1. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3, -1, -2)$ and $\mathbf{b}(1, 2, -1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6, 0, 1)$ and $\overrightarrow{AC} = (1.5, 2, 1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

4. Consider the vectors $\mathbf{a}(2, 3, -1)$ and $\mathbf{b}(1, -1, 3)$ with respect to an orthonormal basis.

1. Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^\perp$.
2. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

5. Consider the points $A(1, 2, 0)$, $B(3, 0, -3)$ and $C(5, 2, 6)$ with respect to an orthonormal coordinate system.

1. Determine the area of the triangle ABC.
2. Determine the distance from C to AB.

6. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.

7. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2, -3, 1)$, $\mathbf{b}(-3, 1, 2)$ and $\mathbf{c}(1, 2, 3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

8. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

9. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi, \psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w} \times \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$ where

1. $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,
2. $\mathbf{w} = \mathbf{i} + \mathbf{k}$,
3. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

10. Prove the following identities:

1. the Jacobi identity,
2. the Lagrange identity,
3. the formula for the cross product of two cross products.

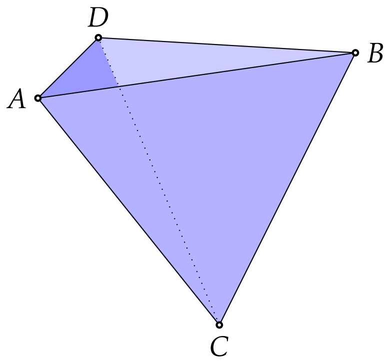
11. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=\mathbf{k}$. Determine if

1. $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis of $\mathbb{V}^3$,
2. if it is a basis, decide if it is left or right oriented.

12. The points $A(1,2-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

13. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{w}=\mathbf{k}$. Determine the matrix of the linear map $\phi:\mathbb{V}^3\to\mathbb{R}$ defined by $\phi(\mathbf{v})=[\mathbf{v},\mathbf{u},\mathbf{w}]$.

14. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.



15. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D\in Oy$. Determine the coordinates of D.

16. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

1. the angles of $\mathbf{d}$ with $\mathbf{a}$ and with $\mathbf{b}$ are congruent,
2. $\mathbf{d}$ is orthogonal to $\mathbf{c}$,
3. $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

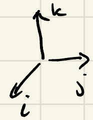
1. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

$$a \times b = (i + 2j - 2k) \times (7i + 4j + 6k)$$

$$= 7 \underbrace{i \times i}_{=0} + 4\, i \times j + 6\, i \times k +$$

$$+ 14\, j \times i + 8 \underbrace{j \times j}_{=0} + 12\, j \times k +$$

$$- 14\, k \times i - 8\, k \times j - 12 \underbrace{k \times k}_{=0}$$



$$= -10 \underbrace{i \times j}_{=k} + 20 \underbrace{i \times k}_{-j} + 20 \underbrace{j \times k}_{= i}$$

$$= 20i - 20j - 10k$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 7 & 4 & 6 \end{vmatrix} = 20i - 20j - 10k$$

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 5i + j + 7k$$

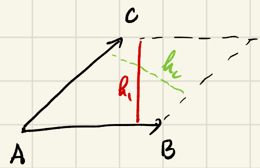
$(2a+b) \times b =$ you can calculate the components of $2a+b$ and a determinant, or

$$= 2a \times b + \underbrace{b \times b}_{=0} = 2a \times b = 10i + 2j + 14k.$$

$$(2a+b) \times (2a-b) = 4 \overbrace{a \times a}^{=0} - 2a \times b + 2 \underbrace{b \times a}_{-a \times b} - \overbrace{b \times b}^{=0} = -4\, a \times b = -20i - 4j - 28k$$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC} = (1.5,2,1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

Let a be the area of the parallelogram



$$\|\overrightarrow{AB}\times\overrightarrow{AC}\| = a = h_1 \cdot \|\overrightarrow{AB}\| = h_2 \cdot \|\overrightarrow{AC}\|$$

$$\Rightarrow h_1 = \frac{\|\overrightarrow{AB}\times\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|} = \frac{\sqrt{673}}{2\sqrt{37}}$$

$$\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \left\| \begin{vmatrix} i & j & k \\ 6 & 0 & 1 \\ 1.5 & 2 & 1 \end{vmatrix} \right\| = \|-2i - 4.5j + 12k\| = \sqrt{4+\frac{81}{4}+144} = \frac{\sqrt{673}}{2}$$

$$\|\overrightarrow{AB}\| = \sqrt{37}$$

$$\|\overrightarrow{AC}\| = \sqrt{\frac{9}{4}+4+1} = \frac{\sqrt{29}}{2}$$

$$\Rightarrow h_2 = \frac{\sqrt{673}}{\sqrt{29}}$$

4. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$ with respect to an orthonormal basis.

1. Determine the vector subspace $\langle \mathbf{a}, \mathbf{b}\rangle^{\perp}$.
2. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p}\cdot(2\mathbf{i}-3\mathbf{j}+4\mathbf{k}) = 51$.

1. Method I take an arbitrary vector $v(x,y,z)$ and impose the conditions: $v \in \langle a,b\rangle^{\perp} \Leftrightarrow v \perp a$ and $v \perp b$

$\Leftrightarrow v\cdot a = 0$ and $v \cdot b = 0$

$$\Leftrightarrow \begin{cases} 2x+3y-z=0 \\ x-y+3z=0 \end{cases}$$

this system allows us to express y and z in terms of x

then $v \in \langle a,b\rangle^{\perp} \Leftrightarrow v = v(x, y(x), z(x))$

Method II $\quad \langle a,b\rangle^{\perp} = \langle a\times b\rangle = \left\langle \begin{bmatrix} 8 \\ -7 \\ -5 \end{bmatrix} \right\rangle = \{(8t, -7t, -5t) : t\in\mathbb{R}\}$

$$a\times b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 8i - 7j - 5k$$

2. $p \in \langle a,b\rangle^{\perp} \iff p = t\begin{bmatrix} 8 \\ -7 \\ -5 \end{bmatrix}$ for some $t\in\mathbb{R}$

$p\cdot(2i-3j+4k) = 51 \iff (t8i - t7j - t5k)\cdot(2i-3j+4k) = 51$

$\iff 16t + 21t - 20t = 51$

$\iff t = \frac{51}{17} = 3$

5. Consider the points $A(1,2,0)$, $B(3,0,-3)$ and $C(5,2,6)$ with respect to an orthonormal coordinate system.

1. Determine the area of the triangle ABC.
2. Determine the distance from C to AB.

$$\vec{BC}\times\vec{BA} = \begin{vmatrix} i & j & k \\ -2 & 2 & 3 \\ 2 & 2 & 9 \end{vmatrix}$$

$\vec{BC}(2,2,9)$

$= 12i - (-24)j + (-8)k$

$\vec{BA}(-2,2,3)$

$= 12i + 24j - 8k = 4(3i + 6j - 2k)$

$\Rightarrow$ area $\Delta ABC = \frac{\|\vec{BC}\times\vec{BA}\|}{2} = \frac{4\cdot\sqrt{49}}{2} = 14$

area $\Delta ABC = \frac{h_1\cdot|BA|}{2} = \frac{d(C,BA)\cdot\|\vec{BA}\|}{2}$

$\Rightarrow d(C,BA) = \frac{\|\vec{BC}\times\vec{BA}\|}{\|\vec{BA}\|}$

$\|\vec{BA}\| = \sqrt{4+4+9} = \sqrt{17}$

$\Rightarrow d(C,BA) = \frac{28}{\sqrt{17}}$

6. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}. \qquad (*)$$

and deduce the law of sines in a triangle.

- we have $u + v + w = 0$

$$\Rightarrow u = -v - w$$

$$\Rightarrow u \times v = (-v-w)\times v = -\underbrace{v\times v}_{=0} - w\times v = -w\times v = v\times w$$

$$\Rightarrow v = -u - w$$

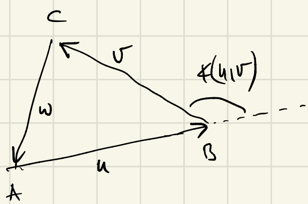
$$\Rightarrow u\times v = u\times(-u-w) = -u\times w = w\times u$$

- from $(*)$ we have $\|u\times v\| = \|v\times w\| = \|w\times u\|$

$$\Leftrightarrow \|u\|\cdot\|v\| \sin\sphericalangle(u,v) = \|v\|\cdot\|w\|\cdot\sin\sphericalangle(v,w) = \|w\|\cdot\|u\|\cdot\sin\sphericalangle(w,u) \quad \Big| : \|u\|\cdot\|v\|\cdot\|w\|$$

$$\Leftrightarrow \frac{\sin\sphericalangle(u,v)}{\|w\|} = \frac{\sin\sphericalangle(v,w)}{\|u\|} = \frac{\sin\sphericalangle(w,u)}{\|v\|}$$

$$\Leftrightarrow \frac{\sin\hat{B}}{\|w\|} = \frac{\sin\hat{C}}{\|u\|} = \frac{\sin\hat{A}}{\|v\|}$$



7. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

$$(a\times b)\times c = (a\cdot c)b - (b\cdot c)a = (2-6+3)b - (-3+2+6)a$$

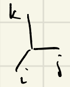
$$= -\begin{bmatrix}-3\\1\\2\end{bmatrix} - 5\begin{bmatrix}2\\-3\\1\end{bmatrix} = \begin{bmatrix}-7\\14\\-3\end{bmatrix}$$

$$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c = -\begin{bmatrix}-3\\1\\2\end{bmatrix} + 7\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}10\\13\\19\end{bmatrix}$$

8. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

- The map is linear. During the lecture we showed Prop 2.5 from which it follows that

$$\phi(\alpha u + \beta w) = v \times (\alpha u + \beta w) = \alpha v \times u + \beta v \times w = \alpha \phi(u) + \beta \phi(w) \quad \forall \alpha, \beta \in \mathbb{R} \ \forall u, w \in \mathbb{V}^3$$



- let $v = v_1 i + v_2 j + v_3 k$ then $\phi(i) = (v_1 i + v_2 j + v_3 k) \times i$

$$= v_1 i \times i + v_2 j \times i + v_3 k \times i$$

$$= -v_2 k + v_3 j$$

similar $\phi(j) = v_1 k - v_3 i$

$\phi(k) = -v_1 j + v_2 i$

$\Rightarrow$ the matrix of ϕ with respect to the basis i, j, k is

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

if we consider $\phi(w) = w \times v$ then, since $w \times v = -v \times w$,

the matrix of ϕ with respect to the basis i, j, k is

$$\begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & v_1 & 0 \end{bmatrix}$$

9. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi, \psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w} \times \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$ where

1. $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,
2. $\mathbf{w} = \mathbf{i} + \mathbf{k}$,
3. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

One can do a similar calculation as in the previous exercise

But since we already computed that matrix one could also just

use $\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$ for ϕ

so, for 1, the matrix of ϕ is $\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

10. Prove the following identities:

1. the Jacobi identity,
2. the Lagrange identity,
3. the formula for the cross product of two cross products.

1.) $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0\ ?$

$$(a \times b) \times c = \cancel{(a \cdot c) b} - \cancel{(b \cdot c)} \cdot a$$

$$(b \times c) \times a = \cancel{(b \cdot a)} c - \cancel{(c \cdot a) \cdot b}$$

$$(c \times a) \times b = \cancel{(c \cdot b)} a - \cancel{(a \cdot b)} \cdot c \quad +$$

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$$

2.) $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (b\cdot c)(a\cdot d)$

let $a=(a_1,a_2,a_3)$ $b=(b_1,b_2,b_3)$ $c=(c_1,c_2,c_3)$ $d=(d_1,d_2,d_3)$

then $a\times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3-b_2a_3)+j(a_3b_1-a_1b_3)+k(a_1b_2-a_2b_1)$

and $c\times d = i(c_2d_3-d_2c_3)+j(c_3d_1-d_3c_1)+k(c_1d_2-d_1c_2)$

So

$$(a\times b)\cdot(c\times d) = (a_2b_3-b_2a_3)(c_2d_3-d_2c_3)+(a_3b_1-b_3a_1)(c_3d_1-d_3c_1)+(a_1b_2-b_1a_2)(c_1d_2-d_1c_2)$$

$$\begin{aligned} &= a_2b_3c_2d_3 - a_2b_3d_2c_3 - b_2a_3c_2d_3 + b_2a_3d_2c_3 + \\ &+ a_3b_1c_3d_1 - a_3b_1d_3c_1 - b_3a_1c_3d_1 + b_3a_1d_3c_1 + \\ &+ a_1b_2c_1d_2 - a_1b_2d_1c_2 - b_1a_2c_1d_2 + b_1a_2d_1c_2 \end{aligned}$$

$$\begin{aligned} (a\cdot c)(b\cdot d)-(b\cdot c)(a\cdot d) &= (a_1c_1+a_2c_2+a_3c_3)(b_1d_1+b_2d_2+b_3\cdot d_3) - \\ &- (b_1c_1+b_2c_2+b_3c_3)(a_1d_1+a_2d_2+a_3d_3) \\ &= \cancel{a_1c_1b_1d_1} + a_1c_1b_2d_2 + a_1c_1b_3d_3 + \\ &+ a_2c_2b_1d_1 + \cancel{a_2c_2b_2d_2} + a_2c_2b_3d_3 \\ &+ a_3c_3b_1d_1 + a_3c_3b_2d_2 + \cancel{a_3c_3b_3d_3} \\ &- \cancel{b_1c_1a_1d_1} - b_1c_1a_2d_2 - b_1c_1a_3d_3 \\ &- b_2c_2a_1d_1 - \cancel{b_2c_2a_2d_2} - b_2c_2a_3d_3 \\ &- b_3c_3a_1d_1 - b_3c_3a_2d_2 - \cancel{b_3c_3a_3d_3} \end{aligned}$$

one checks that these are equal

3) $(a\times b)\times(c\times d) = b\cdot[a,c,d] - a\cdot[b,c,d] = c\cdot[a,b,d] - d\cdot[a,b,c]$

one can do a calculation in coordinates as above

or

$$(a\times b)\times(c\times d) = \underbrace{(a\cdot(c\times d))}_{\substack{= (c\times d)\cdot a \\ = [c,d,a] \\ =}} b - \underbrace{(b\cdot(c\times d))}_{\substack{= (c\times d)\cdot b \\ = [c,d,b] \\ =}}\cdot a$$

$$= [a,c,d]\,b - [b,c,d]\,a$$

and

$$(a\times b)\times(c\times d) = [\underbrace{(a\times b)\cdot d}_{=}]\cdot c - [\underbrace{(a\times b)\cdot c}_{=}]\cdot d$$

$$= [a,b,d]\,c - [a,b,c]\,d$$

11. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=\mathbf{k}$. Determine if

1. $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis of $\mathbb{V}^3$,
2. if it is a basis, decide if it is left or right oriented.

$a=(1,1,0)\quad b=(1,0,-1)\quad c=(0,0,1)$

$$[a,b,c]=\begin{vmatrix}1&1&0\\1&0&-1\\0&0&1\end{vmatrix}=-1$$

$\neq 0 \Rightarrow a,b,c$ is a basis

$< 0 \Rightarrow a,b,c$ is left oriented

12. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

A, B, C, D coplanar $\iff \vec{AB}, \vec{AC}, \vec{AD}$ linearly dependent

$$\vec{AB} = \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}, \quad \vec{AC} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{AD} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\iff \underbrace{\begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix}}_{12-2-8-2=0} = 0 \quad \text{true}$$

13. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u} = -\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v},\mathbf{u},\mathbf{w}]$.

let $v = (v_1, v_2, v_3)$

$$\phi(v) = (v \times u)\cdot w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ -1 & 3 & 1 \end{vmatrix} \cdot k = \begin{vmatrix} v_2 & v_3 \\ 3 & 1 \end{vmatrix} \underbrace{i\cdot k}_{=0} - \begin{vmatrix} v_1 & v_3 \\ -1 & 1 \end{vmatrix} \underbrace{j\cdot k}_{=0} + \begin{vmatrix} v_1 & v_2 \\ -1 & 3 \end{vmatrix} k\cdot k$$

$$= 3v_1 + v_2$$

so $\phi(1,0,0) = 3$

$\phi(0,1,0) = 1$

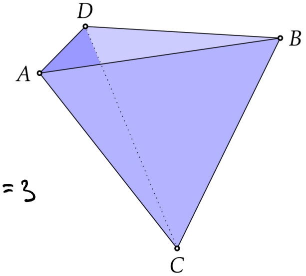
$\phi(0,0,1) = 0$

So the matrix is $[3 \ \ 1 \ \ 0]$

14. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.

$$= \left| \frac{1}{6} [\vec{AB}, \vec{AC}, \vec{AD}] \right|$$

$$= \left| \frac{1}{6} \begin{vmatrix} 3 & 6 & 3 \\ 1 & 3 & -2 \\ 2 & 2 & 2 \end{vmatrix} \right| = |-3| = 3$$



15. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D.

$D \in Oy \Rightarrow D(0,\lambda,0)$

the volume of the tetrahedron is $V = \left| \frac{1}{6} \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & \lambda-1 & 1 \end{vmatrix} \right| = \frac{1}{6} \left| -2 + 8 - 8 - 4(\lambda-1) \right| = \frac{1}{6} |2 - 4\lambda|$

so $5 = \frac{|1-2\lambda|}{3} \iff 15 = |1-2\lambda| \iff \begin{cases} \lambda = -7 \\ \lambda = 8 \end{cases}$

16. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

1. the angles of $\mathbf{d}$ with $\mathbf{a}$ and with $\mathbf{b}$ are congruent,
2. $\mathbf{d}$ is orthogonal to $\mathbf{c}$,
3. $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

Let $d = (d_1, d_2, d_3)$

$\sphericalangle(d,a) = \sphericalangle(d,b) \iff \cos\sphericalangle(d,a) = \cos\sphericalangle(d,b)$

$\iff \frac{a \cdot d}{\|a\| \cdot \cancel{\|d\|}} = \frac{b \cdot d}{\|a\| \cdot \cancel{\|d\|}}$

$\iff \frac{8d_1 + 4d_2 + d_3}{9} = \frac{2d_1 + 2d_2 + d_3}{3}$

$\iff 2d_1 - 2d_2 - 2d_3 = 0$

$\iff \left. \begin{array}{l} d_1 - d_2 - d_3 = 0 \\ d_1 + d_2 + d_3 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} d_1 = 0 \\ d_2 = -d_3 \end{array} \right\} \Rightarrow d = (0, \lambda, -\lambda)$

$d \perp c \iff d \cdot c = 0 \iff d_1 + d_2 + d_3 = 0$

$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 16 + 2 + 4 - 2 - 8 - 8 = 4 \Rightarrow a, b, c$ is right oriented

$$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} = -16\lambda + 2\lambda + 8\lambda - 8\lambda = -14\lambda$$ should be positive

$\Rightarrow$ all possible vectors are $d(0, \lambda, -\lambda)$ with $\lambda < 0$