All objects considered here are in the plane $\mathbb{E}^2$.

1. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,
2. ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,
3. ℓ contains the point $M(1,7)$ and is parallel to Oy,
4. ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2. For the lines ℓ in the previous exercise

1. give a Cartesian equation for ℓ,
2. describe all direction vectors for ℓ.

3. Determine a Cartesian equations for the line ℓ in the following cases:

1. ℓ has slope -5 and contains the point $A(1,-2)$,
2. ℓ has slope 8 and is at distance 2 from the origin,
3. ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,
4. ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

4. For the lines ℓ in the previous exercise

1. give parametric equations for ℓ,
2. describe all normal vectors for ℓ.

5. Consider a line ℓ. Show that

1. if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,
2. if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

6. Consider the points $A(1,1)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

7. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

1. the length of the altitude from A,
2. the line containing the altitude from A.

9. Determine the circumcenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

10. Determine the angle between the lines $\ell_1 : y = 2x + 1$ and $\ell_2 : y = -x + 2$.

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x + 4y - 20 = 0$. Determine the coordinates of A'.

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

1. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,
2. ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,
3. ℓ contains the point $M(1,7)$ and is parallel to Oy,
4. ℓ contains the points $M(2,4)$ and $N(2,-5)$.

1. $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad t \in \mathbb{R} \quad \Leftrightarrow \quad \begin{cases} x = 1 + 3t \\ y = 2 - t \end{cases} \quad t \in \mathbb{R}$

2. $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad s \in \mathbb{R} \quad \Leftrightarrow \quad \begin{cases} x = 4s \\ y = 5s \end{cases} \quad s \in \mathbb{R}$

3. $\ell \parallel Oy \Rightarrow j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a direction vector for $\ell \Rightarrow \ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \lambda \in \mathbb{R} \Leftrightarrow \ell : \begin{cases} x = 1 \\ y = 7 + \lambda \end{cases}$

4. $\ell \ni M, N$, $M \neq N \Rightarrow \overrightarrow{MN}(0,-9)$ is a dir. vect for $\ell \Rightarrow \ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -9 \end{bmatrix} \ \alpha \in \mathbb{R} \Leftrightarrow \ell : \begin{cases} x = 2 \\ y = 4 - 9\alpha \end{cases}$

2. For the lines ℓ in the previous exercise

1. give a Cartesian equation for ℓ,
2. describe all direction vectors for ℓ.

(1.1) $\ell : \frac{x-1}{3} = \frac{y-2}{-1} \ (= t) \Leftrightarrow \ell : -x - 3y + 1 + 6 = 0 \Leftrightarrow \ell : x + 3y - 7 = 0$

$a(3,-1)$ is a dir. vect. for $\ell \Rightarrow$ all dir. vectors for ℓ are

$\{ t a : t \in \mathbb{R} \ t \neq 0 \} = \{ (3t, -t) : t \neq 0 \}$

(1.2) $\ell : \frac{x}{4} = \frac{y}{5} \ (= s) \Leftrightarrow \ell : 5x - 4y = 0$, dir vectors for $\ell = \{ (4\alpha, 5\alpha) : \alpha \in \mathbb{R} \ \alpha \neq 0 \}$

(1.3) $\ell : x = 1$ dir. vect. for $\ell = \{ \beta j : \beta \in \mathbb{R}, \beta \neq 0 \} = \{ (0, \beta) : \beta \in \mathbb{R} \ \beta \neq 0 \}$

(1.4) $\ell : x = 2$ dir. vect. $\{ (0, \gamma) : \gamma \in \mathbb{R} \ \gamma \neq 0 \}$

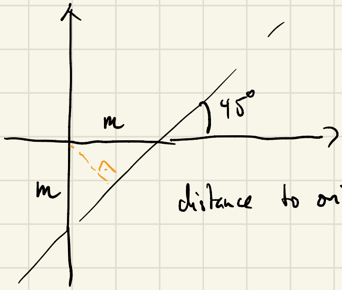
3. Determine a Cartesian equations for the line ℓ in the following cases:

1. ℓ has slope -5 and contains the point $A(1,-2)$,
2. ℓ has slope 1 and is at distance 2 from the origin,
3. ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,
4. ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

1. $\ell: y = -5x + m \ni A(1,-2) \Rightarrow -2 = -5 + m \Rightarrow m = 3$

$\Rightarrow \ell: y = -5x + 3 \iff \ell: 5x + y - 3 = 0$

2.



$\ell: \dfrac{x}{m} + \dfrac{y}{m} = 1$

distance to origin is $\dfrac{\sqrt{2}\cdot m}{2} = 2 \Rightarrow m = \dfrac{4}{\sqrt{2}}$

$\Rightarrow \ell: \sqrt{2}x + \sqrt{2}y - 4 = 0$

$\frac{\sqrt{3}}{2}$

$\frac{2}{2}$

3. $\ell: y - 3 = \overset{\sqrt{3}}{\tan 60°}(x+2) \iff \ell: \sqrt{3}x - y + 2\sqrt{3} - 3 = 0$

4.

n is a normal vector for ℓ

$\ell: 4(x-1) + 3(y-7) = 0$

$\iff \ell: 4x + 3y - 25 = 0$

4. For the lines ℓ in the previous exercise

1. give parametric equations for ℓ,
2. describe all normal vectors for ℓ.

(3.1) $\ell: \begin{cases} y = -5x + 3 \\ x = x \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \end{bmatrix} \quad t \in \mathbb{R}$

$n(5,1)$ is a normal vect. all others are $\alpha \cdot n$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$

(3.2) $\ell: \begin{cases} x = \frac{4}{\sqrt{2}} - y \\ y = y \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda \in \mathbb{R}$, $n(1,1)$ is a normal vect

$\Rightarrow$ all other normal vects are (β, β) $\beta \in \mathbb{R}$, $\beta \neq 0$

(3.3) $\ell: \begin{cases} x = x \\ y = \sqrt{3}x + 2\sqrt{3} - 3 \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{3}-3 \end{bmatrix} + s \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad s \in \mathbb{R}$

$n(\sqrt{3}, 1)$ is a normal vect for $\ell \Rightarrow$ all other normal vectors for ℓ are $(\sqrt{3}t, t)$ $t \in \mathbb{R}$, $t \neq 0$

(3.4) $\ell: \begin{cases} x = \frac{25}{4} - \frac{3}{4}y \\ y = y \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25/4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3/4 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$, $(4, 3)$ is a normal vector for ℓ

5. Consider a line ℓ. Show that

1. if $\mathbf{v}(v_1, v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2, -v_1)$ is a normal vector for ℓ,
2. if $\mathbf{n}(n_1, n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2, -n_1)$ is a direction vector for ℓ.

- If v is a dir. vect. for ℓ then n is a normal vector for ℓ if and only if

$v \perp n \iff v \cdot n = 0$ which is true in our case since

$$v(v_1, v_2) \cdot n(v_2, -v_1) = v_1 v_2 - v_2 v_1 = 0$$

- If n is a normal vector for ℓ then v is a direction vector for ℓ iff

$n \perp v \iff n \cdot v = 0$ gain true if $v = v(n_2, -n_1)$

6. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

$\Leftrightarrow$ give equations

Method I calculate midpoints of sides, then write equations for each median

Method II calculate centroid G, $G = \dfrac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \end{bmatrix}}{3} = \dfrac{\begin{bmatrix} 3 \\ 12 \end{bmatrix}}{3} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

the medians are AG, BG, CG

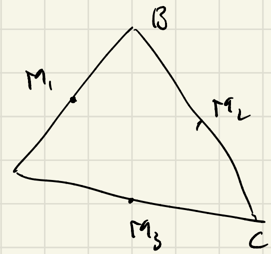
$AG: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{AG}} \quad t \in \mathbb{R}$

$BG: \dfrac{x+2}{1+2} = \dfrac{y-3}{4-3}$

$CG: \ldots$

7. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

Method I



$AB \ni M_1$, $AB \parallel M_2M_3$

$\Downarrow$

$\overrightarrow{M_2M_3}(2,-5)$ is a direction vector for AB

$\Rightarrow AB: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ or $AB: \dfrac{x-1}{2} = \dfrac{y-2}{-5}$

similar for BC and CD

Method II $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$

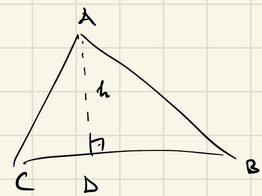
then $\begin{cases} 1 = \dfrac{x_A + x_B}{2} \\ 2 = \dfrac{y_A + y_B}{2} \\ 3 = \dfrac{x_B + x_C}{2} \\ 4 = \dfrac{y_B + y_C}{2} \\ 5 = \dfrac{x_C + x_A}{2} \\ -1 = \dfrac{y_C + y_A}{2} \end{cases}$

solving this system one obtains the coordinates of the vertices of the triangle ABC

so, one can write down the equations required

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

1. the length of the altitude from A,
2. the line containing the altitude from A.



1) h = length of the altitude

$\text{area}_{\Delta ABC} = \frac{1}{2} \underbrace{\|\overrightarrow{AC} \times \overrightarrow{AB}\|}_{\text{area of parallelogram spanned by } \overrightarrow{AC}, \overrightarrow{AB}} = \frac{1}{2} h \cdot \|\overrightarrow{CB}\|$

$\|\overrightarrow{AC} \times \overrightarrow{AB}\| = \left| \begin{vmatrix} 1 & 3 & 1 \\ -4 & 3 & 1 \\ 2 & 9 & 1 \end{vmatrix} \right| = |3 - 36 + 6 - 6 + 12 - 9| = 30$ and $\|\overrightarrow{CB}\| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

since $A, B, C \in Oxy$

$\Rightarrow h = \dfrac{30}{6\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

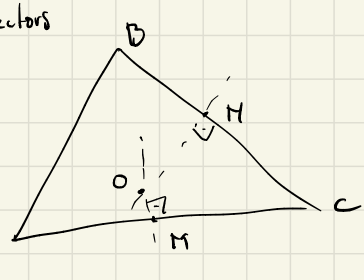
2.) $\ell \perp CB \Rightarrow \vec{CB}$ is a normal vector , $\vec{CB}(-6,6) \Rightarrow n(-1,1)$ is a normal vector

$\ell \ni A \Rightarrow \ell: \; -(x-1)+(y-3)=0$

Rem. one can intersect ℓ with BC to obtain D then $h=|AD|$

9. Determine the circumcenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

O = intersection of perpendicular bisectors



$O = MO \cap NO$ where M is the midpoint of AC

N —— " —— BC

$M=\begin{bmatrix}3\\4\end{bmatrix}$, $N=\begin{bmatrix}4\\2\end{bmatrix}$

$MO \perp AC \Rightarrow \vec{AC}(4,4)$ is a normal vector for MO, and so is $(1,1)$

$NO \perp BC \Rightarrow \vec{BC}(2,8)$ —— " —— NO, and so is $(1,4)$

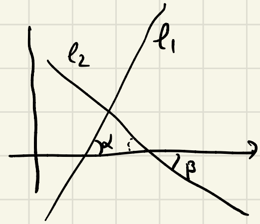
$\Rightarrow MO: \; (x-3)+(y-4)=0 \iff MO: \; x+y-7=0$

$NO: \; (x-4)+4(y-2)=0 \iff NO: \; x+4y-12=0$

$$\Rightarrow MO\cap NO: \begin{cases} x+y-7=0 \\ x+4y-12=0 \end{cases} \ominus$$

$3y-5=0 \Rightarrow y=\frac{5}{3} \Rightarrow x=7-\frac{5}{3}=\frac{16}{3} \Rightarrow O\left(\frac{16}{3},\frac{5}{3}\right)$

10. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.



method I $\tan\alpha = 2$, $\tan\beta=-1$

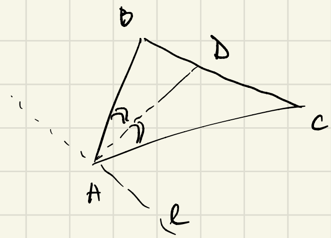
$\Rightarrow$ one angle is $\alpha+\beta \Rightarrow \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2-1}{1+2}=\frac{1}{3}$

$\simeq 18.43°$

method II $\left.\begin{matrix} n_1(2,-1) \text{ normal vector for } \ell_1 \\ n_2(1,1) \text{ —— '' ——} \end{matrix}\right\} \Rightarrow \cos\sphericalangle(n_1,n_2)=\frac{2-1}{\sqrt{5}\sqrt{2}}=\frac{1}{\sqrt{10}} \simeq 71.57°$

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

a direction vector for the interior angle bisector is



$$v = \frac{\vec{AB}}{\|\vec{AB}\|} + \frac{\vec{AC}}{\|\vec{AC}\|} = \frac{1}{2\sqrt{4+9}}\begin{bmatrix}4\\6\end{bmatrix} + \frac{1}{\sqrt{9+4}}\begin{bmatrix}-3\\-2\end{bmatrix}$$

$$\vec{AB}(4,6),\ \vec{AC}(-3,-2)$$

$$\Rightarrow v = \frac{1}{\sqrt{13}}\begin{bmatrix}-1\\1\end{bmatrix} \quad \Rightarrow \tilde{v} = \begin{bmatrix}-1\\1\end{bmatrix} \text{ is also a direction vector for } AD$$

$$\Rightarrow AD: \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} + t\begin{bmatrix}-1\\1\end{bmatrix} \quad \text{or} \quad AD: \frac{x-1}{-1} = \frac{y+2}{1} \iff AD: x+y+1=0$$

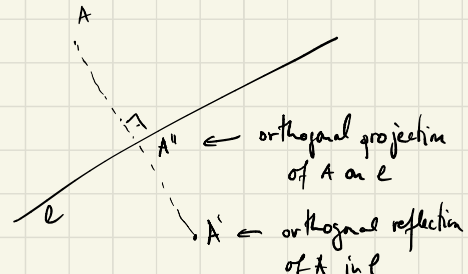
the exterior angle bisector is orthogonal to AD so v is a normal vector for this line

$$\Rightarrow \ell: -1(x-1) + 1(y+2) = 0 \iff \ell: -x+y+3=0$$

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x + 4y - 20 = 0$. Determine the coordinates of A'.

method I

$(3,4)$ is a normal vector for ℓ



$\Rightarrow (3,4)$ is a direction vector for AA'

$\Rightarrow (-4,3)$ is a normal vector for AA'

$$\Rightarrow AA': -4(x-10) + 3(y-10) = 0 \iff AA': -4x+3y+10=0$$

$$\Rightarrow A'': \begin{cases} -4x+3y+10=0 \\ 3x+4y-20=0 \end{cases} \quad \oplus$$

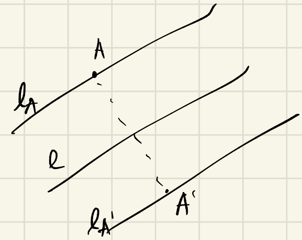
$$-x+7y-10=0 \Rightarrow x = 7y-10$$

$$\Rightarrow 21y - 30 + 4y - 20 = 0 \Rightarrow 25y = 50 \Rightarrow y = 2 \Rightarrow x = 4$$

$$\text{So } A'' = \begin{bmatrix}2\\4\end{bmatrix} \Rightarrow \vec{AA''} = \begin{bmatrix}-8\\-6\end{bmatrix} \Rightarrow \vec{AA'} = \begin{bmatrix}-16\\-12\end{bmatrix} \Rightarrow A' = A + \vec{AA'} = \begin{bmatrix}-6\\-2\end{bmatrix}$$

method II $\quad d(A,\ell) = d(A',\ell) \qquad A' = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\frac{50}{\sqrt{9+16}} = \frac{|3x+4y-20|}{\sqrt{9+16}} \iff |3x+4y-20| = 50$$



$\ell_A : 3x+4y=70 \qquad -3x-4y+20=50$

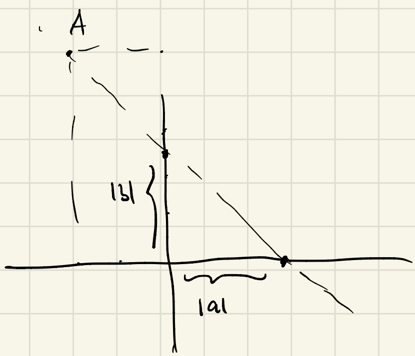
$\ell_{A'} : 3x+4y=-30$

as above $\quad AA' : -4x+3y+10=0$

$\Rightarrow A' : \begin{cases} -4x+3y+10=0 \\ 3x+4y=-30 \end{cases}$ solving we obtain the coordinates of A'

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

Consider ℓ with the eq.



$\ell : \frac{x}{a} + \frac{y}{b} = 1$

$\Rightarrow \ell \cap Ox = (a,0) \qquad \ell \cap Oy = (0,b)$

$\Rightarrow$ the segments are of length $|a|$ and $|b|$

Congruent segment $\Rightarrow |a| = |b|$ so $a = \pm b$

I $\quad a=b$ then $\ell : \frac{x}{a} + \frac{y}{a} = 1$

$\ell \ni A \Rightarrow \frac{-2}{a} + \frac{5}{a} = 1 \iff a=3$ so $\ell : \frac{x}{3} + \frac{y}{3} = 1$

$x+y=3$

II $\quad a=-b$ then $\ell : \frac{x}{a} - \frac{y}{a} = 1$

$\ell \ni A \Rightarrow \frac{-2}{a} - \frac{5}{a} = 1 \iff a=-7$ so $\ell : \frac{x}{-7} - \frac{y}{-7} = 1$

$x-y=-7$