Weak symmetries/Transversal calculus An extension of some notions from Differential Geometry

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September 29, 2020

Abstract

We introduce an extension of some notions from Differential Geometry, giving a natural application to the paths of certain stochastic processes. We introduce first the notion of transversal derivative with the help of which we define the notion of generalized curvature. The definition can be extended to general rough surfaces, by defining a generalized Gaussian type curvature. The same analysis can be performed higher dimensions. The extension of the typical Differential Geometry tools is motivated by the study of Feynman path integral formulation in Quantum Mechanics. In addition, we present some applications, two of which we describe in more detail : generalized curvature flow and geometrical notions in Euclidean Field Theory (EFT). The generalized curvature flow, can be used to model the dynamics of growth process with fractal boundaries. For the second application, it is known that in EFT, the Feynman-Kac formula plays an important role. This approach involves Brownian motion trajectories that do not have a well defined derivative. On the other hand, General Relativity (GR) has a well-established geometric approach using tools from Differential Geometry. Having no classical notion of derivative for the paths used in the study of EFT, it is hard to find a suitable classical geometric interpretation in this setting. The aim of this project, between others, is to find a suitable geometric interpretation of EFT, in order to link it with the approach of General Relativity. We believe that most of the other notions from Differential Geometry can be extended in this language.

1 Preliminaries

In Differential Geometry, one has various ways to define the notion of curvature. One possibility is using the parametrization by arclength of smooth curves and the angle between the derivatives at the curve.

Another possibility is to work directly with general parametrization. One can as well define the curvature for a planar curve, for a general parametrization. In particular, in the case the curve is the graph of a function, i.e. $y = y(x)$, the curvature is defined as

$$k(x) = \frac{y''(x)}{(1 + y'(x)^2)^{3/2}}.$$

There are previous works such as [12], [11] that come from an optimal transport viewpoint (and discuss inequalities that a certain notion of curvature satisfies).

The novelty of this approach consists of considering new geometrical objects (in this case, envelopes instead of lines, in higher dimensions hyper-envelopes instead of hyperplanes) that one uses to study the rough spaces locally. Even though there is not a well defined notion of classical derivative, there are natural transversal notions that are further used to define a notion of curvature.

[To be completed and refined]

2 Generalized curvature

In Stochastic Analysis, some natural self similar processes were so far very well studied. These processes have respective envelopes to describe their asymptotic behavior near the origin given by the corresponding Law of Iterated Logarithm (LIL). One important example is the one-dimensional Brownian motion with an envelope at $t = 0$ given by precise functions. Another class of objects includes the α self-similar Markov processes with envelopes that depend on some parameter $\alpha > 0$ (see [10]).

We start to introduce the objects of our analysis.

Definition 2.1. *A $(\phi(t), \psi(t))$- weak symmetric curve is a curve X_t that evolves with respect to the $(\phi(t), \psi(t))$-envelope (in the sense that $\phi(t)$ and -$\psi(t)$ represent the limsup and liminf of the increments $X_{t+l} - X_t$, as $l \to 0$).*

When $\phi(t) = \psi(t)$, we use the name $\phi(t)$-weak symmetric curves.

In the following, we restrict our attention and write the definitions for ϕ-weak symmetric curves, for a given function ϕ. However, all the notions can be extended to the (ϕ, ψ)-weak symmetric curves. We follow this direction since we are interested in the paths of (certain) stochastic processes that we will show are ϕ-weak symmetric lines for certain choices of ϕ.

Definition 2.2 (Transversal derivative)**.** *For a $\phi(t)$ -weak symmetric curve in the plane, we define the transversal derivative at the point t as the bisector of the envelope of the curve at $t \geqslant 0$.*

We use $X^{\prime,T,\phi}(t)$ to denote the transversal derivative for the envelope given by the function $\phi(t)$ at $t \geqslant 0$. We remark that in the case of the Brownian motion the envelope is symmetric and is given by the curve $\phi(t) = \sqrt{2tloglogt}$. However, there examples of processes for which the envelope is not symmetric.

We are ready to define the notion of generalized curvature.

Definition 2.3 (Generalized curvature)**.** *For a $\phi(t)$ -weak symmetric curve in the plane, we define the generalized curvature at any point $t_0 \geqslant 0$ as*

$$K_\phi(t_0) := \frac{\left.\frac{dX^{\prime,T,\phi}(t)}{dt}\right|_{t=t_0}}{(1 + (X^{\prime,T,\phi}(t_0))^2)^{3/2}}.$$

Remark 2.4. In a similar manner, for stochastic processes, one can define $K_{\phi,\psi}$ for an asymmetric envelope using for the definition of the transversal derivative as the mean of the skewed distribution characterizing the corresponding stochastic process instead of the mean of a symmetric distribution (corresponding geometrically to the bisector of the envelope).

Remark 2.5. Similarly, following the direction of arclength parametrization in Differential Geometry, one can define a generalized curvature using a natural notion of arclength in the fractal setting. One candidate for this is the Hausdorff measure $m^\phi(X_{s,t})$ of the path X_t with the gauge function ϕ that makes it non-trivial such that it grows like $C(t-s)$ for a constant C (that depends on the dimension). For the Brownian motion paths in dimension $d \geqslant 2$ the function is known. See, for example, Theorem 5 for $d \geqslant 3$ in [2]. Alternatively, one can use the Minkowski content when nontrivial in the definition and also its reparametrization such that it grows linearly, see [8].

Remark 2.6. The intuition behind the name weak symmetries comes from the fact that in the case of the lines in the typical Euclidean Geometry, for example, this curve evolves symmetrically with respect to itself (i.e., the envelope has no width and the transversal derivative becomes the derivative).

In general, in the case of smooth curves, the width of the envelopes is zero, and the transversal derivative is the usual derivative. We emphasize that our notion simplifies to the usual curvature when one considers smooth curves.

In comparison, in the case of paths of stochastic processes with self-similarity, the envelope has a non-trivial width. In a certain situations, for example when $M_t - M_s$ has the same distribution as the initial process, then the envelope at zero that one computes from LIL can be found at every point (see, for example, Brownian motion).

Let us introduce the notion of ϕ-weak symmetric line.

Definition 2.7. *A ϕ-weak symmetric line is a curve X_t that evolves with respect to the ϕ-envelope, in the sense that $+\phi$ and $-\phi$ represent the limsup and $liminf$ of the increments $X_{t+l} - X_t$ as $l \to 0$, such that $K_\phi(t) = 0$, for all $t \geqslant 0$.*

We emphasize that, in general, the envelopes are not necessarily symmetric. Thus, one can define a more general notion.

Definition 2.8. *A (ϕ, ψ)-weak symmetric line is a curve X_t that evolves with respect to the (ϕ, ψ)-envelope, in the sense that ϕ and $-\psi$ represent the limsup and $liminf$ of the increments $X_{t+l} - X_t$ as $l \to 0$, such that $K_{\phi,\psi}(t) = 0$, for all $t \geqslant 0$.*

Example 2.9 (Brownian Motion is a.s. a ϕ-weak line for $\phi = \sqrt{2tloglogt}$)**.** Using that $B_t - B_s$ is also a Brownian motion for any fixed time $s \geqslant 0$, we have that the Brownian paths in one-dimension have transversal derivatives that remain horizontal, i.e., we obtain that the Brownian paths are a.s. ϕ-weak symmetric lines.

Other examples include α-stable Markov processes, which have α-dependent envelopes. Also, the same ideas can be naturally extended to surfaces and general rough manifolds.

Example 2.10 (Generalized curves in $d = 2$ that are not ϕ-weak symmetric lines)**.** In dimension $d = 2$ one can use the graph (t, B_t) to analyze the situation. More specifically, one can use the function $exp(z)$ applied to a Brownian Bridge along the vertical direction for convenience and analyze the resulting curves, that will have different generalized curvature given proportional to the inverse of the radius of the circle that is mapped.

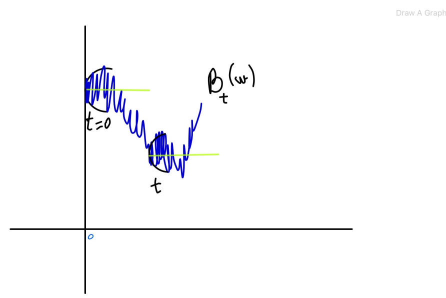

Figure 1: Sample of Brownian Motion as an example of a ϕ-weak symmetric line

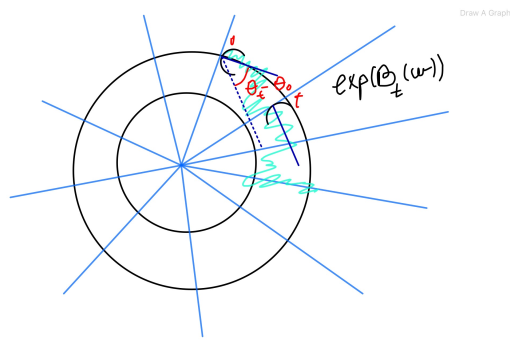

Figure 2: Complex Exponential transformation of the graph of a path of the one-dimensional Brownian motion

Remark 2.11 (Rough circles to describe the generalized curvature)**.** In the smooth geometry case, it is well known that one can use the radius of the osculating circle in order to describe the curvature at any point. One can design a similar approach for the class of generalized circles obtained from the Brownian bridges in the example before using the $exp(z)$ transformation.

[to be completed]

3 Applications: Generalized curvature flow and Euclidean Field Theory

3.1 Generalized curvature flow- 'the one curve' case

In the first application, the interest is in modeling one curve only. In this regard, planar growth models provide a natural setting.

(Planar) Random growth models are a very studied class of models. One is interested then in the shape of the growing profile, which is typically of fractal nature when one considers natural examples coming from Biology, etc. One can model this via a random collection of shapes. A natural approach for this it is via a generalized curvature flow.

$$\frac{dK_\phi}{dt} = f(t, w).$$

In the simplest setting when the function $f(t) = 0$, the solutions are either ϕ-weak symmetric lines when $K_\phi = 0$ and ϕ- weak symmetric circles when $K_\phi = C \neq 0$. More generally, one can use this type of dynamics in order to model naturally the growth via a flow in the generalized curvature of these shapes. The advantage of this approach is given by the fact that this approach also includes also rough shapes that do not have standard notion of curvature well-defined anywhere. [to be completed]

3.2 Euclidean Field Theory- 'the many curves' case

In the second example, the interest is on the dynamics on 'many curves'. A natural setting for this is Feynman's formulation of the paths integrals in the study of Quantum Mechanics. In the following, we will restrict ourselves to the Euclidean Field Theory formulation.

In Euclidean Field Theory, Feynman-Kac Formula along with the other elements of Stochastic Analysis play an important role.

General Relativity is a deterministic theory formulated using the smooth setting offered by Differential Geometry that is so far incompatible with the probabilistic formulation of Quantum Mechanics. The weak symmetries formulation attempts to define a suitable geometric formulation that combines the classical theory of General Relativity with the probabilistic formulation in EFT. We believe that the formulation in terms of weak symmetries will potentially give more insights into this problem.

In light of this, we define the following example: in order to solve Einstein Field Equations in the simplest setting one has to solve $Ric = 0$, where Ric is the Ricci tensor (to be defined). In one-dimension, this becomes $K = 0$, where K is the curvature as in Differential Geometry. In the weak symmetric setting, the equivalent equation will become $K_\phi = 0$, where ϕ is the function describing the envelope. The solutions to this equation are the ϕ-weak symmetric lines. Thus, the flexibility of this formulation offers a setting in which the geometric ideas of the classical General Relativity can be combined with the ideas of Euclidean Field Theory/ Feynman path integral formulation. In Feynman's path integrals approach, the particles move from one point to another on a family of paths, rather than one trajectory as in the classical theory. This is obtained naturally in the weak symmetries setting as a solution to the vacuum (generalized) Einstein Field Equations, in which the usual notions from Differential Geometry, such as the Ricci tensor, are changed with the weakly symmetric version Ric_ϕ (that is naturally the tensorial formulation of the one dimensional K_ϕ discussed previously). This gives that particles in flat space are allowed to move on 'many curves' simultaneously as all of these trajectories are solutions to the $Ric_\phi = 0$ equation.

This formulation can be seen in the picture below, where one can see that this effect is obtained allowing the envelope to have a nontrivial width. [to be completed]

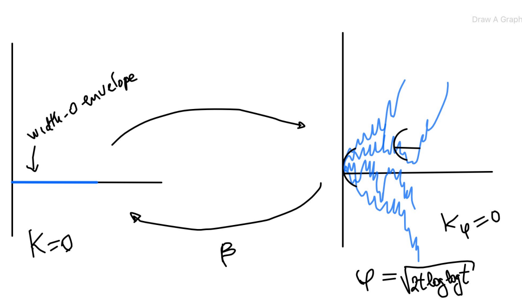

Figure 3: Classical versus weakly symmetric solutions, i.e. solutions to $K = 0$ and $K_\phi = 0$ respectively. The function β controls the width of the envelopes.

Also in the case of EFT applications, if one needs to study ϕ-weak trajectories coming from a certain classical trajectory, one can construct them using the following diagram, where the functions β and η control the width of the envelopes of the corresponding curves.

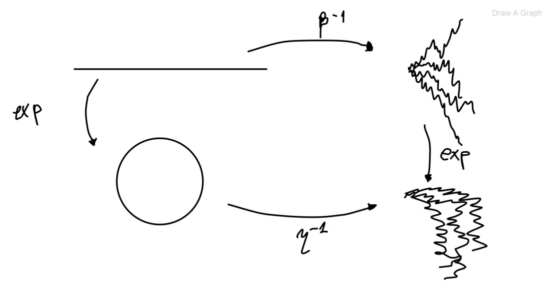

Figure 4: The diagram corresponding to ξ-weak symmetric curves of generalized curvature $k_\xi = \kappa$ corresponding to the classical curvature of the circle $k = \kappa$, for a given function ξ

4 Multi-envelopes

Often in applications the interactions with the environment are not homogenous. For example, a classical particle moving on a fluctuating environment where there is a low decay of the effect of the surroundings of the environment on the particle (changes that happen far away from the particle influence the dynamics of the particle). The width of the envelope, as well as the form of the functions $\phi_{i\in I}(t)$, (where I is an uncountable family of indices) describing the evolutions of the local envelopes, depend on the interaction with the inhomogenous environment. This formulation has enough flexibility to include family of envelopes describing the dynamics of particles in non-homogenous interacting environment.

Example 4.1. Let us consider a particle evolving in an environment in which the interactions increase in one spatial direction. Then, the particle can be modeled as a scaled Brownian motion, i.e. the family of envelopes $\phi_{i\in I}{=}C(i)\sqrt{2t\log\log t}$, with $C(i)$ an increasing sequence.

The previous example can be extended to more general interactions in which the evolution of the envelopes is more complicated (i.e. functions $phi(i)$ change with $i \in I$, i.e. we have a family of envelopes that are not all necessarily symmetric).

5 Higher dimensions

5.1 Applications to random Riemannian metrics

In this section we study the curvature of random non-smooth metrics. These metrics can be thought of as natural models of random metrics in the plane with low-regularity coefficients for which we define a notion of curvature (a more familiar example is the Poincaré upper half-plane model).

The formulation of weak symmetris also gives the possibility to define a notion of (extended) Ricci tensors and (extended) Ricci curvature. In Riemannian Geometry, one can compute the Ricci tensor in terms of the Riemannian metric (with the condition that this one has at least C^2 coefficients). Riemann metrics of low regularity are also a natural framework to be considered (see [1]).

Metrics weighted by (positive) functions of smooth Gaussian fields are natural objects (see Corollary 3.20 in [3], where metrics constructed out of Gaussian fields are linked with the behaviour of percolation clusters).

Using the weak symmetries/transversal calculus, one can compute the Ricci curvature using the notion of transversal derivative since the usual derivative is not well defined. The (extended) Ricci can be computed using the metric and it depends on the second derivatives and products and first derivatives of it (see usual formulas in Differential Geometry, where instead of the derivative (that is infinite) in the chain rule we take the transversal derivative).

One natural application of this is to consider a Schwarschild metric on space-time that is having quantum fluctuations on the low scales and for this we need the notions of tranvsersal derivative and curvature to do the computations. In order to approach this problem see [9], $ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dx^2$ in which we substitute μ and ν with some Hölder fields in the case of two dimensions (one can extend the analysis for any dimensions in particular to the four dimensional case).

6 More applications and future projects

- 1) Connect the generalized curvature flow with the Glauber dynamics.
- 2) Willmore functionals appear naturally in the study of cell membranes. When interfaces are rough and one can define and study the functional for this setting using the generalized notion of curvature. Other examples are extensions of the entropy functionals considered

by Chow ([6]). Also, for students interested in Machine Learning, I foresee applications to noisy data in manifold learning by computing minima of this notion of curvature.

- 3) Study the geometric notions in the QFT version in the cases when one can undo of the Wick rotation from EFT to QFT ([5]).

- 4) For the Gaussian Free Field (that can be understood via testing) there is a LIL computed (see [7]). We plan to study the equivalent notions in that context as well.

- 5) The recent work [3] mentioned earlier is a resource of constructing metrics out of smooth Gaussian Fields ψ i.e. metrics of the form $h(\psi)$*euclidean metric in Corollary 3.20 in [3]. A natural extension is to h($\tilde{\psi}$)*euclidean metrics or metrics on spheres and other geometrical shapes with $\tilde{\psi}$ non-smooth gaussian fields. Another natural application is to define the generalized curvature for these surfaces (also in higher dimensions) for various choices of the function h, via an extension of the formula that allows one to compute the Ricci curvature from the metric from Differential Geometry, as explained in the previous section. The next step is to consider the construction of the Riemannian FPP metric on the LQG surfaces as in [4] and to define the curvature for the LQG surface via a limiting procedure, first having it defined for the molifier.

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