On the continuity of the welding homeomorphisms induced on the real line by the backward SLE for $\kappa \in [0, 8/3)$

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Abstract

We prove that the welding homeomorphism of the real line induced by the backward Loewner differential equation when driven by $\sqrt{\kappa}B_t$ is a.s. point-wise continuous at every real point, for $\kappa \in [0, 8/3)$. The proof uses properties of the real Bessel process obtained from the continuous extension of the family of conformal maps satisfying the backward Loewner differential equation on the real line, such as the Lamperti relation, and the monotonicity of the drifts. In addition, we use Bolzano-Weirstrass and Dini's Theorem, in a pathwise manner.

1 Introduction

The Schramm-Loewner evolution SLE_κ is a one-parameter family of random planar growth processes constructed as a solution to Loewner equation when the driving term is a Brownian motion with diffusivity $\kappa > 0$. It was introduced in [10] by Oded Schramm, in order to give meaning to the scaling limits of Loop-Erased Random Walk and Uniform Spanning Trees.

In this paper, we study the continuity in the parameter κ of the welding homeomorphism on the real line induced by the backward SLE. For more details about the welding homeomorphism induced by the backward SLE, we refer to [9]. We prove almost sure pointwise continuity of the welding homeomorphism induced on the real line by the backward Schramm-Loewner Evolution SLE_κ for $\kappa \in [0, 8/3)$.

The problem is in the same spirit with the one about the continuity of the SLE traces in the parameter κ. The problem of continuity of the traces generated by Lowener chains was studied in the context of chains driven by bounded variation drivers in [13], where the continuity of the traces generated by the Loewner chains was established. Also, the question appeared in [6], where the Loewner chains were driven by Hölder-1/2 functions with norm bounded by σ with $\sigma < 4$. In this context, the continuity of the corresponding traces was established with respect to the uniform topologies on the space of drivers and with respect to the same topology on the space of simple curves in $\mathbb{H}$. Another paper that addressed a similar problem is [12], in which the condition $||U||_{1/2} < 4$ is avoided at the cost of assuming some conditions on the limiting trace. Some stronger continuity results are obtained in [3] under the assumption that the driver has finite energy, in the sense that $\dot{U}$ is square integrable. The question appeares naturally when considering the solution of the corresponding welding problem in [2]. In this paper it is proved that the trace obtained when solving the corresponding welding problem is continuous in a parameter that appears naturally in the setting. In the context of SLE_κ traces the problem was studied in [14], where the continuity in κ of the SLE_κ traces was proved for any $\kappa < 2.1$. A recent update is proved in [4], where the a.s. continuity in κ of the SLE traces is proved for $\kappa < 8/3$.

The paper is divided in several sections. In the first section we introduce versions of Loewner differential equations that we work with and we state the main results. In the second section, we provide as in [9] the setting for studying the extensions of the maps in the backward Loewner differential equation. In the third section we introduce the setting needed for the proofs of the result. In the next section we give an heuristics of the pointwise convergence of the welding homeomorphisms induced on the real line by the backward SLE, and in the final section we give a complete proof of the result.

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2 Introduction

We study chordal SLE_κ , defined in the upper half-plane, by the following equations

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z) \frac{2}{z-\sqrt{\kappa}B_t}\,, \quad f(0,z)=z, z\in\mathbb{H}\,. \tag{2.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z)-\sqrt{\kappa}B_t}\,, \qquad g(0,z)=z, z\in\mathbb{H}\,. \tag{2.2}$$

(iii) Time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z)-\sqrt{\kappa}B_t}\,, \qquad h(0,z)=z, z\in\mathbb{H}\,. \tag{2.3}$$

There are connections between these three formulations for studying families of conformal maps. For every $t\in[0,T]$, the map $g_t(z)$ is the inverse of $f_t(z)$. In other words, the maps $f_t(z)$ "grow" the curve in the reference domain, while $g_t(z)$ maps conformally the slit domain obtained by the growing of the curve up to time t to the reference domain. The connection between the different versions of the Loewner equations is that, for all fixed t, $g_{-t}(z)$ has the same distribution as the maps $f_t(z) - \sqrt{\kappa}B_t$. We prove the a.s. pointwise continuity of the welding homeomorphism induced on the real line by the backward SLE_κ for $\kappa \in [0, 8/3)$.

3 Main result

In this paper, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa}B_t}\,, \qquad h(0,z) = z, \quad z \in \mathbb{H}\,. \tag{3.1}$$

Given the Loewner differential equation, the conformal welding homeomorphism is a homeomorphism of intervals of the real line given by the following rule: two points x and y situated at different sides of the origin are to be identified if they hit zero simultaneously under the backward Loewner differential equation with the same driver. The conformal welding homeomorphism induced by the backward SLE_κ for $\kappa < 4$ was studied in [9] and [11]. In [9] it is proved that the backward SLE_κ trace $\beta(t)$ exists and that is a continuous function of $t \in [0, T]$. The proof uses that the backward Loewner differential equation is driven by $\sqrt{\kappa}(B_T - B_{T-r})$ for $r \in [0, T]$ and that $\sqrt{\kappa}(B_T - B_{T-r})$ has the same distribution as $\sqrt{\kappa}B_t$. Moreover, $\beta(0, T] - \sqrt{\kappa}B_T$ has the same distribution as $\gamma(0, T]$, where $\gamma(0, T]$ is the forward SLE_κ trace. Thus, one can use the backward SLE_κ traces to study the SLE_κ curves. In particular, for $\kappa \in [0, 8/3)$, the backward SLE_κ trace $\beta(t)$ is a simple curve for almost every Brownian driver. In this case, under the backward Loewner flow on the real line, points from opposite sides of the singularity hit the origin and afterwards get mapped to the simple trace $\beta(t)$ for almost every Brownian path. Note that so far we have considered finite time intervals, and consequently the welding homeomorphism is defined on finite intervals of the real line. However, in [9] it is proved that the backward Loewner differential equation induces a conformal welding homeomorphism on the whole real line as well. This corresponds to allowing the hitting time of the origin under backward Loewner flow on the real line to be $T = \infty$. In this paper, we take the same perspective and study the welding homeomorphism induced by the backward Loewner differential equation on the whole real line.

We prove that for $\kappa_i \to \kappa \in [0, 8/3)$ the welding homeomorphism, which we denote by $\phi^\kappa(x, \omega)$, converges a.s. pointwise everywhere except at most countably many points on the real axis. The choice of the in

We are ready to present our main result.

Theorem 3.1. *The welding homeomorphism on the real line induced by the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ is sequentially continuous in κ, for $\kappa \in [0, 8/3)$, a.s., everywhere except at most countably many points on the real axis.*

4 The setting needed for the proof of Theorem 3.1

Let $\mathcal{F}_t = \sigma(B_s, s \in [0, t])$ and let us consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_B)$, where $\mathbb{P}_B$ denotes the Wiener measure. The conformal maps $h_t(z)$ satisfying (3.1) can be continuously extended to the boundary with probability one. In what follows next, we work with these extensions to the real line of the mappings $h_t(z)$. By performing the shift $Z_t(x) := \frac{h_t(x\sqrt{\kappa}) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$, we obtain

$$\begin{aligned} dZ_t &= \frac{-2/\kappa}{Z_t} dt + dB_t, \\ Z_0 &= x \in \mathbb{R}^*. \end{aligned} \tag{4.1}$$

Note that the nullsets of the Brownian paths outside of which the SLE_κ exists and $h_t(z)$ is continuous up to the boundary, depend on κ.

In what follows we consider the Bessel processes of various dimensions to be functions of the Brownian motion driver.

We define also the following quantity that we use extensively in our analysis.

Definition 4.1. *For a Bessel process of dimension $d(\kappa) = 1 - \frac{4}{\kappa}$ starting from $x \in \mathbb{R} \setminus \{0\}$, we define the first hitting time of zero, by*

$$T_x^\kappa := \inf\{t > 0 : Z_t = 0, \text{ where } Z_t \text{ is a Bessel process of dimension } d(\kappa) = 1 - \frac{4}{\kappa} \text{ and } Z_0 = x\}.$$

In our analysis, we naturally obtain the following coupling: we drive the backward Loewner differential equation with the same Brownian motion as the solutions to the Bessel SDE (4.1) on the real line.

5 Proof of the Theorem 3.1

The techniques needed for the proof of the (sequential) continuity in κ of the welding homeomorphisms in the parameter κ, for $\kappa \in [0, 8/3)$ are borrowed from the conformal welding framework that is developed in [9], [11]. The analysis is divided into several steps. In the following, we present the heuristics of the argument along with a detailed list of techniques used in the proof of the result.

5.1 Heuristics of the argument and details on the techniques used in the proof

- An important element is the Lamperti relation (that can be found in [8] or in [5]). It states that the first hitting time of zero by a Bessel process has a certain integral representation. In particular, this gives that for the first hitting time of zero of a Bessel process of dimension $d = d(\kappa)$ we have $T_x^\kappa \stackrel{(d)}{=} \frac{x^2}{2Z_{-\mu(\kappa)}}$, where $Z_{-\mu(\kappa)}$ is a Gamma random variable with index $\mu(\kappa)$ which is an explicit continuous function of κ.

- Another useful tool in the proof is the following order of the hitting times of zero for a fixed sample path of the Brownian motion driver. Let us consider Bessel processes driven by the same sample path of the Brownian motion. Using that the drift depends monotonically on the dimension, we obtain that for a fixed starting point one of the processes (the one with smaller value of the drift) will hit the origin before the other processes with different drifts, a.s.. Thus, in the coupled picture that we consider, we will get that for $\kappa_1 \leqslant \kappa_2$ and any $x_0 \in \mathbb{R}$ we have $T_{x_0}^{\kappa_1} \leqslant T_{x_0}^{\kappa_2}$ for $\kappa_1 \leqslant \kappa_2$ a.s.. We want to understand how these stopping times depend on κ for fixed $x_0 \in \mathbb{R}$. For this, we consider the Laplace transforms of these hitting times. We further use that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$, are coupled random variables, such that $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^{\kappa}$, as $\kappa_i \to \kappa$ and $T_{x_0}^{\kappa_i} \nearrow T_{x_0}^{\kappa}$ or $T_{x_0}^{\kappa_i} \searrow T_{x_0}^{\kappa}$, then $T_{x_0}^{\kappa_i} \to T_{x_0}^{\kappa}$, a.s..

5.2 Proof of Theorem 3.1

We first introduce a probabilistic version of Lemma 3 from [7].

Lemma 5.1. *For any $\kappa \in [0, 8/3)$, with probability one, there are no two points $0 < x_0 < y_0$ or $0 > x_0 > y_0$ such that $T^{\kappa}_{x_0} = T^{\kappa}_{y_0}$.*

Proof. Let us consider the starting points $0 < x_0 < y_0$. For all Brownian paths, the function $\delta(t) = y(t) - x(t)$ is increasing in t, since

$$\frac{d}{dt}\delta(t) = \frac{d}{dt}(y(t) - x(t)) = 2\frac{y(t) - x(t)}{(y(t) - \sqrt{\kappa}B_t)(x(t) - \sqrt{\kappa}B_t)}\,.$$

Thus, with probability one there are no two points at the same side of the singularity that can hit the origin in the same time. $\square$

Combining the previous lemma with the existence of the welding homeomorphism, we obtain that when starting the backward Loewner differential equation from any point $x_0 > 0$ on $\mathbb{R}$, we obtain only one point on the negative real axis that will hit simultaneously with $x_0 > 0$, for almost every Brownian path.

Let us fix a point $x > 0$ and a parameter $\kappa \in [0, 4)$. Let us start the reverse backward Loewner differential equation from x_0 and consider it until the first hitting time of the origin. We know that the dynamics of the reverse Loewner equation restricted to the real line is given by the Bessel SDE for dimensions $d(\kappa) = 1 - \frac{4}{\kappa}$. We recall that we consider some $\kappa \in [0, 8/3)$ that corresponds to $d(\kappa) \leqslant 0$. We consider a family of Bessel processes with indices $\kappa_i \in (0, 4]$, $i \in \mathbb{N}$, that are coupled with the same Brownian paths.

Let us consider a fixed value $\kappa \in (0, 4]$ and a sequence of parameters $\kappa_i \to \kappa$, as $i \to \infty$.

The next lemma is a result about the ordering of hitting times for the Bessel process for the same starting point, same drivers and sequence of different parameters. The argument uses that the time of hitting zero is finite a.s. for dimension $d = 0$ (i.e. $\kappa = 4$) in case of the real Bessel processes. Moreover, since the drift for smaller values is a smaller negative number, then if the Bessel process of dimension $d = 0$ hits the origin in finite time, the process of lower dimension should hit in a shorter time due to the ordering of the drifts (for more details, see [5]). Thus, the hitting time of zero is finite, for $\kappa \in [0, 8/3)$ a.s..

In the following lemma, we use the indexed notation $Z_t^{\kappa_i}(x_0)$ for the solution of the SDE (4.1) on the real line in order to keep track of the sequence of parameters κ_i. We know from [8], chapter XI, that for the SDE (4.1), for $\kappa \in \mathbb{R}_+$, for almost every Brownian path there exists a unique strong solution, until the first hitting time of zero.

Lemma 5.2. *Let us consider $Z_t^{\kappa_i}(x_0)$ to be a collection of Bessel processes started from fixed $x_0 > 0$ coupled as above. Let $(\kappa_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of values inside $(0,4]$. Then for all starting points $x_0 > 0$, it holds with probability one that $T_{x_0}^{\kappa_i} \leqslant T_{x_0}^{\kappa_j}$ for any $i \leqslant j$.*

Proof. Let $a(\kappa_i) := \frac{-2}{\kappa_i}$, for any $i \in \mathbb{N}$. Let us consider the Bessel processes $Z_t^{\kappa_i}(x_0)$ given as solutions to the SDE

$$\begin{aligned} dZ_t^{\kappa_i} &= \frac{a(\kappa_i)}{Z_t^{\kappa_i}}dt + dB_t, \\ Z_0^{\kappa_i} &= x_0 > 0. \end{aligned} \tag{5.1}$$

Let us consider the Bessel processes to be driven by the same collection of Brownian paths. Since $a(\kappa_i) = \frac{-2}{\kappa_i}$ we obtain that if $\kappa_i \leqslant \kappa_j$, then $a(\kappa_i) \leqslant a(\kappa_j)$. Thus, using the fact that the latter drift dominates, we obtain that for almost every Brownian path $T_{x_0}^{\kappa_i} \leqslant T_{x_0}^{\kappa_j}$. □

In the natural coupling that we consider, by looking at the difference of these hitting times, we obtain a random variable

$$0 \leqslant [T_{x_0}^{\kappa_i} - T_{x_0}^{\kappa_j}](\omega)\,.$$

The Lamperti relation gives the law of the hitting time for a fixed value of the parameter and for a fix starting point $x_0 > 0$. Thus, by Lamperti relation $T_{x_0}^{\kappa_j} \overset{d}{=} \frac{x_0}{2Z_{\mu(\kappa_i)}}$, where $Z_{\mu(\kappa_i)}$ is a Gamma random variable with index $\mu(\kappa)$ that is an explicit continuous function of κ. For a fixed value of the starting point $x_0 > 0$, let us consider a sequence of Bessel processes with this starting point driven by the same Brownian driver. Let us assume that the corresponding parameters κ_i lie in $[0,4]$ and κ_i converges to κ. We split $(\kappa_i)_{i\in\mathbb{N}}$ into two subsequences: $\kappa_i^- \to \kappa$ converging from below and $\kappa_i^+ \to \kappa$ converging from above. By the lemma above, we have that

$$T_{x_0}^{\kappa_i^-} \leqslant T_{x_0}^{\kappa} \leqslant T_{x_0}^{\kappa_i^+}\,,\ \ a.s..$$

Taking the Laplace transforms (denoted by $\mathcal{L}$) of these random variables, we obtain the following

$$\mathcal{L}(T_{x_0}^{\kappa_i^-}) \leqslant \mathcal{L}(T_{x_0}^{\kappa}) \leqslant \mathcal{L}(T_{x_0}^{\kappa_i^+}).$$

Thus, using the convergence of the laws in the parameter κ, we obtain via Lévy's continuity theorem that

$$\lim_{\kappa_i \to \kappa} \mathcal{L}(T_{x_0}^{\kappa_i^+}) = \mathcal{L}(T_{x_0}^{\kappa})\,.$$

The same holds for the sequence converging from below. Furthermore, we use that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$, are coupled random variables such that as $\kappa_i \to \kappa$, $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^{\kappa}$, and $T_{x_0}^{\kappa_i} \nearrow T_{x_0}^{\kappa}$ or $T_{x_0}^{\kappa_i} \searrow T_{x_0}^{\kappa}$, then $T_{x_0}^{\kappa_i} \to T_{x_0}^{\kappa}$, a.s..

Note that since the value of κ is chosen arbitrarily, this argument works for any value $\kappa \in [0, 8/3)$.

The next ingredient that we need is Lemma 5.1. We use Lemma 5.1 to argue that for any fixed value κ_i, there is only one point on the left side of the singularity that will hit the origin in time $T_{x_0}^{\kappa_i}$ a.s.. We know from the above that the hitting time is sequentially continuous in κ_i for fixed value of the starting point.

Note that since in our analysis we have as a driver the same collection of Brownian paths, the dynamics on the negative part of the real axis is given by

$$\begin{aligned} d\tilde{X}_t &= \frac{-2}{\kappa \tilde{X}_t} dt + dB_t, \\ \tilde{X}_0 &= x_0 < 0\,. \end{aligned} \tag{5.2}$$

Since $\tilde{X}_t$ is negative (until the first hitting time of zero), multiplying by -1 we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2}{\kappa X_t} dt - dB_t, \\ X_0 &= x_0 > 0\,, \end{aligned} \tag{5.3}$$

with $X_t = -\tilde{X}_t$. Thus, the dynamics on the negative part of the real axis, when coupling with the same Brownian drivers, is equivalent to a dynamics on the positive side of the real axis with the driver $-B_t$ (that is still a Brownian motion). In particular, the Lamperti relation that we have used for the positive part of the real line, holds also for the dynamics on $\mathbb{R}_-$.

Let us consider the backward Loewner flow starting from $x_0 > 0$ and driven by $\sqrt{\kappa} B_t$. Let us take a sequence $\kappa_i \to \kappa$. In [9], there is a proof of the existence of the welding homeomorphism for $\kappa \in [0, 8/3)$. The argument uses the existence of the SLE_κ trace and the fact that the trace is simple a.s. for $\kappa \in [0, 8/3)$. Thus, we consider in our analysis welding homeomorphisms generated by the SLE_κ trace for $\kappa \in [0, 8/3)$.

Lemma 5.3 (Lemma 5 of [1]). *Let X_t be a Bessel process of dimension $d = d(\kappa) = 1 - \frac{4}{\kappa}$, started from $x \geqslant 0$. Then, for any fixed $d \leqslant 2$ and almost every Brownian path, the hitting time of the origin $T_x^\kappa(\omega)$ is continuous in $x \geqslant 0$.*

We will use this lemma for sequences of dimensions. Since the result gives an a.s. continuity of the hitting times in the starting point for a fixed dimension, when considering sequences of dimensions we obtain the same conclusion outside a different nullset formed by the countable union of both the nullsets outside of which the processes are defined and the nullsets where the result of Lemma 5.3 holds. Specifically, in our case we use this lemma for real Bessel processes of dimensions $d_i = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, $i \in \mathbb{N}$ with $d_i < 2, \forall i \in \mathbb{N}$, that are (strong) solutions to the real-valued SDE (4.1).

We also consider as an ingredient in our analysis the following theorem.

Theorem 5.4 (Dini's Theorem). *Let M be a compact topological space, and let us consider monotonic sequence of continuous functions $f_n : M \to \mathbb{R}$ which converge pointwise to a continuous function $f : M \to \mathbb{R}$. Then this convergence is uniform.*

The final step needed in the proof is the following lemma.

Lemma 5.5. *Let us consider an increasing sequence of real parameters $\kappa_i \to \kappa$ with $0 < \kappa_i \leqslant 4$. Let us consider a sequence of Bessel processes of dimension $d = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, with the same starting point $x_0 > 0$. For almost every Brownian motion, we can define $y_i(\omega) < 0$ such that $T_{y_i}^{\kappa_i}(\omega) = T_{x_0}^{\kappa_i}(\omega)$.*

Then, for almost every Brownian path, we can find a convergent subsequence $y_{m_i}(\omega) \to \hat{y}(\omega)$. Moreover, we obtain that a.s.

$$T^{\kappa_i}_{y_{m_i}(\omega)} \to T^{\kappa}_{\hat{y}(\omega)}.$$

Proof. For the sequence $\kappa_i \to \kappa$, we have a sequence of starting points to the left of the origin $y_i = y_i(\omega)$ such that $T^{\kappa_i}_{y_i} = T^{\kappa_i}_{x_0}$. We claim that the sequence y_i is a.s. bounded. For contradiction, let us assume that with positive probability the sequence y_i is not bounded. Then, with positive probability, we can find a subsequence that is diverging to $-\infty$. This is contradicting the fact that the hitting times of the origin for $y_i(\omega)$ are bounded by $T^{\kappa_i}_{x_0} \leqslant T^{\kappa}_{x_0} = T < +\infty$, almost surely, by the monotonicity of the drifts. Indeed, if with positive probability there exists a divergent subsequence, then with positive probability the hitting times of the origin for this subsequence would have $T^{\kappa}_{y_i} \to \infty$. In order to see this, we recall that the Lamperti relation gives that $T^{\kappa}_{y_i} \overset{(d)}{=} \frac{y_i}{Z_\kappa}$, where Z_κ is a Gamma random variable with index depending on κ. Thus, for $T < +\infty$ as above, we have that on the subsequence $y_{m_i} \to \infty$, it holds that $\mathbb{P}\left(T^{\kappa}_{y_{m_i}} > T\right) = \mathbb{P}\left(\frac{y_{m_i}}{Z_\kappa} > T\right) > 0$, contradicting the fact that $T^{\kappa}_{x_0} = T < +\infty$, almost surely.

Thus, the sequence $y_{\kappa_i}(\omega)$ is almost surely bounded. From Bolzano-Weierstrass Theorem, we have a.s. a convergent subsequence $y_{m_i}(\omega) \to \hat{y}(\omega)$.

In order to apply Dini's Theorem for each Brownian path, we consider the splitting

$$|T^{\kappa_i}_{y_i} - T^{\kappa}_{\hat{y}}| \leqslant |T^{\kappa_i}_{y_i} - T^{\kappa}_{y_i}| + |T^{\kappa}_{y_i} - T^{\kappa}_{\hat{y}}|.$$

In order to estimate the first term, we use Dini's Theorem for the family of functions

$$f_j(y) := T^{\kappa_j}_{y}.$$

Note that in order to apply Dini's Theorem for each path of the Brownian driver, we need that for almost every Brownian path the functions $f_j(y)$ are defined on a compact set and are monotonically increasing continuous real valued functions that converge pointwise to a continuous function. Indeed, by the Bolzano-Weierstrass argument we can restrict for any $j \in \mathbb{N}$ the functions $f_j(y)$ on compact sets $\{y_1, y_2, ..., \hat{y}\}$. In this manner we obtain different compacts for each Brownian paths. All these sets are compact sets since for almost every Brownian path they are closed (they contain only countably many points) and bounded (by Bolzano-Weierstrass). By Heine-Borel Theorem we obtain that the functions are indeed defined on compacts for almost every Brownian path. Moreover, they are monotonically increasing a.s. by the ordering of the drifts and they are a.s. continuous at each point by Lemma 5.3.

Using the result from Lemma 5.3, we have that $T_y^{\kappa_i}$ and T_y^{κ} are a.s. sequentially continuous real valued functions at $\hat{y}$.

We have that

$$|T_{y_i}^{\kappa_i} - T_{y_i}^{\kappa}| \leqslant \sup_{y \in \{y_1, y_2, \ldots, \hat{y}\}} |T_y^{\kappa_i} - T_y^{\kappa}|.$$

From Dini's Theorem, we know that $\exists i_0(x_0, \omega, \kappa) > 0$ such that for all $i > i_0$,

$$\sup_{y \in \{y_1, y_2, \ldots, \hat{y}\}} |T_y^{\kappa_i} - T_y^{\kappa}| \leqslant \varepsilon.$$

For the second term, we estimate using the sequential continuity in parameter y_i for a fixed $\kappa \in [0, 8/3)$, as implied by Lemma 5.3. From this, we obtain that for almost all Brownian paths there exists an index $i_1 = i_1(\omega, \kappa) > 0$ such that for all $i \geqslant i_1$, $|T_{y_i}^{\kappa} - T_{\hat{y}}^{\kappa}| \leqslant \varepsilon/2$. Taking $i(x_0, \omega) = \max(i_0(x_0, \omega, \kappa), i_1(\omega, \kappa))$ we obtain the desired conclusion.

By applying Lemma 5.5 together with Lemma 5.1, we obtain that a.s. $\hat{y}(\omega)$ is the unique point with this value of the hitting time. Indeed, otherwise we would have two points on the same side of the singularity that will hit the origin in the simultaneously, contradicting Lemma 5.1.

Note that for any other sequence $\tilde{\kappa}_i \to \kappa$, we obtain a converging subsequence in the same manner as described before using Bolzano-Weierstrass Theorem for almost every Brownian path. Using the sequential continuity in both variables proved in Lemma 5.5 along with Lemma 5.1, we prove that for almost every Brownian path all the subsequences have the unique limit $\tilde{y}(\omega)$ (which depends on the path). □

Performing the same analysis for a countable collection of starting points we obtain the point-wise convergence argument on a dense set of points on the real line. Since the functions $T_y^{\kappa_i}$ are monotonic in y a.s. for all i, and since the limit T_y^{κ} is also monotone, we obtain that they have at most countably many discontinuities. Moreover, since $T_{y_j}^{\kappa_j} \to T_{y_j}^{\kappa}$ on a countable set of points y_j and by monotonicity of the functions, we have that

$$T_{y_m}^{\kappa_j} \leqslant T_y^{\kappa_j} \leqslant T_{y_n}^{\kappa_j}$$

for $y_m \to y$ from below and $y_n \to y$ from above. Thus, we have that

$$T_x^{\kappa} = \lim_{j\to\infty} \lim_{m\to\infty} T_{y_n}^{\kappa_j} \leqslant \liminf_{j\to\infty} T_y^{\kappa_j} \leqslant \limsup_{j\to\infty} T_y^{\kappa_j} \leqslant \lim_{j\to\infty} \lim_{n\to\infty} T_{y_n}^{\kappa_j} = T_x^{\kappa},$$

at points of continuity of T_x^{κ}. We have the desired conclusion on the set of continuity points of T_x^{κ}, a.s.. Since the function is monotone, the set of discontinuities is at most countable.

Note that in our analysis we used a.s. properties of the Bessel processes. In several steps of the proof we considered only countably many changes in either the starting point or the dimension of the process. Thus, the previous estimates hold outside a countable union of nullsets that remain a nullset.

Thus, outside the nullset that is the countable union of all the nullsets where the backward SLE_{κ_i} traces are defined for $(\kappa_i)_{i\in\mathbb{N}}$, along with the nullsets corresponding to the analysis of the Bessel processes described in the previous paragraph, we obtain the point-wise convergence of the welding homeomorphism at the value $\kappa \in [0, 8/3)$. Finally, since this value of κ was chosen arbitrarily, we obtain the almost sure point-wise convergence of the welding homeomorphism induced by the backward SLE_κ for $\kappa \in [0, 8/3)$.

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