Quasi-Sure Stochastic Analysis and SLE_κ Theory - q.s. continuity in κ of the traces

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Abstract

We construct the SLE_κ traces quasi-surely using the ideas developed in [43]. As a by-product of this analysis we prove the q.s. continuity in κ of the SLE_κ traces.

1 Introduction

The Schramm-Loewner evolution, or $SLE(\kappa)$ is a one parameter family of random planar fractal curves introduced by Schramm in [34], that are proved to describe scaling limits of a number of discrete models that appear in planar Statistical Physics. For instance, it was proved in [20] that the scaling limit of loop erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and Uniform spanning trees were proved to converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$ respectively in the series of works [42], [37], [41] and [20]. In fact, the use of Loewner equation along with the techniques of stochastic calculus, provided tools to perform a rigorous analysis of the scaling limits of the discrete models. In this framework it has been established a precise meaning to the passage to the scaling limit and its conformal invariance. We refer to [19] for a detailed study of the object and many of its properties.

When studying the SLE_κ , in the upper half-plane, we study the corresponding families of conformal maps in the formats

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z-\sqrt{\kappa}B_t}\,, \quad f(0,z)=z, z\in\mathbb{H}\,. \tag{1.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z)-\sqrt{\kappa}B_t}\,, \qquad g(0,z)=z, z\in\mathbb{H}\,. \tag{1.2}$$

(iii) Time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z)-\sqrt{\kappa}B_t}\,, \qquad h(0,z)=z, z\in\mathbb{H}\,. \tag{1.3}$$

The paper is divided in several sections. In the first part of the paper, we construct quasi-surely the SLE_κ traces and in the second part we prove the continuity in κ of these objects. We emphasize that this method allows us to construct SLE_κ traces simultaneously for all κ, compared with the typical theory that does it only for fixed κ.

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2 Quasi-sure existence of the SLE_κ trace, polarity of the nullsets outside of which the SLE_κ trace is defined and continuity in κ for $\kappa \neq 8$-Defining the SLE traces simultaneously for all $\kappa \neq 8$

We start by introducing the objects needed in our analysis. The exposure is based on [43] which we refer to, for more details.

2.1. Aggregation and Quasi-sure Stochastic Analysis. A probability measure $\mathbb{P}$ is a local martingale measure if the process B is a local martingale under $\mathbb{P}$. It is proved that there exists an $\mathfrak{F}$-progresively measurable process denoted as $\int_0^t B_s dB_s$ which coincides with the Ito integral $\mathbb{P}$ -a.s. for all local martingale measures $\mathbb{P}$. In particular this provides a pathwise definition of

$$\langle B \rangle_t := B_t B_t^T - 2 \int_0^t B_s dB_s$$

and

$$\hat{a}_t := \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} [\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}].$$

Let us introduce first $\bar{\mathcal{P}}_W$ which denotes the set of all local martingale measures $\mathbb{P}$ such that $\mathbb{P}$-a.s. $\langle B \rangle_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{R}_+$.

Definition 2.1.
- *We say that a property holds $\mathcal{P}$-quasi-surely if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}$.*
- *Denote $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P} \in \mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$*
- *A probability measure $\mathbb{P}$ is called absolutely continuous with respect to $\mathcal{P}$ is $\mathbb{P}(E) = 0$ for all $E \in \mathcal{N}_{\mathcal{P}}$.*

In this approach, we use the following universal filtration $\mathfrak{F}^{\mathcal{P}}$ for the mutually singular measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$.

$$\mathfrak{F}^{\mathcal{P}} := \{\mathcal{F}_t^{\mathcal{P}}\}_{t \geqslant 0}$$

where

$$\mathcal{F}_t^{\mathcal{P}} := \cap_{\mathbb{P} \in \mathcal{P}} \left(\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}} \right) .$$

Note that the construction is suitable for mutually singular probability measures. In the case the measures are absolutely continuous, the situation becomes simpler since one can work under the nullsets of the dominating measure directly.

The next definition introduces the notion of aggregator that we use in our analysis.

Definition 2.2. *Let $\mathcal{P} \subset \bar{\mathcal{P}}_W$. Let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of $\mathfrak{F}^{\mathcal{P}}$ progressively measurable processes. An $\mathfrak{F}^{\mathcal{P}}$ progressively measurable process X is called a $\mathcal{P}$-aggregator of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$, if $X = X^{\mathbb{P}}$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}$.*

2.2. The universal Brownian motion. In this section, we introduce the notion of Universal Brownian motion as in [43].

Let

$$\bar{\mathcal{A}} := \{a : \mathbb{R}_+ \to \mathbb{S}_d^{>0} | \mathbb{F} - \text{progresively measurable and} \int_0^t |a_s| ds < +\infty, \forall t \geqslant 0\}.$$

We consider $d = 1$, and $\mathbb{S}_d^{>0}$ becomes the space of symmetric matrices in $d = 1$ that is $\mathbb{R}_+$.

For a given $\mathbb{P} \in \bar{\mathcal{P}}_W$, let

$$\bar{\mathcal{A}}_W(\mathbb{P}) := \{a \in \bar{\mathcal{A}} : a = \hat{a}, \mathbb{P} - a.s.\}$$

Recall that $\hat{a}$ is the density of the quadratic variation of $< B >$ and is defined pointwise. We define

$$\bar{\mathcal{A}}_W := \cup_{\mathbb{P} \in \bar{\mathcal{P}}_W} \bar{\mathcal{A}}_W(\mathbb{P})$$

Let us define for any $a, b \in \mathcal{A}$, the disagreement time $\theta^{a,b} := \inf\{t \geqslant 0 : \int_0^t a_s ds \neq \int_0^t b_s ds\}$.

Definition 2.3. *A subset $\mathcal{A}_0 \subset \mathcal{A}_W$ is called a generating class of diffusion coefficients if*

- *$\mathcal{A}_0$ satisfies the concatenation property $a\mathbf{1}_{[0,t)} + b\mathbf{1}_{[t,\infty)} \in \mathbb{A}_0$, for $a, b, \in \mathcal{A}_0, t \geqslant 0$*
- *$\mathcal{A}_0$ has constant disagreement times: for all $a, b \in \mathcal{A}_0$, $\theta^{a,b}$ is constant or equivalently $\Omega_t^{a,b} = \emptyset$ or Ω for all $t \geqslant 0$.*

Definition 2.4. *We say $\mathcal{A}$ is a separable class of diffusion coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0 \subset \mathcal{A}_W$ is generated by a class of diffusion coefficients and $\mathcal{A}$ consists of all processes a of the form*

$$a = \sum_0^\infty \sum_{i=1}^\infty a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}$$

where $(a_i^n)_{i,n} \subset \mathcal{A}_0$, $(\tau_n)_n \subset \mathcal{T}$ is non-decreasing with $\tau_0 = 0$.

- *$\inf\{n : \tau_n = \infty\} < \infty$ and $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$ and each τ_n takes at most countably many values.*
- *For each n $\{E_i^n, i \geqslant 1\} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.*

A separable class $\mathcal{A}$ of diffusion coefficients generated by $\mathcal{A}_0$ is said to satisfy the consistency conditions. We denote

$$\mathcal{P} = \{\mathbb{P}^a, a \in \mathcal{A}\}.$$

Let us consider a standard Brownian motion B_t. For any $\mathbb{P} \in \mathcal{P}_W$ and $a \in \bar{\mathcal{A}}_W(\mathbb{P})$ by Levy's characterization, we obtain that the following Ito's stochastic integral under $\mathbb{P}$ is a $\mathbb{P}$- Brownian motion

$$W_t^{\mathbb{P}} := \int_0^t a_s^{-1/2} dB_s$$

For $\mathcal{A}$ satisfying the consistency condition, the family $\{W^{\mathbb{P}_a}, a \in \mathcal{A}\}$ admits a unique $\mathcal{P}$-aggregator W. Since $W^{\mathbb{P}_a}$ is a $\mathbb{P}^a$ Brownian motion for every $a \in \mathcal{A}$, we call W- a universal Brownian motion.

A fundamental result that we use is the aggregate solution to stochastic differential equations. In the paper, they show how to solve a stochastic differential equation simultaneously under all the measures $\mathbb{P} \in \mathcal{P}$. Specifically, they prove the following result:

Proposition 2.5 (Proposition 6.10 of [43])**.** *Let A be satisfying the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s, t \geqslant 0$ has a unique $\mathbb{F}^{\mathbb{P}}$ progresively measurable strong solution. on the interval $[0, \tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution to the equation above.*

We use this result for a stochastic differential equation solved by the process $\tilde{N}_s$, that we will introduce in the next section, that is used in the construction of the SLE trace in order to obtain the estimate the derivative of the conformal map at fixed times.

3 Heuristics of the proof

The main idea is to consider the construction of the aggregated solution to SDE as in [43] and applied to the SDE corresponding to the process $\tilde{N}_s$. Furthermore, we express the derivative of the map $\tilde{h}_t(z0)$ using the aggregated solution $\tilde{N}_s$ and use the typical Lemmas in [20] to obtain the existence of the trace. We have

$$|\tilde{h}_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \frac{\tilde{K}_s^2 + 1}{\tilde{K}_s^2 - 1} ds\right) = e^{at} \exp\left(2a \int_0^t \tilde{N}_s ds\right),$$

where

$$d\tilde{N}_s = (1-\tilde{N}_s)[-4(a+1)\tilde{N}_s + 1]ds + 2\sqrt{\tilde{N}_s}(1-\tilde{N}_s)d\tilde{B}_s$$

.

This formulation is equivalent with viewing the Loewner equation driven by the universal Brownian motion. The SDE for $\tilde{K}_t$ is

$$d\tilde{K}_t = 2a\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}d\tilde{B}_t$$

It can be shown that the $sinh(J_t)$ solves the SDE for $\tilde{K}_t$ where J_t satisfies

$$dJ_t = -(q+r)\tanh(J_t)dt + d\tilde{B}_t\,.$$

with $q = q(\kappa)$ and $r = r(\kappa)$.

Once we have a unique notion of strong solution for the SDE for J_t, $\mathbb{P}$-a.s., we can construct an aggregated solution for this SDE and then express

$$|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right) = e^{-at}\exp\left(2a\int_0^t \frac{\sinh(J_s)^2+1}{\sinh(J_s)^2-1}ds\right).$$

Note that, equivalently we can consider the unique strong solution (checking the conditions of Yamada Watanabe Theorem the SDE $\tilde{K}_t$ and construct an aggregated solution for it. **We use that aggregated solution for the SDE $\tilde{K}_t$ to construct simultaneously the SLE_κ trace for all parameters $\kappa \neq 8$ by relating the aggregated solution for $\tilde{K}_t$ with the derivative of the backward map.**

Note that since the structure of the SDE that $\tilde{N}_s$ satisfies we are naturally lead to consider the problem of aggregation as in the paper [43] since the measures $\mathbb{P}_a(\kappa)$ are not a-priori absolutely continuous with respect to the Wiener measure.

With this definition, we recover the typical construction of the estimate of the Rohde-Schramm Theorem simultaneously under all measures $\mathbb{P}_\kappa$ (uncountably many choices of the parameter). The advantage is that in this formulation, of q.s. analysis we can have a clear definition of the SLE trace quasi surely in which the continuty in κ is a direct consequence of the procedure of constructing the traces and the continous perturbation of the algorithm designed in [44]. In this way we work over all the possible measures simultaneously, i.e. over uncountably many κ.

The quasi-sure construction assures that the typical estimate on the derivative of the map, holds simultaneously under all the choices of the parameter κ since the aggregated solution of $\tilde{N}_s$ is constructed simultaneously under all the measures $\mathbb{P}_\kappa$

4 The quasi-sure existence of the trace of SLE_κ-Defining the SLE trace simultaneously for all parameters κ

Rohde and Schramm proved one fundamental result about the existence of the trace for the SLE_κ process for all values of $\kappa \neq 8$. In order to discuss the result, we introduce the notions and definitions that we use. We say that a continous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$-hulls K_t if $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \geqslant 0$.

Theorem 4.1 (Rohde-Schramm)**.** *Let $(K_t)_{t\geqslant 0}$ be a SLE_κ for $\kappa \neq 8$. We denote g_t to be the Loewner flow and U_t be the Loewner transform. Then, $g_t^{-1} : \mathbb{H} \mapsto H_t$ extends continously to $\bar{\mathbb{H}}$ for all $t \geqslant 0$, almost surely. Morevoer, if we set $\gamma_t = g_t^{-1}(U_t)$, then γ_t is continous and generates $(K_t)_{t\geqslant 0}$ almost surely.*

4.1. Estimates for the mean of the derivative for a fixed κ. In order to provide the technical results for this part, we make a useful time change in the corresponding Loewner equation for the real and imaginary parts. With this new clock at hand, we obtain some technical Lemmas that are crucial in the proof of the existence of the trace for the SLE process. Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2 + Y_t^2} dt - \sqrt{\kappa} dB_t\,, \qquad dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} dt\,, \tag{4.1}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2 + Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$.

We provide a martingale estimate for the backward Loewner differential equation. We start with the following proposition, in which the polynomial condition in the hypothesis comes from the fact that we are searching for martingales of the type $M_t :=$ $\tilde{Y}_t^\alpha (|\tilde{Z}_t|/\tilde{Y}_t)^\beta |h_t'(z_0)|^\gamma$, where α, β, γ depend on each other and $h_t(z)$ is a family of conformal maps satisfying the backward Loewner differential equation. This leads to the fact that they should satisfy a constraint.

Proposition 4.2 (Proposition 7.2 in [20]). *Let r, b such that*

$$r^2 - (2a+1)r + ab = 0\,,$$

then

$$M_t \;:=\; \tilde{Y}_t^{b-(r/a)}(|\tilde{Z}_t|/\tilde{Y}_t)^{2r}|h_t'(z_0)|^b\,,$$

is a martingale. Moreover,

$$\mathbb{P}(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}e^{t(r-ab)}\,.$$

Proof. By taking the complex derivative in z in the Loewner equation and by applying the chain rule for the function $L_t = \log h_t'(z_0)$, we obtain that $L_t = -\int_0^t \frac{a}{Z_s^2}ds$, and in particular, $|\tilde{h}_t'(z_0)| = \exp\left(a\int_0^t \frac{\tilde{Y}_s^2 - \tilde{X}_s^2}{\tilde{X}_s{}^2 + \tilde{Y}_s{}^2}ds\right)$. Moreover, if we consider $\tilde{K}_t = \frac{\tilde{X}_t^2}{\tilde{Y}_t^2}$ and $\tilde{N}_t = \frac{\tilde{K}_t}{1+\tilde{K}_t}$, we obtain that

$$|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \tilde{N}_s ds\right)\,.$$

In the $\sigma(t)$ time parametrization, looking at the equation for $\tilde{Y}_t$ we obtain a deterministic one $d\tilde{Y}_t = -a\tilde{Y}_t dt$, so in this time parametrization Y_t grows deterministically in an exponential manner $\tilde{Y}_t = Y_0 e^{at}$. At this moment, we can rephrase the formula for M_t as

$$M_t = y_0^{b-(r/a)}e^{-rt}(1-\tilde{N}_t)^{-r}\exp(2a\int_0^t \tilde{N}_s ds)\,.$$

and by applying Ito's formula, we obtain that

$$dM_t = 2r\sqrt{\tilde{N}_t}M_t d\tilde{B}_t\,,$$

where $\tilde{B}_t = \int_0^{\sigma(t)}\frac{1}{\sqrt{X_t^2+Y_t^2}}dB_t$ is the Brownian motion that we obtain in the time reparametrization. This shows that M_t is a martingale, hence

$$\mathbb{E}[M_t] = \mathbb{E}[M_0] = y_0^{b-(r/a)}(|z_0|/y_0)^{2r}\,.$$

Note that since for $r \geqslant 0$, $(|\tilde{Z}_t|/Y_t)^{2r} \geqslant 1$, then by Markov inequality, we have that

$$\mathbb{P}(|\tilde{h}_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}e^{t(r-ab)}\,.$$

□

Corollary 4.3 (Corollary 7.3 in [20]). *For every* $0 \leqslant r \leqslant 2a+1$, *there is a finite* $c = c(a,r)$ *such that for all* $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, *we have that*

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda)\,,$$

where $b = [(2a+1)r - r^2]/a \geqslant 0$ *and*

$$\delta(y_0,\lambda) = \begin{cases} \lambda^{(r/a)-b}, & \text{if } r < ab\,, \\ -\log(\lambda y_0), & \text{if } r = ab\,, \\ y_0^{b-(r/a)}, & \text{if } r > ab\,. \end{cases}$$

Proof. From $dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt$, we obtain that $dY_t \leqslant \frac{a}{Y_t}dt$, and hence $Y_t \leqslant \sqrt{2at+y_0^2} \leqslant \sqrt{2a+1}$. In the last inequality, we used that $t \leqslant 1$ and $y_0 \leqslant 1$. Using the exponential growth of Y_t in this time reparametrization, we obtain that $\tilde{Y}_t = \sqrt{2a+1}$ at time $T = \frac{\log\sqrt{2a+1} - \log y_0}{a}$.

Therefore,

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \mathbb{P}(\sup_{0\leqslant s\leqslant T} |\tilde{h}_s'(z_0)| \geqslant \lambda)\,.$$

Using that $|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \tilde{N}_s ds\right)$ we obtain that $|\tilde{h}_{t+s}'(z0)| \leqslant e^{as}|\tilde{h}_t'(z0)|$. So by addition of the probabilities, we have that

$$\mathbb{P}(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^a\lambda) \leqslant \sum_{j=0}^{[T]} \mathbb{P}(|\tilde{h}_j'(z_0)| \geqslant \lambda)\,.$$

Using the Schwarz-Pick Theorem for the upper half-plane we obtain that $|\tilde{h}_t(z_0)| \leqslant \mathrm{Im}\tilde{h}_t'(z_0)/y_0 = e^{at}$. This gives a lower bound for the t that we are summing over and we obtain that via the Proposition 4.2 that

$$\begin{aligned}\mathbb{P}(\sup_{0\leqslant t\leqslant T} |\tilde{h}_t'(z_0)| \geqslant e^a\lambda) &\leqslant \sum_{(1/a)\log\lambda\leqslant j\leqslant T} \mathbb{P}(|\tilde{h}_j'(z_0)| \geqslant \lambda) \\ &\leqslant \lambda^{-b}(|z_0|/y_0)^{2r} \sum_{(1/a)\log\lambda\leqslant j\leqslant T} e^{j(r-ab)} \\ &\leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda)\,.\end{aligned}$$

□

4.2. Estimates on the moments of the derivatives for many κ using aggregation of solutions of a SDE. We consider the universal Brownian motion $W_t^{\mathbb{P}}$ as a driver for the backward Loewner differential equation. We consider also the random time changed universal Brownian motion $\tilde{W}_t^{\mathbb{P}} := \frac{dW_t^{\mathbb{P}}}{\sqrt{X_t^2+Y_t^2}}$

Investigating the real and the imaginary part of the backward SLE, we have that

$$dX_t = \frac{-2X_t}{X_t^2+Y_t^2}dt - dW_t^{\mathbb{P}}\,, \qquad dY_t = \frac{2Y_t}{X_t^2+Y_t^2}dt\,, \tag{4.2}$$

We consider the time change $\sigma(t) = X_t^2 + Y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{X_s^2+Y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$. We prove the same estimates as above using the aggregated solution $\tilde{N}_s$ that gives that

$$M_t = y_0^{b-(r/a)}e^{-rt}(1-\tilde{N}_t)^{-r}\exp(2a\int_0^t \tilde{N}_s ds)\,,$$

is a quasi-sure martingale, i.e. a martingale with respect to all the measures simultaneously. Thus, the same estimate on the absolute value derivative of the maps, can be recovered simultaneously for all probability measures. We obtain the same inequality as in Proposition 4.2 simultaneously for all the measures. Also the Corollary 4.3 is recovered quasi-surely from the Proposition using the aggregated solution, in the same manner.

We obtain from the quasi-sure versions of the Lemmas and Propositions from above, a way of estimating simultaneously under all the measures the absolute value of the derivative of the map with the 'worst' value of the parameter κ. The idea is that since the estimate holds for the 'worst' κ, then by the q.s. construction, gives that it holds simultaneously for all the 'better' κ, i.e. we can estimate uniformly in κ with the same bounds, simultaneously under all the choices of the measure. In particular, this gives that the typical Rohde-Schramm Theorem that is sufficient to prove the existence of the SLE_κ trace, holds quasi surely for all $\kappa \neq 8$.

4.3. Existence of the trace for fixed κ. Before stating the main Theorem of the section, we prove two propositions that together with corollary 4.3 build the argument for the existence of trace of SLE_κ for $\kappa \neq 8$.

Proposition 4.4 (Proposition 4.33 in [20])**.** *Suppose that g_t is a Loewner chain with driving function U_t and assume that there exist a sequence of positive numbers $r_j \to 0$ and a* *constant c such that*

$$|\hat{f}'_{k2^{-2j}}(2^{-j}i)| \leqslant 2^j r_j \, , k = 0, 1, \dots, 2^{2j} - 1 \, ,$$
$$|U_{t+s} - U_t| \leqslant c\sqrt{j} 2^{-j} \, , 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 2^{-2j} \, .$$

and

$$\lim_{j\to\infty} \sqrt{j}/\log r_j = 0 \, .$$

Then $V(y,t) := \hat{f}_t(iy)$ is continuous on $[0,1] \times [0,1]$.

Proof. By differentiating $\partial_t f(t,z) = -\partial_z f(t,z) \frac{2}{z-U(t)}$, $f(0,z) = z, z \in \mathbb{H}$, we obtain that

$$\dot{f'}_t(z) = -f''_t(z)\frac{2}{z-U_t} + f'_t(z)\frac{2}{(z-U_t)^2} \, .$$

Bieberbach Theorem ((3.16), [20]) implies that $|f''_t(z)| \leqslant \frac{6|f'_t(z)|}{\mathrm{Im}(z)^2}$, and that $|f'_{t+s}(z)| \leqslant \exp\left[\frac{6s}{\mathrm{Im}(z)}\right] |f'_t(z)|$. From hypothesis, we get that for $k = 0, 1, \dots, 2^{2j} - 1$

$$|f'_t(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j \, , k2^{-2j} \leqslant t \leqslant (k+1)2^{-2j} \, .$$

Using the Distortion Theorem, we get that for a univalent function on $\mathbb{D}$, we have that $|f'(z)| \leqslant 12|f'(0)|$ for $|z| \leqslant 1/2$. We iterate this estimate on a sequence of intersecting disks that connect $z, w \in \mathbb{H}$ with $\mathrm{Im}(z), \mathrm{Im}(w) \geqslant y > 0$. For the conformal transformation $f : \mathbb{H} \to \mathbb{D}$, we have that

$$|f'(w)| \leqslant 144^{(z-w)/y+1)} |f'(z)| \, .$$

In particular, by combining the hypothesis and $|f'_t(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j$ we obtain that there exist c and β such that

$$|\hat{f}'_t(i2^{-j})| \leqslant e^{\sqrt{j}\beta} 2^j r_j \, , \;\; 0 \leqslant t \leqslant 1, \;\; j = 0, 1, 2, \dots, 2^{-j} \, .$$

Using the Distortion Theorem again but for a point that is not on the lattice of space and time, we get

$$|\hat{f}'_t(iy)| \leqslant e^{\sqrt{j}\beta} 2^j r_j \, , \;\; 0 \leqslant t \leqslant 1, \;\; 2^{-j} < y < 2^{-j+1} \, , \;\; j = 0, 1, 2, \dots, 2^{-j} \, .$$

For $s \leqslant 2^{-2j}$ and $y, y_1 \leqslant 2^{-j}$ we get that $|\hat{f}_t(iy) - \hat{f}_{t+s}(iy)| \leqslant |\hat{f}_t(iy) - \hat{f}_t(i2^{-j})| + |\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j})| + |\hat{f}_{t+s}(i2^{-j}) - \hat{f}_{t+s}(iy)|$. The first and the third term are bounded by the estimate elaborated so far via

$$|\hat{f}_t(iy) - \hat{f}_t(i2^{-j})| \leqslant \sum_{l=j}^{\infty} ce^{\beta\sqrt{l}} r_l \,.$$

From the assumption, the right hand side goes to 0 as $j \to \infty$. Using the estimate and the format of the partial differential equation that f solves, for the middle term, we have that

$$|\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j}) \leqslant 2s2^j \sup_{t \leqslant r \leqslant t+s} |f'(2^{-j})| \leqslant cr_j \,.$$

Since V is continuous already in $(0,\infty)\times[0,\infty)$ to establish the continuity on $[0,\infty)\times[0,\infty)$ it suffices to show that there exists a $\delta(\varepsilon)$ such that $\delta(0+) = 0$ and such that $|V(y,t) - V(y_1, s)| \leqslant \delta(y + y_1 + |t-s|)$, $0 \leqslant t, s, \leqslant t_0, y, y_1 > 0$. So by using the hypothesis, we conclude. □

We need another result in order to conclude the existence of the trace for SLE process. For this we introduce the notion of *accessible point*. We call a point $z \in \hat{K}_t \setminus \cup_{s<t}\hat{K}_s$ $t-$*accessible* if there exists a curve $\eta : [0,1] \to \mathbb{C}$, with $\eta(0) = z$ and $\eta(0,1] \subset H_t$.

Proposition 4.5 (Proposition 4.29 in [20]). *Suppose g_t is a Loewner chain with driving function U_t and let $\hat{f}_t(z) = g_t^{-1}(z + U_t)$. Suppose that for each t, the limit $\gamma(t) = \lim_{y\to 0+} \hat{f}_t(iy)$, exists and the function $t \to \gamma(t)$ is continuous. Then g_t is the Loewner chain generated by γ.*

Proof. The proof relies on Proposition 4.27 from [20] that shows together with the condition from the hypothesis that $\gamma(t)$ is the only t-accesible point. Since $\gamma[0,t]$ is closed, the same Proposition 4.27 from [20] shows that $\partial H_t \cap \mathbb{H}$ is contained in $\gamma[0,t]$. □

In order to prove this result, we need the Lemma from the introduction also.

The following Lemma can be recovered using the quasi-sure analysis.

Lemma 4.6. *[Lemma 7.6 in [20]] For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - \zeta(t)$.*

Theorem 4.7 (Rohde-Schramm). *If $\kappa \neq 8$ the chordal SLE_κ is generated by a path with probability 1.*

Proof. By using the scaling of the SLE_κ, it suffices to prove the Theorem only for $t \in [0,1]$. According to the preliminary propositions it suffices to show that with probability 1 there exists an ε and a random constant c (because this estimate should hold for all $j's$ and $k's$) such that

$$|f'_{k2^{-2j}}(i2^{-j})| \leqslant c2^{j-\varepsilon}, j = 1, 2, \ldots, k = 0, 1, \ldots, 2^{2j},$$
$$|B_t - B_s| \leqslant c|t-s|^{1/2} |\log \sqrt{|t-s|}| \quad 0 \leqslant t \leqslant 1.$$

The second inequality is a consequence of the modulus of continuity for the Brownian motion. For the first inequality, we use a Borel-Cantelli Lemma along with Lemma 4.6 to find c and ε such that for all $0 \leqslant t \leqslant 1$

$$\mathbb{P}(|h'_t(i2^{-j}| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-(2+\varepsilon)j}.$$

Notice that we apply h'_t to points on the imaginary axis and that the corresponding $\lambda = 2^{j-\varepsilon}$.

We consider $r = a + (1/4) < 2a + 1$ and $b = \frac{(1+2a)r - r^2}{a} = a + 1 + \frac{3}{16a}$, according to the Corollary 4.3. Thus, we are in the regime $r < ab$, so by Corollary 4.3 we have that

$$\mathbb{P}(|h'_t(i2^{-j})| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-j(2b-(r/a))(1-\varepsilon)}.$$

Investigating the exponent of 2, we obtain that $2b - (r/a) = 2a + 1 + 1/(8a) > 2$ provided that $a \neq 1/4$. So, we can apply Borel-Cantelli argument provided that $a \neq 1/4$, i.e. $\kappa \neq 8$, and finish the proof.

□

5 Quasi-sure existence of the trace -defining the trace simultaneously for all κ- and proving continuity in κ

Once we obtain the quasi-sure version of the estimates in the previous section, we can continue to go exactly in the same manner to prove the existence of the trace quasi-surely with the tools from above.

For this, we use the aggregated solution of the SDE $\tilde{K}_t$:

$$d\tilde{K}_t = 2a\tilde{K}_t dt + \sqrt{1+\tilde{K}_t^2}\,d\tilde{B}_t$$

constructed via the methods of Quasi-Sure Stochastic Analysis through Aggregation from the following Proposition.

Proposition 5.1 (Proposition 6.10 of [43])**.** *Let A be satisfying the consistency assumption. Assume that for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_s(X_s)dB_s$,$t \geqslant 0$ has a unique $\mathbb{F}^{\mathbb{P}}$ progresively measurable strong solution. on the interval $[0,\tau]$. Then there exists $\mathcal{P}$-q.s. aggregated solution to the equation above.*

In order to perform the analysis and to prove all the estimates (that were done in the previous section for fixed κ) simultaneously for all κ, we use the aggregated solution and relate it with the derivative of the map, via

$$|\tilde{h}_t'(z_0)| = e^{-at}\exp\left(2a\int_0^t \frac{\tilde{K}_s^2+1}{\tilde{K}_s^2-1}ds\right)$$

.

We redo all the proofs of the previous section using the aggregated solution. In this manner we can construct the trace via controlling the derivative estimate simultaneously for all $\kappa \neq 8$ using the aggregated solution, and obtain the existence of the limit and in particular the existence of SLE traces simultaneously for all $\kappa \neq 8$.

Corollary 5.2 (Continuity in $\kappa \neq 8$ of the SLE_κ traces)**.** *Once the Rohde-Schramm estimate is established quasi-surely, we obtain naturally the continuity in κ of the traces, when applying the construction of the algorithm in [44]. The main difficulty in these proofs is to control the nullset outside of which the SLE_κ trace is defined, in order to get the the quasi-sure continuity in κ. We have a quasi-sure definition of the SLE_κ trace (i.e. the SLE_κ trace that we consider is defined simultaneously for all the values of κ) and this allows us directly to consider uncountably many parameters $\kappa \neq 8$.*

Proof of Corollary. Throughout this section, the precise subpower function that we use is changing from line to line. We do not track these changes in order to not burden the reader, since in our analysis all that is needed is that this is a subpower function.

For this, let us fix a value $\kappa_1 > 0$ and let us consider the set of $\omega \in \Omega_{\kappa_1}$. Let us choose any other value $\kappa_\infty > 0$, and let us consider $(\kappa_i)_{i\in\mathbb{R}}$, such that $\kappa_i \to \kappa_\infty$, with starting point of the sequence κ_1. Let us define the following subsets of $\mathbb{H}$ as in [44].

$$A_{n,c,\phi} = \left\{ x + iy \in \mathbb{H} : |x| \leqslant \frac{\phi(n)}{\sqrt{n}}, \frac{1}{\sqrt{n}\phi(n)} \leqslant y \leqslant \frac{c}{\sqrt{n}} \right\}.$$

We need the following Lemmas from [44] as well as the following Proposition from [45] that we use for sequences of parameters $\kappa_n \to \kappa_2$. In the case of Proposition from [45] in order to be sure that we do not change the nullsets that depend on κ_n when we consider $\kappa_n \to \infty$ we use the rigidity given by the absolute continuity of the laws in the parameter κ_n of the Bessel processes with different dimension depending on κ_n that we obtain on the boundary when we consider the continuous extension of the conformal maps. The estimate needed in the proposition is the derivative estimate that is needed to prove the existence of the trace. The nullsets $\mathcal{N}_{\kappa_n}$ outisde of which this estimate holds are included one into another when we consider $\kappa_n \to \kappa_2$.

Lemma 5.3 (Lemma 3.2 of [44])**.** *There exist a subpower function ψ depending only on ϕ, c_0 and β such that for $n \geqslant 1$ and $0 \leqslant k \leqslant n-1$, there exists $s \in [0, \frac{2}{n}]$ such that $\gamma_k(s) \in A_{n,2\sqrt{2},\psi}$.*

The Lemma 5.3 gives information about properties of the SLE traces for fixed κ. We use again the control on the nullsets $\mathcal{N}_{\kappa_n}$ outside of which the trace is defined in order to conserve this a.s. property of the SLE trace for values κ_n (with different subpower functions that depend on κ_n).

Lemma 5.4 (Lemma 3.2 of [44])**.** *There exist a subpower function $\tilde{\psi}$ depending only on ϕ, c_0 and β such that for $n \geqslant 1, k$ and $r \in [\frac{1}{n}, \frac{2}{n}]$, $\gamma_k^n(r)$ is in the box $A_{n,2\sqrt{2},\tilde{\psi}}$.*

This Lemma is applied for the trace obtained when approximating the SLE trace for the fixed value κ_2.

We apply the previous Lemmas for sequences $\kappa_n \to \kappa_2$. The sizes of the boxes depend on κ_n via the dependence of the subpower function that we choose on $\beta = \beta(\kappa_n)$ and on ϕ that depends also on κ_n. However, since the constant $c = 2\sqrt{2}$ is fixed, the upper level of the boxes remains the same as we consider $\kappa_n \to \kappa_2$, their width and lower level changes. The analysis in [44] uses the same idea when comparing the points z and w from different boxes by taking the largest box that contains both points i.e. considering the box $A_{n,2\sqrt{2},\xi}$ with $\xi = \max(\psi(\kappa), \tilde{\psi}(\kappa))$. We do the same thing here as we consider $\kappa_n \to \kappa_2$ by choosing for each n the largest box that contains both points in order to estimate the hyperbolic distance between them, i.e. we make use of the fact that the upper height of the boxes coincides and we work on $A_{n,2\sqrt{2},\xi(\kappa_n)}$ with $\xi = \max(\psi(\kappa_n), \tilde{\psi}(\kappa_2))$. This is a dynamical version (as we varry the index n) of the analysis in [44] that is performed for fix κ. For each fix n, the estimates work in the same manner.

Proposition 5.5 (Proposition 3.8 of [45]). *Let (g_t) be a Loewner chain with driver $\lambda(t)$ that satsfies the conditions*

- $osc(\lambda;\delta) = sup\{|\lambda(t) - \lambda(s)| : s,t \in [0,1], |t-s| \leqslant \delta\| \leqslant \sqrt{\delta}\phi(1/\delta)$.
- $|\hat{f}_t'(iy)| \leqslant c_0 y^{-\beta}$ *for all* $0 \leqslant y \leqslant y_0$, $t \in [0,1]$.

Then there exists a subpower function ϕ_1 such that if $0 \leqslant t \leqslant t+s \leqslant 1$ *and* $s \in [0, y^2]$

$$|\gamma(t+s) - \gamma(t)| \leqslant \phi_1(\frac{1}{y})\frac{2}{1-\beta}y^{1-\beta} \tag{5.1}$$

for $0 \leqslant s \leqslant y^2 \leqslant y_0^2$.

Secondly, let us consider any sequence κ_n, starting from κ_1 with $\lim_{n\to\infty}\kappa_n = \kappa_2$, such that $|\kappa_n - \kappa_2| \leqslant \frac{\phi(n)}{\sqrt{n}}$. The condition is not restrictive, since for convergent sequences $\kappa_n \to \kappa_2$ that have the n-th term outside the circle of radius $\frac{\phi(n)}{\sqrt{n}}$ we can use a relabeling of the terms. We have that

$$|\gamma^{\kappa_n}(t) - \gamma^{\kappa_2}(t)| \leqslant |\gamma^{\kappa_n}(t) - \gamma^n_{\kappa_2}(t)| + |\gamma^n_{\kappa_2}(t) - \gamma^{\kappa_2}(t)|, \tag{5.2}$$

where $\gamma^n_{\kappa_2}(t)$ is the trace obtained form interpolating with square root terms the driver $\sqrt{\kappa_2}B_t$.

The last term in the inequality is estimated by the Theorem 2.2 of [44]. For the first term, we use the analysis performed in [44] again by performing the same type of estimates as the ones used to show the convergence of the traces obtained by interpolating the Brownian driver for the Loewner equation in [44]. Namely, we estimate, for $r \in \left[\frac{1}{n}, \frac{2}{n}\right]$,

$$|\gamma^{\kappa_n}(s+t_k) - \gamma^n_{\kappa_2}(r+t_k)| \leqslant |f^{\kappa_n}{}_{t_k}(z) - f^{\kappa_n}{}_{t_k}(w)| + |f^{\kappa_n}{}_{t_k}(w) - f^{\kappa_2,n}_{t_k}(w)|.$$

As in [44], we estimate **the first term** using

$$|{f^{\kappa_n}}_{t_k}(z) - {f^{\kappa_n}}_{t_k}(w)| \leqslant (2\mathrm{Im} z)|(f^{\kappa_n}_{t_k})'(z)| \exp(4d(z,w)).$$

For a generic point $u = x + iy$, we use the almost sure estimate (note that is essential that we can control the nullsets as we vary κ using the rigidity of the Bessel processes on the real line).

$$|(f^{\kappa_n}_{t_k})'(z)| \leqslant c y^{-\beta(\kappa_n)} .$$

For the second term along with these estimates, we use Lemma 2.3 of [47]. For this we estimate the distance ε between the two driving terms: the square root interpolation of the $\sqrt{\kappa_2}B_t$ and $\sqrt{\kappa_n}B_t$ with

$$|\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_n}B_t| \leqslant |\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_2}B_t| + |\sqrt{\kappa_2}B_t - \sqrt{\kappa_n}B_t| .$$

Thus, by considering an appropriate mesh of the interval where κ takes values in, we obtain combining the estimates with the ones in [44] that

$$|\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_n}B_t| \leqslant \frac{\phi(n)}{\sqrt{n}} + |\sqrt{\kappa_n} - \sqrt{\kappa_2}| \sup_{t\in[0,1]} |B_t| \leqslant \frac{\phi(n)}{\sqrt{n}} + c\frac{\phi(n)}{\sqrt{n}} .$$

The next result that we use is the following Lemma that appeared in [47].

Lemma 5.6 (Lemma 2.3 of [47])**.** *Let $0 < T < \infty$. Suppose that for $t \in [0,T]$, $f^{(1)}_t$ and $f^{(2)}_t$ satisfy the backward Loewner differential equation with drivers $W^{(1)}_t$ and $W^{(2)}_t$. Suppose that $\varepsilon = sup_{s\in[0,T]}|W^{(1)}_s - W^{(2)}_s|$. Then, if $u = x + iy \in \mathbb{H}$, then*

$$|f^{(1)}_T(u) - f^{(2)}_T(u)| \leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log\frac{I_{Ty}|(f^{(1)}_T)'(u)|}{y} \log\frac{I_{Ty}|(f^{(2)}_T)'(u)|}{y} + \log\log\frac{I_{T,y}}{y}\right]\right] ,$$

where $I_{T,y} = \sqrt{4T + y^2}$.

Thus, in our case this gives that

$$|f^{\kappa_n}_{t_k}(w) - f^{\kappa_2,n}_{t_k}(w)| \leqslant \varepsilon \exp\left[\frac{1}{2}\left[\log\frac{I_{t_k,y}|(f^{\kappa_n}_{t_k})'(w)|}{y} \log\frac{I_{t_k,y}|(f^{\kappa 2,n}_{t_k})'(w)|}{y} + \log\log\frac{I_{t_k,y}}{y}\right]\right] ,$$

where $I_{t_k,y} = \sqrt{4t_k+y^2}$ with $\varepsilon = \sup_{t\in[0,t_k]} |\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_n}B_t| \leqslant \frac{2\phi(n)}{\sqrt{n}} + c\frac{2\phi(n)}{\sqrt{n}}$. These estimates are used for points inside the boxes $A_{n,c,\phi}$. Thus, for $y = \mathrm{Im} u \in [\frac{1}{\sqrt{n}\phi(n)}, \frac{2\sqrt{2}}{\sqrt{n}}]$ we have that

$$\frac{I_{t_k,y}}{y} \leqslant C\sqrt{n}\phi(n),$$

for $\phi(n)$ some sub-power function of n . Using the estimates

$$|(f^{\kappa_n}_{t_k})'(u)| \leqslant cy^{-\beta(\kappa_n)} \leqslant c_0\phi(n)^{\beta(\kappa_n)}\sqrt{n}^{\beta(\kappa_n)}\,.$$

and

$$|(f^{\kappa 2,n}_{t_k})'(w)| \leqslant C(1/y+1) \leqslant 2C\phi(n)\sqrt{n}$$

Note that the second estimate holds true for any conformal map of $\mathbb{H}$. Combining these estimates, we obtain that

$$|f^{\kappa_n}_{t_k}(w) - f^{\kappa_2,n}_{t_k}(w)| \leqslant (1+c)\frac{\phi(n)}{\sqrt{n}} \exp\left[\sqrt{\frac{1+\beta(\kappa_n)}{2}}\log(c\phi(n)\sqrt{n}) + \log\log 2\sqrt{2n}\phi(n)\right] \tag{5.3}$$

$$=: \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \tag{5.4}$$

Thus, using the Lemmas before for the sequence of parameters κ_n we obtain that

$$|\gamma^{\kappa_n}(s+t_k) - \gamma^{\kappa_2,n}(r+t_k)| \leqslant \frac{\phi_1(n)}{\sqrt{n}^{1-\beta(\kappa_n)}} + \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \tag{5.5}$$

for all $r \in [\frac{1}{n}, \frac{2}{n}]$. Using Proposition 5.5, for the sequence of parameters κ_n we obtain that

$$|\gamma^{\kappa_n}(s+t_k) - \gamma^{\kappa_2,n}(r+t_k)| \leqslant \frac{\phi_2(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \tag{5.6}$$

for all $r \in [t_{k+1}, t_{k+2}]$ and $0 \leqslant k \leqslant n-2$ and hence for all $r \in [0,1]$.

In order to estimate the first term in 5.2, i.e. the term $|\gamma^{\kappa_n}(t) - \gamma^n_{\kappa_2}(t)|$ we consider the following estimate from [45].

Proposition 5.7 (Proposition 4.2 in [45]). *Let $\beta = \beta(\kappa) < 1$ for $\kappa \neq 8$. Suppose $\beta > \max\{0, \beta_+\}$ where β_+ is an explicit quadratic function of κ. With probability one there exists $y_0(\omega) > 0$, such that for all $t \in [0,1]$ and all $y < y_0$,*

$$|f'_t(iy)| \leqslant y^{-\beta}\,. \tag{5.7}$$

This implies that for constants c_3 and c_4, β' depending on κ that

$$\sum_{m=n}^{\infty}\sum_{j=1}^{2^{2m}}\mathbb{P}\left[|\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \geqslant 2^{\beta' m}\right] \leqslant \frac{c_3}{2^{nc_4}} \tag{5.8}$$

As in [44], this implies (via the change $2^m \to \sqrt{m}$, that there exits $c_5 > 0$ and $\beta \in (\beta', 1)$ such that

$$\mathbb{P}\left[|\hat{f}'_t(iy)| \leqslant y^{-\beta} \text{ for all } 0 \leqslant y \leqslant \frac{1}{\sqrt{n}}, t \in [0,1]\right] \geqslant 1 - \frac{c_3}{n^{c_4/2}}\,. \tag{5.9}$$

Note that the estimate 5.9 holds for fix value of the parameter κ. A weaker form of this estimate along with the modulus of continuity of Brownian motion is used in order to prove the existence of the SLE trace for fixed value of $\kappa \neq 8$ in [36]. When taking the limit $n \to \infty$, the estimate holds outside of a nullset that depends on the fixed value κ.

The constants appearing in the estimate change with κ_n but the nullsets are controlled since have established the previous result also quasi surely (q.s.)
Thus, we can use the estimate

$$Cap\left[|\hat{f}'_t(iy)| \leqslant y^{-\beta(\kappa_n)} \text{ for all } 0 \leqslant y \leqslant \frac{1}{\sqrt{n}}, t \in [0,1]\right] \geqslant 1 - \frac{c_3(\kappa_n)}{n^{c_4(\kappa_n)/2}}\,. \tag{5.10}$$

without changing the whole polar set on which we defined everything when we consider $\kappa_n \to \kappa_2$. Moreover, it follows from [**?**] that there exists constants c_1 (depending on κ) and c_2 such that

$$\mathbb{P}\left[osc(\sqrt{\kappa}B_t, \frac{1}{m}) \geqslant c_1\sqrt{\frac{\log m}{m}}\right] \leqslant \frac{c_2}{n^2}\,. \tag{5.11}$$

Using 5.6, we obtain the following estimate for

$$Cap\left[||\gamma^{k_n} - \gamma^{\kappa_2,n}||_{[0,1],\infty} \leqslant \frac{c_6(\log m)^{c_7}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}} \text{for all } m \geqslant n\right] \geqslant 1 - \left(\frac{c_2(\kappa_n)}{n^2} + \frac{c_3(\kappa_n)}{n^{c_4(\kappa_n)/2}}\right)$$

In [44] it is proved the following result about the SLE trace $\gamma(t)$ for a generic fixed κ.

$$\mathbb{P}\left[||\gamma^{(\kappa)}(t) - \gamma^{(m)}(t)||_{[0,1],\infty} \leqslant \frac{c_1(log(m))^{c_2}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa)}{2}}}} \text{for all } m \geqslant n\right] \geqslant 1 - \frac{c_3}{n^{c_4}},$$

where the constants c_1, c_2, c_3 and c_4 depend on κ. We use this to estimate the second term in 5.2, i.e. we the term $|\gamma^n_{\kappa_2}(t) - \gamma^{\kappa_2}(t)|$ for the fixed value of $\kappa = \kappa_2$.

By letting $n \to \infty$ we obtain the estimate as in [44] quasi surely (i.e. outside the prescribed polar set) and we finish the proof.

$\square$

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