Research Project Proposal

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Remez Algorithm for best polynomial approximation in sup-norm of Brownian Motion

Brownian motion is a fundamental object that appears in various applications, whenever one is interested to model quantities that evolve in a stochastic manner. Examples include but are not restricted to Financial markets, Biological systems, Physical systems, etc.

The **Remez algorithm** is a fundamental tool to obtain the mini-max polynomial that optimally approximates in the strong sup-norm a continuous function. This type of algorithm appears in the Numerical Analysis course offered at NYU Shanghai, see [3].

The aim of this project is to the understand the law of the coefficients of the best approximation polynomial of fixed degree $n > 0$, of the standard one-dimensional Brownian motion. Then we aim at advancing the method originally developed by [1] for approximating brownian paths with random polynomials.

Given the ubiquitous nature of the Brownian motion process in practical problems as well as the need for optimal tools to approximate it, the project is expected to have a very broad impact for numerical methods of stochastic dynamical systems. Depending on the interest of the student, the project can be focused on the simulations of the coefficients and implementing statistical tests to understand the format of the laws of these, or can also be focused on the theoretical aspects of the problem. Matlab implementation should be an important part of the project.

Moreover, one can study a **a randomized version of Remez algorithm** with the goal to obtain faster the optimal approximation polynomial in the sup-norm to a given continuous function.

If time permits: Bringing tools of interest Machine Learning (ML) to Mathematical Physics problems

In Machine Learning, approximation of an intractable integration is often achieved by using the unbiased Monte Carlo estimator, but the variances of the estimation are generally very large. Control variates approaches are well-known to reduce the variance of the estimation. This method was successfully applied to specific diffusion processes ([2]) in order to obtain reduced-variance estimations of functionals of them. Diffusion processes appear very often in computations and reduced-variance estimations potentially will bring new insights in the theory.

In a different direction, we can explore efficient tools ([2]) to obtain optimal (low variance) estimates for approximations of SLE traces that are another universal object of interest in Mathematical Physics, beyond the typical square-root, linear and step functions approximations where the maps and curves are known in one curve case.

References

[1] James Foster, Terry Lyons, and Harald Oberhauser. An optimal polynomial approximation of Brownian motion. *SIAM Journal on Numerical Analysis*, 58(3):1393–1421, 2020.

[2] Josselin Garnier and Laurent Mertz. A control variate method driven by diffusion approximation. *Communications on Pure and Applied Mathematics*, 2019.

[3] Endre Suli and David F. Mayers. *An introduction to numerical analysis.*