Splitting Simulation and Convergence of Stochastic Loewner Curves

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July 2021

Abstract

The main object in SLE_κ with $k \neq 8$ is a conformal map between a simply connected domain in the upper half-plane $\mathbb{H}$ and all of $\mathbb{H}$. The complement of this simply connected domain (*i.e.* the hulls) in the upper half-plane $\mathbb{H}$ is generated by a curve, called SLE_κ trace. And the evolution of the hulls K_t can be described by a Loewner equation.

This article states the simulation of the SLE_κ traces via the Ninomiya-Victoir splitting scheme. We adopt a notion similar to the exponential Lie flow on smooth manifolds, and let the initial point evolve according to the constraint equations imposed on our simulation scheme.

After giving the simulation scheme of SLE_κ via splitting, this article discusses the convergence pattern of the simulation and proves a strong convergence in probability *w.r.t.* $\|\cdot\|_{[0,1],\infty}$ norm to the distance between the original Loewner curve and our numerical simulation. Of course this type of strong convergence in probbility alo implies pointwise convergence, as a corollary. This article also proposes an L^p convergence of the same simulation based on several assumptions. We hope this splitting scheme is grounded in different topology. In the last section we add an extra linear interpolation method simply to complete the picture.

1 Introduction

There are various versions of Loewner equations. This article focuses on the forward Loewner equations defined in the upper half-plane $\mathbb{H}$ and a fixed time interval $[0, T]$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \tag{1.1}$$

with the initial condition $g_0(z) = z$, for all $z \in \mathbb{H}$ and the continuous driving force $\lambda : [0, T] \to \mathbb{R}$. Sometimes we will discriminate λ_t with $\lambda(t)$ and this should be clear from the context.

The family of maps $(g_t)_{0\le t\le T}$ is called the Loewner chain. For all $z\in\mathbb{H}$, the solution of the above forward Loewner equation is uniquely defined up to $T_z=\inf\{t\ge 0,\, g_t(z)=\lambda(t)\}$. Over time, the set $K_t=\{z\in\mathbb{H},\, T_z\le t\}$, called the hull, grows. It is also known that for all $t\in[0,T]$, $g_t:\mathbb{H}\backslash K_t\to\mathbb{H}$ is the unique conformal map satisfying the hydrodynamic normalization at ∞

$$\lim_{z\to\infty}[g_t(z)-z]=0, \tag{1.2}$$

and we additionally require the Loewner chain $(g_t)_{0\le t\le T}$ to be parametrized by upper half-plane capacity

$$g_t(z)=z+\frac{2t}{z}+o(1/|z|),\ \ as\ |z|\to\infty, \tag{1.3}$$

where by ([1], *Lem.* 4.1), the coefficient of the z-term is 1 and each coefficient a_k of the term $z^{-k},\ k\in\mathbb{N}_+$ is real.

We are particularly interested in the case when the Loewner chain $(g_t)_{0\le t\le T}$ is generated by a simple curve $\gamma:[0,T]\to\mathbb{H}\cup\{\lambda(T)\}$, possibly admitting self-touchings. In other words, the domain of $g_t(z)$ is the unbounded component of $\mathbb{H}\backslash\gamma([0,t])$. By ([2], *Thm.* 4.1), this is equivalent to the existence and continuity in $0<t<T$ of

$$\gamma(t)=\lim_{\epsilon\to 0^+}g_t^{-1}(\lambda(t)+i\epsilon). \tag{1.4}$$

Even though the piecewise linear interpolation method is independent to our main topic on splitting simulation. Let us briefly discuss that idea to simulate the driving force $\lambda(t)$ and its corresponding hull K_t for all $t\in[0,T]$. The interpolation algorithm is based on the following observations. Fix $s>0$, let $(\widetilde{g}_t)_{0\le t\le T}$ be the Loewner chain generated by the continuous driving force $\widetilde{\lambda}(t)=\lambda(s+t)$ with $0\le t\le T-s$. In fact, this solution admits the form

$$\partial_t g_{s+t}\circ g_s^{-1}(z)=\frac{2}{g_{s+t}\circ g_s^{-1}(z)-\lambda(s+t)}=\frac{2}{g_{s+t}\circ g_s^{-1}(z)-\widetilde{\lambda}(t)}, \tag{1.5}$$

and $g_s\circ g_s^{-1}(z)=z$ with $z\in\mathbb{H}\backslash K_s$. The uniqueness property of Loewner chains implies that $\widetilde{g}_t(z)=g_{s+t}\circ g_s^{-1}(z)$. And we denote $\widetilde{K}_t$ to be the hull associated with the Loewner chain $\widetilde{g}_t$, indeed

$$\mathbb{H}\cap g_s(K_{s+t})=\widetilde{K}_t\ \text{ and }\ K_{s+t}=K_s\cup g_s^{-1}(\widetilde{K}_t). \tag{1.6}$$

Hence, computing K_s and g_s^{-1} and $\widetilde{K}_t$ would enable us to compute $\widetilde{K}_{s+t}$.

There are methods interpolating $\lambda(t)$ with discretized force $\lambda^n(t)$. Using square-root ([11], *Sec.* 2.) or linear interpolation, there is usually an easier form of $\lambda^n(t)$, which endows computational convenience. The driving force $\lambda^{(n)}$ in the context of square-root and linear interpolation is written below

$$\lambda^n_{square\text{-}root}(t) \coloneqq \sqrt{n}\big(\lambda(t_{k+1}) - \lambda(t_k)\big)\sqrt{t - t_k} + \lambda(t_k) \text{ on } [t_k, t_{k+1}], \tag{1.7a}$$

$$\lambda^n_{linear}(t) \coloneqq n\big(\lambda(t_{k+1}) - \lambda(t_k)\big)(t - t_k) + \lambda(t_k) \text{ on } [t_k, t_{k+1}]. \tag{1.7b}$$

And in fact, we will briefly discuss the linear interpolation method in the last section.

2 Splitting simulation

In this section we propose an algorithm to simulate Lowener chains whose hulls in $\mathbb{H}$ are simple curves, possibly admitting self-touchings. And to simplify notation, we set $T = 1$, which is irrelevant to the convergence analysis in the next section. And we further require our driving force to be *weakly* Hölder-1/2 continuous. In fact, this seemingly strong constraint allows us to extend our result to SLE_κ with $\kappa \neq 8$, which are *stochastic* Loewner chains generated by Brownian motions.

In the context of progressive random functions, the sample paths of $\sqrt{\kappa}B_t$ depend on the element $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the "un-observable" probability space. Of course Ω contains a filtration $(\mathcal{F}_t)_{0\le t\le T}$ which actually defines the Brownian motion. And the space Ω is assumed to be large enough (via tensoring with its copies) to contain all necessary *i.i.d.* random variables.

The intuition of SLE_κ comes from the theory of statistical physics. Given a lattice model with temperature parameter, phase transitions ([1], *Sec.* 5.1) occur when the temperature varies. At a non-trivial point, the system falls into a critical regime with conformally invariant scaling limits ([12], *Sec.* 1.) of interlacements. Their behavior is best understood in the stochastic sense. And the notion of SLE_κ arises in such a realm.

SLE_κ traces are conformal Loewner chains driven by Brownian motions $\sqrt{\kappa}B_t$ with $t \in [0, 1]$. Hence they are constrained by the forward Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \tag{2.1}$$

with $g_0(z) = z$ for all $z \in \mathbb{H}\backslash K_t$. In the following discussion of SLE_κ, we focus on the case where the hulls K_t are generated by a simple curve $\gamma : [0, 1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_{T=1}\}$. To describe its image $\gamma([0, 1])$ driven by Brownian motion, we discretize the Loewner equation before drawing the approximated solution piecewisely.

In the early years of SLE_κ, mathematicians proposed a backward Loewner equation which is related to the forward version *Eqn.* (2.1) and admits the form

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa} B_{1-t}}, \tag{2.2}$$

with $h_0(z) = z$. This backward equation generates a Loewner curve $\eta : [0,1] \to \mathbb{H} \cup \{0\}$. Notice that $g_t(z)$ and $h_t(z)$ are driven by the same Brownian motion at different time. In fact, there is a correspondence ([13], *Sec.* 1.1) bewteen the dynamics constrained by *Eqn.* (2.1) and *Eqn.* (2.2): The random set $\eta([0,1])$ has the same law as the chordal SLE_κ trace $\gamma([0,1])$ modulo a real scalar shift $\sqrt{\kappa} B_{T=1}(\omega)$ with $\omega \in \Omega$. For convenience, we call $\eta(t)$ the shifted Loewner curve, though it is not a simple scalar translation of $\gamma(t)$.

There are many schemes to simulate SLE_κ traces. This article focuses on the Ninomiya-Victoir scheme. In few words, we do not use any interpolation method with piecewise linear or square-root functions. But we adopt a notion similar to the Lie flow of tangent bundles of smooth manifolds. But really we do not emphasize the geometric properties. In this regard we evolve the approximated solutions under the constraint of some stochastic differential equations.

At this point, some new notations are necessary before we formally present the Ninomiya-Victoir scheme. If we set $\widehat{g}_t(z) \coloneqq g_t(z) - \sqrt{\kappa} B_t$ for all $\in \mathbb{H} \backslash K_t$, from the existence and continuity of $\gamma(t)$ discussed in Sec. 1, we write

$$\gamma(t) = \lim_{y \to 0^+} g_t^{-1}(\sqrt{\kappa} B_t + iy) = \lim_{y \to 0^+} \widehat{g}_t^{-1}(iy), \tag{2.3}$$

and the stochastic differential equation

$$\begin{aligned} d\widehat{g}_t(z) &= \frac{2}{\widehat{g}_t(z)} dt - \sqrt{\kappa} dB_t, \\ \widehat{g}_0(z) &= z, \end{aligned} \tag{2.4}$$

with $z \in \mathbb{H} \backslash K_t$. Remember that the Loewner curve $\eta(t)$ has the same law as $\gamma(t)$ up to a translation $\sqrt{\kappa} B_{T=1}(\omega)$ with $\omega \in \Omega$. In fact, we will simulate the Loewner curve $\eta(t)$ instead of $\gamma(t)$ for convenience, since the former curve preserves all randomness of the latter curve. It is now necessary to discretize the backward Loewner equation

$$
\begin{aligned}
dZ_t &= -\frac{2}{Z_t}dt + \sqrt{\kappa}dB_t, \\
Z_0 &= iy,
\end{aligned}
\tag{2.5}
$$

where $y > 0$ is taken sufficiently small and we avoid the choice $Z_0 = 0$, due to the singularity at 0 inherent in the boundary condition of *Eqn.* (2.5). And in fact, from ([3], *Eqn.* 6.5) we know that $Z_t(z) = h_t(z) - \sqrt{\kappa}B_t$, for all $z \in \mathbb{H}$. Then with the choice of initial value $Z_0 = iy$, we employ the Ninomiya-Victoir splitting scheme to derive a numerical solution.

Definition 2.1. ***Ninomiya-Victoir Scheme***
Consider n-dimensional SDE on $\mathbb{R}_+$ with the form

$$
\begin{aligned}
dW_t &= L_0(W_t)dt + L_1(W_t)dB_t, \\
W_0 &= \xi,
\end{aligned}
\tag{2.6}
$$

where $\xi \in \mathbb{R}^n$ and $L_i : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. For all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, let the flow $\exp(tL_i)x$, $i = 1, 2$, denote the unique solution at time $u = 1$ to the ODE

$$
\begin{aligned}
\frac{dy}{du} &= tL_i(y), \\
y(0) &= x.
\end{aligned}
\tag{2.7}
$$

For a fixed iteration step $n \in \mathbb{N}$, we choose an arbitrary, possibly non-uniform, partition $\{t_0 = 0, t_1, \ldots, t_n = 1\}$ with step size $h_k = t_{k+1} - t_k$. And we approximate a numerical solution $\{\widetilde{W}_{t_k}\}_{0\le k\le n}$ in the sense that $\widetilde{W}_0 = \xi$ and

$$
\widetilde{W}_{t_{k+1}} = \exp\left(\frac{1}{2}h_k L_0\right)\exp\left(B_{t_k,t_{k+1}}L_1\right)\exp\left(\frac{1}{2}h_k L_0\right)\widetilde{W}_{t_k}, \tag{2.8}
$$

for all $k = 0, 1, \ldots, n$. In fact, the approximation $\{\widetilde{W}_{t_k}\}_{0\le k\le n}$ enjoys an integral form between every two discretization points

$$
\widetilde{W}_t = \xi + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(2)})ds + \int_0^t L_1(\widetilde{W}_s^{(1)})dB_s + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(0)})ds, \tag{2.9}
$$

where the above three discretization processes defined on each time interval

$[t_k, t_{k+1}]$ *admit the form*

$$\begin{aligned}\widetilde{W}_t^{(0)} &:= \exp\left(\frac{1}{2}(t-t_k)L_0\right)\widetilde{W}_{t_k},\\ \widetilde{W}_t^{(1)} &:= \exp\left(B_{t_k,t}L_1\right)\widetilde{W}_{t_{k+1}}^{(0)},\\ \widetilde{W}_t^{(2)} &:= \exp\left(\frac{1}{2}(t-t_k)L_0\right)\widetilde{W}_{t_{k+1}}^{(1)}.\end{aligned} \tag{2.10}$$

Looking back to our Loewner equation, we write $L_0(z) = -2/z$ and $L_1(z) = \sqrt{\kappa}$. Hence, by ([3], $Thm.$ 6.2) the following form is immediate

$$\begin{aligned}\exp(tL_0)z &= \sqrt{z^2 - 4t},\\ \exp(tL_1)z &= z + \sqrt{\kappa}t.\end{aligned} \tag{2.11}$$

Given the above arbitrary partition $\{t_0 = 0, t_1, \ldots, t_n = 1\}$, we could formulate an approximated solution $\{\widetilde{Z}_{t_k}\}_{0\le k\le n}$ via the Ninomiya-Victoir splitting scheme

$$\widetilde{Z}_{t_{k+1}} := \sqrt{\left(\sqrt{\widetilde{Z}_{t_k}^2 - 2h_k} + \sqrt{\kappa}B_{t_k,t_{k+1}}\right)^2 - 2h_k}\,, \tag{2.12}$$

with the initial value $\widetilde{Z}_0 = iy$ specified at each n^{th} iteration.

3 Strong convergence in probability

In simulating SLE_κ traces, we manually choose a partition on the time interval $[0, 1]$. However, a fixed uniform partition may overlook the effect of some sample Brownian paths and may result in huge numerical errors. Instead, we propose an adaptive method to control step size.

Definition 3.1. *Let $Z_t(iy_n)$ be the solution to Eqn.* (2.5) *started from $iy_n \in \mathbb{H}$, and $\widetilde{Z}_t(iy_n)$ be its approximation following Ninomiya-Victoir Scheme. And let $\eta(t)$ be the shifted Loewner curve defined above.*

The idea is choosing a constant $\tau > 0$, called tolerance, to ensure that

$$\left|\widetilde{Z}_{t_{k+1}} - \widetilde{Z}_{t_k}\right| \le \tau \tag{3.1}$$

for each k. To achieve this, we start by computing $\widetilde{Z}_t$ along a prior uniform partition until $\left|\widetilde{Z}_{t_{k+1}} - \widetilde{Z}_{t_k}\right| > \tau$. If this event occurs, we reduce the step size h_k of the SLE_κ discretization. Then we insert the midpoint of this interval $[t_k, t_{k+1}]$ into the partition.

In other words, we refine the prior partition via midpoint insertion whenever necessary. Notice that the choice of a refined partition actually depends on $\omega \in \Omega$ because the evolution $(\widetilde{Z}_t)_{0\le t\le 1}$ contains Brownian motion. Another aforementioned remark is that such "mid-point" refinement does not come into the proof of our main convergence theorem. But really, it is important when dealing with the boundary singularity of our backward Loewner equation.

But how good is the above simulating scheme? We will give a strong convergence in probability to the decay rate of the $\|\cdot\|_{[0,1],\infty}$ norm (*i.e. supremum norm*) between the original Loewner curve and our simulation. In the next section, we are going to alternatively discuss the general formalism to prove the L^p convergence of our splitting simulation, which is the same object *w.r.t.* a different topology.

Notice that at each iteration step $n \in \mathbb{N}$, we specify an initial condition $y \in \mathbb{R}_+$ and let the approximated sample paths $(\widetilde{Z}_t(iy))_{0\le t\le 1}$ evolve according to the backward Loewner equation (2.5). To ensure a decent $\|\cdot\|_{[0,1],\infty}$ convergence result, we not only require the mesh of the partition tends to 0, but also choose a sequence $\{y_n\} \subset \mathbb{R}_+$ so that $y_n \to 0^+$ monotonically and strictly.

Remark 3.2. *Notice that we cannot let $y_n \equiv y$ for some $y > 0$, otherwise the convergence pattern breaks down and hence strict monotonicity of $\{y_n\}$ is necessary. On the other hand, the decay rate of $\{y_n\}$ should not be too fast to destroy the probability inequality w.r.t. the $\|\cdot\|_{[0,1],\infty}$ norm, which we will see in the following context.*

In this section, we manually set $y_n = n^{-1/2}$ for all, $n \in \mathbb{N}$. This choice of $\{y_n\}$ actually satisfies the requirements in the above *Rmk.* 3.2 for the initial conditions.

For now we choose a prior uniform partition $\mathcal{D}_n$ on $[0,1]$ at each step $n \in \mathbb{N}$, before writing a refined partition $\widetilde{\mathcal{D}}_n$ via the above adaptive method. Notice again that $\widetilde{\mathcal{D}}_n$ generally depends on $\omega \in \Omega$, since the driving force is Brownian motion.

To reach our main object, we make the following observation.

Definition 3.3. *For all $t \in [0,1]$, given an arbitrary uniform partition $\mathcal{D}_n$, we define $\{t_k(t), t_{k+1}(t)\} \subset \mathcal{D}_n$ to be the neighboring two points in the partition $\mathcal{D}_n$ between which t resides, i.e. $t_k(t) \le t < t_{k+1}(t)$. And the same notation $\{t_k(t), t_{k+1}(t)\}$ also applies to the refined partition $\widetilde{\mathcal{D}}_n$.*

We are now reaching our main object: an upper bound for the probability of $\|\cdot\|_{[0,1],\infty}$ norm of $(\eta(t) - \widetilde{Z}_t(iy_n))_{0\le t\le 1}$ to be small in the sense given by the following theorem, which is the main result of the paper

Theorem 3.4. *There exist two non-increasing functions $\varphi_i : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \varphi_i(n) = 0^+$ with $i = 1, 2$. If the mesh $||\widetilde{D}_n|| \to 0^+$ with $n \to \infty$, faster than a proper rate $||\widetilde{D}_n|| \backsim o(n^{-3})$, then*

$$\mathbb{P}\Big(\Big\| \eta(t) - \widetilde{Z}_t(iy_n) \Big\|_{[0,1],\infty} \leq \varphi_1(n) \Big) \geq 1 - \varphi_2(n), \tag{3.2}$$

where $\widetilde{\mathcal{D}}_n$ is the refined partition via tolerance $\tau > 0$.

To prove this theorem, we use

$$\begin{aligned} \Big| Z_t(iy_n) - \widetilde{Z}_t(iy_n) \Big| &\leq \big| Z_t(iy_n) - Z_{t_k(t)}(iy_n) \big| \\ &+ \Big| Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \Big| + \Big| \widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n) \Big|, \end{aligned} \tag{3.3a}$$

in addition to estimate

$$\Big| \eta(t) - \widetilde{Z}_t(iy_n) \Big| \leq |\eta(t) - Z_t(iy_n)| + \Big| Z_t(iy_n) - \widetilde{Z}_t(iy_n) \Big|. \tag{3.3b}$$

And we will start from evaluating *Ineq.* (3.3*a*) and evaluate *Ineq.* (3.3*b*) as a byproduct.

In the following, we investigate *Ineq.* (3.3*a*). It follows from

Lemma 3.5. ([4], *Thm.* 3.4.2) *There exist $c_1, c_2 > 0$ such that if we consider the event*

$$E'_{n,1} := \Big\{ osc(\sqrt{\kappa} B_t, \frac{1}{n}) \leq c_1 \sqrt{\frac{\log(n)}{n}} \Big\}, \tag{3.4}$$

then we have

$$\mathbb{P}\big(E'_{n,1}\big) \geq 1 - \frac{c_2}{n^2}. \tag{3.5}$$

Lemma 3.6. ([11], *Eqn.* 21.) *There exist $c_3, c_4, c_5 > 0$ and $\beta_1 \in (0, 1)$ such that if we consider the event*

$$E''_{n,1} := \Big\{ \big| \partial_z \widehat{g}_t^{-1}(iv) \big| \leq c_3 \cdot v^{-\beta_1} \; for\; all\, t \in [0,1] \; and \; v \in [0, \frac{1}{\sqrt{n}}] \Big\}, \tag{3.6}$$

then we have

$$\mathbb{P}\big(E''_{n,1}\big) \geq 1 - \frac{c_4}{n^{c_3/2}}. \tag{3.7}$$

We have an estimate to the first term to *Ineq.* (3.3*a*) with the form

$$\begin{aligned} \big| Z_t(iy_n) - Z_{t_k(t)}(iy_n) \big| &\leq |Z_t(iy_n) - \eta(t)| + \big| Z_{t_k(t)}(iy_n) - \eta\big(t_k(t)\big) \big| \\ &+ \big| \eta(t) - \eta\big(t_k(t)\big) \big|. \end{aligned} \tag{3.8}$$

To proceed our discussion, we remind our readers of the following definition.

Definition 3.7. *A continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a subpower function if it is non-increasing and satisfies*

$$\lim_{x\to\infty} x^{-\nu}\phi(x) = 0, \text{ for all } \nu > 0. \tag{3.9}$$

Remark 3.8. *A typical subpower function is $\phi(x) = (\log x)^\alpha$, for all real $\alpha > 0$.*

With the notion of a subpower function, we have

Proposition 3.9. *There exists a subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that if we consider the event*

$$E^*_{n,1} := \left\{ \left\|\eta(t) - \eta\big(t_k(t)\big)\right\|_{[0,1],\infty} \le \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} \right\}, \tag{3.10}$$

and if $||\mathcal{D}_n|| \le n^{-1}$, then

$$\mathbb{P}\big(E^*_{n,1}\big) \ge 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}}. \tag{3.11}$$

Proof. In the proof we omit the bracket in $t_k(t)$ and simply write this term as t_k, which will be clear from the context. In the proof, we follow the statement in ([11], *Lem.* 2.5) with some obvious changes of notations. Since $\eta([0,1])$ has identical distribution to $\gamma([0,1])$ modulo a scalar sshift $\sqrt{\kappa}B_1$, it is immediate that $\mathbb{P}\big(E^*_{n,1}\big)$ is equal to the probability of the event with an expression which we substitute $\gamma(t)$ (*and resp.* $\gamma\big(t_k(t)\big)$) into $\eta(t)$ (*and resp.* $\eta\big(t_k(t)\big)$). By ([11], *Lem.* 2.5) there exists a subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, on the event $E'_{n,1} \cap E''_{n,1} \subset \Omega$, provided $0 \le t - t_k \le n^{-1}$ for all $t \in [0,1]$, we have

$$\begin{aligned} |\gamma(t) - \gamma(t_k)| &\le \phi(\sqrt{n}) \bigg(\int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr + \int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr \bigg) \\ &\le \phi(n) \cdot \frac{2}{1-\beta_1} n^{-(1-\beta_1)/2}. \end{aligned} \tag{3.12}$$

Hence $\mathbb{P}\big(E^*_{n,1}\big) \ge \mathbb{P}\big(E'_{n,1} \cap E''_{n,1}\big)$ and the conclusion follows. □

To finish the evaluation of *Eqn.* (3.8) and then finishing the first term in *Ineq.* (3.3*a*), we have the following result

Proposition 3.10. *There exists $\epsilon_0 \in (0,1)$. If we choose $M_n = n^{(1-\epsilon_0)/4}$ and consider the event*

$$E^{**}_{n,1} := \left\{ \|Z_t(iy_n) - \eta(t)\|_{[0,1],\infty} \le M_n \cdot y_n^{1-\epsilon_0} = \frac{1}{n^{(1-\epsilon_0)/4}} \right\}, \tag{3.13}$$

then there exists $\epsilon_n \to 0^+$ *monotonically such that*

$$\mathbb{P}\big(E^{**}_{n,1}\big) \geq 1 - \epsilon_n. \tag{3.14}$$

Proof. It is stated in ([3], *Lem.* 6.7) that there exists $\epsilon_0 \in (0,1)$ so that ω-*almost surely*, we have

$$\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq C_1'(\omega) \cdot y_n^{1-\epsilon_0}, \tag{3.15}$$

where $C_1'(\omega)$ is *almost surely* finite. Guaranteed with the existence of at least one $C_1'(\omega) \in \mathbb{R}_+$ for *almost surely* all $\omega \in \Omega$, we define the collection $\mathcal{A}(\omega) \subset \mathbb{R}_+$ for those $\omega \in \Omega$ with which there exists at least one $C_1'(\omega)$ satisfying *Ineq.* (3.15). Notice that the collection $\mathcal{A}(\omega)$ is defined except for a *measure-zero* event. The well-ordering principle tells us that $\mathcal{A}(\omega)$ has a lower bound. Hence it is legitimate to define

$$C_1(\omega) \coloneqq \inf \mathcal{A}(\omega), \tag{3.16}$$

which is *almost surely* defined. Hence, we could simply assume $C(\omega)$ exists and is finite everywhere via subtracting a *measure-zero* event from Ω. With our choice of $M_n \to \infty$, there exists $\epsilon_n \in [0,1]$ with $\epsilon_n \coloneqq \min\{\epsilon_1, \ldots, \epsilon_{n-1}, \mathbb{P}\big(E^{**}_{n,1}\big)\}$ such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq M_n \cdot y_n^{1-\epsilon_0} \right) \geq 1 - \epsilon_n. \tag{3.17}$$

On the event $E^{**}_{n,1} \subset \Omega$, we know then

$$\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq C_1(\omega) \cdot y_n^{1-\epsilon_0} \text{ and } \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \leq M_n \cdot y_n^{1-\epsilon_0}. \tag{3.18}$$

By the definition of $C_1(\omega)$, it is then clear that on the event $E^{**}_{n,1}$, we have

$$C_1(\omega) \leq M_n. \tag{3.19}$$

Hence, on the event $E^{**}_{n,1}$, it is immediate that

$$\sup_{t\in[0,1]} |Z_t(iy_{n+1}) - \eta(t)| \leq C_1(\omega) \cdot y_{n+1}^{1-\epsilon_0} \leq M_n \cdot y_{n+1}^{1-\epsilon_0} \leq M_{n+1} \cdot y_{n+1}^{1-\epsilon_0}. \tag{3.20}$$

Hence the event $E^{**}_{n+1,1}$ occurs and

$$\mathbb{P}\big(E^{**}_{n+1,1}\big) \geq \mathbb{P}\big(E^{**}_{n,1}\big), \tag{3.21}$$

which justifies our choice of definition of ϵ_n, from which the monotonicity of $\{\epsilon_n\} \subset \mathbb{R}_+$ is easily seen. To show that $\epsilon_n \to 0^+$, we notice that $C_1(\omega)$ is *almost* *surely* finite. Suppose $\epsilon_n \to \sigma > 0$. Then with probability σ, the constant $C_1(\omega)$ is greater than any M_n, $n \in \mathbb{N}_+$. Because $M_n \to \infty$, we are forced to conclude that $C_1(\omega) = \infty$ with positive probability, which is impossible. $\square$

We have now discussed every term in *Eqn.* (3.8), it is time to finalize the estimate of the first term in *Ineq.* (3.3*a*).

Proposition 3.11. *Given the assumptions that* $||\mathcal{D}_n|| \leq n^{-1}$ *and* $y_n = n^{-1/2}$, *if we define the event*

$$E_{n,1} \coloneqq \left\{ \left\| Z_t(iy_n) - Z_{t_k(t)}(iy_n) \right\|_{[0,1],\infty} \leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{2}{n^{(1-\epsilon_0)/4}} \right\}, \tag{3.22}$$

then the following inequality holds

$$\mathbb{P}\big(E_{n,1}\big) \geq 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}} - 2\epsilon_n. \tag{3.23}$$

Proof. On the event $E^*_{n,1} \cap E^{**}_{n,1}$, we know that

$$\begin{aligned} \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| &\leq \frac{1}{n^{(1-\epsilon_0)/4}}, \\ \sup_{t\in[0,1]} \big|\eta(t) - \eta\big(t_k(t)\big)\big| &\leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}}. \end{aligned} \tag{3.24}$$

Looking back to *Eqn.* (3.8), we see $\mathbb{P}\big(E_{n,1}\big) \geq \mathbb{P}\big(E^*_{n,1} \cap E^{**}_{n,1}\big) \geq 1 - c_2 n^{-2} - c_4 n^{-c_3/2} - 2\epsilon_n$. $\square$

Hence we have estimated the $\|\cdot\|_{[0,1],\infty}$ norm of the first term in *Ineq.* (3.3*a*). This is in fact the most complicated term among these three terms. Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the second term.

Inspect *Eqn.* (2.12), the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ resembles a forward Loewner map driven by constant forces on the local time interval $[t_k, t_{k+1}]$. In fact, this is the case. We are going to split the total time interval $[0,1]$ into many local time intervals $[t_k, t_{k+1}]$. And on each local time interval, the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ is a composition of two inverse of some forward Loewner map driven by constant forces with an intermediate parallel translation.

Lemma 3.12. ([10], *Sec.* 2.) *Given a constant driving force* $t \mapsto A$ *on the time interval* $[0,1]$*, the forward Loewner chain admits the form*

$$g_t(z) = A + \big[(z-A)^2 + 4t\big]^{\frac{1}{2}}. \tag{3.25}$$

And its conformal inverse admits the form

$$f_t(z) = A + \left[(z-A)^2 - 4t\right]^{\frac{1}{2}}. \tag{3.26}$$

Lemma 3.13. *On each local time interval $[t_k, t_{k+1}]$, consider the constant force $t \mapsto 0$ and denote the inverse of the forward Loewner chain driven by this constant force as $\iota_{k,1}$. Consider the parallel translation $z \xmapsto{\iota_{k,2}} z + \sqrt{\kappa}B_{t_k,t_{k+1}}$. Then we have the composition*

$$\widetilde{Z}_{t_{k+1}}(iy_n) = \iota_{k,2} \circ \iota_{k,1} \circ \iota_{k,2} \widetilde{Z}_{t_k}(iy_n). \tag{3.27}$$

Proof. Inspect *Eqn* (2.12) and *Eqn.* (3.26) and the conclusion follows. □

The above perspective in *Lem.* 3.14 endows us convenience because now we can introduce some discretized force with the form of a step-function to estimate the second term in *Ineq.* (3.3a). In fact, we have the following proposition.

Proposition 3.14. *If we consider the perturbation event*

$$E_{n,2} \coloneqq \left\{ \left\| Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right\|_{[0,1],\infty} \leq \frac{1}{\sqrt{4n+1}} \right\}, \tag{3.28}$$

and if we further restrict $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, then

$$\mathbb{P}\big(E_{n,2}\big) \geq 1 - 2e^{-(4n+1)/\kappa}. \tag{3.29}$$

Proof. From *Sec.* 2. we already know $Z_t(iy_n) = h_t(iy_n) - \sqrt{\kappa}B_t$. If we further choose $\widehat{B}_t \coloneqq B_{1-t} - B_1$, then *Eqn.* (2.2) can be written as

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa}\widehat{B}_t}, \tag{3.30}$$

with $h_0(z) = z$ and where $\widehat{B}_t$ has the law of a standard Brownian motion because a simple calculation shows that they have the same finite-dimensional marginals. We also comment that our splitting simulation scheme could be formulated in a similar fashion. Define

$$\widetilde{\lambda}(t) \coloneqq 0 \cdot \mathbb{1}_{[0,\frac{t_1}{2})} + \sum_{k \geq 1} \sqrt{\kappa}B_{t_k} \cdot \mathbb{1}_{[t_k - \frac{h_k}{2}, t_k + \frac{h_k}{2} \wedge 1)}. \tag{3.31}$$

The random process $\widetilde{\lambda}(t)$ can be viewed as a step-function interpolation to the sample paths of Brownian motion $\sqrt{\kappa}B_t$ on $[0,1]$. In this regard, we say $\widetilde{Z}_t(iy_n)$ is driven by some discrete force similar to $Z_t(iy_n)$ being driven by $\sqrt{\kappa}\widehat{B}_t$ in the following sense

$$\widetilde{Z}_t(iy_n) = \widetilde{h}_t(iy_n) - \widetilde{\lambda}(t), \tag{3.32}$$

due to *Lem.* 3.13 and where $(\widetilde{h}_t)_{t\in[0,1]}$ is a backward Loewner chain constrained by

$$\partial_t \widetilde{h}_t(z) = \frac{-2}{\widetilde{h}_t(z) - \widehat{\lambda}_t}, \tag{3.33}$$

with $\widetilde{h}_0(z) = z$ and $\widehat{\lambda}_t := \widetilde{\lambda}(1-t) - \widetilde{\lambda}(1)$. The above scheme brings us some consistency to the perturbation term $Z_t(iy_n) - \widetilde{Z}_t(iy_n)$. And our goal is to estimate

$$\left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right| \leq \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right|. \tag{3.34}$$

Define $\epsilon := \sup\limits_{t\in[0,1]} \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right|$, then it is obvious that

$$\left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right| \leq \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \epsilon. \tag{3.35}$$

To achieve this goal, we further define $H(t) := h_t(iy_n) - \widetilde{h}_t(iy_n)$. And we will first estimate $|H(t)|$. Differentiate $H(t)$ *w.r.t.* $t \in [0,1]$ and use *Eqn.* (2.2) and *Eqn.* (3.33) to obtain

$$\frac{d}{dt}H(t) - H(t)\zeta(t) = (\sqrt{\kappa}\widehat{B}_t - \widehat{\lambda}_t)\zeta(t), \tag{3.36}$$

where we define $\zeta(t) := \left(h_t(iy_n) - \sqrt{\kappa}\widehat{B}_t\right)^{-1} \cdot \left(\widetilde{h}_t(iy_n) - \widehat{\lambda}_t\right)^{-1}$. Integrate the above differential equation and choose $u(t) := e^{-\int_0^t \zeta(s)ds}$, we find

$$H(t) = u(t)^{-1}\left[H(0) - \int_0^t (\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s)u(s)\zeta(s)ds\right]. \tag{3.37}$$

Since it is obvious that $H(0) = 0$, we see

$$|H(t)| \leq \int_0^t \left|\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s\right| e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds. \tag{3.38}$$

Then, it is immediate that

$$\begin{aligned} \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| &\leq \epsilon \cdot \int_0^t e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds \\ &\leq \epsilon \cdot \left(e^{\int_s^t |\zeta(r)|dr} - 1\right), \end{aligned} \tag{3.39}$$

where the last inequality is due to ([15], *Lem.* 2.3) and ([15], *Eqn.* 2.12). Now turning attention to *Eqn.* (3.34), we have

$$\Big|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\Big| \leq \Big|h_t(iy_n) - \widetilde{h}_t(iy_n)\Big| + \epsilon \leq \epsilon \cdot e^{\int_s^t \operatorname{Re}\zeta(r)dr}. \tag{3.40}$$

Furthermore, ([15], *Eqn.* 2.12) tells us that $\int_0^t |\zeta(s)|ds \leq \log\Big(\sqrt{4+y_n^2}/y_n\Big)$. Consequently, we have

$$\Big|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\Big| \leq \epsilon \cdot \sqrt{4+y_n^2}/y_n = \epsilon \cdot \sqrt{4n+1}. \tag{3.41}$$

Notice that

$$\epsilon = \sup_{t\in[0,1]} \Big|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\Big| \leq \bigvee_{t_k\in\mathcal{D}_n} \sup_{t\in[0,h_k]} \sqrt{\kappa}|B_t|, \tag{3.42}$$

where the notation "$\vee$" indicates we take the maximal value over all $t_k \in \mathcal{D}_n$. By ([9], *Cor.* 2.2), we encounter the supremum Brownian motion $S_t \coloneqq \sup\limits_{0\leq s\leq t} B_s$ which admits the distribution law

$$\mathbb{P}\Big(S_t \leq x\Big) = 2\Phi\Big(\frac{x}{\sqrt{t}}\Big) - 1, \tag{3.43}$$

for all $x \geq 0$ and where $\Phi(x) \coloneqq e^{-x^2/2}/\sqrt{2\pi}$ is the law of standard normal variable. It follows that

$$\mathbb{P}\Big(\sup_{0\leq t\leq h_k} \big|\sqrt{\kappa}B_t\big| \geq \frac{1}{4n+1}\Big) = 2\mathbb{P}\Big(S_{h_k} \geq \frac{1}{\sqrt{\kappa}\cdot(4n+1)}\Big) \leq 2\sqrt{\frac{2}{\pi}}e^{-\frac{(4n+1)^{-2}}{2h_k\cdot\kappa}}. \tag{3.44}$$

Because we further restrict $h_k \leq n^{-1} \wedge (4n+1)^{-3}$ for all $t_k \in \mathcal{D}_n$, then

$$\Big\{\epsilon > \frac{1}{4n+1}\Big\} \leq \bigcup_{t_k\in\mathcal{D}_n} \Big\{\sup_{0\leq t\leq \frac{h_k}{2}} \big|\sqrt{\kappa}B_t\big| > \frac{1}{4n+1}\Big\}, \tag{3.45a}$$

and we see

$$\begin{aligned}\mathbb{P}\Big(\epsilon > \frac{1}{4n+1}\Big) &\leq \sum_{t_k\in\mathcal{D}_n} \mathbb{P}\Big(\sup_{0\leq t\leq \frac{h_k}{2}} \big|\sqrt{\kappa}B_t\big| > \frac{1}{4n+1}\Big) \\ &\leq 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}.\end{aligned} \tag{3.45b}$$

Conditioned on the event $\{\epsilon > (4n+1)^{-1}\}^c \in \Omega$, following *Eqn.* (3.41), we have

$$\sup_{t\in[0,1]} \Big|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\Big| \leq \frac{1}{\sqrt{4n+1}}. \tag{3.46}$$

Hence, by the strict inclusion of events in probability space, we have our desired result

$$\mathbb{P}\big(E_{n,2}\big) \geq 1 - 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}. \tag{3.47}$$

$\square$

Remark 3.15. *In Sec.* 1. *we mentioned other notions to simulate SLE_κ trace $\gamma(t)$ via interpolating the driving force. In those interpolation cases we also need to measure another type of perturbation term generated by $g_t^{-1}(z)$, inverse of the Loewver map. We use the same result ([15], Lem.* 2.2*) via a "box" method. On the contrary, the perturbation in Prop.* 3.12 *in our splitting scheme is estimated via a probabilistic argument. Notice that the time-reversed Loewner map $h_t(z)$ is different from the inverse of the forward map $g_t^{-1}(z)$, even though we do have the equality $h_{T=1}(z) = g_{T=1}^{-1}(z)$. And in Sec.* 5. *we are going to briefly discuss the linear interpolation of driving force. To prove its convergence, we will need ([15], Lem.* 2.2*) again under a different context.*

Hence we have estimated the sup-norm on $[0,1]$ of the second term in *Ineq.* (3.3*a*). Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the third term. Following *Eqn.* (2.10) with $L_0(z) = -2/z$ and $L_1(z) = \sqrt{\kappa}$, we could explicitly calculate, with $t_k \leq s < t_{k+1}$

$$\begin{aligned}
\widetilde{Z}_s^{(0)} &= \exp\left(\frac{1}{2}(s-t_k)L_0\right)\widetilde{Z}_{t_k} = \sqrt{\widetilde{Z}_{t_k}^2 - 2(s-t_k)}, \\
\widetilde{Z}_s^{(1)} &= \exp\left(B_{t_k,s}L_1\right)\widetilde{Z}_{t_{k+1}}^{(0)} = \sqrt{\widetilde{Z}_{t_k}^2 - 2h_k} + B_{t_k,s}, \\
\widetilde{Z}_s^{(2)} &= \exp\left(\frac{1}{2}(s-t_k)L_0\right)\widetilde{Z}_{t_{k+1}}^{(1)} = \sqrt{\big(\widetilde{Z}_{t_{k+1}}^{(1)}\big)^2 - 2(s-t_k)}.
\end{aligned} \tag{3.48}$$

which could be written into the form via solving *Eqn.* (2.9)

$$\begin{aligned}
&\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \\
&\quad= \frac{1}{2}\int_{t_k(t)}^t L_0(\widetilde{Z}_s^{(2)})ds + \int_{t_k(t)}^t L_1(\widetilde{Z}_s^{(1)})dB_s + \frac{1}{2}\int_{t_k(t)}^t L_0(\widetilde{Z}_s^{(0)})ds, \\
&\quad= \int_{t_k}^t \sqrt{\kappa}dB_s - \int_{t_k}^t \frac{1}{\sqrt{(\widetilde{Z}_{t_k})^2 - 2(s-t_k)}}ds - \int_{t_k}^t \frac{1}{\sqrt{\big(\widetilde{Z}_{t_{k+1}}^{(1)}\big)^2 - 2(s-t_k)}}ds.
\end{aligned} \tag{3.49a}$$

Solving these integrals, we have

$$
\begin{aligned}
&\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \\
&\quad = \sqrt{\kappa} B_{t_k(t),t} - \frac{2(t - t_k(t))}{\sqrt{\widetilde{Z}^2_{t_k(t)} - 2(t - t_k(t))} + \widetilde{Z}_{t_k(t)}} \\
&\quad - \frac{2(t - t_k)}{\sqrt{(\sqrt{\widetilde{Z}^2_{t_k} - 2h_k} + \sqrt{\kappa} B_{t_k,t_{k+1}})^2 - 2(t - t_k)} + \sqrt{\widetilde{Z}^2_{t_k} - 2h_k} + \sqrt{\kappa} B_{t_k,t_{k+1}}}.
\end{aligned}
\tag{3.49b}
$$

Eqn. (3.49*b*) provides an exact form of the approximated process $\widetilde{Z}_t$, which leads us to the following result

Proposition 3.16. *Consider the event*

$$
E_{n,3} \coloneqq \left\{ \left\| \widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right\|_\infty \leq \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}} \right\}. \tag{3.50}
$$

Then as long as $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, *we have*

$$
\mathbb{P}\big(E_{n,3}\big) \geq 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \tag{3.51}
$$

Proof. By ([3], *Sec.* 6.1), we have two general results $\mathrm{Im}(z) \leq \mathrm{Im}(\sqrt{z^2 - c})$ and $\mathrm{Im}(z) = \mathrm{Im}(z + c)$ for all $z \in \mathbb{H}$ and $c \in \mathbb{R}$. Applying this two results to *Eqn.* (3.49*b*), we have

$$
\begin{aligned}
\left| \widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right| &\leq \left| \sqrt{\kappa} B_{t_k(t),t} \right| + \frac{h_k}{\mathrm{Im}\, \widetilde{Z}_{t_k}(iy_n)} + \frac{h_k}{\mathrm{Im}\, \widetilde{Z}_{t_k}(iy_n)}, \\
&\leq \left| \sqrt{\kappa} B_{t_k(t),t} \right| + \frac{2h_k}{y_n},
\end{aligned}
\tag{3.52}
$$

where the last inequality follows from ([1], *Lem.* 4.9) that the map $t \mapsto \mathrm{Im}\, \widetilde{Z}_t(iy)$ is strictly increasing. And by ([9], *Cor.* 2.2), we encounter the supremum Brownian motion $S_t = \sup_{0 \leq s \leq t} B_s$ again. It follows that

$$
\mathbb{P}\left(\sup_{0 \leq t \leq h_k} \left| \sqrt{\kappa} B_t \right| \geq \frac{1}{n^{1/4}} \right) = 2\mathbb{P}\left(S_{h_k} \geq \frac{1}{\sqrt{\kappa} \cdot n^{1/4}} \right) \leq 2\sqrt{\frac{2}{\pi}} e^{-\frac{n^{-1/2}}{2h_k \cdot \kappa}}, \tag{3.53}
$$

by reflection principle. Because we have restricted $h_k \leq n^{-1} \wedge (4n+1)^{-3}$ for all $t_k \in \mathcal{D}_n$. In this regard

$$
\mathbb{P}\left(\sup_{t \in [0,1]} \left| \sqrt{\kappa} B_{t_k(t),t} \right| \geq \frac{1}{n^{1/4}} \right) \leq 2e^{-\sqrt{n}/2\kappa}, \tag{3.54}
$$

and

$$\mathbb{P}\big(E_{n,3}\big) \geq 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \tag{3.55}$$

□

At this point, we have evaluated the $\|\cdot\|_{[0,1],\infty}$ norm *w.r.t.* all the three terms in *Ineq.* (3.3*a*). Therefore, we come to

Proof. (*of Thm. 3.4*) Denote $E_{n,4} := E^{**}_{n,1}$. On the event $E_{n,1} \cap E_{n,2} \cap E_{n,3} \cap E_{n,4} \subset \Omega$, we observe from *Prop.* 3.10, *Prop.* 3.11, *Prop.* 3.14, and *Prop.* 3.16 that

$$\begin{aligned}
&\|\eta(t) - Z_t(iy_n)\|_{[0,1],\infty} \leq \frac{1}{n^{(1-\epsilon_0)/4}}, \\
&\big\|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big\|_{[0,1],\infty} \leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{2}{n^{(1-\epsilon_0)/4}}, \\
&\Big\|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\Big\|_{[0,1],\infty} \leq \frac{1}{(4n+1)^{1/2}}, \\
&\Big\|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\Big\|_{[0,1],\infty} \leq \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}},
\end{aligned} \tag{3.56}$$

given $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$. If we define

$$\begin{aligned}
\varphi_1(n) &:= \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{3}{n^{(1-\epsilon_0)/4}} + \frac{1}{(4n+1)^{1/2}} + \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}} \to 0, \\
\varphi_2(n) &:= \frac{1}{n} + \frac{c_2}{n^2} + \frac{c_4}{n^{c_3/2}} + 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa} + 2e^{-\sqrt{n}/2\kappa} + 3\epsilon_n \to 0,
\end{aligned} \tag{3.57}$$

as $n \to \infty$. Then clearly the following inequality holds

$$\mathbb{P}\bigg(\Big\|\eta(t) - \widetilde{Z}_t(iy_n)\Big\|_{[0,1],\infty} \leq \varphi_1(n)\bigg) \geq \mathbb{P}\bigg(E_{n,1} \cap \cdots \cap E_{n,4}\bigg) \geq 1 - \varphi_2(n). \tag{3.58}$$

Of course the above inequality regarding our approximation $\widetilde{Z}_t(iy_n)$ is only *w.r.t.* the prior uniform partition $\mathcal{D}_n$ on $[0,1]$. But we could always insert midpoints into the partition pointwisely on the event $E_{n,1} \cap \cdots \cap E_{n,4}$ using tolerance $\tau > 0$ and obtain the refined partition $\widetilde{\mathcal{D}}_n$. The above *Eqn.* (3.58) still holds because we have *almost surely* $||\widetilde{\mathcal{D}}_n|| \leq ||\mathcal{D}_n||$. □

Corollary 3.17. *For almost surely all $\omega \in \Omega$, The sample paths almost surely admit*

$$\Big\|\eta(t) - \widetilde{Z}_t(iy_n)\Big\|_{[0,1],\infty} \to 0, \ \textit{with}\, n \to \infty. \tag{3.59}$$

Proof. This corollary immediately follows from the strong convergence in probability in *Thm.* 3.4. □

4 Convergence in L^p norm

Following *Ineq.* (3.3*a*) and *Ineq.* (3.3*b*), we are going to estimate the L^p norm to $\big(\eta(t) - \widetilde{Z}_t(iy_n)\big)$ with $p \geq 2$. Notice that the discussion of this alternative convergence pattern is not completed: it relies on several assumptions, which we have not proved but we will consider them to be of furture work. And in fact, similar to the proof of strong convergence in probability, the first step is to estimate each of the four terms in *Eqn.* (3.56) individually. The L^p convergence would hold if we have the following assumptions

Assumption 4.1. *There exists $p_1 \geq 2$ and $\epsilon_0 \in (0,1)$ such that*

$$\sup_{t\in[0,1]} |\eta(t) - Z_t(iy_n)| \leq C_1(\omega) \cdot y_n^{1-\epsilon_0}, \tag{4.1}$$

where the constant $C_1(\omega)$ is almost surely finite as in ([3], Lem. 6.7). And moreover $C_1(\omega)$ is p_1-integrable, i.e. $C_1(\omega) \in L^{p_1}(\mathbb{P})$.

Assumption 4.2. *There exists $p_2 \geq 2$ and $\beta_2 \in (0,1)$ such that the SLE_κ Loewner chain is generated by a curve when $\kappa \neq 8$ with the following modulus of continuity*

$$|\eta(t+s) - \eta(t)| \leq C_2(\omega) s^{(1-\beta_2)/2} \tag{4.2}$$

where the constant $C_2(\omega)$ is almost surely finite as in ([5], Prop. 4.3). And moreover $C_2(\omega)$ is p_2-integrable, i.e. $C_2(\omega) \in L^{p_2}(\mathbb{P})$.

Assumption 4.3. *The Ninomiya-Victoir splitting scheme satisfies the various regularity assumptions, including the ellipticity condition in* ([7], *Rmk.* 4.1) *and then admits the inequality*

$$\mathbb{E}\bigg[\sup_{t\in[0,1]} \Big|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\Big|^p\bigg] \leq \frac{c_6}{n^p}, \tag{4.3}$$

for some constant $c_6 > 0$, and for all $p \geq 2$.

Of course we expect these above three assumptions to hold, since they are very reasonable and crucial to the idea of L^p convergence of our splitting simulation. In this section, we choose $p \coloneqq p_1 \wedge p_2 \geq 2$. And we have the following propositions.

Proposition 4.4. *Admitting Asmp.* 4.1 *with $||\widetilde{\mathcal{D}}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, there exists a decreasing function $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi_1(n) \to 0$ as $n \to \infty$, and*

$$\mathbb{E}\bigg[\int_0^1 |\eta(t) - Z_t(iy_n)|^p dt\bigg] \leq \psi_1(n). \tag{4.4}$$

Proof. By ([3], *Lem.* 6.7), there exists $\epsilon_0 \in (0,1)$ such that *almost surely*

$$\sup_{t\in[0,1]} |\eta(t) - Z_t(iy_n)| \le C_1(\omega) \cdot y_n^{1-\epsilon_0}. \tag{4.5}$$

It is clear then

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 |\eta(t) - Z_t(iy_n)|^p dt\Big] &\le \mathbb{E}\Big[\|\eta(t) - Z_t(iy_n)\|^p_{[0,1],\infty}\Big] \\ &\le \mathbb{E}\big[C_1(\omega)^p\big] \cdot \frac{1}{n^{(1-\epsilon_0)p/2}} := \psi_1(n) \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.6}$$

□

Proposition 4.5. *Admitting Asmp.* 4.2 *with* $||\widetilde{\mathcal{D}}_n|| \le n^{-1} \wedge (4n+1)^{-3}$*, there exists a decreasing function* $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi_2(n) \to 0$ *as* $n \to \infty$*, and*

$$\mathbb{E}\Big[\int_0^1 \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big|^p dt\Big] \le \psi_2(n). \tag{4.7}$$

Proof. ([5], *Prop.* 3.8) and ([5], *Prop.* 4.3) imply that there exists $\beta_2 \in (0,1)$ such that

$$\sup_{t\in[0,1]} \big|\eta(t) - \eta\big(t_k(t)\big)\big| \le C_2(\omega) \cdot \frac{1}{n^{(1-\beta_2)/2}}. \tag{4.8}$$

By *Prop.* 4.4, it follows that

$$\sup_{t\in[0,1]} \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big| \le \frac{2C_1(\omega)}{n^{(1-\epsilon_0)/2}} + \frac{C_2(\omega)}{n^{(1-\beta_2)/2}}. \tag{4.9}$$

It is clear then

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big|^p dt\Big] &\le \mathbb{E}\Big[\big\|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big\|^p_{[0,1],\infty}\Big] \\ \le \mathbb{E}\big[C_1(\omega)^p\big] \cdot \frac{2^{2p-1}}{n^{(1-\epsilon_0)p/2}} &+ \mathbb{E}\big[C_2(\omega)^p\big] \cdot \frac{2^{p-1}}{n^{(1-\beta_2)p/2}} := \psi_2(n) \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.10}$$

□

Proposition 4.6. *Admitting Asmp.* 4.3 *with* $||\widetilde{\mathcal{D}}_n|| \le n^{-1} \wedge (4n+1)^{-3}$*, it is immediate that*

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \Big|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\Big|^p\Big] \le \frac{c_6}{n^p}. \tag{4.11}$$

If we denote $\psi_3(n) \coloneqq c_6/n^p$, *then*

$$\mathbb{E}\left[\int_0^1 \left|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\right|^p dt\right] \leq \psi_3(n) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.12}$$

Proof. The proposition follows from *Eqn.* (4.9). □

To give an estimate to the last term in *Eqn.* (3.56), we quote the known interpolation inequality from ([8], *Sec.* 6.5) for Lebesgue spaces and another inequality *w.r.t.* supremum Brownian motion.

Lemma 4.7. *For all* $1 < p < r < q$, *suppose* $f \in L^p \cap L^q$. *Then* $f \in L^r$ *with*

$$\|f\|_r \leq \left(\|f\|_p\right)^{(1/r-1/q)/(1/p-1/q)} \left(\|f\|_q\right)^{(1/r-1/q)/(1/p-1/q)}. \tag{4.13}$$

Lemma 4.8. *For all* $m \in \mathbb{N}$, *the standard Brownian motion enjoys*

$$\mathbb{E}\left[\sup_{s\in[0,t]} |B_s|^{2m}\right] = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+m\right)2^m \cdot t^m. \tag{4.14}$$

Proof. The proof follows if we revisit the supremum Brownian motion $S_t = \sup_{0\leq s\leq t} B_s$ from ([9], *Cor.* 2.2). by compute all even order moments. □

Proposition 4.9. *With* $||\widetilde{\mathcal{D}}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, *there exists a decreasing function* $\psi_4 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi_4(n) \to 0$ *with* $n \to \infty$, *and*

$$\mathbb{E}\left[\int_0^1 \left|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\right|^p dt\right] \leq \psi_4(n). \tag{4.15}$$

Proof. By *Prop.* 3.14 and *Eqn.* (3.52), we know *almost surely* that

$$\left|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\right|^p \leq 2\left|\sqrt{\kappa}B_{t_k(t),t}\right|^p + \frac{2^{p+1}}{n^{p/2}}. \tag{4.16}$$

Since $p \geq 2$, there exists $\{m, m+1\} \subset \mathbb{N}$ so that $2m \leq p < 2(m+1)$. Remember that $t - t_k(t) \leq h_k \leq n^{-1}$. Then, by *Lem.* 4.8

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t_k(t),t}\right|^{2m}\right] &\leq \frac{2^m\kappa^m\Gamma\left(\frac{1}{2}+m\right)}{\pi^{\frac{1}{2}} \cdot n^m}, \\
\mathbb{E}\left[\sup_{t\in[0,1]} \left|\sqrt{\kappa}B_{t_k(t),t}\right|^{2m+2}\right] &\leq \frac{2^{m+1}\kappa^{m+1}\Gamma\left(\frac{3}{2}+m\right)}{\pi^{\frac{1}{2}} \cdot n^{m+1}}.
\end{aligned} \tag{4.17}$$

By *Lem.* 4.7, we use the interpolation inequality in Lebesgue spaces and have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k(t),t}\big|^p\bigg]\\
&\le \mathbb{E}\bigg[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k,t}\big|^{2m}\bigg]^{m+1-\frac{p}{2}}\mathbb{E}\bigg[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k,t}\big|^{2m+2}\bigg]^{(m+1-\frac{p}{2})\cdot\frac{m}{m+1}} \qquad (4.18)\\
&\le \frac{c_7}{n^{m(m+1-\frac{p}{2})}} \coloneqq \psi_4(n) \overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where $c_7 > 0$ is a constant depending only on $p \ge 2$. □

We have, at this point, estimated the L^p norm *w.r.t.* the four terms in *Eqn.* (3.52) under several assumptions. Combining them together, we have the following result.

Theorem 4.10. *Admitting Asmp.* 4.1, 4.2, *and* 4.3, *together with* $||\widetilde{\mathcal{D}}_n|| \le n^{-1} \wedge (4n+1)^{-3}$, *there exists a decreasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(n) \to 0$ *as* $n \to \infty$, *and*

$$\mathbb{E}\bigg[\int_0^1 \Big|\eta(t) - \widetilde{Z}_t(iy_n)\Big|^p\bigg] \le \psi(n). \qquad (4.19)$$

Proof. Define $\psi \coloneqq \psi_1 + \psi_2 + \psi_3 + \psi_4$, this result immediately follows. □

5 Linear interpolation of driving force

In *Sec.* 2. we have formulated a splitting simulation of SLE_κ traces via Ninomiya-Victoir Scheme. In *Sec.* 3. we proved the strong convergence in probability of the splitting simulation. In *Sec.* 4. we proposed an L^p convergence of the above splitting simulation under several reasonable assumptions, which we expect to be verified. In this section, we diverge a little to see a variant simulating scheme: linearly interpolating the driving Brownian motion $\sqrt{\kappa}B_t$ pointwisely, and discuss its convergence.

Definition 5.1. *In this section,* $\mathcal{D}_n \coloneqq \{t_0 = 0, t_1, \dots, t_n = 1\}$ *will denote a uniform partition on the time interval* $[0,1]$.

Definition 5.2. *Given the Brownian sample paths* $B_\cdot(\omega) : [0,1] \to \mathbb{R}$, *we write our linear interpolation in the following form*

$$\lambda^n(t) \coloneqq n\big(\sqrt{\kappa}B_{t_{k+1}} - \sqrt{\kappa}B_{t_k}\big)(t-t_k) + \lambda(t_k) \text{ on } [t_k, t_{k+1}]. \qquad (5.1)$$

Definition 5.3. *Following the convention in Sec.* 2. *and Sec.* 3. *we use the noatation* $\gamma : [0,1] \to \mathbb{H}\cup\{\sqrt{\kappa}B_1\}$ *to denote the forward Loewner curve generated* *by driving force $\sqrt{\kappa}B_t$, with forward Loewner chain $(g_t)_{t\in[0,1]}$. And we further let $\gamma^n : [0,1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_1\}$ to denote the forward Loewner curve driven the piecewise-linear force $\lambda^n(t)$, with forward Loewner chain $(g_t^n)_{t\in[0,1]}$.*

Remark 5.4. *Notice that we will interpolate the forward Loewner chain and hence simulate the forward Loewner curve $\gamma(t)$, whereas we have simulated the backward Loewner curve $\eta(t)$ via Ninomiya-Victoir Scheme in Sec.* 2.

Definition 5.5. *With $(g_t^n)_{t\in[0,1]}$ the Loewner chain corresponding to $\lambda^n(t)$, let $f_t^n : \mathbb{H} \to \mathbb{H}\backslash\gamma^n([0,t])$ be the inverse map of $g_t^n(z)$ and denote $\widehat{f}_t^n(z) := f_t^n\big(z + \lambda^n(t)\big)$. Choose $G_k^n := (\widehat{f}_{t_k}^n)^{-1} \circ \widehat{f}_{t_{k+1}}^n$. Then*

$$\widehat{f}_{t_k}^n = G_0^n \circ G_1^n \circ \cdots \circ G_{k-1}^n. \tag{5.2}$$

Definition 5.6. *Choose $\gamma_t^n(s) := g_t^n\big(\gamma^n(t+s)\big)$ with $s \in [0, 1-t]$, for all $t \in [0,1]$.*

Lemma 5.7. *Consider the event $F_{n,1} := E'_{n,1}$ and $F_{n,2} := E''_{n,1}$ as in Eqn.* (3.4) *and in Eqn.* (3.6)*. Then we have*

$$\mathbb{P}\big(F_{n,1}\big) \geq 1 - \frac{c_2}{n^2} \quad \text{and} \quad \mathbb{P}\big(F_{n,2}\big) \geq 1 - \frac{c_4}{n^{c_3/2}}, \tag{5.3}$$

where $c_2, c_3, c_4 > 0$ are constants depending only on $\kappa \neq 8$.

Theorem 5.8. *There exist $c_6, c_7 > 0$ depending only on $\kappa \neq 8$ such that if we consider the event*

$$F_n := \left\{ \|\gamma - \gamma^n\|_{[0,1],\infty} \leq \frac{c_6(\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}} \right\}. \tag{5.4}$$

Then we have $\mathbb{P}\big(F_n\big) \geq 1 - c_2 \cdot n^{-2} - c_3 \cdot n^{-c_4/2}$.

This theorem is our main result in the context of linearly interpolating Brownian driver. We will not give a detailed proof here because the proof is similar to the known result of square-root interpolation in ([11], *Sec.* 2.). Instead, we outline the ideas to estimate the convergence in probability of the linear interpolation method.

On the event $F_{n,1} \cap F_{n,1}$, we want to give a uniform bound to $|\gamma(t) - \gamma^n(t)|$ with $t \in [0,1]$. In fact, for all $t_k \in \mathcal{D}_n$, we write

$$\begin{aligned} |\gamma(r+t_k) - \gamma^n(r+t_k)| &\leq |\gamma(r+t_k) - \gamma(s+t_k)| + \left|\widehat{f}_{t_k}(z) - \widehat{f}_{t_k}^n(w)\right| \\ &\leq |\gamma(r+t_k) - \gamma(s+t_k)| + \left|\widehat{f}_{t_k}(z) - \widehat{f}_{t_k}(w)\right| + \left|\widehat{f}_{t_k}(w) - \widehat{f}_{t_k}^n(w)\right|, \end{aligned} \tag{5.5}$$

where $w := \gamma_k^n(r)$, r is arbitrarily fixed in $[\frac{1}{n}, \frac{2}{n}]$ and $z := \gamma_k(s)$ is chosen to be the highest point in the arc $\gamma_k([0, \frac{2}{n}])$. The first term in *Eqn.* (5.5) is bounded by the uniform continuity of $\gamma(t)$ on the event $F_{n,1}$ where the driving force $\sqrt{\kappa}B_t$ has small oscillation.

The estimate of the second term in *Eqn.* (5.5) is comparing the images of nearby points in $\mathbb{H}$ very close to the real and the imaginary axis, under a conformal map. To proceed our discussion, we introduce, for any subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, constant $c > 0$, and integer $n \in \mathbb{N}_+$ that

$$A_{n,c,\phi} := \left\{ x + iy \in \mathbb{H};\ |x| \leq \frac{\phi(n)}{\sqrt{n}} \text{ and } \frac{1}{\sqrt{n}\phi(n)} \leq y \leq \frac{c}{\sqrt{n}} \right\}, \tag{5.6}$$

which is a box near the origin in complex plane. The reason we introduce this extra object is that the images of nearby points in $A_{n,c,\phi}$ will also be close to each other under certain conformal maps, in the sense of the following lemmas.

Lemma 5.9. ([11], *Lem.* 2.6) *There exist constants $\alpha > 0$, and $c' > 0$, depending only on $c > 0$ in the definition of the box $A_{n,c,\phi}$, such that for all $z_1, z_2 \in A_{n,c,\phi}$ and conformal map $f : \mathbb{H} \to \mathbb{C}$, we have*

$$\begin{aligned} |f'(z_1)| &\leq c'\phi(n)^\alpha \cdot |f'(i \operatorname{Im} z_1)|, \\ d_{\mathbb{H},hyp}(z_1, z_2) &\leq c' \log \phi(n) + c', \end{aligned} \tag{5.7}$$

where $d_{\mathbb{H},hyp}(z_1, z_2)$ denotes the hyperbolic distance between z_1 and z_2 in $\mathbb{H}$.

Lemma 5.10. ([16], *Cor.* 1.5) *Suppose $f : \mathbb{H} \to \mathbb{C}$ is a conformal map, then for all $z_1, z_2 \in \mathbb{H}$, we have*

$$|f(z_1) - f(z_2)| \leq 2|(\operatorname{Im} z_1) f'(z_1)| \cdot \exp\big(4 d_{\mathbb{H},hyp}(z_1, z_2)\big). \tag{5.8}$$

Therefore, it is natural that we want to show $\{z, w\} \in A_{n,c,\phi}$ with proper parameters. Indeed, this is the case in the square-root interpolation ([11], *Lem.* 3.3). The only non-trivial remark is the following.

Remark 5.11. *In the linear interpolation, from ([10], Sec. 3.) we know that for a typical linear force $t \mapsto t$ on the time interval $[0, \infty)$, its line of singularities (i.e. Loewner curve) admits the form*

$$t \mapsto 2 - 2\rho_t \cot \rho_t + 2i\rho_t, \tag{5.9}$$

where ρ_t increases monotonously from $\rho_0 = 0$ to $\rho_\infty = \pi$. Indeed, the Loewner curve of a general linear force $t \mapsto at + b$ requires some change of constant *parameters depending on $a, b \in \mathbb{R}$, possibly with change in signs. Hence, we know the arc $\gamma_k^n : [0, \frac{1}{n}] \to \mathbb{H} \cup \{0\}$ corresponding to the piecewise-linear force $\lambda^n(t_k+t) - \lambda^n(t_k)$ with $t \in [0, \frac{1}{n}]$ has an image which vertically stretches monotonically upward and horizontally either leftward or rightward. Hence, the images $\gamma_k^n([0, \frac{1}{n}])$ attains its maximal height at its tip $\gamma_k^n(\frac{1}{n})$, which justifies our choice of $z = \gamma_k(s)$.*

In this regard we could use *Lem.* 5.9 and *Lem.* 5.10 to give an upper bound to the second term in *Ineq.* (5.5).

Now, let us turn our attention to the third term in *Ineq.* (5.5). This is actually a perturbation term: we need to measure the difference of one point in $\mathbb{H}$ under two conformal maps. This is the place where we need ([15], *Lem.* 2.2) again. Actually, we need the following lemma to estimate the third term in *Ineq.* (5.5).

Lemma 5.12. ([15], *Lem.* 2.2) *Let $0 < T < \infty$. Suppose $f_t^{(1)}$ and $f_t^{(2)}$ are the inverse map to the forward Loewner chain satisfying Eqn.* (2.1) *with driving force $W_t^{(1)}$ and $W_t^{(2)}$, respectively. Define $\epsilon := \sup_{s\in[0,T]} \left|W_s^{(1)} - W_s^{(2)}\right|$. Then if $u = x+iy \in \mathbb{H}$, we have*

$$\left|f_T^{(1)} - f_T^{(2)}\right| \leq \epsilon \exp\left\{\frac{1}{2}\left[\log\frac{I_{T,y}\left|\partial_z f_T^{(1)}(u)\right|}{y}\log\frac{I_{T,y}\left|\partial_z f_T^{(2)}(u)\right|}{y}\right]^{\frac{1}{2}} + \log\log\frac{I_{T,y}}{y}\right\}, \tag{5.10}$$

where $I_{T,y} := \sqrt{4T+y^2}$. Under furthermore conditions, the above term will be dominated by a subpower function, which is written explicitly in the reference.

Applying *Lem.* 5.12 to estimate the third term in *Ineq.* (5.5), combining the estimate to the first and the second terms, on the event $F_{n,1} \cap F_{n,2}$, we see that

$$\sup_{t\in[0,1]} |\gamma(t) - \gamma^n(t)| \leq \frac{c_6(\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}}. \tag{5.11}$$

By definition of the event $F_n \in \Omega$, we know $F_{n,1} \cap F_{n,2} \subset F_n$. And then by measuring the size of $F_{n,1}$ and $F_{n,2}$ in *Ineq.* (5.3), we could verify *Thm.* 5.8.

References

[1] Antti Kemppainen. Shramm-Loewner Evolution. Springer Briefs in Mathematical Physics, 2017.

[2] Steffen Rohde and Oded Schramm. Basic Properties of SLE. Annals of Mathematics, 2005.

[3] James Foster, Terry Lyons and Vlad Margarint. An Asymptotic Radius of Convergence for The Loewner Equation and Simulation of SLE_κ Traces via Splitting. 2020.

[4] Gregory Lawler and Vlada Limic. Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics, 2012.

[5] Fredrik Johansson, Gregory Lawler. Optimal Hölder Exponent for The SLE Path. Duke Math, 2011.

[6] J.G. Gaines and Terry Lyons. Variable Step Size Control in The Numerical Solution of Stochastic Differential Dquations. SIAM Journal on Applied Mathematics, 1997.

[7] Vlad Bally and Clément Rey. Approximation of Markov Semigroups in Total Variation Distance. Electronic Journal of Probability, 2015.

[8] Gerald B. Folland. Real Analysis: Modern Techniques and Their Applications. Pure and Applied Mathematics, 1984.

[9] Ben Boukai. An Explicit Expression for The Distribution of The Supremum of Brownian Motion With a Change Point. Communication in Statistics, Theory and Methods, 1990

[10] Wouter Kager, Bernard Nienhuis, and Leo Kadanoff. Exact Solutions for Loewner Evolutions. Journal of Statistical Physics, 2004.

[11] Huy Tran. Convergence of an Algorithm Simulating Loewner Curves. AAnnales Academiæ Scientiarum Fennicæ, 2015.

[12] Dmitry Beliaev, Terry Lyons,and Vlad Margarint. A New Approach to SLE Phase Transition. 2020.

[13] Steffen Rohde and Dapeng Zhan. Backward SLE and The Symmetry of The Welding. 2018.

[14] Atul Shekhar, Huy Tran and Yilin Wang. Remarks on Loewner Chains Driven by Finite Variation Functions. Annales Academiæ Scientiarum Fennicæ, 2019.

[15] Fredrik Johansson Viklund, Steffen Rohde, and Ccarto Wong. On The Continuity of SLE_κ in κ. 2012.

[16] Christian Pommerenke. Boundary Behaviour of Conformal Maps. Grundlehren der mathematischen Wissenschaften, 1992.