Quasi-Sure Stochastic Analysis through Aggregation and SLE_κ Theory - the quasi-sure continuity in κ of the SLE_κ traces

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Abstract

We study SLE_κ theory with elements of Quasi-Sure Stochastic Analysis through Aggregation. Specifically, we show how the latter can be used to construct the SLE_κ traces quasi-surely for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, for any $\varepsilon > 0$ with $\mathcal{K} \subset \mathbb{R}_+$ a nontrivial interval. We emphasize that this method allows us to construct SLE_κ traces simultaneously for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$. As a by-product of this analysis we prove the quasi-sure continuity in κ for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$ of the SLE_κ traces.

1 Introduction

The Loewner equation (also known as the Loewner evolution) was introduced by Charles Loewner in 1923 in [16] and it played an important role in the proof of the Bieberbach Conjecture [4] by Louis de Branges in 1985 in [6]. In 2000, Oded Schramm introduced in [18] a stochastic version of the Loewner equation, the Stochastic Loewner Evolution (SLE_κ). The SLE_κ describes the evolution of a curve in terms of a driving function that is chosen to be $\sqrt{\kappa}B_t$, with $\kappa \geqslant 0$ a real parameter and $B_t,\ t \in [0,\infty)$, a real-valued standard Brownian motion. This is a one-parameter family of random planar fractal curves that are the only possible conformally invariant scaling limits of interfaces of a number of discrete models that appear in planar Statistical Physics. In several cases, it was proved that indeed the interfaces converge to the SLE_κ curves. We refer to [13] for a detailed study of the object and many of its properties.

The problem of continuity of the traces generated by Lowener chains was studied in the context of chains driven by bounded variation drivers in [21], where the continuity of the traces generated by the Loewner chains was established. Also, the question appeared in [15], where the Loewner chains were driven by Hölder-1/2 functions with norm bounded by σ with $\sigma < 4$. In this context, the continuity of the corresponding traces was established with respect to the uniform topologies on the space of drivers and with respect to the same topology on the space of simple curves in $\mathbb{H}$. Another paper that addressed a similar problem is [20], in which the condition $||U||_{1/2} < 4$ is avoided at the cost of assuming some conditions on the limiting trace. Some stronger continuity results are obtained in [9] under the assumption that the driver has finite energy, in the sense that $\dot{U}$ is square integrable. The question appears naturally when considering the solution of the corresponding welding problem in [2]. In this paper it is proved that the trace obtained when solving the corresponding welding problem is continuous in a parameter that appears naturally in the setting. In the context of SLE_κ traces the problem was studied in [24], where the continuity in κ of the SLE_κ traces was proved for any $\kappa < 2.1$. An update is proved in [10], where the a.s. continuity in κ of the SLE traces is proved for $\kappa < 8/3$.

Our method relies on the Quasi-Sure Stochastic Analysis through Aggregation as constructed in [22]. The construction in [22] is suitable when one works with mutually singular probability measures. In the case the measures are absolutely continuous, the situation becomes simpler since one can work under the nullsets of the dominating measure directly. In [22], the authors work with the canonical process that under $\mathbb{P}_0$ is a Brownian Motion. When considering the family of measures $\mathbb{P}_a$, indexed by a real number a taking values in an interval, the canonical process becomes under each $\mathbb{P}_a$ a local martingale with quadratic variation a. Using Lévy's characterization in [22] it is constructed the Universal Brownian Motion $W_t^{\mathbb{P}_a} = \int a_s^{-1/2} dB_s$ that is a $\mathbb{P}_a$- Brownian motion (since is a local martingale with quadratic variation t under any $\mathbb{P}_a$ in the family of measures). These constructions are natural when one is interested in studying problems related with uncertain volatility in Financial Markets. We use this construction of the Universal Brownian motion (see [22]) to obtain a family of drivers for the Loewner differential equation when changing the measures $\mathbb{P}_a$. In our case, the role of the parameter a will be played by the natural parameter κ in the SLE_κ theory, since this is the volatility in this setting.

Using this, one can construct the SLE traces simultaneously quasi-surely (i.e. simultaneously for a family of measures $\mathbb{P}_\kappa$) for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$ using a notion of aggregated solution to a Stochastic Differential Equation that appears in the analysis, and expressing the derivative of the conformal maps in terms of this aggregated solution. Using the Quasi-sure Stochastic Analysis through Aggregation method, one can view these models in a unified framework. Furthermore, the quasi-sure continuity in κ is obtained in this setting using an estimate between conformal maps solving the Loewner Differential Equation whose drivers are close to each other obtained in [24].

The paper is divided in several sections. In the first part of the paper, we construct quasi-surely the SLE_κ traces and in the second part we prove the q.s. continuity in κ for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$ of these objects. We remark that in the case $\kappa \in [0, \varepsilon]$ the a.s. continuity in κ of the SLE_κ is known from previous works (see [24] and [10]).

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2 Preliminaries

We start by introducing the objects needed in our analysis. For the SLE theory the exposure is based on [13] and [3] and for the Quasi-Sure Stochastic Analysis through Aggregations the exposure is based on [22] which we refer to, for more details.

2.1. Introduction to SLE_κ theory. An important object in the study of the Loewner differential equation is the $\mathbb{H}$-compact hull that is a bounded set in $\mathbb{H}$ such that its complement in $\mathbb{H}$ is simply connected. To every compact $\mathbb{H}$-hull, that we typically denote with

K we associate a canonical conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ that is called the *mapping out function of* K.

Using the Riemann Mapping Theorem, we get uniqueness by imposing the *hydrodynamic normalization at* infinity for g_t (i.e. we require that $g_t(z)$ has no constant term and its complex derivative is 1). The mapping near infinity is of the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \to \infty .$$

The coefficient a_K that appears in the expansion at infinity of the mapping is called *halfplane capacity*. Throughout the paper we use the notation $\mathrm{hcap}(K)$ for the halfplane capacity a_K.

We work with a family of growing compact hulls K_t with $H_t := \mathbb{H} \setminus K_t$. Firstly, we define the radius of a hull to be

$$rad(K) = \inf\{r \geqslant 0 : K \subset r\mathbb{D} + \mathrm{x} \text{ for some } \mathrm{x} \in \mathbb{R}\} .$$

Definition 2.1. *Let $(K_t)_{t\geqslant 0}$ be a family of increasing $\mathbb{H}$-hulls, i.e. K_s is contained in K_t whenever $s < t$. For $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t\geqslant 0}$ has the local growth property if*

$$rad(K_{t,t+h}) \to 0 \ \text{ as } \ h \to 0, \ \text{ uniformly on compacts in } t.$$

The first connection between the family of growing compact $\mathbb{H}$-hulls and the real-valued path $(U_t)_{t\geqslant 0}$ is done in the following proposition.

Proposition 2.2 (Proposition 7.1 of [3])**.** *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then, $K_{t+} = K_t$ for all t. Moreover, the mapping $t \mapsto hcap(K_t)$ is continuous and strictly increasing on $[0,\infty)$. Moreover, for all $t \geqslant 0$, there is a unique $U_t \in \mathbb{R}$ such that $U_t \in \bar{K}_{t,t+h}$, for all $h > 0$, and the process $(U_t)_{t\geqslant 0}$ is continuous.*

When considering SLE_κ, we have that $U_t = \sqrt{\kappa}B_t$. The following map $t \mapsto hcap(K_t)/2$ is a non-decreasing homeomorphism on $[0,T)$ and by choosing τ to be the inverse of this homeomorphism, we obtain a new family of hulls K_t' in a new parametrization such that $\mathrm{hcap}(K_t') = 2t$. This is the canonical parametrization that we use throughout the paper. We use the standard terminology for this, i.e parametrization by halfplane capacity.

When studying the SLE_κ , in the upper half-plane, we analyze the following families of conformal maps

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z-\sqrt{\kappa}B_t}\,, \quad f(0,z)=z, z\in\mathbb{H}\,. \tag{2.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z)-\sqrt{\kappa}B_t}\,, \qquad g(0,z)=z, z\in\mathbb{H}\,. \tag{2.2}$$

(iii) Time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z)-\sqrt{\kappa}B_t}\,, \qquad h(0,z)=z, z\in\mathbb{H}\,. \tag{2.3}$$

There are connections between these three formulations for studying families of conformal maps. For example, at each instance of time $t\in[0,T]$ the map $z\to g_t(z)$ is the inverse of the map $z\to f_t(z)$. The connection between the family of maps $h_t(z)$ and $g_t(z)$ in this case is captured in the following lemma.

Lemma 2.3 (Lemma 5.5 of [12])**.** *Let $h_t(z)$ be the solution to the backward Loewner differential equation with driving function $\sqrt{\kappa}B_t$ and let $f_t(z)$ be the solution of the partial differential equation version of the Loewner differential equation with the same driver. Then, for any $t\in\mathbb{R}_+$, the function $z\to f_t(z+\sqrt{\kappa}B_t)-\sqrt{\kappa}B_t$ and $z\to h_t(z)$ have the same distribution.*

In SLE_κ theory, a fundamental object of study are the SLE_κ traces that we introduce in the following definition.

Definition 2.4. *Let g_t be the conformal maps solving the forward Loewner differential equation with $U_t=\sqrt{\kappa}B_t$. The SLE_κ trace is defined via*

$$\gamma(t) := \lim_{y\to 0}\hat{g}_t^{-1}(iy),$$

where $\hat{g}_t^{-1}(iy) = g_t^{-1}(iy+\sqrt{\kappa}B_t)$.

For general Loewner chains, we have the following definition for hulls generated by a trace.

Definition 2.5. *We say that a continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$-hulls K_t if $H_t = \mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all $t\geqslant 0$.*

When considering the SLE_κ case, we have the following fundamental result.

Theorem 2.6 (Theorem 4.1 of [19])**.** *Let $(K_t)_{t\geqslant 0}$ be a SLE_κ for $\kappa\neq 8$. Then, $\hat{g}_t^{-1}(z) = g_t^{-1}(z+\sqrt{\kappa}B_t) : \mathbb{H}\mapsto H_t$ extends continuously to $\bar{\mathbb{H}}$ for all $t\geqslant 0$, almost surely. Moreover, γ_t is continuous and generates $(K_t)_{t\geqslant 0}$ almost surely.*

Remark 2.7. The same result holds for $\kappa = 8$ as it was showed in [14] using a different approach.

2.2. Introduction to Quasi-sure Stochastic Analysis through Aggregation. In this section, we introduce the Quasi-Sure Stochastic Analysis through Aggregation following [22]. We refer the reader to [22] and [7] for further information. In [22] the interest is to develop stochastic analysis simultaneously under a general family of probability measures (that is not dominated by a single probability measure) driven by applications, between others, in Mathematical Finance (for example Financial Markets with uncertain volatility). In the context of SLE_κ theory the parameter κ it will play the role of the uncertain volatility in our analysis. The results in [22] extend the theory that one naturally has for a fixed probability measure to defining Stochastic Analysis simultaneously for a family of probability measures using the notion of *aggregation* that we define in the following.

Let us consider the probability space $(\Omega,\mathbb{F}^B,\mathbb{P})$ where $\Omega = C(\mathbb{R}_+,\mathbb{R})$ and let $\mathbb{F}=\mathbb{F}^B$ be the filtration generated by the canonical process B. For example, if $\mathbb{P}$ is the Wiener measure then the canonical process is a standard Brownian motion. Throughout our analysis we will consider a family of probability measures on this space indexed by a parameter, that will change the canonical process under each of them. We recall from [22] that a probability measure $\mathbb{P}$ is a local martingale measure if the process B is a local martingale under $\mathbb{P}$. It is proved in [11] that there exists an $\mathfrak{F}$-progresively measurable process denoted as $\int_0^t B_s dB_s$ which coincides with the Itô integral $\mathbb{P}$ -a.s. for all local martingale measures $\mathbb{P}$. In particular this provides a pathwise definition of

$$\langle B \rangle_t := B_t^2 - 2\int_0^t B_s dB_s$$

and

$$\hat{a}_t := \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}[\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}].$$

We first introduce as in [22] the following notions.

Definition 2.8. *$\bar{\mathcal{P}}_W$ is the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s. $\langle B \rangle_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{R}_+$.*

Definition 2.9 (Definition of capacity)**.** *For each $f \in C_b(\Sigma)$- the set of bounded continuous functions on Σ, we put*

$$cap(f) = \sup\{||f||_{L^2(\Sigma,P)} : \mathbb{P} \in \mathcal{P}\}.$$

By definition for a measurable set A, we have $cap(A) = cap(I_A)$. Capacities naturally have applications in the theory of Risk Measures, see [1], [8]. Next, we introduce the notion of polar set.

Definition 2.10. *We say that a property holds $\mathcal{P}$-quasi-surely if it holds $\mathbb{P}$-a.s. for all the probability measures $\mathbb{P} \in \mathcal{P}$. We call a set is polar if $c(A) = 0$, i.e. if $\mathbb{P}(A) = 0$ for all $\mathbb{P} \in \mathcal{P}$.*

Let us denote $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$. We use the following universal filtration $\mathfrak{F}^{\mathcal{P}}$ for the mutually singular measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$.

$$\mathfrak{F}^{\mathcal{P}} := \{\mathcal{F}_t^{\mathcal{P}}\}_{t \geqslant 0}$$

where

$$\mathcal{F}_t^{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}} \left(\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}\right).$$

The next definition introduces the notion of *aggregator* that is fundamental in our analysis as it will allow us to describe objects in SLE_κ theory simultaneously when varying the parameter κ.

Definition 2.11. *Let $\mathcal{P} \subset \bar{\mathcal{P}}_W$. Let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of $\mathfrak{F}^{\mathcal{P}}$ progressively measurable processes. An $\mathfrak{F}^{\mathcal{P}}$ progressively measurable process X is called a $\mathcal{P}$-aggregator of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, if $X = X^{\mathbb{P}}$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$.*

2.3. The universal Brownian motion. In this section, we introduce the notion of Universal Brownian motion as in [22].

We first introduce the required tools to define this notion. We refer the reader to [22] for more details. Let

$$\bar{\mathcal{A}} := \{a : \mathbb{R}_+ \to \mathbb{R}_+ | \mathbb{F} - \text{progresively measurable and} \int_0^t |a_s| ds < +\infty, \forall t \geqslant 0\}\,.$$

For a given $\mathbb{P} \in \bar{\mathcal{P}}_W$, let

$$\bar{\mathcal{A}}_W(\mathbb{P}) := \{a \in \bar{\mathcal{A}} : a = \hat{a}, \mathbb{P} - a.s.\}$$

Recall that $\hat{a}$ is the density of the quadratic variation of $\langle B \rangle$ (where B is the canonical process under the Wiener measure on path space) and is defined point-wise. We define

$$\bar{\mathcal{A}}_W := \cup_{\mathbb{P} \in \bar{\mathcal{P}}_W} \bar{\mathcal{A}}_W(\mathbb{P})$$

In order to construct a process with a given quadratic variation $a \in \bar{\mathcal{A}}$ from the canonical process (by changing the canonical measure on the pathspace) as in [22], we consider the following stochastic differential equation

$$dX_t = a_t^{1/2}(X) dB_t, \qquad \mathbb{P}_0 - a.s. \tag{2.4}$$

Furthermore, if the equation (2.4) has weak uniqueness, we let $\mathbb{P}_a \in \bar{\mathcal{P}}_W$ be the unique solution of (2.4) with initial condition $\mathbb{P}_a(B_0 = 0) = 1$, and we define

$$\mathcal{A}_W := \{a \in \bar{\mathcal{A}}_W : (2.4) \text{ has weak uniqueness}\}$$

$$\mathcal{P}_W := \{\mathbb{P}_a, a \in \mathcal{A}_W\}$$

Let us fix a subset $\mathcal{A} \subset \mathcal{A}_W$. We further denote

$$\mathcal{P} = \{\mathbb{P}_a, a \in \mathcal{A}\}.$$

Let us define for any $a, b \in \mathcal{A}$, the disagreement time

$$\theta^{a,b} := \inf\{t \geqslant 0 : \int_0^t a_s ds \neq \int_0^t b_s ds\}.$$

Definition 2.12. *A subset $\mathcal{A}_0 \subset \mathcal{A}_W$ is called a generating class of diffusion coefficients if*

- $\mathcal{A}_0$ *satisfies the concatenation property* $a\mathbf{1}_{[0,t)} + b\mathbf{1}_{[t,\infty)} \in \mathcal{A}_0$, *for* $a, b, \in \mathcal{A}_0, t \geqslant 0$.

- $\mathcal{A}_0$ *has constant disagreement times: for all* $a, b \in \mathcal{A}_0$, $\theta^{a,b}$ *is constant.*

Definition 2.13. *We say $\mathcal{A}$ is a separable class of diffusion coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0 \subset \mathcal{A}_W$ is generated by a class of diffusion coefficients and $\mathcal{A}$ consists of all processes a of the form*

$$a = \sum_{0}^{\infty} \sum_{i=1}^{\infty} a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}$$

where $(a_i^n)_{i,n} \subset \mathcal{A}_0$, $(\tau_n)_n \subset \mathcal{T}$ is non-decreasing with $\tau_0 = 0$.

- *We have that* $\inf\{n : \tau_n = \infty\} < \infty$ *and* $\tau_n < \tau_{n+1}$ *whenever* $\tau_n < \infty$ *and each* τ_n *takes at most countably many values.*

- *For each* n $\{E_i^n, i \geqslant 1\} \subset \mathcal{F}_{\tau_n}$ *forms a partition of* Ω.

A separable class $\mathcal{A}$ of diffusion coefficients generated by $\mathcal{A}_0$ is said to satisfy the consistency conditions.

Let us consider a standard Brownian motion B_t (the canonical process under the Wiener measures $\mathbb{P}_0$ as in [22]). For any $\mathbb{P}_a \in \mathcal{P}_W$ and $a \in \bar{\mathcal{A}}_W(\mathbb{P})$ by Lévy's characterization, we obtain that the following Itô stochastic integral under $\mathbb{P}_a$ is a $\mathbb{P}_a$- Brownian motion

$$W_t^{\mathbb{P}_a} := \int_0^t a_s^{-1/2} dB_s$$

For $\mathcal{A}$ satisfying the consistency condition, the family $\{W^{\mathbb{P}_a}, a \in \mathcal{A}\}$ admits a unique $\mathcal{P}$-aggregator W (see [22]). Since $W^{\mathbb{P}_a}$ is a $\mathbb{P}_a$ Brownian motion for every $a \in \mathcal{A}$, we call $W_t^{\mathbb{P}_a}$- a universal Brownian motion.

When studying SLE_κ theory, the natural process to be considered is $\sqrt{\kappa} B_t$ with B_t a standard Brownian Motion. Thus, we will work with $a_s^{1/2} dW_t^{\mathbb{P}_a}$, i.e. $a_s = \kappa$, i.e. we are moving a_s to the other side. Since in [22], the process $W_t^{\mathbb{P}_a}$ is defined for all a, and is a $\mathbb{P}_a$ standard BM, we will just modify its quadratic variation by constants.

From now on, for the convenience of notation (since our quantities will depend on κ, for $\kappa \in \mathbb{R}_+, \kappa \neq 8$, we use for the family of probability measures $\mathbb{P}_a$ the indexing $\mathbb{P}_\kappa$ for $\kappa \in \mathbb{R}_+$).

A fundamental result that we use is the aggregate solution to stochastic differential equations. In the paper, they show how to solve a stochastic differential equation simultaneously under all the measures $\mathbb{P} \in \mathcal{P}$. Specifically, they prove the following result:

Proposition 2.14 (Proposition 6.10 of [22]). *Let $\mathcal{T}$ be the set of all $\mathbb{F}$-stopping times taking values in $\mathbb{R}_+ \cup \{\infty\}$. Let $\mathcal{A}$ satisfy the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation*

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s,$$

has a unique $\mathbb{F}^{\mathbb{P}}$ progressively measurable strong solution on the interval $[0, \tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution (see Def. 2.11) to the equation above, i.e.

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s, \quad t \geqslant 0$$

has solution simultaneuously under all prrobability measures $\mathbb{P}_\kappa$.

In the next section, we use this result for the stochastic differential equation corresponding to $\tilde{K}_s$ that is used in the construction of the SLE_κ trace.

3 Heuristics of the quasi-sure construction of the SLE_κ traces

Following the parametrization in [Chapter 7, [13]] we set $\hat{g}_t(z) := \frac{g_t(\sqrt{\kappa}z)}{\sqrt{\kappa}}$ for the maps $g_t(z)$ satisfying the forward Loewner differential equation. Thus, the maps $\hat{g}_t(z)$ satisfy the differential equation $\partial_t \hat{g}_t(z) = \frac{2/\kappa}{\hat{g}_t(z) - B_t}$. We work with this paramatrization of the dynamics in order to keep the exposition of the method in line with the approach from [Chapter 7. [13]]. This formulation is equivalent with the one in which the parameter κ is kept in $\sqrt{\kappa}B_t$. To be precise, the Proposition 2.14 applies for the process $\sqrt{\kappa}B_t$ and then we reparametrize. We avoid doing the details and work directly with this parametrization in order to keep the exposure neat.

The main idea is to consider the construction of the aggregated solution to SDE as in 2.14 and applied to the SDE corresponding to the process $\tilde{K}_s$. Furthermore, we express the derivative of the map $\tilde{h}_t(z_0)$ using the aggregated solution of an SDE and use the lemmas in [14] to obtain the quasi-sure existence of the SLE_κ trace.

We have

$$|\tilde{h}'_t(z_0)| = e^{-\frac{2}{\kappa}t} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right) = e^{\frac{2}{\kappa}t} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right),$$

where

$$d\tilde{N}_s = (1-\tilde{N}_s)[-4(\frac{2}{\kappa}+1)\tilde{N}_s + 1]ds + 2\sqrt{\tilde{N}_s}(1-\tilde{N}_s)d\tilde{B}_s.$$

The SDE for $\tilde{K}_t$ is

$$d\tilde{K}_t = \frac{4}{\kappa}\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion.

Checking the conditions of Yamada-Watanabe Theorem for the SDE $\tilde{K}_t$ (see [17]), we obtain that this stochastic differential equation has a unique strong solution.

Once we have a unique notion of strong solution for the SDE for K_t, $\mathbb{P}$-a.s., we can construct an aggregated solution for this SDE using Proposition 2.14 and then express

$$|\tilde{h}'_t(z_0)| = e^{-2t/\kappa} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right).$$

Using the aggregated solution for the SDE $\tilde{K}_t$ we construct simultaneously the SLE_κ trace for all parameters $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, for any $\varepsilon > 0$, where $\mathcal{K}$ is a nontrivial interval of $\mathbb{R}_+$ by relating the aggregated solution for $\tilde{K}_t$ with the derivative of the backward SLE map $h_t(z)$.

4 The a.s. existence of the SLE_κ trace for fixed $\kappa \in \mathbb{R}_+ \setminus \{8\}$

For the convenience of the reader, we recall the a.s. construction of the SLE_κ trace for any fixed $\kappa \neq 8$ from [19]. The elements used in the proof of the existence of the SLE_κ trace for fixed κ, a.s., are listed in the following. For more details, we refer the reader to [19] and [13].

4.1. Estimates for the mean of the derivative for a fixed κ. Considering the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - \sqrt{\kappa}dB_t\,, \quad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{4.1}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$.

The first elements of the proof of the existence of the SLE_κ trace are the following proposition and the corollary of it.

Proposition 4.1 (Proposition 7.2 in [14]). *Let r, b such that*

$$r^2 - \left(\frac{4}{\kappa} + 1\right) r + \frac{2}{\kappa} b = 0 \,,$$

then

$$M_t \;:=\; \tilde{Y}_t^{b-(r\kappa/2)} (|\tilde{Z}_t|/\tilde{Y}_t)^{2r} |h_t'(z_0)|^b \,,$$

is a martingale. Moreover,

$$\mathbb{P}(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} e^{t(r-2b/\kappa)} \,.$$

Corollary 4.2 (Corollary 7.3 in [14]). *For every $0 \leqslant r \leqslant \frac{4}{\kappa} + 1$, there is a finite $c = c(\kappa, r)$ such that for all $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, we have that*

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} \delta(y_0, \lambda) \,,$$

where $b = \frac{[(\frac{4}{\kappa}+1)r - r^2]\kappa}{2} \geqslant 0$ and

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{(r\kappa/2)-b}, & \text{if } r < \frac{2b}{\kappa} \,, \\ -\log(\lambda y_0), & \text{if } r = \frac{2b}{\kappa} \,, \\ y_0^{b-(r\kappa/2)}, & \text{if } r > \frac{2b}{\kappa} \,. \end{cases}$$

4.2. Existence of the trace for fixed κ.

Proposition 4.3 (Proposition 4.33 in [14]). *Suppose that g_t is a Loewner chain with driving function U_t and assume that there exist a sequence of positive numbers $r_j \to 0$ and a constant c such that*

$$|\hat{f}_{k2^{-2j}}'(2^{-j} i)| \leqslant 2^j r_j \,, k = 0, 1, \ldots, 2^{2j} - 1 \,,$$
$$|U_{t+s} - U_t| \leqslant c\sqrt{j} 2^{-j} \,, 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 2^{-2j} \,.$$

and

$$\lim_{j\to\infty} \sqrt{j}/\log r_j = 0\,.$$

Then $V(y,t) := \hat{f}_t(iy)$ *is continuous on* $[0,1]\times[0,1]$.

Combining the previous results, one obtains Theorem 2.6 that we repeat for convenience.

Theorem 4.4 (Theorem 4.1 of [19])**.** *Let* $(K_t)_{t\geqslant 0}$ *be a* SLE_κ *for* $\kappa\neq 8$. *Then,* $\hat{g}_t^{-1}(z) = g_t^{-1}(z+\sqrt{\kappa}B_t) : \mathbb{H}\mapsto H_t$ *extends continuously to* $\bar{\mathbb{H}}$ *for all* $t\geqslant 0$, *almost surely. Moreover,* γ_t *is continuous and generates* $(K_t)_{t\geqslant 0}$ *almost surely.*

5 Quasi-sure existence of the SLE_κ trace -defining the SLE_κ trace simultaneously for all $\kappa\in\mathcal{K}\cap\mathbb{R}_+\setminus([0,\varepsilon)\cup\{8\})$

In this section, we construct the SLE_κ quasi-surely, i.e. we construct the SLE_κ traces for a sequence of measures $\mathbb{P}_\kappa$, for $\kappa\in\mathcal{K}\cap\mathbb{R}_+\setminus([0,\varepsilon)\cup\{8\})$.

5.1. Estimates on the moments of the derivatives for many κ using aggregation of solutions of a SDE. For the sequence of measures $\mathbb{P}_\kappa$, we consider the universal Brownian motion $W_t^{\mathbb{P}_\kappa}$ constructed in Section 2, as a driver for the backward Loewner differential equation.

Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - \sqrt{\kappa}dW_t^{\mathbb{P}_\kappa}\,, \qquad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{5.1}$$

We recall from the heuristics that we follow the parametrization in [Chapter 7, [13]] and we set $\hat{g}_t(z) := \frac{g_t(\sqrt{\kappa}z)}{\sqrt{\kappa}}$ for the maps $g_t(z)$ satisfying the forward Loewner differential equation. Thus, the maps $\hat{g}_t(z)$ satisfy the differential equation $\partial_t\hat{g}_t(z) = \frac{2/\kappa}{\hat{g}_t(z)-B_t}$. We recall also that this choice is for the convenience of the analysis and does not change the aggregation result. We consider the time change $\sigma(t) = X_t^2+Y_t^2$, $t=\int_0^{\sigma(t)}\frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$. Furthermore, let us consider $\sqrt{\kappa}\tilde{W}_t^{\mathbb{P}_\kappa} := \frac{\sqrt{\kappa}dW_t^{\mathbb{P}_\kappa}}{\sqrt{X_t^2+Y_t^2}}$. Using the Lévy's characterization of Brownian motion (for every $\mathbb{P}_\kappa$) we deduce that the random time changed Brownian motion is also a Brownian motion (in the random time defined above) for all $\mathbb{P}_\kappa$.

Using Yamada-Watanabe Theorem (see [17]), the following SDE has a unique strong solution

$$d\tilde{K}_t = \frac{4}{\kappa}\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}\, d\tilde{W}_t^{\mathbb{P}_\kappa}.$$

Next, we use Proposition 2.14 in order to obtain the aggregated solution of the SDE for $\tilde{K}_t$:

$$d\tilde{K}_t = \frac{4}{\kappa}\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}\, d\tilde{W}_t^{\mathbb{P}_\kappa}$$

In order to prove similar estimates that were obtained in the previous section for fixed κ simultaneously for all κ, we use the aggregated solution and relate it with the derivative of the map, via

$$|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1} ds\right).$$

Using the aggregated solution, we obtain a version of Proposition 5.1 using the family of measures $\mathbb{P}_\kappa$. In this manner we can construct the trace by obtaining an estimate for the derivative of the conformal maps $h_t(z)$ similar to the one in Proposition simultaneously for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$ using the aggregated solution. In this manner we obtain the q.s. existence of SLE traces simultaneously for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, for any $\varepsilon > 0$.

First, we prove a version of Proposition 5.1 for the family of measures $\mathbb{P}_\kappa$.

Proposition 5.1. *Let r, b such that*

$$r^2 - \left(\frac{4}{\kappa}+1\right) r + \frac{2b}{\kappa} = 0\,,$$

then

$$M_t \;:=\; \tilde{Y}_t^{b-(\frac{2r}{\kappa})} (|\tilde{Z}_t|/\tilde{Y}_t)^{2r} |h_t'(z_0)|^b\,,$$

is a martingale. Moreover,

$$\mathbb{P}_\kappa(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r} e^{t(r-\frac{2b}{\kappa})}\,.$$

Proof. By applying the chain rule for the function $L_t = \log h_t'(z_0)$, we obtain that $L_t = -\int_0^t \frac{2/\kappa}{Z_s^2} ds$, and in particular, $|\tilde{h}_t'(z_0)| = \exp\left(\frac{2}{\kappa}\int_0^t \frac{\tilde{Y}_s^2 - \tilde{X}_s^2}{\tilde{X}_s{}^2 + \tilde{Y}_s{}^2} ds\right)$. Moreover, for fixed κ we have $\tilde{K}_t = \frac{\tilde{X}_t^2}{\tilde{Y}_t^2}$ and $\tilde{N}_t = \frac{\tilde{K}_t}{1+\tilde{K}_t}$. Then, for fixed κ, we obtain that

$$|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right).$$

Next, we use the aggregated solution for $\tilde{K}_t$ and we obtain via $\tilde{N}_t = \frac{\tilde{K}_t}{1+\tilde{K}_t}$ an aggregator for $\tilde{N}_t$ as well. Next, we prove similar estimates as one obtains for fixed κ, using the aggregated solution $\tilde{N}_s$.

In the $\sigma(t)$ time parametrization, we have that $d\tilde{Y}_t = -\frac{2}{\kappa}\tilde{Y}_t dt$, so in this time parametrization $\tilde{Y}_t$ grows deterministically $\tilde{Y}_t = \tilde{Y}_0 e^{\frac{2}{\kappa}t}$. At this moment, we can rephrase the formula for M_t as

$$M_t = y_0^{b-(\frac{\kappa r}{2})} e^{-rt}(1-\tilde{N}_t)^{-r} \exp\left(\frac{4b}{\kappa}\int_0^t \tilde{N}_s ds\right) .$$

and by applying Itô formula, we obtain that

$$dM_t = 2r\sqrt{\tilde{N}_t} M_t d\tilde{W}_t^{\mathbb{P}_\kappa} ,$$

where $d\tilde{W}_t^{\mathbb{P}_\kappa} = \int_0^{\sigma(t)} \frac{1}{\sqrt{X_t^2+Y_t^2}} dW_t^{\mathbb{P}_\kappa}$ is the Brownian motion that we obtain in the time reparametrization. This shows that M_t is a martingale, hence

$$\mathbb{E}_\kappa[M_t] = \mathbb{E}_\kappa[M_0] = y_0^{b-(r\kappa/2)}(|z_0|/y_0)^{2r} .$$

Note that since for $r \geqslant 0$, $(|\tilde{Z}_t|/\tilde{Y}_t)^{2r} \geqslant 1$, then by Markov inequality, we have that

$$\mathbb{P}_\kappa(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r} e^{t(r-\frac{2b}{\kappa})} .$$

$\square$

Moreover, we obtain a version of the Corollary 4.2 under the family of measures $\mathbb{P}_\kappa$.

Corollary 5.2. *For every $0 \leqslant r \leqslant \frac{4}{\kappa}+1$, there is a finite $c = c(\kappa, r)$ such that for all $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, we have that*

$$\mathbb{P}_\kappa(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0, \lambda) ,$$

where $b = \frac{[(\frac{4}{\kappa}+1)r - r^2]\kappa}{2} \geqslant 0$ and

$$\delta(y_0,\lambda) = \begin{cases} \lambda^{(r\kappa/2)-b}, & \text{if } r < \frac{2b}{\kappa} , \\ -\log(\lambda y_0), & \text{if } r = \frac{2b}{\kappa} , \\ y_0^{b-(r\kappa/2)}, & \text{if } r > \frac{2b}{\kappa} . \end{cases}$$

Proof. From $dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt$, we obtain that $dY_t \leqslant \frac{2/\kappa}{Y_t}dt$, and hence we obtain in the following that $Y_t \leqslant \sqrt{\frac{4}{\kappa}t + y_0^2} \leqslant \sqrt{\frac{4}{\kappa}+1}$. In the last inequality, we used that $t \leqslant 1$ and $y_0 \leqslant 1$. Using the exponential growth of Y_t in this time reparametrization, we obtain that $\tilde{Y}_t = \sqrt{\frac{4}{\kappa}+1}$ at time $T = \frac{\log\sqrt{\frac{4}{\kappa}+1} - \log y_0}{2/\kappa}$.

Therefore,

$$\mathbb{P}_\kappa(|h_t'(z_0)| \geqslant \lambda) \leqslant \mathbb{P}_\kappa(\sup_{0\leqslant s\leqslant T} |\tilde{h}_s'(z_0)| \geqslant \lambda).$$

Using that $|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right)$ we obtain that $|\tilde{h}_{t+s}'(z_0)| \leqslant e^{2s/\kappa}|\tilde{h}_t'(z_0)|$. So by addition of the probabilities, we have that

$$\mathbb{P}_\kappa(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^{2/\kappa}\lambda) \leqslant \sum_{j=0}^{[T]} \mathbb{P}_\kappa(|\tilde{h}_j'(z_0)| \geqslant \lambda).$$

Using the Schwarz-Pick Theorem for the upper half-plane we obtain that $|\tilde{h}_t(z_0)| \leqslant \mathrm{Im}\tilde{h}_t'(z_0)/y_0 = e^{2t/\kappa}$. This gives a lower bound for the t that we are summing over and we obtain that via the Proposition 5.1 that

$$\begin{aligned}
\mathbb{P}_\kappa(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^{2/\kappa}\lambda) &\leqslant \sum_{(\kappa/2)\log\lambda\leqslant j\leqslant T} \mathbb{P}_\kappa(|\tilde{h}_j'(z_0)| \geqslant \lambda) \\
&\leqslant \lambda^{-b}(|z_0|/y_0)^{2r} \sum_{(\kappa/2)\log\lambda\leqslant j\leqslant T} e^{j(r-ab)} \\
&\leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda).
\end{aligned}$$

□

In order to prove the result, we need the following Lemma, that we recall from the introduction for the convenience of the reader.

Lemma 5.3 (Lemma 5.5 of [12]). *Let $h_t(z)$ be the solution to the backward Loewner differential equation with driving function $\sqrt{\kappa}B_t$ and let $f_t(z)$ be the solution of the partial differential equation version of the Loewner differential equation with the same driver. Then, for any $t \in \mathbb{R}_+$, the function $z \to f_t(z+\sqrt{\kappa}B_t) - \sqrt{\kappa}B_t$ and $z \to h_t(z)$ have the same distribution.*

We note that the previous lemma is also true when one changes the measures $\mathbb{P}_\kappa$, for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, for any $\varepsilon > 0$. We combine the previous results, to obtain the quasi-sure existence of the SLE_κ trace.

Theorem 5.4. *Let $\varepsilon > 0$. Let $\mathcal{K} \subset \mathbb{R}_+$ be a nontrivial interval. Then, for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$ the chordal SLE_κ is q.s. generated by a path.*

Proof. Using the scaling of the SLE_κ, it suffices to prove the Theorem only for $t \in [0,1]$. According to the preliminary results it suffices to show that q.s. there exists an $\varepsilon_1 > 0$ and a random constant c (that depends on the worst κ) such that

$$|f'_{k2^{-2j}}(i2^{-j})| \leqslant c2^{j-\varepsilon_1}\,, j = 1,2,\ldots, k = 0,1,\ldots,2^{2j}\,,$$
$$\sqrt{\kappa}|W_t^{\mathbb{P}_\kappa} - W_s^{\mathbb{P}_\kappa}| \leqslant c_1\sqrt{\kappa}|t-s|^{1/2}|\log\sqrt{|t-s|}| \quad 0 \leqslant t \leqslant 1\,.$$

The second inequality is a consequence of the modulus of continuity for the Brownian motion, for every measure $\mathbb{P}_\kappa$. For the first inequality, we consider $r = \frac{2}{\kappa} + \frac{1}{4} < \frac{4}{\kappa} + 1$ and $b = \frac{(1+\frac{4}{\kappa})r - r^2}{2/\kappa} = \frac{2}{\kappa} + 1 + \frac{3}{32/\kappa}$, according to the Corollary 5.2. Thus, we are in the regime $r < \frac{2b}{\kappa}$, so by Corollary 5.2 we have the following κ dependent bound

$$\mathbb{P}_\kappa(|h'_t(i2^{-j})| \geqslant 2^{j-\varepsilon_1}) \leqslant c2^{-j(2b-(2r/\kappa))(1-\varepsilon_1)} \tag{5.2}$$

In order to obtain the estimate for the family of functions $f_t(z)$, we use for each measure $\mathbb{P}_\kappa$ the Lemma 5.3.

We repeat the optimization procedure for each $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$ and obtain a continuos constant c in κ. We present parts of the analysis in the following. For more details on how to optimize over the parameters r and b for fixed κ we refer the reader to the proof of the existence of the trace in [12]. Following the analysis from Section 5.5 of [12], the parameter b is expressed in terms of r for fixed κ. In this setting, we have $b(r) = \kappa/2((1+4/\kappa)r - r^2)$. Then one studies the quantity $\alpha(r) = 2b(r) - \frac{r\kappa}{2}$ that is maximized by $r_0 = \frac{1}{4} + 2\kappa$. Thus, $\alpha(r_0) = \kappa(1/4 + 2/\kappa)^2 \geqslant 2$ and $\alpha(r_0) = 2$ if and only if $\kappa = 8$. Thus the estimate on the derivative of the conformal map $f_t(iy)$ is obtained for $r = r_0$ and $b = b_0$. In particular, for the choices of $r = r_0 = 1/4 + 2/\kappa$ and $b = b_0 = \kappa/2((1+4/\kappa)r_0 - r_0^2)$, we can express explicitly the constant c in the estimate $\mathbb{P}_\kappa(\sup_{0\leqslant t\leqslant T}|\tilde{h}'_t(z_0)| \geqslant e^{2/\kappa}\lambda) \leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda)$ in the case $b_0 - \kappa r_0/2 \leqslant 0$ i.e. $\kappa > 8$ and simillary when $b_0 - \kappa r_0/2 \geqslant 0$, i.e. when $\kappa < 8$. Thus, this constant is indeed a continuous function of κ in our interval of interest.

Thus in order to obtain the first inequality for all the measures $\mathbb{P}_\kappa$ in our interval of interest, we use Borel-Cantelli Lemma for capacities (see [5], [7], [23]) along with Lemma

5.3 to find c (that can be chosen a continuous function of κ) and $\varepsilon_1 > 0$ such that for all $0 \leqslant t \leqslant 1$

$$\sup_{\kappa} \mathbb{P}_\kappa(|h_t'(i2^{-j})| \geqslant 2^{j-\varepsilon_1}) \leqslant \sup_{\kappa} c2^{-j(2b-(2r/\kappa))(1-\varepsilon_1)}. \tag{5.3}$$

We obtain that $2b - (2r/\kappa) = 4/\kappa + 1 + \kappa/16 > 2$ provided that $2/\kappa \neq 1/4$. So, we can apply Borel-Cantelli argument for capacities provided that $2/\kappa \neq 1/4$, i.e. $\kappa \neq 8$. We restrict to $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$, in order to obtain the control on the constant c in (5.3). Thus,

$$cap(|h_t'(i2^{-j})| \geqslant 2^{j-\varepsilon_1}) \leqslant c2^{-(2+\varepsilon_1)j} ,$$

and the proof is finished. Let us consider the dyadic partition of the time interval $[0, 1]$, $\mathcal{D}_{2n} = \{l2^{-2n} : l \in [0, 2^{2n}]\}$. Then,

$$\sum_{n\in\mathbb{N}} \sum_{t\in\mathcal{D}_{2n}} cap(|h_t'(i2^{-j})| \geqslant 2^{j-\varepsilon_1}) < \infty, \tag{5.4}$$

and we obtain the desired conclusion. □

6 The quasi-sure continuity in κ for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$ of the SLE_κ traces

Once the SLE_κ traces are constructed quasi-surely, we would like to prove the quasi-sure continuity in κ of the traces.

Using quasi-sure definition of the SLE_κ trace allows us directly to consider uncountably many parameters $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$. In this section, we use the notation $[\kappa_m, \kappa_M]$ for $\mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$. Furthermore, we consider the following coupling: we fix a parameter $\kappa_1 \in [\kappa_m, \kappa_M]$ and we consider the canonical process on the path space under the measures $\mathbb{P}_{\kappa_1}$ and $\mathbb{P}_\kappa$ for $\kappa \in [\kappa_m, \kappa_M]$ and we compare the traces obtained for the fixed choice κ_1 with the family of traces obtained for $\kappa \in [\kappa_m, \kappa_M]$, with $\kappa_m > \varepsilon$, for any $\varepsilon > 0$.

Let us consider M to be the space of continuous curves defined on $[0, 1]$ with values in the closed upper half-plane $\bar{\mathbb{H}} = \{z : \mathrm{Im}(z) \geqslant 0\}$. Further, we equip the space M with the supremum norm.

Theorem 6.1 (Quasi-sure continuity in κ of the SLE_κ traces)**.** *Let $\mathcal{K} \subset \mathbb{R}_+$ be a nontrivial interval. Then, the SLE_κ traces are quasi-surely continuous in κ, for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$, i.e. there exists an event Ω^* that happens quasi-surely for which for every $\omega \in \Omega^*$ the following holds: The SLE_κ traces parametrized by capacity exist as elements of M and the function $\kappa \to \gamma^{(\kappa)}(\cdot, \omega)$ is a continuous function from $[\kappa_m, \kappa_M]$ to $(M, || \cdot ||_\infty)$, in the sense that for any $\kappa \in [\kappa_m, \kappa_M]$ for any $\kappa_i \to \kappa$, there exists a function θ with $\theta(\delta) \to 0$ as $\delta \to 0$ such that for $t \in [0, 1]$ we have $|\gamma^{(\kappa_i)}(t) - \gamma^{(\kappa)}(t)| \leqslant \theta(|\kappa_i - \kappa|)$ $\mathbb{P}_{\kappa_i}$ a.s.*

We remark that the quasi-sure setting can allow that (κ_i) in the above result to be uncountable.

Proof of Theorem 6.1. Throughout the proof, we use the notation $F(t, y, \kappa) = f_t^{(\kappa)}(iy)$. We showed in the previous section that one can construct for the sequence of measures $\mathbb{P}_\kappa$ the SLE_κ traces simultaneously and can view them as elements of the metric space M. We use the set-up from [24], in order to define the Whitney-type partition of the (t, y, κ) space. The main idea of this section is to show how we can avoid the typical Borel-Cantelli argument of [24] using the quasi-sure construction of the SLE_κ traces from the previous section.

We also need the following distortion result for conformal maps.

Lemma 6.2 (Distortion Lemma: Lemma 2.2 in [24])**.** *There exists a constant $0 < c < \infty$ such that the following holds. Suppose that f_t satisfies the chordal Loewner PDE 2.1 and that $z = x + iy \in \mathbb{H}$, then for $0 \leqslant s \leqslant y^2$*

$$c^{-1} \leqslant \frac{|f'_{t+s}(z)|}{|f'_t(z)|} \leqslant c$$

and

$$|f_{t+s}(z) - f_t(z)| \leqslant cy|f'_t(z)|\,.$$

We consider the partition of the (t, y, κ) three dimensional space in boxes obtain by partitioning each coordinate. We follow the proof in [24] and we estimate the derivative of the map $(f_t^{(\kappa)})'(iy)$ in the corners of the boxes. Using Distortion Theorems for the conformal maps along with the following Lemma that appears in [24].

Lemma 6.3 (Lemma 2.3 of [24]). *Let $0 < T < \infty$. Suppose that for $t \in [0,T]$, $f_t^{(1)}$ and $f_t^{(2)}$ satisfy the backward Loewner differential equation with drivers $W_t^{(1)}$ and $W_t^{(2)}$. Suppose that $\varepsilon = sup_{s\in[0,T]}|W_s^{(1)} - W_s^{(2)}|$. For $u = x + iy \in \mathbb{H}$, for every $t \in [0,T]$ we have that*

$$|f_t^{(1)}(u) - f_t^{(2)}(u)| \leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log \frac{I_{ty}|(f_t^{(1)})'(u)|}{y} \log \frac{I_{ty}|(f_t^{(2)})'(u)|}{y} + \log\log \frac{I_{t,y}}{y}\right]\right],$$

where $I_{t,y} = \sqrt{4t + y^2}$.

We use this estimate in our analysis. Namely, we take the following approach. We fix parameters κ_1 and κ in $[\kappa_1, \kappa_M]$. Thus, in this manner here we fixed a coupling given by the choice of the initial measure $\mathbb{P}_0$ on the path space and by the relation (2.4), i.e. we couple to the Loewner chains with these drivers $\sqrt{\kappa_1}W_t^{\mathbb{P}_\kappa}$ and $\sqrt{\kappa}W_t^{\mathbb{P}_\kappa}$ when we vary the measures $\mathbb{P}_\kappa$.

Then, we can consider $W_t^{(1)} - W_t^{(2)} = \sqrt{\kappa_1}W_t^{\mathbb{P}_\kappa} - \sqrt{\kappa}W_t^{\mathbb{P}_\kappa}$. Using Lévy's characterization of the Brownian motion, we have that under any measure $\mathbb{P}^\kappa$, the process $\sqrt{\kappa_1}W_t^{\mathbb{P}_\kappa}$ is a Brownian motion multiplied with the diffusivity constant $\sqrt{\kappa_1}$ (indeed since for any measure $\mathbb{P}_\kappa$, the process $\sqrt{\kappa_1}W_t^{\mathbb{P}_\kappa}$ is a local martingale with the quadratic variation $\kappa_1 t$). We have then that for fixed κ_1 the difference $|f_t^{(1)}(u) - f_t^{(2)}(u)|$ is a function of κ.

Thus, we can estimate the difference using the above Lemma and the quasi-sure estimates on the derivatives of the maps $f_t(z)$, i.e. for any $\mathbb{P}_\kappa$ (obtained when choosing the drivers $\sqrt{\kappa_1}W_t^{\mathbb{P}_\kappa}$ and $\sqrt{\kappa}W_t^{\mathbb{P}_\kappa}$). Furthermore, we have the following remark.

Remark 6.4. Let $\beta \in \left(\frac{2}{2b_0 - \kappa r_0/2}, 1\right)$, with $b_0 = \kappa/2((1+4/\kappa)r_0 - r_0^2)$ and $r_0 = \frac{1}{4} + 2\kappa$ (as in the proof in the previous section). Then, it can be shown that (see Section 5.5 in [12]) for any fixed $\kappa \neq 8$, we have that

$$\mathbb{P}(|h_t'(i2^{-j})| \geqslant 2^{n(1-\theta)}) \leqslant c 2^{-(2+\varepsilon_1)n},$$

where c is a random constant that depends on κ and ω.

Thus, we can choose $\beta \in \left(\frac{2}{2b_0 - \kappa r_0/2}, 1\right)$ in order to bound the derivatives of the conformal maps. In order to simplify the analysis we bound the first derivative term using Remark 6.4 for fixed κ_1, i.e. we have $|f_t'(u)| \leqslant c y^{-\beta}$ $\mathbb{P}_\kappa$- *a.s.*, for any $\kappa \in [\kappa_m, \kappa_M]$ for $\beta \in \left(\frac{2}{2b_0 - \kappa r_0/2}, 1\right)$. For the other derivative term since the conformal maps are normalized at infinity there exists a constant $c < \infty$ depending only on T such that $|f_t'(z)| \leqslant c(y^{-1}+1)$ for all $z \in \mathbb{H}$ and all $t \in [0,T]$.

Then the estimate reads for any choice of the measure $\mathbb{P}_\kappa$, on complex numbers u such that their imaginary parts are elements of the dyadic partition of $[0,1]$ (in order to use the estimate (5.4))

$$|f_t^{(1)}(u) - f_t^{(2)}(u)| \leqslant c_3 \varepsilon_1(\kappa_1, \kappa) y^{-\sqrt{\frac{1+\beta}{2}}}$$

$\mathbb{P}_\kappa$-a.s., with $\varepsilon_1(\kappa_1, \kappa)$ a function of κ_1 and κ.

Let us consider $q > 0$ and

$$S_{n,j,k}(q) = \left[\frac{j-1}{2^{2n}}, \frac{j}{2^{2n}}\right] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \left[\frac{k-1}{2^{qn}}, \frac{k}{2^{nq}}\right],$$

and let

$$p_{n,j,k} = \left(\frac{j}{2^{2n}}, \frac{1}{2^n}, \frac{k}{2^{qn}}\right) \in S_{j,n,k},$$

be the corners of the boxes. In the following, we choose $q > 0$ and estimate the derivative of the the Loewner maps in corners of the boxes, as in [24]. Comparaed with the analysis in in [24], one important aspect is that as we change the parameter κ (and implicitly go along the κ axis in the Whitney boxes) we also change the measures $\mathbb{P}^\kappa$. In [24], the typical estimate on the derivative of the map on the corners of the boxes is combined with the application of the Borel-Cantelli Lemma in order to assure the analysis on a unique nullset of the Brownian motion driving the Loewner differential equation. The use of Borel-Cantelli in this approach restricts the applicability of the derivative estimate in the corners of the boxes for the values $\kappa < 2.1$ (more recently up to $\kappa < 8/3$ with new estimates in [10]). The novelty is that we use the polar set outside of which the aggregated solution is defined and then we can vary $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$. In this way, we argue that the estimates on the derivative of the maps in the corners $p_{n,j,k}$ of the Whitney boxes hold q.s. In this new setting, we avoid the restriction to the interval $\kappa \in [0, 8(2-\sqrt{3}))$, since the estimate on the derivative used in the proof of Theorem 2.6 can be used simultaneously for a family of probability measures $\mathbb{P}_\kappa$ for $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0, \varepsilon) \cup \{8\})$.

We give the following version of Lemma 3.3 in [24], that does not contain the restriction on the κ interval, due to the application of the Borel-Cantelli Lemma.

Lemma 6.5. *Let $\varepsilon > 0$. Let $\kappa \in [\kappa_m, \kappa_M] = \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, then q.s. there exists a random constant $c = c(\varepsilon, \beta, q, \omega) < +\infty$ such that $|F'(p_{n,j,k})| \leqslant c2^{n\beta}$ for all pairs $(n,j,k) \in \mathbb{N}^3$ such that $p_{n,j,k} \in [0,1] \times [0,1] \times [\kappa_m, \kappa_M]$.*

Proof. Using the analysis from the previous sections, we have that

$$\sum_{j=1}^{2^{2n}} \mathbb{P}_\kappa \left[|F'(p_{n,j,k})| \geqslant 2^{n\beta} \right] \leqslant c2^{-n\sigma}, \tag{6.1}$$

where the parameter σ depends on κ Following the analysis in the previous section one can show that the constant $c(\omega, \kappa)$ can be chosen to be a continuous function of κ (see the analysis of the parameters using the optimization procedure at Section 5.5 in [12]). Thus, one can take the supremum of this constant in the interval of κ that one considers. According to the analysis in the previous section, we have that the series 6.1 is convergent for all $\kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, i.e.

$$\sum_{j=1}^{2^{2n}} \sup_\kappa \mathbb{P}_\kappa \left[|F'(p_{n,j,k})| \geqslant 2^{n\beta} \right] \tag{6.2}$$

is convergent for any choice of measure $\mathbb{P}_\kappa$ with $\kappa \in [\kappa_m, \kappa_M] = \kappa \in \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$,

□

The next step is to use Distortion Theorem along with Lemma 6.3 in order to push the estimate on the derivative from the corners of the box to all the points inside. The result is captured in the following Lemma. We emphasize that in [24], there are two parts of the analysis in order to obtain this estimate for fixed κ, i.e. the analysis is split into the cases κ near 0 and the complementary regime. In our setting, we discuss only the case $\kappa > \varepsilon$ for any $\varepsilon > 0$ since the a.s. continuity in κ of the traces in the regime $\kappa \in [0, 8(2-\sqrt{3}))$ was proved in [24] already.

Lemma 6.6. *Let $\kappa \in [\kappa_m, \kappa_M] = \mathcal{K} \cap \mathbb{R}_+ \setminus ([0,\varepsilon) \cup \{8\})$, then for every $\varepsilon > 0$ there exists $\delta > 0$ and $q > 0$ and a random constant $c = c(q, \varepsilon, \omega, \beta) < \infty$ such that $diam(F(S_{n,j,k})) \leqslant c2^{-n\delta}$, quasi-surely for all $(n,j,k) \in \mathbb{N}^3$ with $p_{n,j,k} \in [0,1] \times [0,1] \times [\kappa_m, \kappa_M]$.*

Proof. We will show that there exists $\delta > 0$ such that $|F(p) - F(p_{n,j,k})| \leqslant cn2^{-n\delta}$. Let us fix $\kappa_1 \in [\kappa_m, \kappa_M]$. We estimate for $|\Delta t| \leqslant y^2$, using Lemmas 6.5 and 6.2

$$|F(t + \Delta t, y, \kappa_1) - F(t, y, \kappa_1)| \leqslant cy|F'(p_{n,j,k})| \leqslant c'2^{-n(1-\beta)}$$

quasi surely with $c' = c'(\beta, q, \omega)$.

By Koebe Distortion Theorem and Lemmas 6.5 and 6.2, we obtain that

$$|F(t+\Delta t, y+\Delta y, \kappa_1) - F(t+\Delta t, y, \kappa_1)| \leqslant cy|F'(p_{n,j,k})| \leqslant c2^{-n(1-\beta)},$$

quasi-surely.

Let $\phi(\beta) = \sqrt{\frac{1+\beta}{2}}$. Using Lemma 6.3 and estimating

$$\sup_{t\in[0,1]} |\sqrt{\kappa_1 + \Delta\kappa} W_t^{\mathbb{P}_\kappa} - \sqrt{\kappa_1} W_t^{\mathbb{P}_\kappa}| \leqslant c\Delta\kappa \sup_{t\in[0,1]} |W_t^{\mathbb{P}_\kappa}| \leqslant c'\Delta\kappa\,,$$

where $c' = c'(\omega, \varepsilon)$, we obtain that

$$|F(t+\Delta t, y+\Delta y, \kappa_1 + \Delta\kappa) - F(t+\Delta t, y, \kappa_1)| \leqslant c\Delta\kappa y^{-\phi(\beta)} \log(y^{-1}) \leqslant cn2^{-n(q-\phi(\beta)}\,,$$

quasi-surely. We choose $\delta = \min\{1-\beta, q-\phi(\beta)\}$ that is clearly positive for the right choice $q > \phi(\beta)$, and we finish the proof.

$\square$

In order to finish the proof of Theorem 6.1, we redo the exact elements of Theorem 4.1 in [24] under all the measures $\mathbb{P}_\kappa$, in our coupling. We consider the family of measures $\mathbb{P}_\kappa$ for $\kappa \in [\kappa_m, \kappa_M]$. In order to achieve it we estimate under all the measures $\mathbb{P}_\kappa$ for $\kappa \in [\kappa_m, \kappa_M]$ using the previous lemma (i.e. the bound on the diameters of the Whitney boxes) in the following manner $|F(t, y, \kappa_1) - F(t, y, \kappa)| \leqslant |F(t, y, \kappa_1) - F(t, 2^{-N}, \kappa_1)| + |F(t, 2^{-N}, \kappa_1) - F(t, 2^{-N}, \kappa)| + |F(t, 2^{-N}, \kappa) - F(t, y, \kappa)| \leqslant C\sum_{n=N}^{\infty} 2^{-n\delta}$ as we vary the parameter $\kappa \in [\kappa_m, \kappa_M]$.

When comparing the fixed value κ_1 and any other $\kappa_2 \in [\kappa_m, \kappa_M]$, (w.l.o.g $\kappa_2 > \kappa_1$) we obtain that $|F(t, y, \kappa_1) - F(t, y, \kappa_2)| \leqslant |F(t, y, \kappa_1) - F(t, 2^{-N}, \kappa_1)| + |F(t, 2^{-N}, \kappa_1) - F(t, 2^{-N}, \kappa_2)| + |F(t, 2^{-N}, \kappa_2) - F(t, y, \kappa_2)| \leqslant C\sum_{n=N}^{\infty} 2^{-n\delta} \leqslant C2^{-N\delta} \leqslant C|\kappa_1 - \kappa_2|^{\delta/q}$, where we have used the stopping time $N = O(-\log|\kappa_1 - \kappa_2|^{1/q})$ given by the bounds $2^{-qN} < |\kappa_1 - \kappa_2| \leqslant 2^{-q(N-1)}$.

Then, for any choice $\kappa_2 \in [\kappa_m, \kappa_M]$, when taking $y \to 0+$ we get

$$|\gamma^{\kappa_1}(t) - \gamma^{\kappa_2}(t)| \leqslant C|\kappa_1 - \kappa_2|^{\delta/q}$$

under the $\mathbb{P}_{\kappa_2}$ measure.

Furthermore, for a sequence of parameters $(\kappa_i)_{i\in\mathbb{N}}$ started from κ_2 converging to κ_1 w.l.o.g from above, we can obtain the estimate

$$|\gamma^{\kappa_1}(t) - \gamma^{\kappa_i}(t)| \leqslant C|\kappa_1 - \kappa_i|^{\delta/q}$$

under the measures $\mathbb{P}_{\kappa_i}$, for $i \in \mathbb{N}$. Since κ_1 was arbitrary the same analysis works for any choice of $\kappa \in [\kappa_m, \kappa_M]$.

Thus outside a polar set that depends on the choice of the nontrivial interval $[\kappa_m, \kappa_M]$, we obtain the desired result.

$\square$

Remark 6.7. We believe that, as in [24], once we have the decay of the diameter of the boxes simultaneously for all measures $\mathbb{P}_\kappa$, we believe one can also obtain the bound $|\gamma^{(\kappa_1)}(t_1) - \gamma^{(\kappa_2)}(t_2)| = |f_{t_1}^{(\kappa_1)}(0+) - f_{t_1}^{(\kappa_2)}(0+)| = O(|t_1 - t_2|^{\delta/2}) + O(|\kappa_1 - \kappa_2)^{\delta})$, when considers two different times t_1 and t_2 in $[0, T]$. However, we do not provide this analysis in this paper.

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