Proposition 6.1. Let $\phi \in \mathrm{AGL}(\mathbb{E}^n)$ be an affine transformation given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. The following are equivalent:

1. ϕ is an isometry
2. $A^{-1} = A^t$.

$\boxed{1. \Rightarrow 2.}$ Let $\psi(x) = \phi(x) - b$ be the linear map given by $\psi(x) = Ax$

$\left.\begin{array}{l}\psi \text{ is the composition of the isometrie } \phi \text{ with a translation} \\ \text{translations are isometries}\end{array}\right\} \Rightarrow \psi$ is an isometrie

$\Rightarrow \quad \|\psi(v)\| = \|v\| \quad \forall\, v \in \mathbb{V}^n$

Let $O, e_1, \dots, e_n$ be the orthonormal coordinate system w.r.t which ϕ is given

Let $f_1 = \psi(e_1) = Ae_1$, $f_2 = \psi(e_2) = Ae_2$, ..., $f_n = \psi(e_n) = Ae_n$

Then $\langle f_i, f_j \rangle = \frac{1}{4}\left(\|f_i + f_j\|^2 - \|f_i - f_j\|^2\right)$

$$= \frac{1}{4}\left(\|e_i + e_j\|^2 - \|e_i - e_j\|^2\right)$$

$$= \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The components of $f_i = Ae_i$ are the i-th column of the matrix A

$\Rightarrow$ The (i,j)-entry of the matrix A^tA is $\langle f_i, f_j \rangle = \delta_{ij}$

$\Rightarrow \quad A^tA = I_n$

$\boxed{2. \Rightarrow 1.}$ $\forall$ points $x, y \in \mathbb{E}^n$ we have $\|f(x) - f(y)\| = \|(Ax+b) - (Ay+b)\| = \|A(x-y)\|$

$\Rightarrow \quad \|f(x) - f(y)\|^2 = \|A(x-y)\|^2 = \langle A(x-y), A(x-y) \rangle$

$$= \left(A(x-y)\right)^t A(x-y)$$

$$= (x-y)^t A^t A (x-y) = (x-y)^t (x-y)$$

$$= \|x-y\|^2$$

$\Rightarrow \quad d(f(x), f(y)) = d(x, y)$

Proposition 6.2. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a matrix such that $A^t A = I_n$. Then $\det(A) \in \{\pm 1\}$.

$$1 = \det(I_n) = \det(A^t A) = \det(A^t)\det(A) = \det(A)^2$$

Proposition 6.3. A matrix A is in $SO(2)$ if and only if A has the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$.

$\boxed{\Leftarrow}$ • It is an easy check to see that $R_\theta \in SO(2)$

$\boxed{\Rightarrow}$ • Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$

then $\det A = ad - bc = 1.$

and $A^t \cdot A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Bigg\} \Leftrightarrow \begin{cases} a^2+c^2=1 \\ b^2+d^2=1 \\ ab+cd=0 \\ ad-bc=1 \end{cases}$

$a^2+c^2=1 \Rightarrow \exists\, \theta$ s.t $a=\cos\theta$ and $c=\sin\theta$

$b^2+d^2=1 \Rightarrow \exists\, \tilde\theta$ s.t. $d=\cos\tilde\theta$ and $b=\sin\tilde\theta$

So $\left.\begin{aligned} ab+cd &= \cos\theta\sin\tilde\theta + \sin\theta\cos\tilde\theta = \sin(\theta+\tilde\theta)=0 \\ ad-bc &= \cos\theta\cos\tilde\theta - \sin\tilde\theta\sin\theta = \cos(\theta+\tilde\theta)=1 \end{aligned}\right\} \Rightarrow \tilde\theta = -\theta + 2k\pi$

$\Rightarrow A = \begin{pmatrix} \cos\theta & \sin -\theta \\ \sin\theta & \cos -\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Corollary 6.4. A direct isometry ϕ of $\mathbb{E}^2$ that fixes a point is either the identity or a rotation. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\operatorname{tr}(\operatorname{lin}(\phi))}{2}.$$

Let $\phi(x) = Ax + b$. It is a direct isometry if $A \in SO(2) \iff A = R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

So $\phi(x) = R_\theta x + b$

If $\theta = 0$ then ϕ has a fixed point only if $b = 0$ in which case $\phi = \mathrm{Id}$

Assume that $\theta \neq 0$

Let p be a fixed point for ϕ

$$\phi(p) = p \iff R_\theta p + b = p \iff b = (I_2 - R_\theta)p \quad (*)$$

Moreover

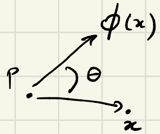
$$\det(I - R_\theta) = \det\begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} = (1-\cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta$$

So, if $\theta \neq 0$ then $\det(I - R_\theta) \neq 0 \Rightarrow$ eq. $(*)$ has a unique solution p

So, in this case ϕ has a unique fixed point

To finish the proof we show that ϕ is a rotation around the point p

$$\begin{aligned} \phi(x) - \phi(p) &= \phi(x) - p = R_\theta x + b - p \\ &\overset{(*)}{=} R_\theta x + (I - R_\theta)p - p \\ &= R_\theta (x - p) \end{aligned}$$



this means that ϕ rotates $\overrightarrow{px}$ with θ and maps it to $\overrightarrow{\phi(p)\phi(x)} = \overrightarrow{p\phi(x)}$

. Finally $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$ so $\operatorname{tr}(\operatorname{lin}\phi) = \operatorname{tr} R_\theta = 2\cos\theta$.

Theorem 6.5 (Euler). A direct isometry ϕ of $\mathbb{E}^3$ that fixes a point is either the identity or a rotation around an axis that passes through that point. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\operatorname{tr}(\operatorname{lin}(\phi)) - 1}{2}.$$

- We may suppose that ϕ fixes the origin. $\Rightarrow$ ϕ is linear given by $\phi(x) = Ax$ with respect to some coordinate system

- Since ϕ is a direct isometry $A \in SO(3)$ $\quad A^tA = I_3$, $\det A = 1$

- A rotation around an axis fixes the axis. We show that A fixes a line.

  For this it suffices to show that A has an eigenvector v for the eigenvalue 1 since then $\phi(tv) = A(tv) = t(Av) = tv \quad \forall t \in \mathbb{R}$

- So, we show that $\det(A - I) = 0$

We have
$$\det(A - I_3) = \underbrace{\det(A^t)}_{1} \det(A - I_3) = \det(A^t(A - I_3))$$
$$= \det(A^tA - A^t) = \det(I_3 - A^t) = \det((I_3 - A)^t)$$
$$= \det(I_3 - A)$$

Since $A - I$ is a 3×3 matrix $\quad \det(A - I_3) = -\det(I_3 - A)$

$$\Rightarrow \det(I_3 - A) = -\det(I_3 - A)$$

$$\Rightarrow \det(I_3 - A) = 0$$

- Next, we show that ϕ is a rotation around the axis $\mathbb{R}a$ where a is an eigenvector for the eigenvalue 1

- Choose a vector $u_1 \perp a$ with $\|u_1\| = 1$ and let $u_2 := a \times u_1$

Then Au_1 and Au_2 are also orthogonal to a (to the axis $\mathbb{R}a$) since

$$\langle Au_i, a \rangle = \langle Au_i, Aa \rangle = \langle u_i, a \rangle = 0$$

$\Rightarrow$ Au_1 and Au_2 are linear combinations of u_1 and u_2

since $\|Au_1\| = \|Au_2\| = 1$ we have

$$Au_1 = \cos(\theta)\, u_1 + \sin(\theta)\, u_2$$
$$Au_2 = -\sin(\theta)\, u_1 + \cos(\theta)\, u_2$$

where θ is the angle between u_1 and Au_1

$\Rightarrow$ the matrix of ϕ with respect to the basis u_1, u_2, a is

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a rotation around the axis $\mathbb{R}a$

- Finally the trace of ϕ is the trace of the above matrix $= 2\cos\theta + 1$.

Proposition 6.8. Any set of mutually orthogonal vectors is linearly independent. In particular, any set of n mutually orthogonal vectors is a basis of $\mathbb{V}^n$.

Let $v_1, \dots, v_t$ be an orthogonal set of vectors

If we have a linear relation $a_1 v_1 + \dots + a_t v_t = 0$

Then, for each $i \in \{1, \dots, t\}$

$$0 = \langle v_i, a_1 v_1 + \dots + a_t v_t \rangle$$
$$= a_1 \langle v_i, v_1 \rangle + \dots + a_t \langle v_i, v_t \rangle$$
$$= a_i \langle v_i, v_i \rangle$$

Since $\langle v_i, v_i \rangle > 0$ it follows that $a_i = 0$

So for all i $a_i = 0$ $\Rightarrow$ $v_1, \dots, v_t$ are linearly independent.

Proposition 6.9. Let $e = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $f = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of the vector space $\mathbb{V}^n$, and suppose that e is orthonormal. The basis f is orthonormal if and only if the base change matrix $\mathrm{M}_{e,f}$ is orthogonal.

- The columns of $M = M_{e,f}(\mathrm{Id}_V)$ are the components of the vectors $f_1, \dots, f_n$ with respect to the basis e
- The basis f is orthogonal if and only if

$$\underset{\substack{\uparrow \\ i\text{-th column}}}{M_i^t} \underset{\substack{\uparrow \\ j\text{-th column}}}{M_j} = \langle f_i, f_j \rangle = \delta_{ij} \quad \forall\, i,j$$

- The above condition translates exactly to $M^t M = I_n \iff M \in O(n)$.

Lemma 6.10. The characteristic polynomial of a symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has only real roots.

- We may consider A as a matrix over $\mathbb{C}$, $A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$
- Consider the operator $\phi_A : \mathbb{C}^n \to \mathbb{C}^n$ defined by A
- Let $\lambda \in \mathbb{C}$ be a root of the characteristic polynomial of A and

  Let $x \in \mathbb{C}^n$ be an eigenvector for λ

$$A x = \lambda x$$

- Taking complex conjugates we have $\overline{Ax} = \bar{A}\bar{x} = A\bar{x} = \bar{\lambda}\bar{x}$

  Then $\quad \bar{x}^t A x = \bar{x}^t (Ax) = \bar{x}^t \lambda x = \lambda \bar{x}^t x$

  and $\quad \bar{x}^t A x = (\bar{x}^t A) x) = (A\bar{x})^t x = (\bar{\lambda} \bar{x}^t) x = \bar{\lambda} \bar{x}^t x$

- Note that $\quad \bar{x}^t x = \bar{x}_1 x_1 + \dots + \bar{x}_n x_n \underset{\neq}{\geq} 0 \quad$ since $x \neq 0$

$\Rightarrow \quad \bar{\lambda} = \lambda \quad \Rightarrow \lambda \in \mathbb{R}$

Theorem 6.11. (Spectral Theorem) Let $T : \mathbb{V}^n \to \mathbb{V}^n$ be a symmetric operator. There is an orthonormal basis of $\mathbb{V}^n$ with respect to which the matrix of T is diagonal.

Proof by induction on $n = \dim(V)$

- If $n=1$ there is nothing to prove
- Suppose $n \geq 2$ and that the theorem holds for a space of dimension $n-1$
- Lem. 5.16 $\Rightarrow$ all eigenvalues for T are real
- Let λ be an eigenvalue of T and let e_1 be an eigenvector
  - we may choose e_1 of length 1.
- Consider $U = e_1^\perp$. For each $u \in U$ we have

$$\langle T(u), e_1\rangle \overset{\text{since } T \text{ is symmetric}}{=} \langle u, T(e_1)\rangle = \langle u, \lambda e_1\rangle = \lambda\langle u, e_1\rangle = \lambda \cdot 0 = 0$$

$\Rightarrow T(u) \in U$

$\Rightarrow T$ induces an operator $T_U : U \to U$

Since $T_U(u) = T(u)$ $\forall u \in U$ the operator T_U is symmetric

- By induction, U has an orthonormal basis $\{e_2, \dots, e_n\}$ which diagonalizes T_U
- Thus $\{e_1, \dots, e_n\}$ is an orthonormal basis of V which diagonalizes T.

Theorem 6.12. For every real symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ there is an orthogonal matrix $M \in O(n)$ such that $M^{-1}AM$ is diagonal.

- A is the matrix of a symmetric operator ϕ_A on $\mathbb{R}^n$ w.r.t the canonical basis $e_1, \dots, e_n$

- By the Spectral Theorem, there is an orthonormal basis $f_1, \dots, f_n$ w.r.t which the matrix of T_A is diagonal

$$[T_A]_f = M_{ef}^t \, [T_A]_e \, M_{ef} \qquad M_{ef} \in O(n)$$

$$\begin{array}{cccc} [T_A]_f & M_{ef}^t & [T_A]_e & M_{ef} \\ \| & \| & \| & \| \\ \text{diagonal} & M^{-1} & A & M \end{array}$$

Proposition 6.13. Let $T : \mathbb{V}^n \to \mathbb{V}^n$ be a symmetric operator on a Euclidean vector space. If λ and μ are two distinct eigenvalues of T then every eigenvector with eigenvalue λ is orthogonal to every eigenvector with eigenvalue μ.

- Let v be an eigenvector for λ
  w —— '' —— μ

- Then $\langle T(v), w\rangle = \langle \lambda v, w\rangle = \lambda\langle v, w\rangle$
  $\langle v, T(w)\rangle = \langle v, \mu w\rangle = \mu\langle v, w\rangle$

Since T is symmetric $\langle T(v), w\rangle = \langle v, T(w)\rangle$

$\Rightarrow \quad \lambda\langle v, w\rangle = \mu\langle v, w\rangle$

$\Rightarrow \quad \langle v, w\rangle = 0$ since $\lambda \neq \mu$.