On the perturbations of the multiple SLE simultaneous growth model

Jiaming Chen and Vlad Margarint

September 2021

Abstract

In this work we study multiple SLE_κ, for $\kappa \in (0,4]$, driven by Dyson Brownian motion. This model was introduced in the unit disk by Cardy [18] in connection with the Quantum Calogero-Sutherland model. We study using a version of Caratheodory convergence the perturbations on the initial value and diffusivity parameter $\kappa \in (0,4]$ for the case of $N = 2$ drivers. Our proofs use the analysis of Bessel processes and estimates on Loewner differential equation with multiple drivers. In the last section, we study the Hausdorff distance convergence under natural assumptions on the modulus of the derivative of the multiple SLE maps.

1 Introduction

The forward multiple Loewner chain encodes the dynamics of a family of conformal maps $g_t(z)$ defined on simply connected domains $\mathbb{H}\backslash K_t$ of the upper-half plane $\mathbb{H}$, where K_t are growing hulls ([5] *Sec.* 4.1.2) in the sense that $K_s \subset K_t$ for all $0 \leq s \leq t$. In this work we study a Loewner chain generated by $N \in \mathbb{N}$ continuous driving forces $\{\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)\}$ from $\mathbb{R}$ to $\mathbb{R}$. We denote these driving functions by $\lambda_j : [0,T] \to \mathbb{R}$, $j = 1, \dots, N$. We have

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_j(t)}, \tag{1.1}$$

with $g_0(z) = z$. This work is motivated by [18], where the author establishes a connection between the Quantum Calogero-Sutherland Model and the Multiple SLE with Dyson Brownian Motion driver. This model was studied in more detail recently in [43]. In this paper, we study further the multiple SLE with Dyson Brownian Motion as a driver. In order to define this object, we consider the Weyl chamber ([2] *Sec.* 4.) defined by

$$\mathcal{M}_N := \{\mathbf{x} \in \mathbb{R}^N;\ x_1 < x_2 < \cdots < x_N\}. \tag{1.2}$$

Throughout the paper we work with $(\Omega, \mathcal{F}_t, \mathbb{P})$ a standard probability space. Let $B_j(t)$, $j = 1, \ldots, N$ be one-dimensional standard independent Brownian motions. The Dyson Brownian notions with diffusivity parameter $\kappa \in (0, 4]$ are defined by a system of differential equations in the following

$$d\lambda_j(t) = \frac{1}{\sqrt{2}} dB_j(t) + \frac{2}{\kappa} \sum_{1 \le k \le N, k \neq j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \tag{1.3}$$

with $(\lambda_1(0), \ldots \lambda_N(0)) \in \mathcal{M}_N$, for all $t \in \mathbb{R}_+$ and $j = 1, \ldots, N$. In the Dyson Brownian Motion literature the natural parameter $\beta > 0$ is connected with the natural parameter $\kappa \in \mathbb{R}_+$ in SLE theory through $\beta = \frac{8}{\kappa}$. The Dyson Brownian Motion it is known (Theorem 12.2 in [33])it has a unique strong solution, that we use as a driver for our multiple SLE. An intuitive picture is that $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}$ describes an ensemble of diffusing particles ([3] *Rmk.* 2.4) in which particles repel each other via a Coulomb force.

It is known that when $\kappa \in (0, 4]$, no two Dyson Brownian particles will collide (*i.e.* touch $\partial\mathcal{M}_N$) almost surely. To be precise, denote by

$$\tau_N := \inf\{0 \le t \le T;\ \exists\, i, j \ s.t.\ |\lambda_i(t) - \lambda_j(t)| = 0\}. \tag{1.4}$$

Then $\tau_N = \infty$ almost surely as in ([16] *Prop.* 3.1). This result also justifies our choice of an arbitrary time interval $[0, T]$. Also, it is known ([1] *Thm.* 1.3) that when $\kappa \in (0, 4]$, the transformations $g_t(z)$ map a simply connected subset $\mathbb{H} \backslash K_t$ conformally onto the upper-half plane $\mathbb{H}$, where K_t consists of the image of N non-intersecting simple curves, that is $g_t : \mathbb{H} \setminus \cup_{j=1}^{n} \gamma_t^j \to \mathbb{H}$. Each curve corresponds to a driving force $\lambda_j(t)$, $j \in \mathbb{N}$. We focus on this case and throughout this article we assume $\kappa \in (0, 4]$.

The disk version of the multiple SLE with Dyson Brownian motion driver was introduced by Cardy in connection with the Quantum Calogero-Sutherland Model. In the last years, there many papers on both the disk and the upper-half plane version of this model (see [19], [20], [21], [22], [24], [25], [26], [27], [28], [29], [31], [32], [30] for a non-exhaustive list of papers where the model is studied in the upper-half plane, unit disk, either in the simultaneous growth case, or in the non-simultaneous growth case). There is also literature on the connection betweem Multiple SLE and Conformal Field Theory (CFT). We refer the reader to [20], and [23].

In this paper, we mainly focus on how the forward Loewner chain $g_t(z)$ behaves when the system is under different perturbations. In the following sections, we propose an estimate of such perturbations in the sense of Carathéodory convergence. Then we study via a version of Caratheodory convergence what happens when either the initial value of the driving forces $\lambda_j(t)'s$ or the diffusivity parameter κ are perturbed in the $N = 2$ case. The analysis in this paper can be thought as a first-step towards the general N case, and asymptotic $N \to \infty$. In the case $N \to \infty$ there are techniques that involve the study of local statistical properties (such as as the local study of the gaps between particles, and k-point correlations) of the Dyson Brownian motion developed in Random Matrix Theory (see [37], [36], [35], [34], [33]), for a non-exhaustive list). We plan to investigate this direction in future work. The paper is divided in several sections, the first one being the Introduction. In Section two, we introduce the Carathéodory convergence of Multiple Loewner Chains and obtain some preliminary estimates useful later on. In section three, we analyze two types of perturbations of the Multiple SLE. Firstly we study the perturbation in the initial value, and secondly we study the perturbation in the parameter $\kappa \in (0.4)]$. The first type perturbation can be though as an initial step in the study of the general N curves drivers with techniques that were developed for the proof of the Universality of certain random matrices ensembles in Random Matrix Theory. In that context, Dyson Brownian motion appears as a tool in the proof of the universality using the three-steps strategy. We refer the reader to[33] for a detailed exposition of these methods. The second type of perturbation, that is the perturbation in the parameter $\kappa \in (0, 4]$ is studied in a sequence of papers in the one-curve case (see [38], [39], [40], [41], and [42] for the continuity in the parameter $\kappa \leq 4$ of the welding homemorphism) in stronger topologies. In this paper, we study the perturbation in the parameter $\kappa \in (0, 4]$ in the Carathéodory sense. We plan to study the problem considering stronger topologies in the future. The final section is a variant of this analysis in which under a natural assumption on the derivative of the maps, we estimate the Hausdorff distance between the perturbation of the hulls under the multiple backward Loewner Differential Equation.

Acknowledgements We would like to kindly thank Henri Elad-Altman for his valuable comments on analyzing stochastic processes and to V. Healey for useful discussions and for looking over the previous versions of this manuscript. We also kindly thank A. Swan, and L. Schoug for helpful discussions. VM acknowledges the support of the NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai.

2 Carathéodory convergence of Loewner chains

Throughout this paper, we use $\|\cdot\|_{[0,T]}$ for the uniform norm on the interval $[0, T]$, and denote by $\|\cdot\|_{[0,T]\times G}$ the uniform norm on the product space $[0, T]\times G$, where $G \in \mathbb{H}$ is compact. Also, throughout the paper we consider the coupling of the Loewer chains in which both of the chains are driven by Dyson Brownian motion with the same Brownian motions.

In this section, we propose an estimate to the perturbation of forward Loewner chain $g_t(z)$ in the sense of Carathéodory convergence. The central idea is convergence on compact sets. This type of convergence is useful. For example in complex analysis, we know ([4] *Thm.* 10.28) that when a sequence of holomorphic functions Carathéodory converges to a limit function, then taking the limit preserves the holomorphicity, hence the limit function is holomorphic. This article follows the convention in ([5] *Sec.* 6.1.1).

One can study the Caratheodory convergence in the setting when G is a subset of $\mathbb{H}$. Indeed, the result in ([7] *Lem.* 3.1) tells us that height $K_t \leq 2\sqrt{t}$ for all $t \in [0, T]$. Hence we could simply subtract the box $\mathbb{R} \times [0, iT]$, which we denote by K_0, from $\mathbb{H}$. We know that $K_t \subset K_0$ for all $t \in [0, T]$ almost surely. And then one can restrict $g_t(z)$ to the simply connected domain $\mathbb{H}\backslash K_0$. This methodology gives a uniform bound to perturbation of forward Loewner chains. But if we want a Carathéodory estimate where compact sets G could run freely over a larger domains (depending on $\omega \in \Omega$), then the results must be discussed pathwisely.

Definition 2.1. *Denote by $D \subset \mathbb{H}$ a simply connected domain. Let $f_n(t, z)$: $[0, T] \times D \to \mathbb{H}$ be a sequence of conformal maps, and let $f(t, z) : [0, T] \times D \to \mathbb{H}$ be a conformal map. We say f_n converges in the Carathéodory sense to f, or $f_n \xRightarrow{Cara} f$, if for each compact $G \subset D$, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on $[0, T] \times G$.*

The estimate on the $g_t(z) : \mathbb{H}\backslash K_t \to \mathbb{H}$ corresponding to different inputs is based on Definition 3.1. We will give a proposition regarding estimating the difference between two forward Loewner chains. For notational convenience, we will write $g_t(z)$ as $g(t, z)$ from now on. Consider two forward Loewner chains $g_1(t, z)$ and $g_2(t, z)$ defined on $[0, T] \times \mathbb{H}\backslash K_t$, where K_t is the union of the hulls of $g_1(t, z)$ and $g_2(t, z)$, that is $K_t = \cup_{i=1}^n K_t^i$. Suppose they are generated respectively by N continuous driving forces $\{V_{k,1}(t), \ldots, V_{k,N}(t)\}$ with $k = 1, 2$, and restrict $g_k(t, z)$ to the domain $[0, T] \times K_T$. Then we have the following estimate.

Proposition 2.2. *For an arbitrary compact $G \subset \mathbb{H}\backslash K_T$, there exists a constant*

$C(T, G) > 0$ *such that we have*

$$\|g_1(t,z) - g_2(t,z)\|_{[0,T]\times G} \le C(T,G) \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]}. \tag{2.1}$$

Proof. It is by *Eqn.* (1.1) that we have the constraint

$$\partial_t g_k(t,z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_k(t,z) - V_{k,j}(t)}, \tag{2.2}$$

with $k = 1, 2$. Choose arbitrarily $z_1, z_2 \in G$. Let $\psi(t) := g_1(t, z_1) - g_2(t, z_2)$. And we have

$$\begin{aligned} \frac{d}{dt}\psi(t) &= \partial_t g_1(t,z_1) - \partial_t g_2(t,z_2) \\ &= \frac{1}{N} \sum_{j=1}^{N} \left(\frac{2}{g_1(t,z_1) - V_{1,j}(t)} - \frac{2}{g_2(t,z_2) - V_{2,j}(t)} \right) \\ &= \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) \Big(g_1(t,z_1) - V_{1,j}(t) - g_2(t,z_2) + V_{2,j}(t) \Big), \end{aligned} \tag{2.3}$$

where we define

$$\xi_j(t) := \frac{-2}{\big(g_1(t,z_1) - V_{1,j}(t)\big) \cdot \big(g_2(t,z_2) - V_{2,j}(t)\big)}, \tag{2.4}$$

for $j = 1, \ldots, N$. Additionally we define $D_j(t) := V_{1,j}(t) - V_{2,j}(t)$ for each j. Combined with *Eqn.* (2.3), then we have

$$\frac{d}{dt}\psi(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) \big(\psi(t) - D_j(t)\big). \tag{2.5}$$

At this moment, we observe that

$$\frac{d}{dt}\left(e^{-\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds} \cdot \psi(t) \right) = -\frac{1}{N} \sum_{j=1}^{N} \xi_j(t) D_j(t) \cdot e^{-\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}, \tag{2.6}$$

and consequently

$$\psi(t) = e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds} \cdot \psi(0) - \frac{1}{N} \sum_{j=1}^{N} \int_0^t du \cdot \xi_j(u) D_j(u) \cdot e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^u \xi_j(s)ds}. \tag{2.7}$$

On the other hand, we have the following inequality

$$\left| e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^t \xi_j(s)ds} \right| \leq e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^t |\xi_j(s)|ds}. \tag{2.8}$$

Then, we know that

$$\begin{aligned}
&\left| \frac{1}{N}\sum_{j=1}^{N}\int_0^t du \cdot \xi_j(u) D_j(u) \cdot e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^t \xi_j(s)ds} \right| \\
&\qquad \leq \frac{1}{N}\sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \cdot \int_0^t du \cdot |\xi_j(u)| e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^u |\xi_j(s)|ds} \\
&\qquad \leq \left(\sum_{j=1}^{N} \|D_j(t)\|_{[0,T]}\right) \cdot \int_0^t du \cdot \frac{1}{N}\sum_{j=1}^{N} |\xi_j(u)| \cdot e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^u |\xi_j(s)|ds} \\
&\qquad = \left(\sum_{j=1}^{N} \|D_j(t)\|_{[0,T]}\right) \cdot \left(e^{\frac{1}{N}\sum_{j=1}^{N}\int_0^t |\xi_j(s)|ds} - 1\right).
\end{aligned} \tag{2.9}$$

Moreover, by the Cauchy-Schwartz inequality, we have

$$\frac{1}{N}\sum_{j=1}^{N}\int_0^t |\xi_j(s)|ds \leq \frac{1}{N}\sum_{j=1}^{N}\sqrt{I_{1,j} \cdot I_{2,j}}, \tag{2.10}$$

where we define $I_{k,j}$ for $k = 1, 2$ and $j = 1, \ldots, N$ in the following

$$I_{k,j} := \int_0^t \frac{2}{|g_k(s, z_k) - V_{k,j}(s)|^2} ds = \log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(t, z_k)}, \tag{2.11}$$

by ([5] *Sec.* 4.2.2). In fact, with the compact $G \subset \mathbb{H}\backslash K_T$, there exists $\delta_1(G) > 0$ such that $\operatorname{Im} g_k(T, z) \geq \delta_1(G)$ for all $z \in G$, $k = 1, 2$. Hence, we have

$$I_{k,j} \leq \log \frac{\operatorname{Im} z_k}{\max\left\{\delta_1(G), \sqrt{\left((\operatorname{Im} z_k)^2 - 4t\right)^+}\right\}}, \tag{2.12}$$

where $x^+ = \max\{x, 0\}$. Since $t \in [0, T]$ and $z_1, z_2 \in G$ where G is compact in $\mathbb{H}\backslash K_0$, we could choose $\delta_2(G) := \operatorname{dist}(G, \mathbb{R}) > 0$ and define

$$C(T, G) := \frac{\delta_2(G)}{\max\left\{\delta_1(G), \sqrt{\left(\delta_2(G)^2 - 4t\right)^+}\right\}}. \tag{2.13}$$

Here we have $I_{k,j} \leq \log C(T,G)$ for all k and j. Hence, we know that

$$|\psi(t)| \leq e^{\frac{1}{N}\sum\limits_{j=1}^{N} \log C(T,G)} \cdot |\psi(0)| + \left(\sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \right) \cdot \big(C(T,G) - 1\big). \quad (2.14)$$

Therefore, we conclude

$$|g_1(t,z_1) - g_2(t,z_2)| \leq C(T,G) \cdot \big(\sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]} + |z_1 - z_2| \big), \quad (2.15)$$

for all $t \in [0,T]$ and $z_1, z_2 \in G$. Now choose $z_1 = z_2 = z$ and take supremum over the left side, we arrive at our final result

$$\|g_1(t,z) - g_2(t,z)\|_{[0,T]\times G} \leq C(T,G) \cdot \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]}. \quad (2.16)$$

□

Remark 2.3. *With slight changes, the above argument can be adapted to the multiple backward Loewner maps, and a similar Carathéodory estimate ([5] Lem.* 6.1*) for the deterministic curves can be obtained.*

3 Perturbations

We are particular interested in the forward Loewner map driven by Dyson Brownian motions. In this section, we restrict our attention to the $N = 2$ case. We plan to study the general N-curve case in future works. When $N = 2$, we have two driving forces $\{\lambda_1(t), \lambda_2(t)\}$ that are interacting diffusions modelled by Dyson Brownian motion. Their evolution is described in the following equation

$$\begin{aligned} d\lambda_1(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_1(t) - \lambda_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\lambda_2(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_2(t) - \lambda_1(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \quad (3.1)$$

with $\lambda_1(0) = a_1,\ \lambda_2(0) = a_2,\ a_1 > a_2$, where $B_1(t)$ and $B_2(t)$ are independent one-dimensional Brownian motions. Here we consider only the phase $\kappa \in (0,4]$. In this case, these two particles $\lambda_1(t)$ and $\lambda_2(t)$ never collide on $\mathbb{R}$. In other words, the stopping time defined in *Eqn.* (1.4) satisfies $\tau_2 = \infty$ almost surely as in ([6] *Prop.* 1.) because $d = 1 + \frac{8}{\kappa} \geq 3$ for $\kappa \geq 4$.

Let $X_t \coloneqq \lambda_1(t) - \lambda_2(t)$. Based on the above observations, we know $X_t > 0$

for all $t \in [0,T]$ almost surely. We further observe that

$$dX_t = \frac{4}{\kappa} \cdot \frac{dt}{X_t} + dW_t, \tag{3.2}$$

with $X_0 = a_1 - a_2$ and $W_t := \frac{1}{\sqrt{2}}\big(B_1(t) - B_2(t)\big)$ is a Wiener process. Choose $d = 1 + \frac{8}{\kappa}$, then X_t admits the canonical form of n-dimensional Bessel process with

$$dX_t = \frac{d-1}{2} \cdot \frac{dt}{X_t} + dW_t. \tag{3.3}$$

In this section we discuss two types of perturbations. The first type of perturbation is varying the initial value of driving forces. The second type is varying the diffusivity parameter $\kappa \in (0,4]$. The study in both cases involves the analysis of transient Bessel processes with dimension $d \geq 3$.

3.1 Perturbation of the initial value

The first type of perturbation is to slightly change the initial value of $\lambda_k(0)$ for $k = 1,2$. With the initial value under perturbation, we get a different set of Dyson Brownian motions. Our goal is to estimate the difference of the forward Loewner chains driven by these varying forces. Dyson Brownian motions with different initial conditions appear naturally in the context of the study of the universality of certain random matrices ensembles, as mentiond in the Introduction. The analysis presented in this section can be thought as a first step in this direction, as the analysis in the case of general N Dyson particles, and random intiial conditions is much more involved. We plan to do this in future work.

To be precise, choose $0 < \epsilon < \frac{1}{3}(a_1 - a_2)$ and select b_k in the ϵ-ball of a_k for $k = 1,2$ to be the perturbed initial value of the two Dyson Brownian motions. It is obvious then $b_1 > b_2$ and we arrive at another set of perturbed Dyson Brownian motions $\{\eta_1(t), \eta_2(t)\}$ with

$$\begin{aligned} d\eta_1(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\eta_1(t) - \eta_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\eta_2(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\eta_2(t) - \eta_1(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \tag{3.4}$$

with $\eta_1(0) = b_1$ and $\eta_2(0) = b_2$. Notice that the process $\eta_k(t)$ is still driven by the same Brownian motion $B_k(t)$, because we consider perturbation only on the initial value.

In this two-force case, we denote by $g_\lambda(t,z)$ the original forward Loewner chain generated by forces $\{\lambda_1(t), \lambda_2(t)\}$ and by $g_\eta(t,z)$ the perturbed forward

Loewner chain generated by forces $\{\eta_1(t), \eta_2(t)\}$. Hence, we have

$$\begin{aligned}\partial_t g_\lambda(t,z) &= \frac{1}{g_\lambda(t,z)-\lambda_1(t)} + \frac{1}{g_\lambda(t,z)-\lambda_2(t)},\\ \partial_t g_\eta(t,z) &= \frac{1}{g_\eta(t,z)-\eta_1(t)} + \frac{1}{g_\eta(t,z)-\eta_2(t)},\end{aligned} \tag{3.5}$$

with $g_{\lambda/\eta}(0,z) = z$ for all $z \in \mathbb{H}$. We continue using $X_t = \lambda_1(t) - \lambda_2(t)$ to denote the gap between two interacting Brownian forces $\lambda_k(t),\ k = 1,2$. As shown in *Eqn.* (3.2), X_t is a Bessel process with dimension $1 + \frac{8}{k}$ and initial value $X_0 = a_1 - a_2$. Denote by $Y_t \coloneqq \eta_1(t) - \eta_2(t)$ the gap between $\eta_k(t),\ k = 1,2$. Then Y_t is a Bessel process with the same dimension $d = 1 + \frac{8}{\kappa}$ and satisfies

$$dY_t = \frac{4}{\kappa} \cdot \frac{dt}{Y_t} + dW_t, \tag{3.6}$$

with $Y_0 = b_1 - b_2$. Observe the Bessel processes X_t and Y_t are driven by the same Wiener process W_t. Hence their difference $X_t - Y_t$ satisfies

$$d(X_t - Y_t) = -\frac{4}{\kappa} \cdot \frac{X_t - Y_t}{X_t Y_t} dt. \tag{3.7}$$

Denote by $a \coloneqq a_1 - a_2$ and $b \coloneqq b_1 - b_2$. Integrate both sides on *Eqn.* (3.7) and we see that

$$X_t - Y_t = (a-b) \cdot e^{-\frac{4}{\kappa}\int_0^t \frac{1}{X_s Y_s} ds}. \tag{3.8}$$

Notice that we cannot ascertain $X_t - Y_t$ to be whether deterministic at this moment. In fact, the term $\frac{1}{X_t Y_t}$ might evolve stochastically. Still, for $k = 1,2$, we could observe that

$$\begin{aligned} d\lambda_k(t) - d\eta_k(t) &= \frac{2}{\kappa}\left(\frac{1}{\lambda_k(t) - \lambda_{3-k}(t)} - \frac{1}{\eta_k(t) - \eta_{3-k}(t)}\right) dt \\ &= (-1)^k \frac{2}{\kappa} \cdot \frac{X_t - Y_t}{\big(\lambda_k(t) - \lambda_{3-k}(t)\big) \cdot \big(\eta_k(t) - \eta_{3-k}(t)\big)} dt. \end{aligned} \tag{3.9}$$

Hence, we have

$$d\lambda_k(t) - d\eta_k(t) = (-1)^k (a-b) \frac{2}{\kappa} \cdot e^{-\frac{4}{\kappa}\int_0^t \frac{1}{X_s Y_s} ds} \cdot \frac{1}{X_t Y_t} dt, \tag{3.10}$$

for $k = 1,2$. The above equation admits an integral form

$$\begin{aligned} \lambda_k(t) - \eta_k(t) &= a_k - b_k + (-1)^k (a-b) \frac{2}{\kappa} \int_0^t e^{-\frac{4}{\kappa}\int_0^s \frac{1}{X_u Y_u} du} \cdot \frac{1}{X_s Y_s} ds \\ &= a_k - b_k + \frac{1}{2}(-1)^{3-k}(a-b)\left(e^{-\frac{4}{\kappa}\int_0^t \frac{1}{X_s Y_s} ds} - 1\right). \end{aligned} \tag{3.11}$$

At this point, we have an explicit form to $\lambda_k(t) - \eta_k(t)$. Looking back to Proposition 2.3, we naturally want to have an estimate to $g_\lambda(t,z) - g_\eta(t,z)$ in the Carathéodory sense.

Proposition 3.1. *For all $0 < \epsilon < \frac{a}{3}$, let $H_T = \mathbb{H}\backslash K_T(\omega)$, where $K_T(\omega) = \cup_{j=1}^{2} K_T^j$. Choose $b_k \in \mathbb{R}$ with $|a_k - b_k| < \epsilon$ for $k = 1, 2$. Let $g_\lambda(z)$ and $g_\eta(z)$ be two multiple Loewner chains induced by Dyson Brownian motions $\{\lambda_1(t), \lambda_2(t)\}$ and $\{\eta_1(t), \eta_2(t)\}$, respectively. Suppose $\lambda_k(0) = a_k$ and $\eta_k(0) = b_k$ for $k = 1, 2$. Then almost surely we have*

$$\|g_\lambda(t,z) - g_\eta(t,z)\|_{[0,T]\times G} < 4C(T,G)\cdot\epsilon, \ \forall \underset{compact}{G \subseteq H_T}. \tag{3.12}$$

Proof. At this moment, we already know $X_t, Y_t > 0$ for all $t \in [0,T]$ almost surely. Inspect *Eqn.* (3.11), we know for $k = 1, 2$ that

$$|\lambda_k(t) - \eta_k(t)| \leq |a_k - b_k| + \frac{1}{2}|a-b|\cdot\left(1 - e^{-\frac{4}{\kappa}\int_0^T \frac{1}{X_t Y_t}dt}\right) < 2\epsilon. \tag{3.13}$$

By Proposition 2.3, we know that

$$\begin{aligned}\|g_\lambda(t,z) - g_\eta(t,z)\|_{[0,T]\times G} &\leq C(T,G)\cdot\sum_{k=1}^{2}\|\lambda_k(t) - \eta_k(t)\|_{[0,T]} \\ &< 4C(T,G)\cdot\epsilon.\end{aligned} \tag{3.14}$$

And the proposition is verified. $\square$

So far we have estimated $g_\lambda(t,z) - g_\eta(t,z)$ in the Carathéodory sense under a perturbation of initial value of driving forces. In practice, when we compute a multiple forward Loewner chain driven by Dyson Brownian motions, we could approximate its initial value and it turns out the approximated Loewner chains converge in a version of the Carathéodory sense. Indeed, we have the following result.

Corollary 3.2. *Suppose $g_t(z): \mathbb{H}\backslash K_T(\omega) \to \mathbb{H}$, where $K_T(\omega) = \cup_{j=1}^{n} K_T^j(\omega)$ is a forward Loewner chain generated by two Dyson Brownian motions $\{\lambda_1(t), \lambda_2(t)\}$ with initial value $\lambda_1(0) > \lambda_2(0)$. Suppose there is a sequence of forward Loewner chains $g_t^n(z): \mathbb{H}\backslash K_T(\omega) \to \mathbb{H}$, generated by Dyson Brownian motions $\{\lambda_1^n(t), \lambda_2^n(t)\}$ with $\lambda_1^n(0) > \lambda_2^n(0)$ and approaching initial value $\lambda_k^n(0) \xrightarrow{n} \lambda_k(0)$. Then we have*

$$\mathbb{P}\big(g_T^n(z) \xrightarrow{Cara} g_T(z)\big) = 1. \tag{3.15}$$

3.2 Perturbation of the diffusivity parameter $\kappa \in (0, 4]$

The second type of perturbation is with respect to the diffusivity parameter $\kappa \in (0.4]$.This type of perturbation is a natural problem and was considered extensively in the one-curve case where many results have been proved in the recent years in various topologies, as mentioned in the Introductiom. Remember that we have always chosen $\kappa \in (0, 4]$ so that there is no phase transition ([8] *Sec.* 3.) corresponding to the $(1 + \frac{8}{\kappa})$-dimensional Bessel process. When there is perturbation, κ is varied and we have a new diffusivity parameter $\kappa^* \in (0, 4]$ such that $\kappa^* \neq \kappa$. The difference in parameter results in different Dyson Brownian motions, and therefore different forward Loewner chains.

To simplify the model, we assume $\kappa^* > \kappa$ without loss of generality. Denote by $\{\lambda_1(t), \lambda_2(t)\}$ the original Dyson Brownian motions. Their dynamics is described in *Eqn.* (3.1) with initial value $\lambda_1(0) = a_1,\ \lambda_2(0) = a_2,\ a_1 > a_2$. Denote by $\{\lambda_1^*(t), \lambda_2^*(t)\}$ the perturbed Dyson Brownian motions. They respect the following equations

$$\begin{aligned} d\lambda_1^*(t) &= \frac{2}{\kappa^*} \cdot \frac{dt}{\lambda_1^*(t) - \lambda_2^*(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\lambda_2^*(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_2^*(t) - \lambda_1^*(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \tag{3.16}$$

with initial value $\lambda_k^*(0) = \lambda_k(0) = a_k,\ k = 1, 2$. Let $K_T(\omega) = \cup_{j=1}^2 K_T^j(\omega)$, with $j = 1$ corresponding to the parameter $\kappa \in (0, 4]$ and $j = 2$ corresponding to the parameter $\kappa^* \in (0, 4]$. We have $g(t, z) : [0, T] \times \mathbb{H} \backslash K_T(\omega) \to \mathbb{H}$ the original Loewner chain generated by forces $\{\lambda_1(t), \lambda_2(t)\}$. And we denote by $g^*(t, z) : [0, T] \times \mathbb{H} \backslash K_T(\omega) \to \mathbb{H}$ the perturbed Loewner chain generated by $\{\lambda_1^*(t), \lambda_2^*(t)\}$.

The evolution respects

$$\partial_t g^*(t, z) = \frac{1}{g^*(t, z) - \lambda_1^*(t)} + \frac{1}{g^*(t, z) - \lambda_2^*(t)}, \tag{3.17}$$

with $g^*(0, z) = z$ for all $z \in \mathbb{H} \backslash K_T(\omega)$. Denote by X_t the gap between $\lambda_1(t)$ and $\lambda_2(t)$. Then X_t is a $(1 + \frac{8}{\kappa})$-dimensional Bessel process with initial value $X_0 = a$. Its evolution is described in *Eqn.* (3.2). At the same time, let $X_t^* := \lambda_1^*(t) - \lambda_2^*(t)$ the gap of the two perturbed driving forces. The the gap respects the following equations

$$dX_t^* = \frac{4}{\kappa^*} \cdot \frac{dt}{X_t^*} + dW_t, \tag{3.18}$$

with $X_0^* = X_0 = a$ and where W_t is the Wiener process defined in *Eqn.* (3.2).

Notice here X_t^* is a $(1+\frac{8}{\kappa^*})$-dimensional Bessel process.

Our main goal is to give an probabilistic estimate of $g(t,z)-g^*(t,z)$ in the Carathéodory sense. Following Proposition 2.3, we need first estimate the sup-norm of $\lambda_k(t)-\lambda_k^*(t)$ for $k=1,2$. Indeed, define the indices of the Bessel processes by $\nu := \frac{4}{\kappa}-\frac{1}{2}$, $\nu^* := \frac{4}{\kappa^*}-\frac{1}{2}$. Before proving Proposition 2.3, we have the following lemma. Elements of this lemma were kindly provided by H. Elad-Altman in a private communication.

Lemma 3.3. *Given a $(1+\frac{8}{\kappa^*})$-dimensional Bessel process X_t^* and a $(1+\frac{8}{\kappa})$-dimensional Bessel process X_t with $4\geq\kappa^*>\kappa>0$ and the same initial value $X_0^*=X_0=a>0$, we have almost surely that*

$$\sup_{0\leq s\leq t}(X_s^*-X_s)^2\leq\frac{4t}{\kappa^2}(\kappa^*-\kappa). \tag{3.19}$$

Proof. Observe *Eqn.* (3.2) and *Eqn.* (3.17), we see

$$X_t^*-X_t=\frac{4}{\kappa^*}\int_0^t\frac{ds}{X_t^*}-\frac{4}{\kappa}\int_0^t\frac{ds}{X_t}. \tag{3.20}$$

Using Itô's lemma, we have

$$\begin{aligned} d(X_t^*-X_t)^2 &= 2(X_t^*-X_t)\cdot\left(\frac{4}{\kappa^*X_t^*}-\frac{4}{\kappa X_t}\right)dt \\ &= \frac{4}{\kappa^*\kappa}(\kappa-\kappa^*)\cdot\frac{X_t^*-X_t}{X_t^*}dt+\frac{8}{\kappa}(X_t^*-X_t)\cdot(\frac{1}{X_t^*}-\frac{1}{X_t})dt. \end{aligned} \tag{3.21}$$

At the same time, we have that $(X_t^*-X_t)\cdot(\frac{1}{X_t^*}-\frac{1}{X_t})\leq 0$ for all $t\in[0,T]$ almost surely. Integrating both sides and we obtain

$$(X_t^*-X_t)^2\leq(\kappa-\kappa^*)\frac{4}{\kappa^*\kappa}\cdot\int_0^t\frac{(X_s^*-X_s)_+}{X_s^*}ds. \tag{3.22}$$

On the other hand, it is obvious that $(X_s^*-X_s)_+\leq X_s^*$. By considering $\kappa^*\leq\kappa$, we have the conclusion

$$\sup_{0\leq s\leq t}(X_s^*-X_s)^2\leq\frac{4t}{\kappa^2}(\kappa-\kappa^*). \tag{3.23}$$

$\square$

We denote by $S_t=\sup_{0\leq s\leq t}W_t$ the supremum Brownian motion. We are ready to state main result.

Proposition 3.4. *Let $g(t,z)$ and $g^*(t,z)$ be two multiple Loewner chains for the* *parameters $\kappa, \kappa^* \in (0,4]$, resp. Choose arbitrary compacts $G \subset H_T = \mathbb{H}\backslash K_T(\omega)$, where $K_T(\omega) = \cup_{j=1}^2 K_T^j(\omega)$, with $K_T^1(\omega)$ corresponding to the parameter $\kappa \in (0,4]$ and K_T^2 to the parameter $\kappa* \in (0,4]$. There exist $\alpha_1, \alpha_2, \alpha_3 > 0$ depending on (T, G, α, κ) such that if we further define*

$$\begin{aligned} \varphi(x) &:= \alpha_1 x^{1/8} + \alpha_2 x^{1/4} + \alpha_3 x^{7/8}, \\ \zeta(x) &:= 2\frac{x^{\nu/8}}{a^{2\nu}} + 2x^{3/4} e^{-1\big/2x^{3/2}}, \end{aligned} \tag{3.24}$$

for all $x \in \mathbb{R}_+$. Then $\lim\limits_{x\to 0^+} \varphi(x) = 0$, $\lim\limits_{x\to 0^+} \zeta(x) = 0$ almost surely and we have

$$\mathbb{P}\bigg(\|g(t,z) - g^*(t,z)\|_{[0,T]\times G} > \varphi(\kappa^* - \kappa),\ \forall \underset{compact}{G \subseteq H_T} \bigg) < \zeta(\kappa^* - \kappa). \tag{3.25}$$

Proof. From *Eqn.* (3.1) and *Eqn.* (3.15), we see for $k = 1, 2$ that

$$\begin{aligned} d\lambda_k(t) - d\lambda_k^*(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_k(t) - \lambda_{3-k}(t)} - \frac{2}{\kappa^*} \cdot \frac{dt}{\lambda_k^*(t) - \lambda_{3-k}^*(t)} \\ &= (-1)^{3-k} \frac{2}{\kappa^* \kappa} \cdot \frac{\kappa^* X_t^* - \kappa X_t}{X_t^* X_t} dt. \end{aligned} \tag{3.26}$$

To obtain an expression of $\lambda_k(t) - \lambda_k^*(t)$, we need to express the process $\kappa^* X_t^* - \kappa X_t$. Indeed, we have

$$\kappa^* dX_t^* - \kappa dX_t = 4\bigg(\frac{1}{X_t^*} - \frac{1}{X_t}\bigg)dt + (\kappa^* - \kappa)dW_t. \tag{3.27}$$

Integrate both sides, we write

$$\kappa^* X_t^* - \kappa X_t = (\kappa^* - \kappa)\cdot a + 4\int_0^t \frac{X_s - X_s^*}{X_s^* X_s} ds + (\kappa^* - \kappa) W_t. \tag{3.28}$$

On the other hand, inspecting the above equation, we see another term $X_t^* - X_t$ appears in the integrand. Based on Lemma 3.2, we have

$$\sup_{0\le s\le t} |X_s^* - X_s| \le \frac{2\sqrt{t}}{\kappa} (\kappa^* - \kappa)^{1/2}. \tag{3.29}$$

At this moment, we have obtained an explicit form of $\kappa^* X_t^* - \kappa X_t$, which is contained in the expression of $d\lambda_k(t) - d\lambda_k^*(t)$. Define $M_t := \inf\limits_{0\le s\le t} X_s$ as the running infimum of the Bessel process X_t. The running infimum M_t^* of X_t^* is similarly defined. We further denote by $M_\infty = \lim\limits_{t\to\infty} M_t$ the infimum of X_t. And similarly, we denote by $M_\infty^* = \lim\limits_{t\to\infty} M_t^*$ the infimum of X_t^*. Indeed, from ([9]

Eqn. 2.1) we know that

$$\begin{aligned}\mathbb{P}(M_\infty < y) &= \frac{y^{2\nu}}{a^{2\nu}} \cdot \mathbb{1}_{y\in[0,a]},\\ \mathbb{P}(M_\infty^* < y) &= \frac{y^{2\nu^*}}{a^{2\nu^*}} \cdot \mathbb{1}_{y\in[0,a]}.\end{aligned} \tag{3.30}$$

Combining *Eqn.* (3.20) and *Eqn.* (3.26), then

$$\begin{aligned}|\lambda_k(t) - \lambda_k^*(t)| \le (\kappa^* - \kappa)\frac{2a}{\kappa^*\kappa} \cdot \frac{t}{M_t^* M_t} + (\kappa^* - \kappa)^{\frac{1}{2}}\frac{16}{\kappa^*\kappa^2} \cdot \frac{t^{\frac{5}{2}}}{(M_t^* M_t)^2}\\ + (\kappa^* - \kappa)\frac{2}{\kappa^*\kappa} \cdot \frac{t}{M_t^* M_t}\sup_{0\le s\le t}|W_s|.\end{aligned} \tag{3.31}$$

Considering $\kappa^* > \kappa$, it is then obvious that

$$\begin{aligned}\sup_{t\in[0,T]}|\lambda_k(t) - \lambda_k^*(t)| \le (\kappa^* - \kappa)\frac{2a}{\kappa^2} \cdot \frac{T}{M_\infty^* M_\infty} + (\kappa^* - \kappa)^{\frac{1}{2}}\frac{16}{\kappa^3} \cdot \frac{T^{\frac{5}{2}}}{(M_\infty^* M_\infty)^2}\\ + (\kappa^* - \kappa)\frac{2}{\kappa^2} \cdot \frac{T}{M_\infty^* M_\infty}\sup_{0\le s\le T}|W_s|\end{aligned} \tag{3.32}$$

Based on *Eqn.* (3.27), define the following events

$$\begin{aligned}E_1 &:= \left\{M_\infty \ge (\kappa^* - \kappa)^{\frac{1}{16}}\right\},\\ E_2 &:= \left\{M_\infty^* \ge (\kappa^* - \kappa)^{\frac{1}{16}}\right\},\\ E_3 &:= \left\{\sup_{0\le s\le T}|W_s| \le \frac{1}{(\kappa^* - \kappa)^{\frac{3}{4}}}\right\}.\end{aligned} \tag{3.33}$$

Since $\kappa^* > \kappa$ and by *Eqn.* (3.27), we know that

$$\begin{aligned}\mathbb{P}\big(E_1\big) &= 1 - \frac{(\kappa^* - \kappa)^{\frac{\nu}{8}}}{a^{2\nu}},\\ \mathbb{P}\big(E_2\big) &= 1 - \frac{(\kappa^* - \kappa)^{\frac{\nu^*}{8}}}{a^{2\nu^*}}.\end{aligned} \tag{3.34}$$

From ([10], *Cor.* 2.2), we know the supremum Brownian motion S_t admits the distribution

$$\mathbb{P}\bigg(S_t \le x\bigg) = 2\Phi\bigg(\frac{x}{\sqrt{t}}\bigg) - 1, \tag{3.35}$$

for all $x \ge 0$ and where $\frac{d}{dx}\Phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ is the density of standard normal variable. It follows from the reflection principle that

$$1 - \mathbb{P}\big(E_3\big) = 2\mathbb{P}\big(S_T \ge (\kappa^* - \kappa)^{-\frac{1}{2}}\big) \le 2\sqrt{\frac{2}{\pi}}(\kappa^* - \kappa)^{\frac{3}{4}} \cdot e^{-\frac{1}{2(\kappa^* - \kappa)^{3/2}}}. \tag{3.36}$$

Choose $\alpha_1 := C(T,G)\frac{4T}{\kappa^2}$, $\alpha_2 := C(T,G)\frac{32T^{\frac{5}{2}}}{\kappa^3}$, $\alpha_3 := C(T,G)\frac{4Ta}{\kappa^2}$. It follows from Proposition 2.3 and *Eqn.* (3.32) that on the event $E_1 \cap E_2 \cap E_3 \subset \Omega$, we have the estimate

$$\begin{aligned} \|g(t,z) - g^*(t,z)\|_{[0,T]\times G} &\leq C(T,G) \sum_{k=1}^{2} \|\lambda_k(t) - \lambda_k^*(t)\|_{[0,T]} \\ &\leq \alpha_1(\kappa^* - \kappa)^{1/8} + \alpha_2(\kappa^* - \kappa)^{1/4} + \alpha_3(\kappa^* - \kappa)^{7/8}. \end{aligned} \tag{3.37}$$

On the other hand, from *Eqn.* (3.33) and *Eqn.* (3.35) we have

$$\mathbb{P}\big(E_1 \cap E_2 \cap E_3\big) \geq 1 - 2\frac{(\kappa^* - \kappa)^{\frac{\nu}{8}}}{a^{2\nu}} - 2(\kappa^* - \kappa)^{\frac{3}{4}} \cdot e^{-\frac{1}{2(\kappa^* - \kappa)^{3/2}}}, \tag{3.38}$$

where $a = a_1 - a_2 > 0$. Hence, the result is verified. $\square$

Corollary 3.5. *Suppose there is a sequence of forward Loewner chains* $g_t^n(z) : \mathbb{H}\backslash K_T(\omega) \to \mathbb{H}$ *generated by Dyson Brownian motions* $\{\lambda_1^n(t), \lambda_2^n(t)\}$ *with parameter* κ_n *such that* $\lim_{n\to\infty} \kappa_n = \kappa \in (0,4]$, *and let* $K_T(\omega) = \cup_{j=1}^n K_T^n(\omega)$. *Suppose* $g_t(z) : \mathbb{H}\backslash K_T(\omega) \to \mathbb{H}$ *is a forward Loewner chain generated by two Dyson Brownian motions* $\{\lambda_1(t), \lambda_2(t)\}$ *with difusivity parameter* $\kappa \in (0,4]$.

Then, for all $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that with* $n > N$, *we have*

$$\mathbb{P}\big(g_t^n(z) \xrightarrow{Cara} g_t(z)\big) \geq 1 - \epsilon. \tag{3.39}$$

4 Variant estimate on the Hausdorff distance

In this section, we analyze the Hausdorff convergence under assumptions on the behaviour of the derivative of the map. The method follows and extends the one curve strategy from [14]. First, we define the notion of Hausdorff distance. For any two compacts $A, B \subset \mathbb{C}$, define the Hausdorff metric ([12] *Sec.* 6.1) by

$$d_H(A,B) := \inf\{\epsilon > 0;\, A \subset \bigcup_{z\in B} \mathcal{B}(z,\epsilon),\, B \subset \bigcup_{z\in A} \mathcal{B}(z,\epsilon)\}, \tag{4.1}$$

where $\mathcal{B}(z,\epsilon)$ is the ϵ-ball centered at $z \in \mathbb{C}$. In this section, we prove a variant pathwise perturbation estimate in Hausdorff distance of the hulls K_t generated by the forward Loewner flow. Following [1], we have that for $\kappa \in (0,4]$, the multiple SLE curves are a.s. simple and non-intersecting. This serves as a motivation to understand the Hausdorff distance convergence following the analysis of the Caratheodory type convergence. In general, Hausdorff distance convergence is stronger than the Caratheodory convergence, however we are a.s. in the case of simple non-intersecting curves.

Going back to the N-curve case. First consider a forward Loewner chain $g_t(z) : \mathbb{H}\backslash K_t \to \mathbb{H}$ driven by forces $t \mapsto \big(\lambda_1(t), \dots, \lambda_N(t)\big)$. Denote the inverse map corresponding to $g_t(z)$ by $f_t(z) : \mathbb{H} \to \mathbb{H}\backslash K_t$, with $g_t\big(f_t(z)\big) = z$, for all $z \in \mathbb{H}$. On the other hand, consider the time-reversed forward Loewner chain generated by the time-reversed forces $t \mapsto \big(\lambda_1(T-t), \dots, \lambda_N(T-t)\big)$. Denote this forward Loewner chain by $h_t(z)$ for $z \in \mathbb{H}$. Then it satisfies

$$\partial_t h_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{-2}{h_t(z) - \lambda_j(T-t)}, \tag{4.2}$$

with $h_0(z) = z$ for all $z \in \mathbb{H}$. Similar to the $N = 1$ case as in ([13] *Sec.* 2.), we could verify that $f_T(z) = h_T(z)$ for all $z \in \mathbb{H}$. When the system is under perturbation, we need to compare a Loewner chain with its perturbed counterpart. Indeed, denote by $f_k(t,z)$ and $g_k(t,z)$ the Loewner chains driven by $\{V_{k,1}(t), \dots, V_{k,N}(t)\}$ for $k = 1, 2$. Denote by $h_k(t,z)$ the backward Loewner chains driven by $\{V_{k,1}(T-t), \dots, V_{k,N}(T-t)\}$ for $k = 1, 2$. The following lemma estimates pathwisely the backward Loewner chain.

Lemma 4.1. *For all $\delta > 0$, there exists a constant $C(\delta, T) = \sqrt{1 + 4T/\delta^2}$ such that whenever $\operatorname{Im} z \geq \delta$, we have*

$$|h_1(T,z) - h_2(T,z)| \leq C(\delta, T) \sum_{j=1}^{N} \|V_{1,j}(T-t) - V_{2,j}(T-t)\|_{[0,T]}. \tag{4.3}$$

Proof. The proof is similar to Proposition 2.3. Take $I_{k,j} = \log \frac{\operatorname{Im} h_k(t,z)}{\operatorname{Im} z}$. □

We also need the following Koebe distortion theorem, see ([14] *Lem.* 2.1)

Lemma 4.2. *Let D be a simply connected domain and assume $f : D \to \mathbb{C}$ is conformal map. Let $d = dist(z, \partial D)$ for $z \in D$. If $|z - w| \leq rd$ for some $0 < r < 1$, then*

$$\frac{|f'(z)|}{(1+r)^2}|z-w| \leq |f(z) - f(w)| \leq \frac{|f'(z)|}{(1-r)^2}|z-w|. \tag{4.4}$$

Proposition 4.3. *Let $g_k(t,k) : [0,T] \times \mathbb{H}\backslash K_{k,t}$ be two forward Loewner chains driven by forces $t \mapsto \big(V_{k,1}(t), \dots, V_{k,N}(t)\big)$ with hulls $K_{k,t}$, for $k = 1, 2$. Let $f_k(t,z)$ be their inverse so that $g_k\big(t, f_k(t,z)\big) = z$. Write $f_k(z) \coloneqq f_k(T,z)$, for*

$k = 1, 2$. *Suppose that*

$$\sum_{j=1}^{N} \sup_{0 \le t \le T} |V_{1,j}(t) - V_{2,j}(t)| < \epsilon, \tag{4.5}$$

where $\epsilon > 0$ *is taken sufficiently small. Suppose further there exists* $\beta \in (0, 1)$ *such that for all* $\zeta \in \mathbb{R}$, *we have*

$$|f_1'(\zeta + i\delta)| \le \delta^{-\beta}, \tag{4.6}$$

for all $\delta \le 4\sqrt{T\epsilon}$. *Then, we have the Hausdorff metric estimate*

$$d_H(K_{1,T} \cup \mathbb{R}, K_{2,T} \cup \mathbb{R}) \le 8(T\epsilon)^{\frac{1-\beta}{2}} + 3\sqrt{\epsilon(1+\epsilon)}. \tag{4.7}$$

Proof. Denote by $h_k(t, z)$ the time-reversed Loewner chains driven by $\{V_{k,1}(T - t), \ldots, V_{k,N}(T - t)\}$ for $k = 1, 2$. Based on Lemma 4.1 and the observation that $f_k(z) = h_k(T, z)$, we know

$$|f_1(z) - f_2(z)| \le \epsilon \cdot \sqrt{1 + 4T/\delta^2}, \tag{4.8}$$

whenever $\operatorname{Im} z \ge \delta$. Take $\delta_0 = 4\sqrt{T\epsilon}$, we have

$$\sup_{\operatorname{Im} z \ge \frac{\delta_0}{2}} |f_1(z) - f_2(z)| \le \sqrt{\epsilon(1+\epsilon)}. \tag{4.9}$$

Hence, Cauchy's integral formula implies

$$\begin{aligned}\sup_{\operatorname{Im} z \ge \delta_0} |f_1'(z) - f_2'(z)| &\le \sqrt{\epsilon(1+\epsilon)} \sup_{\operatorname{Im} z \ge \frac{\delta_0}{2}} \frac{1}{2\pi i} \oint_{\partial \mathcal{B}(z, \delta_0/2)} \frac{d\zeta}{|z - \zeta|^2} \\ &\le \sqrt{\frac{1+\epsilon}{4T}}.\end{aligned} \tag{4.10}$$

For notational convenience, we write $\widehat{K_k} \coloneqq K_{k,T} \cup \mathbb{R}$, for $k = 1, 2$. Fix $\zeta \in \mathbb{R}$, by Lemma 4.2, we have

$$\left|f_1(\zeta + i0^+) - f_1(\zeta + i\delta)\right| \le \delta \cdot |f_1'(\zeta + i\delta)| \le \delta^{1-\beta} \le (16T\epsilon)^{\frac{1-\beta}{2}}. \tag{4.11}$$

Hence, we have

$$f_1\big(\{\operatorname{Im} z \le \delta_0\}\big) \subset \bigcup_{z \in \widehat{K}_1} \mathcal{B}\big(z, (16T\epsilon)^{\frac{1-\beta}{2}}\big). \tag{4.12}$$

It is obvious that

$$\widehat{K}_2 \subset f_2\big(\{\operatorname{Im} z \le \delta_0\}\big). \tag{4.13}$$

For the above fixed $\zeta \in \mathbb{R}$, write $w \coloneqq f_1(\zeta + i0^+) \in \widehat{K}_1$. Choose $\widehat{w} \in \widehat{K}_2$ be the point in $\widehat{K}_2$ nearest to $f_2(\zeta + i\delta_0)$. Notice that this point is admissible in $\widehat{K}_2$, because $K_{2,T} \subset \mathbb{H}$ is compact. By Lemma 4.2 again

$$\begin{aligned}
|\widehat{w} - f_2(\zeta + i\delta_0)| &\le \big|f_2(\zeta + i0^+) - f_2(\zeta + i\delta_0)\big| \\
&\le \delta_0 \cdot |f_2'(\zeta + i\delta_0)| \\
&\le \delta_0 \cdot \Big(|f_1'(\zeta + i\delta_0)| + |f_1'(\zeta + i\delta_0) - f_2'(\zeta + i\delta_0)|\Big) \\
&\le (16T\epsilon)^{\frac{1-\beta}{2}} + \sqrt{4\epsilon(1+\epsilon)}.
\end{aligned} \tag{4.14}$$

Hence, we see that

$$\begin{aligned}
|w - \widehat{w}| &\le |w - f_1(\zeta + i\delta_0)| + |f_1(\zeta + i\delta_0) - f_2(\zeta + i\delta_0)| + |\widehat{w} - f_2(\zeta + i\delta_0)| \\
&\le (16T\epsilon)^{\frac{1-\beta}{2}} + \sqrt{\epsilon(1+\epsilon)} + (16T\epsilon)^{\frac{1+\beta}{2}} + \sqrt{4\epsilon(1+\epsilon)} \\
&\le 8(T\epsilon)^{\frac{1-\beta}{2}} + 3\sqrt{\epsilon(1+\epsilon)}.
\end{aligned} \tag{4.15}$$

Hence the result is verified. □

References

[1] Makoto Katori and Shinji Koshida. Three Phases of Multiple SLE Driven by Non-colliding Dyson's Brownian Motions. Journal of Physics A: Mathematical and Theoretical, 2021.

[2] David Grabiner. Brownian Motion in a Weyl Chamber, Non-colliding Particles, and Random Matrices. Annales de l'Institut Henri Poincare, 1999.

[3] Martin Bender. Global Fluctuations in General β Dyson's Brownian Motion. Stochastic Processes and their Applications, 2008.

[4] Walter Rudin. Real and Complex Analysis. Higher Mathematics Series, 1966.

[5] Antti Kemppainen. Shramm-Loewner Evolution. Springer Briefs in Mathematical Physics, 2017.

[6] Henri Elad Altman. Bismut-Elworthy-Li Formulae for Bessel Processes, 2017.

[7] Carto Wong. Smoothness of Loewner Slits, 2012.

[8] Dmitry Beliaev, Terry J. Lyons, and Vlad Margarint. A New Approach to SLE Phase Transition, 2020.

[9] Zhan Shi. How Long Does It Take a Transient Bessel process To Reach Its Future Infimum. Séminaire de probabilités Strasbourg, 1996.

[10] Ben Boukai. An Explicit Expression for The Distribution of The Supremum of Brownian Motion With a Change Point. Communication in Statistics, Theory and Methods, 1990

[11] Scott Sheffield and Nike Sun. Strong Path Convergence from Loewner Driving Function Convergence. Annals of Probability, 2012.

[12] Alexey A. Tuzhilin. Lectures on Hausdorff and Gromov-Hausdorff Distance Geometry, 2020.

[13] Fredrik Johansson and Gregory F. Lawler. Optimal Holder Exponent for The SLE Path. Duke Math 2009.

[14] Christian Benes, Fredrik Johansson Viklund, and Michael J. Kozdron. On The Rate of Convergence of Loop-erased Random Walk to SLE(2). Communications in Mathematical Physics, 2013.

[15] Steffen Rohde and Oded Schramm. Basic Properties of SLE. Annals of Mathematics, 2005.

[16] Vadim Gorin and Mykhaylo Shkolnikov. Multilevel Dyson Brownian Motions via Jack Polynomials. Probability Theory and Related Fields, 2015.

[17] Gregory F. Lawler. Conformally Invariant Processes in the Plane.Mathematical Surveys and Monographs, 2005.

[18] John. Cardy. Stochastic Loewner evolution and Dyson's circular ensembles, Journal of Physics A: Mathematical and General, 2003

[19] Katori, Makoto and Koshida, Shinji. Conformal welding problem, flow line problem, and Multiple Schramm–Loewner Evolution,

[20] J. Lenells, F. Viklund. Schramm's formula and Green's function for multiple SLE, 2017.

[21] Vincent Beffara, Eveliina Peltola, and Hao Wu. On the uniqueness of global multiple SLEs, 2018.

[22] Eveliina Peltola and Hao Wu.Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities for Level Lines of GFF . *Communications in Mathematical Physics*. 2019.

[23] Sakai, Kazumitsu. Multiple Schramm–Loewner evolutions for conformal field theories with Lie algebra symmetries. Nuclear Physics B 867.2 (2013): 429-447.

[24] Hotta, Ikkei, and Sebastian Schleissinger. Limits of Radial Multiple SLE and a Burgers–Loewner Differential Equation. Journal of Theoretical Probability 34.2 (2021): 755-783.

[25] del Monaco, Andrea, and Sebastian Schleißinger. Multiple SLE and the complex Burgers equation." Mathematische Nachrichten 289.16 (2016): 2007-2018.

[26] Katori, Makoto, and Shinji Koshida. Conformal welding problem, flow line problem, and multiple Schramm–Loewner evolution. Journal of Mathematical Physics 61.8 (2020): 083301.

[27] Dubédat, Julien. Commutation relations for Schramm-Loewner evolutions. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 60.12 (2007): 1792-1847.

[28] Beffara, Vincent, Eveliina Peltola, and Hao Wu. "On the uniqueness of global multiple SLEs." The Annals of Probability 49.1 (2021): 400-434.

[29] Karrila, Alex. Multiple SLE type scaling limits: from local to global. arXiv preprint arXiv:1903.10354 (2019).

[30] Kytölä, Kalle, and Eveliina Peltola. Pure partition functions of multiple SLEs. Communications in Mathematical Physics 346.1 (2016): 237-292.

[31] del Monaco, Andrea, Ikkei Hotta, and Sebastian Schleissinger. Tightness results for infinite-slit limits of the chordal Loewner equation. Computational Methods and Function Theory 18.1 (2018): 9-33.

[32] Hotta, Ikkei, and Makoto Katori. Hydrodynamic limit of multiple SLE. Journal of Statistical Physics 171.1 (2018): 166-188.

[33] Erdős, László, and Horng-Tzer Yau. A dynamical approach to random matrix theory. Vol. 28. American Mathematical Soc., 2017.

[34] Erdős, László, Benjamin Schlein, and Horng-Tzer Yau. Universality of random matrices and local relaxation flow. Inventiones mathematicae 185.1 (2011): 75-119.

[35] Bourgade, Paul, László Erdős, and Horng-Tzer Yau. Universality of general β-ensembles. Duke Mathematical Journal 163.6 (2014): 1127-1190.

[36] Landon, Benjamin, Philippe Sosoe, and Horng-Tzer Yau. Fixed energy universality of Dyson Brownian motion. Advances in Mathematics 346 (2019): 1137-1332.

[37] Arous, Gérard Ben, and Paul Bourgade. Extreme gaps between eigenvalues of random matrices. The Annals of Probability 41.4 (2013): 2648-2681.

[38] Viklund, Fredrik Johansson, Steffen Rohde, and Carto Wong. On the continuity of SLE_κ in κ. Probability Theory and Related Fields 159.3-4 (2014): 413-433.

[39] Friz, Peter K., Huy Tran, and Yizheng Yuan. Regularity of SLE in (t, κ) and refined GRR estimates. Probability Theory and Related Fields 180.1 (2021): 71-112.

[40] Beliaev, Dmitry, Terry J. Lyons, and Vlad Margarint. Continuity in κ in SLE theory using a constructive method and Rough Path Theory. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques. Vol. 57. No. 1. Institut Henri Poincaré, 2021.

[41] Margarint, Vlad. Quasi-Sure Stochastic Analysis through Aggregation and SLE_κ Theory. arXiv preprint arXiv:2005.03152 (2020).

[42] Beliaev, D., V. Margarint, and A. Shekhar. Continuity of zero-hitting times of Bessel processes and welding homeomorphisms of SLE. ALEA: Latin American Journal of Probability and Mathematical Statistics.

[43] Healey, Vivian Olsiewski, and Gregory F. Lawler. N-sided radial Schramm–Loewner evolution. Probability Theory and Related Fields (2021): 1-38.