Solve the following optimization problem with KKT conditions and submit your detailed solution:

Task 1. Page 1.
HW4
Optimization

Convexity analysis:

$$\min\ x_1 - 2x_2 + x_3 \rightarrow \text{convex}$$

$$\text{s.t.}\quad -x_2 + x_2 + x_3 \le -2 \rightarrow \text{convex}$$

$$x_1 + 2x_2 + x_3 \le 10 \rightarrow \text{convex}$$

$$x_1 + x_2 - x_3 = 4 \rightarrow \text{affine}$$

$$x_i \ge 0 \rightarrow \text{convex}$$

~~and~~ feasible set and objective function is convex

$\rightarrow$ We have a convex problem and this is a LP-problem.

KKT conditions:

1. $g_i(x^*) = 0$ $\qquad x_1 + x_2 - x_3 - 4 = 0$ $\qquad$ 1.1.
2. $h_j(x^*) \le 0$ $\qquad -x_1 + x_2 + x_3 + 2 \le 0$ $\qquad$ 2.1

   $x_1 + 2x_2 + x_3 - 10 \le 0$ $\qquad$ 2.2

   $-x_1 \le 0,\ -x_2 \le 0\ \ -x_3 \le 0$ $\qquad$ 2.3, 2.4, 2.5
3. $\mu_j^* \ge 0$ $\qquad \mu_1^*, \mu_2, \mu_3, \mu_4, \mu_5 \ge 0$
4. $\mu_j^* h_j(x^*) = 0$ $\qquad \mu_1(-x_1 + x_2 + x_3 + 2) = 0$ $\qquad$ 4.1

   $\mu_2(x_1 + 2x_2 + x_3 - 10) = 0$ $\qquad$ 4.2

   $\mu_3 x_1 = 0$ $\qquad$ 4.3

   $\mu_4 x_2 = 0$ $\qquad$ 4.4

   $\mu_5 x_3 = 0$ $\qquad$ 4.5
5. $L'_x(x^*, \lambda^*, \mu^*) = 0$

$$L = (x_1 - 2x_2 + x_3) + \lambda(x_1 + x_2 - x_3 - 4) + \mu_1(-x_1 + x_2 + x_3 + 2) + \mu_2(x_1 + 2x_2 + x_3 - 10) - \mu_3 x_1 - \mu_4 x_2 - \mu_5 x_3$$

$$\frac{\partial L}{\partial x_1} = 1 + \lambda - \mu_1 + \mu_2 - \mu_3 = 0 \qquad 5.1$$

$$\frac{\partial L}{\partial x_2} = -2 + \lambda + \mu_1 + 2\mu_2 - \mu_4 = 0 \qquad 5.2$$

$$\frac{\partial L}{\partial x_3} = 1 - \lambda + \mu_1 + \mu_2 - \mu_5 = 0 \qquad 5.3$$

Task 1. Page 2.
HW4
Optimization

Let $\lambda \neq 0$, $\mu_1 \neq 0$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = 0$, $\mu_5 \neq 0$

(1.1) $x_1 + x_2 - x_3 - 4 = 0$ on

(4.1) $-x_1 + x_2 + x_3 + 2 = 0$ on

(4.2), (4.3), (4.4) - off $\mu_2, \mu_3, \mu_4 = 0$

(4.5) $\mu_5 \neq 0$ $x_3 = 0$

(5.1) $1 + \lambda + \mu_1 + \mu_2 - \mu_3 = 0$

(5.2) $-2 + \lambda + \mu_1 + 2\mu_2 - \mu_4 = 0$

(5.3) $1 - \lambda + \mu_1 + \mu_2 - \mu_5 = 0$

$1 + \lambda - \mu_1 = 0$ $\quad -1 + 2\lambda = 0$ $\quad \lambda = \frac{1}{2}$

$-2 + \lambda + \mu_1 = 0$ $\quad -2 + \frac{1}{2} + \mu_1 = 0$ $\quad \mu_1 = \frac{3}{2}$

$1 - \lambda + \mu_1 - \mu_5 = 0$ $\quad \mu_5 = 2$

$\lambda = \frac{1}{2}, \mu_1 = \frac{3}{2}, \mu_5 = 2$

$x_3 = 0$ $\quad$ (1.1) $x_1 + x_2 - 4 = 0$ $\quad \boxed{x_1 = 3}$

(4.1) $-x_1 + x_2 + 2 = 0$ $\quad 2x_2 = 2$ $\quad \boxed{x_2 = 1}$

For $\lambda = 1/2$
$\mu_1 = 3/2$
$\mu_2, \mu_3 = 0$
$\mu_4 = 0$
$\mu_5 = 0$
$x_1 = 3$
$x_2 = 1$
$x_3 = 0$

Let's check it:

1.1: $3+1-0-4=0$ ok

2.1: $-3+1+0+2 \leq 0$ ok

2.2 $3+2+3-10 \leq 0$ ok

2.3, 2.4, 2.5 $-x_1, -x_2, -x_3 \leq 0$ ok

4.1: $\frac{3}{2} \cdot 0 = 0$ | 4.2 $0 \cdot \ldots = 0$ | 4.3 $0 \cdot 3 = 0$ | 4.4 $0 \cdot 1 = 0$ ok

4.5 $2 \cdot 0 = 0$ ok

3 $\mu_1 = \frac{3}{2} \geq 0, \mu_5 = 2 \geq 0, \mu_2, \mu_3, \mu_4 = 0 \geq 0$ ok

5.1: $1 + \frac{1}{2} - \frac{3}{2} + 0 - 0 = 0$ ok

5.2: $-2 + \frac{1}{2} + \frac{3}{2} + 2 \cdot 0 - 0 = 0$

5.3: $1 - \frac{1}{2} + \frac{3}{2} + 0 - 2 = 0$ ok

$\Rightarrow$ KKT conditions hold and we have a convex problem $\Rightarrow$ we have the necessary and sufficient conditions for global minimum $\Rightarrow$

$\Rightarrow$ $\boxed{x^* = (3, 1, 0)}$ is the solution of our convex optimization problem (LP-problem)