Splitting Simulation and Convergence of Stochastic Loewner Curves

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Abstract

In this paper, we study the approximations of the SLE_κ traces via the Ninomiya-Victoir splitting scheme. We prove a strong convergence in probability *w.r.t.* the *sup*-norm to the distance between the SLE trace and the output of the Ninomiya-Victoir splitting scheme. We show also that an L^p convergence of the scheme can be achieved under a set of assumptions. In the last section we show the uniform convergence of the approximation of the SLE trace obtained using a different scheme that is based on the linear interpolation of the Brownian driver.

1 Introduction

The Loewner equation was introduced by Charles Loewner in 1923 in and it was one of the important ingredients in the proof of the Bieberbach Conjecture that was done by Louis de Branges, years later in 1985 in. In 2000, Oded Schramm introduced a stochastic version of the Loewner equation. The stochastic version of the Loewner evolution, i.e. the Schramm-Loewner evolution, SLE_κ, generates a one parameter family of random fractal curves that are proved to describe scaling limits of a number of discrete models that appear in two-dimensional statistical physics.

In the last years, results on numerical schemes to approximate the SLE_κ traces, along with mathematical proofs of their convergence, appeared in the body of literature on the topic. For example, V. Beffara used a Euler Scheme to produce approximations of the SLE_κ hulls.

There are various versions of Loewner equations. This article focuses on the forward Loewner equations defined in the upper half-plane $\mathbb{H}$ and a fixed time interval $[0, T]$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \tag{1.1}$$

with the initial condition $g_0(z) = z$, for all $z \in \mathbb{H}$ and the continuous driving force $\lambda : [0, T] \to \mathbb{R}$.

The family of maps $(g_t)_{0 \le t \le T}$ is called the Loewner chain. For all $z \in \mathbb{H}$, the solution of the above forward Loewner equation is uniquely defined up to $T_z = \inf\{t \ge 0,\, g_t(z) = \lambda(t)\}$. Over time, the set $K_t = \{z \in \mathbb{H},\, T_z \le t\}$, called the hull, grows. It is also known that for all $t \in [0, T]$, $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ is the unique conformal map satisfying the hydrodynamic normalization at ∞

$$\lim_{z \to \infty} [g_t(z) - z] = 0. \tag{1.2}$$

In addition, we work with Loewner chains $(g_t)_{0 \le t \le T}$ parametrized by upper half-plane capacity

$$g_t(z) = z + \frac{2t}{z} + o(1/|z|), \;\; as\ |z| \to \infty, \tag{1.3}$$

where by ([1], *Lem.* 4.1), the coefficient of the z-term is 1 and each coefficient a_k of the term z^{-k}, $k \in \mathbb{N}_+$ is real.

We are particularly interested in the case when the Loewner chain $(g_t)_{0 \le t \le T}$ is generated by a simple curve $\gamma : [0, T] \to \mathbb{H} \cup \{\lambda(T)\}$. In other words, the domain of $g_t(z)$ is the unbounded component of $\mathbb{H}\backslash\gamma([0, t])$. By ([2], *Thm.* 4.1), this is equivalent to the existence and continuity in $0 < t < T$ of

$$\gamma(t) = \lim_{\epsilon \to 0^+} g_t^{-1}(\lambda(t) + i\epsilon). \tag{1.4}$$

Even though the piecewise linear interpolation method is independent to our main topic on splitting simulation, let us briefly discuss that idea to simulate the driving force $\lambda(t)$ and its corresponding hull K_t for all $t \in [0, T]$. The interpolation algorithm is based on the following observations. Fix $s > 0$, let $(\widetilde{g}_t)_{0 \le t \le T}$ be the Loewner chain generated by the continuous driving force $\widetilde{\lambda}(t) = \lambda(s + t)$ with $0 \le t \le T - s$. In fact, this solution admits the form

$$\partial_t g_{s+t} \circ g_s^{-1}(z) = \frac{2}{g_{s+t} \circ g_s^{-1}(z) - \lambda(s + t)} = \frac{2}{g_{s+t} \circ g_s^{-1}(z) - \widetilde{\lambda}(t)}, \tag{1.5}$$

and $g_s \circ g_s^{-1}(z) = z$ with $z \in \mathbb{H}\backslash K_s$. The uniqueness property of Loewner chains implies that $\widetilde{g}_t(z) = g_{s+t} \circ g_s^{-1}(z)$. And we denote $\widetilde{K}_t$ to be the hull associated with the Loewner chain $\widetilde{g}_t$, indeed

$$\mathbb{H} \cap g_s(K_{s+t}) = \widetilde{K}_t \;\text{ and }\; K_{s+t} = K_s \cup g_s^{-1}(\widetilde{K}_t). \tag{1.6}$$

Hence, computing K_s and g_s^{-1} and $\widetilde{K}_t$ would enable us to compute $\widetilde{K}_{s+t}$.

There are methods interpolating $\lambda(t)$ with discretized force $\lambda^n(t)$. Using square-root ([11], *Sec.* 2.) or linear interpolation, there is usually an easier form of $\lambda^n(t)$, which endows computational convenience. The driving force $\lambda^{(n)}$ in the context of square-root and linear interpolation is written below

$$\lambda^n_{square\text{-}root}(t) \coloneqq \sqrt{n}\big(\lambda(t_{k+1}) - \lambda(t_k)\big)\sqrt{t - t_k} + \lambda(t_k) \text{ on } [t_k, t_{k+1}], \tag{1.7a}$$

$$\lambda^n_{linear}(t) \coloneqq n\big(\lambda(t_{k+1}) - \lambda(t_k)\big)(t - t_k) + \lambda(t_k) \text{ on } [t_k, t_{k+1}]. \tag{1.7b}$$

We will briefly discuss the linear interpolation method in the last section.

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2 Splitting simulation

In this section we propose an algorithm to simulate Lowener chains whose hulls in $\mathbb{H}$ are simple curves. To simplify notation, we set $T = 1$, which is irrelevant to the convergence analysis in the next section. We further require our driving force to be *weakly* Hölder-1/2 continuous. In fact, this seemingly strong constraint allows us to extend our result to SLE_κ with $\kappa \neq 8$, which are *stochastic* Loewner chains generated by Brownian motions.

Throughout our paper, we work with $(\Omega, \mathcal{F}, \mathbb{P})$ that is a standard probability space, where the Brownian motion is defined.

The intuition of SLE_κ comes from the theory of statistical physics. In ([17], *Sec.* 1.1) it is shown that the scaling limit of loop erased random walk, with the loops erased in a chronological order, converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from statistical mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and uniform spanning trees were proved to converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$, and $\kappa = 8$ respectively in the series of works by Chelkak, Duminil-Copin, Hongler, Izyurov, Smirnov, Schramm, and Lawler.

SLE_κ traces are generated by the Loewner chains driven by Brownian motions $\sqrt{\kappa}B_t$ with $t \in [0, 1]$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \tag{2.1}$$

with $g_0(z) = z$ for all $z \in \mathbb{H} \backslash K_t$. In the following discussion of SLE_κ, we focus on the case where the hulls K_t are generated by a simple curve $\gamma : [0, 1] \to \mathbb{H} \cup \{0\}$. Throughout the proof we will use the backward Loewner equation which is related to the forward version *Eqn.* (2.1) and admits the form

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa} B_{1-t}}, \tag{2.2}$$

with $h_0(z) = z$. This backward equation generates a Loewner curve $\eta : [0, 1] \to \mathbb{H} \cup \{0\}$. Notice that $g_t(z)$ and $h_t(z)$ are both driven by Brownian motions with different time directions. In fact, there is a correspondence ([13], *Sec.* 1.1) bewteen the dynamics constrained by *Eqn.* (2.1) and *Eqn.* (2.2): The random set $\eta([0, 1])$ has the same law as the chordal SLE_κ trace $\gamma([0, 1])$ modulo a real scalar shift $\sqrt{\kappa} B_{T=1}(\omega)$ with $\omega \in \Omega$. For convenience, we call $\eta(t)$ the shifted Loewner curve.

There are many schemes to simulate SLE_κ traces. This article focuses on the Ninomiya-Victoir scheme.

At this point, some new notations are necessary before we formally present the Ninomiya-Victoir scheme. If we set $\widehat{g}_t(z) := g_t(z) - \sqrt{\kappa} B_t$ for all $\in \mathbb{H} \backslash K_t$, from the existence and continuity of $\gamma(t)$ discussed in Sec. 1, we write

$$\gamma(t) = \lim_{y \to 0^+} g_t^{-1}(\sqrt{\kappa} B_t + iy) = \lim_{y \to 0^+} \widehat{g}_t^{-1}(iy), \tag{2.3}$$

and the stochastic differential equation

$$\begin{aligned} d\widehat{g}_t(z) &= \frac{2}{\widehat{g}_t(z)} dt - \sqrt{\kappa} dB_t, \\ \widehat{g}_0(z) &= z, \end{aligned} \tag{2.4}$$

with $z \in \mathbb{H} \backslash K_t$. Remember that the Loewner curve $\eta(t)$ has the same law as $\gamma(t)$ up to a translation $\sqrt{\kappa} B_{T=1}(\omega)$ with $\omega \in \Omega$. In fact, we will simulate the Loewner curve $\eta(t)$ instead of $\gamma(t)$ for convenience, since the former curve preserves statistical properties of the latter curve. Let us consider

$$\begin{aligned} dZ_t &= -\frac{2}{Z_t} dt + \sqrt{\kappa} dB_t, \\ Z_0 &= iy, \end{aligned} \tag{2.5}$$

where $y > 0$ is taken sufficiently small and we avoid the choice $Z_0 = 0$, due to the singularity at 0 inherent in the boundary condition of *Eqn.* (2.5). The format of (2.5) follows from $Z_t(z) := h_t(z) - \sqrt{\kappa} B_t$, for all $z \in \mathbb{H}$ (see [3], *Eqn.* 6.5).

Then with the choice of initial value $Z_0 = iy$, we employ the Ninomiya-Victoir splitting scheme to derive a numerical solution.

Definition 2.1. ***Ninomiya-Victoir Scheme***
Consider n-dimensional SDE on $\mathbb{R}_+$ with the form

$$\begin{aligned} dW_t &= L_0(W_t)dt + L_1(W_t)dB_t, \\ W_0 &= \xi, \end{aligned} \tag{2.6}$$

where $\xi \in \mathbb{R}^n$ and $L_i : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. For all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, let the flow $\exp(tL_i)x$, $i = 1, 2$, denote the unique solution at time $u = 1$ to the ODE

$$\begin{aligned} \frac{dy}{du} &= tL_i(y), \\ y(0) &= x. \end{aligned} \tag{2.7}$$

For a fixed iteration step $n \in \mathbb{N}$, we choose an arbitrary, possibly non-uniform, partition $\{t_0 = 0, t_1, \dots, t_n = 1\}$ with step-size $h_k = t_{k+1} - t_k$. We approximate a numerical solution $\{\widetilde{W}_{t^-}\}_{0\le k\le n}$ in the sense that $\widetilde{W}_0 = \xi$ and

$$\widetilde{W}_{t_{k+1}} = \exp\left(\frac{1}{2}h_k L_0\right) \exp\left(B_{t^-,t_{k+1}} L_1\right) \exp\left(\frac{1}{2}h_k L_0\right) \widetilde{W}_{t^-}, \tag{2.8}$$

for all $k = 0, 1, \dots, n$ and where $B_{t^-,t_{k+1}}$ is the abbreviation for $B_{t_{k+1}} - B_{t^-}$. In fact, the approximation $\{\widetilde{W}_{t^-}\}_{0\le k\le n}$ enjoys an integral form between every two discretization points

$$\widetilde{W}_t = \xi + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(2)})ds + \int_0^t L_1(\widetilde{W}_s^{(1)})dB_s + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(0)})ds, \tag{2.9}$$

where the above three discretization processes defined on each time interval $[t^-, t_{k+1}]$ admit the form

$$\begin{aligned} \widetilde{W}_t^{(0)} &:= \exp\left(\frac{1}{2}(t - t^-)L_0\right)\widetilde{W}_{t^-}, \\ \widetilde{W}_t^{(1)} &:= \exp\left(B_{t^-,t} L_1\right)\widetilde{W}_{t_{k+1}}^{(0)}, \\ \widetilde{W}_t^{(2)} &:= \exp\left(\frac{1}{2}(t - t^-)L_0\right)\widetilde{W}_{t_{k+1}}^{(1)}. \end{aligned} \tag{2.10}$$

Looking back to our Loewner equation, we write $L_0(z) = -2/z$ and $L_1(z) =$

$\sqrt{\kappa}$. Hence, by ([3], *Thm.* 6.2) the following form is immediate

$$\begin{aligned} \exp(tL_0)z &= \sqrt{z^2 - 4t}, \\ \exp(tL_1)z &= z + \sqrt{\kappa}t. \end{aligned} \tag{2.11}$$

Given the above arbitrary partition $\{t_0 = 0, t_1, \ldots, t_n = 1\}$, we could formulate an approximated solution $\{\widetilde{Z}_{t^-}\}_{0\le k\le n}$ via the Ninomiya-Victoir splitting scheme

$$\widetilde{Z}_{t_{k+1}} \coloneqq \sqrt{\left(\sqrt{\widetilde{Z}^2_{t^-} - 2h_k} + \sqrt{\kappa}B_{t^-,t_{k+1}}\right)^2 - 2h_k}\,, \tag{2.12}$$

with the initial value $\widetilde{Z}_0 = iy$ specified at each n^{th} iteration.

3 Strong convergence in probability

When simulating SLE_κ traces, we manually choose a partition on the time interval $[0, 1]$. However, a fixed uniform partition may overlook the effect of some sample Brownian paths and may result in huge numerical errors. Instead, we propose an adaptive method to control step size.

Definition 3.1. *Let $Z_t(iy_n)$ be the solution to Eqn.* (2.5) *started from $iy_n \in \mathbb{H}$, and $\widetilde{Z}_t(iy_n)$ be its approximation following Ninomiya-Victoir Scheme, and let $\eta(t)$ be the shifted Loewner curve defined above.*

The main scheme is choosing some partitions on $[0, 1]$ and some point in the positive imaginary axis. We will discuss how close the evolution of this point under splitting simulation in *Sec.* 2. resembles the original backward Loewner map. In fact, under each iteration we choose an initial point in the imaginary axis slightly closer to the origin point, which is important to ensure the simulation works. We work with a uniform partition in the proof that is further refined via a so-called "mid-point" insertion.

The idea is choosing a constant $\tau > 0$, called tolerance, to ensure that

$$\left|\widetilde{Z}_{t_{k+1}} - \widetilde{Z}_{t_k}\right| \le \tau \tag{3.1}$$

for each k. To achieve this, we start by computing $\widetilde{Z}_t$ along a prior uniform partition until $\left|\widetilde{Z}_{t_{k+1}} - \widetilde{Z}_{t_k}\right| > \tau$. If this event occurs, we reduce the step size h_k of the SLE_κ discretization, that is we insert the mid-point of this interval $[t^-, t_{k+1}]$ into the partition.

In other words, we refine the prior partition via mid-point insertion whenever necessary. Notice that the choice of a refined partition actually depends on

$\omega \in \Omega$ because the evolution $(\widetilde{Z}_t)_{0 \le t \le 1}$ contains Brownian motion.

In this paper we study the above simulating scheme. We will give a strong convergence in probability to the decay rate of the $\|\cdot\|_{[0,1],\infty}$ norm (*i.e. supremum norm*) between the original Loewner curve and our simulation. In the next section, we are going to alternatively discuss the general formalism to prove the L^p convergence of our splitting simulation, which is the same object *w.r.t.* a different topology.

Notice that at each iteration step $n \in \mathbb{N}$, we specify an initial condition $y \in \mathbb{R}_+$ and let the approximated sample paths $(\widetilde{Z}_t(iy))_{0 \le t \le 1}$ evolve according to the backward Loewner equation (2.5). To ensure a decent $\|\cdot\|_{[0,1],\infty}$ convergence result, we not only require the mesh of the partition tends to 0, but also choose a sequence $\{y_n\} \subset \mathbb{R}_+$ so that $y_n \to 0^+$ strictly monotonically.

Remark 3.2. *Notice that we cannot let $y_n \equiv y$ for some $y > 0$, otherwise the convergence pattern breaks down and hence strict monotonicity of $\{y_n\}$ is necessary. On the other hand, the decay rate of $\{y_n\}$ should not be too fast to destroy the probability inequality w.r.t. the $\|\cdot\|_{[0,1],\infty}$ norm, which we will see in the following context.*

In this section, we manually set $y_n = n^{-1/2}$ for all $n \in \mathbb{N}$. This choice of $\{y_n\}$ actually satisfies the requirements in the above *Rmk*. 3.2 for the initial conditions.

For now we choose a prior uniform partition $\mathcal{D}_n$ on $[0, 1]$ at each step $n \in \mathbb{N}$, before writing a refined partition $\widetilde{\mathcal{D}}_n$ via the above adaptive method. Notice again that $\widetilde{\mathcal{D}}_n$ generally depends on $\omega \in \Omega$, since the driving force is Brownian motion.

To reach our main object, we make the following observation.

Definition 3.3. *For all $t \in [0, 1]$, given an arbitrary uniform partition $\mathcal{D}_n$, we define $\{t^-, t_{k+1}(t)\} \subset \mathcal{D}_n$ to be the neighboring two points in the partition $\mathcal{D}_n$ between which t resides, i.e. $t^- \le t < t_{k+1}(t)$. We also use the same notation $\{t^-, t_{k+1}(t)\}$ for the refined partition $\widetilde{\mathcal{D}}_n$.*

In the following, we use $\|\cdot\|_{[0,1],\infty}$ for the sup-norm on the interval $[0, 1]$. We are now reaching our main object: an upper bound for the probability of $\|\cdot\|_{[0,1],\infty}$ norm of $(\eta(t) - \widetilde{Z}_t(iy_n))_{0 \le t \le 1}$ to be small in the sense given by the following theorem, which is the main result of the paper

Theorem 3.4. *Let $\eta(t)$ be the backward SLE trace for $\kappa \le 4$ and let $\tau \in \mathbb{R}_+$. There exist two non-increasing functions $\varphi_i : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \varphi_i(n) = 0^+$ with $i = 1, 2$. If the mesh $||\widetilde{D}_n|| \to 0^+$ with $n \to \infty$, faster than a proper* *rate* $||\widetilde{D}_n|| = o(n^{-3})$, *then*

$$\mathbb{P}\left(\left\|\eta(t) - \widetilde{Z}_t(iy_n)\right\|_{[0,1],\infty} \le \varphi_1(n)\right) \ge 1 - \varphi_2(n), \tag{3.2}$$

where $\widetilde{\mathcal{D}}_n$ *is the refined partition via tolerance* $\tau > 0$.

To prove this theorem, we use

$$\begin{aligned} \left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right| &\le |Z_t(iy_n) - Z_{t^-}(iy_n)| \\ &+ \left|Z_{t^-}(iy_n) - \widetilde{Z}_{t^-}(iy_n)\right| + \left|\widetilde{Z}_{t^-}(iy_n) - \widetilde{Z}_t(iy_n)\right|, \end{aligned} \tag{3.3a}$$

in addition to estimate

$$\left|\eta(t) - \widetilde{Z}_t(iy_n)\right| \le |\eta(t) - Z_t(iy_n)| + \left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right|. \tag{3.3b}$$

We study *Ineq.* (3.3*a*) and *Ineq.* (3.3*b*). In the following, we investigate *Ineq.* (3.3*a*). This inequality follows from the lemma below.

Lemma 3.5. ([4], *Thm.* 3.4.2) *There exist* $c_1, c_2 > 0$ *such that if we consider the event*

$$E'_{n,1} := \left\{ osc(\sqrt{\kappa}B_t, \frac{1}{n}) \le c_1\sqrt{\frac{\log(n)}{n}} \right\}, \tag{3.4}$$

then we have

$$\mathbb{P}\big(E'_{n,1}\big) \ge 1 - \frac{c_2}{n^2}. \tag{3.5}$$

Lemma 3.6. ([11], *Eqn.* 21.) *There exist* $c_3, c_4 > 0$ *and* $\beta_1 \in (0,1)$ *such that if we consider the event*

$$E''_{n,1} := \left\{ \left|\partial_z \widehat{g}_t^{-1}(iv)\right| \le c_3 \cdot v^{-\beta_1} \text{ for all } t \in [0,1] \text{ and } v \in [0, \frac{1}{\sqrt{n}}] \right\}, \tag{3.6}$$

then we have

$$\mathbb{P}\big(E''_{n,1}\big) \ge 1 - \frac{c_4}{n^{c_3/2}}. \tag{3.7}$$

We have an estimate to the first term to *Ineq.* (3.3*a*) with the form

$$\begin{aligned} |Z_t(iy_n) - Z_{t^-}(iy_n)| \le |Z_t(iy_n) - \eta(t)| + \big|Z_{t^-}(iy_n) - \eta\big(t^-\big)\big| \\ + \big|\eta(t) - \eta\big(t^-\big)\big|. \end{aligned} \tag{3.8}$$

To proceed our discussion, we remind our readers of the following definition.

Definition 3.7. *A continuous function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is called a subpower* *function if it is non-increasing and satisfies*

$$\lim_{x\to\infty} x^{-\nu}\phi(x) = 0,\ \textit{for all}\ \nu > 0. \tag{3.9}$$

Remark 3.8. *A typical subpower function is* $\phi(x) = (\log x)^\alpha$, *for all real* $\alpha > 0$.

With the notion of a subpower function, we have the following result.

Proposition 3.9. *Let* $\beta_1 \in (0,1)$. *There exists a subpower function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that if we consider the event*

$$E^*_{n,1} := \left\{ \left\|\eta(t) - \eta\big(t^-\big)\right\|_{[0,1],\infty} \leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} \right\}, \tag{3.10}$$

and if $||\mathcal{D}_n|| \leq n^{-1}$, *then*

$$\mathbb{P}\big(E^*_{n,1}\big) \geq 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}}. \tag{3.11}$$

Proof. In the proof we omit the bracket in t^- and simply write this term as t^-, which will be clear from the context. In the proof, we follow the statement in ([11], *Lem.* 2.5) with some obvious changes of notations. Since $\eta([0,1])$ has identical distribution to $\gamma([0,1])$ modulo a scalar shift $\sqrt{\kappa}B_1$, it is immediate that $\mathbb{P}\big(E^*_{n,1}\big)$ is equal to the probability of the event with an expression which we substitute $\gamma(t)$ (*and resp.* $\gamma\big(t^-\big)$) into $\eta(t)$ (*and resp.* $\eta\big(t^-\big)$). By ([11], *Lem.* 2.5) there exists a subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, on the event $E'_{n,1} \cap E''_{n,1} \subset \Omega$, provided $0 \leq t - t^- \leq n^{-1}$ for all $t \in [0,1]$, we have

$$\begin{aligned} \big|\gamma(t) - \gamma\big(t^-\big)\big| &\leq \phi(\sqrt{n})\bigg(\int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr + \int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr\bigg) \\ &\leq \phi(n) \cdot \frac{2}{1-\beta_1} n^{-(1-\beta_1)/2}, \end{aligned} \tag{3.12}$$

where $\beta_1 \in (0,1)$. Hence $\mathbb{P}\big(E^*_{n,1}\big) \geq \mathbb{P}\big(E'_{n,1} \cap E''_{n,1}\big)$ and the conclusion follows. $\square$

To finish the evaluation of *Eqn.* (3.8) and then finishing the first term in *Ineq.* (3.3*a*), we have the following result

Proposition 3.10. *Let* $\epsilon_0 \in (0,1)$. *If we choose* $M_n = n^{(1-\epsilon_0)/4}$ *and consider the event*

$$E^{**}_{n,1} := \left\{ \|Z_t(iy_n) - \eta(t)\|_{[0,1],\infty} \leq M_n \cdot y_n^{1-\epsilon_0} = \frac{1}{n^{(1-\epsilon_0)/4}} \right\}, \tag{3.13}$$

then there exists $\epsilon_n \to 0^+$ monotonically such that

$$\mathbb{P}\big(E^{**}_{n,1}\big) \geq 1 - \epsilon_n. \tag{3.14}$$

Proof. It is stated in ([3], *Lem.* 6.7) that there exists $\epsilon_0 \in (0,1)$ so that *almost surely*, we have

$$\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq C_1'(\omega) \cdot y_n^{1-\epsilon_0}, \tag{3.15}$$

where $C_1'(\omega)$ is *almost surely* finite. Guaranteed with the existence of at least one $C_1'(\omega) \in \mathbb{R}_+$ for almost all $\omega \in \Omega$, we define the collection $\mathcal{A}(\omega) \subset \mathbb{R}_+$ for those $\omega \in \Omega$ with which there exists at least one $C_1'(\omega)$ satisfying *Ineq.* (3.15). Notice that the collection $\mathcal{A}(\omega)$ is defined except for a *measure-zero* event. The well-ordering principle tells us that $\mathcal{A}(\omega)$ has a lower bound. Hence it is legitimate to define

$$C_1(\omega) := \inf \mathcal{A}(\omega), \tag{3.16}$$

which is *almost surely* defined. Hence, we could simply assume $C(\omega)$ exists and is finite everywhere via subtracting a *measure-zero* event from Ω. With our choice of $M_n \to \infty$, there exists $\epsilon_n \in [0,1]$ with $\epsilon_n := \min\{\epsilon_1, \ldots, \epsilon_{n-1}, \mathbb{P}\big(\Omega - E^{**}_{n,1}\big)\}$ such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq M_n \cdot y_n^{1-\epsilon_0}\right) \geq 1 - \epsilon_n. \tag{3.17}$$

On the event $E^{**}_{n,1} \subset \Omega$, we know then

$$\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq C_1(\omega) \cdot y_n^{1-\epsilon_0} \text{ and } \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq M_n \cdot y_n^{1-\epsilon_0}. \tag{3.18}$$

By the definition of $C_1(\omega)$, it is then clear that on the event $E^{**}_{n,1}$, we have

$$C_1(\omega) \leq M_n. \tag{3.19}$$

Hence, on the event $E^{**}_{n,1}$, it is immediate that

$$\sup_{t\in[0,1]} |Z_t(iy_{n+1}) - \eta(t)| \leq C_1(\omega) \cdot y_{n+1}^{1-\epsilon_0} \leq M_n \cdot y_{n+1}^{1-\epsilon_0} \leq M_{n+1} \cdot y_{n+1}^{1-\epsilon_0}. \tag{3.20}$$

Hence the event $E^{**}_{n+1,1}$ occurs and

$$\mathbb{P}\big(E^{**}_{n+1,1}\big) \geq \mathbb{P}\big(E^{**}_{n,1}\big), \tag{3.21}$$

which justifies our choice of definition of ϵ_n, from which the monotonicity of $\{\epsilon_n\} \subset \mathbb{R}_+$ is easily seen. To show that $\epsilon_n \to 0^+$, we notice that $C_1(\omega)$ is *almost* *surely* finite. Suppose $\epsilon_n \to \sigma > 0$. Then with probability σ, the constant $C_1(\omega)$ is greater than any $M_n,\ n \in \mathbb{N}_+$. Since $M_n \to \infty$, we are forced to conclude that $C_1(\omega) = \infty$ with positive probability, which is impossible. $\square$

We have now discussed every term in *Eqn.* (3.8), it is time to finalize the estimate of the first term in *Ineq.* (3.3*a*).

Proposition 3.11. *Let $\beta_1 \in (0,1)$. Given the assumptions that $||\mathcal{D}_n|| \leq n^{-1}$ and $y_n = n^{-1/2}$, if we define the event*

$$E_{n,1} := \left\{ \|Z_t(iy_n) - Z_{t^-}(iy_n)\|_{[0,1],\infty} \leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{2}{n^{(1-\epsilon_0)/4}} \right\}, \tag{3.22}$$

then the following inequality holds

$$\mathbb{P}\big(E_{n,1}\big) \geq 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}} - 2\epsilon_n. \tag{3.23}$$

Proof. On the event $E^*_{n,1} \cap E^{**}_{n,1}$, we know that for $\beta_1 \in (0,1)$, we have

$$\begin{aligned} \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| &\leq \frac{1}{n^{(1-\epsilon_0)/4}}, \\ \sup_{t\in[0,1]} \big|\eta(t) - \eta\big(t^-\big)\big| &\leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}}. \end{aligned} \tag{3.24}$$

Looking back to *Eqn.* (3.8), we see $\mathbb{P}\big(E_{n,1}\big) \geq \mathbb{P}\big(E^*_{n,1} \cap E^{**}_{n,1}\big) \geq 1 - c_2 n^{-2} - c_4 n^{-c_3/2} - 2\epsilon_n$. $\square$

Hence we have estimated the $\|\cdot\|_{[0,1],\infty}$ norm of the first term in *Ineq.* (3.3*a*). This is in fact the most complicated term among these three terms. Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the second term.

Inspecting *Eqn.* (2.12), we observe that the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ resembles a forward Loewner map driven by constant forces on the local time interval $[t^-, t_{k+1}]$. In fact, this is the case. We are going to split the total time interval $[0,1]$ into time sub-intervals $[t^-, t_{k+1}]$. On each time sub-interval, the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ is a composition of two local backward Loewner maps driven by constant forces with an intermediate parallel translation.

Lemma 3.12. ([10], *Sec.* 2.) *Given a constant driving force $t \mapsto A$ on the time interval $[0,T]$, the forward Loewner chain admits the form*

$$g_t(z) = A + \big[(z-A)^2 + 4t\big]^{\frac{1}{2}}. \tag{3.25}$$

And this forward Loewner chain induces a time-reversed (i.e. backward) Loewner *chain at the final moment $t = T$ with the form*

$$h_T(z) = A + \left[(z - A)^2 - 4T\right]^{\frac{1}{2}}. \tag{3.26}$$

Proof. We know $g_T(z) \circ h_T(z) = z$, for all $z \in \mathbb{H}$ by ([1], *Lem.* 4.10). The result immediately follows. $\square$

Lemma 3.13. *On each local time interval $[t^-, t_{k+1}]$, we consider the constant force $t \mapsto 0$ on time interval $[t^-, t^- + \frac{h_k}{2}]$ and the constant force given by the corresponding value of the Brownian path at the end of the time sub-interval on $[t^- + \frac{h_k}{2}, t_{k+1}]$. We denote the time-reversed (i.e. backward) Loewner chain driven by these constant forces as $\iota'_{k,1}$ and $\iota''_{k,1}$, respectively. Consider the parallel translation $z \xmapsto{\iota_{k,2}} z + \sqrt{\kappa} B_{t^-, t_{k+1}}$. Then we have the composition*

$$\widetilde{Z}_{t_{k+1}}(iy_n) = \iota''_{k,1} \circ \iota_{k,2} \circ \iota'_{k,1} \widetilde{Z}_{t_k}(iy_n). \tag{3.27}$$

Proof. Inspect *Eqn* (2.12) and *Eqn.* (3.26) and the conclusion follows. $\square$

The above perspective in *Lem.* 3.14 endows us convenience because now we can introduce some discretized force with the form of a step-function to estimate the second term in *Ineq.* (3.3*a*). In fact, we have the following proposition.

Proposition 3.14. *If we consider the perturbation event*

$$E_{n,2} := \left\{ \left\| Z_{t^-}(iy_n) - \widetilde{Z}_{t^-}(iy_n) \right\|_{[0,1],\infty} \leq \frac{1}{\sqrt{4n+1}} \right\}, \tag{3.28}$$

and if we further restrict $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, then

$$\mathbb{P}\big(E_{n,2}\big) \geq 1 - 2e^{-(4n+1)/\kappa}. \tag{3.29}$$

Proof. From *Sec.* 2. we already know $Z_t(iy_n) = h_t(iy_n) - \sqrt{\kappa} B_t$. If we further choose $\widehat{B}_t := B_{1-t} - B_1$, then *Eqn.* (2.2) can be written as

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa}\widehat{B}_t}, \tag{3.30}$$

with $h_0(z) = z$ and where $\widehat{B}_t$ has the law of a standard Brownian motion. We also comment that our splitting simulation scheme could be formulated in a similar fashion. Define the driver

$$\widetilde{\lambda}(t) := 0 \cdot \mathbb{1}_{[0, \frac{t_1}{2})} + \sum_{k \geq 1} \sqrt{\kappa} B_{t^-} \cdot \mathbb{1}_{[t^- - \frac{h_k}{2}, t^- + \frac{h_k}{2} \wedge 1)}. \tag{3.31}$$

The random process $\widetilde{\lambda}(t)$ can be viewed as a step-function interpolation to the sample paths of Brownian motion $\sqrt{\kappa}B_t$ on $[0,1]$. In this regard, we denote by $\widetilde{Z}_t^*(iy_n)$ the trajectory driven by the above driver similar to $Z_t(iy_n)$ being driven by $\sqrt{\kappa}\widehat{B}_t$ in the following sense

$$\widetilde{Z}_t^*(iy_n) = \widetilde{h}_t(iy_n) - \widetilde{\lambda}(t), \tag{3.32}$$

due to *Lem.* 3.13 and where $(\widetilde{h}_t)_{t\in[0,1]}$ is a backward Loewner chain constrained by

$$\partial_t \widetilde{h}_t(z) = \frac{-2}{\widetilde{h}_t(z) - \widehat{\lambda}_t}, \tag{3.33}$$

with $\widetilde{h}_0(z) = z$ and $\widehat{\lambda}_t \coloneqq \widetilde{\lambda}(1-t) - \widetilde{\lambda}(1)$. The above scheme brings us some consistency to some perturbation term $Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)$. And our first goal is to estimate

$$\left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right|. \tag{3.34}$$

Define $\epsilon \coloneqq \sup_{t\in[0,1]} \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right|$, then it follows that

$$\left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \epsilon. \tag{3.35}$$

To achieve this goal, we further define $H(t) \coloneqq h_t(iy_n) - \widetilde{h}_t(iy_n)$. And we will first estimate $|H(t)|$. Differentiate $H(t)$ *w.r.t.* $t \in [0,1]$ and use *Eqn.* (2.2) and *Eqn.* (3.33) to obtain

$$\frac{d}{dt}H(t) - H(t)\zeta(t) = (\sqrt{\kappa}\widehat{B}_t - \widehat{\lambda}_t)\zeta(t), \tag{3.36}$$

where we define $\zeta(t) \coloneqq \left(h_t(iy_n) - \sqrt{\kappa}\widehat{B}_t\right)^{-1} \cdot \left(\widetilde{h}_t(iy_n) - \widehat{\lambda}_t\right)^{-1}$. Notice that the derivative of $H(t)$ *w.r.t.* time t is defined except for finitely many points because the driving force $\widetilde{\lambda}(t)$ is piecewise continuous. Integrating the above differential equation and choose $u(t) \coloneqq e^{-\int_0^t \zeta(s)ds}$, we find

$$H(t) = u(t)^{-1}\left[H(0) - \int_0^t (\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s)u(s)\zeta(s)ds\right]. \tag{3.37}$$

Since $H(0) = 0$, we obtain

$$|H(t)| \leq \int_0^t \left|\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s\right| e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds. \tag{3.38}$$

Then, it is immediate that

$$\begin{aligned}\left|h_t(iy_n)-\widetilde{h}_t(iy_n)\right| &\le \epsilon\cdot\int_0^t e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds\\ &\le \epsilon\cdot\left(e^{\int_s^t|\zeta(r)|dr}-1\right),\end{aligned} \tag{3.39}$$

where the last inequality is due to ([15], *Lem.* 2.3) and ([15], *Eqn.* 2.12). Now turning attention to *Eqn.* (3.34), we have

$$\left|Z_t(iy_n)-\widetilde{Z}_t^*(iy_n)\right| \le \left|h_t(iy_n)-\widetilde{h}_t(iy_n)\right|+\epsilon \le \epsilon\cdot e^{\int_s^t \operatorname{Re}\zeta(r)dr}. \tag{3.40}$$

Furthermore, ([15], *Eqn.* 2.12) tells us that $\int_0^t|\zeta(s)|ds \le \log\left(\sqrt{4+y_n^2}/y_n\right)$. Consequently, we have

$$\left|Z_t(iy_n)-\widetilde{Z}_t^*(iy_n)\right| \le \epsilon\cdot\sqrt{4+y_n^2}/y_n = \epsilon\cdot\sqrt{4n+1}. \tag{3.41}$$

Notice that

$$\epsilon = \sup_{t\in[0,1]}\left|\sqrt{\kappa}B_t-\widetilde{\lambda}(t)\right| \le \bigvee_{t^-\in\mathcal{D}_n}\sup_{t\in[0,h_k]}\sqrt{\kappa}|B_t|, \tag{3.42}$$

where the notation "$\vee$" indicates we take the maximal value over all $t^-\in\mathcal{D}_n$. By ([9], *Cor.* 2.2), for the supremum Brownian motion $S_t := \sup\limits_{0\le s\le t} B_s$ we have that

$$\mathbb{P}\left(S_t\le x\right) = 2\Phi\left(\frac{x}{\sqrt{t}}\right)-1, \tag{3.43}$$

for all $x\ge 0$ and where $\frac{d}{dx}\Phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ is the law of standard normal variable. It follows from the reflection principle that

$$\mathbb{P}\left(\sup_{0\le t\le h_k}\left|\sqrt{\kappa}B_t\right| \ge \frac{1}{4n+1}\right) = 2\mathbb{P}\left(S_{h_k}\ge\frac{1}{\sqrt{\kappa}\cdot(4n+1)}\right) \le 2\sqrt{\frac{2}{\pi}}e^{-\frac{(4n+1)^{-2}}{2h_k\cdot\kappa}}, \tag{3.44}$$

if we restrict $h_k \le n^{-1}\wedge(4n+1)^{-3}$ for all $t^-\in\mathcal{D}_n$. Then

$$\left\{\epsilon > \frac{1}{4n+1}\right\} \le \bigcup_{t^-\in\mathcal{D}_n}\left\{\sup_{0\le t\le\frac{h_k}{2}}\left|\sqrt{\kappa}B_t\right| > \frac{1}{4n+1}\right\}, \tag{3.45a}$$

and we see

$$\begin{aligned}\mathbb{P}\left(\epsilon > \frac{1}{4n+1}\right) &\leq \sum_{t^- \in \mathcal{D}_n} \mathbb{P}\left(\sup_{0\leq t\leq \frac{h_k}{2}} \left|\sqrt{\kappa}B_t\right| > \frac{1}{4n+1}\right) \\ &\leq 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}.\end{aligned} \tag{3.45b}$$

Conditioned on the event $\{\epsilon > (4n+1)^{-1}\}^c \in \Omega$, following *Eqn.* (3.41), we have

$$\sup_{t\in[0,1]} \left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \frac{1}{\sqrt{4n+1}}. \tag{3.46}$$

Hence, by the strict inclusion of events in probability space, we have our estimate to the perturbation term

$$\mathbb{P}\left(\sup_{t\in[0,1]} \left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \frac{1}{\sqrt{4n+1}}\right) \geq 1 - 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}. \tag{3.47a}$$

We further observe that the splitting simulation $\widetilde{Z}_t(iy_n)$ coincides with the trajectory $\widetilde{Z}_t^*(iy_n)$ at the times $t^- \in \mathcal{D}_n$, by virtue of *Lem.* 3.13. Hence we have the desired result

$$\mathbb{P}\big(E_{n,2}\big) \geq 1 - 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}. \tag{3.47b}$$

□

Remark 3.15. *The perturbation in Prop.* 3.14 *in our splitting scheme is estimated via a probabilistic argument using ([15], Lem.* 2.2*). Notice that the time-reversed Loewner map $h_t(z)$ is different from the inverse of the forward map $g_t^{-1}(z)$, even though we do have the equality $h_{T=1}(z) = g_{T=1}^{-1}(z)$. In Sec.* 5. *we are going to briefly discuss the linear interpolation of driving force. To prove its convergence, we will need ([15], Lem.* 2.2*) again under a different context.*

Hence we have estimated the sup-norm on $[0,1]$ of the second term in *Ineq.* (3.3*a*). Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the third term. Following *Eqn.* (2.10) with $L_0(z) = -2/z$ and $L_1(z) = \sqrt{\kappa}$, we could explicitly calculate, with $t^- \leq s < t_{k+1}$

$$\begin{aligned}\widetilde{Z}_s^{(0)} &= \exp\left(\frac{1}{2}(s-t^-)L_0\right)\widetilde{Z}_{t^-} = \sqrt{\widetilde{Z}_{t^-}^2 - 2(s-t^-)}, \\ \widetilde{Z}_s^{(1)} &= \exp\left(B_{t^-,s}L_1\right)\widetilde{Z}_{t_{k+1}}^{(0)} = \sqrt{\widetilde{Z}_{t^-}^2 - 2h_k} + B_{t^-,s}, \\ \widetilde{Z}_s^{(2)} &= \exp\left(\frac{1}{2}(s-t^-)L_0\right)\widetilde{Z}_{t_{k+1}}^{(1)} = \sqrt{\left(\widetilde{Z}_{t_{k+1}}^{(1)}\right)^2 - 2(s-t^-)},\end{aligned} \tag{3.48}$$

which could be written into the form via solving *Eqn.* (2.9)

$$\begin{aligned}
&\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t^-}(iy_n) \\
&\quad = \frac{1}{2}\int_{t^-}^{t} L_0(\widetilde{Z}_s^{(2)})ds + \int_{t^-}^{t} L_1(\widetilde{Z}_s^{(1)})dB_s + \frac{1}{2}\int_{t^-}^{t} L_0(\widetilde{Z}_s^{(0)})ds, \\
&\quad = \int_{t^-}^{t} \sqrt{\kappa} dB_s - \int_{t^-}^{t} \frac{1}{\sqrt{(\widetilde{Z}_{t_k})^2 - 2(s-t^-)}} ds - \int_{t^-}^{t} \frac{1}{\sqrt{\left(\widetilde{Z}_{t_{k+1}}^{(1)}\right)^2 - 2(s-t^-)}} ds.
\end{aligned} \tag{3.49a}$$

Solving these integrals, we have

$$\begin{aligned}
&\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t^-}(iy_n) \\
&\quad = \sqrt{\kappa} B_{t^-,t} - \frac{2(t-t^-)}{\sqrt{\widetilde{Z}_{t^-}^2 - 2(t-t^-)} + \widetilde{Z}_{t^-}} \\
&\quad - \frac{2(t-t^-)}{\sqrt{(\sqrt{\widetilde{Z}_{t^-}^2 - 2h_k} + \sqrt{\kappa}B_{t^-,t_{k+1}})^2 - 2(t-t^-)} + \sqrt{\widetilde{Z}_{t^-}^2 - 2h_k} + \sqrt{\kappa}B_{t^-,t_{k+1}}},
\end{aligned} \tag{3.49b}$$

where t^- abbreviates $t_k(t)$ in the above expression. *Eqn.* (3.49*b*) provides an exact form of the approximated process $\widetilde{Z}_t$, which leads us to the following result.

Proposition 3.16. *Let $\kappa \in (0,4)$. Let us consider the event*

$$E_{n,3} \coloneqq \left\{ \left\| \widetilde{Z}_t(iy_n) - \widetilde{Z}_{t^-}(iy_n) \right\|_{[0,1],\infty} \le \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}} \right\}. \tag{3.50}$$

Then as long as $||\mathcal{D}_n|| \le n^{-1} \wedge (4n+1)^{-3}$, we have

$$\mathbb{P}\big(E_{n,3}\big) \ge 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \tag{3.51}$$

Proof. By ([3], *Sec.* 6.1), we have two general results $\mathrm{Im}(z) \le \mathrm{Im}(\sqrt{z^2-c})$ and $\mathrm{Im}(z) = \mathrm{Im}(z+c)$ for all $z \in \mathbb{H}$ and $c \in \mathbb{R}$. Applying this two results to *Eqn.* (3.49*b*), we have

$$\begin{aligned}
\left|\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t^-}(iy_n)\right| &\le \left|\sqrt{\kappa}B_{t^-,t}\right| + \frac{h_k}{\mathrm{Im}\,\widetilde{Z}_{t^-}(iy_n)} + \frac{h_k}{\mathrm{Im}\,\widetilde{Z}_{t^-}(iy_n)}, \\
&\le \left|\sqrt{\kappa}B_{t^-,t}\right| + \frac{2h_k}{y_n},
\end{aligned} \tag{3.52}$$

where the last inequality follows from ([1], *Lem.* 4.9), where it is shown that the map $t \mapsto \operatorname{Im} \widetilde{Z}_t(iy)$ is strictly increasing. Using ([9], *Cor.* 2.2), we obtain that

$$\mathbb{P}\Big(\sup_{0 \le t \le h_k} \big|\sqrt{\kappa} B_t\big| \ge \frac{1}{n^{1/4}} \Big) = 2\mathbb{P}\Big(S_{h_k} \ge \frac{1}{\sqrt{\kappa} \cdot n^{1/4}} \Big) \le 2\sqrt{\frac{2}{\pi}} e^{-\frac{n^{-1/2}}{2h_k \cdot \kappa}}, \quad (3.53)$$

by reflection principle. Note that we have restricted $h_k \le n^{-1} \wedge (4n+1)^{-3}$ for all $t^- \in \mathcal{D}_n$. In this regard

$$\mathbb{P}\Big(\sup_{t \in [0,1]} \big|\sqrt{\kappa} B_{t^-,t}\big| \ge \frac{1}{n^{1/4}} \Big) \le 2e^{-\sqrt{n}/2\kappa}, \quad (3.54)$$

and

$$\mathbb{P}\big(E_{n,3}\big) \ge 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \quad (3.55)$$

□

At this point, we have evaluated the $\|\cdot\|_{[0,1],\infty}$ norm *w.r.t.* all the three terms in *Ineq.* (3.3*a*). Therefore, we come to prove the main result.

Proof. (*of Thm. 3.4*) Denote $E_{n,4} := E_{n,1}^{**}$. On the event $E_{n,1} \cap E_{n,2} \cap E_{n,3} \cap E_{n,4} \subset \Omega$, we observe from *Prop.* 3.10, *Prop.* 3.11, *Prop.* 3.14, and *Prop.* 3.16 that

$$\begin{aligned}
&\|\eta(t) - Z_t(iy_n)\|_{[0,1],\infty} \le \frac{1}{n^{(1-\epsilon_0)/4}}, \\
&\|Z_t(iy_n) - Z_{t^-}(iy_n)\|_{[0,1],\infty} \le \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{2}{n^{(1-\epsilon_0)/4}}, \\
&\Big\|Z_{t^-}(iy_n) - \widetilde{Z}_{t^-}(iy_n)\Big\|_{[0,1],\infty} \le \frac{1}{(4n+1)^{1/2}}, \\
&\Big\|\widetilde{Z}_{t^-}(iy_n) - \widetilde{Z}_t(iy_n)\Big\|_{[0,1],\infty} \le \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}},
\end{aligned} \quad (3.56)$$

given $||\mathcal{D}_n|| \le n^{-1} \wedge (4n+1)^{-3}$ and $\beta_1 \in (0,1)$. If we define

$$\begin{aligned}
\varphi_1(n) &:= \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{3}{n^{(1-\epsilon_0)/4}} + \frac{1}{(4n+1)^{1/2}} + \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}} \to 0, \\
\varphi_2(n) &:= \frac{1}{n} + \frac{c_2}{n^2} + \frac{c_4}{n^{c_3/2}} + 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa} + 2e^{-\sqrt{n}/2\kappa} + 3\epsilon_n \to 0,
\end{aligned} \quad (3.57)$$

as $n \to \infty$. Then the following inequality holds

$$\mathbb{P}\bigg(\Big\|\eta(t) - \widetilde{Z}_t(iy_n)\Big\|_{[0,1],\infty} \le \varphi_1(n) \bigg) \ge \mathbb{P}\bigg(E_{n,1} \cap \cdots \cap E_{n,4} \bigg) \ge 1 - \varphi_2(n). \quad (3.58)$$

We note that the above inequality regarding our approximation $\widetilde{Z}_t(iy_n)$ is only *w.r.t.* the prior uniform partition $\mathcal{D}_n$ on $[0,1]$. But we could always insert mid-points into the partition pointwisely on the event $E_{n,1} \cap \cdots \cap E_{n,4}$ using tolerance $\tau > 0$ and obtain the refined partition $\widetilde{\mathcal{D}}_n$. The above *Eqn.* (3.58) still holds because we have *almost surely* $||\widetilde{\mathcal{D}}_n|| \leq ||\mathcal{D}_n||$. $\square$

Corollary 3.17. *For almost all* $\omega \in \Omega$*, we have that*

$$\left\| \eta(t) - \widetilde{Z}_t(iy_n) \right\|_{[0,1],\infty} \to 0, \text{ with } n \to \infty. \tag{3.59}$$

Proof. This corollary immediately follows from the strong convergence in probability in *Thm.* 3.4. $\square$

4 Convergence in L^p norm

Following *Ineq.* (3.3*a*) and *Ineq.* (3.3*b*), we are going to estimate the L^p norm to $\big(\eta(t) - \widetilde{Z}_t(iy_n)\big)$ with $p \geq 2$. Notice that the discussion of this alternative convergence pattern is not completed: it relies on several assumptions, which we have not proved but we will consider them to be of furture work. Either these assumptions hold or not would be a contribution to the convergence analysis of the splitting simulation. Similar to the proof of strong convergence in probability, the first step is to estimate each of the four terms in *Eqn.* (3.56) individually. In this section we show that the L^p convergence would hold if we have the following assumptions.

Assumption 4.1. *There exists* $p_1 \geq 2$ *and* $\epsilon_0 \in (0, 1)$ *such that*

$$\sup_{t \in [0,1]} |\eta(t) - Z_t(iy_n)| \leq C_1(\omega) \cdot y_n^{1-\epsilon_0}, \tag{4.1}$$

where the constant $C_1(\omega)$ *is almost surely finite as in ([3], Lem.* 6.7*). Moreover, we assume that* $C_1(\omega)$ *is* p_1*-integrable, i.e.* $C_1(\omega) \in L^{p_1}(\mathbb{P})$.

Assumption 4.2. *There exists* $p_2 \geq 2$ *and* $\beta_2 \in (0, 1)$ *such that the* SLE_κ *Loewner chain is generated by a curve when* $\kappa \neq 8$ *with the following modulus of continuity*

$$|\eta(t+s) - \eta(t)| \leq C_2(\omega) s^{(1-\beta_2)/2} \tag{4.2}$$

where the constant $C_2(\omega)$ *is almost surely finite as in ([5], Prop.* 4.3*). And moreover* $C_2(\omega)$ *is* p_2*-integrable, i.e.* $C_2(\omega) \in L^{p_2}(\mathbb{P})$.

Assumption 4.3. *The Ninomiya-Victoir splitting scheme satisfies the various regularity assumptions, including the ellipticity condition in* ([7], *Rmk.* 4.1) *and* *then admits the inequality*

$$\mathbb{E}\left[\sup_{t\in[0,1]}\left|Z_t(iy_n)-\widetilde{Z}_t(iy_n)\right|^p\right]\le\frac{c_6}{n^p},\tag{4.3}$$

for some constant $c_6>0$, and for all $p\ge 2$.

If these assumptions hold as expected, we could take a crucial step near the idea of L^p convergence of our splitting simulation. In this section, we choose $p:=p_1\wedge p_2\ge 2$. We have the following propositions.

Proposition 4.4. *Admitting Asmp. 4.1 with $||\widetilde{\mathcal{D}}_n||\le n^{-1}\wedge(4n+1)^{-3}$ where $||\widetilde{\mathcal{D}}_n||$ is refined partition, then there exists a decreasing function $\psi_1:\mathbb{R}_+\to\mathbb{R}_+$ such that $\psi_1(n)\to 0$ as $n\to\infty$, and*

$$\mathbb{E}\left[\int_0^1|\eta(t)-Z_t(iy_n)|^p dt\right]\le\psi_1(n).\tag{4.4}$$

Proof. By ([3], *Lem.* 6.7), there exists $\epsilon_0\in(0,1)$ such that *almost surely*

$$\sup_{t\in[0,1]}|\eta(t)-Z_t(iy_n)|\le C_1(\omega)\cdot y_n^{1-\epsilon_0}.\tag{4.5}$$

It is clear then

$$\begin{aligned}\mathbb{E}\left[\int_0^1|\eta(t)-Z_t(iy_n)|^p dt\right]&\le\mathbb{E}\left[\|\eta(t)-Z_t(iy_n)\|^p_{[0,1],\infty}\right]\\&\le\mathbb{E}\big[C_1(\omega)^p\big]\cdot\frac{1}{n^{(1-\epsilon_0)p/2}}:=\psi_1(n)\overset{n\to\infty}{\longrightarrow}0.\end{aligned}\tag{4.6}$$

□

Proposition 4.5. *Admitting Asmp. 4.2 with $||\widetilde{\mathcal{D}}_n||\le n^{-1}\wedge(4n+1)^{-3}$, there exists a decreasing function $\psi_2:\mathbb{R}_+\to\mathbb{R}_+$ such that $\psi_2(n)\to 0$ as $n\to\infty$, and*

$$\mathbb{E}\left[\int_0^1|Z_t(iy_n)-Z_{t^-}(iy_n)|^p dt\right]\le\psi_2(n).\tag{4.7}$$

Proof. ([5], *Prop.* 3.8) and ([5], *Prop.* 4.3) imply that there exists $\beta_2\in(0,1)$ such that

$$\sup_{t\in[0,1]}\left|\eta(t)-\eta\big(t^-\big)\right|\le C_2(\omega)\cdot\frac{1}{n^{(1-\beta_2)/2}}.\tag{4.8}$$

By *Prop.* 4.4, it follows that

$$\sup_{t\in[0,1]}|Z_t(iy_n)-Z_{t^-}(iy_n)|\le\frac{2C_1(\omega)}{n^{(1-\epsilon_0)/2}}+\frac{C_2(\omega)}{n^{(1-\beta_2)/2}}.\tag{4.9}$$

It is clear then

$$\mathbb{E}\left[\int_0^1 |Z_t(iy_n) - Z_{t^-}(iy_n)|^p dt\right] \leq \mathbb{E}\left[\|Z_t(iy_n) - Z_{t^-}(iy_n)\|^p_{[0,1],\infty}\right]$$
$$\leq \mathbb{E}\big[C_1(\omega)^p\big] \cdot \frac{2^{2p-1}}{n^{(1-\epsilon_0)p/2}} + \mathbb{E}\big[C_2(\omega)^p\big] \cdot \frac{2^{p-1}}{n^{(1-\beta_2)p/2}} \coloneqq \psi_2(n) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.10}$$

□

Proposition 4.6. *Admitting Asmp.* 4.3 *with* $||\widetilde{\mathcal{D}}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$*, it is immediate that*

$$\mathbb{E}\left[\sup_{t\in[0,1]} \left|Z_{t^-}(iy_n) - \widetilde{Z}_{t^-}(iy_n)\right|^p\right] \leq \frac{c_6}{n^p}. \tag{4.11}$$

If we denote $\psi_3(n) \coloneqq c_6/n^p$*, then*

$$\mathbb{E}\left[\int_0^1 \left|Z_{t^-}(iy_n) - \widetilde{Z}_{t^-}(iy_n)\right|^p dt\right] \leq \psi_3(n) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.12}$$

Proof. The proposition follows from *Eqn.* (4.9). □

To give an estimate to the last term in *Eqn.* (3.56), we quote the known interpolation inequality from ([8], *Sec.* 6.5) for Lebesgue spaces and another inequality *w.r.t.* supremum Brownian motion.

Lemma 4.7. *For all* $1 < p < r < q$*, suppose* $f \in L^p \cap L^q$*. Then* $f \in L^r$ *with*

$$\|f\|_r \leq \big(\|f\|_p\big)^{(1/r-1/q)/(1/p-1/q)}\big(\|f\|_q\big)^{(1/r-1/q)/(1/p-1/q)}. \tag{4.13}$$

Lemma 4.8. *For all* $m \in \mathbb{N}$*, we have*

$$\mathbb{E}\left[\sup_{s\in[0,t]} |B_s|^{2m}\right] = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+m\right)2^m \cdot t^m, \tag{4.14}$$

where B_t *is a standard one-dimensional Brownian motion.*

Proof. The proof follows if we revisit the supremum Brownian motion $S_t = \sup_{0\leq s\leq t} B_s$ from ([9], *Cor.* 2.2), by computing all even order moments. □

Proposition 4.9. *With* $||\widetilde{\mathcal{D}}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$*, there exists a decreasing function* $\psi_4 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi_4(n) \to 0$ *with* $n \to \infty$*, and*

$$\mathbb{E}\left[\int_0^1 \left|\widetilde{Z}_{t^-}(iy_n) - \widetilde{Z}_t(iy_n)\right|^p dt\right] \leq \psi_4(n). \tag{4.15}$$

Proof. By *Prop.* 3.14 and *Eqn.* (3.52), we know *almost surely* that

$$\left|\widetilde{Z}_{t^-}(iy_n) - \widetilde{Z}_t(iy_n)\right|^p \le 2\left|\sqrt{\kappa}B_{t^-,t}\right|^p + \frac{2^{p+1}}{n^{p/2}}. \tag{4.16}$$

Since $p \ge 2$, there exists $\{m, m+1\} \subset \mathbb{N}$ so that $2m \le p < 2(m+1)$. Remember that $t - t^- \le h_k \le n^{-1}$. Then, by *Lem.* 4.8

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t^-,t}\right|^{2m}\right] &\le \frac{2^m \kappa^m \Gamma\left(\frac{1}{2}+m\right)}{\pi^{\frac{1}{2}} \cdot n^m}, \\ \mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t^-,t}\right|^{2m+2}\right] &\le \frac{2^{m+1} \kappa^{m+1} \Gamma\left(\frac{3}{2}+m\right)}{\pi^{\frac{1}{2}} \cdot n^{m+1}}. \end{aligned} \tag{4.17}$$

By *Lem.* 4.7, we use the interpolation inequality in Lebesgue spaces and have

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t^-,t}\right|^{p}\right] \\ &\le \mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t^-,t}\right|^{2m}\right]^{m+1-\frac{p}{2}} \mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t^-,t}\right|^{2m+2}\right]^{(m+1-\frac{p}{2})\cdot\frac{m}{m+1}} \\ &\le \frac{c_7}{n^{m(m+1-\frac{p}{2})}} \coloneqq \psi_4(n) \overset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{4.18}$$

where $c_7 > 0$ is a constant depending only on $p \ge 2$. □

We have, at this point, estimated the L^p norm *w.r.t.* the four terms in *Eqn.* (3.52) under several assumptions. Combining them together, under the specified assumptions, we obtain the following result.

Theorem 4.10. *Admitting Asmp.* 4.1, 4.2, *and* 4.3, *together with* $||\widetilde{\mathcal{D}}_n|| \le n^{-1} \wedge (4n+1)^{-3}$, *there exists a decreasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(n) \to 0$ *as* $n \to \infty$, *and*

$$\mathbb{E}\left[\int_0^1 \left|\eta(t) - \widetilde{Z}_t(iy_n)\right|^p\right] \le \psi(n). \tag{4.19}$$

Proof. Define $\psi \coloneqq \psi_1 + \psi_2 + \psi_3 + \psi_4$, then the result follows. □

5 Linear interpolation of driving force

In *Sec.* 2. we have formulated a splitting simulation of SLE_κ traces via Ninomiya-Victoir Scheme. In *Sec.* 3. we proved the strong convergence in probability of the splitting simulation. In *Sec.* 4. we proposed an L^p convergence of the above splitting simulation under several assumptions (whose investigation will expand in future work). In this section, we diverge a little to see a variant simulating scheme: linearly interpolating the driving Brownian motion $\sqrt{\kappa}B_t$ pointwisely, and discuss its convergence.

Definition 5.1. *In this section, $\mathcal{D}_n := \{t_0 = 0, t_1, \ldots, t_n = 1\}$ will denote a uniform partition on the time interval $[0, 1]$.*

Definition 5.2. *Given the Brownian sample paths $B_\cdot(\omega) : [0, 1] \to \mathbb{R}$, we write our linear interpolation in the following form*

$$\lambda^n(t) := n\big(\sqrt{\kappa}B_{t_{k+1}} - \sqrt{\kappa}B_{t^-}\big)(t - t^-) + \sqrt{\kappa}B_{t^-} \text{ on } [t^-, t_{k+1}]. \tag{5.1}$$

Definition 5.3. *Following the convention in Sec. 2. and Sec. 3. we use the notation $\gamma : [0, 1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_1\}$ for the forward Loewner curve generated by the forward Loewner chain $(g_t)_{t\in[0,1]}$ with driving force $\sqrt{\kappa}B_t$. We further let $\gamma^n : [0, 1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_1\}$ to denote the forward Loewner curve generated by the forward Loewner chain $(g^n_t)_{t\in[0,1]}$, driven by the piecewise-linear force $\lambda^n(t)$.*

Remark 5.4. *Notice that in this section we will interpolate the forward Loewner chain and hence simulate the forward Loewner curve $\gamma(t)$, whereas we have simulated the backward Loewner curve $\eta(t)$ via Ninomiya-Victoir Scheme in Sec. 2.*

Definition 5.5. *With $(g^n_t)_{t\in[0,1]}$ the Loewner chain corresponding to $\lambda^n(t)$, let $f^n_t : \mathbb{H} \to \mathbb{H}\backslash\gamma^n([0, t])$ be the inverse map of $g^n_t(z)$ and denote $\widehat{f}^n_t(z) := f^n_t\big(z + \lambda^n(t)\big)$. Choose $G^n_k := (\widehat{f}^n_{t^-})^{-1} \circ \widehat{f}^n_{t_{k+1}}$. Then*

$$\widehat{f}^n_{t^-} = G^n_0 \circ G^n_1 \circ \cdots \circ G^n_{k-1}. \tag{5.2}$$

Definition 5.6. *Choose $\gamma^n_t(s) := g^n_t\big(\gamma^n(t + s)\big)$ with $s \in [0, 1 - t]$, for all $t \in [0, 1]$.*

Lemma 5.7. *Consider the event $F_{n,1} := E'_{n,1}$ and $F_{n,2} := E''_{n,1}$ as in Eqn. (3.4) and in Eqn. (3.6). Then we have*

$$\mathbb{P}\big(F_{n,1}\big) \geq 1 - \frac{c_2}{n^2} \quad \text{and} \quad \mathbb{P}\big(F_{n,2}\big) \geq 1 - \frac{c_4}{n^{c_3/2}}, \tag{5.3}$$

where $c_2, c_3, c_4 > 0$ are constants depending only on $\kappa \neq 8$.

Theorem 5.8. *Let γ be the SLE_κ trace and let γ^n be the trace obtained by the linear interpolation of the Brownian driver. There exist $c_6, c_7 > 0$ depending* *only on $\kappa \neq 8$ such that if we consider the event*

$$F_n := \left\{ \|\gamma - \gamma^n\|_{[0,1],\infty} \leq \frac{c_6 (\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}} \right\}. \tag{5.4}$$

Then we have $\mathbb{P}(F_n) \geq 1 - c_2 \cdot n^{-2} - c_3 \cdot n^{-c_4/2}$.

This theorem is our main result in the context of linearly interpolating Brownian driver. We will not give a detailed proof here because the proof is similar to the known result of square-root interpolation obtained in ([11], *Sec.* 2.). Instead, we outline the ideas to estimate the convergence in probability of the linear interpolation method.

On the event $F_{n,1} \cap F_{n,1}$, we want to give a uniform bound to $|\gamma(t) - \gamma^n(t)|$ with $t \in [0,1]$. In fact, for all $t^- \in \mathcal{D}_n$, we write

$$\begin{aligned} \left|\gamma(r+t^-) - \gamma^n(r+t^-)\right| &\leq \left|\gamma(r+t^-) - \gamma(s+t^-)\right| + \left|\widehat{f}_{t^-}(z) - \widehat{f}^n_{t^-}(w)\right| \\ &\leq \left|\gamma(r+t^-) - \gamma(s+t^-)\right| + \left|\widehat{f}_{t^-}(z) - \widehat{f}_{t^-}(w)\right| + \left|\widehat{f}_{t^-}(w) - \widehat{f}^n_{t^-}(w)\right|, \end{aligned} \tag{5.5}$$

where $w := \gamma^n_k(r)$, r is arbitrarily fixed in $[\frac{1}{n}, \frac{2}{n}]$ and $z := \gamma_k(s)$ is chosen to be the highest point in the arc $\gamma_k([0, \frac{2}{n}])$. The first term in *Eqn.* (5.5) is bounded by the uniform continuity of $\gamma(t)$ on the event $F_{n,1}$.

The estimate of the second term in *Eqn.* (5.5) is comparing the images of nearby points in $\mathbb{H}$ very close to the real and the imaginary axis, under a conformal map. To proceed our discussion, we introduce, for any subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, constant $c > 0$, and integer $n \in \mathbb{N}_+$ that

$$A_{n,c,\phi} := \left\{ x + iy \in \mathbb{H};\ |x| \leq \frac{\phi(n)}{\sqrt{n}} \text{ and } \frac{1}{\sqrt{n}\phi(n)} \leq y \leq \frac{c}{\sqrt{n}} \right\}, \tag{5.6}$$

which is a box near the origin in the upper half-plane. The reason we introduce this extra object is that the images of nearby points in $A_{n,c,\phi}$ will also be close to each other under certain conformal maps, in the sense of the following lemmas.

Lemma 5.9. ([11], *Lem.* 2.6) *There exist constants $\alpha > 0$, and $c' > 0$, depending only on $c > 0$ in the definition of the box $A_{n,c,\phi}$, such that for all $z_1, z_2 \in A_{n,c,\phi}$ and conformal map $f : \mathbb{H} \to \mathbb{C}$, we have*

$$\begin{aligned} |f'(z_1)| &\leq c' \phi(n)^\alpha \cdot |f'(i \operatorname{Im} z_1)|, \\ d_{\mathbb{H},hyp}(z_1, z_2) &\leq c' \log \phi(n) + c', \end{aligned} \tag{5.7}$$

where $d_{\mathbb{H},hyp}(z_1, z_2)$ denotes the hyperbolic distance between z_1 and z_2 in $\mathbb{H}$.

Lemma 5.10. ([16], *Cor.* 1.5) *Suppose $f : \mathbb{H} \to \mathbb{C}$ is a conformal map, then* *for all* $z_1, z_2 \in \mathbb{H}$, *we have*

$$|f(z_1) - f(z_2)| \leq 2|(\operatorname{Im} z_1) f'(z_1)| \cdot \exp\big(4 d_{\mathbb{H},hyp}(z_1, z_2)\big). \tag{5.8}$$

Therefore, it is natural that we want to show $\{z, w\} \in A_{n,c,\phi}$ with proper parameters. Indeed, this is the case in the square-root interpolation ([11], *Lem.* 3.3). The only non-trivial remark is the following.

Remark 5.11. *In the linear interpolation, from ([10], Sec.* 3.*) we know that for a typical linear force* $\lambda(t) = t$ *on the time interval* $[0, \infty)$*, the Loewner curve admits the form*

$$t \mapsto 2 - 2\rho_t \cot \rho_t + 2i\rho_t, \tag{5.9}$$

where ρ_t *increases monotonously from* $\rho_0 = 0$ *to* $\rho_\infty = \pi$*. Indeed, the Loewner curve of a general linear force* $t \mapsto at + b$ *requires some change of constant parameters depending on* $a, b \in \mathbb{R}$*, possibly with change in signs. Hence, we know the arc* $\gamma_k^n : [0, \frac{1}{n}] \to \mathbb{H} \cup \{0\}$ *corresponding to the piecewise-linear force* $\lambda^n(t^- + t) - \lambda^n(t^-)$ *with* $t \in [0, \frac{1}{n}]$ *has an image which vertically stretches monotonically upward and horizontally either leftward or rightward. Hence, the images* $\gamma_k^n([0, \frac{1}{n}])$ *attains its maximal height at its tip* $\gamma_k^n(\frac{1}{n})$*, which justifies our choice of* $z = \gamma_k(s)$*.*

In this regard we could use *Lem.* 5.9 and *Lem.* 5.10 to give an upper bound to the second term in *Ineq.* (5.5).

Now, let us turn our attention to the third term in *Ineq.* (5.5). This is actually a perturbation term: we need to measure the difference of one point in $\mathbb{H}$ under two conformal maps. In order to estimate the third term in *Ineq.* (5.5), we use the following result.

Lemma 5.12. ([15], *Lem.* 2.2) *Let* $0 < T < \infty$*. Suppose* $f_t^{(1)}$ *and* $f_t^{(2)}$ *are the inverse map to the forward Loewner chain satisfying Eqn.* (2.1) *with driving force* $W_t^{(1)}$ *and* $W_t^{(2)}$*, respectively. Define* $\epsilon := \sup_{s \in [0,T]} \left| W_s^{(1)} - W_s^{(2)} \right|$*. Then if* $u = x + iy \in \mathbb{H}$*, we have*

$$\left| f_T^{(1)} - f_T^{(2)} \right| \leq \epsilon \exp\left\{ \frac{1}{2} \left[\log \frac{I_{T,y} \left| \partial_z f_T^{(1)}(u) \right|}{y} \log \frac{I_{T,y} \left| \partial_z f_T^{(2)}(u) \right|}{y} \right]^{\frac{1}{2}} + \log \log \frac{I_{T,y}}{y} \right\}, \tag{5.10}$$

where $I_{T,y} := \sqrt{4T + y^2}$*.*

Applying *Lem.* 5.12 to estimate the third term in *Ineq.* (5.5), combining the estimate to the first and the second terms, on the event $F_{n,1} \cap F_{n,2}$, we see that

$$\sup_{t\in[0,1]} |\gamma(t) - \gamma^n(t)| \leq \frac{c_6(\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}}, \tag{5.11}$$

for $\beta_1 \in (0,1)$. By definition of the event $F_n \in \Omega$, we know $F_{n,1} \cap F_{n,2} \subset F_n$. Moreover using the probability bounds on the events $F_{n,1}$ and $F_{n,2}$ in *Ineq.* (5.3), we obtain *Thm.* 5.8.

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