Pathwise and probabilistic analysis in the context of Schramm-Loewner Evolutions (SLE)

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Chapter 1

Overview and literature review

The Loewner equation was introduced by Charles Loewner in 1923 in [26] and it played an important role in the proof of the Bieberbach Conjecture [8] by Louis de Branges in 1985 in [12]. In 2000, Oded Schramm introduced in [35] a stochastic version of the Loewner equation, the stochastic Loewner evolution -SLE_κ- in which he considered Loewner evolution with driving function $\sqrt{\kappa}B_t$, with $\kappa \geqslant 0$ a real parameter and B_t, $t \in [0, \infty)$, a real valued standard Brownian motion. The SLE_κ, as introduced by O.Schramm, is a one-parameter family of random planar fractal curves that are proved to describe the scaling limits of a number of discrete models that appear in planar Statistical Physics. For instance, it was proved in [21] that the scaling limit of loop-erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ, with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and Uniform spanning trees converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$ respectively in the series of works [42], [37], [41] and [21]. In fact, the use of Loewner equation along with the techniques of stochastic calculus provided tools to perform a rigorous analysis of the scaling limits of the discrete models. Moreover, in this framework a precise meaning of the passage to the scaling limit could be established and used to prove the conformal invariance of the limits. We refer to [20] and [33] for a detailed study of the object and many of its properties.

Rough Paths Theory was introduced in 1998 by Terry Lyons in [27]. The theory provides a deterministic platform to study stochastic differential equations which extends both Young's and stochastic integration theory beyond regular functions and semimartingales. Also, Rough Paths Theory provides a method of constructing solutions to differential equations driven by paths that are not of bounded variation but have controlled roughness. Step by step, we introduce the ingredients and terminology necessary to characterize the roughness of a path and to give precise meaning to natural objects that appear in the study of rough paths. We also give precise meaning to the notion of solution of a differential equation with rough driver.

Recently, many papers were written at the interface between the aforementioned domains. The paper of Brent Werness [50] provides new ideas about how we can use a version of Green formula for rough paths and a certain observable for SLE_κ to compute the first three gradings of the expected signature of SLE_κ in the regime $\kappa \in [0, 4]$. An extension to this computation is provided in [9], where the authors show ways of computing the fourth grading of the signature (and do it explicitly for $SLE_{8/3}$, where the required observable is known), along with several other parts of the higher grading. From a different perspective, in [14] the authors question the existence of the trace for a general class of processes (such as semimartingales) as a driver function in the Loewner differential equation. These ideas are also developed in a Rough Paths Theory flavour. More recently, Peter Friz with Huy Tran in [15] revisited the regularity of the SLE traces and obtained a clear result using Besov spaces type analysis. Also, Atul Shekhar and Yilin Wang studied the stability of the traces generated by Loewner chains driven by bounded variation drivers in [40].

This Thesis is divided in several chapters. In Chapters 2 and 3 we give an introduction to both SLE Theory and Rough Paths Theory.

The fourth Chapter addresses the uniqueness/non-uniqueness of solution backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ started from the origin with applications to the structural behaviour of the backward SLE traces (obtained from Excursion Theory of the underlying Bessel process). In general, when studying the stochastic version of the Loewner differential equation as a differential equation, the uniqueness/non-uniqueness of the solutions of the backward Loewner differential equation starting from the origin is a natural question. These type of questions appear also in the study of Rough Differential Equations (RDE) in Rough Paths Theory. In the framework of Rough Paths Theory, the uniqueness/non-uniqueness of solutions depends on the regularity of the vector field compared with the roughness of the driving process. For more details, we refer to [16]. The backward Loewner differential equation started from the singularity can not be interpreted directly as a RDE due to the singularity of the vector field. However, questions about uniqueness/non-uniqueness of solutions started from the singularity can be asked. In this Thesis, we adapt these type of questions to the study of the backward Loewner Differential Equation in the upper half-plane and we use the information provided by the extension of the conformal maps to the boundary to perform the analysis. One main feature of the analysis is that uniqueness/non-uniqueness of solutions of the backward Loewner equation driven by Brownian Motion, started from the singularity, can be understood using (only) some property of the dynamics on the boundary. We consider the same setting for the study of the backward Loewner differential equation and the corresponding backward Loewner trace as the one in [34]. The analysis consists of two parts: in the first part we study the existence of solutions of the backward Loewner differential equation starting from the origin by constructing a solution while in the second part we question the uniqueness/non-uniqueness of solutions in this setting. In order to show the existence of the solutions, we use standard techniques in Complex Analysis along with information about the SLE_κ hulls. The uniqueness/non-uniqueness of the solutions analysis is done using the extensions of the conformal maps to the boundary. When performing this extension, we obtain a Bessel process of dimension $d(\kappa) = 1 - \frac{4}{\kappa}$, for $\kappa \geqslant 0$. The Bessel processes are studied extensively in a number of papers and monographs. For more details about the various properties and characterizations of these processes, we refer to [31] and also to [18]. As a main ingredient in our analysis of the uniqueness/non-uniqueness of solutions, we use the phase transition in the boundary behaviour of the Bessel process along the real line for the extended maps, i.e. absorbtion in the origin of the Bessel process for $\kappa \leqslant 4$, and reflection at the origin for $\kappa > 4$. In this manner, when comparing our method with the typical forward flow analysis, using the backward flow we recover also new structural information about the SLE_κ traces via Excursion Theory of the underlying Bessel process.

In the fifth Chapter, we are interested in the continuity in the parameter κ for $\kappa \in [0, 4]$ of the welding homeomorphism on the real line induced by the backward Loewner differential equation. For more details about the welding homeomorphism induced by the backward Loewner differential equation, we refer to [34].
The problem of continuity of welding homeomorphisms in the parameter κ, for $\kappa \in [0, 4]$, is in the same spirit with the problem of the continuity of the traces generated by the Loewner differential equation with respect to perturbations of the driving term. The problem of continuity of the traces generated by Lowener chains was studied in the context of chains driven by bounded variation drivers in [40], where the continuity of the traces

generated by the Loewner chains was established. Also, the question appeared in [23], where the Loewner chains were driven by Hölder-$1/2$ functions with norm bounded by σ with $\sigma < 4$. In this context, the continuity of the corresponding traces was established w.r.t. to the uniform topologies on the space of drivers and on the space of simple curves in $\mathbb{H}$. Another paper that addressed a similar problem is [39], in which the condition $||U||_{1/2} < 4$ is avoided at the cost of assuming some conditions on the limiting trace. Some stronger continuity results are obtained in [14] under the assumption that the driver has finite energy, in the sense that $\dot{U}$ is square integrable. The question appeared naturally also when considering the solution of the corresponding welding problem in [4]. In this paper it is proved that the trace obtained when solving the corresponding welding problem is continuous in the parameter β that naturally appears in the setting.
In the context of SLE_κ traces, the problem was studied in [47] where the continuity in κ of the SLE_κ traces was proved for any $\kappa < 2.1$.

In this Chapter, we prove the sequential continuity of the welding homeomorphisms in the parameter κ induced by the backward Loewner differential equation when driven by $\sqrt{\kappa}B_t$, for $\kappa \in [0,4]$, for uncountably many points, with respect to the Euclidean norm, a.s.. The restriction to this interval is a consequence of the connection between the SLE_κ traces and the welding homeomorphisms in this regime. [to be further completed]

In Chapter 6, we provide a closed form expression relevant for the study of the backward Loewner flow. In the last years, ergodic properties of the Loewner flow were studied in [51], in which the ergodic properties of the tip of the SLE_κ trace are computed in capacity time parametrisation.
In our analysis, we perform a random time change in the context of the backward Loewner differential equation. This random time change was used for the forward Loewner differential equation in [36] to obtain the probability that the SLE trace passes to the left of a fixed point in the upper half plane. Compared with the approach in [36], we use it for the backward Loewner differential equation.

After performing the random time change, we obtain a one dimensional diffusion, that describes, via a time change, the cotangent of the argument of the points in the in $\mathbb{H}$ under the backward Loewner flow.

$$dT_u = -4\frac{T_u}{1+\kappa T_u^2}du - dB_u, \quad T_0 = 0. \tag{1.0.1}$$

We use this diffusion process in order to characterize the law of the arguments of the backward Loewner flow. In order to achieve this, we show the convergence in a random ergodic average sense of the law of the diffusion T_u described by (1.0.1) to its stationary measure. The density of this measure has the closed-form expression

$$\rho(T) = \left(\kappa T^2 + 1\right)^{-4/\kappa} .$$

We also show that this stationary measure can be identified with the distribution of the argument of points under the backward Loewner flow in the total variation sense, solving in this manner a Skorokhod-like Embedding Problem for the law of the arguments of the points under backward Loewner flow. For $\kappa = 4$, we recover a precise construction of the random times on which this law is uniform. Moreover, we show that this laws are part of a family of distributions that are stationary for the process T_u, for various choices of $\kappa \in [0, 8)$. We also show that this analysis provides a phase transition in the behaviour of the backward Loewner flow at $\kappa = 8$, from the integrability of the density of the stationary measure for $\kappa < 8$ (positive recurrence of the process T_u) to non-integrability of the density of this quantity for $\kappa \geqslant 8$ (null-recurrence of the process T_u) .

The work in Chapter 7 is situated in an interface between Numerical Analysis and Rough Paths Theory in the study of a class of singular Rough Differential Equations (with applications to the Loewner equation and SLE_κ traces). Together with my collaborator, James Foster, we proved results about truncated Taylor approximations of the backward Loewner differential equation near the origin. Specifically, we have obtained closed form expressions for the second order of the truncated Taylor approximation of the Loewner Differential Equation near the origin, at the natural Brownian scaling in space and time. Another major result obtained in this project was the best order of convergence of Taylor approximations, at these specific scales, for the Loewner dynamics, near the singularity. Questions about the best order of approximation of a numerical scheme are of broad interest, see [11]. We have answered this question for a class of singular RDEs, including the backward Loewner Differential Equation.

Chapter 2

Introduction to SLE

2.1 The Loewner Differential Equation and SLE_κ

An important object in the study of the Loewner differential equation is the $\mathbb{H}$-compact hull that is a bounded set in $\mathbb{H}$ such that its complement in $\mathbb{H}$ is simply connected. To every compact $\mathbb{H}$-hull, that we typically denote with K , we associate a canonical conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ that is called the *mapping out function of* K. Note that the theory that we introduce has, at its core, a conformal invariance structure. Thus, the general study of the objects of this theory can be mapped into a domain that is more convenient from the mathematical point of view. Most of the time, the choice is the upper half-plane $\mathbb{H}$ with ∞ as a boundary point.

Given any compact $\mathbb{H}$-hull K, we construct the conformal mapping out function $g_K : \mathbb{H} \setminus K \to \mathbb{H}$. Using the Riemann mapping theorem, we get uniqueness by imposing the *hydrodynamic normalization at* ∞ for g_t (i.e. we require that $g_t(z)$ has no constant term and its complex derivative is 1). Thus, the mapping at ∞ is of the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}) \,, \quad |z| \to \infty \,.$$

The coefficient a_K that appears in the expansion at ∞ of the mapping has the traditional name *halfplane capacity*.

We start by introducing the Chordal Loewner Theory which establishes a one-to-one correspondence between continuous valued paths $(U_t)_{t\geqslant 0}$ and an increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain growth property. Throughout the Thesis, we typically denote the family of growing compact hulls with K_t and the complements with

$H_t := \mathbb{H} \setminus K_t$. Firstly, we define the radius of a hull to be

$$rad(K) = \inf\{r \geqslant 0 : K \subset r\mathbb{D} + \text{x for some x} \in \mathbb{R}\} .$$

This is a very useful notion to estimate the distance between the points and their image for the mapping out function g_K, associated with the compact hull K .

Let $(K_t)_{t\geqslant 0}$ be a family of increasing $\mathbb{H}$-hulls, i.e. K_s is contained in K_t whenever $s < t$. For $K_{t+} = \cap_{s>t} K_s$ and for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t\geqslant 0}$ has the *local growth property* if

$$rad(K_{t,t+h}) \to 0 \text{ as } h \to 0, \text{ uniformly on compacts in t.}$$

The first connection between the family of growing compact $\mathbb{H}$-hulls and the real-valued path $(U_t)_{t\geqslant 0}$ is done in the following proposition.

Proposition 2.1.1 (Proposition 7.1 of [5]). *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then, $K_{t+} = K_t$ for all t . Moreover, the mapping $t \mapsto hcap(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t \geqslant 0$, there is a unique $U_t \in \mathbb{R}$ such that $U_t \in \bar{K}_{t,t+h}$, for all $h > 0$, and the process $(U_t)_{t\geqslant 0}$ is continuous.*

The process $(U_t)_{t\geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t\geqslant 0}$. The map $t \mapsto hcap(K_t)/2$ is a homeomorphism on $[0, T)$ and by choosing τ to be the inverse of this homeomorphism, we obtain a new family of hulls K'_t in a new parametrization such that $\text{hcap}(K'_t) = 2t$. This is the canonical parametrization that we use throughout the thesis. We use the standard terminology for this, i.e. *parametrization by halfplane capacity*.

In the following proposition, we introduce the Loewner differential equation starting from the family of growing compact hulls. The main idea is that the local growth property of the hulls gives a description in terms of a specific differential equation for the associated mapping out functions. Throughout the thesis, we are mostly interested in the cases where the hull $K_t = \gamma([0, t])$, i.e. in the case when the hull is a curve. As such, although we state the theorems in the general form (i.e. for compact $\mathbb{H}$-hulls), we draw in the associated pictures the hulls as simple curves in $\mathbb{H}$.

Proposition 2.1.2 (Proposition 7.3 of [5]). *Let $(K_t)_{t\geqslant 0}$ be a family of increasing compact hulls in $\mathbb{H}$ satisfying the local growth property and that are parametrized by the halfplane capacity. Let $(U_t)_{t\geqslant 0}$ be its Loewner transform. Set $g_t = g_{K_t}$ and $\zeta(z) = \inf\{t \geqslant 0 : z \in$*

$K_t\}$. *Then, for all* $z \in \mathbb{H}$, *the function* $(g_t(z) : t \in [0, \zeta(z))$ *is differentiable and satisfies the Loewner differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}\,.$$

Moreover, if $\zeta(z) < \infty$ *then* $g_t(z) - U_t \to 0$ *as* $t \to \zeta(z)$.

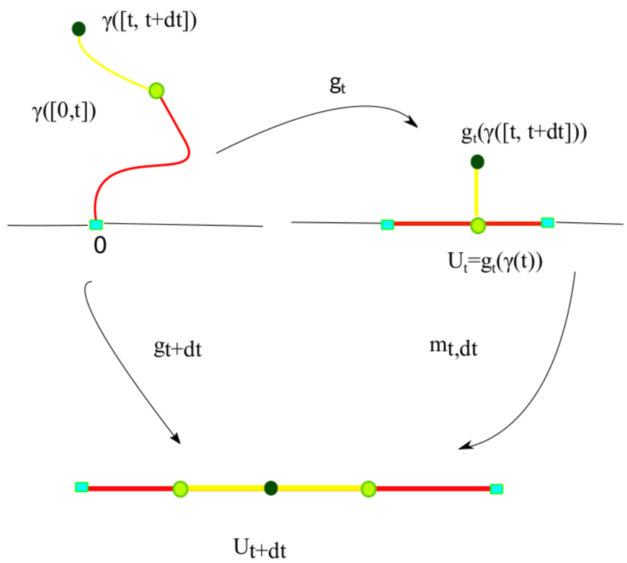

FIGURE 2.1.1. The idea behind Loewner differential equation in the case $K_t = \gamma[0,t]$.

The reverse situation is also true, i.e. from the driving function U_t, we recover the family of growing compact $\mathbb{H}$-hulls.

Proposition 2.1.3 (Proposition 8.1 of [5])**.** *For all* $z \in \mathbb{H} \setminus \{\zeta_0\}$, *there is a unique* $\zeta(z) \in (0, \infty]$ *and a unique continuous map* $(g_t(z) : t \in [0, \zeta(z))$ *in* $\mathbb{H}$ *such that, for all* $t \in [0, \zeta(z))$ *we have* $g_t \neq U_t$ *and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - U_s} ds\,,$$

and such that $|g_t(z) - U_t| \to 0$ *as* $t \to \zeta(z)$ *whenever* $\zeta(z) < \infty$. *Set* $\zeta_0 = 0$ *and define*

$$H_t = \{z \in \mathbb{H} : \zeta(z) > t\}\,.$$

Then, for all $t \geqslant 0$ H_t *is open and* $g_t : H_t \to \mathbb{H}$ *is holomorphic. Moreover, the family of sets* $K_t = (z \in \mathbb{H} : \zeta(z) \leqslant t)$ *is an increasing family of compact* $\mathbb{H}$*-hulls having the local growth property. Moreover, hcap*(K_t)=$2t$, *and* $g_{K_t} = g_t$, *for all* t. *Moreover, the driving function* U_t *is the Loewner transform of* $(K_t)_{t \geqslant 0}$.

The process $g_t(z) : t \in [0, \zeta(z))$ is called *the maximal solution starting from* z and $\zeta(z)$ is called *lifetime of the solution*.

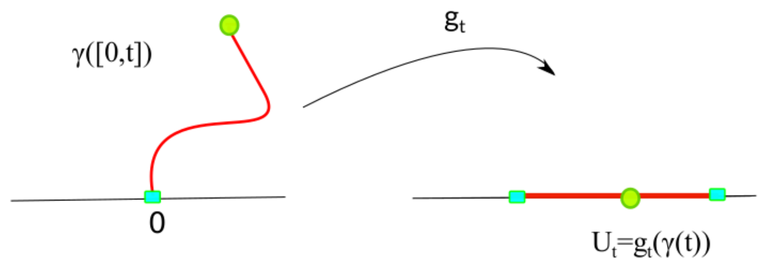

FIGURE 2.1.2. The conformal map that removes the curve grown up to time t.

Candidates for the SLE driving function In this subsection, we give a brief sketch of the proof that given the principles that our curves should satisfy, then the right candidate for the driver in Loewner equation is $\sqrt{\kappa}B_t$ where $\kappa \in \mathbb{R}$. When SLE was defined by Oded Schramm, the purpose of it was to be the suitable candidate for a family of random curves in a domain $D \subset \mathbb{C}$ that respected two principles. The principles were motivated by the study of Schramm to give a precise meaning of the scaling limits of planar loop erased random walk with loops erased in the chronological order. When studying the planar loop erased random walk, Schramm realized that the scaling limit (if existed any) should manifest some *Domain Markov Property* and *Conformal Invariance*, that are presented in the following. Note that, chordal SLE law can be defined in every simply connected domain, but we do not insist on the general definition in this thesis. This concept is used to make precise the following proprieties.

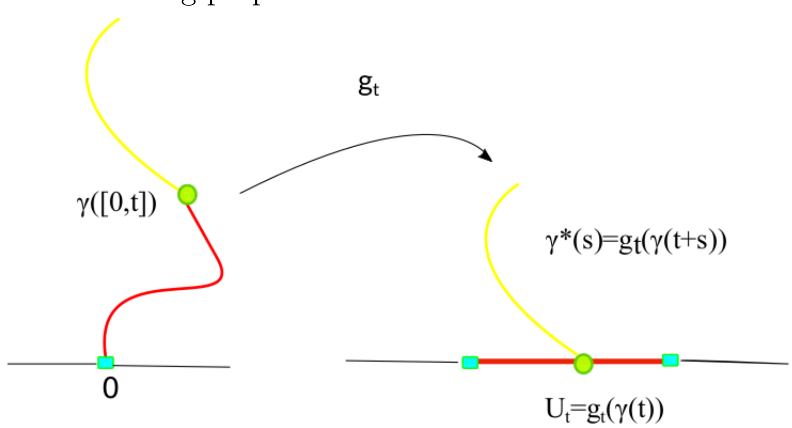

FIGURE 2.1.3. Conformal invariance of the curve γ.

- [Domain Markov] Given the curve $\gamma[0,T]$, then the law of $\gamma[T,\infty]$ is a chordal SLE in $\mathbb{H}\setminus\gamma[0,T]$, from $\gamma(t)$ to ∞.
- [Conformal Invariance] The chordal SLE law in $\mathbb{H}\setminus\gamma[0,t]$ from $\gamma(t)$ to ∞ is the pull-back of the chordal SLE law in $\mathbb{H}$ from U_t to ∞.

If we require these two properties and a certain symmetry of the law on curves with respect to the imaginary axes, then the driving function in Loewner equation should be a $\sqrt{\kappa}B_t$. In the following, we give a sketch of the proof for this result of Oded Schramm that marks the introduction of SLE.

Sketch of the proof. Given the Domain Markov Property, the curve from U_t to ∞ that is denoted by $\gamma^*(s)$ has the law of a chordal SLE curve in $\mathbb{H}$ from U_t to $+\infty$ that is independent of $\gamma[0,t]$. Note that $\gamma^*(s)$ is determined by the driving function $U_{t+s}-U_t$, $s\geqslant 0$, and $\gamma[0,t]$ by the driving function U_s, $0\leqslant s\leqslant t$. We also have by the domain Markov property that $\gamma^*(s)$ is independent of $\gamma[0,t]$ means that the driving function after time t should be independent of the driving function up to time t, i.e. U_t is a Markov process.

Moreover, by conformal invariance (since g_t^{-1} is a conformal map), we have that $\gamma^*[0,t]$ is identical in law to $\gamma[0,t]$, so U_s ,$0\leqslant s\leqslant t$, should have the same law as $U_{t+s}-U_t$, $0\leqslant s\leqslant t$. So the increments of the U_t process have the same distribution and are independent, hence are i.i.d. Moreover, by writing $dU_t=bdt+\sigma dB_t$ then by the remarks that we obtain before about U_t, then a and b are forced to be constants. Thus, the conformal invariance and the Domain Markov property force the driving process U_t to be a Brownian motion with drift. If we use our last consideration, i.e. the law of the SLE curve should be invariant under reflections about the imaginary axis (i.e. U_t should be invariant under negations), we obtain that $b=0$ leaving $dU_t=\sigma dB_t$ for some constant $\sigma>0$ (we typically denote this by $\sqrt{\kappa}$, due to reasons that will become obvious later). □

2.2 SLE definition, existence of the trace and first properties

The Schramm-Loewner evolution (SLE) is a one-parameter (usually denoted by κ) family of random planar growth processes constructed as a solution to Loewner equation when the driving term is a re-scaled Brownian motion. Thus, when studying the SLE_κ, in the upper half-plane, we study the corresponding families of conformal maps in the formats

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z-\sqrt{\kappa}B_t}\,, \quad f(0,z)=z, z\in\mathbb{H}\,. \tag{2.2.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z)-\sqrt{\kappa}B_t}\,, \qquad g(0,z)=z, z\in\mathbb{H}\,. \tag{2.2.2}$$

(iii) Time reversal differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z)-\sqrt{\kappa}B_t}\,, \qquad h(0,z)=z, z\in\mathbb{H}\,. \tag{2.2.3}$$

There are connections between these three formulations for studying families of conformal maps. The solution to the equation (7.2.2), i.e. the family of conformal maps satisfying (7.2.2), is related with the family of conformal maps satisfying (7.2.1) by the fact that at each instance of time t, the map $g_t(z)$ is the inverse of the map $f_t(z)$. In other words, the maps $f_t(z)$ "grow" the curve in the reference domain, while $g_t(z)$ maps conformally the slit domain obtained by the growing of the curve up to time t to the reference domain. The connection between the different versions of the Loewner equations is that, for all fixed t, $g_{-t}(z)$ has the same distribution as the maps $f_t(z)-U_t$ that satisfy equation (7.2.3). This is proved in the next Lemma.

Lemma 2.2.1 (Lemma 7.6 of [20])**.** *For all fixed $t\in\mathbb{R}$, the mappings $z\to g_{-t}(z)$ has the same distribution as the map $z\to f_t(z)-U_t$.*

Proof. Fix a time $s\in\mathbb{R}$, and let

$$U_s(t) = U_{s+t} - U_s\,.$$

By the shifting property of Brownian motion, we have that $U_s(t)$ has the same distribution with U. Let us consider the mapping

$$\hat{g}_t(z) := g_{s+t}\circ g_s^{-1}(z+U_s) - U_s\,,$$

Since g_t maps-down the curve up to time t and g_{t+s} maps down the curve obtain up to a later time $t+s$, we have that $\hat{g}_{-s}(z) = f_s(z) - U_s$. Since

$$\partial_t \hat{g}_t = \frac{2}{\hat{g}_t + U_s - U_{t+s}} = \frac{2}{\hat{g}_t - U_s(t)}\,,$$

the Lemma follows. $\square$

Rohde and Schramm proved in [33] one fundamental result about the existence of the trace for the SLE_κ process for all values of $\kappa \neq 8$. In order to discuss the result, we introduce the notions and definitions that we use. We say that a continous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$-hulls K_t if $H_t = \mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all $t\geqslant 0$. The main result for which we give a complete proof in this section is the following theorem.

Theorem 2.2.2 (Rohde-Schramm)**.** *Let $(K_t)_{t\geqslant 0}$ be a SLE_κ for $\kappa \neq 8$. We denote g_t to be the Loewner flow and U_t be the Loewner transform. Then, $g_t^{-1} : \mathbb{H} \mapsto H_t$ extends continously to $\bar{\mathbb{H}}$ for all $t \geqslant 0$, almost surely. Morevoer, if we set $\gamma_t = g_t^{-1}(U_t)$, then γ_t is continous and generates $(K_t)_{t\geqslant 0}$ almost surely.*

We use the following scaling of the SLE traces and of the Loewner maps.

Proposition 2.2.3 (Proposition 6.5 of [20])**.** *Let us consider g_t as being the solution to the chordal Loewner equation and let us take $r > 0$. Then $\hat{g}_t(z) := r^{-1}g_{r^2t}(rz)$ has the same distribution as chordal SLE_κ, i.e. if γ is an SLE_κ path, then $\hat{\gamma}(t) := r^{-1}\gamma_{r^2t}$ has the same distribution as γ.*

2.3 Forward Bessel processes and the phases of SLE_κ

In this subsection, we analyze the phases of SLE_κ as a function of the parameter κ. We show that there exists two phase transitions for values of $\kappa > 4$ and $\kappa \geqslant 8$. We determine these regimes not directly, but by studying the Loewner flow.

Consider the Loewner flow $(g_t(x) : t \in [0, \tau(x)),\ x \in \mathbb{R}\setminus\{0\}$, associated with the Loewner differential equation when the driving function is $\sqrt{\kappa}B_t$, i.e. the flow associated with SLE_κ. Thus, we have that $g_t(x) - U_t \to 0$ as $t \to \zeta(x)$, whenever $\zeta(x) < \infty$. We introduce the notation

$$D = \frac{2}{\kappa}, \quad B_t = \frac{-U_t}{\sqrt{\kappa}}, \quad \tau(x) = \zeta(x\sqrt{\kappa}).$$

and set

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - U_t}{\sqrt{\kappa}},$$

for $t \in [0, \zeta(x))$.

With this notation, B_t becomes a standard Brownian motion that starts from 0 and we have that for $X_t(x) \neq 0$ for $t \in [0, \tau(x))$ we have that

$$X_t(x) = x + B_t + \int_0^t \frac{D}{X_s(x)} ds\,,$$

i.e. $X_t(x)$ is a Bessel process.

The Bessel processes manifest some phase transition that we capture in the following proposition. These results give an interpretation of the phase transition of the SLE processes in terms of the understanding of the Loewner flow on the real axis as a Bessel process.

Proposition 2.3.1 (Proposition 10.1 of [5]). *Let $x, y \in (0, \infty)$ with $x < y$. Then*

- *For $D \in (0, 1/4]$, we have*

$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = 1\,.$$

- *For $D \in (1/4, 1/2)$, we have that*

$$\mathbb{P}(\tau(x) < \infty) = 1\,.$$

$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = \phi\left(\frac{y - x}{x}\right)\,,$$

  where ϕ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}\,.$$

- *For $D \in [1/2, \infty)$, we have*

$$\mathbb{P}(\tau(x) < \infty) = 0,$$

  and moreover for $D \in [1/2, \infty)$, we have $X_t(x) \to \infty$ as $t \to \infty$ almost surely.

These results translate from Bessel flow in terms of the path γ of an SLE_κ, in terms of hitting probabilities of the real line.

Proposition 2.3.2 (Proposition 10.3 of [5]). *Let γ be an SLE_κ. Then we have the following behaviors in terms of κ.*

- *For $\kappa \in (0,4]$, we have that $\gamma(0,\infty) \cap \mathbb{R} = 0$ almost surely.*
- *For $\kappa \in (4,8)$ and all $x, y \in (0,\infty)$, γ hits $[x,\infty)$ and*

$$\mathbb{P}(\gamma \ \text{hits} \ [x, x+y)) = \phi\left(\frac{y}{x+y}\right).$$

- *For $\kappa \in [8,\infty)$ then $\mathbb{R} \subseteq \gamma[0,\infty)$ almost surely.*

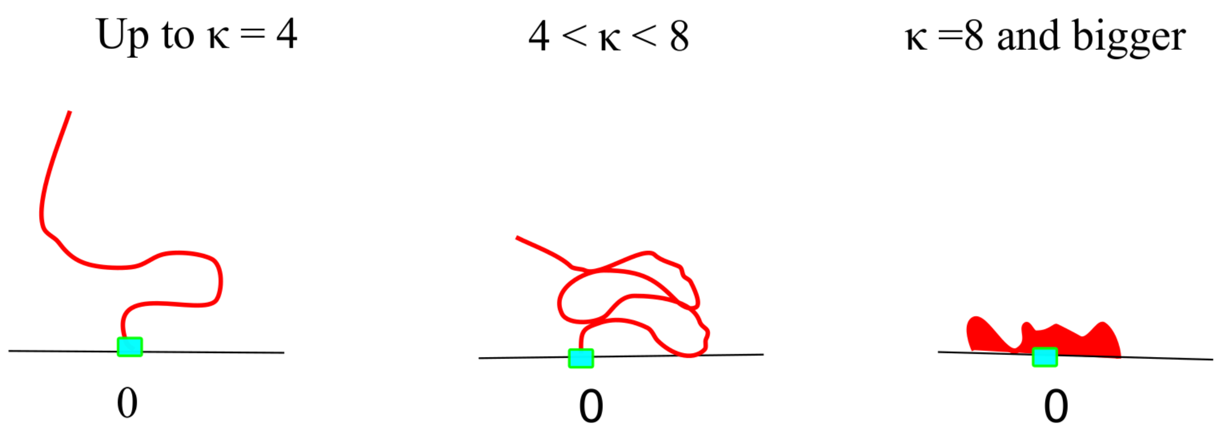

FIGURE 2.3.1. The phases of SLE_κ.

Concerning the phases of SLE_κ, there are the following results.

Theorem 2.3.3 (Theorem 11.1 of [5]). *Let $(\gamma_t)_{t\geqslant 0}$ be a SLE. Then γ is almost surely transient for all κ, i.e. $|\gamma_t| \to \infty$ as $t \to \infty$.*

Theorem 2.3.4 (Theorem 11.2 of [5]). *Let $(\gamma_t)_{t\geqslant 0}$ be a SLE. Then, we have the following behaviors when the parameter κ changes.*

- *For $\kappa \in [0,4]$, $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely.*
- *For $\kappa \in (4,8)$, $\bigcup_{t\geqslant 0} K_t = \mathbb{H}$, almost surely and for each $z \in \bar{\mathbb{H}} \setminus \{0\}$, $(\gamma_t)_{t\geqslant 0}$ does not hit z almost surely.*
- *For $\kappa \in [8,\infty)$, $\gamma[0,\infty) = \bar{\mathbb{H}}$, almost surely.*

Chapter 3

Introduction to Rough Paths Theory

In this Chapter we introduce the basic notions in the Theory of Rough Paths such as p-variation , Young's Theory of integration, the notion of signature of a path and the underlying tensor algebra, the definition of a Rough Path and concepts related with Rough Differential Equations. The main references for the material presented in this Chapter are Lyons, Lévy and Caruana [28] along with Friz and Victoir [16].

The signature of a path is a way of summarizing the information needed to solve differential equations driven by paths. The collection of these iterated integrals of the path is the *Signature of the Path.* The Riemann-Stieltjes integral gives meaning to the object $\int XdY$, where $X, Y : [0,T] \to \mathbb{R}$ are continuous functions (paths) with Y having bounded variation. Moreover, due to Young this result may be extended to a much richer class of regular functions.

Let $X_{[s,t]}$ denote the restriction of the path X to the compact interval $[s,t]$. We introduce the notion of p-variation.

Definition 3.0.1. *Let (E,d) be a metric space. The p-variation of a path $X : [0,T] \to E$ is defined by*

$$||X_{[0,T]}||_{p-var} := \sup_{\mathcal{D}=(t_0,t_1,\ldots,t_n)\subset[0,T]} \left(\sum_{i=0}^{n-1} d(X_{t_i}, X_{t_{i+1}})^p \right)^{\frac{1}{p}} ,$$

where the supremum is taken over all finite partitions of the interval $[0,T]$.

Unless stated, the metric space E is a finite dimensional real vector space V with dimension d and basis vectors $e_1, \ldots, e_d$. Throughout the thesis, we use the notation $X_{s,t} = X_t - X_s$. The notion of control is used in order to understand the p-variation of paths. Also, this notion is of great importance in the definition of rough paths.
Let us define $\Delta_T = \{(s,t)|0 \leqslant s \leqslant t \leqslant T\}$.

Definition 3.0.2. *A control on* $[0,T]$ *is a non-negative continuous function* $\omega : \Delta_T \to [0,\infty)$ *for which*

$$\omega(s,t) + \omega(t,u) \leqslant \omega(s,u),$$

for all $0 \leqslant s \leqslant t \leqslant u \leqslant T$, *and* $\omega(t,t) = 0$, *for all* $t \in [0,T]$.

For a path $X : [0,T] \to \mathbb{E}$ of finite p-variation, define $\omega_X(s,t) := ||X_{[s,t]}||^p_{p-var}$. To check that this quantity indeed defines a control we need to check both proprieties along with the continuity (the continuity is proved in Section 1.2.2 of [28] and Proposition 5.8 of [16]). Moreover, this specific control is used to provide a reparametrization of the interval $[0,T]$ such that X becomes Hölder continuous with exponent $\frac{1}{p}$. To see this, consider the function $\tau : [0,T] \to [0,T]$ to be the inverse of the function $t \to \omega_X(0,t)\frac{T}{\omega_X(0,T)}$. Using this inverse function, we obtain that $d(X_{\tau(s)}, X_{\tau(t)})^p \leqslant \frac{\omega_X(0,T)}{T}(t-s)$.

3.1 Young integration

In what follows next, let us consider E to be a Banach space and X, Y to be continuous functions (paths) with values in a Banach spaces. A first extension to the usual Riemann-Stieltjes integral is given by Young's Theory of Integration. Using the results of Young, we can make sense of $\int_0^t Y_s dX_s$ provided that X and Y have finite p-variation and finite q-variation respectively with $\frac{1}{p} + \frac{1}{q} > 1$. Note that the bounded variation (i.e. $p = 1$) of X is contained in Young's theory of integration. To be precise, we have the result

Theorem 3.1.1 (Young, Theorem 1.16 of [28])**.** *Let* V *and* W *be Banach spaces and* $X : [0,T] \to V$ *and* $Y : [0,T] \to \mathbf{L}(V,W)$ *be two paths of finite p-variation and q-variation respectively with* $\frac{1}{p} + \frac{1}{q} > 1$. *Then, the limit*

$$\lim_{|\mathcal{D}|\to 0, \mathcal{D}\subset[0,t]} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_i,t_{i+1}})$$

exists for all $t \in [0,T]$ *and we define* $\int_0^t Y_s dX_s$ *as this limit. Furthermore, as a path in* W, $\int_0^\cdot Y_s dX_s$ *has finite p-variation.*

Note that the Theorem holds independent of the sequence of partitions chosen provided that the size of the mesh $|\mathcal{D}| = \max_{0\leqslant i\leqslant n-1}(t_{i+1} - t_i) \to 0$.

3.2 Differential equations driven by signals with finite p-variation for $p < 2$

In the classical theory of ordinary differential equations, the Cauchy-Peano Theorem asserts that when X has bounded variation, if the vector field f is a continuous function, then the differential equation $dY_t = f(Y_t)dX_t$ has solutions. However, this result does not in general guarantee uniqueness. In order to have uniqueness of solution, the Theorem of Picard-Lindelöf tells us that we have to impose more conditions on the function f, i.e. we need f to be Lipschitz continuous. In order to extend these results for signals with finite p-variation for $1 \leqslant p < 2$, presumably we need to have to use a smoother class of vector fields f than the continuous ones. For example, in order to make sense of the integral $\int f(Y)dX$, we need that $f(Y)$ has finite q-variation for some q such that $\frac{1}{p} + \frac{1}{q} > 1$. It is clear that not every continuous function does that, but we can restrict to the specific class of functions with this property.

The main result of this Section, presented in [28] is a version of the classical Picard Fix Point Theorem in the context of differential equations driven by signals with finite p-variation for some $p < 2$. In order to state it, we need to introduce the notion of *Lip*(γ) function. For this we consider again the metric space E to be a Banach space.

Definition 3.2.1. *Let V and W be two Banach spaces. Let $k \geqslant 0$ be an integer. Let $\gamma \in (k, k+1]$ be a real number. Let F be a closed subset of V. Let $f : F \to W$ be a function. For each integer $j = 1, \dots, k$ let $f^j : F \to \mathbf{L}(V^{\otimes j}, W)$ be a function which takes its values in the space of j-linear mappings from V to W. The collection $(f = f^0, f^1, \dots, f^k)$ is an element of Lip(γ, F) if the following condition holds.*

There exists a constant M such that, for each $j = 0, \dots, k$,

$$\sup_{x \in F} |f^j(x)| \leqslant M$$

and there exists a function $R_j : V \times V \to \mathbf{L}(V^{\otimes j}, W)$ such that, for each $x, y \in F$ and each $v \in V^{\otimes j}$, we have

$$f^j(y)(v) = \sum_{l=0}^{k-j} \frac{1}{l!} f^{j+l}(x)(v \otimes (y-x)^{\otimes l}) + R_j(x,y)(v)\,,$$

and

$$|R_j(x,y)| \leqslant M|x-y|^{\gamma - j}\,.$$

The smallest M for which the inequalities hold for all j is called the Lip(γ, F)-norm of f.

With this definition at hand, we can state the version of Picard Fix Point Theorem for differential equations driven by signals with finite p-variation for some $p < 2$.

Theorem 3.2.2 (Picard, Theorem 1.28 of [28])**.** *Let p and γ be such that $1 \leqslant p < 2$ and $p \leqslant \gamma$. Assume that X has finite p-variation and that f is Lip(γ). Then, for every $\zeta \in W$, the differential equation $dY_t = f(Y_t)dX_t$, admits a unique solution, i.e. for every $\zeta \in W$ there exist a unique path $Y : [0, T] \to W$ of finite p-variation which satisfies $Y_0 = \zeta$ and*

$$Y_t = \int_0^t f(Y_s)dX_s .$$

In order to prove this Theorem , the technique is similar with the classical proof of the Picard's fix point Theorem. We consider the map F that sends a path $Y : [0, T] \to W$ to a new path defined via $F(Y_t) = \zeta + \int_0^t f(Y_s)dX_s$ which we prove that is a contraction under suitable conditions.

3.3 The signature of a path

In this section, we introduce one of the fundamental ingredients of the theory, the collection of its iterated integrals, i.e. the *signature of the path*. This object appears naturally when one uses iteration in order to solve linear differential equations. Moreover, the signature of a path carries very interesting algebraic properties.

In order to simplify the computations and to introduce the signature of a path in a clear manner, throughout this section we consider the driver X to be of bounded variation. For this specific class of drivers, we consider the linear differential equation $dY_t = \bar{f}(Y_t)dX_t$, where $\bar{f} : W \to \bar{L}(V, W)$ is a linear map.

The collection of iterated integrals appears naturally when one tries to apply an iterative procedure in order to find a solution to the ordinary differential equation via fix point arguments. To make this precise, we remark that $\bar{f}(Y_t)dX_t$ can be understood in two ways. Every linear map $\bar{f} \in \mathbf{L}(W, \mathbf{L}(V, W))$ in a natural way also induces a linear map $\bar{f} : V \to \mathbf{L}(W)$, (we call it the same using a slight abuse of notation). Another abuse of notation that we use is that we have elements of $\mathbf{L}(W)$ act on W on the right instead of the left as this will simplify notation in the following.

We start with the constant path $Y_t^0 = F(Y^0)_t = Y_0(I + \bar{f}(\int_0^t dX_s))$. By re-iterating the procedure, we obtain that

$$Y_t^2 = F(Y^1)_t = Y_0 + \int_0^t \bar{f}(Y_s^1)dX_s = Y_0 \left(I + \bar{f} \int_0^t dX_s + \bar{f}^{\otimes 2} \int_0^t (\int_0^s dX_u) \otimes dX_s \right)$$

After k steps, we obtain that

$$Y_t^n = Y_0 \left(\sum_{k=0}^{n} \bar{f}^{\otimes k} \int_{0<u_1<\ldots<u_k<t} dX_{u_1} \otimes dX_{u_2} \ldots \otimes \dot{X}_{u_k} \right),$$

where we define $\bar{f}^{\otimes k}(x_1 \otimes \ldots \otimes x_k) = \bar{f}(x_1) \circ \ldots \circ \bar{f}(x_k)$ and then we extend linearly. In the next part of the section, we study the tensor vector spaces where this iterated integrals live and we state a result that ensures that the iterated integrals do converge.

Let $e_1, e_2, \ldots, e_d$ be a basis for V. The space $V^{\otimes k}$ is a d^k dimensional vector space with basis elements of the form $(e_{i_1} \otimes e_{i_2} \ldots \otimes e_{i_k})_{(i_1,\ldots,i_k)\in\{1,\ldots,d\}^k}$. We store the indices $(i_1, \ldots, i_k) \in \{1, 2, \ldots d\}^k$ in a multi-index I and let $e_I = e_{i_1} \otimes e_{i_2} \otimes \ldots e_{i_k}$. From the several possible norms on the tensor space $V^{\otimes k}$, see Definition 1.25 of [28] for the so-called *admissible norms*, due to its properties, we chose the *projective* or the l_1 norm defined for any element $x = \sum_{|I|=k} \lambda_I e_I \in V^{\otimes k}$ via $||x|| = \sum_{|I|=k} |\lambda_I|$. The two main properties of this norm are that for any $x \in V^{\otimes k}$ and $y \in V^{\otimes m}$ we have that $||x \otimes y|| = ||x||||y||$ and that for $x = \sum_{|I|=k} \lambda_I e_I$ we have $||M^{\otimes k} x|| = ||\sum_{|I|=k} \lambda_I M^{\otimes k} e_I|| \leqslant \sum\limits_{|I|=k} |\lambda_I|||M||^k = ||M||^k||x||$.

By considering the basis of $V^{\otimes k}$ given by elements of the form e_I, we can take the dual basis in $(V^{\otimes k})^*$ as e_I^* that we can further associate the element $e_{i_1}^* \otimes e_{i_2}^* \otimes \ldots \otimes e_{i_k}^* \in (V^*)^{\otimes k}$. Furthermore, for any I, e_I^* is an element of $T(V*)$, i.e. the sub-algebra of $T((V^*))$ consisting of elements with finitely many non-zero coordinates and running through all possible multi indices with $|I| > 0$, we find a basis for $T(V^*)$ consisting of e_I. We are now ready to introduce the following result that guarantees the convergence of the Picard iterates.

Proposition 3.3.1 (Proposition 2.2 of [28]). *Let $X : [0,T] \to V$ be a path of bounded variation. Then it follows that*

$$|| \int_{0<u_1<\ldots u_k<t} dX_{u_1} \otimes dX_{u_2} \ldots dX_{u_k} || \leqslant \frac{||X_{[0,t]}||_{1-var}^k}{k!}.$$

3.4 The tensor algebra

Given that the collection of iterated integrals is such an important object in the study of the linear differential equations, it is natural to consider it as a separate object which we call the signature of a path. This object takes value in the tensor algebra $T(V)$, which is described in this subsection.

The tensor algebra $T((V)) = \bigoplus_{k\geqslant 0} V^{\otimes_k}$ is the infinite sum of all tensor products of V. For two elements $a = (a^0, a^1 \ldots)$ and $b = (b^0, b^1, \ldots)$ we define

$$a + b = (a^0 + b^0, a^1 + b^1, \ldots)$$

and

$$a \otimes b = (c^0, c^1, \ldots),$$

where $c^k = \sum_{i=0}^{k} a^i \otimes b^{k-i}$. These operations together with the typical component by component multiplication by scalars $\lambda a = (\lambda a^0, \lambda a^1 \ldots)$, turn $T((V))$ into a real non-commutative algebra with unit $\mathbf{1} = (1, 0, \ldots)$. An important aspect of this Tensor Algebra is that a generic element $d \in T((V))$ is invertible if and only if $d^0 \neq 0$, and moreover there is an explicit way to compute the inverse

$$d^{-1} = \frac{1}{d_0} \sum_{k\geqslant 0} \left(\mathbf{1} - \frac{1}{d_0} d\right)^{\otimes k}. \tag{3.4.1}$$

For all $d \in T((V))$, we define the exponential and the logarithm maps via

$$\exp d = \sum_{k\geqslant 0} \frac{1}{k!} d^{\otimes k}. \tag{3.4.2}$$

$$\log d = \log d^0 + \sum_{k\geqslant 1} \frac{(-1)^k}{k} \left(\mathbf{1} - \frac{1}{d_0} d\right), \tag{3.4.3}$$

whenever $a^0 > 0$. Like the inverse mapping, the logarithm is well defined since finitely many terms of d contribute to each term of $\log(d)$. For the exponential map in the finite dimensional case it suffices to ensure the convergence of terms of each degree according to Lemma 2.19 of [28]. An important subspace of $T((V))$ that is denoted by B_0 is the is the subspace for which the term of degree zero is 0. Another important object is the multiplicative subgroup of $T((V))$ which is denoted by $T((\tilde{V}))$, for which the term of degree zero is 1. Then the maps $\exp : B_0 \to T((\tilde{V}))$ and $\log : T((\tilde{V})) \to B_0$ are bijections and inverses of each other.

3.5 The signature of a path, shuffle product and group-like elements

In this subsection, we introduce the signature of a path with p-variation for some $p < 2$ and describe some computation tools that are extremely useful when computing terms in the signature along with some fundamental theorems.

Definition 3.5.1. *Let $X : [0,T] \to V$ be a path of finite p-variation for some $p < 2$. For $k \geqslant 0$, let $\tilde{S}_k([s,t]) = \int_{s<u_1<\ldots<u_k<t} dX_{u_1} \otimes \ldots \otimes dX_{u_k}$. Then the signature of the path $X_{[s,t]}$ is given by $S(X_{[s,t]}) = (1, \tilde{S}_1([s,t]), \tilde{S}_2([s,t]), \ldots)$.*

Note that the signature of a path X lies in the subgroup $\tilde{T}((V))$ of $T((V))$. Note that the integral in the definition is a Young integral since $p < 2$. One of the properties of the signature is that is not sensitive to the starting point or to the speed that a path is traversed. That is, $S(X_{[s,t]}) = S(Y_{[u,v]})$ for $Y_w = X_{\tau w} + x$, where $\tau : [s,t] \to [u,v]$ is an increasing bijection and $x \in V$. For two paths $X_{[s,t]}$ and $Y_{[u,v]}$, we define the concatenated path $(X * Y) : [s, t+v-u] \to V$ via

$$(X * Y)_w = \begin{cases} X_w, & \text{if w} \in [s,t] \\ X_t + Y_w - Y_t, & \text{if w} \in [t, t+v-u]. \end{cases}$$

In this setting, we have one result about how the signature map behaves for the concatenated path in terms of the paths, attributed to Chen.

Theorem 3.5.2 (Chen, Theorem 2.9 and Corollary 2.13 of [28])**.** *For two paths $X_{[s,t]}$ and $Y_{[u,v]}$ of finite p-variation for some $p < 2$, we have that*

$$S(X * Y) = S(X) \otimes S(Y).$$

Chen's Theorem also gives that the range of the signature is closed in $T((V))$. Moreover, if we consider the reversal of the path $X : [s,t] \to E$ given by $X^r : [s,t] \to E$ given by $X^r_u = X_{s+t-u}$, then we have

Proposition 3.5.3 (Proposition 2.14 of [28])**.** *Let $X : [s,t] \to V$ be a path of finite p-variation for some $p < 2$, and $X^r_{[s,t]}$ be its reversal. Then it follows that $S(X^r) = S(X)^{-1}$.*

Using the previous Proposition and the fact that $S(x_{[s,t]}) = \mathbf{1}$ with $x \in V$ ($x_{[s,t]}$ is understood as the constant path at x), we conclude that the range of the signature map forms a multiplicative subgroup of $\tilde{T}((V))$.

We now define another useful property when it comes to dealing with signatures of paths. For $i, j \in \mathbb{Z}$, we define the *shuffle product* $Sh(i,j) \subset S_{i+j}$ as the permutations σ on the set $\{1, 2, \ldots, i+j\}$ which preserve the order of the sets $\{1, 2, \ldots, i\}$ and $\{i+1, \ldots, i+j\}$, that is $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(i+j)$. For two multi-indexes $U = (u_1, \ldots, u_i)$,and $V = (v_1, \ldots, v_j)$, we define the joint multi-index $H = (h_1, \ldots, h_{i+j}) = (u_1, \ldots, u_i, v_1, \ldots, v_j)$, and for $\sigma \in S_{i+j}$ we define $\sigma(H) = (h_{\sigma_1}, \ldots, h_{\sigma(i+j)})$. Then, we define for e^*_U and $e^*_V \in T(V*)$ the shuffle product by

$$e^*_U \sqcup e^*_V = \sum_{\sigma \in Sh(i+j)} e^*_{\sigma^{-1}(H)} \in T(V^*).$$

We further define the *group-like elements* of $\tilde{T}((V))$, denoted by $G^{(*)}$, consisting of all elements $a \in \tilde{T}((V))$ for which $f(a)g(a) = (f \sqcup g)(a) \quad \forall f, g \in T(V^*)$.

The verification that $G^{(*)}$ forms indeed a subgroup of $\tilde{T}((V))$ is proved in Lemma 2.17 in [28]. One of the most important results about this group is that this group is actually the subspace where the signature takes values.

Theorem 3.5.4 (Theorem 2.15 of [28]). *For a path $X_{[s,t]}$ of finite p-variation with $p < 2$, it holds that $S(X) \in G^{(*)}$.*

We put on $T((V))$ the product topology (i.e. the topology for which a sequence of elements $(a_n)_{n\geqslant 0} \in T((V))$ converges to $a \in T((V))$ in the product topology if and only if it converges coordinate-wise, i.e. for every $k \geqslant 0$, the limit $\lim_{n\to\infty} a_n^k = a^k$. Also, all the maps and operations discussed so far, that is $+, \otimes : T((V)) \times T((V)) \to T((V))$, inversion, exp and log are continuous with respect to the product topology on $T((V)) \times T((V)$.

The (associative) tensor algebra $T((V))$ can be viewed as a Lie algebra with the Lie bracket given by $[a, b] = a \otimes b - b \otimes a$. When studying the Lie algebra $T((V))$, two Lie subalgebras are of interest. First the Lie subalgebra generated by the elements $V \subset T((V))$, which we denote by $\mathcal{L}(V)$, and $\mathcal{L}((V))$ that is the completion of $\mathcal{L}(V)$ in $T((V))$ with respect to the product topology. In order to describe $\mathcal{L}(V)$ and $\mathcal{L}((V))$, we have the following procedure. For the subspace $U, W \subset T((V))$, we define $[U, W] \subset T((V))$ as the subspace spanned by all the elements of the form $[a, b]$ with $a \in U$ and $b \in W$. We define inductively for $k \in \mathbb{N}$, $L_k = [V, L_{k-1}] \subset V^{\otimes k}$, with $L_0 = 0$ and $L_1 = V$. It follows that

$$\mathcal{L}((V)) = \bigoplus_{k\geqslant 0} L_k$$

and naturally $\mathcal{L}(V)$ consists of those elements of $\mathcal{L}((V))$ which have finitely many non-zero terms. The elements of $\mathcal{L}((V))$ are typically called Lie series and those of $\mathcal{L}(V)$ Lie polynomials. As a complete classification of group-like elements, we have the following result

Theorem 3.5.5 (Theorem 2.23 of [28]). *An element is group-like if and only if its logarithm is a Lie series. That is $\log G^{(*)} = \mathcal{L}((V))$.*

3.6 Free Nilpotent Lie Group and Rough Paths

All the spaces that we defined so far in our analysis, have also a finite dimensional version (that we call the truncated version) which consists of terms up to a fixed degree

n . In order to define them precisely, we consider

$$B_n = \{a = (a^0, a^1, \ldots)|a^0 = a^1 = \ldots = a^n = 0\}$$

that is an ideal of $T((V))$. We define the truncated tensor algebra $T^{(n)}(V) = T((V))/B_n$ with $\pi_n : T((V)) \to T^{(n)}(V)$ the natural quotient map. As an algebraic structure, $T^{(n)}(V)$ is an algebra where multiplication $\otimes$ is the same as in $T((V))$, but truncated at level n (i.e. all the elements with degree higher than n are automatically null in this space). Between the truncated spaces, we can define truncated versions of the log map or of the exponential map. We can also view $T^{(n)}(V) = \bigoplus_{0\leqslant k\leqslant n} V^{\otimes k}$ - as a finite sum of finite dimensional vector spaces. Thus, when V is of dimension d , $T^{(n)}(V)$ is of dimension $1 + d + \ldots + d^n$. We also remark that we consider $T^{(n)}(V)$ endowed with the product topology from $T((V))$. In this setting, as before, the operations remain continuous.

We consider also the sets of truncated group-like elements and Lie polynomials of degree n , given by $G^{(n)} = \pi_n G^{(*)}$ and $\mathcal{L}^{(n)}(V) = \pi_n \mathcal{L}(V) = \pi_n \mathcal{L}((V))$. Also, remark that $\tilde{T}^{(n)}(V) = \pi_n \tilde{T}((V))$ is a simply connected n-step nilpotent Lie group with Lie algebra $\pi_n B_0$ for which $\exp^{(n)} : \pi_n B_0 \to \tilde{T}^{(n)}(V)$ is a diffeomorphism .

We have also that

Proposition 3.6.1 (Lemma 2.24 and Proposition 2.25 of [28])**.** *An element $a \in T((V))$ is group-like if and only if $\pi_n a \in G^{(n)}$ for all $n \geqslant 0$ if and only if $\log^{(n)} \pi_n a \in \mathcal{L}^{(n)}(V)$ for all $n \geqslant 0$ .Furthermore, $G^{(n)}$ is a closed , simply connected n-step nilpotent Lie subgroup of $\tilde{T}^{(n)}(V)$, for which $\mathcal{L}^{(n)}(V)$ is the Lie algebra and $\exp^{(n)} : \mathcal{L}^{(n)} \to G^{(n)}$ is the diffeomorphism.*

We call $G^{(n)}$and $\mathcal{L}((V))$ the free-step nilpotent Lie group and algebra over V . Also, we can naturally consider the notion of truncated signature $S^{(n)}(X_{[s,t]}) = \pi_n S^{(n)}(X_{[s,t]})$ as an element of $G^{(n)}$. We also have that for two paths X and Y, $S^{(n)}(X*Y) = S^{(n)}(X) \otimes S^{(n)}(Y)$ and $S^{(n)}(X)^{-1} = S^{(n)}(X^r)$, with X^r the reversal of X . An important result that gives a description of elements in $G^{(n)}$ is the Chow-Rashevskii Theorem.

Theorem 3.6.2 (Chow-Rashevskii Theorem, Theorem 2.26 of [28])**.** *For every $a \in G^{(n)}$ there exists a piecewise linear path $X : [0, T] \to V$ such that $a = S^{(n)}(X_{[0,T]})$.*

Remarkably, each element of $G^{(n)}$ is in fact the signature of some piecewise linear path. With this result at hand, we can define the Carnot-Caratheodory (CC) norm on the group $G^{(n)}$ by $||a||_{CC} = \inf_{S(n)(X_{[0,T]})|} ||X||_{1-var}$ which is guaranteed to be finite. Since $G^{(n)}$ is not a vector space, the CC norm is not a norm in the real sense but gives rise to a metric on $G^{(n)}$ by $d_{CC}(a,b) = ||a^{-1}b||_{CC}$. We are now ready to define the notion of *p-rough path*. This notion builds naturally on previous results. These paths are taking values in the Lie group $T^{(n)}(V)$ and satisfy an algebraic condition- Chen's identity- and an analytic bound that is very similar with the factorial decay of the norms of the signature levels presented in Proposition 3.3.1 . We now define the notion of *multiplicative functional.*

Definition 3.6.3. *Let $n \geqslant 1$ be an integer and let $X : \Delta_T \to T^{(n)}(V)$ be a continuous map. Denote with $X_{s,t}$ the image of the interval (s,t) by X , and write*

$$X_{s,t} = (X^0_{s,t}, \dots X^n_{s,t}) \in \mathbb{R} \oplus V \oplus V^{\otimes 2} \dots \oplus V^{\otimes n} \, .$$

The function X is called multiplicative functional of degree n in V if $X^0_{s,t} = 1$ and for all $(s,t) \in \Delta_t$ we have

$$X_{s,u} \otimes X_{u,t} = X_{s,t} \ \ \forall s,u,t \in [0,T] \, .$$

The multiplicative property is called *Chen's identity* (see connection with Theorem-Chen) . To get familiar with the object, let us take the following example.

Example 3.6.4 (Example from [28])**.** Let X be a multiplicative functional of degree 1 that is for all $(s,t) \in \Delta_T, X_{s,t} \in T^{(1)}(V)\mathbb{R} \oplus V$. That is $X_{s,t} = (1, X^1_{s,t})$, with $X^1_{s,t} \in V$. Then Chen's identity reads

$$(1, X^1_{s,t}) = (1, X^1_{s,u}) \otimes (1, X^1_{u,t}) = (1, X^1_{s,u} + X^1_{u,t}) \, .$$

So, in $T^{(1)}(V)$ (at 'first level') the multiplication condition in $T^{(1)}(V)$ reduces to the additivity in V of the mapping $(s,t) \to X^1_{s,t}$. This fact is equivalent with the existence of a path $X^1_{s,t} = x_t - x_s$, for all $(s,t) \in \Delta_T$. This path is unique up to an addition of a constant element in V . Note that if X is a multiplicative functional of degree $n \geqslant 2$ in V then we have that $\pi_1 X : \Delta_T \to T^{(1)}(V)$ is a multiplicative functional of degree 1.This implies that there exists a classical path $x : [0,T] \to V$, which underlies X (i.e. $X^1_{s,t} = x_t - x_s$,) but X is not the signature of X in general. First of all, X may have p-variation for some $p \geqslant 2$, in which case the signature does not exist. Even if the signature does indeed exist, consider for example the path $X_{s,t} = (1, 0, (t-s)w)$ that is a multiplicative functional of degree 2 . If we look only at the first level, then the underlying path x is a constant path whose signature is simply $(1,0,0)$ which is different from X . We are now ready to define the central object of Rough Paths Theory.

Definition 3.6.5. *A $p-$rough path of degree n is a map $X : \Delta_T \to \tilde{T}^{(n)}(V)$ which satisfies Chen's identity $X_{s,t} \otimes X_{t,u} = X_{s,u}$ and the following 'level dependent' analytic bound*

$$||X^i_{s,t}|| \leqslant \frac{w(s,t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!} ,$$

where $y! = \Gamma(y+1)$ whenever y is a positive real number and β_p , is a positive constant.

We call w a p-variation control of X . The factors $(\frac{i}{p})!$ and β_p in the definition of the p-rough path are not important due to the possibility of recalling; however, they become important in the following Extension Theorem that states that there is only one way to extend a p-rough path to the entire group $\tilde{T}((V))$.

Theorem 3.6.6 (Extension Theorem, Theorem 3.7 of [28])**.** *Let $p \geqslant 1$ be a real number and $n \geqslant 1$ be an integer. Let $X : \Delta_T \to T^{(n)}(V)$ be a multiplicative functional with finite $p-$variation controlled by w . Assume that $n \geqslant [p]$. Then there exists a unique extension of X to a multiplicative functional $\Delta_T \to T((V))$ which has finite $p-$ variation, i.e. for every $m \geqslant [p]+1$, there exists a unique continuous function $X^m : \Delta_T \to V^{\otimes m}$ such that $(s,t) \to X_{s,t} = (1, X^1_{s,t}, \ldots, X^{[p]}_{s,t}, \ldots, X^m_{s,t}, \ldots) \in T((V))$, is a multiplicative functional with finite $p-$variation controlled by w , in the sense of definition 3.6.5 , with* $\beta_p = p^2 \left(1 + \sum\limits_{r=3}^{\infty} \left(\frac{2}{r-2}\right)^{\frac{[p]+1}{p}}\right)$.

We denote the space of all $p-$rough paths of degree $[p]$ by $\Lambda_p(V)$. A first remark is that for every continuous map $X : \Delta_T \to \tilde{T}((V))$, we associate the path $x_t = X_{0,t}$, and conversely for every $x \in C([0,T], \tilde{T}^{(n)}(V))$ we can define the mapping $X : \Delta_T \to \tilde{T}((V))$ by $X_{s,t} = x_s^{-1} x_t$, which indeed satisfies Chen's identity. Thus, every map $X : \Delta_T \to \tilde{T}((V))$ that satisfies Chen's identity is completely characterized by its associated path $x : t \to X_{0,t}$. For a path with $p-$ variation smaller than 2, that is canonically a $p-$rough path of degree $[p] = 1$, we have the signature $S(X) : (s,t) \to S(X_{[s,t]})$, that satisfies Chen's identity also. Furthermore, a modification of the factorial decay of the signature gives that the signature satisfies the required analytic bound also. To sum up, we are in the position to apply Theorem 3.6.6 to obtain that the signature is the unique extension of a $p-$ rough path for $p < 2$. Also, the Extension map is a continuous mapping as stated in the following important result.

Theorem 3.6.7 (Continuity of the Extension Map, Theorem 3.10 of [28])**.** *Let $X, Y \in \Lambda_p(V)$ with $p-$variation control w and let $\varepsilon \in (0,1)$. Then the bound*

$$||X^i_{s,t} - Y^i_{s,t}|| \leqslant \varepsilon \frac{w(s,t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!} ,$$

holds for all $(s,t) \in \Delta_t$ and $i \geqslant 1$, provided that it holds for all $1 \leqslant i \leqslant [p]$.

3.7 The spaces of p-Rough Paths and Geometric Rough Paths

We introduce a metric on $\Lambda_p(V)$ which transform the space $\Lambda_p(V)$ in a complete metric space. For $X, Y \in \Lambda_p(V)$ we define

$$d_p(X,Y) = \max_{1 \leqslant i \leqslant [p]} \sup_{\mathcal{D} \subset [0,T]} \left(\sum_{\mathcal{D}} ||X^i_{t_i,t_{i+1}} - Y^i_{t_i,t_{i+1}}||^{\frac{p}{i}} \right)^{\frac{i}{p}} .$$

In general, this quantity might be infinite for general mappings from $\Delta_t \to T((V))$, but in the context of $p-$rough paths this is finite due to Theorem 3.6.7. Related to this notion is a notion of convergence (stronger notion) that is the *convergence in the p-variation topology*. Formally, this is defined in terms of converging sequences. A sequence $(X(n))_{n \geqslant 1} \in \Lambda_p(V)$ is said to converge to $X \in \Lambda_p(V)$ in $p-$ variation topology if there exists a $p-$ control w of X and $X(n)$ for all $n \geqslant 1$, and a sequence $(a(n))_{ngeq1}$ of positive reals such that $\lim_{n\to\infty} a(n) = 0$ and

$$||X(n)^i_{s,t} - X^i_{s,t}|| \leqslant a(n) w(s,t)^{\frac{i}{p}} ,$$

for all $(s,t) \in \Delta_T$ and $1 \leqslant i \leqslant [p]$.

A path in V with finite $q-$ variation for some $q < 2$ defines q-rough path. For $p \geqslant q$, this $q-$rough path can be extended by the Extension Theorem to a multiplicative functional of degree $[p]$ with finite $q-$ variation, hence finite $p-$variation. This means that in particular, a path with bounded variation can be considered canonically as a $p-$rough path for every $p \geqslant 1$, where the extension is given by the signature, as discussed before. We are now ready to define the notion of a geometric rough path.

Definition 3.7.1. *A geometric p-rough path is a p-rough path that can be expressed as a limit of* 1*-rough paths in the p-variation metric.*

The space of geometric rough paths in V is denoted by $G\Omega_p(V)$.

Example 3.7.2 (Example from [28])**.** Let us take $V = \mathbb{R}^d, (d > 2)$. Let $(x^i)_{i=1,\ldots,d}$ be an element of V. The $2-$tensors are pictured as arrays $(x^{ij})_{i,j\in 1,\ldots,d}$. Let us take x_u to be a path of finite $1-$ variation in $\mathbb{R}^d$. We define $X^i_{s,t} = x^i_t - x^i_s$, and $X^{i,j}_{s,t} = \int\int_{s<u_1<u_2<t} dx^i_{u_1} dx^j_{u_2}$. Then $X_{s,t} = (1, (X^i_{s,t})^d_{i=1}, (X^{i,j}_{s,t})^d_{i,j=1})$, is a multiplicative functional in $T^{(2)}(\mathbb{R}^d)$, with finite $p-$ variation for some $p < 3$. This functional belongs to $G\Omega_p(\mathbb{R}^d)$

and is called the canonical extension of the path x_u . Writing $(X^{i,j}_{s,t})^d_{i,j=1}$ into its symmetric and anti-symmetric parts, we obtain

$$X^{i,j}_{s,t} = \frac{1}{2}(x^i_t - x^i_s)(x^j_t - x^j_s) + A^{i,j}_{s,t}\,,$$

where $A^{i,j}_{s,t} = \frac{1}{2}\int\int_{s<u_1<u_2<t} dx^i_{u_1}dx^j_{u_2} - dx^j_{u_1}dx^i_{u_2}$. The $A^{i,j}_{s,t}$ has the geometric interpretation of the area enclosed by the path concatenated with the chord between the points given by the indices i and j . Thus, by integrating the winding number over the plane gives $A^{i,j}_{s,t}$.

3.8 Rough Differential Equations and Rough Differential equations with drift

Throughout this section we work with a suitable space and a suitable notion of distance needed in order to set up exactly the definitions for Rough Differential Equations (RDE's) .

Definition 3.8.1. *A weakly geometric p-rough path is p-rough path which takes its values in $G^{[p]}$, the free nilpotent group of step $[p]$. The space of weakly geometric rough paths is denoted by $WG\Omega_p(V)$.*

This space is indeed related to the space of geometric rough path via the inclusion $G\Omega_p(V) \subset WG\Omega_p(V)$, and this inclusion is strict. For futher details, see 3.22, [28]. Also, we work with the notion of distance given by $d_{0;[0,1]}(\mathbf{x},\mathbf{y}) := \sup_{0\leqslant s<t\leqslant 1} d(x_{s,t}, y_{s,t})$. Rough Differential Equations introduction with all the detailes needed can be found in Chapter 10 in [16] . For now let $x \in C^{1-var}([0,1],\mathbb{R}^d)$. For the solution y to the following controlled differential equations

$$dy = V(y)dx \ := \ \sum_{i=1}^{d} V_i(y)dx^i\,, \ \ y_0 \in \mathbb{R}^q\,,$$

we use the notation $\pi_{(V)}(0;y_0,x)$. In fact, the notation $\pi_{(V)}(s,y_s;x)$ stands for solutions of controlled ODE with vector fields $V = (V_i)_{i=1,\ldots,d}$ started at time s from a point $y_s \in \mathbb{R}^q$.

We extend our study to the Rough Differential equations with drift term as this fits perfectly with the study of the Loewner differential equation as a Rough Differential Equation. We start with the corresponding definition of RDE's with drift.

Definition 3.8.2. *Let $p,q \geqslant 1$ and such that $1/p+1/q>1$. Let $\mathbf{x} \in C^{p-var}([0,T],G^{[p]}(\mathbb{R}^d))$ be a weak geometric p-rough path and $\mathbf{h} \in C^{q-var}([0,T],G^{[p]}(\mathbb{R}^{d'}))$ be a weakly geometric q-rough path. We say that $y \in C([0,T],\mathbb{R}^e)$ is a a solution to the rough differential equation* *with drift driven by* (x,h) *along the collection of* $\mathbb{R}^e$ *vector fields* $((V_i)_{1\leqslant i\leqslant d},(W_j)_{1\leqslant j\leqslant d'})$ *and started at* y_0 *if there exists a sequence* $(x^n,h^n)\subset C^{1-var}([0,T],\mathbb{R}^d)\times C^{1-var}([0,T],\mathbb{R}^{d'})$ *,such that*

- $\sup_n ||S_{[p]}(x_n)||_{p-var;[0,T]}+||S_{[p]}(h_n)||_{q-var;[0,T]}\leqslant\infty$.
- $\lim_{n\to+\infty} d_{0;[0,T]}(S_{[p]}(x_n),\mathbf{x})=0$, *and* $\lim_{n\to+\infty} d_{0;[0,T]}(S_{[p]}(h_n),\mathbf{h})=0$,

and ODE solutions $y_n=\pi_{(V)}(0,y_0;(x_n,h_n))$ *such that*

$$y_n\to y \text{ uniformly on [0,1] as } n\to\infty$$

When this happens, we write the formal equation:

$$dY=V(y)d\mathbf{x}+W(y)d\mathbf{h}\,,y_0\in\mathbb{R}^q\,,$$

which is called rough differential equation with drift.

A very important set of theorems regarding RDE's with drift are the ones that give existence, uniquness of solutions and continuity estimates in terms of the regularity of the vector fields involved in the RDE with drift. The theorems follow the spirit of the *Universal Limit Theorem* from [28].

The first result, is giving existence of solution along with some continuity estimates. However, this result does not guarantee uniqueness of solution.

Lemma 3.8.3 (Lemma 12.3 of [16])**.** *Let* $V=(V_i)_{1\leqslant i\leqslant d}$ *a collection of vector fields in* $Lip^{\gamma-1}(\mathbb{R}^d)$ *with* $\gamma\geqslant 1$, *and* $W=(W_j)_{1\leqslant i\leqslant d'}$ *a collection of vector fields in* $Lip^{\beta-1}(\mathbb{R}^d)$ *with* $\beta>1$. *Let* $x,\tilde{x}$ *be two paths in* $C^{1-var}([s,u],\mathbb{R}^d)$ *such that* $S_{[\gamma]}(x)_{s,u}=S_{[\gamma]}(\tilde{x})_{s,u}$, *and* $h,\tilde{h}$ *be two paths in* $C^{1-var}([s,u],\mathbb{R}^d)$ *such that* $S_{[\beta]}(h)_{s,u}=S_{[\beta]}(\tilde{h})_{s,u}$. *We then have that the two corresponding RDE's with drift have solutions and moreover we have the following estimate for some* $C=C(\gamma,\beta)$,

$$|\pi_{V,W}(s,y_s;(x,h))_{s,u}-\pi_{V,W}(s,y_s;(\tilde{x},\tilde{h}))_{s,u}|\leqslant C(l_h^\beta+l_xl_h^{\gamma-1}+l_xl_h+l_x^{\beta-1}l_h+l_x^\gamma)exp(C(l_x+l_h))\,,$$

where l_x *and* l_h *are bounds for* $|V|_{Lip^{\gamma-1}}\max\{\int_s^u|dx|,\int_s^u|d\tilde{x}|\}$ *and* $|W|_{Lip^{\beta-1}}\max\{\int_s^u|dh|,\int_s^u|d\tilde{h}|\}$, *respectively.*

In order to provide existence and uniqueness of solution, the collection of vector fields defining the RDE with drift should be in Lip_γ respectively Lip_β. See Theorem 12.10 and Theorem 12.11 in [16]. So, the fundamental difference between the two results is in the regularity of the vector fields that give the transition between existence and uniqueness and guaranteed existence only. In the classical theory of ordinary differential equations this results are in the spirit of Cauchy-Peano Theorem and Picard-Lindelöf Theorem, i.e. in the transition between just continuous vector field versus Lipschitz vector field in the space variable.

Chapter 4

Phase transition at $\kappa = 4$ in terms of uniqueness/non-uniqueness of solutions of the backward Loewner differential equation started from the singularity: Excursion Theory of the real Bessel process and its impact on the behaviour of the SLE traces

4.1 Preliminaries and main result

We recall from the Introduction, the forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa} B_t}\,, \qquad g(0,z) = z, z \in \mathbb{H}\,. \tag{4.1.1}$$

The functional inverses of this maps satisfy the partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z) \frac{2}{z - \sqrt{\kappa} B_t}\,, \quad f(0,z) = z, z \in \mathbb{H}\,. \tag{4.1.2}$$

In this Chapter, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa} B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H} \tag{4.1.3}$$

In order to perform the analysis, let us consider $Z_t(z) := \frac{h_t(z\sqrt{\kappa})-\sqrt{\kappa}B_t}{\sqrt{\kappa}}$ with $z \in \bar{\mathbb{H}}$, where $h_t(z)$ are the continuous extensions to the real line of the conformal maps solving (4.1.3).

Thus, when restricting the analysis to the real line, we obtain the following SDE

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t$$
$$Z_0 = x_0 \in \mathbb{R}. \tag{4.1.4}$$

This SDE is the same as the ones governing real Bessel processes started from $x_0 \in \mathbb{R}$,

$$dX_t = \frac{d-1}{2}\frac{1}{X_t}dt + dB_t,$$

with $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leqslant 0$ for $\kappa \leqslant 4$ and $d > 0$ for $\kappa > 4$. We show throughout this Chapter that for for $\kappa \leqslant 4$, there is no solution of the SDE (4.1.4) starting from the origin for almost every Brownian path, since $\{0\}$ is absorbing boundary point a.s.. For $\kappa > 4$, we show that there exists more than one strong solution for the SDE (4.1.4) that is instantaneous reflecting at zero. In this way we give meaning (abstractly) to a notion of solution for the Loewner equation started from the origin (along the real line). In [3] there is a proof that there exists a unique strong non-negative solution to (4.1.4) that starts at zero and that spends zero Lebesgue measure time there. We use this solution to construct another strong solution that is not positive and has the same characteristics (spends null Lebesgue measure time at zero).

We introduce the following notions needed to state our main result.

Definition 4.1.1. *An accessible boundary point η of a domain $D \in \mathbb{C}$ is an equivalence class of continuous curves $\gamma : [0,1] \to \bar{D}$ which join a given point $\eta \in \partial D$ with an arbitrary interior point. We assume that γ lies completely inside D with the exception $\gamma(1) = \eta$. Two curves are equivalent if for an arbitrary neighbourhood V of η, parts of the curves that are inside of $D \cap V$ could be joined by a continuous curve.*

Definition 4.1.2. *A function $\phi(x) : \mathbb{R} \to \mathbb{R}$ is a sub-power function, if for any $\nu > 0$,*

$$\lim_{x\to\infty} x^{-\nu}\phi(x) = 0\,.$$

Definition 4.1.3. *For $m > 0$, an m-macroscopic excursion is an excursion of the Bessel process of length at least $m > 0$.*

Next, we introduce the notion of macroscopic hull.

Definition 4.1.4. *We call macroscopic hull, a bounded compact set in $\bar{\mathbb{H}}$, with a simply connected complement, such that its intersection with the real axis is an interval $I \subset \mathbb{R}$ with strictly positive Lebesgue measure. We call the intersection of macroscopic hull with the real axis the base of the hull.*

We also introduce the following definition.

Definition 4.1.5. *Let $m > 0$. For the (backward) SLE trace, we call an m-macroscopic double point, a double point that corresponds to self-touching of the trace after time at least $m > 0$.*

The origin in $\mathbb{H}$ is a singularity for the backward Loewner differential equation, since the vector field $\frac{-2/\kappa}{Z_0}$ has infinite modulus. For almost every Brownian path, a notion of solution for $t \in [0, T]$ for this differential equation is defined by the limit $Z_t(0) := \lim_{y \to 0+} Z_t(iy)$. The existence of the limit and the continuity in time, is a consequence of the Rohde-Schramm Theorem. We give a detailed explanation for this fact, in the following section of the paper. We are now ready to state our main result.

Theorem 4.1.6. *For $\kappa \in [0, 4]$, for any $t \in [0, T]$, there is a unique solutions of the backward Loewner differential equation started from the origin, a.s.. For $\kappa > 4$, there are at least two solutions for the backward Loewner differential equation started from the origin, a.s..*
Let $m > 0$ be a positive real number. For $\kappa \in (4, \infty)$, on m-macroscopic excursions from the origin of the squared Bessel process obtained from extensions of the backward SLE_κ maps on the real line, we obtain macroscopic hulls with base depending on $m > 0$ and $m-$ macroscopic double points of the backward SLE_κ trace.

Remark 4.1.7. For $\kappa \in [0, 4]$, we show how we can deduce a.s. structural properties about the SLE traces using only the information on the boundary about Bessel processes,. We also show how the second technique of proof can be used in the case of the forward flow in the same regime in order to characterize the two points on the real line where the root of the SLE trace is mapped, for almost every Brownian path.

In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behaviour of the SLE trace? The previous analysis suggests that the classes of points along a Brownian driver that are responsible for the non-simpleness of the SLE_κ trace are the ones responsible for the reflective behaviour of the origin for the backward Bessel process. Compared with the case when the parameter $\kappa \in [0,4]$, where the solution along the positive real line is stuck at the origin, in the case $\kappa \in (4,8)$ there is a unique notion of strong solution along the real line . Equivalently, when the diffusivity parameter $\kappa \in [0,4]$ is not large enough to allow the solution of the backward Bessel SDE to escape the origin, there is no solution along the real line.

In a nutshell, in our analysis, we extend the conformal maps to the boundary and obtain that the dynamics of boundary points satisfy the SDE (4.1.4). We use known results about the solutions of the SDE (4.1.4) along with theorems about the backward Loewner evolution in order to study the start of the backward Loewner evolution from the origin. For this, we use the result from [3], in which there is a proof of the existence and uniqueness of strong positive solutions to the Bessel SDE started from the origin for $0 < d < 1$. In our case, this gives the existence and uniqueness of a strong non-negative solution for the SDE (4.1.4) on the real line for $\kappa > 4$, since $d(\kappa) = 1 - \frac{4}{\kappa}$.

The details of this analysis are performed in the last section of the paper.

Remark 4.1.8. For dimensions $d \in (0,1]$ ($\kappa > 4$) there exists a construction of the solutions to the SDE using Excursion Theory. Using this and the previous analysis, we obtain that beginning of m-macroscopic excursions for the SDE (4.1.4) give macroscopic hulls for the backward Loewner differential equation started from the origin. Thus, another structural information that we obtain about the dynamics of the backward SLE hulls, and implicitly backward SLE traces is that the m-macroscopic excursions of the Bessel process started from the origin, create in this context, macroscopic hulls of the backward SLE. In particular, when starting the backward Loewner differential equation from the beginning of m-macroscopic excursions of the Bessel processes, we obtain a macroscopic hull, a.s. For further details about the Excursion Theory construction of the Bessel processes for $d \in (0,1]$, see [6].

Backward Loewner differential equation and the Backward Loewner trace. In the following, we show how using the a.s. existence of the forward chordal SLE_κ, we obtain the a.s. existence of the backward Loewner trace. For $\kappa > 0$, the backward chordal SLE_κ is defined by solving the backward Loewner differential equation with the driver $\sqrt{\kappa}B_t$, $0 \leqslant t < \infty$. For any $t_0 > 0$ $(\sqrt{\kappa}B_{t_0-t} - \sqrt{\kappa}B_{t_0}, 0 \leqslant t \leqslant t_0)$ has the same distribution as $(\sqrt{\kappa}B_t, 0 \leqslant t \leqslant t_0)$. Using the result from [33], we obtain that every $t_0 \in [0,T)$, $\sqrt{\kappa}B_{(t_0-t)}$, $0 \leqslant t \leqslant t_0$ generates a forward Loewner trace which we denote by $\beta_{t_0}(t_0 - t), 0 \leqslant t \leqslant t_0$.

Then, using the identity in distribution for the Brownian motion from above, we obtain that $\sqrt{\kappa}B_t$ generates the backward SLE traces β_{t_0} for $0 \leqslant t_0 \leqslant T$.

Note that β_t is a continuous function defined on $[0, t]$. The parametrizations of the backward traces β_t is different from the usual parametrisation of the chordal SLE trace in the sense that the backward traces β_t is a continous function defined on $[0, t]$ such that β_0 is the tip and β_t is the root that is an element of $\mathbb{R}$. The difference with the chordal SLE is that the chordal trace is parametrized such that the root of it is β_0. For further details about this construction, we refer to [34].

We also use in our analysis that for fixed time $T > 0$, the law of the curves $\beta_T = \beta(0, T]$ generated by the backward Loewner differential equation is the same as the forward chordal SLE trace $\gamma(0, T]$ (modulo a translation with the driver $\sqrt{\kappa}B_T$). Thus, we use a.s. results about the SLE traces from the forward chordal setting, by just shifting with $\sqrt{\kappa}B_T$ the backward trace. It is known that the forward SLE traces are simple for $\kappa \in [0, 4]$, a.s. (see [20]). We borrow this result also for the backward Loewner traces.

4.2 Heuristics of the proof of the main result

- We consider a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ equipped with the Wiener measure. We consider the backward Loewner Differential Equation driven by a standard Brownian motion that lives on $(\Omega, \mathcal{F}_t, \mathbb{P})$. We consider also the extension of the conformal maps $h_t(z)$ to the real line.

- On the real line, when extending continously the maps solving the backwar Lowener differential equation with $\sqrt{\kappa}B_t$ driver, we obtain a real valued SDE (4.1.4). This real valued SDE, has the origin as a boundary point. Moreover, the behaviour of this real-valued SDE at this boundary point dependes on the dimension $d = d(\kappa)$. For dimension $d = 0$ and $d < 0$, the origin is an absorbing boundary and for $d > 0$, zero is a reflecting boundary. For the backward flow we have $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leqslant 0$ for $\kappa \leqslant 4$ and $d > 0$ for $\kappa > 4$, i.e. zero has different behaviour as a boundary point for the SDE (4.1.4). For $\kappa > 4$, there is a way to give meaning to a notion of (strong) solution to the SDE (4.1.4), started from $\{0\}$ for almost every Brownian path. This idea appeared in [3], where it is proved that there exists a strong non-negative solution for this process, that spends null Lebesgue measure time at the origin, for almost every Brownian path. In contrast, for $d = 0$ ($\kappa = 4$) and $d < 0$ (i.e. $\kappa < 4$) there is no notion of solution that exits the origin along the real line a.s., so if there is a notion of solution started from the origin, the only possibility is that the dynamics of the origin under this solution, should happen only in $\mathbb{H}$.

- First, we show how we can define a notion a solution for the backward Loewner differential equation started from the origin, when driven by $\sqrt{\kappa}B_t$, for all κ and for almost every Brownian path.

  We first show that the behavior of the real-valued SDE (4.1.4) at the origin is absorbing for $d = d(\kappa) \leqslant 0$.

  Using this, we argue that for $\kappa \in [0, 4]$, there is a unique solution a.s.. The argument is the developed in the following section.

- For $d > 0$, i.e. $\kappa > 4$, the picture differs. In this case, we prove that there are at least two solutions with the same starting point and with the same Brownian path as a driver. We use the existence of the strong solution result for the SDE (4.1.4) of dimension $d > 0$ proved in [3], to construct a new (negative) strong solution (with the same starting point and the same Brownian path as a driver). Thus, we conclude that in this setting, the origin gets mapped to at least two different points for $t \in [0, T]$, a.s.. The result in [3], gives also the uniqueness of the strong solution on the positive semi-axis.

  Extending the conformal maps $(h_t(\sqrt{(z)} - \sqrt{\kappa}B_t)^2$ to the real line and considering the SLE trace $\gamma_2(t) = (g_t^{-1}(\sqrt{\kappa}B_t))^2$, we obtain that the left-most and the right-most points of the closure of backward SLE hull intersected with the real line, solve the squared Bessel SDE for almost every Brownian path.

  This perspective helps us to recover the phase transition for the backward SLE_κ traces from simple for $\kappa \in [0, 4]$ to non-simple curves for $\kappa \in (4, 8)$ using the phase transition in terms of the behavior of the squared Bessel SDE at $\{0\}$, seen as boundary point for this SDE. This analysis, compared with the typical phase transition in terms of the behavior of the forward Bessel process, gives additional information about the structure of the backward SLE_κ traces, by making use of the fact that there are at least two strong solutions a.s. when starting from the origin (to be compared with the forward flow case, where the proof uses only information on the positive part of the real line).

4.3 Proof of the main result

In order to argue the existence of solutions for the backward Loewner differential equation started from the origin when driven by $\sqrt{\kappa}B_t$, we use the fact that the SLE_κ hulls K_t are locally connected for all $t \in [0,T]$, a.s.. We also use the identity in law between the backward hulls and the SLE_κ hulls. The fact that these families of hulls have the same law (modulo a shift with $\sqrt{\kappa}B_T$) is based on the same ideas as the analysis presented in the subsection 2.1. For any κ, the backward chordal SLE_κ is defined by solving the backward Loewner differential equation with the driver $\sqrt{\kappa}B_t$, $0 \leqslant t < \infty$. For any $T > 0$ $(\sqrt{\kappa}B_{T-t} - \sqrt{\kappa}B_T, 0 \leqslant t \leqslant T)$ has the same distribution as $(\sqrt{\kappa}B_t,\ 0 \leqslant t \leqslant T)$. Thus, using that the forward SLE_κ hulls are locally connected for any κ a.s., we obtain that the backward SLE hulls are also locally connected for any value of κ, a.s..
Thus, using that the backward SLE_κ hulls are locally connected for almost every Brownian driver, along with the fact that a.s. every boundary point of $\mathbb{H}$ is an accessible point, we have that $\lim_{z\to 0} h_t(z)$ exists along any curve and is independent of the curve.

Part 1: Proof of the uniqueness for $\kappa \leqslant 4$.

Next, we provide proof by contradiction that the non-uniqueness can not happen only in the upper half plane without including the boundary, i.e. we show that for $\kappa \leqslant 4$, the origin is mapped under the backward Loewner differential equation to only one point in $\mathbb{H}$, for any fixed time, a.s.. For this, we use the following theorem from [32].

Theorem 4.3.1 (Theorem 51.2 of [32])**.** *Let X_t be a real valued diffusion in the natural scale on $I = [0, +\infty)$ with speed measure m. Then the boundary $\{0\}$ must be absorbing if $\int_I m(dx) = \infty$.*

For the SDE (4.1.4), using the formula for the scale function in terms of the coefficients of the SDE, we obtain that

$$S(\zeta) = \int_0^\zeta \eta^{1-d} d\eta.$$

Thus, $S(\zeta) = \frac{1}{2-d}\zeta^{2-d}$, when $d \neq 2$ and $S(\zeta) = \log(\zeta)$ for $d = 2$. Furthermore, when computing the speed measure, from the scale function, we obtain that its value is given by $m(\eta) = \eta^{d-1}$. Thus, $\int_I m(dx) = +\infty$ for $d < 0$ (i.e. $\kappa < 4$) For $d = 0$, we refer to Proposition 1.5 in Chapter IX of [31] that is stated in the following.
We first need the following definition.

Definition 4.3.2. *The real squared Bessel process for any $\delta \geqslant 0$ and $x \geqslant 0$ as the unique strong solution for the equation $\tilde{X}_t = x + 2\int_0^t \sqrt{\tilde{X}_s} dB_s + \delta t$.*

For this process, we have the following result

Proposition 4.3.3 (Proposition 1.5 of [31])**.** *For $\delta = 0$, the point 0 is absorbing and for $0 < \delta < 2$ the point 0 is instantaneously reflecting.*

We use this result, in order to argue that the point 0 is also absorbing for the squared Bessel process of dimension $d = 0$ (i.e. $\kappa = 4$). Indeed, if with positive probability the point $\{0\}$ would not be absorbing then would imply that with positive probability we could construct solutions for the squared Bessel SDE that are different from the identical zero solution, that is a contradiction.

Thus, since for $\kappa \in [0, 4]$, for almost every Brownian path, the boundary behavior of the origin is absorbing, we obtain the the only possible notion of solution for the backward Loewner differential equation started from the origin is the one constructed before from 'inside the domain' (seeing the origin as an accessible point for the family of conformal maps satisfying the backward Loewner differential equation).
In the next subsection, we give an application of the result about the boundary behavior of the origin for the real Bessel process, in the study of the simpleness of the SLE_κ trace using (only) information on the boundary of the domains.

Application: Almost sure simpleness of the SLE_κ trace, for $\kappa \in [0, 4]$ using just the information on the boundary. In this section, we show how the information on the dynamics of boundary for points under the backward Loewner differential equation, gives information about the structure of the SLE traces for $\kappa \in [0, 4]$ (that are defined via a limiting procedure from the interior of the domain).
Essentially, we show that the a.s. behavior of the origin for the Bessel SDE 4.1.4, gives information about the behaviour of the SLE_κ traces.

Proof by contradiction of the uniqueness for $\kappa \leqslant 4$. On the real line, the dynamics is governed by the following SDE

$$dZ_t = \frac{-2/\kappa}{Z_t} dt + dB_t \,, \quad Z_0 = x \in \mathbb{R}.$$

The forward Loewner differential equation when driven by $\sqrt{\kappa} B_t$ is generated by a trace a.s..

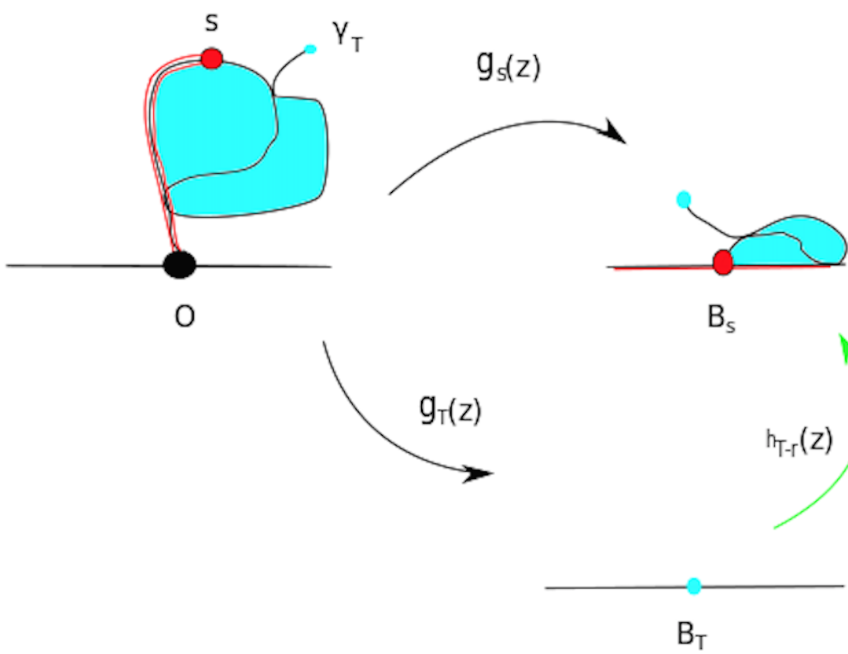

FIGURE 4.3.1. The uniqueness for $\kappa \leqslant 4$.

For the backward Loewner differential equation, we have the that on the real line, the dynamics of points is governed by the following SDE

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t\,, \quad Z_0 = x \in \mathbb{R}.$$

We obtain that with positive probability $\exists s(\omega) < T, \gamma(s) \cap \mathbb{R} \neq 0$. Thus, for the SDE

$$dZ_r = \frac{-2/\kappa}{Z_r}dr + d\tilde{B}_r\,, \quad Z_0 = x \in \mathbb{R}$$

with dimensions $d \leqslant 0$, where $\tilde{B}_r = B_T - B_{T-r}$ we have a collection with strictly positive mass of paths of Brownian motion that allow the origin to be mapped along the real line. Thus, when studying the maps $h_r(z)$ driven by $\sqrt{\kappa}B_{T-r}, r \in [0,T]$ we obtain a contradiction with the fact that $\{0\}$ is absorbing boundary (i.e. when $X_0 = 0$, we have $X_t = 0$ for all $t > 0$, a.s.) for the backward Loewner differential equation for almost every Brownian path.

□

Remark 4.3.4. An alternative approach to define the solution is to use the a.s. existence of the backward SLE trace as proved in [34]. Using this result, we can give meaning to a notion of solution for all times $t > 0$, given by the image as we run time $t \in [0, T]$ of the tip of the backward SLE trace, for almost every Brownian path.

Part 2: Proof of the non-uniqueness when $\kappa > 4$. In this section, we prove that in the case $\kappa > 4$, there at least two types of solutions for the backward Loewner differential equation started from the origin a.s. (i.e. solutions obtained from the same starting point and the same Brownian motion path).

In order to show this we consider again the maps $h_t(z)$ solving the backward Loewner differential equation driven by the Brownian driver $\sqrt{\kappa}B_t$ for $t \in [0, T]$. We extend continuously the conformal mappings to the real line.

In this section we show that we can give meaning to a notion of solution to SDE (4.1.4) that spends null Lebesgue measure time at the origin. Moreover, in the case $d > 0$ (i.e. $\kappa > 4$) we use this strong solution to construct a different solution, that gives the non-uniqueness in this regime.

The main result that we use is proved in [3] and is phrased as follows:

Let $L_a^Z(t)$ be the local time of the process Z_t at the point $a \in \mathbb{R}_+$. Let us introduce the principal value correction for $d \in (0, 1]$, i.e.

$$Z_t = Z_0 + B(t) + \frac{d-1}{2} k(t)\,, Z_0 \geqslant 0,$$

where

$$k(t) = P.V. \int_0^t \frac{1}{Z_s} ds := \int_0^\infty a^{d-2} (L_a^Z(t) - L_0^Z(t)) da.$$

In [3] there is a proof for the existence of a weak solution spending zero time at the origin implies the existence and uniqueness of a non-negative strong solution spending zero time at the origin for the above equation. Note that this analysis allows us to view the strong solutions on the real line starting from the origin as functions of the Brownian motion paths.

In this setting, we are ready to prove the second part of the main result.

Proof of the non-uniqueness for $\kappa > 4$.

In our setting, the dynamics on the real line is obtained by the continuous extension of the conformal maps $h_t(z)$, to the whole real line. Thus, in this setting, we consider the following version of the previous equation (since the dynamics is defined on both the positive part of the real line and on the negative part of the real line) We consider the following

$$\tilde{k}(t) = P.V. \int_0^t \frac{dt}{Z_t} = \int_{-\infty}^{\infty} a^{d-2}(L_a^Z(t) - L_0^Z(t))da \tag{4.3.1}$$

Furthermore, we use the result from [3] in order to obtain a unique positive strong solution for the 'extended' SDE

$$Z_t = Z_0 + B(t) + \frac{d-1}{2}\tilde{k}(t)\,, Z_0 \geqslant 0, \tag{4.3.2}$$

Note that this is a strong solution and thus can be thought as a measurable function $\phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ such that $\rho(t) = \phi(B_t(\omega))$. Note that this solution is unique, non-negative and starts from the origin, according to the result from [3].
The next step is to consider $\tilde{B}_t(\omega) = -B_t(\omega)$, and as before $Z_0 = 0$. We consider

$$\tilde{\rho}(t) := -\phi(-B_t(\omega)). \tag{4.3.3}$$

We obtain that

$$-\phi(-B_t(\omega)) = \frac{d-1}{2}\tilde{k}(t) + B_t(\omega). \tag{4.3.4}$$

Thus, $\tilde{\rho}(t)$ is another solution to the equation (4.3.2), with the same starting point and the same Brownian motion path. Note that the two solutions are different since one is positive and the other one is negative. A similar idea is used in [10] to construct two different solutions for the Bessel SDE for the parameter d taking bigger values than in our case. Thus, for all times $t \in [0, T]$, for $\kappa > 4$, we have at least two different real solutions with the same starting point and with the same Brownian driver.
In particular, at each fixed time $t > 0$, for $\kappa > 4$ we obtain for the backward Loewner differential equation started from the origin at least two solutions, a.s.. $\square$

Remark 4.3.5. The previous analysis can be also used for identifying the points on the real line, where the root $\gamma(0)$ of the SLE_κ trace is mapped under the forward flow (i.e. where the origin is mapped under forward Bessel process) for $\kappa \in [0, 4]$ as the two strong solutions of the transient forward Bessel started from the origin, i.e. by the uniqueness of the strong solution for the Bessel process started from the origin, we have that the solutions $g_t(0)$ on $\mathbb{R}+$ and on $\mathbb{R}-$ coincide with the points of where the root of the SLE trace is moved on the the boundary. In particular, we find that we can find the position of one from the position of the other one as above using that $g_t^l(0)(B_t(\omega) = -g_t^r(0)(-B_t(\omega))$.

4.4 Applications to the study of the (backward) SLE_κ traces: Excursion Theory of the Bessel processes and behaviour of the SLE traces

We first introduce basic notions about Excursion Theory for Markov processes, following [7].

Excursion theory for Markov processes with applications to the Squared Bessel process. In this subsection, we follow the approach in [7], that we refer to for more details. We define the terminology for a general real-valued Markov process M_t, keeping in mind that in our setting, we apply the techniques for the solutions to the squared Bessel SDE. An open interval $(g, d) \subset \mathbb{R}$ is called an excursion interval for a stochastic process M_t if $M_t \neq 0$, for all $t \in (g, d)$.

We fix a real number $c > 0$. If we are dealing with a process that is not identically zero, then with positive probability there exists at least one excursion interval with length $l > c$. We introduce as in [7], the right continuous function $\bar{\Pi} : (0, \infty] \to (0, \infty)$ which is meant to describe the distribution of the lengths of excursion intervals. The function is defined, for every $a \in (0, \infty]$ as,

$$\bar{\Pi}(a) = \begin{cases} \frac{1}{\mathbb{P}(l_1(a)>c)} & a \leqslant c \\ \mathbb{P}(l_1(c) > a) & a > c \end{cases}$$

Following section 4 of Chapter IV in [7], we are concerned with $(M_{g+t}, 0 \leqslant t \leqslant d - g)$ corresponding to each excursion interval (g, d).

For every $a > 0$, denote by

$$\mathcal{E}^{(a)} = \{\omega \in \Omega : \tau > a \text{ and } \omega(t) \neq 0, \forall 0 < t < \tau\}.$$

Let us consider $\mathcal{E} = \cup_{a>0} \mathcal{E}^{(a)}$. It is proved in [6], that there exists a unique measure n on $\mathcal{E} = \cup_{a>0} \mathcal{E}^{(a)}$ that we call the excursion measure, such that

$$n(\lambda) = \bar{\Pi}(a) n(\lambda | \tau > a)$$

for every $\lambda \subset \mathcal{E}^{(a)}$.

We introduce the excursion process e_t on $\mathcal{E} \cup \{\eta\}$, where $\{\eta\}$ is an additional isolated point, associated with the stochastic process M_t

$$e_t = (M_{s+L^{-1}(t-)}, 0 \leqslant s < L^{-1}(t) - L^{-1}(t-)) \text{ if } L^{-1}(t) - L^{-1}(t-) > 0$$

and $e_t = \eta$ otherwise, where $L^{-1}(t)$ is the inverse of the local time at zero of the process M_t up to time $t > 0$ defined by $L^{-1}(t) = \inf\{s \geqslant 0 : L(s) > t\}$ and $L^{-1}(t-)$ is defined by $L^{-1}(t) = \inf\{s \geqslant 0 : L(s) \geqslant t\}$. We have the following result that is due to Itô (1970)

Theorem 4.4.1 (Theorem 10, Chapter IV of [6])**.** *If* 0 *is recurrent then,* $(e_t, t \geqslant 0)$ *is a Poisson Point Process with characteristic measure* n*.*

Using that for $0 < d < 2$, for the squared Bessel process the origin is recurrent, we obtain from the above result the fact that there exists an excursion decomposition for this process.
We will use this in the following subsections to obtain results about the backward SLE trace for $\kappa \in (4, 8)$.

SLE traces and the squared Bessel processes. In this subsection, we discuss as applications of the analysis developed so far, the a.s. non-simpleness of the SLE traces for $\kappa \in (4, 8)$. We consider, as in [22], the definition of the SLE_κ trace using the mappings $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$, where $h_t(z) : \mathbb{H} \to \mathbb{H} \setminus K_t$ are solving the backward Loewner differential equation (4.1.3).

The family of maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ are conformal maps from $\mathbb{C} \setminus [0, \infty)$ to $\mathbb{C} \setminus [0, \infty)$. Using the fact that the hulls of the backward SLE are locally connected, we obtain that the maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ can be extended continuously to the real line.

When studying the extensions of the conformal maps satisfying the backward Loewner differential equation (4.1.3), we obtain on the real line two SDEs (one corresponding to the positive part of the real line, and one for the negative part), that are driven by the same Brownian motion. When multiplying the SDE governing the dynamics on the negative part of the real line with -1, we obtain a SDE on the positive part of the real line driven by $-B_t$. Thus, in order to consider the solutions on both the positive and negative part of the real line for the maps satisfying (4.1.3), when considering the extension of the maps$(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ to the real line, we need to consider two SDEs driven by $B_t(\omega)$ and $-B_t(\omega)$ respectively. The square roots of the unique strong solutions to these SDEs give the two solutions of the backward Loewner differential equation started from the origin that we have considered in the previous section.
These SDEs are the one satisfied by Squared Bessel processes. Following [31], by using the

Yamada-Watanabe Theorem, we obtain that the Squared Bessel process of any dimension, starting from $x \geqslant 0$ has a strong unique solutions a.s..
Thus, in our case, we have the following SDEs

$$d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt + \sqrt{\tilde{Z}_t} dB_t \tag{4.4.1}$$

and

$$d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt - \sqrt{\tilde{Z}_t} dB_t. \tag{4.4.2}$$

We couple these processes with the same Brownian driver $B_t(\omega)$, in the sense that we drive (4.4.1) with $B_t(\omega)$ and (4.4.2) with $-B_t(\omega)$. First, we restrict our attention to the solution of (4.4.1) . The analysis for the lower squared Bessel process will be performed in the same manner.

As in [22], we denote the SLE_κ trace in this new setting by $\gamma_2(t) := (g_t^{-1}(\sqrt{\kappa}B_t))^2$.

For $d = 1 - \frac{4}{\kappa} > 0$, there is a unique positive strong solution for this process starting from the origin such that $X_t \geqslant 0$, a.s. Also, for $2 > d = 1 - \frac{4}{\kappa} > 0$ the squared Bessel process is recurrent. Thus, we can apply elements of Excursion Theory developed in the previous section.

m-Macroscopic excursions of the squared real Bessel process and application to the study of the backward SLE traces. -to be completed-

Fix $m > 0$. In this section, we analyze the backward Loewner flow at times between a m-macroscopic excursion from the origin of the (upper) squared Bessel process on $\mathbb{R}$. The same analysis holds for the (lower) squared Bessel process. We can obtain the stochastic process corresponding to the unique strong solution of the squared Bessel process using the Excursion Theory of Markov processes described in the previous section, by gluing together the excursions of this process out of the origin. Let us consider the collection of m-macroscopic excursions of the solution to the real (squared) Bessel SDE. Let us consider the backward Loewner differential equation from the origin, for time $[0, +\infty)$. Among these times, there exist set of times which correspond to beginning of m-macroscopic excursions of the Squared Bessel process from the origin. Using that the SLE_κ hulls are locally connected a.s. and the identity in distribution between the shifted forward SLE_κ traces and the backward discussed in the introduction, we obtain that the conformal maps satisfying the backward Loewner differential equation can be continuosly extended to the boundary, for all times and for almost every Brownian path.
Let us consider time running in $t \in [0, +\infty)$,

We further consider $(h_t(\sqrt{z}) - \sqrt{\kappa}\hat{B}_t)^2$. When extending these maps to the real line, we obtain a squared Bessel process. We can do this for any time $t \in [0, +\infty)$. Among these times, there will be times $r = r(\omega)$ (that depend on the path of the Brownian motion) that will correspond to beginning of m-macroscopic excursions of the real squared Bessel process, started from the origin. Let us consider $\tilde{h}_r(z) := h_{s+r} \circ h_r^{-1}(z)$, for $s \geqslant 0$. In particular, for times between $r(\omega)$ and $r(\omega) + m$ which correspond to the duration of a m-macroscopic excursions, we obtain that the extended conformal maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$ for times in $[r(\omega), r(\omega) + m)$, map the origin to the real line. In order to see this, let us assume that the image of the origin under the maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$ is mapped in $\mathbb{H}$. Then, for any $\tilde{m} > 0$ with $\tilde{m} < m$, we would have two images of the origin, for the extended conformal maps, one in $\mathbb{H}$ and the unique strong solution of the squared Bessel SDE at time $r(\omega) + \tilde{m}$. Using Rohde-Schramm Theorem (see [33]) we know that the maps $g_t^{-1}(z)$ can be extended for all $t \in [0, \infty)$, for almost every Brownian path (since the SLE_κ hulls K_t are locally connected, for all $t \in [0, T]$ for almost every Brownian path). Using that the backward SLE_κ hulls agree in distribution with the forward SLE_κ hulls (module a shift with Brownian motion at fixed time), we obtain that the backward SLE_κ hulls are locally connected as well. Thus, for all times $t \in [0, \infty)$, for almost every Brownian path, the maps $h_t(z)$ (and their functional inverses) are continuously extended to the boundary. This gives that the maps $\tilde{h}_r(z)$ are continuously extended to the boundary (since they are obtained from compositions of maps that are continuously extended to the boundary). Hence, since the maps $\tilde{h}_r(z)$ should map the origin to a unique point, we obtain a contradiction. Thus, the two images of the origin should coincide. Since the backward SLE_κ trace grows from the origin, under the backward Loewner flow, we obtain that when starting the backward Loewner flow from the set of m-macroscopic excursions of the underlying Bessel, we obtain points of intersection of the trace with the real line. We can redo the argument for a full measure set of Brownian paths, by identifying the set of times on which m-macroscopic excursions of the squared Bessel process obtained by continuously extending these maps to the real line. Thus, we obtain that the closure of backward SLE_κ hulls are macroscopic hulls whenever we have m-macroscopic excursions of the squared Bessel process. Moreover, since the squared Bessel process is recurrent and the backward SLE trace starts from the origin, we obtain also double points (that correspond to self touchings of the curve after at least $m > 0$) time.

Moreover, from topological considerations, we obtain at the end of a m-macroscopic excursion of the unique strong solution of the squared Bessel process, a closed 'large bubble' of the backward SLE trace formed from the outer boundary of a portion of the curve, where $t(\omega)$ is the beginning of a large excursion that collects the self intersections of the backward SLE trace with itself and with real line, that happened on $[t(\omega), t(\omega) + m)$. In contrast with the forward flow approach, this approach not only recovers the phase transition in the behaviour of the SLE traces but also it provides structural information about the backward SLE traces.

The same analysis holds also for the solution to the SDE (4.4.2). When driving them simultaneously with $B_t(\omega)$ and $-B_t(\omega)$ we obtain the complete picture of the evolution of the backward Loewner hulls in $\mathbb{C} \setminus [0, \infty)$.

Brownian motions and the behaviour of the SLE_κ trace for $\kappa \in (4, \infty)$. In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behaviour of the backward SLE trace? The previous analysis suggests that the classes of points along a Brownian driver that are responsible for the non-simpleness of the backward SLE_κ trace are the ones responsible for the reflective behaviour of the origin for the backward Bessel process. Compared with the case when the parameter $\kappa \in [0, 4]$, where the solution along the positive real line is stuck at the origin, in the case $\kappa \in (4, \infty)$ there is a unique notion of strong solution along the real line. Equivalently, when the diffusivity parameter $\kappa \in [0, 4]$ is not large enough to allow the solution of the backward Bessel SDE to escape the origin, there is no solution along the real line.

In general, the change in behavior of the backward Bessel process started from the origin, gives also the following additional information: the collection of points that represent the beginning of macroscopic excursions of the real Bessel process of dimensions $d \in (0, 1)$ give also the formation of macroscopic bubbles for the backward SLE hulls, since in our case, the real Bessel process that we consider comes from the extensions of the conformal maps to the real line.

Chapter 5

Sequential continuity in the parameter κ for the welding homeomorphisms induced on the real line by the backward SLE for $\kappa \in [0, 4]$; SLE traces and Quasi-Sure Stochastic Analysis in the context of SLE: Defining the SLE traces simultaneously for all parameters κ

5.1 Preliminaries and main result

In this Chapter, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa} B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H}\,. \tag{5.1.1}$$

Given the Loewner differential equation, the conformal welding homeomorphism is a homeomorphism of intervals of the real line given by the following rule: two points x and y situated at different sides of the origin, are to be identified if they hit zero simultaneously under the backward Loewner differential equation with the same driver. The conformal welding homeomorphism induced by the backward SLE for $\kappa < 4$, was studied in [34] and [38]. First, in [34] it is proved the existence of the backward SLE trace $\beta(t)$ and that is a continuous function of $t \in [0, T]$, using the fact that the backward Loewner differential equation is driven by $\sqrt{\kappa}(B_T - B_{T-r})$ for $r \in [0, T]$ and that $\sqrt{\kappa}(B_T - B_{T-r})$ has the same distribution with $\sqrt{\kappa} B_t$. Moreover, $\beta(0, T] - \sqrt{\kappa} B_T$ has the same distribution as $\gamma(0, T]$, where $\gamma(0,T]$ is the forward SLE trace. Thus, statistical properties of the forward SLE traces can be borrowed in the analysis of the backward Loewner trace. In particular, for $\kappa \in [0,4]$, the backward SLE trace $\beta(t)$ is a simple curve for almost every Brownian driver. The existence of the backward SLE trace proved in [34], gives the existence of the conformal welding homeomorphism of intervals of $\mathbb{R}$ for the backward Loewner differential equation, for almost every Brownian driver, in the following manner: for $\kappa \in [0,4]$, under the backward Loewner flow on the real line, points from opposite sides of the singularity hit the origin and afterwards get mapped to the simple trace $\beta(t)$, for almost every Brownian path. Note that so far we have considered finite time intervals, and consequently the welding homeomorphism is defined on finite intervals of the real line. However, it is proved in [34], that the backward Loewner differential equation induces a conformal welding homeomorphism on the whole real line. In this Chapter, we take the same perspective and we study the welding homeomorphism induced by the backward Loewner differential equation on the whole real line.

We prove that the welding homeomorphisms depending on the parameters κ_i, $i \in \mathbb{N}$, induced by the backward Loewner differential equation, as (random) functions from $\mathbb{R}$ to $\mathbb{R}$ are point-wise convergent with respect to the euclidean norm, for almost every Brownian path. We interpret this as sequential continuity in the parameter κ of (random) functions from $\mathbb{R}$ to $\mathbb{R}$, for every point, a.s., in the following sense. For all $x \in \mathbb{R}$, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x, \omega)$ such that $|\phi^{\kappa_i}(x,\omega) - \phi^{\kappa_j}(x,\omega)| \leqslant \varepsilon$ whenever $|\kappa_i - \kappa_j| \leqslant \delta$, for almost every Brownian path.
We are ready to present our main result.

Theorem 5.1.1. *The welding homeomorphism on the real line induced by the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ is sequentially continuous in κ, for $\kappa \in [0,4]$, a.s., on an uncountable set of points.*

5.2 The setting needed for the proof of Theorem 5.1.1

Let us consider the probability space $(\Omega, \mathcal{F}_t = \sigma(B_s, s \in [0,t]), \mathbb{P}_B)$, where $\mathbb{P}_B$ denotes the Wiener measure. We consider the backward Loewner equation driven by $\sqrt{\kappa}B_t$ with $t \in [0,T]$ as described in the previous section. Via Rohde-Schramm Theorem, we have the a.s. existence of the trace for the forward Loewner differential equation driven by $\sqrt{\kappa}B_t$. Via the identity in law between the forward and backward drivers, we obtain the existence of the backward SLE traces $\beta[0,T]$ up to some fixed $T > 0$ for a fixed value of the parameter $\kappa \in \mathbb{R}_+$ a.s.. We extend continuously the backward Loewner maps $h_t(z)$

satisfying (7.2.3) to the real line. By performing the shift $Z_t(x) := \frac{h_t(x\sqrt{\kappa})-\sqrt{\kappa}B_t}{\sqrt{\kappa}}$, we obtain

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t.$$
$$Z_0 = x \in \mathbb{R}^*. \tag{5.2.1}$$

Note that the nullsets of the Brownian paths outside of which the trace generated by the above Loewner chain exists, depends on κ.

The Bessel processes of various dimensions that appear as solutions of the equation (5.2.1), are thought as functions of the Brownian paths since they appear naturally in our setting by the continuous extension of the conformal maps to the boundary.

Thus, in our analysis, we obtain naturally the following coupling: we drive the backward Loewner differential equation with the same Brownian motion as the solutions to the Bessel SDE (5.2.1) on the real line.

5.3 Proof of the Theorem 5.1.1

The techniques needed for the proof of the (sequential) continuity of the welding homeomorphisms in the parameter κ, for $\kappa \in [0, 4]$ are borrowed from the conformal welding framework as developed in [34], [38]. The analysis is divided in several steps. We give the heuristics of the argument along with a detailed list of techniques used in the proof of the result, in the following.

Heuristics of the argument and details on the techniques used in the proof.

- In order to argue the uniqueness of the conformal welding homeomorphsim, we need a probabilistic version of Lemma 3 from [24], that we prove in the beginning of the next section.

- An important element is the Lamperti relation (that can be found in [31] or in [18]). This gives that the first hitting time of zero by a Bessel process has a certain integral representation. In particular, this gives that $T_x^\kappa \overset{(d)}{=} \frac{x^2}{2Z_{-\mu(\kappa)}}$, where $Z_{-\mu(\kappa)}$ is a Gamma random variable with index $\mu(\kappa)$ that depends continuously on κ.

- Another useful tool in the proof along with the Lamperti relation, is the following order of the hitting times of zero for a fixed sample path of the Brownian motion driver. If we consider Bessel processes driven by the same sample path of the Brownian motion, since the drift depends monotonically on the dimension, we obtain

that for a fixed starting point one ,of the processes (the one with smaller value of the drift) will hit before the other process the origin a.s.. Thus, in the coupled picture that we consider, we will get that $T_{x_0}^{\kappa_1} \leqslant T_{x_0}^{\kappa_2}$ for $\kappa_1 \leqslant \kappa_2$ a.s.. We would like to investigate what is happening in the limit as we take κ_1 to κ_2 and fixing the starting point. For this, we consider the Laplace transforms of this hitting times, i.e. we consider $\mathbb{E}[e^{-T_{x_0}^{\kappa_1}}]$ and $\mathbb{E}[e^{-T_{x_0}^{\kappa_1}}]$.
We use the fundamental result about random variables stating that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$ are coupled, and that $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^{\kappa_\infty}$, and that $T_{x_0}^{\kappa_i} \uparrow T_{x_0}^{\kappa_\infty}$ or $T_{x_0}^{\kappa_i} \downarrow T_{x_0}^{\kappa_\infty}$, then $T_{x_0}^{\kappa_i} \to T_{x_0}^{\kappa_\infty}$ for almost every Brownian path.

Proof of the point-wise convergence result. We first introduce a probabilistic version of Lemma 3 from [24].

Lemma 5.3.1. *No two points on the same side of the singularity can give rise to solutions to the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ that will hit the position of the driver $\sqrt{\kappa}B_t$, at the same time, for almost every Brownian path. Thus, when starting the backward Loewner differential equation from any point $x_0 > 0$ on $\mathbb{R}$, we obtain only one point on the negative real axis that will hits simultaneously with $x_0 > 0$, for almost every Brownian path.*

Proof. Let us consider the starting points $0 < x_0 < y_0$. For almost all Brownian paths, the function $\delta(t) = y(t) - x(t)$ is increasing in t, since

$$\frac{d}{dt}\delta(t) = \frac{d}{dt}(y(t) - x(t)) = 2\frac{y(t) - x(t)}{(y(t) - \sqrt{\kappa}B_t)(x(t) - \sqrt{\kappa}B_t)}\,.$$

Thus, there are no two points at the same side of the singularity that can hit the origin in the same time, a.s..
Equivalently, there can not be more than one solution that hits the origin at a certain time, for almost all Brownian paths.

□

Proof of the main result. Let us fix a point $x > 0$ to the left of the singularity 0 and let us fix a parameter $\kappa_1 \in [0, 4)$. Let us start the reverse backward Loewner differential equation from x_0 and consider it until the first hitting time of the origin. We know that the dynamics of the reverse Loewner equation restricted to the real line is given by the Bessel SDE for dimensions $d(\kappa) = 1 - \frac{4}{\kappa}$. We recall that we consider $\kappa \in [0, 4]$ that corresponds $d(\kappa) \leqslant 0$. We consider a family of Bessel processes with indices $\kappa_i \in [0, 4]$, $i \in \mathbb{N}$, that are coupled with the same Brownian paths.

Let us consider a fixed value $4 \geqslant \kappa > 0$. Let us consider a sequence of parameters $\kappa_i \to \kappa$ as $i \to \infty$. For this sequence, we will make sense of the following Lemma about hitting times of Bessel processes in the coupling considered. The next Lemma is a result about the ordering of times of the Bessel process for the same starting point, same drivers and sequence of different parameters. The argument uses that the time of hitting zero is finite a.s. for dimension $d = 0$ (i.e. $\kappa = 4$) in case of the real Bessel processes. Moreover, since the drift for smaller values is a smaller negative number, then if the Bessel process of dimension $d = 0$ hits the origin in finite time, the process of lower dimension should hit in a shorter time due to the ordering of the drifts (for more details, see [18]). Thus, the hitting time of zero is finite, for $\kappa \in [0, 4]$ a.s.. In the following Lemma, we use the indexed notation $Z_t^{\kappa_i}(x_0)$ for the solution of the SDE (5.2.1) on the real line in order to keep track of the sequence of parameters κ_i, $i \in \mathbb{N}$, in the analysis. We know from [31], Chapter XI, that for the SDE (5.2.1), for $\kappa \in \mathbb{R}_+$, for almost every Brownian path there exists a unique strong solution, until the first hitting time of zero. Note that in the next Lemma, we consider a countable sequence of parameters and thus we do not modify the null-set of Brownian paths.

Lemma 5.3.2. *Let us consider $Z_t^{\kappa_i}(x_0)$ a collection of Bessel processes started from fixed $x_0 > 0$ coupled, in the sense that these processes are driven by the same Brownian paths. Let $(\kappa_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of values of $\kappa \in [0, 4]$ and let $a(\kappa_i) = \frac{-2}{\kappa_i}$, for $i \in \mathbb{N}$. Then for all starting points $x_0 > 0$, for almost every Brownian path, $T_{x_0}^{\kappa_i} \leqslant T_{x_0}^{\kappa_j}$ for $i \leqslant j$.*

Proof. Let us consider the Bessel processes $Z_t^{\kappa_i}(x_0)$ given as solutions to the *SDE*

$$dZ_t^{\kappa_i} = \frac{a(\kappa_i)}{Z_t^{\kappa_i}} dt + dB_t,$$

$$Z_0^{\kappa_i} = x_0 > 0\,.$$

Let us consider the Bessel processes to be driven by the same collection of Brownian paths. Since $a(\kappa_i) = \frac{-2}{\kappa_i}$ we obtain that if $\kappa_i \leqslant \kappa_j$, then $a(\kappa_i) \leqslant a(\kappa_j)$. Thus, using the fact that the latter drift dominates, we obtain that for almost every Brownian path $T_{x_0}^{\kappa_i} \leqslant T_{x_0}^{\kappa_j}$. $\square$

In the natural coupling that we consider, by looking at the difference of these hitting times, we obtain a random variable

$$0 \leqslant [T_{x_0}^{\kappa_i} - T_{x_0}^{\kappa_j}](\omega)\,.$$

The Lamperti relation gives the law of the hitting time for a fixed value of the parameter and for a fix starting point $x_0 > 0$. Thus, $T_{x_0}^{\kappa_j} \overset{d}{=} \frac{x_0}{2Z_{\mu(\kappa_i)}}$, where $Z_{\mu(\kappa_i)}$ is a Gamma random variable with index $\mu(\kappa)$ that depends continuously on κ. For a fixed value of the starting point $x_0 > 0$, let us consider a sequence of Bessel processes with this starting point, the same Brownian drivers and countably many parameters that are elements of a sequence that converges to a fixed value κ of the parameter. We extract two subsequences $\kappa_i - \to \kappa$ converging from below and $\kappa_i + \to \kappa$ converging from above. By the Lemma above, we have that

$$T_{x_0}^{\kappa_i -} \leqslant T_{x_0}^{\kappa} \leqslant T_{x_0}^{\kappa_i +},$$

for almost every Brownian path. We consider the Laplace transform of a random variable $X : \Omega \to \mathbb{R}$ to be given by

$$\mathcal{L}(X) := \mathbb{E}(e^X).$$

Taking the Laplace transforms of these random variables, we obtain the the following

$$\mathcal{L}(T_{x_0}^{\kappa_i -}) \leqslant \mathcal{L}(T_{x_0}^{\kappa}) \leqslant \mathcal{L}(T_{x_0}^{\kappa_i +}).$$

Thus, using the convergence of the laws in the parameter κ, we obtain that

$$\lim_{\kappa_i \to \kappa} \mathcal{L}(T_{x_0}^{\kappa_i +}) = \mathcal{L}(T_{x_0}^{\kappa}).$$

The same holds for the sequence converging from below. We use the fundamental result about random variables stating that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$ are coupled, and that $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^{\kappa}$, and that $T_{x_0}^{\kappa_i} \uparrow T_{x_0}^{\kappa}$ or $T_{x_0}^{\kappa_i} \downarrow T_{x_0}^{\kappa}$, then $T_{x_0}^{\kappa_i} \to T_{x_0}^{\kappa}$, a.s..

Note that the value of κ is chosen arbitrarily, thus this argument works for any value that we choose in the interval $\kappa \in [0, 4]$.
The next ingredients that we need is Lemma 5.3.1. We use Lemma 5.3.1 to argue that for the fixed value κ_1, the first term in the sequence $\kappa_i \to \kappa$ there is only one point on the left side of the singularity that will hit the origin in time $T_{x_0}^{\kappa_i}$ a.s.. We know from the above that the hitting time is sequentially continuous in the parameter κ_i for fixed value of the starting point.

Note that since in our analysis we have as a driver a unique collection of Brownian paths, the dynamics on the negative part of the real axis is given by

$$d\tilde{X}_t = \frac{-2}{\kappa \tilde{X}_t} dt + dB_t,$$

$$\tilde{X}_0 = x_0 < 0.$$

Since $\tilde{X}_t$ is negative (until the first hitting time of zero), we obtain by multiplication with -1, the following SDE

$$dX_t = \frac{-2}{\kappa X_t}dt - dB_t,$$
$$X_0 = x_0 > 0\,,$$

with $X_t = -\tilde{X}_t$. Thus, the dynamics on the negative part of the real axis, when coupling with the same Brownian drivers, is a equivalent with a dynamics on the positive side of the real axis with the driver $-B_t$ (that is still a Brownian motion). In particular, the Lamperti relation that we have used for the positive part of the real line, holds also for the dynamics on $\mathbb{R}_-$.

Let us consider the backward Loewner flow starting from $x_0 > 0$ and driven by$\sqrt{\kappa}B_t$. Let us take a sequence $\kappa_i \to \kappa$. For $i = 1$, we know from Lemma 5.3.1 that for almost every Brownian path, there exists a unique point on the left of the singularity such that $T^{\kappa_1}_{x_0} = T^{\kappa_1}_{y_0}$.

Lemma 5.3.3 (Lemma 5 of [2])**.** *Let X_t be a Bessel process of dimension $d \leqslant 2$, started from $x \geqslant 0$. Then, for any fixed $d \leqslant 2$, for almost every Brownian path, the hitting time of the origin $T^d_x(\omega)$ is continuous at the starting point $x \geqslant 0$.*

We will make use of this Lemma for sequences of dimensions. Since the result gives an a.s. convergence of the hitting times in the starting point for a fix dimension, when considering sequences of dimensions we obtain the same conclusion outside a different nullset formed by the countable union of the nullsets outside of which the processes are defined. Specifically, in our case we use this Lemma for real Bessel processes of dimensions $d_i = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, $i \in \mathbb{N}$ with $d_i < 2, \forall i \in \mathbb{N}$, that are (strong) solutions to the real-valued SDE (5.2.1).

We also consider as an ingredient in our analysis the following Theorem.

Theorem 5.3.4 (Dini's Theorem)**.** *Let M be a compact topological space, and let us consider monotonic sequence of continuous functions $f_n : M \to \mathbb{R}$ which converge pointwise to a continuous function $f : M \to \mathbb{R}$, then the convergence is uniform.*

The final step needed in the proof is in the following Lemma.

Lemma 5.3.5. *Let us consider an increasing sequence of real parameters $\kappa_i \to \kappa$ with $0 < \kappa_i \leqslant 4$. Let us consider a sequence of Bessel processes of dimension $d = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, with the same starting point $x_0 > 0$. We construct a sequence of real negative points $y_i(\omega)$ such that for almost every Brownian path $T^{\kappa_i}_{y_i}(\omega) = T^{\kappa_i}_{x_0}(\omega)$.*

Then, for almost every Brownian path, we can find a convergent subsequence $y_{m_i}(\omega)$ with limit $\hat{y}(\omega)$. Moreover, we obtain that for almost every Brownian path

$$T(\omega)^{\kappa_i}_{y_{m_i}} \to T^{\kappa}_{\hat{y}(\omega)}.$$

Proof. For the sequence $\kappa_i \to \kappa$, we have a sequence of starting points on the left of the origin $y_i = y_i(\omega)$ that by construction have the property that for each Brownian path, $T^{\kappa_i}_{y_i} = T^{\kappa_i}_{x_0}$. Let us assume that with positive probability the sequence y_i is not bounded. Then, with positive probability, we can find a subsequence that is diverging to $+\infty$. This is a contradiction with the fact that the hitting times of the origin for $y_i(\omega)$ are bounded by $T^{\kappa_i}_{y_i} \leqslant T^{\kappa_{max}}_{x_0} = T < +\infty$, almost surely, by the monotonicity of the drifts. Indeed, if with positive probability there exists a divergent subsequence, then with positive probability, the hitting times of the origin for this subsequence would have $T^{\kappa_{max}}_{y_i} \to \infty$. In order to see this, we recall that Lamperti relation gives that $T^{\kappa_{max}}_{y_i} \overset{(d)}{=} \frac{y_i}{Z_{\kappa_{max}}}$, where $Z_{\kappa_{max}}$ is a Gamma Random variable with index depending on κ_{max}. Thus for $T < +\infty$ as above, we have that on the subsequence $y_{r_i} \to \infty$, $\mathbb{P}\left(T^{\kappa_{max}}_{y_{r_i}} > T\right) = \mathbb{P}\left(\frac{y_{r_i}}{Z_{\kappa_{max}}} > T\right) > 0$ contradicting the fact that $T^{\kappa_{max}}_{x_0} = T < +\infty$, almost surely).

Thus, the sequence $y_{\kappa_i}(\omega)$ is almost surely bounded. From Bolzano-Weirstrass Theorem, we have for almost every Brownian path a convergent subsequence $y_{m_i}(\omega) \to \hat{y}(\omega)$. Let us call the limit of this subsequence $\hat{y} = \hat{y}(\omega)$.

In order to apply Dini's Theorem, for each Brownian path, we consider the splitting

$$|T^{\kappa_i}_{y_i} - T^{\kappa}_{\hat{y}}| \leqslant |T^{\kappa_i}_{y_i} - T^{\kappa}_{y_i}| + |T^{\kappa}_{y_i} - T^{\kappa}_{\hat{y}}|.$$

In order to estimate the first term, we use Dini's Theorem for the family of functions

$$f_j(y) := T^{\kappa_j}_{y}.$$

Using the result from Lemma 5.3.3, we have that $T^{\kappa_i}_{y}$ and T^{κ}_{y} are a.s. sequentially continuous real valued functions at $\hat{y}$.

We have that

$$|T^{\kappa_i}_{y_i} - T^{\kappa}_{y_i}| \leqslant \sup_{y \in \{y_1, y_2, \ldots, \hat{y}\}} |T^{\kappa_i}_{y} - T^{\kappa}_{y}|.$$

From Dini's Theorem, we know that $\exists i_0(x_0, \omega, \kappa) > 0$ such that for all $i > i_0$,

$$\sup_{y \in \{y_1, y_2, \ldots, \hat{y}\}} |T^{\kappa_i}_{y} - T^{\kappa}_{y}| \leqslant \varepsilon$$

.

For the second term, we estimate using the sequential continuity in parameter y_i for a fixed parameter, implied by 5.3.3. From this, we obtain that for almost all Brownian paths, there exists an index $i_1 = i_1(\omega, \kappa) > 0$ such that for all $i \geqslant i_1$, $|T^{\kappa}_{y_i} - T^{\kappa}_{\hat{y}}| \leqslant \varepsilon/2$. Taking $i(x_0, \omega) = \max(i_0(x_0, \omega, \kappa), i_1(\omega, \kappa))$ we obtain the desired conclusion.

We apply Lemma 5.3.5 with Lemma 5.3.1, we obtain that for each Brownian path $\hat{y}(\omega)$ is the unique point with this value of the hitting time. Indeed, since otherwise we would have two points on the same side of the singularity that will hit the origin in the simultaneously, contradicting the Lemma 5.3.1.

Note that for any other sequences $\tilde{\kappa_i} \to \kappa$, we obtain in the same manner as described before using Bolzano-Weirstrass Theorem for almost every Brownian path, a converging subsequence. Using the sequential continuity in both variables proved in Lemma 5.3.5 along with Lemma 5.3.1, we prove that for almost every Brownian path all the subsequences have unique limit $\tilde{y}(\omega)$ (that depends on the path), and conclude. □

Performing the same analysis for a countable collection of starting points we obtain the point-wise convergence argument on a dense set of points on the real line. Since the functions $T^{\kappa_i}_y$ are monotonic in y a.s., for all i, and since the limit T^{κ}_y is also monotone, we obtain that they have countable many discontinuities. Moreover, since $T^{\kappa_j}_{y_j} \to T^{\kappa}_{y_j}$ on a countable set of points y_j and by monotonicity of the functions, we have that

$$T^{\kappa_j}_{y_m} \leqslant T^{\kappa_j}_{y} \leqslant T^{\kappa_j}_{y_n}$$

for $y_m \to y$ from below and $y_n \to y$ from above. Thus, we have that

$$T^{\kappa}_x = \lim_{j\to\infty} \lim_{m\to\infty} T^{\kappa_j}_{y_n} \leqslant \liminf T^{\kappa_j}_{y} \leqslant \limsup T^{\kappa_j}_{y} \leqslant \lim_{j\to\infty} \lim_{n\to\infty} T^{\kappa_j}_{y_n} = T^{\kappa}_x,$$

at all the continuity points of T^{κ}_x. We have the desired conclusion on the set of continuity points of T^{κ}_x, a.s.. Since the function is monotone, the set of points where it is continuous is uncountable.

Thus, outside the nullset that is the countable union of all the nullsets where the backward SLE_{κ_i} traces are defined for $(\kappa_i)_{i\in\mathbb{N}}$, we obtain the point-wise convergence of the welding homemorphism at the value κ, that was chosen arbitrarily in $[0, 4]$. Finally, we obtain the almost sure point-wise convergence of the welding homeomorphism induced by the backward SLE_{κ} for $\kappa \in [0, 4]$.

□

5.4 Quasi-sure existence of the SLE_κ trace, polarity of the nullsets outside of which the SLE_κ trace is defined and continuity in κ for $\kappa \neq 8$-Defining the SLE traces simultaneously for all $\kappa \neq 8$

Aggregation and Quasi-sure Stochastic Analysis. A probability measure $\mathbb{P}$ is a local martingale measure if the process B is a local martingale under $\mathbb{P}$. It is proved that there exists an $\mathfrak{F}$-progresively measurable process denoted as $\int_0^t B_s dBS$ which coincides with the Ito integral $\mathbb{P}$ -a.s. for all local martingale measures $\mathbb{P}$. In particular this provides a pathwise definition of

$$< B >_t := B_t B_t^T - 2\int_0^t B_s dB_s$$

and

$$\hat{a}_t := \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}[< B >_t - < B >_{t-\varepsilon}].$$

Let us introduce first $\bar{\mathcal{P}}_W$ which denotes the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s. $< B >_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{R}_+$.

Definition 5.4.1.
- *We say that a property holds $\mathcal{P}$-quasi-surely if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}$.*
- *Denote $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$*
- *A probability measure $\mathbb{P}$ is called absolutely continuous with respect to $\mathcal{P}$ is $\mathbb{P}(E) = 0$ for all $E \in \mathcal{N}_{\mathcal{P}}$.*

In this approach, we use the following universal filtration $\mathfrak{F}^{\mathcal{P}}$ for the mutually singular measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$.

$$\mathfrak{F}^{\mathcal{P}} := \{\mathcal{F}_t^{\mathcal{P}}\}_{t\geqslant 0}$$

where

$$\mathcal{F}_t^{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}} \left(\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}\right) .$$

Note that the construction is suitable for mutually singular probability measures. In the case the measures are absolutely continuous, the situation becomes simpler since one can work under the nullsets of the dominating measure directly.

The next definition introduces the notion of aggregator that we use in our analysis.

Definition 5.4.2. *Let $\mathcal{P} \subset \bar{\mathcal{P}}_W$. Let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of $\mathfrak{F}^{\mathcal{P}}$ progressively measurable processes. An $\mathfrak{F}^{\mathcal{P}}$ progressively measurable process X is called a $\mathcal{P}$-aggregator of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, if $X = X^{\mathbb{P}}$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$.*

The universal Brownian motion. In this section, we introduce the notion of Universal Brownian motion as in [43].
Let

$$\bar{\mathcal{A}} := \left\{a : \mathbb{R}_+ \to \mathbb{S}_d^{>0} | \mathbb{F} - \text{progresively measurable and } \int_0^t |a_s| ds < +\infty, \forall t \geqslant 0\right\}.$$

We consider $d = 1$, and $\mathbb{S}_d^{>0}$ becomes the space of symmetric matrices in $d = 1$ that is $\mathbb{R}_+$.

For a given $\mathbb{P} \in \bar{\mathcal{P}}_W$, let

$$\bar{\mathcal{A}}_W(\mathbb{P}) := \{a \in \bar{A} : a = \hat{a}, \mathbb{P} - a.s.\}$$

Recall that $\hat{a}$ is the density of the quadratic variation of $< B >$ and is defined pointwise. We define

$$\bar{\mathcal{A}}_W := \cup_{\mathbb{P} \in \bar{\mathcal{P}}_W} \bar{\mathcal{A}}_W(\mathbb{P})$$

Let us define for any $a, b \in \mathcal{A}$, the disagreement time $\theta^{a,b} := \inf\{t \geqslant 0 : \int_0^t a_s ds \neq \int_0^t b_s ds\}$.

Definition 5.4.3. *A subset $\mathcal{A}_0 \subset \mathcal{A}_W$ is called a generating class of diffusion coefficients if*

- *$\mathcal{A}_0$ satisfies the concatenation property $a\mathbf{1}_{[0,t)} + b\mathbf{1}_{[t,\infty)} \in \mathbb{A}_0$, for $a, b, \in \mathcal{A}_0, t \geqslant 0$*
- *$\mathcal{A}_0$ has constant disagreement times: for all $a, b \in \mathcal{A}_0$, $\theta^{a,b}$ is constant or equivalently $\Omega_t^{a,b} = \emptyset$ or Ω for all $t \geqslant 0$.*

Definition 5.4.4. *We say $\mathcal{A}$ is a separable class of diffusion coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0 \subset \mathcal{A}_W$ is generated by a class of diffusion coefficients and $\mathcal{A}$ consists of all processes a of the form*

$$a = \sum_0^{\infty} \sum_{i=1}^{\infty} a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}$$

where $(a_i^n)_{i,n} \subset \mathcal{A}_0$, $(\tau_n)_n \subset \mathcal{T}$ is non-decreasing with $\tau_0 = 0$.

- $\inf\{n : \tau_n = \infty\} < \infty$ *and $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$ and each τ_n takes at most countably many values.*
- *For each n $\{E_i^n, i \geqslant 1\} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.*

A separable class $\mathcal{A}$ of diffusion coefficients generated by $\mathcal{A}_0$ is said to satisfy the consistency conditions. We denote

$$\mathcal{P} = \{\mathbb{P}^a, a \in \mathcal{A}\}.$$

Let us consider a standard Brownian motion B_t. For any $\mathbb{P} \in \mathcal{P}_W$ and $a \in \bar{\mathcal{A}}_W(\mathbb{P})$ by Levy's characterization, we obtain that the following Ito's stochastic integral under $\mathbb{P}$ is a $\mathbb{P}$- Brownian motion

$$W_t^{\mathbb{P}} := \int_0^t a_s^{-1/2} dB_s$$

For $\mathcal{A}$ satisfying the consistency condition, the family $\{W^{\mathbb{P}_a}, a \in \mathcal{A}\}$ admits a unique $\mathcal{P}$-aggregator W. Since $W^{\mathbb{P}_a}$ is a $\mathbb{P}^a$ Brownian motion for every $a \in \mathcal{A}$, we call W- a universal Brownian motion.

A fundamental result that we use is the aggregate solution to stochastic differential equations. In the paper, they show how to solve a stochastic differential equation simultaneously under all the measures $\mathbb{P} \in \mathcal{P}$. Specifically, they prove the following result:

Proposition 5.4.5 (Proposition 6.10 of [43])**.** *Let A be satisfying the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s$,$t \geqslant 0$ has a unique $\mathbb{F}^{\mathbb{P}}$ progresively measurable strong solution. on the interval $[0, \tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution to the equation above.*

We use this result for a stochastic differential equation solved by the process $\tilde{N}_s$, that we will introduce in the next section, that is used in the construction of the SLE trace in order to obtain the estimate the derivative of the conformal map at fixed times.

5.5 Heuristics of the proof

The main idea is to consider the construction of the aggregated solution to SDE as in [43] and applied to the SDE corresponding to the process $\tilde{N}_s$. Furthermore, we express the derivative of the map $\tilde{h}_t(z0)$ using the aggregated solution $\tilde{N}_s$ and use the typical Lemmas in [20] to obtain the existence of the trace. We have

$$|\tilde{h}_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \frac{\tilde{K}_s^2 + 1}{\tilde{K}_s^2 - 1} ds\right) = e^{at} \exp\left(2a \int_0^t \tilde{N}_s ds\right),$$

where

$$d\tilde{N}_s = (1 - \tilde{N}_s)[-4(a+1)\tilde{N}_s + 1]ds + 2\sqrt{\tilde{N}_s}(1 - \tilde{N}_s)d\tilde{B}_s$$

.

This formulation is equivalent with viewing the Loewner equation driven by the universal Brownian motion. The SDE for $\tilde{K}_t$ is

$$d\tilde{K}_t = 2a\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}\, d\tilde{B}_t$$

It can be shown that the $sinh(J_t)$ solves the SDE for $\tilde{K}_t$ where J_t satisfies

$$dJ_t = -(q+r)\tanh(J_t)dt + d\tilde{B}_t\,.$$

with $q = q(\kappa)$ and $r = r(\kappa)$.

Once we have a unique notion of strong solution for the SDE for J_t, $\mathbb{P}$-a.s., we can construct an aggregated solution for this SDE and then express

$$|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right) = e^{-at}\exp\left(2a\int_0^t \frac{\sinh(J_s)^2+1}{\sinh(J_s)^2-1}ds\right).$$

Note that, equivalently we can consider the unique strong solution (checking the conditions of Yamada Watanabe Theorem the SDE $\tilde{K}_t$ and construct an aggregated solution for it. **We use that aggregated solution for the SDE $\tilde{K}_t$ to construct simultaneously the SLE_κ trace for all parameters $\kappa \neq 8$ by relating the aggregated solution for $\tilde{K}_t$ with the derivative of the backward map.**

Note that since the structure of the SDE that $\tilde{N}_s$ satisfies we are naturally lead to consider the problem of aggregation as in the paper [43] since the measures $\mathbb{P}_a(\kappa)$ are not a-priori absolutely continuous with respect to the Wiener measure.

With this definition, we recover the typical construction of the estimate of the Rohde-Schramm Theorem simultaneously under all measures $\mathbb{P}_\kappa$ (uncountably many choices of the parameter). The advantage is that in this formulation, of q.s. analysis we can have a clear definition of the SLE trace quasi surely in which the continuty in κ is a direct consequence of the procedure of constructing the traces and the continous perturbation of the algorithm designed in [44]. In this way we work over all the possible measures simultaneously, i.e. over uncountably many κ.

The quasi-sure construction assures that the typical estimate on the derivative of the map, holds simultaneously under all the choices of the parameter κ since the aggregated solution of $\tilde{N}_s$ is constructed simultaneously under all the measures $\mathbb{P}_\kappa$

5.6 The quasi-sure existence of the trace of SLE_κ-Defining the SLE trace simultaneously for all parameters κ

Rohde and Schramm proved one fundamental result about the existence of the trace for the SLE_κ process for all values of $\kappa \neq 8$. In order to discuss the result, we introduce the notions and definitions that we use. We say that a continous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$-hulls K_t if $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \geqslant 0$.

Theorem 5.6.1 (Rohde-Schramm). *Let $(K_t)_{t\geqslant 0}$ be a SLE_κ for $\kappa \neq 8$. We denote g_t to be the Loewner flow and U_t be the Loewner transform. Then, $g_t^{-1} : \mathbb{H} \mapsto H_t$ extends continously to $\bar{\mathbb{H}}$ for all $t \geqslant 0$, almost surely. Morevoer, if we set $\gamma_t = g_t^{-1}(U_t)$, then γ_t is continous and generates $(K_t)_{t\geqslant 0}$ almost surely.*

Estimates for the mean of the derivative for a fixed κ. In order to provide the technical results for this part, we make a useful time change in the corresponding Loewner equation for the real and imaginary parts. With this new clock at hand, we obtain some technical Lemmas that are crucial in the proof of the existence of the trace for the SLE process. Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2 + Y_t^2}dt - \sqrt{\kappa}dB_t\,, \quad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{5.6.1}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$.

We provide a martingale estimate for the backward Loewner differential equation. We start with the following proposition, in which the polynomial condition in the hypothesis comes from the fact that we are searching for martingales of the type $M_t :=$ $\tilde{Y}_t^\alpha(|\tilde{Z}_t|/\tilde{Y}_t)^\beta |h_t'(z_0)|^\gamma$, where α, β, γ depend on each other and $h_t(z)$ is a family of conformal maps satisfying the backward Loewner differential equation. This leads to the fact that they should satisfy a constraint.

Proposition 5.6.2 (Proposition 7.2 in [20]). *Let r, b such that*

$$r^2 - (2a+1)r + ab = 0\,,$$

then

$$M_t \;:=\; \tilde{Y}_t^{b-(r/a)}(|\tilde{Z}_t|/\tilde{Y}_t)^{2r}|h_t'(z_0)|^b\,,$$

is a martingale. Moreover,

$$\mathbb{P}(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}e^{t(r-ab)}\,.$$

Proof. By taking the complex derivative in z in the Loewner equation and by applying the chain rule for the function $L_t = \log h_t'(z_0)$, we obtain that $L_t = -\int_0^t \frac{a}{Z_s^2} ds$, and in particular, $|\tilde{h}_t'(z_0)| = \exp\left(a \int_0^t \frac{\tilde{Y}_s^2 - \tilde{X}_s^2}{\tilde{X}_s^2 + \tilde{Y}_s^2} ds\right)$. Moreover, if we consider $\tilde{K}_t = \frac{\tilde{X}_t^2}{\tilde{Y}_t^2}$ and $\tilde{N}_t = \frac{\tilde{K}_t}{1+\tilde{K}_t}$, we obtain that

$$|\tilde{h}_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \tilde{N}_s ds\right).$$

In the $\sigma(t)$ time parametrization, looking at the equation for $\tilde{Y}_t$ we obtain a deterministic one $d\tilde{Y}_t = -a\tilde{Y}_t dt$, so in this time parametrization Y_t grows deterministically in an exponential manner $\tilde{Y}_t = Y_0 e^{at}$. At this moment, we can rephrase the formula for M_t as

$$M_t = y_0^{b-(r/a)} e^{-rt} (1 - \tilde{N}_t)^{-r} \exp(2a \int_0^t \tilde{N}_s ds).$$

and by applying Ito's formula, we obtain that

$$dM_t = 2r\sqrt{\tilde{N}_t} M_t d\tilde{B}_t,$$

where $\tilde{B}_t = \int_0^{\sigma(t)} \frac{1}{\sqrt{X_t^2 + Y_t^2}} dB_t$ is the Brownian motion that we obtain in the time reparametrization. This shows that M_t is a martingale, hence

$$\mathbb{E}[M_t] = \mathbb{E}[M_0] = y_0^{b-(r/a)} (|z_0|/y_0)^{2r}.$$

Note that since for $r \geqslant 0$, $(|\tilde{Z}_t|/Y_t)^{2r} \geqslant 1$, then by Markov inequality, we have that

$$\mathbb{P}(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} e^{t(r-ab)}.$$

$\square$

Corollary 5.6.3 (Corollary 7.3 in [20])**.** *For every* $0 \leqslant r \leqslant 2a+1$, *there is a finite* $c = c(a,r)$ *such that for all* $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, *we have that*

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} \delta(y_0, \lambda),$$

where $b = [(2a+1)r - r^2]/a \geqslant 0$ *and*

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{(r/a)-b}, & \text{if } r < ab, \\ -\log(\lambda y_0), & \text{if } r = ab, \\ y_0^{b-(r/a)}, & \text{if } r > ab. \end{cases}$$

Proof. From $dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt$, we obtain that $dY_t \leqslant \frac{a}{Y_t}dt$, and hence $Y_t \leqslant \sqrt{2at+y_0^2} \leqslant \sqrt{2a+1}$. In the last inequality, we used that $t \leqslant 1$ and $y_0 \leqslant 1$. Using the exponential growth of Y_t in this time reparametrization, we obtain that $\tilde{Y}_t = \sqrt{2a+1}$ at time $T = \frac{\log\sqrt{2a+1}-\log y_0}{a}$.

Therefore,

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \mathbb{P}(\sup_{0\leqslant s\leqslant T} |\tilde{h}_s'(z_0)| \geqslant \lambda) .$$

Using that $|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \tilde{N}_s ds\right)$ we obtain that $|\tilde{h}_{t+s}'(z0)| \leqslant e^{as}|\tilde{h}_t'(z0)|$. So by addition of the probabilities, we have that

$$\mathbb{P}(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^a\lambda) \leqslant \sum_{j=0}^{[T]} \mathbb{P}(|\tilde{h}_j'(z_0)| \geqslant \lambda) .$$

Using the Schwarz-Pick Theorem for the upper half-plane we obtain that $|\tilde{h}_t(z_0)| \leqslant \mathrm{Im}\tilde{h}_t'(z_0)/y_0 = e^{at}$. This gives a lower bound for the t that we are summing over and we obtain that via the Proposition 5.6.2 that

$$\begin{aligned}\mathbb{P}(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^a\lambda) &\leqslant \sum_{(1/a)\log\lambda\leqslant j\leqslant T} \mathbb{P}(|\tilde{h}_j'(z_0)| \geqslant \lambda) \\ &\leqslant \lambda^{-b}(|z_0|/y_0)^{2r} \sum_{(1/a)\log\lambda\leqslant j\leqslant T} e^{j(r-ab)} \\ &\leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda) .\end{aligned}$$

□

Estimates on the moments of the derivatives for many κ using aggregation of solutions of a SDE. We consider the universal Brownian motion $W_t^{\mathbb{P}}$ as a driver for the backward Loewner differential equation. We consider also the random time changed universal Brownian motion $\tilde{W}_t^{\mathbb{P}} := \frac{dW_t^{\mathbb{P}}}{\sqrt{X_t^2+Y_t^2}}$

Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - dW_t^{\mathbb{P}}, \quad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt , \tag{5.6.2}$$

We consider the time change $\sigma(t) = X_t^2+Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$. We prove the same estimates as above using the aggregated solution $\tilde{N}_s$ that gives that

$$M_t = y_0^{b-(r/a)}e^{-rt}(1-\tilde{N}_t)^{-r}\exp(2a\int_0^t \tilde{N}_s ds) ,$$

is a quasi-sure martingale, i.e. a martingale with respect to all the measures simultaneously. Thus, the same estimate on the absolute value derivative of the maps, can be recovered simultaneously for all probability measures. We obtain the same inequality as in Proposition 5.6.2 simultaneously for all the measures. Also the Corollary 5.6.3 is recovered quasi-surely from the Proposition using the aggregated solution, in the same manner.

We obtain from the quasi-sure versions of the Lemmas and Propositions from above, a way of estimating simultaneously under all the measures the absolute value of the derivative of the map with the 'worst' value of the parameter κ. The idea is that since the estimate holds for the 'worst' κ, then by the q.s. construction, gives that it holds simultaneously for all the 'better' κ, i.e. we can estimate uniformly in κ with the same bounds, simultaneously under all the choices of the measure. In particular, this gives that the typical Rohde-Schramm Theorem that is sufficient to prove the existence of the SLE_κ trace, holds quasi surely for all $\kappa \neq 8$.

Existence of the trace for fixed κ. Before stating the main Theorem of the section, we prove two propositions that together with corollary 5.6.3 build the argument for the existence of trace of SLE_κ for $\kappa \neq 8$.

Proposition 5.6.4 (Proposition 4.33 in [20])**.** *Suppose that g_t is a Loewner chain with driving function U_t and assume that there exist a sequence of positive numbers $r_j \to 0$ and a constant c such that*

$$|\hat{f}'_{k2^{-2j}}(2^{-j}i)| \leqslant 2^j r_j \,, k = 0, 1, \ldots, 2^{2j} - 1 \,,$$
$$|U_{t+s} - U_t| \leqslant c\sqrt{j}2^{-j} \,, 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 2^{-2j} \,.$$

and

$$\lim_{j\to\infty} \sqrt{j}/\log r_j = 0 \,.$$

Then $V(y, t) := \hat{f}_t(iy)$ is continuous on $[0, 1] \times [0, 1]$.

Proof. By differentiating $\partial_t f(t, z) = -\partial_z f(t, z)\frac{2}{z-U(t)}\,, \quad f(0, z) = z, z \in \mathbb{H}\,,$ we obtain that

$$\dot{f'}_t(z) = -f''_t(z)\frac{2}{z - U_t} + f'_t(z)\frac{2}{(z - U_t)^2}\,.$$

Bieberbach Theorem ((3.16), [20]) implies that $|f_t''(z)| \leqslant \frac{6|f_t'(z)|}{\mathrm{Im}(z)^2}$, and that $|f_{t+s}'(z)| \leqslant \exp\left[\frac{6s}{\mathrm{Im}(z)}\right]|f_t'(z)|$. From hypothesis, we get that for $k = 0, 1, \ldots, 2^{2j} - 1$

$$|f_t'(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j \, , k2^{-2j} \leqslant t \leqslant (k+1)2^{-2j} \, .$$

Using the Distortion Theorem, we get that for a univalent function on $\mathbb{D}$, we have that $|f'(z)| \leqslant 12|f'(0)|$ for $|z| \leqslant 1/2$. We iterate this estimate on a sequence of intersecting disks that connect $z, w \in \mathbb{H}$ with $\mathrm{Im}(z), \mathrm{Im}(w) \geqslant y > 0$. For the conformal transformation $f : \mathbb{H} \to \mathbb{D}$, we have that

$$|f'(w)| \leqslant 144^{(z-w)/y+1)}|f'(z)| \, .$$

In particular, by combining the hypothesis and $|f_t'(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j$ we obtain that there exist c and β such that

$$|\hat{f}_t'(i2^{-j})| \leqslant e^{\sqrt{j}\beta} 2^j r_j \, , \;\; 0 \leqslant t \leqslant 1, \;\; j = 0, 1, 2, \ldots, 2^{-j} \, .$$

Using the Distortion Theorem again but for a point that is not on the lattice of space and time, we get

$$|\hat{f}_t'(iy)| \leqslant e^{\sqrt{j}\beta} 2^j r_j \, , \;\; 0 \leqslant t \leqslant 1, \;\; 2^{-j} < y < 2^{-j+1} \, , \;\; j = 0, 1, 2, \ldots, 2^{-j} \, .$$

For $s \leqslant 2^{-2j}$ and $y, y_1 \leqslant 2^{-j}$ we get that $|\hat{f}_t(iy) - \hat{f}_{t+s}(iy)| \leqslant |\hat{f}_t(iy) - \hat{f}_t(i2^{-j})| + |\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j})| + |\hat{f}_{t+s}(i2^{-j}) - \hat{f}_{t+s}(iy)|$. The first and the third term are bounded by the estimate elaborated so far via

$$|\hat{f}_t(iy) - \hat{f}_t(i2^{-j})| \leqslant \sum_{l=j}^{\infty} c e^{\beta\sqrt{l}} r_l \, .$$

From the assumption, the right hand side goes to 0 as $j \to \infty$. Using the estimate and the format of the partial differential equation that f solves, for the middle term, we have that

$$|\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j}) \leqslant 2s2^j \sup_{t \leqslant r \leqslant t+s} |f'(2^{-j})| \leqslant c r_j \, .$$

Since V is continuous already in $(0, \infty) \times [0, \infty)$ to establish the continuity on $[0, \infty) \times [0, \infty)$ it suffices to show that there exists a $\delta(\varepsilon)$ such that $\delta(0+) = 0$ and such that $|V(y, t) - V(y_1, s)| \leqslant \delta(y + y_1 + |t - s|)$, $0 \leqslant t, s, \leqslant t_0, y, y_1 > 0$. So by using the hypothesis, we conclude. $\square$

We need another result in order to conclude the existence of the trace for SLE process. For this we introduce the notion of *accessible point*. We call a point $z \in \hat{K}_t \setminus \cup_{s<t}\hat{K}_s$ $t-accessible$ if there exists a curve $\eta : [0,1] \to \mathbb{C}$, with $\eta(0) = z$ and $\eta(0,1] \subset H_t$.

Proposition 5.6.5 (Proposition 4.29 in [20])**.** *Suppose g_t is a Loewner chain with driving function U_t and let $\hat{f}_t(z) = g_t^{-1}(z+U_t)$. Suppose that for each t, the limit $\gamma(t) = \lim_{y\to 0+} \hat{f}_t(iy)$, exists and the function $t \to \gamma(t)$ is continuous. Then g_t is the Loewner chain generated by γ.*

Proof. The proof relies on Proposition 4.27 from [20] that shows together with the condition from the hypothesis that $\gamma(t)$ is the only t-accesible point. Since $\gamma[0,t]$ is closed, the same Proposition 4.27 from [20] shows that $\partial H_t \cap \mathbb{H}$ is contained in $\gamma[0,t]$. □

In order to prove this result, we need the Lemma from the introduction also.

The following Lemma can be recovered using the quasi-sure analysis.

Lemma 5.6.6. *[Lemma 7.6 in [20]] For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - \zeta(t)$.*

Theorem 5.6.7 (Rohde-Schramm)**.** *If $\kappa \neq 8$ the chordal SLE_κ is generated by a path with probability 1.*

Proof. By using the scaling of the SLE_κ, it suffices to prove the Theorem only for $t \in [0,1]$. According to the preliminary propositions it suffices to show that with probability 1 there exists an ε and a random constant c (because this estimate should hold for all $j's$ and $k's$) such that

$$|f'_{k2^{-2j}}(i2^{-j})| \leqslant c2^{j-\varepsilon}\,, j = 1,2,\ldots, k = 0,1,\ldots,2^{2j}\,,$$
$$|B_t - B_s| \leqslant c|t-s|^{1/2}|\log\sqrt{|t-s|}| \quad 0 \leqslant t \leqslant 1\,.$$

The second inequality is a consequence of the modulus of continuity for the Brownian motion. For the first inequality, we use a Borel-Cantelli Lemma along with Lemma 5.6.6 to find c and ε such that for all $0 \leqslant t \leqslant 1$

$$\mathbb{P}(|h'_t(i2^{-j}| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-(2+\varepsilon)j}\,.$$

Notice that we apply h'_t to points on the imaginary axis and that the corresponding $\lambda = 2^{j-\varepsilon}$.

We consider $r = a + (1/4) < 2a+1$ and $b = \frac{(1+2a)r-r^2}{a} = a + 1 + \frac{3}{16a}$, according to the Corollary 5.6.3. Thus, we are in the regime $r < ab$, so by Corollary 5.6.3 we have that

$$\mathbb{P}(|h_t'(i2^{-j})| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-j(2b-(r/a))(1-\varepsilon)}.$$

Investigating the exponent of 2, we obtain that $2b-(r/a) = 2a+1+1/(8a) > 2$ provided that $a \neq 1/4$. So, we can apply Borel-Cantelli argument provided that $a \neq 1/4$, i.e. $\kappa \neq 8$, and finish the proof.

□

5.7 Quasi-sure existence of the trace -defining the trace simultaneously for all κ- and proving continuity in κ

Once we obtain the quasi-sure version of the estimates in the previous section, we can continue to go exactly in the same manner to prove the existence of the trace quasi-surely with the tools from above.

For this, we use the aggregated solution of the SDE $\tilde{K}_t$:

$$d\tilde{K}_t = 2a\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}d\tilde{B}_t$$

constructed via the methods of Quasi-Sure Stochastic Analysis through Aggregation from the following Proposition.

Proposition 5.7.1 (Proposition 6.10 of [43]). *Let A be satisfying the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s, t \geqslant 0$ has a unique $\mathbb{F}^{\mathbb{P}}$ progresively measurable strong solution. on the interval $[0,\tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution to the equation above.*

In order to perform the analysis and to prove all the estimates (that were done in the previous section for fixed κ) simultaneously for all κ, we use the aggregated solution and relate it with the derivative of the map, via

$$|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right)$$

.

We redo all the proofs of the previous section using the aggregated solution. In this manner we can construct the trace via controlling the derivative estimate simultaneously for all $\kappa \neq 8$ using the aggregated solution, and obtain the existence of the limit and in particular the existence of SLE traces simultaneously for all $\kappa \neq 8$.

Corollary 5.7.2 (Continuity in $\kappa \neq 8$ of the SLE_κ traces). *Once the Rohde-Schramm estimate is established quasi-surely, we obtain naturally the continuity in κ of the traces, when applying the construction of the algorithm in [44]. The main difficulty in these proofs is to control the nullset outside of which the SLE_κ trace is defined, in order to get the the quasi-sure continuity in κ. We have a quasi-sure definition of the SLE_κ trace (i.e. the SLE_κ trace that we consider is defined simultaneously for all the values of κ) and this allows us directly to consider uncountably many parameters $\kappa \neq 8$.*

Proof of Corollary. -To be completed-

□

Chapter 6

Closed-form expression of a stationary law, Skorokhod embedding type problem for the backward SLE flow and the phase transition at $\kappa = 8$

6.1 Preliminaries

We recall from the Introduction the formats of the Loewner equation, that we consider.

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z) \frac{2}{z - \sqrt{\kappa} B_t}\,, \quad f(0,z) = z, z \in \mathbb{H}\,. \tag{6.1.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa} B_t}\,, \qquad g(0,z) = z, z \in \mathbb{H}\,. \tag{6.1.2}$$

(iii) Time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa} B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H}\,. \tag{6.1.3}$$

The connection between the different versions of the Loewner equations is that, for all fixed t, $g_{-t}(z)$ has the same distribution as the maps $f_t(z) - \sqrt{\kappa} B_t$. This is the content of the next Lemma.

Lemma 6.1.1. *For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ have the same distribution as the map $z \to f_t(z) - \sqrt{\kappa} B_t$.*

6.2 A time change of Loewner differential equation and main result

We consider $\Omega = C([0,+\infty),\mathbb{R})$. On this space, we consider the coordinate process $X_t(\omega) = \omega(t)$ as the canonical Brownian motion, and we equip this probability space with the sigma algebra $\sigma(B_s, s \in [0,+\infty))$. Throughout the proof we use the Levy's characterizations of Brownian motion for a local martingale that we obtain in our analysis, in order to obtain a new Brownian motion. Note that under this operation, the Wiener measure $\mathbb{P}$ on the path space is not changing. The filtration that we consider change when we do this procedure, but it is still included in the large sigma algebra $\sigma(B_s, s \in [0,+\infty))$. We show that we can identify the stationary law of the diffusion with the law of the tip of the SLE trace at a fixed time, Before going to the proof of the result in this section, we define a time change of the (backward) Loewner differential equation that is going to be the framework of study for our problem.
First, we rewrite the equation

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa}B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H} \tag{6.2.1}$$

in the format

$$dz_t = \frac{-2}{z_t}dt - \sqrt{\kappa}dB_t\,, \qquad z_0 = 1, z \in \mathbb{H}\,, \tag{6.2.2}$$

where we make the identification $z_t = x_t + iy_t = h_t(z) - \sqrt{\kappa}B_t$. Furthermore, we split this flow in the upper half plane into its real and imaginary components. This gives

$$dx_t = \frac{-2x_t}{x_t^2+y_t^2}dt - \sqrt{\kappa}dB_t\,, \quad dy_t = \frac{2y_t}{x_t^2+y_t^2}dt\,. \tag{6.2.3}$$

We consider $D_t = x_t/y_t$. Applying Ito's formula for the function $f(x,y) = \frac{x}{y}$ and using the fact that the quadratic variation terms all vanish due to the fact that $f_{xx}(x) = 0$, we obtain that

$$dD_t = -\frac{\sqrt{\kappa}dB_t}{y_t} - \frac{D_t}{x_t^2+y_t^2}dt\,.$$

We study the random time change given by

$$\tilde{u}(s) = \int_0^s \frac{dt}{y_t^2}\,,$$

We consider also the inverse

$$c(u) = \inf\{t \geqslant 0 : u^1(s) \geqslant u\}\,. \tag{6.2.4}$$

Alternatively, we can write $dc(u) = y^2_{c(u)} du$.
By setting

$$\tilde{B}(u) = \int_0^u \frac{dB_{c(r)}}{y_{c(r)}} \,,$$

we obtain a new SDE in $T_u = D_{c(u)}, u \geqslant 0$.

$$dT_u = -4\frac{T_u}{1+\kappa T_u^2} du + d\tilde{B}_u \,. \tag{6.2.5}$$

The continuous local martingale $\tilde{B}_u$ has quadratic variation u. Thus, by Lévy's characterization, we obtain that $\tilde{B}_u$ is a Brownian motion on $(\Omega, \mathcal{F}_u, \mathbb{P})$. Thus, when performing this random time change in the study of the backward Loewner differential equation, we obtain a radius-independent stochastic differential equation.

Main result. We are now ready to state the result.

Theorem 6.2.1. *For $\kappa > 0$, let us consider the random time change $\tilde{u}(S, \omega) = \int_0^S \frac{dt}{y_t^2(\omega)}$, where y_t is the imaginary value of the backward Loewner flow driven by $\sqrt{\kappa} B_t$ in $\mathbb{H}$. Let us consider the positive functions $f \in L^1(\mu)$, where $L^1(\mu)$ represents the space of functions which are integrable with respect to the measure $\mu(dx) = \frac{dx}{(1+x^2)^{4/\kappa}}$.*
We consider the process T_u started with initial value $T_0 = 0$ given by the stochastic differential equation

$$dT_u = -4\frac{T_u}{1+\kappa T_u^2} du + d\tilde{B}_u \,. \tag{6.2.6}$$

This process has stationary distribution with density

$$\rho(T) = C\left(\kappa T^2 + 1\right)^{-4/\kappa} \,.$$

For $\kappa < 8$, the process $T(u)$ convergence in the sense of random ergodic averages towards its stationary law, i.e.

$$\left| \frac{1}{\tilde{u}(S,\omega)} \int_0^{\tilde{u}(S,\omega)} f(T_s(\omega)) ds - \mu(f) \right| \xrightarrow{S \to \infty} 0$$

in probability, with respect to the Wiener measure.
Moreover, for $\kappa < 8$ the total variation convergence of the diffusion process T_u gives a precise family of random times $s(\omega)$ on which $ctg(\arg(h_{s(\omega)}(i))) \to \mu(f)$.

Corollary 6.2.2. *For $\kappa = 4$, the random times $s(\omega)$ are the times on which the distribution of the argument of the tip of the backward SLE trace is the uniform distribution.*

Remark 6.2.3. From the definition of our process, we have that $D_c(u) = T_u$ and $D_S = T_{\tilde{u}(S)}$, where S is a deterministic time. Thus, we consider the format of the random ergodic average as in the Theorem, in order to relate it with the law of the backward Loewner flow at a fixed time.

Remark 6.2.4. We remark that there is a phase transition at the value $\kappa = 8$ in terms of the integrability of the density of the stationary distribution of the process T_u, i.e. T_u has stationary distribution that is a probability measure for all $\kappa < 8$ and is not a probability measure for all $\kappa \geqslant 8$. Moreover, by a change of coordinates, when we plot the density of the stationary measure of $\arg(\hat{h}_t(i))$ we get another phase transition at $\kappa = 4$, from convex to concave function.

6.3 The set-up needed for the proof of the main result

The proof is divided in several subsections culminating with the last subsection that contains the proof.

We consider the typical SLE scaling of the maps, we have that

$$\frac{h_t(i)}{\sqrt{t}} = h_1(i/\sqrt{t}) \tag{6.3.1}$$

in distribution, as processes for $t \in [0, +\infty)$.
We can recover the same identity in distribution as processes for the shifted mappings also:

$$\frac{h_t(i) - \sqrt{\kappa} B_t}{\sqrt{t}} = h_1(i/\sqrt{t}) - \sqrt{\kappa} B_1 \,.$$

Throughout this Chapter, we use the notation $\hat{h}_t(i) := h_t(i) - \sqrt{\kappa} B_t = X_t(i) + iY_t(i)$.
In order to study the distribution at a fixed capacity time t of $ctg(arg(\hat{h}_t(0)))$, using the scaling (6.3.1), we study the process $D_t = \frac{x_t(i)}{y_t(i)}$. We also use that the law of $ctg(arg(\frac{\hat{h}_t(i)}{\sqrt{t}}))$ is the same as the law of the cotangent of $ctg(arg(\hat{h}_t(i))$.
By Itô formula, we have that $\frac{x_t(i)}{y_t(i)}$ as t varies in $[0, +\infty)$ satisfies SDE

$$d\frac{x_t(i)}{y_t(i)} = -4\frac{\frac{x_t(i)}{y_t(i)}}{1 + \kappa\left(\frac{x_t(i)}{y_t(i)}\right)^2}\frac{dt}{y_t^2} + \frac{dB_t}{y_t} \tag{6.3.2}$$

Thus, we can rephrase (6.3.2) as before, to the following SDE

$$\begin{aligned} dT_u &= -4\frac{T_u}{1+\kappa T_u^2}du + d\tilde{B}_u \\ T_0 &= 0 \end{aligned} \tag{6.3.3}$$

The starting point of the diffusion is $T_0 = 0$ since $\hat{h}_0(i) = i$.

We consider the following useful Lemma that characterizes the random time change $\tilde{u}(S, \omega)$ defined in the previous section.

Lemma 6.3.1. *Let $\kappa > 0$ and let y_t be the imaginary part of the solution of the Loewner equation driven by $\sqrt{\kappa}B_t$. The random time change $\tilde{u}(S, \omega) = \int_0^S \frac{dt}{y_t^2}(\omega)$ is bounded from below, for all $S > 0$, uniformly in ω, by*

$$\tilde{u}(S, \omega) \geqslant \frac{\kappa}{4}\log\left(1 + \frac{4S}{\kappa}\right) .$$

Proof. Using the equation for the dynamics of y_t from the Loewner differential equation, we obtain that $\partial_t[y_t(z)^2] = 2y_t(z)\partial_t y_t(z) = \frac{4}{\kappa}\frac{y_t(z)^2}{|z_t(z)|^2} \leqslant \frac{4}{\kappa}$. Thus, we obtain that

$$y_t^2 \leqslant y_0^2 + \frac{4t}{\kappa} ,$$

i.e.

$$\frac{dt}{y_t^2} \geqslant \frac{dt}{(y_0^2 + 4t/\kappa)}.$$

By integrating up to time S, we obtain that

$$\int_0^S \frac{dt}{y_t^2} \geqslant \int_0^S \frac{dt}{(y_0^2 + 4t/\kappa)} = \frac{\kappa}{4}\log\left(1 + \frac{4S}{\kappa y_0^2}\right) .$$

□

Cotangent of the argument of the tip of the SLE trace. In this subsection, we study the law of the cotangent of the argument of the backward Loewner flow from the origin up to time 1. Furthermore, we make precise the connection between this law and the law of the process T_u at large random times.

Since the SLE trace is defined a.s. with respect to the initial Wiener measure, its tip is a well defined random variable. Thus, the limit $\lim_{t\to\infty} ctg(arg(\hat{h}_1(i/\sqrt{t})))$ exists for almost every Brownian motion. Thus, the limit $\lim_{t\to\infty} ctg(arg(\hat{h}_1(i/\sqrt{t})))$ is a well-defined random variable.

Using that $\hat{h}_1(i/\sqrt{t}) > 0$ a.s. (since y_t for the backward Loewner differential equation is increasing in time a.s.), along with the continuity of the cotangent and argument in $\mathbb{C}$ at z for $|z| \neq 0$, we obtain that

$$\lim_{t\to\infty} ctg(arg(\hat{h}_1(i/\sqrt{t}))) = ctg(arg(\lim_{t\to\infty} \hat{h}_1(i/\sqrt{t}))) = ctg(arg(\hat{h}_1(0)))\,. \tag{6.3.4}$$

The $\lim_{t\to\infty} ctg(arg(\hat{h}_t(i)))$ does not exist a.s., however as a random variable its distribution converges to the stationary distribution of T_u started from $T_0 = 0$, since according to the Lemma 6.3.1, $u \to \infty$ as $t \to \infty$, for almost every realization of the Brownian driver. Thus, we can identify the distribution of $\lim_{t\to\infty} ctg(arg(\hat{h}_t(i)))$ with the distribution of $\lim_{t\to\infty} ctg(arg(\hat{h}_t(i/\sqrt{t})))$.

6.4 The stationary measure of the diffusion process T_u

In this section, we return to the study of

$$dT_u = -4\frac{T_u}{1+\kappa T_u^2}du + d\tilde{B}_u\,. \tag{6.4.1}$$

We recall that from Lemma 6.3.1, we obtain that $u \geqslant \log(1 + 4S/\kappa)$, for almost all the realizations of the Brownian motion. Since $\tilde{u}(S,\omega) \to \infty$ as $S \to \infty$, uniformly in $\omega \in \Omega$, questions about the long term behavior of the SDE (6.4.1) become natural. Throughout this section, we show that the there is an explicit stationary law for the SDE (6.4.1).

The density of the stationary measure for the process T_u can be computed directly via solving the stationary Kolmogorov forward equation. For the process T_u, we obtain

$$\frac{1}{2}\rho''(T) + \frac{4T}{\kappa T^2+1}\rho'(T) + \frac{(4-4\kappa T^2)}{(\kappa T^2+1)^2}\rho(T) = 0$$

The solution to the above equation process is

$$\rho(T) = C_1 T\left(\kappa T^2+1\right)^{-4/\kappa}\,{}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}; \frac{3}{2}; -T^2\kappa\right) + 2C_2\left(\kappa T^2+1\right)^{-4/\kappa}\,,$$

where C_1 and C_2 are constants to be chosen.

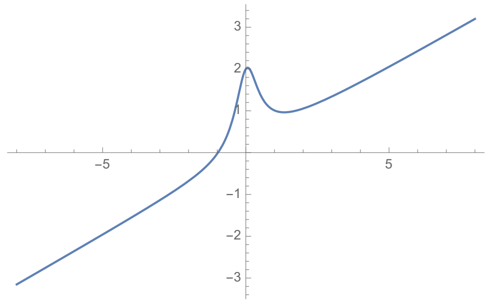

FIGURE 6.4.1. Solution of the Kolmogorov forward equation for T_u for $C_1 = C_2 = 1$ and $\kappa = 1.2$.

We observe that the explicit solution contains a function of the form

$$T\left(\kappa T^2+1\right)^{-4/\kappa}\ {}_2F_1\left(\frac{1}{2},-\frac{4}{\kappa};\frac{3}{2};-T^2\kappa\right).$$

Studying the behaviour of the Hypergeometric function near its singularities, we obtain that $\lim_{T\to\infty} T\left(\kappa T^2+1\right)^{-4/\kappa}\ {}_2F_1\left(\frac{1}{2},-\frac{4}{\kappa};\frac{3}{2};-T^2\kappa\right) \to \frac{T}{1-8\kappa}$ if $\kappa > 0$. Indeed in order to show this we have used the following splitting of hypergeometric functions

$$\begin{aligned} {}_2F_1\left(a,b;c;z\right) = &\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^a_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right) \\ &+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}{}_2F_1\left(b,b-c+1;-a+b+1;\frac{1}{z}\right) \end{aligned}$$

given that $a-b \neq \mathbb{Z}$ and $z \neq (0,1)$. Since we are searching for the solution that is a density of a (probability) distribution, we are forced to take the first constant to be zero, since otherwise due to the presence of the odd part, the solution takes negative values when defined on the whole $\mathbb{R}$ (and a density of a probability distribution is always positive). Thus, we obtain an integrable probability density on $\mathbb{R}$ for $\kappa \in [0,8)$, that has the form

$$\rho(T) = C\left(\kappa T^2+1\right)^{-4/\kappa}.$$

where $C = \left(\int_{-\infty}^{+\infty}(\kappa T^2+1)^{-4/\kappa}dT\right)^{-1}$.

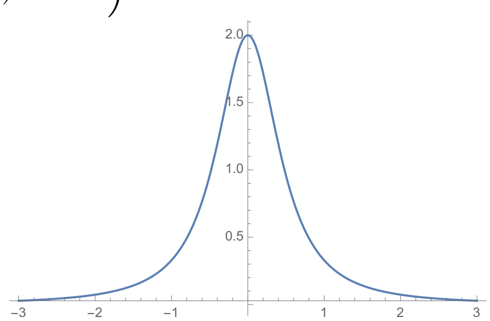

FIGURE 6.4.2. The stationary measure for the process T_u for $\kappa = 3$.

Remark 6.4.1. Changing the coordinates of from cotangent of the argument to the argument, we obtain that the density of the stationary measure can be written as $d\tilde{\rho}(\theta) = \tilde{C}\sin^{\frac{8}{\kappa}-2}(\theta)d\theta$. This function changes its behaviour at $\kappa = 4$, being concave for $\kappa < 4$, while being convex for $\kappa > 4$ At $\kappa = 4$ the function is constant. Thus, for $\kappa = 4$, we obtain a uniform distribution on $[0, \pi]$.

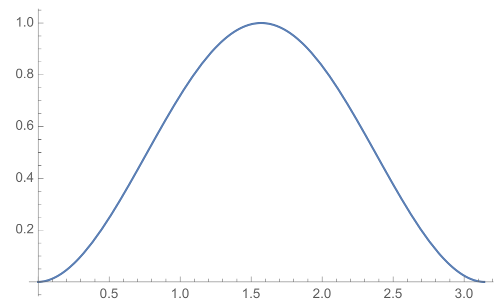

FIGURE 6.4.3. Density of the stationary measure for the argument via change of variable from the stationary measure of T_u for $\kappa = 2.3$

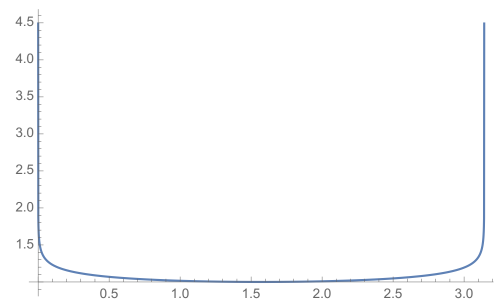

FIGURE 6.4.4. Density of the stationary measure for the argument via change of variable from the stationary measure of T_u for $\kappa = 4.5$

6.5 Proof of the main result

In this section, we give the proof of our main result. The analysis consists of two parts. First, we use a result in [32] that gives convergence to the stationary distribution of the laws of a one dimensional diffusion as time goes to infinity, in the sense of ergodic averages, for almost every Brownian path. We stress that this result gives the convergence towards the stationary law for large deterministic times, for almost every Brownian path. However, in order to link the stationary distribution of the diffusion T_u with the law of the SLE tip at a fixed capacity time, we need an estimate on the ergodic average up to a random time.

The first result that we use is the following Theorem. The process T_u satisfies the conditions of the Theorem for $\kappa < 8$.

Theorem 6.5.1 (Theorem 53.1 in [32])**.** *Let T_s be a recurrent diffusion on the real line in natural scale with speed measure m, such that $\int_{\mathbb{R}} \mu(dx) < \infty$, where $\mu(dx)$ is the stationary measure of the diffusion, that is up to a constant equal to the speed measure. Let $f \in L^1(\mu)$ and $f \geqslant 0$, then*

$$\frac{1}{t} \int_0^t f(T_s) ds \xrightarrow{a.s.} \mu(f),$$

as $t \to \infty$.

In order to obtain the control on random times, we use the previous Theorem combined with a technical estimate, captured in the next Lemma.

Lemma 6.5.2. *Let $Z_u(f) := \frac{1}{u} \int_0^u f(T_s)\mathrm{d}s$, then for $\varepsilon > 0$, we have*

$$\lim_{N \to \infty} \mathbb{P}\left[\sup_{u > N} |Z_u - \mu(f)| \geqslant \varepsilon\right] = 0.$$

Proof. By Theorem 6.5.1, we have that for a.e. ω and $\forall \varepsilon > 0$, $\exists S(\omega) > 0$ such that

$$\sup_{u > S(\omega)} |Z_u(\omega) - \mu(f)| \leqslant \varepsilon.$$

Therefore, by continuity of measures

$$\begin{aligned} 0 &= \mathbb{P}\left[\bigcap_{S=1}^{\infty} \left\{\sup_{u > S} |Z_u - \mu(f)| \geqslant \varepsilon\right\}\right] \\ &= \lim_{N \to \infty} \mathbb{P}\left[\bigcap_{S=1}^{N} \left\{\sup_{u > S} |Z_u - \mu(f)| \geqslant \varepsilon\right\}\right] \\ &= \lim_{N \to \infty} \mathbb{P}\left[\sup_{u > N} |Z_u - \mu(f)| \geqslant \varepsilon\right]. \end{aligned}$$

□

Combining the two results, we obtain the proof of the main result.

Proof of the main result. Denote $a_S := \kappa/4 \log(1 + 4S/\kappa)$. Recall that by the Lemma 6.3.1 we have $\tilde{u}(S, \omega) > a_S$, $\mathbb{P} - a.s.$ Note that $a_S \to \infty$ as $S \to \infty$. Let $\varepsilon > 0$, then by the Lemma 6.5.2, we have

$$\mathbb{P}[|Z_{\tilde{u}(S,\omega)} - \mu(f)| \geqslant \varepsilon] \leqslant \mathbb{P}\left[\sup_{u \in [a_S, \infty)} |Z_u - \mu(f)| \geqslant \varepsilon\right] \to 0$$

as $S \to +\infty$.

$\square$

In particular, this convergence result underpins a general phenomena valid for one dimensional diffusion processes and can be phrased as follows. If a one dimensional diffusion has finite mass stationary measure (speed measure), and if the a.s. convergence of the ergodic averages towards the spatial average holds when running the diffusion up to large deterministic times, then we obtain convergence in probability for the random ergodic average to the same spatial average if we consider the processes running up to random times that are diverging for almost every realization of the Brownian motion. In our case, this condition holds for the random times $u(s, \omega)$ that we consider.

6.6 The phase transition at $\kappa = 8$ and the law of the backward Loewner flow at asymptotic large times

In this section we show how stronger notions of convergence (total variation norm convergence) for Stochastic Differential Equations towards their stationary measure give information about the law of the backward SLE trace at certain times. For $\kappa = 4$, these times are exactly the times when argument of points under the backward SLE flow has a uniform distribution. This analysis is a Skorokhod type embedding problem for the backward SLE_κ flow, in the sense that we find and show how to construct times on which the law of the argument of points under this flow is uniform. Moreover, we show that this belongs to a general family of laws that are obtained in the same manner for the SLE trace for other values of $\kappa < 8$.

In this section, we consider the dynamics of the imaginary value of the backward Loewner differential equation. The random time change that we consider is a natural process that is written in terms of the imaginary value of the backward Loewner flow. Moreover, is a stochastic process that is increasing, a.s.. Thus, the natural hitting times to be considered for the process $u(s, \omega)$ are constant value levels.

For the diffusion process T_u defined by the SDE (6.4.1) we have stronger convergence results towards the stationary measure (at sequences of large deterministic times).

We use the main result from [17], that we state in the following. Let us consider the following solution of a one-dimensional SDE:

$$X_t = x + B_t - \frac{1}{2} \int_0^t b(X_s) ds,$$

with $b : \mathbb{R} \to \mathbb{R}$ a continuos function. When imposing conditions on the behavior of the function $b(x)$, we obtain various convergence results for the law of the stochastic process X_t towards its stationary measure in stronger senses than the ergodic convergence.

In [17], it is proved that if $|b(x)| \sim_{|x|\to\infty} \frac{C}{x}, C > 1, d = 1$ then, we have that there exists $\gamma > 0$ and there exists $t_0 > 0$ such that for any compact set K, there exists $C(K) > 0$ such that for any $f \in L_\infty$ and $t \geqslant t_0$, we have

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C(K)\frac{1}{t^\gamma}|f|_\infty\,, \tag{6.6.1}$$

where $P_t f(x)$ is the transition semigroup of the one dimensional diffusion X_t.

For $n \in \mathbb{N}$ let us define

$$a_n := \log\left(1 + \frac{4n}{\kappa}\right),$$

and let us consider

$$s_n(\omega) = \inf\left\{t \in [0,\infty) : \int_0^t \frac{ds}{y_s^2}(\omega) = a_n\right\}.$$

For the diffusion process T_u, we meet the required conditions on the drift for $\kappa < 8$. Thus, in our case, we take $t = a_n$, and from the Lemma 6.3.1, we have that

$$\sup_{x\in K} |P_{a_n}(f(T(x))) - \mu(f)| \leqslant C(K)\frac{1}{{a_n}^\gamma}|f|_\infty \leqslant C(K)\frac{1}{(\log(1+4n/\kappa))^\gamma}|f|_\infty.$$

This quantity converges to zero, as $n \to \infty$. Thus, on large deterministic a_n-times, we obtain in the limit the stationary law. Finally, in terms of the backward Loewner differential equation, we obtain this behavior for a sequence of large random times $s_n(\omega)$ for the process $\frac{X_s(i)}{Y_s(i)}$. In particular, on this sequence of times we converge in the total variation sense towards the uniform distribution for $\kappa = 4$. Thus, this method provides us with a resolution to a Skorokhod type embedding problem for the backward Loewner differential equation, in the sense that it identifies a sequence of random times along the backward SLE flow started from $y_0 = 1$ on which the argument of points is uniformly distributed for $\kappa = 4$. In fact this is a closed-form expression family of distributions that are recovered using the stationary law of the diffusion process T_u.

A better domain to visualize this feature is the vertical strip that is obtain from the upper half-plane by applying the Logarithm conformal transformation. In this geometry, this method provides a way to recover the sequence of times on which the law on each horizontal bar on the infinite strip is given by a closed form expression for all $\kappa < 8$. For $\kappa = 4$, this law is the uniform law on horizontal lines (since is the uniform law for the argument)

-to be further completed-

Chapter 7

Approximation via Rough Paths Theory of the backward Loewner dynamics in $\mathbb{H}$

7.1 Simulating the SLE trace with piece-wise driving function

Consider the following ODE

$$\begin{aligned}\frac{dg}{dt} &= \frac{2}{g_t - ct},\\ g(0) &= g_0,\end{aligned}$$

where $c \in \mathbb{R} \setminus \{0\}$ and $g_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 \geqslant 0$.

Consider the "shifted" process $\hat{g}_t := g_t - ct$. Then

$$\begin{aligned}\frac{d\hat{g}}{dt} &= \frac{2}{\hat{g}} - c\\ &= \frac{2 - c\hat{g}}{\hat{g}}\\ \frac{d\hat{g}}{dt} &= -\frac{d\hat{g}}{dF(\hat{g})},\end{aligned}$$

where F is defined as

$$F(x) := \frac{x}{c} + \frac{2}{c^2}\log(2 - cx).$$

It follows that

$$\frac{d}{dt}(F(\hat{g}) + t) = \frac{d\hat{g}}{dt}\frac{dF}{d\hat{g}} + 1 = -\frac{d\hat{g}}{dF(\hat{g})}\frac{dF(\hat{g})}{d\hat{g}} + 1 = 0.$$

So $F(\hat{g}) + t$ must be constant, i.e.

$$F(\hat{g}_t) = -t + c(\hat{g}_0),$$

where $c(\hat{g}_0)$ is some function of the initial condition $\hat{g}_0$. Setting $t = 0$ gives $c(\hat{g}_0) = F(\hat{g}_0)$. Thus

$$F(\hat{g}_0) = F(\hat{g}_t) + t.$$

Rearranging the above gives

$$\frac{\hat{g}_t}{c} + \frac{2}{c^2} \log(2 - c\hat{g}_t) = \frac{\hat{g}_0}{c} + \frac{2}{c^2} \log(2 - c\hat{g}_0) - t.$$

Multiplying both sides by $\frac{1}{2}c^2$ gives

$$\log(2 - c\hat{g}_t) = -\frac{c}{2}\hat{g}_t + \frac{c}{2}\hat{g}_0 + \log(2 - c\hat{g}_0) - \frac{1}{2}c^2 t.$$

After exponentiating, we have

$$2 - c\hat{g}_t = (2 - c\hat{g}_0) \exp\left(-\frac{c}{2}\hat{g}_t\right) \exp\left(\frac{c}{2}\hat{g}_0 - \frac{1}{2}c^2 t\right).$$

Diving by $2\exp\left(-\frac{1}{2}c\hat{g}_t\right)$ gives

$$\left(1 - \frac{c}{2}\hat{g}_t\right) \exp\left(\frac{c}{2}\hat{g}_t\right) = \left(1 - \frac{c}{2}\hat{g}_0\right) \exp\left(\frac{c}{2}\hat{g}_0 - \frac{1}{2}c^2 t\right).$$

Therefore

$$\left(\frac{c}{2}\hat{g}_t - 1\right) \exp\left(\frac{c}{2}\hat{g}_t - 1\right) = \left(\frac{c}{2}\hat{g}_0 - 1\right) \exp\left(\frac{c}{2}\hat{g}_0 - 1\right) \exp\left(-\frac{1}{2}c^2 t\right).$$

This can be rearranged to

$$\left(\frac{c}{2}\hat{g}_0 - 1\right) + \log\left(\frac{c}{2}\hat{g}_0 - 1\right) = \log\left(\frac{c}{2}\hat{g}_t - 1\right) + \left(\frac{c}{2}\hat{g}_t - 1\right) + \left(\frac{1}{2}c^2 t\right).$$

Note that the Wright Omega function $\omega : \mathbb{C} \to \mathbb{C}$ is defined so that

$$\omega(z) + \log(\omega(z)) = z,$$

where log denotes the principal branch of the complex logarithm. Hence

$$\frac{c}{2}\hat{g}_0 - 1 = \omega\left(\log\left(\frac{c}{2}\hat{g}_t - 1\right) + \left(\frac{c}{2}\hat{g}_t - 1 + \frac{1}{2}c^2 t\right)\right),$$

Finally, we have the formula for $\hat{g}_0$ in terms of $\hat{g}_t$.

$$\hat{g}_0 = \frac{2}{c}\left(1 + \omega\left(\log\left(\frac{c}{2}\hat{g}_t - 1\right) + \left(\frac{c}{2}\hat{g}_t - 1\right) + \left(\frac{1}{2}c^2 t\right)\right)\right)$$

There appears to be a state of the art implementation of the Wright ω function in Python.

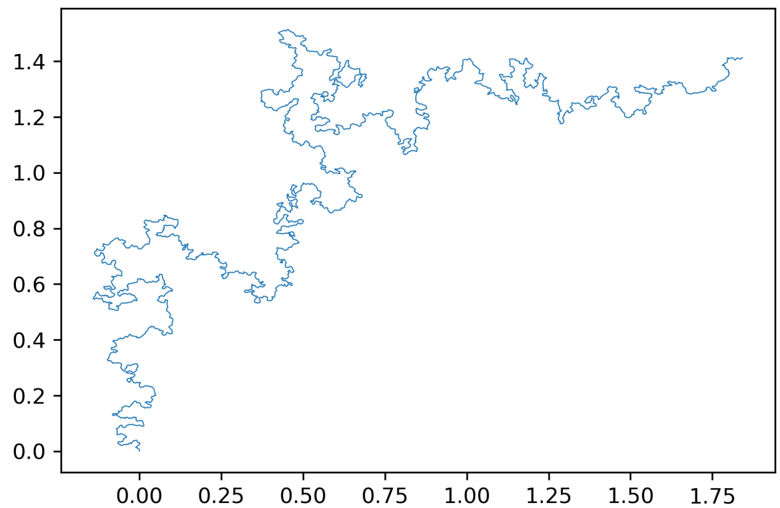

Figure 7.1.1. A sample of SLE_κ trace for $\kappa = 8/3$ for 2793 steps. Credit to James Foster.

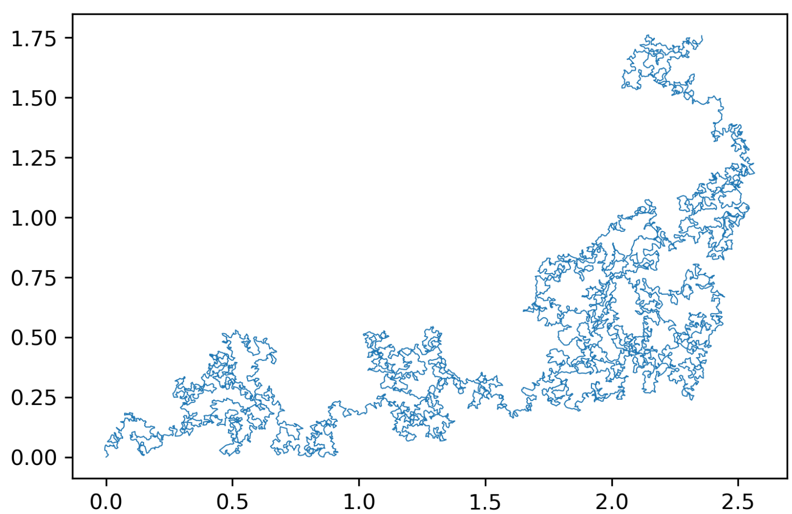

Figure 7.1.2. A sample of SLE_κ trace for $\kappa = 6$ for 9763 steps. Credit to James Foster.

7.2 Approximation of the backward Loewner flow

Let us consider the Loenwer differential equation with reference domain the upper half-plane. We first list the possible formulations of the Loewner equation in this case. The typical format of the equation is the forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa}B_t}\,, \qquad g(0,z) = z, z \in \mathbb{H}\,. \tag{7.2.1}$$

The functional inverses of this maps satisfy the partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z - \sqrt{\kappa}B_t}\,, \quad f(0,z) = z, z \in \mathbb{H}\,. \tag{7.2.2}$$

In this paper, we work mainly with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa}B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H} \tag{7.2.3}$$

By performing the identification $h_t(z) - \sqrt{\kappa}B_t = Z_t$, we obtain the following dynamics in $\mathbb{H}$, that we consider throughout this section

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t, \quad Z_0 = z_0.$$

Let $\varepsilon > 0$. Let us consider the starting point of the backward Loewner differential equation $|z_0| = \sqrt{x^2+y^2} = \varepsilon$ and let us run the equation for $t = \varepsilon^2$ amount of time.

Definition 7.2.1. *For fixed t, the quantity $A_t = \frac{1}{2}\left(\int_0^t B_s ds - \int_0^t s dB_s\right)$ is the increment of the area process between t and B_t.*

We are now ready to state our first result.

Proposition 7.2.2. *For $z = x + iy$, we have that $[\frac{-2x}{x^2+y^2}\frac{\partial}{\partial x} + \frac{2y}{x^2+y^2}\frac{\partial}{\partial y}, \sqrt{\kappa}\frac{\partial}{\partial x}] = \frac{-2\sqrt{\kappa}}{z^2}\frac{\partial}{\partial z}$. For $\varepsilon > 0$, at space scales ε and time scale ε^2 the increment of the horizontal Brownian motion B_t and the increment of A_t are uncorrelated. Moreover, they give the same order $\varepsilon > 0$ contribution in the approximation.*

The main result is the following

Theorem 7.2.3. *For $\varepsilon > 0$, the second order Taylor approximation to the backward Loewner Differential Equation at fixed time and space scales $(\varepsilon^2, \varepsilon)$ is described by the field of ellipses with semi-axis*

$$a_{1,2}(\kappa,\theta,\varepsilon) = \pm\frac{1}{\sqrt{\frac{1}{2}\left(\frac{1}{\kappa\varepsilon^2} + \frac{3}{\kappa\varepsilon^6 Im^2\frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa\varepsilon^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\frac{1}{\kappa\varepsilon^2} + \frac{3}{\kappa\varepsilon^6 Im^2\frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa\varepsilon^2}}{2}\right)^2 - \frac{12}{\kappa^2\varepsilon^8 Im^2\frac{1}{z^2}}}}}\,.$$

Moreover, at these fixed scales, the best order of convergence for the one-step Taylor approximation up to any level $N > 0$ is $O(\varepsilon)$.

Proof of the Proposition. In the following subsections, we prove the Proposition.

The exact formula for the Lie bracket. By definition,

$$\begin{aligned}
&[\frac{-2x}{x^2+y^2}\frac{\partial}{\partial x}+\frac{2y}{x^2+y^2}\frac{\partial}{\partial y},\sqrt{\kappa}\frac{\partial}{\partial x}] \\
&=\left(\frac{-2x}{x^2+y^2}\frac{\partial}{\partial x}+\frac{2y}{x^2+y^2}\frac{\partial}{\partial y}\right)\sqrt{\kappa}\frac{\partial}{\partial x}-\sqrt{\kappa}\frac{\partial}{\partial x}\left(\frac{-2x}{x^2+y^2}\frac{\partial}{\partial x}+\frac{2y}{x^2+y^2}\frac{\partial}{\partial y}\right) \\
&=-2\sqrt{\kappa}\frac{x^2-y^2}{(x^2+y^2)^2}\frac{\partial}{\partial x}+2\sqrt{\kappa}\frac{2xy}{(x^2+y^2)^2}\frac{\partial}{\partial y}\,.
\end{aligned}$$

The lack of correlation between the two normal random variables B_t and A_t for fixed time t.

When performing the chain rule, we obtain that

$$\begin{aligned}
A_t &= \frac{1}{2}\left(\int_0^t B_s ds - \int_0^t s dB_s\right) \\
&= \frac{1}{2}\left(\int_0^t B_s ds - tB_t + \int_0^t B_s ds\right) \\
&= \int_0^t B_s ds - \frac{1}{2}tB_t \\
&= \int_0^t (B_s - \frac{s}{t}B_t) ds\,.
\end{aligned}$$

Proposition 7.2.4 (Distribution of Brownian time integral)**.** *Let B be a standard one-dimensional Brownian motion. Then $\int_0^t B_s - \frac{s}{t}B_t\, ds$ and B_t are independent random variables and*

$$\int_0^t B_s - \frac{s}{t}B_t\, ds \sim \mathcal{N}\left(0, \frac{1}{12}t^3\right).$$

Proof. Recall the following well known properties of Brownian motion:

Lemma. *$\left(B_s - \frac{s}{t}B_t\right)_{0\leqslant s\leqslant t}$ and B_t are independent.*

Proof. The covariance between the two random variables is given by

$$\begin{aligned}
\operatorname{cov}\left(B_s - \frac{s}{t}B_t, B_t\right) &= \operatorname{cov}(B_s, B_t) - \operatorname{cov}\left(\frac{s}{t}B_t, B_t\right) \\
&= s - \frac{s}{t}\cdot t \\
&= 0.
\end{aligned}$$

Since $B_s - \frac{s}{t}B_t$ and B_t are jointly normal, they must be independent. $\square$

Lemma. *$\left(B_s - \frac{s}{t}B_t\right)_{0 \leqslant s \leqslant t}$ is a Gaussian process*

Proof. Let $s_1, \cdots, s_N \in [0, t]$ be a sequence of increasing points. Consider the linear combination

$$\sum_{i=1}^{N} \lambda_i \left(B_{s_i} - \frac{s_i}{t}B_t\right),$$

where $\lambda_1, \cdots, \lambda_N \in \mathbb{R}$. This can be rewritten as

$$\sum_{i=1}^{N} \lambda_i \left(B_{s_i} - \frac{s_i}{t}B_t\right) = \sum_{i=1}^{n} \lambda_i B_{s_i} - \left(\sum_{i}^{N} \frac{s_i}{t}\right) B_t.$$

Since Brownian motion is a Gaussian process, it follows that the linear combination is Gaussian. $\square$

Lemma 7.2.5. *$\left(B_s - \frac{s}{t}B_t\right)_{0 \leqslant s \leqslant t}$ has covariance function*

$$cov\left(B_s - \frac{s}{t}B_t, B_u - \frac{u}{t}B_t\right) = \min(s, u) - \frac{su}{t}.$$

Proof. As $\mathbb{E}\left[B_r - \frac{r}{t}B_t\right] = 0$ for all $r \in [0, t]$, we have

$$\begin{aligned}
cov\left(B_s - \frac{s}{t}B_t, B_u - \frac{u}{t}B_t\right) &= \mathbb{E}\left[\left(B_s - \frac{s}{t}B_t\right)\left(B_u - \frac{u}{t}B_t\right)\right] \\
&= \mathbb{E}\left[B_s B_u\right] - \frac{s}{t}\mathbb{E}\left[B_t B_u\right] - \frac{u}{t}\mathbb{E}\left[B_s B_t\right] + \frac{su}{t^2}\mathbb{E}\left[B_t^2\right] \\
&= \min(s, u) - \frac{s}{t} \cdot u - \frac{u}{t} \cdot s + \frac{su}{t^2} \cdot t \\
&= \min(s, u) - \frac{su}{t}.
\end{aligned}$$

$\square$

Since $\left(B_s - \frac{s}{t}B_t\right)_{0 \leqslant s \leqslant t}$ is a Gaussian process, it follows that $\int_0^t B_s - \frac{s}{t}B_t \, ds$ is also Gaussian.

Using Fubini's theorem, we can now compute the first two moments of $\int_0^t B_s - \frac{s}{t}B_t\,ds$:

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t B_s - \frac{s}{t}B_t\,ds\right] &= \int_0^t \mathbb{E}\left[B_s - \frac{s}{t}B_t\right]\,ds = 0,\\
\mathbb{E}\left[\left(\int_0^t B_s - \frac{s}{t}B_t\,ds\right)^2\right] &= \mathbb{E}\left[\int_0^t\int_0^t \left(B_s - \frac{s}{t}B_t\right)\left(B_u - \frac{u}{t}B_t\right)\,ds\,du\right]\\
&= \int_0^t\int_0^t \mathbb{E}\left[\left(B_s - \frac{s}{t}B_t\right)\left(B_u - \frac{u}{t}B_t\right)\right]\,ds\,du\\
&= \int_0^t\int_0^t \min(s,u) - \frac{su}{t}\,ds\,du\\
&= \int_0^t\int_0^t \min(s,u)\,ds,du - \int_0^t\int_0^t \frac{su}{t}\,ds,du\\
&= 2\int_0^t\int_0^u s\,ds\,du - \int_0^t \frac{t^2u}{2t}\,du\\
&= \int_0^t u^2\,du - \frac{t}{2}\int_0^t u\,du\\
&= \frac{1}{3}t^3 - \frac{1}{4}t^3 = \frac{1}{12}t^3.
\end{aligned}
$$

As $\left(B_s - \frac{s}{t}B_t\right)_{0\leqslant s\leqslant t}$ is independent of B_t, it follows that $\int_0^t B_s - \frac{s}{t}B_t\,ds$ is independent of B_t. $\square$

The contribution in the approximation of the dynamics for fixed $\varepsilon > 0$. Let us fix $\varepsilon > 0$. We investigate the first level terms at the fixed $(\varepsilon, \varepsilon^2)$ scales in $(|z_0|, t)$. We obtain that

$$\int_0^{\varepsilon^2} \frac{-2}{Z_t}dt \text{ is } O(\varepsilon)\,,$$

and

$$\int_0^{\varepsilon^2} \sqrt{\kappa}dB_t \text{ is } O(\varepsilon)\,.$$

Using the numerical scheme approximation we obtain at the fixed scales in space and time as before that the contribution of the second order approximation term is

$$\int_0^{\varepsilon^2} \frac{-2\sqrt{\kappa}}{Z_t^2}dA_t \text{ is } O(\varepsilon)\,,$$

since $\frac{1}{|Z_0|^2}$ is $O\left(\frac{1}{\varepsilon^2}\right)$ and $A_{\varepsilon^2} = \frac{1}{2}\left(\int_0^{\varepsilon^2} B_s ds - \int_0^t s dB_s\right)$ is $O(\varepsilon^3)$. Thus, the second order approximation term gives the same $O(\varepsilon)$ contribution at these specific scales in space and time.

The description of the ellipses corresponding to the two normal random variables B_t and A_t at a fixed time t.

Proof of the first part of the result. In this section we compute the smaller and bigger semi-axis of the ellipses (seen as the confidence intervals corresponding to the two normal random variables) as functions of ε, the argument θ and the parameter κ.

Let us consider the following linear transformation that maps the canonical orthonormal frame to the two diffusion directions in our approximation, i.e. the horizontal direction corresponding to the real Brownian motion that drives the backward Loewner differential equation and the diffusion given by the process A_t in the direction of the Lie bracket.

The linear transformation that performs this change is represented can be represented with the following matrix $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T = \begin{bmatrix} \sqrt{\kappa\varepsilon^2} & -\mathrm{Re}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} \\ 0 & -\mathrm{Im}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} \end{bmatrix}.$$

In order to find the corresponding ellipses (confidence intervals) we use the following identity

$$\left(T\begin{bmatrix} u \\ v \end{bmatrix}\right)^t \begin{bmatrix} A & B \\ C & D \end{bmatrix} T \begin{bmatrix} u \\ v \end{bmatrix} = 1$$

And we obtain that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (TT^t)^{-1}$.

We further obtain that

$$TT^t = \begin{bmatrix} \sqrt{\kappa\varepsilon^2} & -\mathrm{Re}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} \\ 0 & -\mathrm{Im}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} \end{bmatrix} \begin{bmatrix} \sqrt{\kappa\varepsilon^2} & 0 \\ -\mathrm{Re}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} & -\mathrm{Im}\frac{1}{z^2}\sqrt{\frac{\varepsilon^6}{3}\kappa} \end{bmatrix} = \begin{bmatrix} \kappa\varepsilon^2 + \mathrm{Re}^2\frac{1}{z^2}\frac{\varepsilon^6}{3}\kappa & \mathrm{Re}\frac{1}{z^2}\mathrm{Im}\frac{1}{z^2}\frac{\varepsilon^6}{3}\kappa \\ \mathrm{Re}\frac{1}{z^2}\mathrm{Im}\frac{1}{z^2}\frac{\varepsilon^6}{3}\kappa & \mathrm{Im}^2\frac{1}{z^2}\frac{\varepsilon^6}{3}\kappa \end{bmatrix}$$

Computing the determinant of TT^t we obtain $det(TT^t) = \frac{\kappa^2\varepsilon^8\mathrm{Im}^2\frac{1}{z^2}}{3}$.

We further obtain that

$$(TT^t)^{-1} = \begin{bmatrix} \frac{1}{\kappa\varepsilon^2} & \frac{-ctg(-2\theta)}{\kappa\varepsilon^2} \\ \frac{-ctg(-2\theta)}{\kappa\varepsilon^2} & \frac{3}{\kappa\varepsilon^6\mathrm{Im}^2\frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa\varepsilon^2} \end{bmatrix}.$$

In the next part we compute the eigenvalues of the matrix $(TT^t)^{-1}$. For this we compute the roots of

$$det\begin{bmatrix} \frac{1}{\kappa\varepsilon^2} - \lambda & \frac{-ctg(-2\theta)}{\kappa\varepsilon^2} \\ \frac{-ctg(-2\theta)}{\kappa\varepsilon^2} & \frac{3}{\kappa\varepsilon^6\mathrm{Im}^2\frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa\varepsilon^2} - \lambda \end{bmatrix} = 0.$$

We get that

$$\lambda^2 - \lambda \left(\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2} \right) + \frac{3}{\kappa^2 \varepsilon^8 \mathrm{Im}^2 \frac{1}{z^2}} = 0 .$$

The roots of the above polynomial are

$$\lambda_{1,2} = \frac{1}{2} \left(\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2} \right) \pm \frac{1}{2} \sqrt{ \left(\frac{\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}}{2} \right)^2 - \frac{12}{\kappa^2 \varepsilon^8 \mathrm{Im}^2 \frac{1}{z^2}} } .$$

Thus, the semi-axis of the ellipses are

$$a_{1,2}(\kappa, \theta, \varepsilon) = \pm \frac{1}{\sqrt{ \frac{1}{2} \left(\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2} \right) \pm \frac{1}{2} \sqrt{ \left(\frac{\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}}{2} \right)^2 - \frac{12}{\kappa^2 \varepsilon^8 \mathrm{Im}^2 \frac{1}{z^2}} } }} .$$

The eigenvectors directions are in the subspace given by

$$V_{1,2} = \begin{bmatrix} l \\ \frac{l \frac{-ctg(-2\theta)}{\kappa \varepsilon^2}}{\lambda_{1,2} - \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}} \end{bmatrix}$$

with $l \in \mathbb{R}$ a real parameter. Without loss of generality, we choose $l = 1$, and we obtain that

$$\begin{aligned} V_{1,2} &= \begin{bmatrix} 1 \\ \frac{\frac{-ctg(-2\theta)}{\kappa \varepsilon^2}}{\lambda_{1,2} - \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{\frac{-ctg(-2\theta)}{\kappa \varepsilon^2}}{\frac{1}{2} \left(\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2} \right) \pm \frac{1}{2} \sqrt{ \left(\frac{\frac{1}{\kappa \varepsilon^2} + \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}}{2} \right)^2 - \frac{12}{\kappa^2 \varepsilon^8 \mathrm{Im}^2 \frac{1}{z^2}} } - \frac{3}{\kappa \varepsilon^6 \mathrm{Im}^2 \frac{1}{z^2}} + \frac{ctg^2(-2\theta)}{\kappa \varepsilon^2}} \end{bmatrix} . \end{aligned}$$

$\square$

We have the following schematic representation of the field of the ellipses around the origin. The green direction is the direction of the drift and the blue and red represent the directions of the axis of the ellipses.

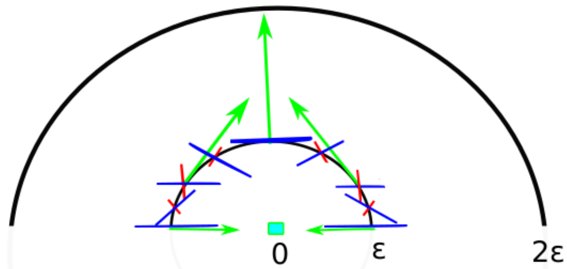

FIGURE 7.2.1. A schematic representation of the field of ellipses.

7.3 The best order of convergence for the Taylor approximation of the backward Loewner Differential Equation at fixed scales near the origin

In this section, we prove the best order of convergence for the Taylor approximation of the backward Loewner Differential Equation near the origin at fixed scales in time and space.

Rigid lower bound in the local truncation error at the natural scale $(|z_0|, t) = (\varepsilon, \varepsilon^2)$ **for strong Euler approximations via the Law of Iterated Logarithm for Brownian motion.** - to be completed- For a general controlled differential equation of the form $dY_t = f(Y_t)dX_t$, with $Y_0 = y_0 > 0$ we are interested in the quantity

$$Y_t - Y_0 - \sum_{k=1}^{N} f^{\circ k}(Y_s) \int_{s<s_1<\ldots<s_k<t} dX_{s_1} \otimes \ldots \otimes dX_{s_k},$$

where $f^{\circ k} : \mathbb{R} \to L(\mathbb{R}, \mathbb{R})$ is defined inductively as $f^{\circ 1} = f$ and $f^{\circ k+1} = D(f^{\circ k})f$. In our case, the rough path $dX_t = (dt, dB_t)$, where dB_t is a one-dimensional Brownian motion and the vector fields are $\frac{-2/\kappa}{z}\frac{\partial}{\partial z}$ and $\sqrt{\kappa}\frac{d}{dx}$ (since the real Brownian motion on the boundary) gives an horizontal diffusive movement of points under the backward Loewner flow.

The main idea of the proof is the following.

Proof of the second part of the Theorem. For $\varepsilon > 0$, let us consider the scales $(\varepsilon,\, \varepsilon^2)$ in space and time fixed as in the previous section. When considering the N-th term in the Taylor approximation for the backward Loewner Differential Equation near the origin in

$\mathbb{H}$, there will be terms of the form

$$\int_0^{\varepsilon^2} \ldots \int_0^{\varepsilon^2} dB_t dB_t ... dB_t dt$$

where dB_t appears $N-1$ times. These are the terms of the signature that give the $O(\varepsilon)$ upper bound when multiplied with the magnitude vector fields (see previous section).

The claim is that in order to obtain convergence for the Taylor approximation at these fixed scales for almost every Brownian path, the best order of convergence is $O(\varepsilon)$. In order to argue this, we use the Law of Iterated Logarithm for Brownian Motion, in the following way:

Using the Law of Iterated Logarithm for the Brownian Motion, we obtain that as $\varepsilon \to 0$, B_ε is a.s. inside the $\sqrt{\varepsilon loglog 1/\varepsilon}$ envelope. If we would have a better order of convergence at these scales, then it should be because of the iterated integrals and not because of the derivative of the vector field $\frac{-1}{z}\frac{\partial}{\partial z}$ at fixed scales $\varepsilon > 0$ in space. At these fixed scales, the infinity norm of the derivative of the vector field $\frac{-1}{z}\frac{\partial}{\partial z}$ gives $1/\varepsilon^N$ factor in front of the iterated integral $\int_0^{\varepsilon^2} \ldots \int_0^{\varepsilon^2} dB_t dB_t ... dB_t dt$. Thus, in order to obtain a better order of convergence, the contribution of the iterated integral of the term $\int_0^{\varepsilon^2} \ldots \int_0^{\varepsilon^2} dB_t dB_t ... dB_t dt$ instead of being ε^{N+1} should be $\varepsilon^{N+1+\delta}$, for $\delta > 0$.

Since the time scale is fixed to be ε^2 for $\varepsilon > 0$, then in order to obtain a better bound for $\int_0^{\varepsilon^2} \ldots \int_0^{\varepsilon^2} dB_t dB_t ... dB_t dt$ we would need that a.s., Brownian motion stays inside a 'tighter envelope' than the one given by the Law of Iterated Logarithm near the origin. This is a contradiction with the pathwise behaviour of the Brownian motion paths as $\varepsilon \to 0$. In this manner, we obtain the lower bound for the order of convergence and to this end, the best a.s. order of convergence at these natural scales for the backward Loewner Differential Equation near the origin.

□

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