + \frac{1}{c^2}\right) = 1$$

$$\Leftrightarrow z_0 = \pm \frac{c^2}{\sqrt{a^2+b^2+c^2}}$$

so $P = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} \pm a^2 \\ \pm b^2 \\ \pm c^2 \end{bmatrix}$

we obtain 8 points where the property holds. In fact, if you find one such point in one octant then with the symmetries of $\mathcal{E}$ you obtain such points also in the other 7 octants.

7. Show that the line

$$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \left\langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\rangle \text{ is tangent to the quadric } \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the tangency point.

$$\ell : \begin{cases} x = 2 \\ y = -3 - t \\ z = 6 + 2t \end{cases} \qquad t \in \mathbb{R}$$

ℓ is tangent to $\mathcal{E}$ if it intersects $\mathcal{E}$ in a double point

this happens if

$$\frac{2^2}{4} + \frac{(-3-t)^2}{9} + \frac{(6+2t)^2}{16} - 1 = 0 \qquad (*)$$

has a double solution in t

$$(*) \iff (3+t)^2 = 0$$

so, yes, ℓ is tangent to $\mathcal{E}$.

The point of tangency (where ℓ touches $\mathcal{E}$)

corresponds to $t = -3$, it is $P = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

8. Prove that the intersection of a quadric in $\mathbb{E}^3$ with a plane is either the empty set or a point or a line or two lines or an ellipse or a hyperbola or a parabola.

A quadric is the set of solutions to a quadratic equation given w.r.t a coordinate system $K = (O, i, j, k)$

Let S be the quadric and let π be a plane

Fix a second coordinate system $K' = (O', i', j', k')$ with $i', j' \parallel \pi$ and $O \in \pi$

Changing the coordinate system from K to K' will change the equation of S to some other quadratic equation in three variables

But in this coordinate system K' the intersection with π is obtained by setting the third coordinate equal to zero, since $\pi = Ox'y' : z' = 0$

This gives a quadratic equation in the plane π which, by the classification of quadratic curves is one of the possibilities in the statement.

9. Prove that the intersection of an ellipsoid with a plane is either the empty set or a point or an ellipse.

As in the previous exercise

With the extra observation that an ellipsoid is bounded (you can put it in a box)

$\Rightarrow$ $S \cap \pi$ is also bounded

$\Rightarrow$ the possibilities for $S \cap \pi$ are the quadratic curves (possibly degenerate) which are bounded

10. Show that the ellipsoid $\mathcal{E}_{a,b,b}$ is the locus of points for which the sum of the distances to two given points is constant. Such a surface is called *ellipsoid of revolution*.

- View $\mathcal{E}_{a,b,b}$ as the union of ellipses $\Pi \cap \mathcal{E}_{a,b,b}$ where Π is a plane which contains the x-axis

- All such ellipses have the same semi-major and semi-minor axes

  $\Rightarrow$ they have the same focal points.

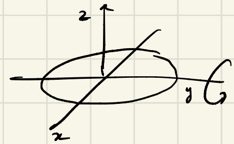
We need to assume $a > b$

$\Rightarrow$ the focal points are on the x-axis

- Since $\mathcal{E}_{a,b,b}$ is the union of such ellipses, all points on $\mathcal{E}_{a,b,b}$ have the required property. (if $a > b$)

11. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution.

ellipse in Oxy
$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \\ z = 0 \end{cases}$$



$$\begin{bmatrix} \cos\theta' & 0 & \sin\theta' \\ 0 & 1 & 0 \\ -\sin\theta' & 0 & \cos\theta' \end{bmatrix} \begin{bmatrix} a\cos\theta \\ b\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} a\cos\theta\cos\theta' \\ b\sin\theta \\ -a\cos\theta\sin\theta' \end{bmatrix}$$

$\uparrow$ rotation around y-axis $\qquad\qquad$ $\uparrow$ ellipsoid of revolution

12. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

- Give an equation of $\mathcal{S}$.
- Find the parameters of the point $P(3, \sqrt{3}, 1)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give an equation of $T_P\mathcal{S}$.

- $S : \frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{4} = 1$

- $P(3, \sqrt{3}, 1)$ $\qquad \sin(t) = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\Rightarrow \begin{cases} 3 = 4\frac{\sqrt{3}}{2}\cos(s) \\ \sqrt{3} = 4\frac{\sqrt{3}}{2}\sin(s) \end{cases} \Rightarrow \begin{cases} \cos(s) = \frac{\sqrt{3}}{2} \\ \sin(s) = \frac{1}{2} \end{cases} \Rightarrow s = \frac{\pi}{6}$$

so P is obtained with the parameters $(t, s) = (\frac{\pi}{6}, \frac{\pi}{6})$

- $T_pS = p + \langle \frac{\partial\sigma}{\partial t}(p), \frac{\partial\sigma}{\partial s}(p) \rangle$ where $\sigma(t,s) = \begin{bmatrix} 4\cos(s)\cos(t) \\ 4\sin(s)\cos(t) \\ 2\sin(t) \end{bmatrix}$

$$\frac{\partial\sigma}{\partial s}(p) = \begin{bmatrix} -4\sin(s)\cos(t) \\ 4\cos(s)\cos(t) \\ 0 \end{bmatrix}(p) = \begin{bmatrix} -\sqrt{3} \\ 3 \\ 0 \end{bmatrix}$$

$$\frac{\partial r}{\partial t}(p) = \begin{pmatrix} -4\cos(s)\sin(t) \\ -4\sin(s)\sin(t) \\ 2\cos(t) \end{pmatrix}(p) = \begin{pmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{pmatrix}$$

$$\Rightarrow T_pS : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ \sqrt{3} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -\sqrt{3} \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{pmatrix}$$

- $T_pS : \frac{3x}{16} + \frac{\sqrt{3}y}{16} + \frac{z}{4} = 1$

13. Prove that the intersection of an elliptic cone with a plane is either a point or a line or an ellipse or a hyperbola or a parabola.

Similar to 8 and 9