An asymptotic radius of convergence for the Loewner equation and simulation of SLE_k traces via splitting

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Abstract

In this paper, we shall study the convergence of Taylor approximations for the backward Loewner differential equation (driven by Brownian motion) near a singularity. More concretely, if the initial value is sufficiently small and has size ε, we show these approximations exhibit an $O(\varepsilon)$ error provided the time horizon is $O(\varepsilon^{2+\delta})$ for $\delta > 0$. Statements of this theorem will be given using both Rough Path Theory and $L^2(\mathbb{P})$ estimates. This scaling comes naturally from the Loewner equation when growing vector field derivatives are balanced against decaying iterated integrals of the Brownian path. Ultimately, this result highlights the limitations of using stochastic Taylor methods (such as the Euler-Maruyama and Milstein methods) for approximating SLE_κ traces. Due to the analytically tractable vector fields of the Loewner equation, we will show Ninomiya-Victoir (or Strang) splitting is particularly well suited for SLE simulation. As the singularity at zero can lead to large numerical errors, we shall employ the adaptive step size control proposed in [**?**] to discretize SLE_κ traces with this splitting.

1 Introduction

The Schramm-Loewner evolution, or $SLE(\kappa)$ is a one parameter family of random planar fractal curves introduced by Schramm in [11], that are proved to describe scaling limits of a number of discrete models that appear in planar Statistical Physics. For instance, it was proved in [7] that the scaling limit of loop erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and Uniform spanning trees were proved to converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$ respectively in the series of works [15], [12], [14] and [7]. In fact, the use of Loewner equation along with the techniques of stochastic calculus, provided tools to perform a rigorous analysis of the scaling limits of the discrete models. In this framework it has been established a precise meaning to the passage to the scaling limit and its conformal invariance. We refer to [6] and [10] for a detailed study of the object and many of its properties.

Rough Path Theory was introduced in 1998 by Terry Lyons in [8]. The theory provides a deterministic platform to study stochastic differential equations which extends both Young's integration and stochastic integration theory beyond regular functions and semi-martingales. Also, Rough Path Theory provides a method of constructing solutions to differential equations driven by paths that are not of bounded variation but have controlled roughness. Step by step, we introduce the ingredients and terminology necessary to characterize the roughness of a path and to give precise meaning to natural objects that appear in the study of rough paths. We also give precise meaning to the notion of solution of a differential equation with rough driver.

Recently, many papers were written at the interface between the aforementioned domains. The paper of Brent Werness [16] defines the expected signature for the SLE_κ traces, that is the expected values of iterated integrals of the path against itself. This approach provides new ideas about how one can use a version of Green's formula for rough paths and a certain observable for SLE_κ to compute the first three elements of the expected signature of SLE_κ in the regime $\kappa \in [0, 4]$. An extension to this computation is provided in [1], where the authors show ways of computing the fourth grading of the signature (and do it explicitly for $SLE_{8/3}$, where the required observable is known), along with several other parts of the higher grading. From a different perspective, in [2] the authors question the existence of the trace for a general class of processes (such as semimartingales) as a driving function in the Loewner differential equation. These ideas are also developed in a Rough Path Theory flavour. More recently, Peter Friz with Huy Tran in [3] revisited the regularity of the SLE_κ traces and obtained a clear result using Besov spaces type analysis. Also, Atul Shekhar, Huy Tran and Yilin Wang studied the continuity of the traces generated by Loewner chains driven by bounded variation drivers in [13] .

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2 Rough Path Theory overview

Let $X_{[s,t]}$ denote the restriction of the path X to the compact interval $[s,t]$. We introduce the notion of p-variation.

Definition 2.1. *Let V be a finite dimensional real vector space with dimension d and basis vectors $e_1, \ldots, e_d$.. The p-variation of a path $X : [0,T] \to V$ is defined by*

$$||X_{[0,T]}||_{p-var} := \sup_{\mathcal{D}=(t_0,t_1,\ldots,t_n)\subset[0,T]} \left(\sum_{i=0}^{n-1} d(X_{t_i}, X_{t_{i+1}})^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all finite partitions of the interval $[0,T]$.

Throughout the paper we use the notation $X_{s,t} = X_t - X_s$. For $T > 0$, let us define $\Delta_T = \{(s,t)|0 \leqslant s \leqslant t \leqslant T\}$. We introduce next the fundamental notion of control.

Definition 2.2. *A control on $[0,T]$ is a non-negative continuous function*

$$\omega : \Delta_T \to [0,\infty)$$

for which

$$\omega(s,t) + \omega(t,u) \leqslant \omega(s,u),$$

for all $0 \leqslant s \leqslant t \leqslant u \leqslant T$, and $\omega(t,t) = 0$, for all $t \in [0,T]$.

We introduce also the following

Definition 2.3. *Let $T((V)) := \{\boldsymbol{a} = (a_0, a_1, \cdots) : a_n \in V^{\otimes n} \ \ \forall n \geqslant 0\}$ denote the set of formal series of tensors of V .*

Definition 2.4. *The tensor algebra* $T(V);=\bigoplus_{k\geqslant 0}V^{\otimes k}$ *is the infinite sum of all tensor products of* V.

Let $e_1,e_2,\ldots,e_d$ be a basis for V. The space $V^{\otimes k}$ is a d^k dimensional vector space with basis elements of the form $(e_{i_1}\otimes e_{i_2}\ldots\otimes e_{i_k})_{(i_1,\ldots,i_k)\in\{1,\ldots,d\}^k}$. We store the indices $(i_1,\ldots,i_k)\in\{1,2,\ldots d\}^k$ in a multi-index I and let $e_I=e_{i_1}\otimes e_{i_2}\otimes\ldots e_{i_k}$. The metric $||\cdot||$ on $T((V))$ is the projective norm defined for

$$x=\sum_{|I|=k}\lambda_I e_I\in V^{\otimes k}$$

via

$$||x||=\sum_{|I|=k}|\lambda_I|.$$

Thus, the bound $||X^i_{s,t}||\leqslant\frac{w(s,t)^{i/p}}{\beta(\frac{i}{p})!}$, $\forall i\geqslant 1$, $\forall(s,t)\in\Delta_T$, gives control on the sum of i-iterated integrals. We collect all the iterated integrals in the following way. We consider for

$$X:\Delta_T\to T((\mathbb{R}))$$

the collection of iterated integrals as

$$(s,t)\to\mathbf{X}_{s,t}=(1,X^1_{s,t},\ldots,X^{\lfloor p\rfloor}_{s,t},\ldots,X^m_{s,t},\ldots)\in T((V)).$$

We call the collections of iterated integrals the signature of the path X. Throughout our analysis, we use the following fundamental result regarding paths with bounded variation.

Proposition 2.5 (Proposition 2.2 of [9])**.** *Let* $X:[0,T]\to V$ *be a path of bounded variation. Then it follows that*

$$||\int_{0<u_1<\ldots u_k<t}dX_{u_1}\otimes dX_{u_2}\ldots dX_{u_k}||\leqslant\frac{||X_{[0,t]}||^k_{1-var}}{k!}.$$

We now define the notion of *multiplicative functional.*

Definition 2.6. *Let* $n\geqslant 1$ *be an integer and let* $X:\Delta_T\to T^{(n)}(V)$ *be a continuous map. Denote by* $X_{s,t}$ *the image of the interval* (s,t) *by* X, *and write*

$$X_{s,t}=(X^0_{s,t},\ldots X^n_{s,t})\in\mathbb{R}\oplus V\oplus V^{\otimes 2}\ldots\oplus V^{\otimes n}.$$

The function X *is called multiplicative functional of degree* n *in* V *if* $X^0_{s,t}=1$ *and for all* $(s,t)\in\Delta_t$ *we have*

$$X_{s,u}\otimes X_{u,t}=X_{s,t}\ \ \forall s,u,t\in[0,T].$$

We will use the notion of p-rough path that we define in the following.

Definition 2.7. *A p-rough path of degree n is a map $X : \Delta_T \to \tilde{T}^{(n)}(V)$ which satisfies Chen's identity $X_{s,t} \otimes X_{t,u} = X_{s,u}$ and the following 'level dependent' analytic bound*

$$||X^i_{s,t}|| \leqslant \frac{w(s,t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!},$$

where $y! = \Gamma(y+1)$ whenever y is a positive real number and β_p, is a positive constant.

In general, for a Rough Differential Equation of the form

$$dZ_t = V(Z_t)dX_t,$$

with $Z_0 = z_0 > 0$ with $X_t : [0,T] \to \mathbb{R}^d$ a finite p-variation path for any $p > 2$, we use the following compact notation for the first r terms of the Taylor approximation.

Definition 2.8. *Given the continuously differentiable vector fields $(V_1, \ldots, V_d)$ on $\mathbb{R}^e$, and a multiplicative functional with finite p-variation, $\mathbf{X} \in T((\mathbb{R}^d))$, we define*

$$\mathcal{E}^r_{(V)}(Z, \mathbf{X}_{0,t}) := \sum_{k=1}^{r} \sum_{i_1,\ldots,i_k} V_{i_1} \ldots V_{i_k} \mathbf{Id}(Z) \mathbf{X}^{k,i_1,\ldots,i_k}_{0,t}$$

as the increment of the step r-truncated Taylor approximation on the interval $[0,t]$. The notation $V_{i_1} \ldots V_{i_k} \mathbf{Id}(Z)$ stands for the composition of differential operators associated with the vector fields

$$V_{i_1} \circ V_{i_2} \circ \ldots \circ V_{i_k} \circ \mathbf{Id}(Z),$$

and $\mathbf{X}^{k,i_1,\ldots,i_k}_{0,t}$ stands for the terms obtained from the iterated integrals

$$\int_{0<s_1<..<s_k<t} dX_{s_1} \otimes ... \otimes dX_{s_k}.$$

3 Asymptotic growth of vector fields for the Loewner Equation

In order to prove the main result, we first prove the following lemmas in this section.

Lemma 3.1. *At fixed level r there are 2^r terms obtained from all the possible ways of composing the vector fields $V_1 = \frac{2/\kappa}{Z}\frac{d}{dz}$ and $V_2 = \frac{d}{dz}$.*

Proof. We prove this using induction. For $r = 1$, there are $2 = 2^1$ possible terms obtained from either of the vector fields. For $r = 2$, the possible compositions are $V_1 \circ V_1$, $V_2 \circ V_2$, $V_2 \circ V_1$ and $V_1 \circ V_2$ given 2^2 possibilities. Let us assume that at level $k > 0$ there are 2^k possible combinations. To obtain all the possible compositions at level $k + 1$, we have to consider $V_1 \circ V^k$ and $V_2 \circ V^k$ where V^k are all the possible compositions at level k. Thus, at level $k + 1$ we obtain in total $2^k + 2^k = 2^{k+1}$ possibilities, and the argument follows by induction. □

Lemma 3.2. *Let $r = m + n$, with n being the number of dB_t entries and m being the number of dt entries in the r-level iterated integral $\int_{0<s_1<\ldots<s_r<t} dX_{s_1}\ldots dX_{s_r}$. Then, for $Z_0 = \varepsilon > 0$, we have that*

$$|V_{i_1} \ldots V_{i_r}\mathbf{Id}(Z_0)| = O\left(\frac{1}{\varepsilon^{2m-1+n}}\right).$$

Proof. Given the format of the forward Loewner differential equation extended on the real line, we have that the vector fields that can appear are either $V_1 = \frac{2/\kappa}{Z}\frac{d}{dz}$ or $V_2 = \frac{d}{dz}$. For fixed values of m and $n = r - m$ we have, by definition m time entries in the iterated integrals and m times the vector field $V_1 = \frac{2/\kappa}{Z}\frac{d}{dz}$ and n dA_t entries together with n times the vector field $V_2 = \frac{d}{dz}$. We also note that in order for $V_{i_1} \ldots V_{i_r}\mathbf{Id}(z)$ to be non zero then $V_{i_r} = \frac{1}{Z}$, otherwise $V_{i_r} = 1$ and applying any other choice of $V_{i_{r-1}}$ will give the derivative of a constant that is zero.

Then, we provide the following rules when considering the composition of vector fields $V_{i_1} \ldots V_{i_r}\mathbf{Id}(z)$ (up to some absolute constants that change, but we avoid keeping their dependence in our analysis). These rules are specific to the structure of the vector fields of the Loewner differential equation and they give a way to transform the composition of the differential operators associated with vector fields in the left into multiplication with the function on the right up to some constants (that we do not keep track of since they do not influence the analysis)

- $\frac{2/\kappa}{Z}\frac{d}{dz} \longleftrightarrow \frac{-2/\kappa}{Z^2}$
- $\frac{d}{dz} \longleftrightarrow \frac{-1}{Z}$

in the following sense: $V_1 \circ V^k = \frac{-2/\kappa}{Z^2} f(Z)$ and $V_2 \circ V^k = \frac{-1}{Z} f(Z)$, where we have that $f(z) = V_{i_1} \circ V_{i_2} \circ \ldots \circ V_{i_k} \circ \mathbf{Id}(Z)$ for any $k > 0$.

To illustrate this, we consider

$$V_1 \circ V_1 \circ \mathbf{Id}(Z) = \frac{1}{Z}\frac{d}{dz}\frac{1}{Z}.$$

Then, $\frac{d}{dz}\frac{1}{Z} = \frac{-1}{Z^2}$ and $V_1 \circ V_1 \circ \mathbf{Id}(Z) = \frac{1}{Z^3}$ up to some constants. In general, the analysis is similar. Indeed, the result of the composition is obtained by iteration using either of the vector fields applied to the initial $\frac{2/\kappa}{Z}$ value (since otherwise we would obtain zero since we apply differential operators to a constant), and since the differential operators either act by differentiation or by differentiation and multiplication with $\frac{1}{Z}$ up to some constants, the rules hold. Using these rules, independent of the order of applications of the vector fields we have that for $Z_0 = \varepsilon > 0$

$$|V_{i_1} \ldots V_{i_r}\mathbf{Id}(Z_0)| = O\left(\frac{1}{\varepsilon^n}\frac{1}{\varepsilon^{2m-1}}\right) = O\left(\frac{1}{\varepsilon^{2m-1+n}}\right).$$

□

4 Truncated Taylor approximation for the backward/forward Loewner differential equation driven by Brownian motion

4.1. Taylor approximations for the backward Loewner differential equation. In this section, we work with the backward Loewner differential equation driven by Brownian motion.

By performing the identification $h_t(z) - B_t = Z_t$, we obtain the following dynamics in $\mathbb{H}$, that we consider throughout this section

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t, \qquad Z_0 = z_0 \in \mathbb{H}.$$

Let $\varepsilon > 0$. Let us consider the starting point of the backward Loewner differential equation $z_0 \in \mathbb{H}$ with $|z_0| = \sqrt{x_0^2 + y_0^2} = \varepsilon$ and let us run the equation for $t = \varepsilon^2$ amount of time.

We note that we can redo the estimate in the Lemma (3.2) in a straightforward manner: The iterated integrals that appear in the remainder are the same, so the same estimates hold for them. The only difference is that the vector fields are $V_1 = \frac{-2/\kappa}{z}\frac{d}{dz}$ and $V_2 = -\frac{d}{dx}$. Note that from the Cauchy-Riemann equations we deduce that for complex differentiable functions the linear differential operators $\frac{d}{dx}$ and $\frac{d}{dz}$ are equivalent when acting on complex differentiable functions, where $\frac{d}{dz}$ denotes the complex differentiation.

In particular, we have the same rules as in Lemma (3.2) written in the language of complex differentiation. As in the previous case, these rules give a way to transform the composition of the vector field in the left equivalent with multiplication with the function on the right.

- $\frac{2/\kappa}{z}\frac{\partial}{\partial z} \longleftrightarrow \frac{-2/\kappa}{z^2}$
- $\frac{\partial}{\partial z} \longleftrightarrow \frac{-1}{z}$

Thus, independent of the order of applications of the vector fields we have that for $Z_0 = \varepsilon > 0$

$$|V_{i_1} \dots V_{i_r} \mathbf{Id}(Z_0)| = O\left(\frac{1}{\varepsilon^n}\frac{1}{\varepsilon^{2m}}\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\varepsilon^{2m+n+1}}\right).$$

Thus, using the estimates on the iterated integrals along with the estimates on the vector fields we obtain the desired conclusion.

Remark 4.1. Note that since the imaginary part of the backward Loewner differential equation satisfies the equation $dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt$, we obtain that the imaginary part is always increasing. Thus, we can concatenate the estimates and obtain from the one-step Taylor approximation a multi-step Taylor approximation. Indeed since the imaginary part increases, the contribution of the absolute value of the vector fields in the remainder becomes smaller, and thus the same time step works. We refer to Section 10.3.5 in [4] for more details.

4.2. Taylor approximations for the forward Loewner differential equation. Let us consider the following dynamics on the real line

$$dZ_t = \frac{2/\kappa}{Z_t}dt - dB_t \quad Z_0 = \varepsilon > 0, \tag{4.1}$$

where B_t is a standard Brownian motion.

The first order differential operators $\frac{2/\kappa}{Z}\frac{d}{dz}$ and $\frac{d}{dz}$ are associated to the vector fields of (4.1). Note that since our process Z_t is one-dimensional (and therefore there is a unique notion of directional derivative), we abuse the notation and use for the $\mathbb{R}$-differentiation the notation $\frac{d}{dz}$, in order to match it with the notation of the functions that is acting on.

5 Asymptotic decay of the iterated integrals and error estimates

5.1. Rough Path Theory analysis. We first show that the pair (t, B_t) is a p-rough path for $p > 2$ in the next section.

5.2. The pair $X_t = (t, B_t)$ as a p-rough path for $p > 2$. In this subsection we show that the pair $X_t = (t, B_t)$ can be lifted to a p-rough path for $p > 2$ and moreover the lift is uniquely determined by the first two levels.

The main ingredient that we use is the Extension Theorem, Theorem 3.7 of [9], that we state in the following.

Theorem 5.1 (Extension Theorem, Theorem 3.7 of [9])**.** *Let $p \geqslant 1$ be a real number and $n \geqslant 1$ be an integer. Let $X : \Delta_T \to T^{(n)}(V)$ be a multiplicative functional with finite p-variation controlled by w. Assume that $n \geqslant \lfloor p \rfloor$. Then there exists a unique extension of X to a multiplicative functional $\Delta_T \to T((V))$ that has finite p-variation, i.e. for every $m \geqslant \lfloor p \rfloor + 1$, there exists a unique continuous function*

$$X^m : \Delta_T \to V^{\otimes m}$$

such that $(s,t) \to \mathbf{X}_{s,t} = (1, X^1_{s,t}, \dots, X^{\lfloor p \rfloor}_{s,t}, \dots, X^m_{s,t}, \dots) \in T((V))$, is a multiplicative functional with finite p-variation controlled by the same w, i.e.

$$||X^i_{s,t}|| \leqslant \frac{w(s,t)^{i/p}}{\beta(\frac{i}{p})!} \quad \forall i \geqslant 1, \;\; \forall (s,t) \in \Delta_T,$$

with $\beta_p = p^2 \left(1 + \sum\limits_{r=3}^{\infty} \left(\frac{2}{r-2} \right)^{\frac{\lfloor p \rfloor + 1}{p}} \right)$.

Using the definition

Definition 5.2. *For the k-multi index $R = (r_1, \dots, r_l)$ we denote the length by $||R|| = l$. Furthermore, we define the function $n_j(R) := card\{i : r_i = j, r_i \in R\}$. We introduce the $\Pi -$ degree of R as $deg_\Pi(R) = \sum_{j=1} k \frac{n_j(R)}{p_j}$.*

in [5] is obtained a more precise estimate that takes into account the different regularities of the Brownian motion and of the time component.

In the last part, for the real numbers $\varepsilon \geqslant 0$ and $\alpha > 0$ we will consider pairs $s, t \in [0, \varepsilon^\alpha]$ and we will estimate the corresponding iterated integrals.

In order to check that the pair $X_t = (t, B_t)$ can be lifted to a p-rough path for any $p > 2$, we have to check according to the definition, the algebraic identities (Chen's relations) and the analytic bounds. The algebraic identities follow from Fubini's Theorem which Riemann-Stieltjes and Itô integrals satisfy. Since we consider iterated integrals of a pair consisiting of a bounded variation path with some other p-variation path for any $p > 2$ the algebraic requirements are straightforward. We check the analytic bounds in the following. For the pair $X_t = (t, B_t)$, we have that the second component has clearly finite p-variation for $p > 2$ since it is of bounded variation. Using standard result in Rough Path Theory (see [4]) we have that B_t has finite p-variation for $p = 2 + q$ for any $q > 0$.

We show that there exists a control $\omega(s,t)$ such that

$$\left| \int\int_{s<u_1<u_2<t} dB_{u_1} du_2 \right| \leqslant C_1 \omega(s,t)^{2/p}$$

$$\left| \int\int_{s<u_1<u_2<t} du_1 dB_{u_2} \right| \leqslant C_2 \omega(s,t)^{2/p}$$

and

$$\left| \int\int_{s<u_1<u_2<t} dB_{u_1} dB_{u_2} \right| \leqslant C_3 \omega(s,t)^{2/p}$$

where C_1, C_2 and C_3 are a.s. finite random constants.

Since the Brownian path is weakly 1/2-Hölder continuous, we will take as a control the function $w(s,t) = (t-s)$. In particular, we know from standard properties of Brownian motion that it is of finite p-variation for $p = 2+q$ for any $q > 0$. For any pair $s, t \in \Delta_T$, the first iterated integral has finite $p/2$-variation since it is a Riemann-Stieltjes integral due to the fact that u_2 is of bounded variation. Moreover, the integral is controlled by $w(s,t) = (t-s)^{2/p}$, since we have $(t-s) = (t-s)^{2/p}(t-s)^{1-2/p}$. For the second iterated integral, for any pair $s, t \in \Delta_T$ we use the integration by parts for the Itô integrals. Since one of the terms is of finite variation, the formula gives

$$\int_s^t (u_2 - s) dB_{u_2} = (t-s)B_t - \int_s^t (B_{u_2} - B_s) du_2. \tag{5.1}$$

We further have that

$$\left| \int_s^t (u_2 - s) dB_{u_2} \right| \leqslant |(t-s)B_t| + \left| \int_s^t (B_{u_2} - B_s) du_2 \right| \tag{5.2}$$

The control for the second term in the right hand side of (5.2) is obtained as in the previous case,since it is an example of a Riemann-Stieltjes integral bounded by $\omega(s,t) = (t-s)^{2/p}$. The first term in the right hand side of (5.2) is bounded directly by $\omega(s,t) = (t-s)^{2/p}$ using that $(t-s)$ is a bounded variation function that in particular has finite p-variation for any $p > 2$, since $(t-s)B_t \leqslant (t-s)^{2/p}(t-s)^{1-2/p}B_t$. Using that B_t is an a.s. finite random variable, we obtain the required estimate.

For the last iterated integral, we obtain via direct computations

$$\begin{aligned}
\left|\int_s^t (B_r - B_s)dB_r\right| &= \left|\int_s^t B_r dB_r - B_s(B_t - B_s)\right| \\
&= \left|\frac{B_t^2 - B_s^2}{2} - \frac{t-s}{2} - B_s B_t + B_s^2\right| \\
&= \left|\frac{B_t^2}{2} + \frac{B_s^2}{2} - \frac{2B_s B_t}{2} - \frac{t-s}{2}\right| \\
&= \left|\frac{(B_t - B_s)^2}{2} - \frac{t-s}{2}\right| \\
&< C_3(t-s)^{2/p}.
\end{aligned} \tag{5.3}$$

The last inequality holds since $|B_t - B_s| \leqslant C\omega(s,t)^{1/p}$ and thus

$$|B_t - B_s|^2 \leqslant C^2 \omega(s,t)^{2/p},$$

where C is a random constant that is a.s. finite. Finally, $(t-s) \leqslant (t-s)^{1-\frac{2}{p}}(t-s)^{2/p}$.

Taking

$$c_m(\omega) := \max(C_1(\omega)^{1/2}, C_2(\omega)^{1/2}, C_3(\omega)^{1/2}, C_0(\omega), C_1(\omega)),$$

where $C_0(\omega)$ and $C_1(\omega)$ are the a.s. finite constants obtained by bounding the increments of the path (t, B_t) at the first level, we define the control

$$w_1(s,t) = c_m(\omega)(t-s).$$

Note that the choice of the constants β_p and $(\frac{i}{p})!$ does not change the analysis. Their choice is usually related with the fact that they simplify computations when Extension Theorem is applied.

With the previous estimates and the choice of control $w_1(s,t) = c_m(\omega)(t-s)$, we check the conditions of Theorem 5.1. In this manner, we obtain the unique Itô lift up to any order $n \geqslant 2$ for the path $X_t = (t, B_t)$.

6 Weak stability of the Truncated Taylor approximation associated with the backward Loewner Differential Equation and Real Bessel Process on the real line

In this section, we will be focusing on the truncated Taylor approximation of the backward and forward Loewner differential equation up to some fixed finite $r > 0$ only. We will analyze the Taylor approximation obtained in this context and we prove a property that we call weak stability of this Taylor approximation series truncated at a finite order. This property of the Taylor approximation in the context of singular SDEs is related with the structure of the singularity of the vector field, the roughness of the driver, and the symmetry of the equation The analysis is based on all the estimates obtained in the previous sections of the paper.

Definition 6.1. *Let $\alpha > 0$. We call a one time step r-truncated Taylor approximation for a differential equation started from $y_0 = 0$ weakly α stable if for pairs consisting of initial point and time step used in the approximation $(y_0^{\varepsilon}, t^{\varepsilon}) = (\varepsilon, \varepsilon^{\alpha})$ we have that the approximation converges as $\varepsilon \to 0$ to the initial value $y_0 = 0$, a.s..*

We prove our main result :

Theorem 6.2. *The one time step r-truncated Taylor approximation for the backward Loewner differential equation and for the Bessel process started from $\varepsilon > 0$ obtained by continuously extending the conformal maps solving the forward Loewner differential equation to the real line it is a.s. weakly α stable for time steps $t < \varepsilon^{2+\delta}$, for any $\delta > 0$. Moreover, the r-truncated Taylor approximations are weakly α-stable for time steps $t < \varepsilon^2$.*

7 Proof of Theorem 6.2

Using as before that these rules, independent of the order of applications of the vector fields we have that for $Z_0 = \varepsilon > 0$

$$|V_{i_1} \dots V_{i_r} \mathbf{Id}(Z_0)| = O\left(\frac{1}{\varepsilon^n} \frac{1}{\varepsilon^{2m-1}}\right) = O\left(\frac{1}{\varepsilon^{2m-1+n}}\right).$$

Furthermore, we consider $\Delta_T = \Delta_{\varepsilon^\alpha}$ for $\alpha > 0$. Thus, at any fixed level k, $k = m + n$, $k \in [0, r]$, we have

$$\sum_{k=1}^{r} \sum_{i_1,\dots,i_k} V_{i_1} \dots V_{i_k} \mathbf{Id}(Z) \mathbf{X}_{0,t}^{k,i_1,\dots,i_k} = \sum_{k=1}^{r} 2^k \frac{C(k)}{\varepsilon^{2m+n-1}} \int \dots \int_{0 \leqslant s_1 \leqslant s_2 .. \leqslant \varepsilon^\alpha} dX_{s_1} \dots dX_{s_k}.$$

For each $k \in [0, r]$, there are terms that are zero (since they correspond to a particular order of application of the vector fields as explained before).

Using the estimate on the iterated integrals for $s = 0$ and $t = \varepsilon^\alpha$ from Theorem 5.1[gyurko], we obtain that

$$\sum_{k=1}^{r} 2^k \frac{C(k)}{\varepsilon^{2m+n-1}} \int \dots \int_{0 \leqslant s_1 \leqslant s_2 .. \leqslant \varepsilon^\alpha} dX_{s_1} \dots dX_{s_k} \leqslant \sum_{k=1}^{r} \frac{\tilde{C}(k)}{\varepsilon^{2m+n-1}} \varepsilon^{\frac{n\alpha}{2+\delta} + m\alpha},$$

where $\tilde{C}(k)$ is a constant obtained by merging all the constants obtained from the iterated integrals estimates and from the operation of composition of the vector fields. Thus, we obtain that

$$\frac{n\alpha + m(2+\delta)\alpha - (2+\delta)(2m+n-1)}{2+\delta} > 0 \Leftrightarrow \alpha > \frac{(2+\delta)(2m+n-1)}{(2+\delta)m + n}$$

In order to guarantee the control of all terms in the Taylor approximation, we need

$$\alpha > \limsup_{m,n \to \infty} \frac{(2+\delta)(2m+n-1)}{(2+\delta)m + n} = 2 + \tilde{\delta},$$

with $\tilde{\delta} = 2\delta$ and the conclusion of the theorem follows.

Remark 7.1. Note that for pairs of the form $(y_0, t^\varepsilon) = (u, \varepsilon^\alpha)$ for some fixed constant $u > 0$, the convergence holds for any value of $\alpha > 0$. However, when dealing with singular SDEs, one is interested in considering starting the equation closer to the singularity. The estimates that we use show that the time step one needs to take has to be small enough to be able to control the output of the approximation as $\varepsilon \to 0$, for almost every Brownian path.

Remark 7.2 (The typical estimates from Rough Path Theory do not work for the non-truncated Taylor approximation in our setting)**.** Note that the usual estimates on the iterated integrals from Rough Path Theory do not ensure convergence for the Taylor approximation in the case of singular differential equations. Let us consider the absolute value of the r-level Taylor approximation remainder

$$R_r(t, Z_t, \mathbf{X}) = \pi(0, Z_0; X)_{0,t} - \mathcal{E}_{(V)}(Z_0, \mathbf{X}_{0,t}^{(r-1)}).$$

Using the previous analysis, we have that the remainder is equal to

$$\sum_{\tilde{r}\geqslant r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n-1}} \int \dots \int_{0\leqslant s_1 \leqslant s_2 .. \leqslant \varepsilon^{\alpha}} dX_{s_1} \dots dX_{s_{\tilde{r}}}.$$

Using the estimate on the Extension Theorem 5.1 for the iterated integrals we have that

$$\sum_{\tilde{r}\geqslant r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n+1}} \int \dots \int_{0\leqslant s_1 \leqslant s_2 .. \leqslant \varepsilon^{\alpha}} dX_{s_1} \dots dX_{s_{\tilde{r}}} \leqslant \sum_{\tilde{r}\geqslant r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n-1}} \varepsilon^{\frac{(m+n)\alpha}{2+\delta}}$$

Thus, the estimates do not work since the series corresponding to the upper bound diverges. Indeed, the factor $C(\tilde{r})$ grows like $\frac{\tilde{r}!}{(\frac{\tilde{r}}{p})!}$ for $p > 2$. The $\tilde{r}!$ term is obtained taking $\tilde{r}$ derivatives of the function $\frac{1}{z}$ while the term $(\frac{\tilde{r}}{p})!$ comes from the bound on the iterated integrals.

However, the weak stability of the Truncated Taylor Approximation in the context of singular differential equations proved before is interesting in its own right when considering the Taylor approximations associated with rough differential equations with singularities.

7.1. $L^2(\mathbb{P})$ error analysis. Consider the backwards Loewner equation

$$dZ_t = -\frac{2}{Z_t}\, dt + \sqrt{\kappa}\, dB_t\,,$$

where $\varepsilon := \mathrm{Im}(Z_0) > 0$. Since the imaginary part of the solution is increasing for all $t \geqslant 0$, the dt vector field becomes smooth and bounded on the domain $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant \varepsilon\}$. Moreover, this argument also shows that the derivatives of the dt vector field are bounded. We shall denote these vector fields on $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant \varepsilon\}$ as follows:

$$V_0(z) := -\frac{2}{z},$$
$$V_1(z) := \sqrt{\kappa}.$$

Given a multi-index $I = (i_1, \cdots, i_k) \in \{0,1\}^*$, we define the degree of I by

$$\deg(I) = \sum_{j=1}^{k} (2 - i_j).$$

This means we can actually quote the following result (Prop 1.1 from [**?**]) that

$$Z_t = \sum_{\substack{I=(i_1,\cdots,i_k)\in\{0,1\}^* \\ \deg(I)\leqslant m, k\in\mathbb{N}}} V_{i_1} \cdots V_{i_k} \mathrm{Id}(Z_0) B_t^I + R_m(t, Z_0, f),$$

where

$$B_t^I := \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} \circ\, dB_t^{i_1} \circ \cdots \circ dB_t^{i_k}$$

Here we use the fact that some of the entries $dB_t^{i_j}$ are dt entries. and there exists a positive constant C such that for $t \leqslant 1$ we have

$$\|R_m(t, Z_0, f)^2\|_{L^2(\mathbb{P})} \leqslant Ct^{\frac{m+1}{2}} \sup_{\substack{I=(i_1,\cdots,i_k)\in\{0,1\}^* \\ m<\deg(I)\leqslant m+2, k\in\mathbb{N}}} \|V_{i_1}\cdots V_{i_k}\mathrm{Id}\|_\infty.$$

We then have to prove that there exists a positive constant C_I, which depends only on $I = (i_1, \cdots, i_k) \in \{0,1\}^*$, such that

$$\sup_{\substack{I=(i_1,\cdots,i_k)\in\{0,1\}^* \\ m<\deg(I)\leqslant m+2, k\in\mathbb{N}}} \|V_{i_1}\cdots V_{i_k}\mathrm{Id}\|_\infty \leqslant C_I \cdot \frac{1}{\mathrm{Im}(Z_0)^m} = O(\varepsilon^{-m}).$$

8 Simulation of SLE traces using the Ninomiya-Victoir splitting

In order to simulate an SLE trace, we must first discretize the backward Loewner equation,

$$\begin{aligned} dZ_t &= -\frac{2}{Z_t}\,dt + \sqrt{\kappa}\,dB_t\,, \\ Z_0 &= \varepsilon\,, \end{aligned} \tag{8.1}$$

Since the above SDE gives an explicit solution in the zero noise case (i.e. when $\kappa = 0$), it is natural to apply a splitting method to approximate its solution. Moreover, as (8.1) can be viewed in Stratonovich form, such a method can be interpreted as the solution of an ODE / RDE governed by the same vector fields but driven by a piecewise linear path. Unfortunately, the convergence results of [13] are not applicable if this has vertical pieces. A well-known splitting method for (Stratonovich) SDEs is the Ninomiya-Victoir scheme, originally proposed in [**?**], which in our setting directly corresponds to the Strang splitting.

Definition 8.1 (Ninomiya-Victoir scheme for SDEs driven by a single Brownian motion)**.** *Consider an n-dimensional Stratonovich SDE on the interval $[0, T]$ with the following form*

$$\begin{aligned} dY_t &= V_0(Y_t)\,dt + V_1(Y_t)\circ dB_t\,, \\ Y_0 &= \xi\,, \end{aligned} \tag{8.2}$$

where $\xi \in \mathbb{R}^n$ and the vector fields $V_i : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be Lipschitz continuous. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\exp(tV_i)\,x$ denote the unique solution at time $u = 1$ of the ODE

$$\frac{dy}{du} = tV_i(y)\,,$$
$$y(0) = x\,.$$

For a fixed number of steps N we can construct a numerical solution $\{\widetilde{Y}_{t_k}\}_{0\leqslant k\leqslant N}$ of (8.2) by setting $\widetilde{Y}_0 := \xi$ and for each $k \in [0\,..\,N-1]$, defining $\widetilde{Y}_{t_{k+1}}$ using a sequence of ODEs:

$$\widetilde{Y}_{t_{k+1}} := \exp\left(\frac{1}{2}hV_0\right)\exp\left(B_{t_k,t_{k+1}}V_1\right)\exp\left(\frac{1}{2}hV_0\right)\widetilde{Y}_{t_k}\,, \tag{8.3}$$

where $h := \frac{T}{N}$ and $t_k := kh$.

It was shown by Bally and Rey in [**?**] that if the SDE (8.2) has smooth bounded vector fields satisfying an ellipticity condition, then the Ninomiya-Victoir scheme converges in total variation distance with order 2. That is, for $t \in (0,T]$ there exists $C_t < \infty$ such that

$$\forall N \geqslant 1,\ \forall f : \mathbb{R}^n \to \mathbb{R} \text{ measurable and bounded, } \sup_{t_k \geqslant t}\left|\mathbb{E}\left[f\big(\widetilde{Y}_{t_k}\big)\right] - \mathbb{E}\left[f\big(Y_{t_k}\big)\right]\right| \leqslant \frac{C_t\|f\|_\infty}{N^2}.$$

Furthermore, the strong convergence properties of this scheme were surveyed in [**?**]. Since the SDE (8.2) satisfies a commutativity condition, it was shown under fairly weak assumptions that the Ninomiya-Victoir scheme converges in an $L^p(\mathbb{P})$ sense with order 1:

$$\text{For } p \geqslant 2, \text{ there exists } C > 0 \text{ such that for all } N \geqslant 1,\ \mathbb{E}\left[\sup_{t\in[0,T]}\big\|Y_t - \widetilde{Y}_t\big\|^p\right] \leqslant \frac{C}{N^p}\,,$$

where the approximation $\widetilde{Y}_t$ is obtained by interpolating between the discretization points,

$$\widetilde{Y}_t := \xi + \frac{1}{2}\int_0^t V_0\big(\widetilde{Y}_s^{(0)}\big)\,ds + \int_0^t V_1\big(\widetilde{Y}_s^{(1)}\big)\circ dB_s + \frac{1}{2}\int_0^t V_0\big(\widetilde{Y}_s^{(2)}\big)\,ds\,,$$

with the three (piecewise) processes $\widetilde{Y}^{(i)}$ defined over each interval $[t_k, t_{k+1}]$ according to

$$\widetilde{Y}_t^{(0)} := \exp\left(\frac{1}{2}(t-t_k)V_0\right)\widetilde{Y}_{t_k}\,,$$
$$\widetilde{Y}_t^{(1)} := \exp\left(B_{t,t_k}V_1\right)\widetilde{Y}_{t_{k+1}}^{(0)}\,,$$
$$\widetilde{Y}_t^{(2)} := \exp\left(\frac{1}{2}(t-t_k)V_0\right)\widetilde{Y}_{t_{k+1}}^{(1)}\,.$$

Turning our attention back to the Loewner differential equation (8.1), we see that the imaginary part of the solution $\mathrm{Im}(Z_t)$ is increasing for all $t \geqslant 0$. So provided $\mathrm{Im}(\varepsilon) > 0$, the dt vector field becomes smooth and bounded on the domain $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant \mathrm{Im}(\varepsilon)\}$. Moreover, this argument also shows that the derivatives of the dt vector field are bounded. Since the dB_t vector field is constant, it will satisfy the various regularity assumptions (including the ellipticity condition in [**?**]). Hence, when applied to the backward Loewner equation, the Ninomiya-Victoir scheme converges with the above strong and weak rates. In particular, this implies the numerical method achieves a high order of weak convergence. The second feature is that the ODEs required to compute (8.3) can be resolved explicitly.

Theorem 8.2. *When $V_0(z) = -\frac{2}{z}$ and $V_1(z) = \sqrt{\kappa}$ for $z \in \mathbb{C}$, one can explicitly show that*

$$\exp(tV_0)\, z = \sqrt{z^2 - 4t}\,,$$

$$\exp(tV_1)\, z = z + \sqrt{\kappa}\, t\,.$$

Therefore, the proposed high order numerical method for discretizing (8.1) is given by

Definition 8.3 (Ninomiya-Victoir scheme for the backward Loewner differential equation)**.** *For a fixed number of steps N we can construct a numerical solution $\big\{\widetilde{Z}_{t_k}\big\}_{0\leqslant k\leqslant N}$ of (8.1) on $[0,T]$ by setting $\widetilde{Z}_0 := \varepsilon$ and for each $k \in [0\,..\,N-1]$, defining $\widetilde{Z}_{t_{k+1}}$ using the formula,*

$$\widetilde{Z}_{t_{k+1}} := \sqrt{\left(\sqrt{\widetilde{Z}_{t_k}^2 - 2h} + \sqrt{\kappa}B_{t_k,t_{k+1}}\right)^2 - 2h}\,, \tag{8.4}$$

where $h := \frac{T}{N}$ and $t_k := kh$.

To simulate SLE traces we shall incorporate the above numerical scheme into the adaptive step size methodology proposed in [**?**]. That is, instead of "tilted" or "vertical" slits, we use the Ninomiya-Victoir scheme described above to approximate the SLE trace.

An SLE_κ trace can be built from the conformal maps g_t given by Loewner's equation,

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}\, B_t}\,, \tag{8.5}$$

$$g_0(z) = z\,.$$

The SLE curve z_t is then defined to have the property that $g_t(z_t) = \sqrt{\kappa}\, B_t$ for $t \geqslant 0$. So after applying the change of variables $h_t = g_t - \sqrt{\kappa}\, B_t$, we see that $z_t = h_t^{-1}(0)$ where

$$\frac{dh_t(z)}{dt} = \frac{2}{h(z)} - \sqrt{\kappa}\, dB_t, \tag{8.6}$$

$$h_0(z) = z\,.$$

As h_t^{-1} satisfies the backward Loewner equation (8.1), it can be discretized using (8.4). For simulations, the challenge is that the driving Brownian motion must be run backwards. More concretely, if we fix a partition $0 = t_0 < t_1 < \cdots < t_N = T$, then we can construct a numerical SLE trace $\{\tilde{z}_{t_k}\}_{0\leqslant k\leqslant N}$ by setting $\tilde{z}_0 := 0$ and for $k \in [1 \,..\, N]$ defining $\tilde{z}_{t_k}$ by

$$\tilde{z}_{t_k} := f_0 \circ f_1 \circ \cdots \circ f_{k-1}(0)\,, \tag{8.7}$$

where

$$f_i(z) := \sqrt{\left(\sqrt{z^2 - 2(t_{i+1} - t_i)} + \sqrt{\kappa} B_{t_{i+1}, t_i}\right)^2 - 2(t_{i+1} - t_i)}\,.$$

As discussed in [**?**], due to the singularity at 0 inherent in the conformal maps g_t, simulating SLE traces using a fixed uniform partition can lead to huge numerical errors. Instead, an adaptive step size methodology was recommended, especially when κ is large. The idea is to ensure that $|\tilde{z}_{t_{k+1}} - \tilde{z}_{t_k}| < C$ for each k, where C is a user-specified tolerance. To achieve this, we start by computing $\tilde{z}$ along a uniform partition until $|\tilde{z}_{t_{k+1}} - \tilde{z}_{t_k}| \geqslant C$. If this occurs, it indicates that we should reduce the step size for the SLE discretization. Therefore, we shall sample the Brownian path at the midpoint of the interval $[t_k, t_{k+1}]$. (This can be done using a Brownian bridge conditioned on the values of B at t_k and t_{k+1}) We now proceed as before, except we have added the midpoint of $[t_k, t_{k+1}]$ to the partition. This process continues (recursively) until each value of $|\tilde{z}_{t_{k+1}} - \tilde{z}_{t_k}|$ is strictly less than C.

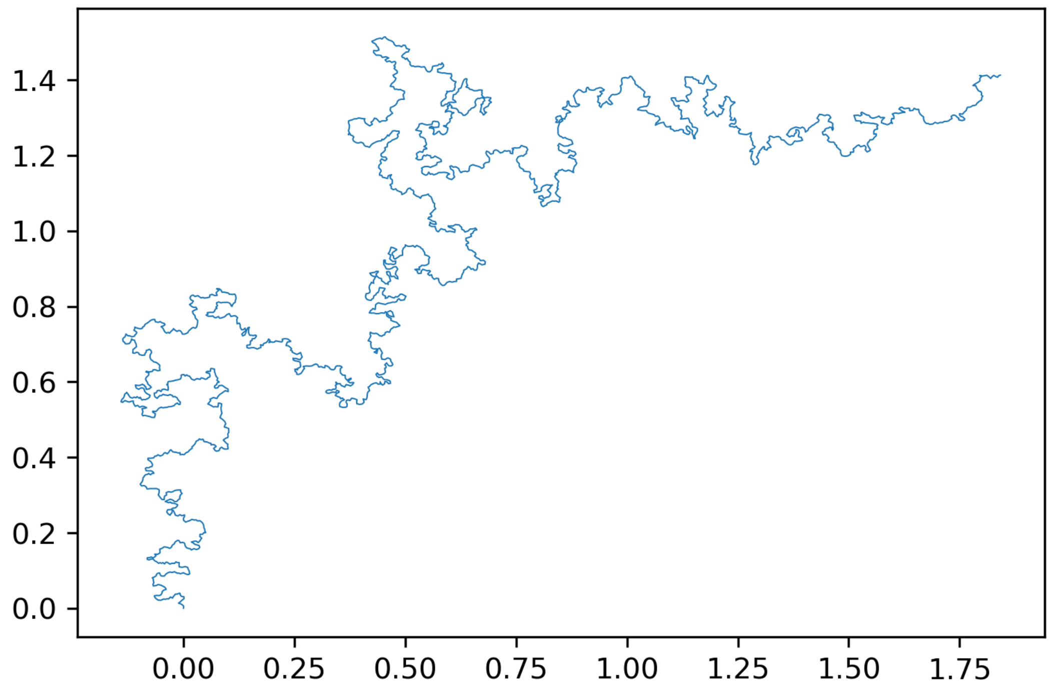

FIGURE 8.1. A sample of SLE_κ trace for $\kappa = 8/3$ with 5103 points.

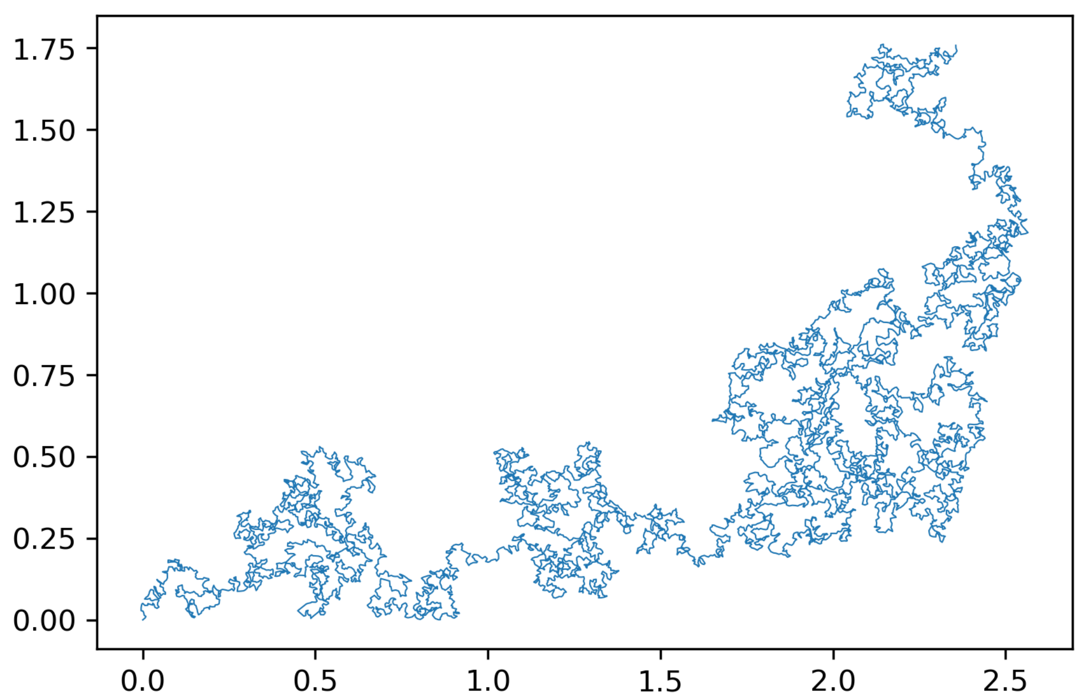

FIGURE 8.2. A sample of SLE_κ trace for $\kappa = 6$ with 17884 points.

This approach naturally leads to the open problem of whether alternative high order "ODE-based" methods can be applied to SLE simulation (such as those presented in [**?**]).

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