The phase transition in terms of uniqueness/non-uniqueness of solutions of the backward Loewner Differential Equation started from the origin in $\mathbb{H}$ and double points along the backward SLE traces

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Abstract

We study the properties of the backward Loewner differential equation, as a differential equation started from the singularity. When the backward Loewner differential equation is driven by $\sqrt{\kappa}B_t$, we show that there is a phase transition at $\kappa = 4$, in terms of uniqueness/non-uniqueness of solutions starting from the origin. In the regime $\kappa > 4$, we study some features of the non-uniqueness in terms of the macroscopic excursions of the backward Bessel process. Compared with the typical proof of the phase transition in the behavior of the SLE_κ trace at $\kappa = 4$, that uses information of the behavior on the boundary for the maps satisfying the forward Loewner differential equation, we use the behavior on the boundary of a different family of mappings and we obtain new structural information about the backward SLE_κ traces, by identifying some classes of double points along the backward SLE_κ traces for $\kappa \in (4, \infty)$.

1 Introduction

The Loewner equation was introduced by Charles Loewner in 1923 in [11] and it played an important role in the proof of the Bieberbach Conjecture [4] by Louis de Branges in 1985 in [5]. In 2000, Oded Schramm introduced a stochastic version of the Loewner equation in [16]. The stochastic version of the Loewner evolution, i.e. the Schramm-Loewner evolution -$SLE(\kappa)$- generates a one parameter family of random fractal curves that are proved to describe scaling limits of a number of discrete models that appear in planar Statistical Physics. We refer to [9] for a detailed study of the object and many of its properties.

When studying the stochastic version of the Loewner differential equation as a differential equation, a natural question is the uniqueness/non-uniqueness of the solutions (if there exists any) of the backward Loewner differential equation starting from its singulairity (the origin). In the typical theory of differential equations, this phase transition is understood via the regularity of the vector fields of the ODE using Picard-Lindelöf Theorem. These type of ideas are extended to differential equations driven by Brownian motion in Rough Paths Theory, introduced by Terry Lyons in [12], as a deterministic platform to study differential equations driven by rough signals. In the framework of Rough Paths Theory, the uniqueness/non-uniqueness of solutions depends on the regularity of the vector field compared with the roughness of the driving process. For more details, we refer to [6]. The backward Loewner differential equation started from the singularity can not be interpreted directly as a Rough Differential Equation due to the blow-up of one of the vector fields that appears in this equation. The situation is more delicate also, since we are dealing with a differential equation started from its singularity. One of the goals of this paper is to motivate that the backward Loewner differential equation carries a rich structure, that allows us to study such questions, using information on a lower subspace: the real line.

The analysis consists of two parts: in the first part we study the existence of solutions to the backward Loewner differential equation starting from the origin by constructing a solution, while in the second part we question the uniqueness/non-uniqueness of solutions in this setting. In order to show the existence of the solutions as well as to study their uniqueness/non-uniqueness, we use the fact that the conformal maps desribing SLE_κ can be continuously extended to the boundary. When performing this extension, we obtain a Bessel process of dimension $d(\kappa) = 1 - \frac{4}{\kappa}$, for $\kappa > 0$. The Bessel processes are studied extensively in a number of papers and monographs. For more details about the various properties and characterizations of these processes, we refer to [13] and also to [7]. As a main ingredient in our analysis of the uniqueness/non-uniqueness of solutions, we use the phase transition in the boundary behavior of the Bessel process along the real line for the extended maps, i.e. absorption at the origin of the Bessel process for $\kappa \leqslant 4$, and reflection at the origin for $\kappa > 4$. In the current literature related to the phase transition of the SLE_κ curves there is used as a main tool the forward Bessel process obtained by the extension of conformal maps to the boundary. In this work we study the backward Bessel process obtained from the extension to the real line of the conformal maps that satisfy the backward Loewner differential equation. The study of phase transition phenomena using this process is new and reveals new ways of obtaining structural information about the backward SLE_κ traces via Excursion Theory of the underlying Bessel process, such as a characterization of a subspace of its double points for $\kappa > 4$.

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2 Preliminaries and main result

The Schramm-Loewner evolution (SLE) is a one-parameter family of random planar growth processes constructed as a solution to Loewner equation when the driving term is a re-scaled Brownian motion. The typical format of the equation is the forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa} B_t}\,, \qquad g(0,z) = z, z \in \mathbb{H}\,. \tag{2.1}$$

The functional inverses of this maps satisfy the partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z-\sqrt{\kappa}B_t}\,, \quad f(0,z)=z, z\in\mathbb{H}\,. \tag{2.2}$$

In this paper, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z)-\sqrt{\kappa}B_t}\,, \qquad h(0,z)=z, z\in\mathbb{H} \tag{2.3}$$

Throughout the paper, we will use also the following Lemma.

Lemma 2.1 (Lemma 5.5 of [8])**.** *Let $h_t(z)$ be the solution to the backward Loewner differential equation with driving function $\sqrt{\kappa}B_t$ and let $f_t(z)$ be the solution of the partial differential equation version of the Loewner differential equation with the same driver. Then, for any $t\in\mathbb{R}_+$, the function $z\to f_t(z+\sqrt{\kappa}B_t)-\sqrt{\kappa}B_t$ and $z\to h_t(z)$ have the same distribution.*

Next, we prove the following result that we will use in our analysis.

Lemma 2.2. *The maps $h_t(z):\mathbb{H}\to\mathbb{H}\setminus K_t$ solving the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ can be extended continuously to the boundary for all times $t\in[0,+\infty)$, a.s..*

Proof. Using Rohde-Schramm Theorem (see [17]) we obtain that the maps $g_s^{-1}(z)$ can be extended for all $s\in[0,\infty)$, for almost every Brownian path (since the SLE_κ hulls K_s are locally connected, for all $s\in[0,T]$ for almost every Brownian path). Let us consider the Loewner chain for $h_{T-s}\circ h_T^{-1}(z)-\sqrt{\kappa}B_T$

$$\partial_s(h_{T-s}\circ h_T^{-1}(z)) = \frac{2}{(h_{T-s}\circ h_T^{-1}(z))-(\sqrt{\kappa}B_{T-s}-\sqrt{\kappa}B_T)}.$$

This is a forward Loewner chain with driver $\tilde{B}_s:=\sqrt{\kappa}B_{T-s}-\sqrt{\kappa}B_T$ that is a Brownian motion. The maps $g_s(z)$ also satisfy this chain. Since the Loewner chain has a unique solution a.s., we obtain that fixed $T>0$ the maps $g_s(z)=h_{T-s}\circ h_T^{-1}(z)$. Thus, $h_{T-s}(z)=g_s\circ h_T(z)$ can be continuously extended to the boundary. Indeed since for any fixed $T>0$, the maps $h_T(z)$ have the same distribution as $g_T^{-1}(z)$ (according to Lemma 2.1) and the maps $g_s(z)$ can be extended continuously to the boundary. Let us choose $(T_n)_{n\in\mathbb{N}}$ an increasing sequence of times such that $T_n \to \infty$, as $n \to \infty$. Let

$$A_{T_n} := \{h_{T_n - s}(z) \text{ can be continosly extended to the boundary for all } s \in [0, T_n]\}.$$

For each $T_n > 0$, the events A_{T_n} happen with probability one from the previous argument. Using an union bound argument, we obtain that $\mathbb{P}(\cup_{n\in\mathbb{N}} A^c_{T_n}) = 0$, and the conclusion follows. $\square$

In the last sections of the paper, we will work with the square of the (shifted) conformal mappings, i.e. we will consider

$$\tilde{h}_t(z) = (h_t(\sqrt{z}) - \sqrt{\kappa} B_t)^2,$$

which is a conformal map from $\mathbb{C} \setminus [0, \infty)$ into $\mathbb{C} \setminus [0, \infty)$. We emphasize that $\sqrt{z}$ is not the principal part, and we work with the branch where we measure the angles of the Argument from the positive semiaxis. These mappings can be extended continuously to the boundary using the same idea as in Lemma 2.2. They form a Squared Bessel process of certain dimension that we analyze in the next sections.

From Lemma 2.1, it follows that for fixed $t \in [0, \infty)$, $h_t(z) - \sqrt{\kappa} B_t$ is equal in distribution to $g_t^{-1}(z + \sqrt{\kappa} B_t)$. Let $x > 0$. Following Theorem 4 of [10], we obtain that the mappings

$$H(x, t) = (g_t^{-1}(i\sqrt{x} + \sqrt{\kappa} B_t))^2$$

are continuously extended a.s. on $[0, \infty) \times [0, \infty)$ and they also generate a trace

$$\gamma_2(t) = H(0, t) = (g_t^{-1}(\sqrt{\kappa} B_t))^2.$$

We introduce the following definitions that we use in our analysis as well:

Definition 2.3. *For a one-dimensional diffusion X_t on the interval $I \subset \mathbb{R}$, let us write $H_y := \inf\{t : X_t = y\}$. A point $c \in I = [0, \infty)$ is called absorbing if*

$$\mathbb{P}^c(H_y < \infty) = 0$$

for all $y \in I \setminus \{c\}$.

A point $c \in I = [0, \infty)$ is called reflective if

$$\mathbb{P}^c(H_y < \infty) > 0$$

for some $y \in I \setminus \{c\}$.

We call a point $c \in [0, \infty)$ instantaneously reflecting if the point is reflecting and the Lebesgue measure of the time spent at point $c \in I$ is zero.

Definition 2.4. *A diffusion is called regular if for all $x \in Int(I)$, $y \in I$, we have that*

$$\mathbb{P}^x(H_y < \infty) > 0.$$

Definition 2.5 (Scale function). *For a (regular) diffusion X_t on I with the following laws $\{\mathbb{P}^x : x \in I\}$, a scale function is a continuous strictly increasing function $s : I \to \mathbb{R}$ such that for all $a < x < b \in I$,*

$$\mathbb{P}^x(H_b < H_a) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

Next, we introduce the following definition.

Definition 2.6. *If $s(x) = x$ is a scale function for X_t, we say that X_t is in natural scale.*

For a diffusion X_t with drift μ and variance σ^2, the scale function can be computed via $s(x) = \int_{x_0}^{x} \exp(-\int_{\eta}^{y} \frac{2\mu(z)}{\sigma^2(z)} dz) dy$, where x_0, η are points fixed (arbitrarily) in (a, b). We further define

Definition 2.7. *The function $m(\zeta) = \frac{1}{\sigma^2(\zeta) s'(\zeta)}$ is the density of the speed measure.*

Let us consider the Borel functions

$$\sigma : \mathbb{R} \to \mathbb{R}$$

and $b : \mathbb{R} \to \mathbb{R}$ bounded on compact sets. Consider the equation (that we write shortly $Eq(\sigma, b)$) given by

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt.$$

Definition 2.8. *A solution to the equation $Eq(\sigma, b)$ consists of:*

- *a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.*
- *an $\mathcal{F}_t$-Brownian motion.*

- *an $\mathcal{F}_t$ adapted continuous process X such that*

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds.$$

When $X_0 = x$, we say that X_t started in x.

We are ready to introduce the notion of strong solutions.

Definition 2.9. *We say that a solution (X, B) to $Eq(\sigma, b)$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a strong solution on this space if X is adapted to the complete filtration generated by the Brownian motion B.*

In order to perform the analysis, let us consider $Z_t(z) := \frac{h_t(z\sqrt{\kappa}) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$ with $z \in \bar{\mathbb{H}}$, where $h_t(z)$ are the continuous extensions to the real line of the conformal maps solving (**??**).

Thus, when restricting the analysis to the real line, we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2/\kappa}{X_t}dt + dB_t \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned} \tag{2.4}$$

This SDE is the same as the ones governing real Bessel processes started from $x_0 \in \mathbb{R}$,

$$dX_t = \frac{d-1}{2}\frac{1}{X_t}dt + dB_t,$$

with $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leqslant 0$ for $\kappa \leqslant 4$ and $d > 0$ for $\kappa > 4$. We show throughout this paper that for $\kappa \leqslant 4$, there is no solution to the SDE (2.4) starting from the origin for almost every Brownian path, since $\{0\}$ is absorbing boundary point a.s.. For $\kappa > 4$, we show that there exists more than one strong solution to the SDE (2.4) that is instantaneous reflecting at zero. In this way we give meaning (abstractly) to a notion of solution for the Loewner equation started from the origin (along the real line). In [1] there is a proof that there exists a unique strong non-negative solution to (2.4) that starts at zero and that spends zero Lebesgue measure time there. We use this solution to construct another strong solution that is not positive and has the same characteristics (spends null Lebesgue measure time at zero).

2.1. Excursion theory for Markov processes. We continue with a subsection on Excursion Theory for Markov processes. First, we introduce the notion of local time.

Definition 2.10. *For a stochastic process $X_t : \Omega \to \mathbb{R}$, for every $t \geqslant 0$ and $x \in \mathbb{R}$, the quantity*

$$L_{x,t} := \limsup_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{|X_s - x| \leqslant \varepsilon} ds$$

is called the local time at level x and time t.

When we refer to the local time of the process at zero (i.e. when $x = 0$), we simplify the notation $L_t = L(t)$.

In this subsection, we follow the approach in [3], that we refer to for more details. We define the terminology for a general real-valued Markov process M_t, keeping in mind that in our setting, we apply the techniques for the solutions to the Squared Bessel SDE.

Definition 2.11. *An open interval $(g, d) \subset \mathbb{R}$ is called an excursion interval for a stochastic process M_t if $M_t \neq 0$, for all $t \in (g, d)$, with $M_t = M_d = 0$.*

We fix a real number $c > 0$. If we are dealing with a process that is not identically zero, then with positive probability there exists at least one excursion interval with length $l > c$. We introduce as in [3], the right continuous function

$$\bar{\Pi} : (0, \infty] \to (0, \infty)$$

which describes the distribution of the lengths of excursion intervals. Let us consider the natural ordering of excursion intervals with length $l > a$. Let us consider the length of the first such interval to be $l_1(a)$. The function is defined for every $a \in (0, \infty]$ as

$$\bar{\Pi}(a) = \begin{cases} \frac{1}{\mathbb{P}(l_1(a) > c)} & a \leqslant c, \\ \mathbb{P}(l_1(c) > a) & a > c. \end{cases}$$

We are interested in the behavior of $(M_{g+t}, 0 \leqslant t \leqslant d - g)$ corresponding to each excursion interval (g, d).

For every $a > 0$, let us consider the set of excursions of length $\tau > a$

$$\mathcal{E}^{(a)} = \{\omega \in \Omega, \omega(0) = 0 : \tau > a \text{ and } \omega(t) \neq 0, \forall\, 0 < t < \tau\}.$$

Let us consider the set of all excursions $\mathcal{E} = \cup_{a>0}\mathcal{E}^{(a)}$. Following the approach in [2] for each $a > 0$ with $\bar{\Pi}(a) > 0$, we denote by $n(\cdot|\tau > a)$ the probability measure on $\mathcal{E}^{(a)}$ corresponding the law of the process M_t on the first excursion with length $l > a$, i.e. the law of the process $(M_{g_1(a)+t}, 0 \leqslant t \leqslant l_1(a))$ under $\mathbb{P}$, where we used that the assumption $\bar{\Pi}(a) > 0$ guarantees $g_1(a) < \infty$ a.s.. It is proved in [2], that there exists a unique measure n on $\mathcal{E} = \cup_{a>0}\mathcal{E}^{(a)}$ that we call the excursion measure, such that

$$n(\lambda) = \bar{\Pi}(a)n(\lambda|\tau > a)$$

for every $\lambda \subset \mathcal{E}^{(a)}$.

Let us define $L^{-1}(t)$ as the inverse of the local time at zero of the process M_t up to time $t > 0$, i.e.

$$L^{-1}(t) = \inf\{s \geqslant 0 : L(s) > t\}.$$

Moreover, let us consider

$$L^{-1}(t-) = \inf\{s \geqslant 0 : L(s) \geqslant t\}.$$

We introduce the excursion process $e_t(\omega)$ on $\mathcal{E} \cup \{\eta\}$, where $\{\eta\}$ is an additional isolated point, associated with the stochastic process M_t

$$e_t(\omega) = (M_{s+L^{-1}(t-)}(\omega), 0 \leqslant s < L^{-1}(t) - L^{-1}(t-)) \text{ if } L^{-1}(t) - L^{-1}(t-) > 0$$

and $e_t(\omega) = \eta$ otherwise.

We have the following result.

Theorem 2.12 (Theorem 10, chapter IV of [2])**.** *If* 0 *is recurrent then,* $(e_t(\omega), t \geqslant 0)$ *is a Poisson Point Process with characteristic measure* n*.*

We will use that for $0 < d < 2$, for the Squared Bessel process the origin is recurrent. We obtain from the above result the fact that there exists an excursion decomposition for this process. We will use this in the following subsections to obtain results about the backward SLE_κ trace for $\kappa > 4$.

3 Main result

Before stating our main result, we introduce the following concepts.

Definition 3.1. *For $m > 0$, an m-macroscopic excursion is an excursion of the Bessel process of length at least $m > 0$, where the length of an excursion is understood as the length of the time interval corresponding to the excursion.*

Next, we introduce the notion of macroscopic hull.

Definition 3.2. *A macroscopic hull is a compact set in $\bar{\mathbb{H}}$ with a simply connected complement, such that its intersection with the real axis is an interval $I \subset \mathbb{R}$ with strictly positive Lebesgue measure. We call the intersection of macroscopic hull with the real axis the base of the hull.*

Definition 3.3. *Let $m > 0$ be a real number. Let $\gamma(t)$ be the SLE_κ curve. If there exists two times r_1, r_2 in the time interval such that $\gamma(r_1) = \gamma(r_2)$ and $|r_1 - r_2| \geqslant m$, then we say that there is a self-touching of the curve after time at least m.*

We also introduce the following definition.

Definition 3.4. *Let $m > 0$. For the (backward) SLE_κ trace, an m-macroscopic double point is a double point that corresponds to self-touching of the trace after time at least $m > 0$.*

The origin in $\mathbb{H}$ is a singularity for the backward Loewner differential equation, since the vector field $\frac{-2/\kappa}{Z_0}$ has infinite modulus. For almost every Brownian path, a notion of solution for $t \in [0, T]$ for this differential equation is defined by the limit $Z_t(0) := \lim_{y \to 0+} Z_t(iy)$.

We are now ready to state our main result.

Theorem 3.5. *For $\kappa \in (0, 4]$ and for all times in a compact time interval, there is a unique solution of the backward Loewner differential equation started from the origin, a.s.. Moreover, for $\kappa \leqslant 4$, the backward SLE_κ trace is a simple curve a.s..*
For $\kappa > 4$, there are at least two real solutions for the backward Loewner differential equation started from the origin, a.s.. Let $m > 0$ be a positive real number. For $\kappa \in (4, \infty)$, on m-macroscopic excursions from the origin of the Squared Bessel process obtained from

extensions of the backward SLE_κ maps on the real line, we obtain macroscopic hulls with base depending on $m > 0$ and m-macroscopic double points of the backward SLE_κ trace.

Remark 3.6. Compared with the proof of the a.s. simpleness of the SLE_κ trace for $\kappa \leqslant 4$ that already exists in the literature, we show that this is implied by the behavior at the origin the backward Bessel process obtained from the extension of the conformal maps to the boundary.

In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behavior of the SLE_κ trace. We will study this in more detail in the last section of this paper.

In a nutshell, in our analysis, we extend the conformal maps to the boundary and obtain that the dynamics of boundary points satisfy the SDE (2.4). We use known results about the solutions of the SDE (2.4) along with theorems about the backward Loewner evolution in order to study the start of the backward Loewner evolution from the origin. For this, we use the result from [1], in which there is a proof of the existence and uniqueness of strong positive solutions to the Bessel SDE started from the origin for $0 < d < 1$. In our case, this gives the existence and uniqueness of a strong non-negative solution for the SDE (2.4) on the real line for $\kappa > 4$, since $d(\kappa) = 1 - \frac{4}{\kappa}$.

Remark 3.7. For dimensions $d \in (0, 1]$ ($\kappa > 4$) there exists a construction of the solutions to the SDE using Excursion Theory. Using this and the previous analysis, we obtain that beginning of m-macroscopic excursions for the SDE (2.4) give macroscopic hulls for the backward Loewner differential equation started from the origin. Thus, we obtain another structural information about the dynamics of the backward SLE_κ hulls, and implicitly backward SLE_κ traces. That is, the m-macroscopic excursions of the Bessel process started from the origin, create in this context, macroscopic hulls of the backward SLE_κ. In particular, when starting the backward Loewner differential equation from the beginning of m-macroscopic excursions of the Bessel processes, we obtain a macroscopic hull, a.s. For further details about the Excursion Theory construction of the Bessel processes for $d \in (0, 1]$, see [2].

3.1. Backward Loewner differential equation and the Backward Loewner trace.

In the following we show how using the a.s. existence of the forward chordal SLE_κ, we can obtain the a.s. existence of the backward Loewner trace. For $\kappa > 0$, the backward chordal SLE_κ is defined by solving the backward Loewner differential equation with the driver $\sqrt{\kappa}B_t$, $0 \leqslant t < \infty$. For any $t_0 > 0$, we have that the process $(\sqrt{\kappa}B_{t_0-t} - \sqrt{\kappa}B_{t_0}, 0 \leqslant t \leqslant t_0)$ has the same distribution as $(\sqrt{\kappa}B_t,\ 0 \leqslant t \leqslant t_0)$. Rohde and Schramm showed in [17] that for every $t_0 \in [0,T)$, $\sqrt{\kappa}B_{(t_0-t)}$, $0 \leqslant t \leqslant t_0$ generates a forward Loewner trace which we denote by $\beta_{t_0}(t_0 - t), 0 \leqslant t \leqslant t_0$. Then, using the identity in distribution for the Brownian motion from above, it is shown in [15] that $\sqrt{\kappa}B_t$ generates the backward SLE_κ traces β_{t_0} for $0 \leqslant t_0 \leqslant T$.

Note that β_t is a continuous function defined on $[0,t]$. The parametrizations of the backward traces β_t is different from the usual parametrization of the chordal SLE_κ trace in the sense that the backward traces β_t is a continuous function defined on $[0,t]$ such that β_0 is the tip and β_t is the root that is an element of $\mathbb{R}$. The difference with the chordal SLE_κ is that the chordal trace is parametrized such that the root of it is β_0. For further details about this construction, we refer to [15].

We also use in our analysis that for fixed time $T > 0$, the law of the curves $\beta_T = \beta(0,T]$ generated by the backward Loewner differential equation is the same as the forward chordal SLE_κ trace $\gamma(0,T]$ (modulo a translation with the driver $\sqrt{\kappa}B_T$).

4 Heuristic of the proof of the main result

- We consider a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ equipped with the Wiener measure. We consider the backward Loewner differential equation driven by a standard Brownian motion that is defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$. We consider also the extension of the conformal maps $h_t(z)$ to the real line.

- On the real line, when extending continously the maps solving the backward Lowener differential equation with $\sqrt{\kappa}B_t$ driver, we obtain a real-valued SDE (2.4). This real valued SDE, has the origin as a boundary point. Moreover, the behaviour of this real-valued SDE at this boundary point dependes on the dimension $d = d(\kappa)$. For dimension $d = 0$ and $d < 0$, the origin is an absorbing boundary and for $d > 0$, zero is a reflecting boundary. For the backward flow we have $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leqslant 0$ for $\kappa \leqslant 4$ and $d > 0$ for $\kappa > 4$, i.e. zero has a different behaviour as a boundary point for the SDE (2.4). For $\kappa > 4$, there is a way to give meaning to a notion of (strong) solution to the SDE (2.4), started from $\{0\}$ for almost every Brownian path. This idea appeared in [1], where it is proved that there exists a strong non-negative solution for this process, that spends null Lebesgue measure time at the origin, for almost every Brownian path. In contrast, for $d = 0$ ($\kappa = 4$) and $d < 0$ (i.e. $\kappa < 4$) there is no notion of solution that exits the origin along the real line a.s., so if there is a notion of solution started from the origin, the only possibility is the one constructed from the interior of $\mathbb{H}$.

- First, we show how we can define a solution for the backward Loewner differential equation started from the origin, when driven by $\sqrt{\kappa}B_t$, for all κ and for almost every Brownian path.

  Then, we show that the behavior of the real-valued SDE (2.4) at the origin is absorbing for $d = d(\kappa) \leqslant 0$.

  Using this, we argue that for $\kappa \in [0, 4]$, there is a unique solution a.s.. The argument is the developed in the following section.

- For $d > 0$, i.e. $\kappa > 4$, the picture differs. In this case, we prove that there are at least two real solutions with the same starting point and with the same Brownian path as a driver. We use the existence of the strong solution result for the SDE (2.4) of dimension $d > 0$ proved in [1], to construct a new (negative) strong solution (with the same starting point and the same Brownian path as a driver). Thus, we conclude that in this setting, the origin gets mapped to at least two different points for $t \in [0, T]$, a.s.. The result in [1], gives also the uniqueness of the strong solution on the positive semi-axis.

  Extending the conformal maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ to the real line and considering the trace $\gamma_2(t) = (g_t^{-1}(\sqrt{\kappa}B_t))^2$, we obtain using the analysis of the Squared Bessel SDE new structural information about the SLE_κ traces.

  This perspective helps us to recover the phase transition for the backward SLE_κ

traces from simple for $\kappa \in [0,4]$ to non-simple curves for $\kappa \in (4,+\infty)$ using the phase transition in terms of the behavior of the Squared Bessel SDE at $\{0\}$, seen as boundary point for this SDE. This analysis, compared with the typical phase transition in terms of the behavior of the forward Bessel process, gives additional information about the structure of the backward SLE_κ traces (to be compared with the forward flow case, where the proof uses only information on the positive part of the real line).

5 Proof of the main result

5.1. Proof of Theorem 3.5: the existence and uniqueness of solutions for $\kappa \leqslant 4$.

In this section, we prove the first part of the result. First, we use Lemma 2.2 in order to assure that for the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$, we have that $\lim_{y\to 0} h_t(iy)$ exists for all times $t \in [0,+\infty)$, a.s.. We postpone to show the fact that the limit indeed satisfies the backward Loewner differential equation until the end of the section and we first show the uniqueness.

5.2. Application: Almost sure simpleness of the SLE_κ trace, for $\kappa \in [0,4]$ using just the information on the boundary.

In this section, we show how the information about the dynamics of boundary points under the backward Loewner differential equation, gives information about the structure of the SLE_κ traces for $\kappa \in [0,4]$. We recall that the SLE_κ traces are defined via a limiting procedure from the interior of the domain. Essentially, we show that the a.s. behavior of the origin for the Bessel SDE (2.4), gives information about the behavior of the SLE_κ traces.

We first need the following definition.

Definition 5.1. *The real Squared Bessel process for any $\delta \geqslant 0$ and $x \geqslant 0$ is defined as the unique strong solution for the equation $X_t = x + 2\int_0^t \sqrt{X_s}dB_s + \delta t$.*

For this process, we have the following result

Proposition 5.2 (Proposition 1.5, Chapter XI of [13]). *For $\delta = 0$, the point 0 is absorbing and for $0 < \delta < 2$ the point 0 is instantaneously reflecting.*

We use this result, in order to argue that the point 0 is also absorbing for the Bessel process of dimension $d = 0$ (i.e. $\kappa = 4$). Indeed, if with positive probability the point $\{0\}$ would not be absorbing then this would imply that with positive probability we could construct solutions for the Squared Bessel SDE that are different from the identical zero solution, that is a contradiction.

In addition, we have the following general theorem from [14].

Theorem 5.3 (Theorem 51.2 of [14]). *Let X_t be a real valued diffusion in the natural scale on $I = [0, \infty)$ with speed measure m. Then the boundary $\{0\}$ must be absorbing if $\int_I m(dx) = \infty$.*

When computing the speed measure for the Squared Bessel process of dimension $d = d(\kappa) = 1 - \frac{4}{\kappa}$, we obtain $m_d(dy) = K_d y^{2(d-1)/(2-d)} dy$, where K_d depends on d. We obtain that for $0 < d < 2$

$$\frac{2(d-1)}{(2-d)} \in (-1, \infty).$$

Since $\frac{2(d-1)}{(2-d)} \leqslant -1$ for $d \leqslant 0$ we obtain via Theorem 5.3 that for $d \leqslant 0$ the origin is reached a.s. and the process is absorbing. Thus, when started from $x_0 = 0$ the solutions of the Squared Bessel process of dimension $d > 0$, 'escape' the origin and reflect when reaching the origin back, while the process is absorbed at the origin for $d \leqslant 0$.

We showed that the Squared Bessel process has an absorbing origin for $d \leqslant 0$, i.e. $\kappa \leqslant 4$. Since the Bessel process is the square root of the Squared Bessel process we obtain that the same result for the Bessel process, that we use in the following.

For $0 < d < 2$, the process is also reaching the origin, a.s. but the origin is not absorbing anymore.

Let us consider the maps $g_t(z)$ solving forward Loewner differential equation, and let us assume that they generate a trace (when the diver is $\sqrt{\kappa} B_t$ this happens with probability 1). Let us consider fixed $s < t$, and let us consider $g_{s,s+t} = g_{s+t} \circ g_s^{-1}$. Then the Loewner chain $g_t^{(s)} = g_{s,s+t}$ has driving function $U_t^{(s)} = U_{s+t}$. When $U_t = \sqrt{\kappa} B_t$, the process

$U_{s+t} - U_s$ is again a Brownian motion. Then, the Loewner chain for $g_t^{(s)}(z)$ is

$$\partial_t g_{s,s+t}(z) = \frac{2}{g_{s,s+t}(z) - \sqrt{\kappa} B_{t+s}}, \qquad g_{s,s}(z) = z.$$

Thus, the format of the equation is the same. We have the following definition.

Definition 5.4. *Suppose g_t is a Loewner chain with driving function U_t, and let $\hat{f}_t(z) = g_t^{-1}(z + U_t)$. Suppose that for each t the limit*

$$\gamma(t) = \lim_{y to 0+} \hat{f}_t(iy)$$

exists and the function $t \to \gamma(t)$ is continuous. Then the Loewner chain g_t is generated by the curve γ.

We use the notation $\gamma(t)$ for the trace generated by $g_t(z)$ and $\gamma^{(s)}(t)$ for the trace generated by $g_t^{(s)}(z)$ for $s > 0$.

Lemma 5.5 (Lemma 4.34 of [9])**.** *Suppose $g_t(z)$ is generated by the curve γ. Then γ is a simple curve with $\gamma(0, \infty) \in \mathbb{H}$ if and only if for all $s \geqslant 0$, $\gamma^{(s)}(0, \infty) \cap \mathbb{R} = \emptyset$.*

The SLE_κ traces $\gamma^{(s)}(t)$ have the scaling property. The argument is identical to the one for the forward Loewner differential equation for $s = 0$. Using the deterministic Lemma 5.5, we see that in general if the Loewner chain is generated by a curve γ, then γ is not simple if and only if there exists s such that $\gamma^{(s)}(0, \infty) \cap \mathbb{R} \neq \emptyset$. For SLE_κ traces we obtain that γ is a.s. non simple if and only if with probability one there exists s such that $\gamma^{(s)}(0, 1) \cap \mathbb{R} \neq \emptyset$. Using the scaling property of the SLE_κ traces, we obtain that probability of events is either 0 or 1. Indeed define for $t > 0$, $E_t = \{\text{there exists } s \text{ such that } \gamma^{(s)}(0, t) \cap \mathbb{R} \neq \emptyset\}$. By scaling the sets E_t have the same probability, that we denote by p. Then $p = \mathbb{P}(\cap_t E_t)$ and $\cap_t E_t \in \mathcal{F}_{0+}$ where $\mathcal{F}_{0+} = \cap_{t>0} \mathcal{F}_t$. By Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only nullsets and their complements. Hence, $p \in \{0, 1\}$.

Thus, it is enough to show that such $s > 0$ exists with positive probability, i.e. we have that the SLE_κ trace $\gamma(t)$ is a.s. non-simple if and only if there exists s such that $\gamma^{(s)}(0, 1) \cap \mathbb{R} \neq \emptyset$ with positive probability. In the next proof, we show that the last statement is equivalent to the statement that the origin is non-absorbing for the Bessel process.

In order to show this, we need the following observation. Fix $s > 0$, a deterministic time. We consider the Loewner chain $h_{s,s+t}$, i.e.

$$\partial_t h_{s,s+t}(z) = \frac{-2}{h_{s,s+t}(z) - \sqrt{\kappa} B_{s+t}}, \qquad h_{s,s}(z) = z.$$

For any deterministic time shift $s > 0$, when considering the continuous extension to the boundary, we obtain the same Bessel process with dimension $d = 1 - \frac{4}{\kappa}$ since the format of the shifted chain is the same with the one for $s = 0$. This gives us that for any deterministic shift of time $s > 0$, we obtain a Bessel process that has an absorbing origin. Thus, for a.e. Brownian Motion for all times $s \in [0, 1]$ on these paths the corresponding Bessel process is absorbing. This excludes the possibility of random shifts of starting times $s \in [0, 1]$ on which the Bessel process obtained from the continuous extension of the conformal maps to the real line started from those (random) times has any other solution than $X = 0$, with positive probability.

For the backward Loewner differential equation, we have that on the real line the dynamics of points is governed by the following SDE

$$dZ_r = \frac{-2/\kappa}{Z_r} dr + d\tilde{B}_r\,, \quad Z_0 = x \in \mathbb{R} \setminus \{0\}. \tag{5.1}$$

with dimensions $d \leqslant 0$, where $\tilde{B}_r = B_1 - B_{1-r}$. On the positive part of the real axis this is the equation of a Bessel process. In addition, in our analysis we have as a driver the same collection of Brownian paths, the dynamics on the negative part of the real axis is given by

$$\begin{aligned} d\tilde{X}_t &= \frac{-2}{\kappa \tilde{X}_t} dt + dB_t, \\ \tilde{X}_0 &= x_0 < 0\,. \end{aligned} \tag{5.2}$$

Since $\tilde{X}_t$ is negative (until the first hitting time of zero), multiplying by -1 we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2}{\kappa X_t} dt - dB_t, \\ X_0 &= x_0 > 0\,, \end{aligned} \tag{5.3}$$

with $X_t = -\tilde{X}_t$. Thus, the dynamics on the negative part of the real axis, when coupling with the same Brownian drivers, is equivalent to a dynamics on the positive side of the real axis with the driver $-B_t$ (that is still a Brownian motion).

If with positive probability the SLE_κ trace were to be not simple, then we obtain that with positive probability $\exists \tau(\omega) < 1$, such that $\gamma^{(s)}(\tau) \cap \mathbb{R} \neq 0$. Then, when considering the SDE (5.1), we obtain that the collection of Brownian Motion paths such that the origin is reflective for the equation (5.1) has positive measure. Thus, when studying the maps $h_r(z)$ driven by $\sqrt{\kappa}B_{1-r}, r \in [0,1]$ we obtain a contradiction with the fact that $\{0\}$ is a.s. absorbing boundary : $Z_t \equiv 0$ for all $t > 0$, for the backward Loewner differential equation for almost every Brownian path. Indeed if with positive probability the trace would touch the real line in $[0,1]$, then we could find a positive mass of paths on which there exists times $r(\omega)$ such that $Z_{r(\omega)} \neq 0$, that gives a contradiction with the previous argument that shows that for almost every Brownian path for all times $t \in [0,1]$ on these paths the only possible solution for the Bessel process from the origin is $Z_t \equiv 0$.

This shows that the information on the boundary is enough to determine the behavior of the SLE_κ trace. Indeed, since for $\kappa \in [0,4]$, for almost every Brownian path, the boundary behavior of the origin is absorbing, we obtain as a corollary that the the only possible notion of solution for the backward Loewner differential equation started from the origin is the one constructed before from 'inside the domain'.

Next, we finish the proof of the first part of the main theorem by showing that the mappings $\lim_{y\to 0+} h_t(iy)$ satisfy the backward Loewner equation for $\kappa \leqslant 4$ on compact time intervals, a.s.. Let us consider the sequence of squared mappings

$$\hat{Z_t^n} = \left(\frac{h_t(i/n) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}\right)^2 .$$

We perform the analysis for this sequence but we can redo it for any other choice of sequence in the same manner. Then, the mappings $\hat{Z_t^n}$ satisfy the equation

$$d\hat{Z_t^n} = d(\kappa)dt + 2\sqrt{\hat{Z_t^n}}dB_t. \tag{5.4}$$

where $d(\kappa) = 1 - \frac{4}{\kappa}$. There are two possible extensions of $z^{1/2}$ to the positive real line (i.e. $+x^{1/2}$ and $-x^{1/2}$), depending on which direction the real line is approached.

For $\kappa \leqslant 4$, for all $n \in \mathbb{N}$, $\hat{Z}_t^n$ and $\hat{Z}_t$ do not touch the real line for all times $t \in [0, T]$ a.s.. We choose the principal branch of square root (i.e. the arguments of the complex numbers are measured from the positive semi-axis counterclockwise).

We show that $\lim_{n\to\infty} \hat{Z}_t^n$ satisfies the equation (5.4). In order to show this, it suffices to show that $\lim_{n\to\infty} 2\int_0^t \sqrt{\hat{Z}_s^n} dB_s = 2\int_0^t \sqrt{\hat{Z}_s} dB_s$, a.s..

Furthermore, in order to show this, it suffices to show that

$$\left\| 2\int_0^t \sqrt{\hat{Z}_s^n} dB_s - 2\int_0^t \sqrt{\hat{Z}_s} dB_s \right\|_{L^2}^2 \to 0$$

as $n \to \infty$. By Itô isometry, we have that

$$\left\| 2\int_0^t \sqrt{\hat{Z}_s^n} dB_s - 2\int_0^t \sqrt{\hat{Z}_s} dB_s \right\|_{L^2}^2 = \mathbb{E}\left(2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right).$$

We need to show that $\mathbb{E}\left(2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right) \to 0$. One of the ingredients used to show this is following deterministic result.

Lemma 5.6 (Lemma 2.4 of [18]). *Suppose γ is the curve generated by the Loewner differential equation with the driving function $\lambda(t)$. Then for all $z \in [0, t]$, we have that*

$$|\operatorname{Re} z| \leqslant \sup_{0 \leqslant s \leqslant r \leqslant t} |\lambda(r) - \lambda(s)|.$$

and

$$\operatorname{Im} z \leqslant 2\sqrt{t}.$$

We will apply the Lemma for $\lambda(t) = \sqrt{\kappa} B_t$ and we use the modulus of continuity of Brownian motion to argue that the real value of the solution to the backward Loewner Differential Equation when running time on a compact time interval, is finite a.s.. We emphasize also that in the proof of the Lemma these deterministic estimates (modulo some finite terms that depend on the starting point i/n and vanish in the limit) are first obtained for $Z_s(i/n)$ for fixed $n < \infty$, so we will use them also for them.

Let us define

$$A_t^n := 2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds.$$

In order to apply Lebesgue Dominated Convergence for $\mathbb{E}\left(2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right)$, we need to argue that $A_t^n \to A_t^\infty$ a.s..

For this, we apply Lebesgue Dominated Theorem for $\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds$.

We know that $\left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 \to 0$ pointwise by the fact that for $s \in [0,t]$, a.s. $\hat{Z}_s^n \to \hat{Z}_s$, as $n \to \infty$.

The a.s. integrability of $\left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2$ is a consequence of triangle inequality, after that splitting Z_s^n and Z_s into real and imaginary parts, and applying Lemma 5.6 for $\lambda(t) = \sqrt{\kappa} B_t$ (using that $\sup_{s\in[0,T]} B_s$ is finite a.s.). Thus, from Lebesgue Dominated Convergence, $A_t^n \to 0$ a.s., i.e. for almost every ω we have that $2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \to 0$. In order to argue that $\mathbb{E}\left(2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right) \to 0$, we just need the integrability of $\left(2\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right)$. This is again a consequence of triangle inequality, splitting into real and imaginary parts and applying Lemma 5.6 (using the fact that $\mathbb{E}[\sup_{s\in[0,T]} B_s]$ is finite).

Thus, using the Lebesgue dominated convergence theorem we obtain that the the limit $\hat{Z}_s$ satisfies the equation for $[0,t]$ a.s.. Since for $\kappa \leqslant 4$, the solution takes values in $\mathbb{C} \setminus [0,\infty)$, we have that $\sqrt{\hat{Z}_t}$ (obtained by halving the angles and considering square root of the modulus of $\hat{Z}_t$) satisfies the backward Loewner differential equation.

Finally, combining all the tools from this section we obtain an existence and uniqueness result for the solutions of the (backward) SLE_κ for $\kappa \in [0,4]$ for almost every Brownian path, as well as structural information about the backward SLE_κ traces using information on the boundary.

5.3. Proof of Theorem 3.5: the non-uniqueness when $\kappa > 4$.

In this section, we prove that in the case $\kappa > 4$, almost surely there are at least two types of solutions for the backward Loewner differential equation started from the origin (i.e. solutions obtained from the same starting point and the same Brownian motion path).

In order to show this, we consider again the maps $h_t(z)$ solving the backward Loewner differential equation driven by the Brownian driver $\sqrt{\kappa} B_t$, for $t \in [0,T]$. We extend continuously the conformal mappings to the real line. We show that we can give meaning to a notion of solution to SDE (2.4) that spends null Lebesgue measure time at the origin. Moreover, in the case $d > 0$ (i.e. $\kappa > 4$) we use this strong solution to construct a different solution, that gives the non-uniqueness in this regime.

The main result that we use is proved in [1] and is phrased as follows:

We use indistinguishably the notiations $L_a^Z(t), L_{t,0}^Z$ and $L_{t,0}$ for the local time of a process Z_t at the point $a \in \mathbb{R}_+$. Let us introduce the principal value correction for $d \in (0, 1]$, i.e.

$$Z_t = Z_0 + B(t) + \frac{d-1}{2}k(t)\,, Z_0 \geqslant 0,$$

where

$$k(t) = P.V. \int_0^t \frac{1}{Z_s} ds := \int_0^\infty a^{d-2}(L_a^Z(t) - L_0^Z(t))da.$$

In [1] there is a proof that the existence of a weak solution spending zero time at the origin implies the existence and uniqueness of a non-negative strong solution spending zero time at the origin for the above equation. Note that this analysis allows us to view the strong solutions on the real line starting from the origin as functions of the Brownian motion paths.

We consider the following version of the previous equation (since the dynamics is defined on both the positive part of the real line and on the negative part of the real line). We consider the following :

$$\tilde{k}(t) = P.V. \int_0^t \frac{dt}{Z_t} = \int_{-\infty}^\infty sgn(a)|a|^{d-2}(L_a^Z(t) - L_0^Z(t))da. \tag{5.5}$$

Furthermore, we use the result from [1] to see this positive solution as a unique positive strong solution for the 'extended' SDE

$$Z_t = Z_0 + B(t) + \frac{d-1}{2}\tilde{k}(t)\,, Z_t \geqslant 0. \tag{5.6}$$

Indeed this holds, since $Z_t \geqslant 0$, gives $k(t) = \tilde{k}(t)$. Note that this is a strong solution and thus can be thought as a measurable function $\phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ such that $\rho(t) := \phi(B_t(\omega))$ solves (5.5). Note that this solution is unique, non-negative and starts from the origin, according to the result from [1].

The next step is to consider $\tilde{B}_t(\omega) = -B_t(\omega)$, and as before $Z_0 = 0$. We consider

$$\tilde{\rho}(t) := -\phi(-B_t(\omega)). \tag{5.7}$$

We obtain that

$$-\phi(-B_t(\omega)) = \frac{d-1}{2}\tilde{k}(t) + B_t(\omega). \tag{5.8}$$

Thus, $\tilde{\rho}(t)$ is another solution to the equation (5.6), with the same starting point and the same Brownian motion path. Note that the two solutions are different since one is positive and the other one is negative. Thus, for all times $t \in [0,T]$, for $\kappa > 4$, we have at least two different real solutions with the same starting point and with the same Brownian driver. In particular, at each fixed time $t > 0$, for $\kappa > 4$ we obtain for the backward Loewner differential equation started from the origin at least two solutions, a.s..

6 Excursion Theory of the Bessel processes and behavior of the SLE_κ traces

6.1. SLE_κ **traces and the Squared Bessel processes.**

In this subsection, we discuss applications of the analysis developed so far on the SLE_κ traces for $\kappa > 4$. We consider, as in [10], the definition of the SLE_κ trace using the mappings $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$, where $h_t(z) : \mathbb{H} \to \mathbb{H} \setminus K_t$ are solving the backward Loewner differential equation (**??**).

The family of maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ are conformal maps from $\mathbb{C} \setminus [0, \infty)$ to $\mathbb{C} \setminus [0, \infty) \setminus \tilde{K}_t$, where $\tilde{K}_t$ is the SLE_κ hull corresponding to the maps

$$(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2.$$

Using the fact that the hulls of the backward SLE are locally connected, we obtain that the maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ can be extended continuously to the real line.

When studying the extensions of the conformal maps satisfying the backward Loewner differential equation (**??**), we obtain on the real line two SDEs (that are both running on $\mathbb{R}_+$. These SDEs are obtained by applying Itô's formula for $x \to x^2$.

These SDEs are the one satisfied by Squared Bessel processes. Following [13], by using the Yamada-Watanabe Theorem (see [13]), we obtain that the Squared Bessel process of any dimension, starting from $x \geqslant 0$ has a strong unique solutions a.s..

Thus, in our case, we have the following SDEs. We call the first one, the upper Squared Bessel SDE and the second one the lower Squared Bessel SDE:

$$d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt + 2\sqrt{\tilde{Z}_t} dB_t \tag{6.1}$$

and

$$d\bar{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt - 2\sqrt{\bar{Z}_t} dB_t. \tag{6.2}$$

We couple these processes with the same Brownian driver $B_t(\omega)$, in the sense that we drive (6.1) and (6.2) with $B_t(\omega)$. First, we restrict our attention to the solution of (6.1). The analysis for the lower Squared Bessel process (i.e. the solution to $d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt - 2\sqrt{\tilde{Z}_t} dB_t$) will be performed in the same manner.

As in [10], we denote the SLE_κ trace in this new setting by $\gamma_2(t) := (g_t^{-1}(\sqrt{\kappa}B_t))^2$. Note that, by conformal invariance $\gamma_2(t)$ has the same law as the backward SLE_κ trace in $\mathbb{H}$ (that modulo a shift has the same law with the forward SLE_κ trace in $\mathbb{H}$).

For $d(\kappa) = 1 - \frac{4}{\kappa} > 0$, there is a unique positive strong solution for this process starting from the origin such that $\tilde{Z}_t \geqslant 0$, a.s.. Also, for $2 > d(\kappa) = 1 - \frac{4}{\kappa} > 0$ the Squared Bessel process is recurrent. Thus, we can apply elements of Excursion Theory developed in the previous section.

6.2. The m-macroscopic excursions of the real Squared Bessel process and application to the study of the backward SLE_κ traces.

Fix $m > 0$. In this section, we analyze the backward Loewner flow during a m-macroscopic excursion from the origin of the (upper) Squared Bessel process on $\mathbb{R}$. The same analysis holds for the (lower) Squared Bessel process.

Let us consider the collection of m-macroscopic excursions of the solution to the real Squared Bessel SDE. Let us consider the backward Loewner differential equation from the origin, for time $t \in [0, \infty)$. For the Squared Bessel process which is obtained by extending the conformal maps on the real line, we have the following result (see Section 4.6 of [14]), for $0 < d < 2$

$$\int_0^\infty \mathbf{1}_{\{\tilde{Z}_t = 0\}} dt = 0.$$

Thus, the origin is instantaneously reflecting for this process for all the times $[0, \infty)$. Among these times, there exist a set of times which correspond to beginning of m-macroscopic excursions of the Squared Bessel process from the origin. In order to see this, we prove the following result:

$$\mathbb{P}(\text{there is an excursion of length at least } m) = 1.$$

Since the Bessel process for $d > 0$ is not identically zero, with positive probability there is at least one excursion interval with length $l > m$. The choice of this constant $m > 0$ is arbitrary and the modification of this constant does not change the result. Following the approach in [3], let us consider the event

$$\Lambda_t = \{\text{all the excursion intervals with right-end point d} < t \quad \text{have length } \ l \leqslant m\}.$$

If the first return to $\{0\}$ of the process (that is a stopping time) is infinite, then the Markov process has an excursion interval of length $l > m$. If this stopping time is finite, then we apply the Markov property and we get that $\mathbb{P}(\Lambda_{3t}) \leqslant \mathbb{P}(\Lambda_t)^2$ and by iteration we get that $\mathbb{P}(\Lambda_{3^n t}) \leqslant \mathbb{P}(\Lambda_t)^{2n}$ for $n \in \mathbb{N}$. Thus, we get that $\lim_{s\to\infty} \mathbb{P}(\Lambda_s) = 0$.

We further investigate the continuous extensions $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$, for times $t \in [0,\infty)$. Among these times, there will be times $r = r(\omega)$ (that depend on the path of the Brownian motion) that are beginning of m-macroscopic excursions of the real Squared Bessel process, started from the origin. Let us consider

$$\tilde{h}_r(z) := h_{s+r} \circ h_r^{-1}(z),$$

for $s \geqslant 0$. In particular, we obtain that the family of extended conformal maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$, where $\hat{B}_r := B_{t+r} - B_r$ for times in $[r(\omega), r(\omega) + m)$, map the origin to the real line. In order to see this, let us assume that the image of the origin under the maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$ is mapped in $\mathbb{H}$. Then, for any $\tilde{m} \in (0, m)$ we would have two images of the origin, for the extended conformal maps, one in $\mathbb{H}$ and the unique strong solution of the Squared Bessel SDE at time $r(\omega) + \tilde{m}$.

Using Lemma 2.2, we obtain that for all times $t \in [0,\infty)$, for almost every Brownian path, the maps $h_t(z)$ (and their functional inverses) are continuously extended to the boundary. This gives that the maps $\tilde{h}_r(z)$ are continuously extended to the boundary (since they are obtained from compositions of maps that are continuously extended to the boundary). Hence, since the maps $\tilde{h}_r(z)$ should map the origin to a unique point, we obtain a contradiction. Thus, the two images of the origin should coincide. Since the backward SLE_κ trace grows from the origin, we obtain that when starting the backward Loewner flow from the set of m-macroscopic excursions of the underlying Bessel, we obtain points of intersection of the trace with the real line. Using that a.s. there is an excursion interval of length at least $m > 0$ as shown before, we obtain that the closure of backward SLE_κ hulls are macroscopic hulls whenever we have m-macroscopic excursions of the Squared Bessel process. Moreover, since the Squared Bessel process is recurrent and the backward SLE_κ trace starts from the origin, we obtain also double points (that correspond to self touchings of the curve after at least $m > 0$ time).

Moreover, from topological considerations, we obtain at the end of a m-macroscopic excursion of the unique strong solution of the Squared Bessel process, a closed 'large bubble' of the backward SLE trace formed from the outer boundary of a portion of the curve, where $t(\omega)$ is the beginning of a large excursion that collects the self intersections of the backward SLE_κ trace with itself and with real line, that happened on $[t(\omega), t(\omega)+m)$. In contrast with the forward flow approach, this approach not only recovers the phase transition in the behavior of the SLE_κ traces, but also it provides structural information about the backward SLE_κ traces.

The same analysis holds also for the solution to the SDE (6.2). When driving them simultaneously with $B_t(\omega)$ and $-B_t(\omega)$ we obtain the complete picture of the evolution of the backward Loewner hulls in $\mathbb{C} \setminus [0, \infty)$.

6.3. Brownian motion and the behavior of the SLE_κ trace for $\kappa \in (4, \infty)$.

In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behavior of the backward SLE_κ trace? The previous analysis suggests that the classes of points along a Brownian driver that allow the process to escape the origin (and to further reflect at the origin), give the possibility of the non-simpleness of the backward SLE_κ traces for $\kappa > 4$. Compared with the case when the parameter $\kappa \in [0, 4]$, where the solution along the positive real line is stuck at the origin, in the case $\kappa \in (4, \infty)$ there is a unique notion of strong solution along the real line. Equivalently, when the parameter $\kappa \in [0, 4]$ is not large enough to allow the solution of the backward Bessel SDE to escape the origin, there is no solution along the real line.

In general, the change in behavior of the backward Bessel process started from the origin, gives also the following additional information: the collection of points that represent the beginning of macroscopic excursions of the real Bessel process of dimensions $d \in (0, 1)$ that is obtained from the extensions of the conformal maps to the real line, give also the formation of macroscopic bubbles for the backward SLE_κ hulls.

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