Eigenvalues and eigenvectors.

1. Consider the matrix
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

2. Give the eigenvalues of $\mathrm{lin}(\mathrm{Pr}_{H,\mathbf{v}})$, $\mathrm{lin}(\mathrm{Ref}_{H,\mathbf{v}})$. What can you say about the eigenvectors?

3. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

4. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map
$$\phi(x,y,z) = (x+y-z, y+z, 2x).$$
Find the matrix $\mathrm{M}_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where
$$\boldsymbol{b} = \{(1,1,0),(-1,0,1),(1,1,1)\}.$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:
$$\begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}.$$

Rotations.

6. Show that an isometry is bijective.

7. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation
$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

8. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

10. Find the eigenvectors for each of the following symmetric matrices:
$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

11. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

- $P_A = \det(A - T\cdot I_n) = \det \begin{vmatrix} -T & 1 \\ -1 & -T \end{vmatrix} = T^2 + 1$
- P_A does not have roots in $\mathbb{R} \Rightarrow A$ is not diagonalizable
- P_A has roots in $\mathbb{C}$ so A viewed in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is diagonalizable

$$T^2+1=0 \qquad \lambda_1 = i \qquad \lambda_2 = -i$$

- to find the eigenvectors we solve the system

$$A\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = i v_1 \\ -v_1 = i v_2 \end{cases}$$ the solutions are $\begin{bmatrix} v_1 \\ i v_1 \end{bmatrix}$

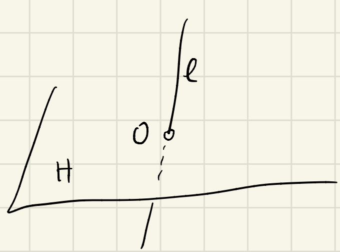
$$\Leftrightarrow V_i(A) = \left\langle \begin{bmatrix} 1 \\ i \end{bmatrix} \right\rangle$$

$$B\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = -i v_1 \\ -v_1 = -i v_2 \end{cases}$$ the solutions are $\begin{bmatrix} v_1 \\ -i v_1 \end{bmatrix}$

$$\Leftrightarrow V_{-i}(B) = \left\langle \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\rangle$$

2. Give the eigenvalues of $\text{lin}(\text{Pr}_{H,\mathbf{v}})$, $\text{lin}(\text{Ref}_{H,\mathbf{v}})$. What can you say about the eigenvectors?

- let $\phi = \text{lin}(\text{Pr}_{H,v})$ then $\phi(\overrightarrow{AB}) = \overrightarrow{\text{Pr}_{H,v}(A)\,\text{Pr}_{H,v}(B)}$



- let W be the vector subspace of vectors represented in H

  let $w = \{w_2, \dots, w_n\}$ be a basis of W

- let ℓ be a line with direction vector v

- let $\{O\} = \ell \cap H$ then $v = \overrightarrow{OP_1}$ $\quad w_i = \overrightarrow{OP_i}$ $\quad i = 2, \dots, n$

$$\Rightarrow \quad \phi(\overrightarrow{OP_1}) = \overrightarrow{\text{Pr}_{H,v}(O)\,\text{Pr}_{H,v}(P_1)} = \overrightarrow{OO} = 0$$

$$\phi(\overrightarrow{OP_i}) = \overrightarrow{\text{Pr}_{H,v}(O)\,\text{Pr}_{H,v}(P_i)} = \overrightarrow{OP_i} = \overrightarrow{OP_i} \quad \forall i = 2, \dots, n$$

- Since $v \nparallel H$, $b = \{v, w_2, \dots, w_n\}$ is a basis of V and

$$[\phi]_b = \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}$$

- ϕ has two eigenvalues: 0 and 1

$$V_0(\phi) = \langle v \rangle \qquad V_1(\phi) = W$$

- let $\phi = \text{lin}\,\text{Ref}_{H,v}$ with W, w_i, P_i, O as above $\phi(\overrightarrow{OP_i}) = \overrightarrow{OP_i}$ $i = 2, n$

  and $\phi(\overrightarrow{OP_1}) = \overrightarrow{\text{Ref}_{H,v}(O)\,\text{Ref}_{H,v}(P_1)} = \overrightarrow{OP'} = -\overrightarrow{OP}$ since O is midpoint of $[PP']$

$$\Rightarrow [\phi]_b = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots\ 1 \end{bmatrix}, \text{ spectrum of } \phi \text{ is } \{1, -1\}, \quad V_{-1}(\phi) = \langle v \rangle \quad V_1(\phi) = W$$

3. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

$A = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ the matrix is already diagonal

- so the eigenvalues are 1 and -1
- $V_1(A) = \langle \begin{bmatrix}1\\0\end{bmatrix} \rangle \qquad V_{-1}(A) = \langle \begin{bmatrix}0\\-1\end{bmatrix} \rangle$

$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \qquad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix}1-T & 1\\ 0 & 1-T\end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$$A\cdot\begin{bmatrix}v_1\\v_2\end{bmatrix} = 1\cdot\begin{bmatrix}v_1\\v_2\end{bmatrix} \iff \begin{cases} v_1+v_2 = v_1\\ v_2 = v_2\end{cases}$$ the solutions are $\begin{bmatrix}v_1\\0\end{bmatrix}$

$\Rightarrow V_1(A) = \langle \begin{bmatrix}1\\0\end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \qquad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix}1-T & 0\\ 1 & 1-T\end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$$A\cdot\begin{bmatrix}v_1\\v_2\end{bmatrix} = 1\cdot\begin{bmatrix}v_1\\v_2\end{bmatrix} \iff \begin{cases} v_1 = v_1\\ v_1+v_2 = v_2\end{cases}$$ the solutions are $\begin{bmatrix}0\\v_2\end{bmatrix}$

$\Rightarrow V_1(A) = \langle \begin{bmatrix}0\\1\end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad P_A = \det(A - T \cdot I_2) = \det \begin{bmatrix} 1-T & 1 \\ 1 & 1-T \end{bmatrix} = (1-T)^2 - 1$

$$= T^2 - 2T = T(T-2)$$

· so A has two eigenvalues $\lambda_1 = 0$ and $\lambda = 2$

$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{cases}$ the solutions are $\begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$

$\Rightarrow V_0(A) = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$

$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 2v_1 \\ v_1 + v_2 = 2v_2 \end{cases}$ the solutions are $\begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$

$\Rightarrow V_2(A) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$

4. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$\phi(x,y,z) = (x+y-z, y+z, 2x).$$

Find the matrix $M_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where

$$\boldsymbol{b} = \{(1,1,0),(-1,0,1),(1,1,1)\}.$$

Let e be the canonical basis of $\mathbb{R}^3$ $\quad e = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right)$

then

$$[\phi_e(x,y,z)]_e = \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 2 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$[\phi]_b = M_{e,b}^{-1}[\phi]_e M_{e,b}$$

$$M_{e,b} = \begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} \Rightarrow M_{e,b}^{-1} = \frac{1}{\underset{\substack{\| \\ 0+1+0 \\ -0-(-1)-1}}{\det M_{e,b}}}\begin{bmatrix}-1 & -1 & 1\\ 2 & 1 & -1\\ -1 & 0 & 1\end{bmatrix}^T = \begin{bmatrix}-1 & 2 & -1\\ -1 & 1 & 0\\ 1 & -1 & 1\end{bmatrix}$$

$$\Rightarrow [\phi]_b = \begin{bmatrix}-1 & 2 & -1\\ -1 & 1 & 0\\ 1 & -1 & 1\end{bmatrix} \cdot \underbrace{\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 2 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}}$$

$$= \begin{bmatrix}-1 & 2 & -1\\ -1 & 1 & 0\\ 1 & -1 & 1\end{bmatrix} \cdot \begin{bmatrix}2 & -2 & 1\\ 1 & 1 & 2\\ 2 & -2 & 2\end{bmatrix} = \begin{bmatrix}-2 & 6 & 1\\ -1 & 3 & 1\\ 3 & -5 & 1\end{bmatrix}$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}.$$

$$P_A = \det(A - T\cdot I_3) = \det \begin{vmatrix} -6-T & 2 & -5 \\ -4 & 4-T & -2 \\ 10 & -3 & 8-T \end{vmatrix} =$$

$$= (-6-T)\begin{vmatrix} 4-T & -2 \\ -3 & 8-T \end{vmatrix} - 2\begin{vmatrix} -4 & -2 \\ 10 & 8-T \end{vmatrix} - 5\begin{vmatrix} -4 & 4-T \\ 10 & -3 \end{vmatrix}$$

$$= -(6+T)(32-12T+T^2-6) - 2(4T-32+20) - 5(12-40+10T)$$

$$= \dots = -T^3 + 6T^2 - 12T + 8$$

$$T^3 - 3\cdot 2T^2 + 3\cdot 2^2 T + 2^3 = 0$$

$$\Longleftrightarrow \quad (T-2)^3 = 0 \quad \Longrightarrow \quad 2 \text{ is the only eigenvalue of } A$$

To find the eigenvectors we solve

$$A\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix} \underset{L_3 + L_1 + \frac{1}{2}L_2}{\sim} \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2 $\Rightarrow$ the solution space has dimension 1

$$\Rightarrow \dim V_2(A) = 1 \lneq h_2(A) = 3$$

• $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}$ We see that 1 is an eigenvalue and that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for 1.

$$P_B = \det \begin{bmatrix} 1-T & 0 & 0 \\ 0 & -3-T & -15 \\ 0 & 2 & 8-T \end{bmatrix} = (1-T)\left(\underbrace{(3+T)(T-8)+30}_{T^2+5T-24+30} \right) = (1-T)(T+2)(T+3)$$

$\Rightarrow$ the spectrum of B is $\{1, -2, -3\}$

B is diagonalizable by prop. 7.9

6. Show that an isometry is bijective.

$\downarrow$

map which preserves distances $f: \mathbb{E}^n \to \mathbb{E}^n$

$d(P,Q) = d(f(P), f(Q))$

injectivity: $f(P) = f(Q) \overset{?}{\Rightarrow} P = Q$

if $f(P) = f(Q) \Rightarrow d(f(P), f(Q)) = 0$

$\Rightarrow d(P,Q) = 0 \Rightarrow P = 0$

surjectivity: isometries are affine maps $f \in AGL(\mathbb{E}^n)$

$\Rightarrow f(x) = Ax + b$ w.r.t some coordinate system

• it is the composition of two maps $f_1(x) = Ax$ and $f_2(x) = x + b$

• f_2 is bijective $\Rightarrow \begin{cases} f \text{ is surjective} \Leftrightarrow f_1(x) \text{ is surj.} \\ f \text{ is injective} \Leftrightarrow f_1(x) \text{ is injective} \end{cases}$

• by the first part f_1 is injective, it is an injective linear map $\mathbb{R}^n \to \mathbb{R}^n$

$\Rightarrow$ it is also surj.

7. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation
$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}\mathbf{x} + \begin{bmatrix}1\\ -2\end{bmatrix}.$$

- that f is a rotation, follows from the classification of isometrie in dim 2
- the center of the rotation f is the point satisfying the equation

$$f(x) = x$$

$$\Leftrightarrow \frac{1}{\sqrt{13}}\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ -2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\Leftrightarrow \begin{cases}\left(\frac{2}{\sqrt{13}}-1\right)x_1 - \frac{3}{\sqrt{13}}x_2 = -1\\ \frac{3}{\sqrt{13}}x_1 + \left(\frac{2}{\sqrt{13}}-1\right)x_2 = 2\end{cases}$$

$$\begin{vmatrix}-1 & -\frac{3}{\sqrt{13}}\\ 2 & \frac{2}{\sqrt{13}}-1\end{vmatrix} = -\frac{2}{\sqrt{13}} + 1 + \frac{6}{\sqrt{13}} = \frac{4}{\sqrt{13}} + 1 = \frac{4+\sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix}\frac{2}{\sqrt{13}}-1 & -1\\ \frac{3}{\sqrt{13}} & 2\end{vmatrix} = \frac{4}{\sqrt{13}} - 2 + \frac{3}{\sqrt{13}} = \frac{7-2\sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix}\frac{2}{\sqrt{13}}-1 & -\frac{3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}-1\end{vmatrix} = \frac{1}{13}\begin{vmatrix}2-\sqrt{13} & -3\\ 3 & 2-\sqrt{13}\end{vmatrix} = \frac{1}{13}\left[4-4\sqrt{13}+13+9\right] = \frac{1}{13}(26-4\sqrt{13})$$
$$= \frac{2\sqrt{13}-4}{\sqrt{13}} = 2\left(\frac{\sqrt{13}-2}{\sqrt{13}}\right)$$

$$\Rightarrow x_1 = \frac{4+\sqrt{13}}{2(\sqrt{13}-2)} = \frac{1}{2}\frac{(4+\sqrt{13})(\sqrt{13}+2)}{(13-4)} = \frac{1}{18}\left(4\sqrt{13}+8+13+2\sqrt{13}\right) = \frac{1}{18}\left(21+6\sqrt{13}\right)$$

$$\Rightarrow x_1 = \frac{1}{6}\left(7+2\sqrt{13}\right)$$

and $$x_2 = \frac{7-2\sqrt{13}}{2(\sqrt{13}-2)} = \frac{1}{18}(7-2\sqrt{13})(\sqrt{13}+2) = \frac{1}{18}\left(7\sqrt{13}+14-26-4\sqrt{13}\right) = \frac{1}{18}\left(-12+3\sqrt{13}\right)$$

$$\Rightarrow x_2 = \frac{1}{6}\left(-4+\sqrt{13}\right)$$

$\Rightarrow$ the center of the rotation f is $C_f = \begin{bmatrix}\frac{7+2\sqrt{13}}{6}\\ \frac{-4+\sqrt{13}}{6}\end{bmatrix}$

A rotation of angle 45° around C_f is obtained by

1. translating C_f in the origin
2. rotating with angle 45° around the origin
3. translating back.

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \underbrace{\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \underbrace{\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix} \right)}_{1}}_{2} + \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}}_{3}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{6\sqrt{2}} \begin{bmatrix} 11+\sqrt{13} \\ 3+3\sqrt{13} \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 7+2\sqrt{13} \\ -4+\sqrt{13} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{6\sqrt{2}} \begin{bmatrix} -11-\sqrt{13}+7\sqrt{2}+2\sqrt{26} \\ -3-3\sqrt{13}-4\sqrt{2}+\sqrt{26} \end{bmatrix}$$

rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

8. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

denote it by θ

then $\cos\theta = \frac{2}{\sqrt{13}}$

$f(x) = Ax + b$ if $f^{-1}(x) = \tilde{A}x + \tilde{b}$ then $x = f^{-1} \circ f(x) = \tilde{A}(Ax+b) + \tilde{b}$

$$= \tilde{A}Ax + \tilde{A}b + \tilde{b}$$

$\Rightarrow \tilde{A} = A^{-1}$ and $\tilde{b} = -\tilde{A}b$

$$\left(\frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \right)^{-1} = \frac{1}{\frac{4+9}{13}} \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}^T = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

we expected this since $A \in O(2)$ so $\det A = 1$

we expected this since $A \in O(2) \Rightarrow A^{-1} = A^T$

$$\tilde{b} = -\frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{13}} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\Rightarrow f^{-1} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{\sqrt{13}} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Rem $\operatorname{tr} A = \operatorname{tr} A^T = \operatorname{tr} A^{-1}$ for $A \in O(n)$

$$\Rightarrow \cos \theta_A = \cos \theta_{A^{-1}}$$

obviously $\theta_{A^{-1}} = -\theta_A$ but with the cosine-trace formula we don't see this

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2, 5)$. Determine the inverse transformation, T^{-1}.

$$T(x) = \mathrm{Rot}_{-\frac{\pi}{3}} \circ T_{(-2,5)}(x)$$

$$\Rightarrow T^{-1}(x) = \left(\mathrm{Rot}_{-\frac{\pi}{3}} \circ T_{(-2,5)}\right)^{-1}(x) = T^{-1}_{(-2,5)} \circ \mathrm{Rot}^{-1}_{-\frac{\pi}{3}}(x)$$

$$= T_{(2,-5)} \circ \mathrm{Rot}_{\frac{\pi}{3}}(x)$$

$$= \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

10. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

A:

$$\begin{vmatrix} 73-T & 36 \\ 36 & 52-T \end{vmatrix} = (73-T)(52-T) - 6^4 = 4\cdot 13\cdot 73 - 125T + T^2 - 6^4$$

$$= T^2 - 125T + 4\underbrace{(949 - \underbrace{2^2\cdot 3^4}_{324})}_{625}$$

$$= T^2 - 5^3 T + 4\cdot 5^4$$

$$\Delta = 5^6 - 2^4\cdot 5^4$$

$$= 5^4(25-16)$$

$$\Rightarrow T_{1,2} = \frac{5^3 \pm 5^2 3}{2}$$

$$= \frac{25(5\pm 3)}{2} \begin{matrix} 100 \\ 25 \end{matrix}$$

B: 338 , −169

C: 578 , 0

11. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_1 & \cdots \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & \cdots \\ \sin(\theta_1+\theta_2) & \cdots \end{bmatrix}$$

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

- $A \in SO(3) \iff AA^t = I_3$ & $\det A = 1$

$$A^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

On checks that $\det A = 1$ so, yeah, $A \in SO(3)$

- the axis of rotation is the line passing through the origin in the direction of the eigenvectors for the eigenvalue 1

this is obtained by solving $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\iff \begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = 0 \Big\} \Rightarrow 2x + z = 0 \Rightarrow z = -2x$$

$\Rightarrow$ eigenspace for $\lambda = 1$ is $V_1 = \{(t, 0, -2t) : t \in \mathbb{R}\}$ ← this is a line passing through the origin, it is the rotation axis

the eigenvectors are the non-zero vectors in V_1

- the angle of rotation θ is determined by

$$\operatorname{tr}(A) = 1 + 2\cos\theta \iff \cos\theta = \frac{\operatorname{tr}A - 1}{2} = \frac{-\frac{1}{3} - 1}{2} = -\frac{2}{3}$$

$$\text{so} \quad \theta = \arccos\left(-\frac{2}{3}\right)$$

the calculation for B is similar.