Quasi-Sure Stochastic Analysis and SLE_κ Theory - the quasi-sure continuity in κ of the SLE_κ traces for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$

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Abstract

We study SLE theory with elements of Quasi-Sure Stochastic Analysis through Aggregation. Specifically, we show how the latter can be used to construct the SLE_κ traces for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ quasi-surely using the ideas developed in [22]. As a by-product of this analysis we prove the quasi-sure continuity in κ for all $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ of the SLE_κ traces.

1 Introduction

The Loewner equation (also known as the Loewner evolution or Loewner chain) was introduced by Charles Loewner in 1923 in [16] and it played an important role in the proof of the Bieberbach Conjecture [4] by Louis de Branges in 1985 in [6]. In 2000, Oded Schramm introduced in [18] a stochastic version of the Loewner equation, the Stochastic Loewner Evolution (SLE_κ). The SLE_κ describes the evolution of a curve in terms of a driving function that is chosen to be $\sqrt{\kappa}B_t$, with $\kappa \geqslant 0$ a real parameter and $B_t,\ t \in [0, \infty)$, a real-valued standard Brownian motion. This is a one-parameter family of random planar fractal curves that are the only possible conformally invariant scaling limits of interfaces of a number of discrete models that appear in planar Statistical Physics. In several cases, it was proved that indeed the interfaces converge to the SLE_κ curves. We refer to [12] for a detailed study of the object and many of its properties.

The problem of continuity of the traces generated by Lowener chains was studied in the context of chains driven by bounded variation drivers in [21], where the continuity of the traces generated by the Loewner chains was established. Also, the question appeared in [15], where the Loewner chains were driven by Hölder-1/2 functions with norm bounded by σ with $\sigma < 4$. In this context, the continuity of the corresponding traces was established with respect to the uniform topologies on the space of drivers and with respect to the same topology on the space of simple curves in $\mathbb{H}$. Another paper that addressed a similar problem is [20], in which the condition $||U||_{1/2} < 4$ is avoided at the cost of assuming some conditions on the limiting trace. Some stronger continuity results are obtained in [9] under the assumption that the driver has finite energy, in the sense that $\dot{U}$ is square integrable. The question appeares naturally when considering the solution of the corresponding welding problem in [2]. In this paper it is proved that the trace obtained when solving the corresponding welding problem is continuous in a parameter that appears naturally in the setting. In the context of SLE_κ traces the problem was studied in [25], where the continuity in κ of the SLE_κ traces was proved for any $\kappa < 2.1$. An update is proved in [10], where the a.s. continuity in κ of the SLE traces is proved for $\kappa < 8/3$.

Our method relies on the Quasi-Sure Stochastic Analysis through Aggregation as constructed in [22]. The construction in [22] is suitable when one works with mutually singular probability measures. In the case the measures are absolutely continuous, the situation becomes simpler since one can work under the nullsets of the dominating measure directly. In [22], the authors work with the canonical process that under $\mathbb{P}_0$ is a Brownian Motion. When changing the measures $\mathbb{P}_a$ for a real number in an interval, the canonical process is a local martingale with quadratic variation a. Using Lévy's characterization in [22] it is constructed $W_t^{\mathbb{P}_a} = \int a_s^{-1/2} dB_s$ that is a $\mathbb{P}_a$- BM (the quadratic variation under $\mathbb{P}_a$ becomes t). We use this construction of a Universal Brownian motion (see [22] to obtain a family of drivers for the Loewner differential equation when changing the measures $\mathbb{P}_a$ on the pathspace. In our case, the role of the parameter a will be played by the natural parameter κ in the SLE_κ theory.

Using this, one can construct the SLE traces simultaneously quasi-surely (i.e. simultaneously for a family of measures $\mathbb{P}_\kappa$) for all $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ using a notion of aggregated solution to a Stochastic Differential Equation that appears in the analysis, and expressing the derivative of the conformal maps in terms of this aggregated solution. Furthermore, the quasi-sure continuity in κ is obtained in this setting using an estimate between conformal maps solving the Loewner Differential Equation whose drivers are close to each other obtained in [25].

The paper is divided in several sections. In the first part of the paper, we construct quasi-surely the SLE_κ traces and in the second part we prove the q.s. continuity in κ for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ of these objects. We emphasize that this method allows us to construct SLE_κ traces simultaneously for all $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.

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2 Preliminaries

We start by introducing the objects needed in our analysis. For the SLE theory the exposure is based on [12] and [3] and for the Quasi-Sure Stochastic Analysis through Aggregations the exposure is based on [22] which we refer to, for more details.

2.1. Introduction to SLE_κ theory. An important object in the study of the Loewner differential equation is the $\mathbb{H}$-compact hull that is a bounded set in $\mathbb{H}$ such that its complement in $\mathbb{H}$ is simply connected. To every compact $\mathbb{H}$-hull, that we typically denote with K we associate a canonical conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ that is called the *mapping out function of K*.

Using the Riemann Mapping Theorem, we get uniqueness by imposing the *hydrodynamic normalization at* infinity for g_t (i.e. we require that $g_t(z)$ has no constant term and its complex derivative is 1). The mapping near infinity is of the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \to \infty .$$

The coefficient a_K that appears in the expansion at infinity of the mapping has the traditional name *halfplane capacity*. Throughout the paper we use the notation $\mathrm{hcap}(K)$ for the halfplane capacity a_K.

We work with a family of growing compact hulls K_t with $H_t := \mathbb{H} \setminus K_t$. Firstly, we define the radius of a hull to be

$$rad(K) = \inf\{r \geqslant 0 : K \subset r\mathbb{D} + \text{x for some x} \in \mathbb{R}\}.$$

Definition 2.1. *Let $(K_t)_{t\geqslant 0}$ be a family of increasing $\mathbb{H}$-hulls, i.e. K_s is contained in K_t whenever $s < t$. For $K_{t+} = \cap_{s>t} K_s$ and for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t\geqslant 0}$ has the local growth property if*

$$rad(K_{t,t+h}) \to 0 \text{ as } h \to 0, \text{ uniformly on compacts in } t.$$

The first connection between the family of growing compact $\mathbb{H}$-hulls and the real-valued path $(U_t)_{t\geqslant 0}$ is done in the following proposition.

Proposition 2.2 (Proposition 7.1 of [3])**.** *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then, $K_{t+} = K_t$ for all t. Moreover, the mapping $t \mapsto hcap(K_t)$ is continuous and strictly increasing on $[0,\infty)$. Moreover, for all $t \geqslant 0$, there is a unique $U_t \in \mathbb{R}$ such that $U_t \in \bar{K}_{t,t+h}$, for all $h > 0$, and the process $(U_t)_{t\geqslant 0}$ is continuous.*

When considering SLE_κ, we have that $U_t = \sqrt{\kappa}B_t$. The following map $t \mapsto hcap(K_t)/2$ is a non-decreasing homeomorphism on $[0,T)$ and by choosing τ to be the inverse of this homeomorphism, we obtain a new family of hulls K_t' in a new parametrization such that $\text{hcap}(K_t') = 2t$. This is the canonical parametrization that we use throughout the paper. We use the standard terminology for this, i.e parametrization by halfplane capacity.

When studying the SLE_κ, in the upper half-plane, we study the corresponding families of conformal maps in the formats

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z - \sqrt{\kappa}B_t}, \quad f(0,z) = z, z \in \mathbb{H}. \tag{2.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa}B_t}, \qquad g(0,z) = z, z \in \mathbb{H}. \tag{2.2}$$

(iii) Time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa} B_t}\,, \qquad h(0,z) = z, z \in \mathbb{H}\,. \tag{2.3}$$

Next, we introduce the SLE_κ trace.

Definition 2.3. *Let g_t be the conformal maps solving the forward Loewner differential equation with $U_t = \sqrt{\kappa} B_t$. The SLE_κ trace is defined via*

$$\gamma(t) := \lim_{y \to 0} \hat{g}_t^{-1}(iy),$$

where $\hat{g}_t^{-1}(iy) = g_t^{-1}(iy + \sqrt{\kappa} B_t)$.

In general we have the following definition for hulls generated by a trace.

Definition 2.4. *We say that a continuous path $(\gamma_t)_{t \geqslant 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$-hulls K_t if $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \geqslant 0$.*

One of the important results in the SLE_κ literature is the following.

Theorem 2.5 (Theorem 4.1 of [19])**.** *Let $(K_t)_{t \geqslant 0}$ be a SLE_κ for $\kappa \neq 8$. Then, $\hat{g}_t^{-1}(z) = g_t^{-1}(z + \sqrt{\kappa} B_t) : \mathbb{H} \mapsto H_t$ extends continuously to $\bar{\mathbb{H}}$ for all $t \geqslant 0$, almost surely. Moreover, γ_t is continuous and generates $(K_t)_{t \geqslant 0}$ almost surely.*

Remark 2.6. The same result holds for $\kappa = 8$ as it was showed in [14] using a different approach.

2.2. Aggregation and Quasi-sure Stochastic Analysis. We introduce the Quasi-Sure Stochastic Analysis through Aggregation following [22]. We refer the reader to [22] and [7] for further information.

Let us consider $\Omega = C(\mathbb{R}_+, \mathbb{R})$ and let $\mathbb{F} = \mathbb{F}^B$ be the filtration generated by the canonical process B.

We recall from [22] that a probability measure $\mathbb{P}$ is a local martingale measure if the process B is a local martingale under $\mathbb{P}$. It is proved that there exists an $\mathfrak{F}$-progresively measurable process denoted as $\int_0^t B_s dB_s$ which coincides with the Itôintegral $\mathbb{P}$ -a.s. for all local martingale measures $\mathbb{P}$. In particular this provides a pathwise definition of

$$\langle B \rangle_t := B_t^2 - 2 \int_0^t B_s dB_s$$

and

$$\hat{a}_t := \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}[\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}].$$

We first introduce as in [22] the following notions.

Definition 2.7. *$\bar{\mathcal{P}}_W$ is the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s. $\langle B\rangle_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{R}_+$.*

Definition 2.8 (Definition of capacity)**.** *For each $f \in C_b(\Sigma)$- the set of bounded continuous functions on Σ, we put*

$$cap(f) = \sup\{||f||_{L^2(\Sigma,P)} : \mathbb{P} \in \mathcal{P}\}.$$

By definition for a measurable set A, we have $cap(A) = cap(I_A)$. Capacities naturally have applications in the theory of Risk Measures, see [1], [8].

Definition 2.9.

- *We say that a property holds $\mathcal{P}$-quasi-surely if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}$. We call a set is polar if $c(A) = 0$, i.e. if $\mathbb{P}(A) = 0$ for all $\mathbb{P} \in \mathcal{P}$.*
- *Denote $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P}\in\mathcal{P}}\mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$*
- *A probability measure $\mathbb{P}$ is called absolutely continuous with respect to $\mathcal{P}$ is $\mathbb{P}(E) = 0$ for all $E \in \mathcal{N}_{\mathcal{P}}$.*

Throughout the paper we say that a property holds quasi-surely if it holds outside a polar set.

Furthermore, we use the following universal filtration $\mathfrak{F}^{\mathcal{P}}$ for the mutually singular measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$.

$$\mathfrak{F}^{\mathcal{P}} := \{\mathcal{F}^{\mathcal{P}}_t\}_{t\geqslant 0}$$

where

$$\mathcal{F}^{\mathcal{P}}_t := \cap_{\mathbb{P}\in\mathcal{P}}\left(\mathcal{F}^{\mathbb{P}}_t \vee \mathcal{N}_{\mathcal{P}}\right).$$

The next definition introduces the notion of aggregator that we use in our analysis.

Definition 2.10. *Let $\mathcal{P} \subset \bar{\mathcal{P}}_W$. Let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of $\mathfrak{F}^{\mathcal{P}}$ progressively measurable processes. An $\mathfrak{F}^{\mathcal{P}}$ progressively measurable process X is called a $\mathcal{P}$-aggregator of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, if $X = X^{\mathbb{P}}$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$.*

2.3. The universal Brownian motion. In this section, we introduce the notion of Universal Brownian motion as in [22].

We first introduce the required notions to define this notion. We refer the reader to [22] for more details. Let

$$\bar{\mathcal{A}} := \{a : \mathbb{R}_+ \to \mathbb{R}_+ | \mathbb{F} - \text{progresively measurable and} \int_0^t |a_s| ds < +\infty, \forall t \geqslant 0\}\,.$$

For a given $\mathbb{P} \in \bar{\mathcal{P}}_W$, let

$$\bar{\mathcal{A}}_W(\mathbb{P}) := \{a \in \bar{\mathcal{A}} : a = \hat{a}, \mathbb{P} - a.s.\}$$

Recall that $\hat{a}$ is the density of the quadratic variation of $\langle B \rangle$ (where B is the canonical process under the Wiener measure on path space) and is defined point-wise. We define

$$\bar{\mathcal{A}}_W := \cup_{\mathbb{P} \in \bar{\mathcal{P}}_W} \bar{\mathcal{A}}_W(\mathbb{P})$$

In order to construct a process with a given quadratic variation $a \in \bar{\mathcal{A}}$ from the canonical process (by changing the canonical measure on the pathspace) as in [22], we consider the following stochastic differential equation

$$dX_t = a_t^{1/2}(X) dB_t, \qquad \mathbb{P}_0 - a.s. \tag{2.4}$$

Furthermore, if the equation (2.4) has weak uniqueness, we let $\mathbb{P}_a \in \bar{\mathcal{P}}_W$ be the unique solution of (2.4) with initial condition $\mathbb{P}_a(B_0 = 0) = 1$, and we define

$$\mathcal{A}_W := \{a \in \bar{\mathcal{A}}_W : (2.4) \text{ has weak uniqueness}\}$$

$$\mathcal{P}_W := \{\mathbb{P}_a, a \in \mathcal{A}_W\}$$

Let us fix a subset $\mathcal{A} \subset \mathcal{A}_W$. We further denote

$$\mathcal{P} = \{\mathbb{P}_a, a \in \mathcal{A}\}.$$

Let us define for any $a, b \in \mathcal{A}$, the disagreement time

$$\theta^{a,b} := \inf\{t \geqslant 0 : \int_0^t a_s ds \neq \int_0^t b_s ds\}.$$

Definition 2.11. *A subset $\mathcal{A}_0 \subset \mathcal{A}_W$ is called a generating class of diffusion coefficients if*

- ▸ *$\mathcal{A}_0$ satisfies the concatenation property $a\mathbf{1}_{[0,t)} + b\mathbf{1}_{[t,\infty)} \in \mathcal{A}_0$, for $a, b, \in \mathcal{A}_0, t \geqslant 0$.*

- ▸ *$\mathcal{A}_0$ has constant disagreement times: for all $a, b \in \mathcal{A}_0$, $\theta^{a,b}$ is constant.*

Definition 2.12. *We say $\mathcal{A}$ is a separable class of diffusion coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0 \subset \mathcal{A}_W$ is generated by a class of diffusion coefficients and $\mathcal{A}$ consists of all processes a of the form*

$$a = \sum_{0}^{\infty} \sum_{i=1}^{\infty} a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}$$

where $(a_i^n)_{i,n} \subset \mathcal{A}_0$, $(\tau_n)_n \subset \mathcal{T}$ is non-decreasing with $\tau_0 = 0$.

- ▸ *We have that $\inf\{n : \tau_n = \infty\} < \infty$ and $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$ and each τ_n takes at most countably many values.*

- ▸ *For each n $\{E_i^n, i \geqslant 1\} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.*

A separable class $\mathcal{A}$ of diffusion coefficients generated by $\mathcal{A}_0$ is said to satisfy the consistency conditions.

Let us consider a standard Brownian motion B_t (the canonical process under the Wiener measures $\mathbb{P}_0$ as in [22]). For any $\mathbb{P} \in \mathcal{P}_W$ and $a \in \bar{\mathcal{A}}_W(\mathbb{P})$ by Lévy's characterization, we obtain that the following Itô stochastic integral under $\mathbb{P}$ is a $\mathbb{P}$- Brownian motion

$$W_t^{\mathbb{P}} := \int_0^t a_s^{-1/2} dB_s$$

For $\mathcal{A}$ satisfying the consistency condition, the family $\{W^{\mathbb{P}_a}, a \in \mathcal{A}\}$ admits a unique $\mathcal{P}$-aggregator W. Since $W^{\mathbb{P}_a}$ is a $\mathbb{P}_a$ Brownian motion for every $a \in \mathcal{A}$, we call $W_t^{\mathbb{P}_a}$- a universal Brownian motion.

Our natural process is $\sqrt{\kappa} B_t$ with B_t a standard Brownian Motion. Thus, we will work with $a_s^{1/2} dW_t^{\mathbb{P}_a}$, i.e. $a_s = \kappa$, i.e. we are moving a_s to the other side. Since in [22], the process $W_t^{\mathbb{P}_a}$ is defined for all a, and is a $\mathbb{P}_a$ standard BM, we will just modify its quadratic variation by constants.

From now on, for the convenience of notation (since our quantities will depend on κ, for $\kappa \in \mathbb{R}_+, \kappa \neq 8$, we use for the family of probability measures $\mathbb{P}_a$ the indexing $\mathbb{P}_\kappa$ for $\kappa \in \mathbb{R}_+$).

Moreover, we use this process to drive an SDE with $a_s^{1/2}dW_t^{\mathbb{P}_a}$ and to aggregate solutions for all a.

A fundamental result that we use is the aggregate solution to stochastic differential equations. In the paper, they show how to solve a stochastic differential equation simultaneously under all the measures $\mathbb{P} \in \mathcal{P}$. Specifically, they prove the following result:

Proposition 2.13 (Proposition 6.10 of [22]). *Let $\mathcal{T}$ be the set of all $\mathbb{F}$-stopping times taking values in $\mathbb{R}_+ \cup \{\infty\}$. Let A satisfy the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation*

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s,$$

has a unique $\mathbb{F}^{\mathbb{P}}$ progressively measurable strong solution. on the interval $[0,\tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution (see Def. 2.10) to the equation above, i.e.

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s, \quad t \geqslant 0$$

has solution simultaneuously under all prrobability measures $\mathbb{P}_\kappa$.

In the next section, we use this result for the stochastic differential equation corresponding to $\tilde{N}_s$ that is used in the construction of the SLE trace in order to obtain the estimate the derivative of the conformal map at fixed times.

3 Heuristics of the quasi-sure construction of the SLE_κ traces

The main idea is to consider the construction of the aggregated solution to SDE as in [22] and applied to the SDE corresponding to the process K_s. Furthermore, we express the derivative of the map $\tilde{h}_t(z_0)$ using the aggregated solution of an SDE and use the lemmas in [14] to obtain the quasi-sure existence of the SLE_κ trace.

We have

$$|\tilde{h}_t'(z_0)| = e^{-\frac{2}{\kappa}t} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right) = e^{\frac{2}{\kappa}t} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right),$$

where

$$d\tilde{N}_s = (1-\tilde{N}_s)[-4(\frac{2}{\kappa}+1)\tilde{N}_s + 1]ds + 2\sqrt{\tilde{N}_s}(1-\tilde{N}_s)d\tilde{B}_s.$$

The SDE for $\tilde{K}_t$ is

$$d\tilde{K}_t = \frac{4}{\kappa}\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}\, d\tilde{B}_t.$$

Checking the conditions of Yamada-Watanabe Theorem for the SDE $\tilde{K}_t$ (see [17]), we obtain that this stochastic differential equation has a unique strong solution.

Once we have a unique notion of strong solution for the SDE for K_t, $\mathbb{P}$-a.s., we can construct an aggregated solution for this SDE using Proposition 2.13 and then express

$$|\tilde{h}_t'(z_0)| = e^{-2t/\kappa} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1} ds\right).$$

The main idea is that we use that aggregated solution for the SDE $\tilde{K}_t$ to construct simultaneously the SLE_κ trace for all parameters $\kappa \neq 8$ by relating the aggregated solution for $\tilde{K}_t$ with the derivative of the backward map.

With this definition, we will recover the typical construction of the estimate of the Theorem 2.5 simultaneously under all measures $\mathbb{P}_\kappa$, for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ in the next section. The quasi-sure construction assures that the typical estimate on the derivative of the map, holds simultaneously under all the choices of the parameter κ using the aggregated solution of an SDE simultaneously under all the measures $\mathbb{P}_\kappa$ for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.

4 The existence of the SLE_κ trace for fixed $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ a.s.

In [19] it is proved one fundamental result about the existence of the trace for the SLE_κ process for all values of $\kappa \neq 8$.

The elements used in the proof of the existence of the SLE_κ trace for fixed κ, a.s., are listed in the following. For more details, we refer the reader to [19] and [12].

4.1. Estimates for the mean of the derivative for a fixed κ. Considering the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - \sqrt{\kappa}dB_t\,, \qquad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{4.1}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$.

The first elements of the proof of the existence of the SLE_κ trace are the following proposition and the corollary of it.

Proposition 4.1 (Proposition 7.2 in [14]). *Let r, b such that*

$$r^2 - (\frac{4}{\kappa} + 1)r + \frac{2}{\kappa}b = 0\,,$$

then

$$M_t \; := \; \tilde{Y}_t^{b-(r\kappa/2)}(|\tilde{Z}_t|/\tilde{Y}_t)^{2r}|h'_t(z_0)|^b\,,$$

is a martingale. Moreover,

$$\mathbb{P}(|\tilde{h}'_t(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}e^{t(r-2b/\kappa)}\,.$$

Corollary 4.2 (Corollary 7.3 in [14]). *For every $0 \leqslant r \leqslant \frac{4}{\kappa} + 1$, there is a finite $c = c(\frac{2}{\kappa}, r)$ such that for all $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, we have that*

$$\mathbb{P}(|h'_t(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0, \lambda)\,,$$

where $b = [(\frac{4}{\kappa} + 1)r - r^2]/(2/\kappa) \geqslant 0$ and

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{(r/(2/\kappa))-b}, & \text{if } r < \frac{2b}{\kappa}\,, \\ -\log(\lambda y_0), & \text{if } r = \frac{2b}{\kappa}\,, \\ y_0^{b-(2r/\kappa)}, & \text{if } r > \frac{2b}{\kappa}\,. \end{cases}$$

4.2. Existence of the trace for fixed κ.

Proposition 4.3 (Proposition 4.33 in [14]). *Suppose that g_t is a Loewner chain with driving function U_t and assume that there exist a sequence of positive numbers $r_j \to 0$ and a constant c such that*

$$|\hat{f}'_{k2^{-2j}}(2^{-j}i)| \leqslant 2^j r_j\,, k = 0, 1, \ldots, 2^{2j} - 1\,,$$
$$|U_{t+s} - U_t| \leqslant c\sqrt{j}2^{-j}\,, 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 2^{-2j}\,.$$

and

$$\lim_{j\to\infty} \sqrt{j}/\log r_j = 0\,.$$

Then $V(y, t) := \hat{f}_t(iy)$ is continuous on $[0, 1] \times [0, 1]$.

Combining the previous results, one obtains Theorem 2.5 that we repeat for convenience.

Theorem 4.4 (Theorem 4.1 of [19])**.** *Let* $(K_t)_{t\geqslant 0}$ *be a* SLE_κ *for* $\kappa \neq 8$. *Then,* $\hat{g}_t^{-1}(z) = g_t^{-1}(z+\sqrt{\kappa}B_t) : \mathbb{H} \mapsto H_t$ *extends continuously to* $\bar{\mathbb{H}}$ *for all* $t \geqslant 0$, *almost surely. Moreover,* γ_t *is continuous and generates* $(K_t)_{t\geqslant 0}$ *almost surely.*

5 Quasi-sure existence of the trace -defining the trace simultaneously for all $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$

In this section, we construct the SLE_κ quasi-surely, i.e. we construct the SLE_κ traces for a sequence of measures $\mathbb{P}_\kappa$, for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.

5.1. Estimates on the moments of the derivatives for many κ using aggregation of solutions of a SDE. We consider the universal Brownian motion $W_t^{\mathbb{P}_\kappa}$ as a driver for the backward Loewner differential equation. We consider also the random time changed universal Brownian motion $\sqrt{\kappa}\tilde{W}_t^{\mathbb{P}_\kappa} := \frac{\sqrt{\kappa}dW_t^{\mathbb{P}_\kappa}}{\sqrt{X_t^2+Y_t^2}}$. For every $\mathbb{P}_\kappa$ we have the Lévy's characterization of Brownian motion and thus we deduce that the time changed Brownian motion is also a Brownian motion for all $\mathbb{P}_\kappa$.

Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - \sqrt{\kappa}dW_t^{\mathbb{P}_\kappa}\,, \qquad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{5.1}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$. We prove the same estimates as above using the aggregated solution $\tilde{N}_s$ that gives that

$$M_t = y_0^{b-(2r/\kappa)} e^{-rt}(1-\tilde{N}_t)^{-r} \exp(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds)\,,$$

is a quasi-sure martingale, i.e. a martingale with respect to all the measures $\mathbb{P}_\kappa$.

Once we obtain the quasi-sure version of the estimates in the previous section, we can continue to go exactly in the same manner to prove the existence of the trace quasi-surely with the tools from above (from Yamada-Watanabe Theorem (see [17]), the following SDE has a unique strong solution).

For this, we use the aggregated solution of the SDE $\tilde{K}_t$:

$$d\tilde{K}_t = \frac{4}{\kappa}\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2} dW_t^{\mathbb{P}_\kappa}$$

constructed via the methods of Quasi-Sure Stochastic Analysis through Aggregation from the following Proposition.

Proposition 5.1 (Proposition 6.10 of [22]). *Let A be satisfying the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s, t \geqslant 0$ has a unique $\mathbb{F}^{\mathbb{P}}$ progresively measurable strong solution. on the interval $[0,\tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution to the equation above.*

In order to perform the analysis and to prove all the estimates (that were done in the previous section for fixed κ) simultaneously for all κ, we use the aggregated solution and relate it with the derivative of the map, via

$$|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right).$$

Using the aggregated solution, we obtain a version of Proposition 4.1 using the family of measures $\mathbb{P}_\kappa$. In this manner we can construct the trace via controlling the derivative estimate simultaneously for all $\kappa \in \mathbb{R}_+ \setminus \{0,8\}$ using the aggregated solution, and obtain the q.s. existence of SLE traces simultaneously for all $\kappa \in \mathbb{R}_+ \setminus \{0,8\}$.

First, we prove a version of Proposition 4.1 for the family of measures $\mathbb{P}_\kappa$.

Proposition 5.2. *Let r, b such that*

$$r^2 - (\frac{4}{\kappa}+1)r + \frac{2b}{\kappa} = 0\,,$$

then

$$M_t \;:=\; \tilde{Y}_t^{b-(\frac{2r}{\kappa})}(|\tilde{Z}_t|/\tilde{Y}_t)^{2r}|h_t'(z_0)|^b\,,$$

is a martingale. Moreover,

$$\mathbb{P}_\kappa(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}e^{t(r-\frac{2b}{\kappa})}\,.$$

Proof. By applying the chain rule for the function $L_t = \log h_t'(z_0)$, we obtain that $L_t = -\int_0^t \frac{2/\kappa}{Z_s^2}ds$, and in particular, $|\tilde{h}_t'(z_0)| = \exp\left(\frac{2}{\kappa}\int_0^t \frac{\tilde{Y}_s^2-\tilde{X}_s^2}{\tilde{X}_s{}^2+\tilde{Y}_s{}^2}ds\right)$. Moreover, if we consider

$\tilde{K}_t = \frac{\tilde{X}_t^2}{\tilde{Y}_t^2}$ and $\tilde{N}_t = \frac{\tilde{K}_t}{1+\tilde{K}_t}$, we obtain that

$$|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right) .$$

In the $\sigma(t)$ time parametrization, we have that $d\tilde{Y}_t = -\frac{2}{\kappa}\tilde{Y}_t dt$, so in this time parametrization $\tilde{Y}_t$ grows deterministically $\tilde{Y}_t = \tilde{Y}_0 e^{\frac{2}{\kappa}t}$. At this moment, we can rephrase the formula for M_t as

$$M_t = y_0^{b-(\frac{2r}{\kappa})} e^{-rt}(1-\tilde{N}_t)^{-r} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right) .$$

and by applying Itô formula, we obtain that

$$dM_t = 2r\sqrt{\tilde{N}_t} M_t d\tilde{B}_t ,$$

where $\tilde{B}_t = \int_0^{\sigma(t)} \frac{1}{\sqrt{X_t^2+Y_t^2}} dB_t$ is the Brownian motion that we obtain in the time reparametrization. This shows that M_t is a martingale, hence

$$\mathbb{E}_\kappa[M_t] = \mathbb{E}_\kappa[M_0] = y_0^{b-(2r/\kappa)}(|z_0|/y_0)^{2r} .$$

Note that since for $r \geqslant 0$, $(|\tilde{Z}_t|/Y_t)^{2r} \geqslant 1$, then by Markov inequality, we have that

$$\mathbb{P}_\kappa(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r} e^{t(r-\frac{2b}{\kappa})} .$$

□

Moreover, we obtain a version of the Corollary 4.2 under the family of measures $\mathbb{P}_\kappa$.

Corollary 5.3. *For every* $0 \leqslant r \leqslant \frac{4}{\kappa}+1$, *there is a finite* $c = c(\frac{2}{\kappa}, r)$ *such that for all* $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, *we have that*

$$\mathbb{P}_\kappa(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda) ,$$

where $b = [(\frac{4}{\kappa}+1)r - r^2]/(2/\kappa) \geqslant 0$ *and*

$$\delta(y_0,\lambda) = \begin{cases} \lambda^{(2r/\kappa)-b}, & \text{if } r < \frac{2b}{\kappa} , \\ -\log(\lambda y_0), & \text{if } r = \frac{2b}{\kappa} , \\ y_0^{b-(r/a)}, & \text{if } r > \frac{2b}{\kappa} . \end{cases}$$

Proof. From $dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt$, we obtain that $dY_t \leqslant \frac{2/\kappa}{Y_t}dt$, and hence $Y_t \leqslant \sqrt{\frac{4}{\kappa}t + y_0^2} \leqslant \sqrt{\frac{4}{\kappa}+1}$. In the last inequality, we used that $t \leqslant 1$ and $y_0 \leqslant 1$. Using the exponential growth of Y_t in this time reparametrization, we obtain that $\tilde{Y}_t = \sqrt{\frac{4}{\kappa}+1}$ at time $T = \frac{\log\sqrt{\frac{4}{\kappa}+1} - \log y_0}{2/\kappa}$.

Therefore,

$$\mathbb{P}_\kappa(|h_t'(z_0)| \geqslant \lambda) \leqslant \mathbb{P}_\kappa(\sup_{0\leqslant s\leqslant T} |\tilde{h}_s'(z_0)| \geqslant \lambda)\,.$$

Using that $|\tilde{h}_t'(z_0)| = e^{-\frac{2t}{\kappa}} \exp\left(\frac{4}{\kappa}\int_0^t \tilde{N}_s ds\right)$ we obtain that $|\tilde{h}_{t+s}'(z_0)| \leqslant e^{2s/\kappa}|\tilde{h}_t'(z_0)|$. So by addition of the probabilities, we have that

$$\mathbb{P}_\kappa(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^{2/\kappa}\lambda) \leqslant \sum_{j=0}^{[T]} \mathbb{P}_\kappa(|\tilde{h}_j'(z_0)| \geqslant \lambda)\,.$$

Using the Schwarz-Pick Theorem for the upper half-plane we obtain that $|\tilde{h}_t(z_0)| \leqslant \mathrm{Im}\tilde{h}_t'(z_0)/y_0 = e^{2t/\kappa}$. This gives a lower bound for the t that we are summing over and we obtain that via the Proposition 4.1 that

$$\begin{aligned}
\mathbb{P}_\kappa(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^{2/\kappa}\lambda) &\leqslant \sum_{(\kappa/2)\log\lambda\leqslant j\leqslant T} \mathbb{P}_\kappa(|\tilde{h}_j'(z_0)| \geqslant \lambda) \\
&\leqslant \lambda^{-b}(|z_0|/y_0)^{2r} \sum_{(\kappa/2)\log\lambda\leqslant j\leqslant T} e^{j(r-ab)} \\
&\leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda)\,.
\end{aligned}$$

□

In order to prove the result, we need the following Lemma.

Lemma 5.4 (Lemma 7.6 in [14])**.** *For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - \sqrt{\kappa}B_t$.*

We combine the previous results, to obtain the quasi-sure existence of the SLE_κ trace.

Theorem 5.5. *If $\kappa \in \mathbb{R}_+ \setminus \{0,8\}$ the chordal SLE_κ is q.s. generated by a path.*

Proof. By using the scaling of the SLE_κ , it suffices to prove the Theorem only for $t \in [0,1]$. According to the preliminary results it suffices to show that q.s. there exists an ε and a random constant c (that depends on the worst κ) such that

$$|f'_{k2^{-2j}}(i2^{-j})| \leqslant c2^{j-\varepsilon}\,, j=1,2,\dots,k=0,1,\dots,2^{2j}\,,$$
$$\sqrt{\kappa}|W_t^{\mathbb{P}_\kappa}-W_s^{\mathbb{P}_\kappa}| \leqslant c_1\sqrt{\kappa}|t-s|^{1/2}|\log\sqrt{|t-s|}| \quad 0\leqslant t\leqslant 1\,.$$

The second inequality is a consequence of the modulus of continuity for the Brownian motion. For the first inequality, we use Borel-Cantelli Lemma for capacities (see [5], [7]) along with Lemma 5.4 to find c (that depends on the worst κ, in the sense of the κ that gives the worst upper bound on the convergent sequence of probabilities) and ε such that for all $0\leqslant t\leqslant 1$

$$\sup_\kappa \mathbb{P}_\kappa(|h'_t(i2^{-j}| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-(2+\varepsilon)j}\,.$$

We consider $r=\frac{2}{\kappa}+\frac{1}{4}<\frac{4}{\kappa}+1$ and $b=\frac{(1+\frac{4}{\kappa})r-r^2}{2/\kappa}=\frac{2}{\kappa}+1+\frac{3}{32/\kappa}$, according to the Corollary 4.2. Thus, we are in the regime $r<\frac{2b}{\kappa}$, so by Corollary 4.2 we have that

$$\sup_\kappa \mathbb{P}_\kappa(|h'_t(i2^{-j})| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-j(2b-(2r/\kappa))(1-\varepsilon)}\,.$$

Thus,

$$cap(|h'_t(i2^{-j})| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-j(2b-(2r/\kappa))(1-\varepsilon)}\,.$$

We obtain that $2b-(2r/\kappa)=\frac{4}{\kappa}+1+\kappa/16>2$ provided that $2/\kappa\neq 1/4$. So, we can apply Borel-Cantelli argument for capacities provided that $2/\kappa\neq 1/4$, i.e. $\kappa\in\mathbb{R}_+\setminus\{0,8\}$, and finish the proof. [For more details on how to optimize over the parameters r and p for fixed κ (we do the same for all $\kappa\in\mathbb{R}_+\setminus\{0,8\}$)and obtain a continuos constant c in κ we refer the reader to the proof of the existence of the trace in [11].]

□

6 The quasi-sure continuity in κ for $\kappa\in\mathbb{R}_+\setminus\{0,8\}$ of the SLE_κ traces

Once the SLE_κ traces are constructed quasi-surely, we would like to prove the quasi-sure continuity in κ of the traces. For this, we are applying the construction of the algorithm in [23].

Using quasi-sure definition of the SLE_κ trace allows us directly to consider uncountably many parameters $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$,

Theorem 6.1 (Quasi-sure continuity in κ of the SLE_κ traces)**.** *The SLE_κ traces are quasi-surely continuous in κ, for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.*

Proof. We use the set-up from [25], in order to define the Whitney-type partition of the (t, y, κ) space. The main idea of this section is to show how we can avoid the typical Borel-Cantelli argument of [25] using the quasi-sure construction of the SLE_κ traces from the previous section.

We also need the following distortion result for conformal maps.

Lemma 6.2 (Distortion Lemma)**.** *There exists a constant $0 < c < \infty$ such that the following holds. Suppose that f_t satisfies the chordal Loewner PDE 2.1 and that $z = x + iy \in \mathbb{H}$, then for $0 \leqslant s \leqslant y^2$*

$$c^{-1} \leqslant \frac{|f'_{t+s}(z)|}{|f'_t(z)|} \leqslant c$$

and

$$|f_{t+s}(z) - f_t(z)| \leqslant cy|f'_t(z)|\,.$$

We consider the partition of the (t, y, κ) three dimensional space in boxes obtain by partitioning each coordinate. We follow the typical proof in [25] and we estimate the derivative of the map $(f_t^\kappa)'(iy)$ in the corners of the boxes. Using Distortion Theorems for the conformal maps along with the following Lemma that appeares in [25].

Lemma 6.3 (Lemma 2.3 of [25])**.** *Let $0 < T < \infty$. Suppose that for $t \in [0, T]$, $f_t^{(1)}$ and $f_t^{(2)}$ satisfy the backward Loewner differential equation with drivers $W_t^{(1)}$ and $W_t^{(2)}$. Suppose that $\varepsilon = sup_{s\in[0,T]}|W_s^{(1)} - W_s^{(2)}|$. For $u = x + iy \in \mathbb{H}$, we have that*

$$|f_T^{(1)}(u) - f_T^{(2)}(u)| \leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log\frac{I_{Ty}|(f_T^{(1)})'(u)|}{y}\log\frac{I_{Ty}|(f_T^{(2)})'(u)|}{y} + \log\log\frac{I_{T,y}}{y}\right]\right],$$

where $I_{T,y} = \sqrt{4T + y^2}$.

Using the Quasi-sure formulation, we have the estimate on the derivative of the maps simultaneously on the measures $\mathbb{P}_\kappa$, i.e. the estimates hold quasi surely for $\mathbb{P}_\kappa, \kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.

Let us consider

$$S_{n,j,k} = \left[\frac{j-1}{2^{2n}}, \frac{j}{2^{2n}}\right] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \left[\frac{k-1}{2^{qn}}, \frac{k}{2^{nq}}\right],$$

and let

$$p_{n,j,k} = \left(\frac{j}{2^{2n}}, \frac{1}{2^n}, \frac{k}{2^{qn}}\right) \in S_{j,n,k},$$

be the corners of the boxes. We estimate the derivative of the the Loewner maps in corners of the boxes, as in [25]. In [25], the typical estimate on the derivative of the map on the corners of the boxes is combined with the application of the Borel-Cantelli Lemma in order to assure the analysis on a unique nullset of the Brownian motion driving the Loewner differential equation. The use of Borel-Cantelli in this approach restricts the applicability of the derivative estimate in the corners of the boxes for the values $\kappa < 2.1$. The novelty is that we use the polar set outside of which the aggregated solution is defined and then we can vary $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$. In this way, we argue that the estimates on the derivative of the maps in the corners $p_{n,j,k}$ of the Whitney boxes hold q.s. In this way, we avoid the restriction to the interval $\kappa \in [0, 2.1]$, since the typical estimate on the derivative used in the proof of Theorem 2.5 works for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$.

Throughout the proof, we use the notation $F(t, y, \kappa) = f_t^{(\kappa)}(iy)$.

We give the following version of Lemma 3.3 in [25], that does not contain the restriction on the κ interval, due to the application of the Borel-Cantelli Lemma.

Lemma 6.4. *Let $\kappa \neq 8$, then q.s. there exists a random constant $c = c(\beta, q, \omega) < +\infty$ such that $|F'(p_{n,j,k})| \leqslant c2^{n\beta}$ for all pairs $(n, j, k) \in \mathbb{N}^3$ such that $p_{n,j,k} \in [0,1] \times [0,1] \times [0, \kappa]$.*

Proof. For every measure $\mathbb{P}_\kappa$ that gives that

$$\sum_{j=1}^{2^{2n}} \mathbb{P}_\kappa \left[|F'(p_{n,j,k})| \geqslant 2^{n\beta}\right] \leqslant c_2 2^{-n\sigma}$$

with $\sigma = \sigma(\kappa, \beta)$ and $c_2 = c_2(\kappa, \beta) < \infty$.

Following the analysis in [11], one can show that the constant $c_2(\omega, \kappa)$ can be chosen to be a continuous function of κ (see the analysis of the parameters using the optimization procedure at page 99 in [11]). There it is is shown how the constant $c_2(\kappa, \beta)$ depends on κ and one can take the supremum of this constant in the interval of κ that one considers.

For fixed κ, this expression is summable over n, giving the a.s. (outside of a nullset that depends on κ) estimate needed in the proof of the existence of the trace.

Using the quasi-sure stochastic analysis setting, we obtain

$$\sum_{j=1}^{2^{2n}} \sup_{\kappa} \mathbb{P}_\kappa \left[|F'(p_{n,j,k})| \geqslant 2^{n\beta} \right] \leqslant c2^{-n\sigma}$$

i.e. the same summability condition holds for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ with a constant c that depends on the worst κ, as in the κ that gives the worst upper bound for the series of probabilities. □

The next step is to use Distortion Theorem along with Lemma 6.3 in order to push the estimate on the derivative from the corners of the box to all the points inside.

Lemma 6.5. *Let $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$, then there exists $\delta > 0$ and $q > 0$ and quasi surely a random constant $c = c(q, \varepsilon, \omega) < \infty$ such that diam$(F(S_{n,j,k})) \leqslant c2^{-n\delta}$, for all $(n, j, k) \in \mathbb{N}^3$ with $p_{n,j,k} \in [0,1] \times [0,1] \times [0,\kappa]$.*

Proof. We will show that there exists $\delta > 0$ such that $|F(p) - F(p_{n,j,k})| \leqslant cn2^{-n\delta}$.

We estimate for $|\Delta t| \leqslant y^2$, using Lemmas 6.4 and 6.2

$$|F(t + \Delta t, y, \kappa) - F(t, y, \kappa)| \leqslant cy|F'(p_{n,j,k})| \leqslant c'2^{-n(1-\beta)}$$

with $c' = c'(\beta, q, \omega)$ q.s.

By Koebe Distortion Theorem and Lemmas 6.4 and 6.2, we obtain that

$$|F(t + \Delta t, y + \Delta y, \kappa) - F(t + \Delta t, y, \kappa)| \leqslant cy|F'(p_{n,j,k})| \leqslant c2^{-n(1-\beta)}.$$

Finally, using Lemma 6.3 and estimating

$$\sup_{t\in[0,1]} |\sqrt{\kappa + \Delta\kappa} B_t - \sqrt{\kappa}| \leqslant c'\Delta\kappa \sup_{t\in[0,1]} |B_t| \leqslant c'\Delta\kappa\,,$$

we obtain that

$$|F(t + \Delta t, y + \Delta y, \kappa + \Delta\kappa) - F(t + \Delta t, y, \kappa)| \leqslant c\Delta\kappa y^{-\phi(\beta)} \log(y^{-1}) \leqslant cn2^{-n(q-\phi(\beta)}\,.$$

Choosing $\delta = \min\{1 - \beta, q - \phi(\beta)\}$ we finish the proof.

□

As in [25], once we have the decay of the diameter of the boxes, we obtain the bound $|\gamma^{(\kappa_1)}(t_1 - \gamma^{\kappa}(t_2)| = |f_{t_1}^{(\kappa_1)}(0+) - f_{t_1}^{(\kappa_2)}(0+)| = O(|t_1 - t_2|^{\delta/2}) + O(|\kappa_1 - \kappa_2)^{\delta})$. Arguing outside the unique polar set for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$, we obtain the q.s. continuity of the SLE_κ traces.

Or equivalently, we have the following alternative proof.

It is known that if the driving function is of the form $\lambda(t) = c\sqrt{t} + d$ then the Loewner evolution can be solved explicitly and g_t is a relatively simple Christoffel-Schwarz type function whose explicit form is not that important. What is important, is that the corresponding trace is a straight interval. Its length is proportional to $\sqrt{t}$ with constant which explicitly depends on c and it makes an angle $\alpha\pi$ with the positive real axis where

$$\alpha = \frac{1}{2} - \frac{1}{2}\frac{c}{\sqrt{16 + c^2}}.$$

From now on we make some assumptions about the regularity of the driving function.

Assumption 6.6. *A function λ is weakly Hölder-$1/2$ continuous. This means if there exists a subpower function ϕ (that is the function growing at infinity slower that any positive power) such that for all $\delta > 0$*

$$osc(\lambda, \delta) := \sup\{|\lambda(t) - \lambda(s)| : s, t \in [0, 1], |t - s| \leqslant \delta\} \leqslant \sqrt{\delta}\phi\left(\frac{1}{\delta}\right). \tag{6.1}$$

Assumption 6.7. *There exist $c_0 > 0$, $y_0 > 0$ and $0 < \beta < 1$ such that for all*

$$|\hat{f}_t'(iy)| \leq c_0 y^{-\beta}, \quad \forall y \leq y_0,$$

where $\hat{f}_t(z) = g_t^{-1}(z + \lambda(t))$.

It is known [24] that if the Loewner evolution satisfies Assumptions 6.6 and 6.7, then there is a trace. Under the same assumptions Tran proved that LE trace generated by the driving function λ^n converges to the trace generated by λ.

Theorem 6.8 (Theorem 2.2. of [23])**.** *Let us assume that we have a Loewner evolution with the driving function $\lambda(t)$ which satisfies Assumptions 6.6 and 6.7. Let λ^n be a square root approximation. Let γ and γ^n be the corresponding traces. Then there exists a sub-power* *function $\tilde{\phi}(n)$ which depends on ϕ, c_0 and β (from Assumptions mentioned above), such that for all $n \geqslant \frac{1}{y_0^2}$ and $t \in [0,1]$ we have that*

$$|\gamma^n(t) - \gamma(t)| \leqslant \frac{\tilde{\phi}(n)}{n^{\frac{1}{2}\left(1-\sqrt{\frac{1+\beta}{2}}\right)}}.$$

This theorem shows that γ^n converges uniformly to γ, moreover, we have a control of the rate of convergence in terms of β.

Theorem 6.8 is one of the main ingredients in our proof. Beyond it, we will also need several technical statements that were proved before. We reproduce them here for readers' convenience.

First, following [23], we define

$$A_{n,c,\phi} = \left\{ x + iy \in \mathbb{H} : |x| \leqslant \frac{\phi(n)}{\sqrt{n}}, \frac{1}{\sqrt{n}\phi(n)} \leqslant y \leqslant \frac{c}{\sqrt{n}} \right\}. \tag{6.2}$$

To shorten some many formulas we will use the following notations. Recall that $t_k = k/n$. We define γ_k to be the image of γ under $g_{t_k} - \lambda$, namely,

$$\gamma_k(s) = g_{t_k}(\gamma(t_k + s)) - \lambda(t_k), \quad 0 \leq s \leq 1 - t_k.$$

In the same way we define

$$\gamma_k^n(s) = g_{t_k}^n(\gamma^n(t_k + s)) - \lambda^n(t_k), \quad 0 \leq s \leq 1 - t_k.$$

We would like to claim that $\gamma_k(1/n)$ is in some $A_{n,c,\psi}$, which is unfortunately might be false. Instead we have the following Lemma.

Lemma 6.9 (Lemma 3.2 of [23]). *There exists a subpower function ψ depending only on ϕ, c_0 and β (as in Assumptions 6.6 and 6.7) such that for $n \geqslant 1$ and $0 \leqslant k \leqslant n-1$, there exists $s \in [0, \frac{2}{n}]$ such that $\gamma_k(s) \in A_{n,2\sqrt{2},\psi}$.*

For γ_k^n we have a similar, but slightly simpler, estimate.

Lemma 6.10 (Lemma 3.3 of [23]). *There exists a subpower function $\tilde{\psi}$ depending only on ϕ, c_0 and β (as in Assumptions 6.6 and 6.7) such that $\gamma_k^n(r)$ is in the box $A_{n,2\sqrt{2},\tilde{\psi}}$ for $n \geqslant 1$, $0 \leqslant k \leqslant n-1$ and $r \in [\frac{1}{n}, \frac{2}{n}]$.*

We will need a result describing the uniform continuity of traces.

Lemma 6.11 (Proposition 3.8 of [24]). *Let us consider a Loewner evolution satisfying Assumptions 6.6 and 6.7. Then, there exists a subpower function ϕ_1 such that if $0 \leqslant t \leqslant t+s \leqslant 1$, we have that*

$$|\gamma(t+s) - \gamma(t)| \leqslant \phi_1\left(\frac{1}{y}\right)\frac{2}{1-\beta}y^{1-\beta} \tag{6.3}$$

for $0 \leqslant s \leqslant y^2 \leqslant y_0^2$.

Finally, we will need a result stating that Loewner evolutions with close drivers are close away from the real line.

Lemma 6.12 (Lemma 2.3 of [25]). *Let $0 < T < \infty$. Suppose that for $t \in [0,T]$, $h_t^{(1)}$ and $h_t^{(2)}$ satisfy the backward Loewner differential equation (**??**) with drivers $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$. Let*

$$\varepsilon = \sup_{s\in[0,T]} |\lambda_s^{(1)} - \lambda_s^{(2)}|.$$

Then for $z = x + iy \in \mathbb{H}$ we have

$$|h_T^{(1)}(u) - h_T^{(2)}(u)|$$
$$\leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log\frac{I_{T,y}|(h_T^{(1)})'(u)|}{y}\log\frac{I_{T,y}|(h_T^{(2)})'(u)|}{y} + \log\log\frac{I_{T,y}}{y}\right]\right],$$

where $I_{T,y} = \sqrt{4T + y^2}$.

7 Proof of the main result

In this section we would like to apply the results of the previous section in the case when the driving function is of the form $\sqrt{\kappa}B_t$. We are also interested in how things change when κ is changing.

We start by discussing Assumptions 6.6 and 6.7. It is easy to see that the first one is satisfied for sufficiently small δ since $osc(B_t,\delta)/\sqrt{2\delta\log(1/\delta)} \to 1$. For all δ we use the following result.

Proposition 7.1 (Theorem 3.2.4 in [13]). *Let B_t be the standard Brownian motion on $[0,1]$. There is an absolute constant $c < \infty$ such that for all $0 < \delta \leq 1$ and $r > c$*

$$\mathbb{P}\{osc(B_t,\delta) \geq r\sqrt{\delta\log(1/\delta)}\} \leq c\delta^{(r/c)^2}.$$

This means that if we take r large enough, then we have a uniform bound on osc with very high probability. Alternatively, for almost every B_t there is (random) r such that $osc \le r\sqrt{\delta \log(1/\delta)}$. Throughout our analysis the driver is $\sqrt{\kappa}B_t$, where $\kappa \in J$ a subset of the real axis, with $\kappa \neq 0$ and $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$. Thus, we can merge the constant κ in the modulus of continuity of the driver $\sqrt{\kappa}B_t$ and estimate it directly with the biggest value. We will do the probabilistic version of this estimate in the next section.

Assumption 6.7 was established for SLE in [19]. We follow the detailed analysis presented in [11]. Let $\theta_0(\kappa) = (\kappa/16 + 4/\kappa - 1)/(\kappa/16 + 4/\kappa + 1)$.

Proposition 7.2 (Proposition 5.10 in [11])**.** *For each $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$, there exists $\theta_0(\kappa) > 0$ such that the following holds: for any $\theta \in (0, \theta_0(\kappa))$ there exists a random variable $C(\kappa, \omega)$ such that $C(\kappa, \omega) < +\infty$ a.s. and*

$$|f_t'(i2^{-n})| \leqslant C 2^{n(1-\theta)} .$$

for all $t \in \{k2^{-2n} : k \in [0, 2^{2n}]\}$ and any $n \in \mathbb{N}$.

Thus, the Assumption 6.7 is satisfied for $\beta = 1 - \theta$. Also, following the statement of the above result, we deduce that the Assumption 6.7 is satisfied for all $\kappa \neq 8$.

Let us consider two parameters $\kappa_1, \kappa_2 \notin \{0, 8\}$ and two Loewner evolutions driven by $\sqrt{\kappa_1}B_t$ and $\sqrt{\kappa_1}B_t$. The corresponding maps and curves will be denoted by superscripts (1) and (2) correspondingly.

Throughout this section, the precise subpower function that we use is changing from line to line. Unless it might lead to a confusion, we do not track these changes in order to simplify notations.

Our goal is to estimate the supremum of $|\gamma^{(1)}(t) - \gamma^{(2)}(t)|$. By the triangle inequality

$$|\gamma^{(1)}(t) - \gamma^{(2)}(t)| \leqslant |\gamma^{(1)}(t) - \gamma^{n,(2)}(t)| + |\gamma^{n,(2)}(t) - \gamma^{(2)}(t)| \tag{7.1}$$

where $\gamma^{n,(j)}$ is the trace obtained form interpolating with square root terms the driver $\sqrt{\kappa_j}B_t$.

In order to control this, we first fix an arbitrary interval $I = [t_k, t_{k+2}]$, with $0 \leqslant k \leqslant n-2$. We will estimate $|\gamma^{(1)}(s + t_k) - \gamma^{n,(2)}(r + t_k)|$ for all $r \in \left[\frac{1}{n}, \frac{2}{n}\right]$, and for the specific point s obtained in the Lemma 6.9. Combining with the uniform continuity of γ from

Lemma 6.11, we will have an estimate for $|\gamma^{(1)}(r+t_k) - \gamma^{n,(2)}(r+t_k)|$ for all $r \in \left[\frac{1}{n}, \frac{2}{n}\right]$. Redoing the same analysis on each interval in the time discretization, we obtain the desired estimate.

Starting the analysis, the second term in the inequality is estimated by Theorem 6.8. For the first term, let $z = \gamma^{(1)}(s), w = \gamma^{n,(2)}(r)$, with s and r, as before.

$$|\gamma^{(1)}(s+t_k) - \gamma^{n,(2)}(r+t_k)| \leqslant |{f^{(1)}}_{t_k}(z) - {f^{(1)}}_{t_k}(w)| + |{f^{(1)}}_{t_k}(w) - f^{n,(2)}_{t_k}(w)| \qquad (7.2)$$

As in [23], we estimate **the first term** in (7.2) using

$$|{f^{(1)}}_{t_k}(z) - {f^{(1)}}_{t_k}(w)| \leqslant (2\mathrm{Im}z)|(f^{(1)}_{t_k})'(z)| \exp(4d_{\mathbb{H},hyp}(z,w)),$$

where $d_{\mathbb{H},hyp}(z,w) = \mathrm{Arccosh}\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}\,z\,\mathrm{Im}\,w}\right)$.

For this, we use the almost sure estimate

$$|(f^{(1)}_{t_k})'(z)| \leqslant c(\omega,\kappa_1) y^{-\beta(\kappa_1)} .$$

For the second term along with these estimates, we use Lemma 2.3 of [25]. For this we estimate the distance ε between the two driving terms: the square root interpolation of the $\sqrt{\kappa_2}B_t$ and $\sqrt{\kappa_1}B_t$ with

$$|\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_1}B_t| \leqslant |\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_2}B_t| + |\sqrt{\kappa_2}B_t - \sqrt{\kappa_1}B_t| .$$

Thus, we obtain combining the estimates with the ones in [23], that

$$|\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_1}B_t| \leqslant \frac{\phi(n)}{\sqrt{n}} + |\sqrt{\kappa_1} - \sqrt{\kappa_2}| \sup_{t\in[0,1]} |B_t| \leqslant \frac{\phi(n)}{\sqrt{n}} + c|\sqrt{\kappa_1} - \sqrt{\kappa_2}| .$$

The next result that we use is the following lemma that appeared in [25].

Thus, in our case this gives that

$$\begin{aligned} &|f^{(1)}_{t_k}(w) - f^{n,(2)}_{t_k}(w)| \\ &\qquad \leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log\frac{I_{t_k,y}|(f^{(1)}_{t_k})'(w)|}{y} \log\frac{I_{t_k,y}|(f^{n,(2)}_{t_k})'(w)|}{y} + \log\log\frac{I_{t_k,y}}{y}\right]\right] , \qquad (7.3)\end{aligned}$$

where $I_{t_k,y} = \sqrt{4t_k + y^2}$ with

$$\varepsilon = \sup_{t\in[0,t_k]} |\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_1}B_t| \leqslant \frac{2\phi(n)}{\sqrt{n}} + c|\sqrt{\kappa_1} - \sqrt{\kappa_2}|\,.$$

These estimates are used for points inside the boxes $A_{n,c,\phi}$. Thus, for $y = \mathrm{Im} u \in [\frac{1}{\sqrt{n}\phi(n)}, \frac{2\sqrt{2}}{\sqrt{n}}]$ we have that

$$\frac{I_{t_k,y}}{y} \leqslant 2\sqrt{2}\sqrt{n}\phi(n),$$

for $\phi(n)$ some sub-power function of n. We use the estimates

$$|(f^{(1)}_{t_k})'(u)| \leqslant cy^{-\beta(\kappa_1)} \leqslant c_0\phi(n)^{\beta(\kappa_1)}\sqrt{n}^{\beta(\kappa_1)}\,.$$

and

$$|(f^{n,(2)}_{t_k})'(w)| \leqslant C(1/y+1) \leqslant 2C\phi(n)\sqrt{n}$$

Note that the second estimate holds true for any conformal map of $\mathbb{H}$. Combining these estimates, we obtain that

$$\begin{aligned}
&|f^{(1)}_{t_k}(w) - f^{n,(2)}_{t_k}(w)| \\
&\quad\leqslant \frac{\phi(n)}{\sqrt{n}} \exp\left[\sqrt{\frac{1+\beta(\kappa_1)}{2}}\log(c\phi(n)\sqrt{n}) + \log\log 2\sqrt{2n}\phi(n)\right] \\
&\quad+ c|\sqrt{\kappa_1} - \sqrt{\kappa_2}| \exp\left[\sqrt{\frac{1+\beta(\kappa_1)}{2}}\log(c\phi(n)\sqrt{n}) + \log\log 2\sqrt{2n}\phi(n)\right] \\
&\quad\leqslant \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_1)}{2}}}} + \Phi(|\sqrt{\kappa_1} - \sqrt{\kappa_2}|, \kappa_1, n),
\end{aligned} \tag{7.4}$$

where $\Phi(|\sqrt{\kappa_1}-\sqrt{\kappa_2}|, \kappa_1, n) := c|\sqrt{\kappa_1}-\sqrt{\kappa_2}| \exp\left[\sqrt{\frac{1+\beta(\kappa_1)}{2}}\log(c\phi(n)\sqrt{n}) + \log\log 2\sqrt{2n}\phi(n)\right]$.

Thus, using the lemmas before, we obtain that

$$|\gamma^{(1)}(s+t_k) - \gamma^{n,(2)}(r+t_k)| \leqslant \frac{\phi_1(n)}{\sqrt{n}^{1-\beta(\kappa_1)}} + \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_1)}{2}}}} + \Phi(|\sqrt{\kappa_1} - \sqrt{\kappa_2}|, \kappa_1, n), \tag{7.5}$$

for all $r \in [\frac{1}{n}, \frac{2}{n}]$. Using Proposition 6.11, we obtain that

$$|\gamma^{(1)}(s+t_k) - \gamma^{n,(2)}(r+t_k)| \leqslant \frac{\phi_2(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_1)}{2}}}} + \Phi(|\sqrt{\kappa_1} - \sqrt{\kappa_2}|, \kappa_1, n). \tag{7.6}$$

for all $r \in [t_{k+1}, t_{k+2}]$ and $0 \leqslant k \leqslant n-2$ and hence for all $r \in [0,1]$.

In order to estimate the second term in (7.1), i.e. $|\gamma^{(2)}(t) - \gamma^{n,(2)}(t)|$, we use directly the result from Theorem 6.8.

8 Probabilistic estimates

In this section, we finish the proof providing the probabilistic versions of the estimates in the previous sections. For this, we first consider the Rohde-Schramm estimate.

Proposition 8.1 (Proposition 5.10 in [11])**.** *For each $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$, there exists $\theta_0(\kappa) > 0$ such that the following holds: for any $\theta \in (0, \theta_0(\kappa))$ there exists a random variable $C(\kappa, \omega)$ such that $C(\kappa, \omega) < +\infty$ a.s. and*

$$|f_t'(i2^{-n})| \leqslant C 2^{n(1-\theta)} .$$

for all $t \in \{k2^{-2n} : k \in [0, 2^{2n}]\}$ and any $n \in \mathbb{N}$.

Let as before, $\beta := 1 - \theta$. Then, we have that for constants c_3 and c_4, θ depending on κ,

$$\sum_{m=n}^{\infty} \sum_{j=1}^{2^{2m}} \mathbb{P}\left[|\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \geqslant 2^{m\beta} \right] \leqslant \frac{c_3}{2^{nc_4}}. \tag{8.1}$$

As in [23], applying the union bound, we have that there exists c_3 c_4 and β depending on κ such that

$$\mathbb{P}\left[|\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \geqslant 2^{m\beta} \text{ for all } 1 \leqslant j \leqslant 2^m, m \geqslant n \right] \leqslant \frac{c_3}{2^{nc_4}}.$$

Applying $2^m \to \sqrt{m}$, we further obtain that

$$\mathbb{P}\left[|\hat{f}'_t(iy)| \leqslant y^{-\beta} \text{ for all } 0 \leqslant y \leqslant \frac{1}{\sqrt{n}}, t \in [0,1] \right] \geqslant 1 - \frac{c_3}{n^{c_4/2}} . \tag{8.2}$$

Thus, we can use the previous estimate for the fixed value κ_1

$$\mathbb{P}\left[|\hat{f}'_t(iy)| \leqslant y^{-\beta(\kappa_1)} \text{ for all } 0 \leqslant y \leqslant \frac{1}{\sqrt{n}}, t \in [0,1] \right] \geqslant 1 - \frac{c_3(\kappa_1)}{n^{c_4(\kappa_1)/2}} . \tag{8.3}$$

Moreover, it follows from [13] that there exists constants c_1 (depending on κ) and c_2 such that

$$\mathbb{P}\left[osc(\sqrt{\kappa} B_t, \frac{1}{m}) \geqslant c_1 \sqrt{\frac{\log m}{m}} \right] \leqslant \frac{c_2}{n^2} . \tag{8.4}$$

Notice that in Theorem 6.8 the subpower function is $\phi(n) = \sqrt{\log(n)}$. Then as in [23], by going through the proof, one sees that the subpower functions are changed by adding, multiplying and exponentiating constants. Hence the if we merge the dependence on κ in the initial subpower function, i.e. we start with $\sqrt{\kappa \log n}$, then we end up with $c(\sqrt{\kappa \log n})^{c'}$ for some constants c and c'. Using (7.6) and the probabilistic estimate from before, we obtain that

$$\mathbb{P}\left[\|\gamma^{(1)} - \gamma^{n,(2)}\|_{[0,1],\infty} \leqslant \frac{c_6(\kappa \log m)^{c_7}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa_1)}{2}}}} + \Phi(|\sqrt{\kappa_1} - \sqrt{\kappa_2}|, \kappa_1, m) \text{ for all } m \geqslant n\right]$$
$$\geqslant 1 - \left(\frac{c_2(\kappa_1)}{n^2} + \frac{c_3(\kappa_1)}{n^{c_4(\kappa_1)/2}}\right). \tag{8.5}$$

In order to find the probabilistic estimate for the second term in (7.1), i.e. the term $|\gamma^{n,(2)}(t) - \gamma^{(2)}(t)|$, we adapt the following result from [23] for $\kappa = \kappa_2$

$$\mathbb{P}\left[\|\gamma^{\kappa}(t) - \gamma^{m}(t)\|_{[0,1],\infty} \leqslant \frac{c_1(\kappa \log(m))^{c_2}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa)}{2}}}} \text{for all } m \geqslant n\right] \geqslant 1 - \frac{c_4}{n^{c_5}},$$

where the constants, c_4 and c_5 depend on κ.

Thus, there are c_4 and c_5 depend on κ_2. such that

$$\mathbb{P}\left[\|\gamma^{(2)}(t) - \gamma^{m,(2)}(t)\|_{[0,1],\infty} \leqslant \frac{c_1(\kappa_2 \log(m))^{c_2}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa_2)}{2}}}} \text{for all } m \geqslant n\right] \geqslant 1 - \frac{c_4}{n^{c_5}},$$

The previous analysis performed for the two values κ_1 and κ_2 can be extended for sequences $\kappa_j \to \kappa$.

We emphasize that we work outside the polar set that is the nullset with respect to all the measures $\mathbb{P}_\kappa$.

First, we apply the previous Lemmas 6.9 and 6.10 for sequences $\kappa_j \to \kappa$.

Next, we use the quasi sure estimate

$$|(f^{(j)}_{t_k})'(z)| \leqslant C(\omega) y^{-\beta(\kappa_j)}\,,$$

for the sequence κ_j. The constant $c(\kappa, \omega)$ in the Borel-Cantelli argument can be shown to be an a.s. continuous function of κ that blows up as $\kappa \to 0$ (see the analysis in [11]). Choosing $\kappa \in J$ such that $\kappa \neq 0$ and $\kappa \neq 8$, we can bound this constant for almost every Brownian path by taking the supremum of the function over the interval J in which κ takes values in.

Continuing the analysis, the sizes of the boxes depend on κ_j via the dependence of the subpower function that we choose on $\beta = \beta(\kappa_j)$ and on ϕ (that depends also on κ_j, since the driver is $\sqrt{\kappa_j}B_t$). However, since the constant $c = 2\sqrt{2}$ is fixed, the upper level of the boxes remains the same as we consider $\kappa_j \to \kappa$, only their width and lower level changes.

We consider $\kappa_j \to \kappa$ by choosing for each j the largest box that contains both points z and w in order to estimate the hyperbolic distance between them, i.e. we make use of the fact that the upper height of the boxes coincides and we work on $A_{n,2\sqrt{2},\xi(\kappa_j)}$ with $\xi(\kappa_j) = \max(\psi(\kappa_j), \tilde{\psi}(\kappa))$. This is a dynamical version (as we vary the index j) of the analysis in [23] that is performed for fixed κ. For each fixed j, the estimates work in the same manner.

In order to assure that $\Phi(|\sqrt{\kappa} - \sqrt{\kappa_j}|, \kappa, n)$ converges to zero as $j \to \infty$, we choose $n = n(\kappa_j)$ such that as $j \to \infty$

$$c|\sqrt{\kappa} - \sqrt{\kappa_j}| \exp\left[\sqrt{\frac{1+\beta(\kappa)}{2}} \log(c\phi(n)\sqrt{n}) + \log\log 2\sqrt{2n}\phi(n)\right] \to 0.$$

Combining the previous estimates and using a union bound, we obtain the result.

For the second part of the result, the sequential continuity in κ for $\kappa \in \mathbb{R}_+ \setminus \{0, 8\}$ of the curves generated by the algorithm is obtained by estimating

$$|\gamma^{n,(1)}(t) - \gamma^{n,(2)}(t)| \leqslant |\gamma^{n,(2)}(t) - \gamma^{(2)}(t)| + |\gamma^{(2)}(t) - \gamma^{(1)}(t)| + |\gamma^{(1)}(t) - \gamma^{n,(1)}(t)|. \tag{8.6}$$

The first and the last term can be directly estimated using Theorem 6.8, since these are terms that compare the SLE_{κ_1} and SLE_{κ_2} traces with the corresponding approximated traces. The middle term is estimated using the analysis performed in the proof so far, and the conclusion follows.

Remark 8.2. The algorithm uses estimates on the derivative of the conformal maps. We remark that the derivative of the composition of the conformal maps obtained when solving Loewner equation on each element of the partition of the time interval $[t_k, t_{k+1}]$ (where $t_k = \frac{k}{n}$, $0 \leqslant k \leqslant n$) with $c\sqrt{t} + d$ with c, $d \in \mathbb{R}$, is not easy to estimate directly. That is why we used in our proof the Rohde-Schramm estimate on the derivative of the Loewner map : $|(f_t^{\kappa})'(y_0)| \leqslant c(\omega) y^{-\beta(\kappa)}$ with $\beta(\kappa) < 1$, $\forall \kappa \neq 8$.

□

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