Research Statement

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1 Introduction and past research

Oded Schramm introduced in [2] a stochastic version of the Loewner equation in order to study the scaling limits of planar loop-erased random walks and uniform spanning trees. I consider the following version of the stochastic Loewner Differential Equation

$$dg_t(z) = \frac{2}{g_t(z)}dt - \sqrt{\kappa}dB_t\,, \qquad g_0(z) = z, z \in \mathbb{H}\,. \tag{1.1}$$

where $\sqrt{\kappa}B_t$, with $\kappa \in \mathbb{R}_+$, is a real-valued Brownian motion. The SLE_κ trace is defined via

$$\gamma(t) := \lim_{y \to 0+} g_t^{-1}(iy).$$

The above limit is shown to exist and to be continuous in time by the Rohde-Schramm Theorem (see [3]). One of the motivations to study SLE_κ curves is their success in describing scaling limits of some discrete models in planar Statistical Physics.

All of the projects that I will mention can be found in a compact document that is my Confirmation Thesis (that is a preliminary version of my PhD Thesis to be submitted by May 2019) and that can be accessed at https://people.maths.ox.ac.uk/margarint/T.pdf. Previously, I have been working in ETH Zürich with Prof. Antti Knowles in Random Matrix Theory. This work can be found at the following link https://link.springer.com/article/10.1007/s10955-018-2065-2 and at https://arxiv.org/pdf/1808.07092.pdf.
The following list of projects represent my personal research agenda concerning SLE Theory. Also, I am eager to continue the aforementioned projects in Random Matrix Theory as well as explore new directions.

1.Closed-form expression measures for the backward Loewner flow and a phase transition at $\kappa = 8$

I have studied a one dimensional diffusion obtained from the backward Loewner differential equation in the upper half-plane using a random time change.

$$dT(u) = -4\frac{T(u)}{1+\kappa T(u)^2}du - dB_u, \quad T(0) = 0.$$

For this process, I proved the convergence in the sense of random ergodic averages to its stationary law, that has explicit density $\rho(T) = C(\kappa T^2 + 1)^{-4/\kappa}$, $T \in \mathbb{R}$, where C is the normalizing factor. I have also used stronger convergence results to identify the random times along the backward Loewner flow where the law of the arguments of points has this closed-form expression, for all $\kappa < 8$. This analysis provides a solution to a Skorokhod embedding type problem for the backward Loewner flow. For instance, when $\kappa = 4$ this method provides a precise construction of the sequence of times on which the law of the arguments of points under the backward Loewner flow in $\mathbb{H}$ converges to the uniform law on $[0, \pi]$. In addition, I used this framework to show a phase transition in the behaviour of the backward SLE_κ flow at $\kappa = 8$.

2. Phase transition at $\kappa = 4$ for the backward SLE via Excursion Theory of the backward Bessel processes

I have proven the almost sure phase transition in terms of the uniqueness/non-uniqueness of solutions for the backward Loewner differential equation, seen as a differential equation started from its singularity, using pathwise properties of the Bessel processes obtained from the continuous extensions of the backward SLE_κ maps on the real line. As an application, I showed how the excursions of these backward Bessel process give the non-simpleness of the backward SLE_κ traces. In addition, I showed how one can use this analysis to identify some classes of double points along SLE_κ traces for $\kappa \in (4, +\infty)$. In this setting, I showed that for $\kappa \in [0, 4]$ one could argue that SLE_κ traces are simple using only the behavior of the origin for the real backward Bessel process.

3. Sequential continuity in κ of the welding homeomorphisms induced by the backward SLE_κ for $\kappa \in [0, 4]$

I have proven the sequential continuity of the welding homeomorphisms induced on the real line by the backward Loewner differential equation driven by $\sqrt{\kappa} B_t$ for $\kappa \in [0, 4]$, for almost every Brownian motion, on an uncountable collection of points. The tools used in the proof are the Lamperti relation and other properties of the Bessel processes obtained from the extensions of the backward SLE_κ maps to the real line, along with Dini's Theorem applied pathwise.

4. The maximal convergence rate for the Loewner equation and simulation of the SLE_κ trace

I have studied approximation schemes from Rough Path Theory for the backward Loewner differential equation near the singularity and obtained closed form expressions for the second order truncated Taylor approximation. The major result of this project is the best order of convergence of Taylor approximations, for a general class of singular drift differential equations, that includes the backward Loewner equation. In addition, I obtained a new way to simulate the SLE_κ trace. This method allows us to fix a Brownian path and to perform further refinements of the step size for this fixed sample. The proof of this project is complete and currently I am finishing the writing of the paper.

2 Future research

Loewner dynamics in a change of coordinates: Using the coordinate change from the dynamics of the real and imaginary parts under the backward Loewner flow to the dynamics of the cotangent of the argument studied in 1) and of the radial part, I plan to study an open problem from the early days of the field: the proof of the existence of the SLE_κ trace in the case $\kappa = 8$ by direct methods. In this case, the existence of the trace is proved via a different method from the one that works in the cases $\kappa \neq 8$. I plan to use the analysis for the one dimensional diffusion $T(u)$ and express the other quantities, such as the derivative of the conformal map, as functions of this process. Using similar ideas, I plan to implement this technique in order to study if the SLE_4 domains are Hölder. Another direction of research with this method is to study an excursion theory for the cotangent of the argument of the tip of the SLE_κ trace.

Fine measure theoretic analysis: A fundamental question that I would like to answer is: what pathwise properties of the Brownian Motion influence the behavior of the SLE_κ traces in κ? For this, I aim to build a framework based on the Quasi-Sure Stochastic Analysis through Aggregation developed in [4]. With this method, we will obtain a concrete analytic characterization of the influence of the pathwise characteristics and their dependence on the parameter κ.

Hölder domains and Rough Path Theory: I would like to continue also the study of Hölder domains with elements of Rough Path Theory, started in [1]. Furthermore, I plan to apply the techniques to a specific choice of Hölder domains - the SLE_κ hulls for $\kappa \neq 4$ - which will further explore the interface between Rough Paths and SLE Theory. I would like to use this correspondence to bring new tools for understanding the convergence of discrete models from Statistical Physics in the scaling limit to SLE_κ curves. The analysis in [1] gives new insights on the nature of the boundaries of Hölder domains seen as a limit of bounded variation paths. Specifically, when studying the Hölder domains given by the SLE_κ hulls for $\kappa \neq 4$, this technique gives an approximation with domains with boundaries C^1 loops that is useful for understanding the convergence of discrete models to the SLE_κ traces. This method brings new choices of topology such as the p-variation one that carries the independence of the parametrization of the boundaries of domains.

Singular equations and other directions: I plan to study the best order of convergence of classes of numerical methods at fixed scales, for a wider class of singular differential equations. The goal of the project is to understand how the singularity of the vector fields and the roughness of the driver influence the lower bounds on the convergence towards the solution for various approximations.
In a different direction, using Rough Path Theory, I am interested in developing new ideas concerning dynamics of particles in physical systems that are subjected to small perturbations. For example, these types of systems appear in Theoretical Physics when considering perturbed dynamics of electrons around a central charge. These trajectories can be modeled as Rough Paths.

In addition to these projects, having explored a variety of tools during my PhD projects, I became interested in many areas of Stochastic Analysis and its applications. Given this, I would like to start collaborations within your department in order to explore further these areas.

Further details at https://people.maths.ox.ac.uk/margarint/indexnew.html

References

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