GRAMMARS

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1. Given the grammar $G = (N,\ \Sigma,\ P,\ S)$

$$N = \{S, C\}$$
$$\Sigma = \{a, b\}$$
$$\begin{aligned} P:\ S &\to ab \mid aCSb \\ C &\to S \mid bSb \\ CS &\to b, \end{aligned}$$

prove that $w\ =\ ab(ab^2)^2\ \in L(G)$.

$Obs.:\ (ab)^2 = abab\ \neq a^2b^2 = aabb$

Sol.

$$S \underset{(2)}{\Rightarrow} aCSb \underset{(4)}{\Rightarrow} abSbSb \overset{2}{\underset{(1)}{\Rightarrow}} ababbabb$$

$$\Rightarrow S \overset{4}{\Rightarrow} ababbabb = w \Rightarrow\ w\ \in L(G)$$

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2. Given the grammar $G = (N,\ \Sigma,\ P,\ S)$

$$N = \{S\}$$
$$\Sigma = \{a, b, c\}$$
$$P:\ S\ \to a^2S \mid bc,$$

find $L(G)$.

Sol.

Let $L\ =\ \{a^{2n}bc \mid n\ \in\ \mathbb{N}\}$

? $L\ =\ L(G)$

(1) ? $L\ \subseteq\ L(G)$ (all sequences of that shape are generated by G)

? $\forall\, n \in \mathbb{N},\ a^{2n}bc \in L(G)$

Take P(n): $a^{2n}bc \in L(G)$ and prove P(n) true, $\forall\, n \in \mathbb{N}$
We'll prove by mathematical induction

(a) Verification step: ? P(0): $a^0bc \in L(G)$ is true

$S \underset{(2)}{\Rightarrow} bc = a^0bc$ => P(0) true

(b) Proof step: We suppose P(k) is true and then prove that P(k+1) is also true, where $k \in \mathbb{N}$

P(k) true => $a^{2k}bc \in L(G)$ => S $\overset{*}{\Rightarrow}$ $a^{2k}bc$ (induction hypothesis)

S $\underset{(1)}{\Rightarrow}$ a^2S $\underset{\text{(ind. hypo.)}}{\overset{*}{\Rightarrow}}$ $a^2a^{2k}bc = a^{2(k+1)}bc$

=> S $\overset{*}{\Rightarrow}$ $a^{2(k+1)}bc$ => P(k+1) is true

(a) + (b) => (1)

(2) ? $L \supseteq L(G)$ (G generates only sequences of that shape)

S => bc = a^0bc
=> a^2S => a^2bc
=> a^4S => a^4bc
=> a^6S => …

We notice that starting from S and using all grammar productions in all possible combinations, we only get, as sequences of terminals, sequences of the shape $a^{2n}bc$ where $n \in \mathbb{N}$. It follows that the grammar doesn't generate anything else.

Obs.: This inclusion may also be discharged by induction.

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3. Find a grammar that generates $L = \{0^n 1^n 2^m \mid n, m \in \mathbb{N}^*\}$

Sol.

$G = (N, \Sigma, P, S)$
$N = \{S, V, C\}$
$\Sigma = \{0, 1, 2\}$
P : S -> VC
V -> 0V1 | 01
C -> 2 | 2C

(1) ? $L \subseteq L(G)$
? $\forall\, n, m \in N^*,\ 0^n 1^n 2^m \in L(G)$

Let $n, m \in N^*$

$$S \underset{(1)}{\Rightarrow} VC \overset{n}{\underset{(a)}{\Rightarrow}} 0^n 1^n C \overset{m}{\underset{(b)}{\Rightarrow}} 0^n 1^n 2^m \Rightarrow S \overset{*}{\Rightarrow} 0^n 1^n 2^m \Rightarrow 0^n 1^n 2^m \in L(G)$$

(a) $V \overset{n}{\Rightarrow} 0^n 1^n,\ \forall\, n \in N^*$

(b) $C \overset{m}{\Rightarrow} 2^m,\ \forall\, m \in N^*$

HW: Prove (a) and (b) above by induction
Justify the reverse inclusion