Remarks on Loewner chains driven by Semimartingales and Complex Bessel-type SDEs

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Abstract

In this work we define the Loewner Differential Equation driven by Semi-martingales that satisfy certain conditions. The first condition gives a bound on the increments of the quadratic variation process of the local martingale part, while the second one is a finite energy estimate for the bounded variation part of the driver. Under these conditions, we show that the Loewner Differential Equation is almost surely generated by a simple curve, extending the class of random drivers considered so far in the stochastic Loewner theory. We emphasize that the first condition is optimal, as it is known that in the case of the Brownian driver the SLE trace is a.s. simple for $\kappa \leq 4$.

1 Introduction

In this article we study Loewner chains driven by semimartingales. This is part of a joint project with Y. Yuan (TU Berlin) as a co-author. This result will be combined with the work in [17]. Loewner theory is a key ingredient in the construction of Schramm-Loewner Evolutions (SLEs) and a Loewner chain is a family $\{K_t\}_{t\in[0,T]}$ of subsets of the upper half-plane $\mathbb{H}$ satisfying certain local growth property. Loewner chains are in a one-to-one correspondence with real-valued continuous functions $\{W_t\}_{t\in[0,T]}$ which we refer to as driving functions. When W_t is chosen to be $\sqrt{\kappa}B_t$, where B_t is a standard Brownian motion, it gives rive to the SLE_κ curves.

More precisely, it was prove in [1] that the Loewner chain driven by $\sqrt{\kappa}B_t$ is generated by a continous curve γ^κ in $\bar{\mathbb{H}}$ in the sense that for $t > 0$, $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. The curve is γ^κ is defined as SLE_κ. The curve γ^κ is also simple and $\gamma^\kappa_t \in \mathbb{H}$, $\forall t > 0$, for $\kappa \leq 4$. In this article, we consider we consider driving functions which are continuous semimartingales. We will actually only require the time-reversal of W to be a semimartingale. While the time-reversal of Brownian motion is again a Brownian motion, the time-reversal of a general semimartingale might not be a semimartingale. However, in many cases, e.g. for diffusions the semimartingale property can be established via 'expansion of the filtration technique', see [3] for details. Let us consider a driving function $\{W_t\}$ such that the time-reversal $W_T - W_{T-t}$ is a continuous semimartingale defined on a filtered

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probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfying the usual hypothesis. We ask ourselves whether the corresponding Loewner chain is generated by a curve, and whether it is simple. The main result of this paper is a positive answer to the above question under some natural conditions on the semimartingale U. To state our conditions, we admit that a semimartingale U admits the decomposition $U_t = M_t + A_t$, where M_t is a local martingale and A_t is a finite variation process. We will further assume the following conditions.

- Condition 1 : For some $\kappa < 4$, $|[M]_t - [M]_s| \leqslant \kappa|t-s|$, for all $s, t \in [0, T]$, where $[M]_t$ is the quadratic variation process of M.
- Condition 2 : For some α large enough depending only on κ, (which can be explicitly computed by looking into the proof)

$$\mathbb{E}\left[\exp\left\{\alpha \int_0^T \dot{A}_r^2 dr\right\}\right] < \infty$$

Our main result is the following theorem.

Theorem 1.1. *If U is a semimartingale satisfying Condition 1 then the Loewner chain $\{K_t\}_{t\in[0,T]}$ driven by U is given by $K_t = \gamma[0,t]$ where γ is a simple curve such that $\gamma \in \mathbb{H}$, $\forall t > 0$.*

Some remarks are in order:

- The condition 'Condition 1)' forces the local martingale M to be an actual martingale. The condition 'Condition* $\to$ 2)' implies that A is a finite energy process. This might suggest the role of the Girsanov Theorem in the proof of Theorem 1.1 but since we do not assume any lower bound on $\frac{d[M]_t}{dt}$ it rules out the use of Girsanov Theorem. (Using Rademacher's Theorem the Condition 1 implies that $\frac{d[M]_t}{dt}$ exists for almost every t and $\frac{d[M]_t}{dt} \leq \kappa$.)
- The existence of the curve in Theorem 1.1 for $\frac{d[M]_t}{dt} \leq \kappa < 2$ was also proved in [16]. The simpleness of γ for the whole range $0 \leq \frac{d[M]_t}{dt} \leq \kappa < 4$ is new to this article. The condition $\frac{d[M]_t}{dt} \leq \kappa < 4$ is also optimal for simpleness because it is known that SLE_κ is simple only for $\kappa \leq 4$.
- Our proof of the existence part in Theorem 1.1 is an argument in the same vein as that of [16]. Comparing to the work in [1] we remark that the method of [1] for SLE_κ is based on exact computations of certain 'martingale observables'. When the driving function is a general semimartingale, these exact computations are not feasible. Our main contribution in the proof of Theorem 1.1 is an idea which bypasses these exact computations. While being more general in terms of non-constant $\frac{d[M]_t}{dt}$, one drawback of our method is that it only works for $\kappa < 4$ while the exact computations method of [1] works for all $\kappa \neq 8$.
- The proof of the simpleness of γ in Theorem 1.1 is also completely different as compared to the proof of simpleness of SLE_κ, $\kappa \leq 4$ in [1]. The proof of [1] is based on the fact

that the Bessel process of dimension $\delta \geq 2$ [1][2] is started from $x > 0$ doesn't hit zero in finite time. Our condition $\frac{d[M]_t}{dt} \leq \kappa < 4$ might suggest that we can use stochastic comparison techniques to conclude simpleness in Theorem 1.1 (and this was also our first attempt to prove this), but the presence of the finite variation term A coupled with the fact that we do not assume any lower bound on $\frac{d[M]_t}{dt}$ forces us to avoid stochastic comparison techniques.

We therefore find and argument which does not rely on Bessel processes. While this method is general enough to handle non constant $\frac{d[M]_t}{dt}$, we do have to pay the price of not being able to handle $\kappa = 4$ case. For using stochastic domination techniques there is another issue while considering $[M]^{-1}(t)$. In principle, there can be intervals where $[M]^{-1}(t)$ is constant, so that $[M]^{-1}(t)$ is discontinuous. In this interval of constancy, M_t is constant and the A part dominates. So the stochastic tools fail to apply here and we need deterministic tools at work.

The outlook of this paper is to understand the problem of the existence of the curve γ for more and more general driving functions W. A related question asked by Avelio Sepulveda (motivated from reflection/two-valued sets ?) is whether the Loewner chain driven by $|B_t|$ is generated by a curve. While $|B_t|$ is a semimartingale, its finite variation part (local time) is not of finite energy. Therefore, our Theorem 1.1 does not answer this particular question. But our premise behind proving Theorem 1.1 is to precisely find a technique which allows us to answer questions such as the one above. Another motivation to consider functions with non-constant $\frac{d[M](t)]}{dt}$ is to study scaling limits of gluing of two different statistical mechanics models, e.g. gluing percolation on the upper half-plane with the self-avoiding walk on the lower half-plane (credit Avelio Sepulveda). In view of these relations being only heuristic, we choose to not pursue it here and detail, and content ourselves with the note that Theorem 1.1 may have possible applications as the field of random geometry progresses. Our proof of Theorem 1.1 is based on the following condition which is equivalent with the existence of γ. IT is proved in [1] that the curve γ exists if and only if $\lim_{y\to 0+} f_t\,(iy + W_t)$ exists and the limit is continuous in t, where f_t is the conformal map mapping $\mathbb{H}$ to $\mathbb{H} \setminus K_t$ such that $f_t(z) = z + O(1)$ as $z \to \infty$. To prove Theorem 1.1 we will verify the above conditions. Moreover, to identify the $\lim_{y\to 0+} f_t\,(iy + W_t)$ we construct a canonical object as follows. It is well known that (see [1] for a precise statement) that a description of $f_t(iy+W_t)$ for each fixed t can be obtained by solving the reverse time Loewner Differential Equation (LDE). For a given real-valued continuous function V_s with $V_0 = 0$ the LDE driven by V and started at iy is the equation

$$dh_s = dV_s - \frac{2}{h_s} ds, \qquad h_0 = iy. \tag{1.1}$$

Therefore, to characterize/identify the $\lim_{y\to 0+} f_t\,(iy + W_t)$ is natural to consider (1.1) started from 0. But of course, when $h_0 = 0$, the equation (1.1)is a singular equation and it is a priori not well defined. The second main result of this paper, gives a precise

[1][1] works with the forward flow to establish simpleness. Since we do not have any property of the driving function W and only assume semimartingale property of the time-reversal U, we have to work with the backward flow.

[2]For the forward flow $\delta \geq 2$ corresponds to $\kappa \leq 4$ using the relation $\delta = 1 + \frac{4}{\kappa}$.

sense to this singular equation when V is a semimartingale satisfying the conditions from the introduction. To do so, we follow the approach of [7] which deals with a similar situation while making sense of real Bessel processes starting from zero. The idea is to consider the squared equation, i.e. the equation satisfied by $\varphi_s = h_s^2$. To recover h from ϕ, it is natural to take the square-root. But since we are constructing a candidate for $\lim_{y\to 0+} f_t\,(iy + W_t)$, we expect the solution ϕ or h to be complex valued. Therefore, to take square-root of ϕ, one has to choose a branch. This prompts us to the following definition.

Definition 1.2. *For a continuous function (respectively continuous adapted process $\varphi : [0,T] \to \mathbb{C}$, a branch square-root of φ is a measurable (respectively adapted) ϕ process $\theta : [0,T] \to \mathbb{H}$ such that $\theta_t^2 = \varphi_t$, $\forall t \in [0,T]$. We then write $\theta_t = \sqrt{\varphi}^b$.*

Note that we are here working with the square-root function

$$\sqrt{z} = \operatorname{sgn}(\operatorname{Im}(z))\sqrt{\frac{|z| + \operatorname{Re}(\tau)}{2}} + i\sqrt{\frac{|z| - \operatorname{Re}(z)}{2}}$$

which is a bijection from $\mathbb{C} \setminus [0,\infty)$ to $\mathbb{H}$. Note that there are two possible extensions of $\sqrt{z}$ as z approaches $x \in (0,\infty)$, $+\sqrt{x}$ or $\sqrt{x}$. An Itó formula computation suggests that $\varphi_s = h_s^2(0)$ should satisfy the equation (which is a square Bessel type equation except the vector field $\sqrt{|x|}$)

$$d\phi_s = 2\sqrt{\varphi_s}^b dV_s + d[V]_s - 4ds, \;\; \phi_0 = 0, \tag{1.2}$$

where $\sqrt{\varphi}^b$ is a branch of square-root. Note that by the adapted assumption on $\sqrt{\varphi}^b$, the above SDE is well-defined. Therefore, to make sense of $h_s(0)$ we first uniquely solve for ϕ using equation (1.2) and then define h as its square-root. Note that the solution ϕ of (1.2) can be as well real-valued in general situations. But, when $d[V]_s$ is dominated by $4ds$, the negative drift in (1.2) will push the solution away from $[0,\infty)$. Also, it can be shown that once the solution enters $\mathbb{C} \setminus [0,\infty)$, it stays there for all future times [1]. We utilize this idea to prove that in this case, the equation (1.2) admits a unique strong solution.

Theorem 1.3. *If V is a semimartingale satisfying conditions* 1) *and* 2) *from the introduction, then*

- *a) If ϕ is a solution to (1.2), then a.s. $\forall t > 0$, $\phi_t \in \mathbb{C} \setminus [0,+\infty)$. In particular, $\sqrt{\phi_t}^b = \sqrt{\phi_t}$, and (1.2) is equivalent to*

$$d\phi_s = 2\sqrt{\varphi_s}^b dV_s + d[V]_s - 4ds \tag{1.3}$$

- *b) There exists a continuous adapted process satisfying (1.5). Moreover, if ϕ and $\tilde{\phi}$ are two solution processes, then*

$$\mathbb{P}\left[\varphi_t = \tilde{\varphi}_t, \;\; \forall t \geqslant 0\right] = 1.$$

[1]This negative drift explains also way one should expect a simple curve γ in this case.

Note however that in the absence of this negative drift, the uniqueness of solution of equation (1.2) is not expected to hold. For example, when $\frac{d[\mathrm{U}]_s}{ds} = 4$, one trivial solution is $\varphi = 0$. One can also construct a non-zero solution to equation (1.2) by examining SLE4. When $\frac{d[\mathrm{U}]_s}{ds} = \kappa > 4$, the situation is even more complicated. In this case, real Bessel processes of dimension $\kappa - 4$ are solutions to (1.2) and so are negative of these processes. One can also construct $\mathbb{C} \setminus [0, \infty)$ solutions by examining SLE_κ, for $\kappa > 4$. One can then construct then other solutions that stay on $[0, \infty)$ until some positive time (where it behaves like $\pm$ Bessel processes). and then it escapes to $\mathbb{C} \setminus [0, \infty)$. One such choice of solutions describes the SLE_κ curve for $\kappa > 4$, and the time until which the solution stays on $[0, \infty)$ is deeply connected to the double points of SLE_κ for $\kappa > 4$. One of our future goal is to investigate this in further details. For the purpose of the present, we content ourselves with Theorem 1.3. The main message of this theorem is that by passing from $\mathbb{H}$ to $\mathbb{R}$, there is no loss of information if $\frac{d[\mathrm{U}]_s}{ds} \leq \kappa < 4$.

Some remarks are in order :

- A similar half-plane values solutions to Bessel SDEs has also been considered in [9] Prop. 3.8, in connection to Liouville-Quantum Gravity. but the paper [9] only established the existence and uniqueness of weak solutions. Our Theorem 1.3 however establishes the strong existence and uniqueness.
- The proof of uniqueness of solution to SDE (1.2) is similar to the proof of the existence of the limit $\lim_{y \to 0+} f_t (iy + W_t)$ in the sense that they both use the same underlying technique. However, we remark that even if the underlying proof is similar both of these statements are not completely equivalent. The uniqueness of solution to (1.2) in fact implies that $\lim_{z \to 0+} f_t (iy + U_t)$ exists (including tangential limits) as compared to the existence of $\lim_{y \to 0+} f_t (iy + U_t)$ (non-tangential limit). In general situations, these two are not equivalent, see for example ([4]). Moreover, when $\frac{d[[V]_t]}{dt} = \kappa = 4$, the uniqueness of the solution fails. However, the limit $\lim_{y \to 0+} f_t (iy + W_t)$ still exists.

Related works:

- In the case of real Bessel SDEs, the uniqueness of solution is a consequence of the Yamada-Watanabe Theorem. This technique fails to apply in the complex case because the complex square root $\sqrt{z}$ is not a $\frac{1}{2}-$Hölder function. Our Theorem 1.3 is related to the work of Krylov-Röckner which considers multi-dimensional SDEs with singular drifts. Equation (1.1) started from zero can be viewed as a 2-dimensioal SDE with a singular drift. A distinction between Theorem 1.3 and the paper [11] is that the noise term in Theorem 1.3 is only one-dimensional.
- In case of $\kappa \geq 4$, the non-uniqueness of solution is related to the work of Bass-Burdzy-Chen where they establish the uniqueness of certain degenerate SDE under the assumption that the solution spends zero time at zero. We believe that a similar uniqueness result should also hold in our case.

We define the solution to $h_s(0)$ of (1.1) started from zero as $h_s(0) = \sqrt{\phi_s}$. Note that it follows that $\forall s \geq s_0 > 0$

$$h_s(0) = h_{s_0}(0) + V_s - V_{s_0} - \int_{s_0}^{s} \frac{2}{h_r(0)} dr.$$

This, in particular implies that (using the continuity of h_s)

$$\int_{0+}^{s} \frac{2}{h_r} dr = \lim_{s_0 \to 0+} \int_{s_0}^{s} \frac{2}{h_r} dr, \tag{1.4}$$

exists, and

$$h_s = V_s - \int_{0+}^{s} \frac{2}{h_r} dr. \tag{1.5}$$

But, we are a priori not sure whether $\int_{0+}^{s} \frac{2}{|h_r|} dr \leq \infty$, or not.

The conditional convergence (and not the absolute convergence) of the integral (1.4) makes the equation 1.5 inconvenient to deal with directly. That is why we use the trick of considering the squared equation. However, comparing with the real Bessel process suggests that the absolute convergence should hold at least for small κ.

To represent the curve γ in terms of the solution obtained in Theorem (1.3) we consider the stochastic flow of equation (1.1) in $\bar{\mathbb{H}}$. More precisely, let $h(s,t,z)$ denote the solution of

$$h(s,t,z) = z + U_t - U_s - \int_{s}^{t} \frac{2}{h(s,r,z)} dr, \quad h(s,s,z) = z, \tag{1.6}$$

where $0 \leq s \leq t \leq T$ and $z \in \bar{\mathbb{H}}$. When $z \in \mathbb{H}$ the solution is classically well-defined. When $z = 0$, the solution is constructed by Theorem 1.3. When $z \in \mathbb{R} \setminus \{0\}$, the solution is classically well-defined until it hits zero. For future time, after the hitting time of zero, it continues as constructed by Theorem (1.2). The object $h(\cdot,\cdot,\cdot)$ is the stochastic flow associated with the equation (1.1) and it satisfies the so called flow property : $\forall s \leq u \leq t$

$$h(s,t,z) = h(u,t,h(s,u,z)).$$

Corollary 1.4 (SLE as Stochastic Flow)**.** *The curve γ is given by*

$$\gamma_t = h(T-t,T,0).$$

In particular, the law of the SLE_κ curve restricted to $[0,T]$ is the same as that of $h^\kappa(T-t,T,0)$, where h^κ is the stochastic flow driven by $\sqrt{\kappa}B$.

2 Preliminaries and Definitions

In this section, we recall some of the definitions and well known facts which we will need throughout the paper. A Loewner chain is a family $\{K_t\}_{t\in[0,T]}$ of subsets of $\mathbb{H} \cup \{0\}$ such that $K_0 = \{0\}$ and

– $\forall t > 0$, K_t is bounded and $\mathbb{H} \setminus K_t$ is a simply connected domain.

- $\{K_t\}$ is increasing, i.e $K_s \subseteq K_t$, $\forall s < t$.
- $\{K_t\}_{t\geqslant 0}$ satisfies the local growth property, i.e. $\mathrm{rad}\left(g_t\left(K_{t+h}\backslash K_t\right)\right) \to 0$, uniform in t, as $h \to 0+$, where $g_t : \mathbb{H} \setminus K_s \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$, as $z \to \infty$. The map g_t is called the mapping-out function of K_t. We choose to work with the half-plane capacity parametrisation of $\{K_t\}$, i.e.

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad \text{as } z \to \infty.$$

The driving function of $\{K_t\}$ is defined as

$$W_t := \cap_{h\to 0}\overline{g_t\left(K_{t+h}\backslash K_t\right)}.$$

Note that the right hand side is a single point due to the local growth property. The driving function is a continuous real-valued function with $W_0 = 0$. The maps g_t satisfies the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z. \tag{2.1}$$

Furthermore, if $[0, T^z)$ is the maximal interval where the solution of (LDE) exists, then

$$K_t = \{z \in \mathbb{H} \mid T^z \leq t\}. \tag{2.2}$$

The correspondence $\{K_t\}_{t\in[0,T]}$ to $\{W_t\}_{t\in[0,T]}$, is bijective and for a given continuous real-valued function W with $W_0 = 0$, the relation (2.2) defines a Loewner chain. For this article, we are interested in the backward direction, i.e. starting from W and then recovering K_t. For this purpose, we look at $f_t = g_t^{-1}(\cdot + W)$. The following lemmas allows us to recover f_t from W. Let $h(s,t,z)$ be the solution to

$$h(s,t,z) = z + U_t - U_s - \int_s^t \frac{2}{h(s,r,z)}dr.$$

Lemma 2.1 (Lemma 2.1 of [8])**.**

$$f_t(z) = h(T-t, T, z).$$

It was proven in [1] that the existence of the curve γ is equivalent to the existence of the limit $\lim_{y\to 0+} f_t(iy)$ and its continuity in t. It was also proven in [1] that to verify this condition, it suffices to obtain an estimate as in the following lemma (see [1] or [15] for details).

Lemma 2.2.

$$\sup_{t\in[0,T]} |f_t'(iy)| \leqslant y^{-\theta}, \quad \textit{for some } \theta \in (0,1). \tag{2.3}$$

Furthermore, for a weakly $1/2$*-Hölder driving function the estimate* (2.3) *follows a.s. from a moment estimate*

$$\sup_{y>0} \sup_{t\in[0,T]} \mathbb{E}\left[|f_t'(iy)|^b\right] < \infty,$$

for some $b > 2$.

We will follow this strategy to prove Theorem 1.1. To this end, we recall that (see [1]) if $h(s,t,z) = X_t + iY_t$ (we will drop the index s, z to avoid confusion in the cumbersome notation and its dependence on s, z will be understood from the context), then

$$|h'(s,t,z)| = \exp\left\{\int_s^t \frac{2\left(X_r^2 - Y_r^2\right)}{\left(X_r^2 + Y_r^2\right)^2} dr\right\}, \tag{2.4}$$

and

$$dX_t = dU_t - \frac{2X_t}{X_t^2 + Y_t^2} dt, \;\; dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} dt. \tag{2.5}$$

Another key tool which will be important for our proof is the Gronwall inequality. We will use in the following slightly unconventional form.

Lemma 2.3 (Gronwall inequality)**.** *Let $F(t,x)$ be a bounded continuous function which is continuously differentiable in x with $\partial_x F(t,x) \geq 0$. Let L_t be a continuous function such that*

$$L_t \leq \int_0^t F\left(r, L_r\right) dr,$$

and R_t be a continuous function satisfying

$$R_t = \int_0^t F\left(r, R_r\right) dr. \tag{2.6}$$

Then,

$$L_t \leq R_t, \;\; \forall t \geq 0.$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1. In this section, we prove Theorem 1.1. We will rely on the strategy of Lemma 2.2 stated above. The existence of the curve γ will be a consequence of the following Proposition.

Proposition 3.1. *It U is a semimartingale satisfying the conditions* 1) *and* 2)*, then there exists $b > 2$ depending only on κ, α such that*

$$\sup_{s,y} \mathbb{E}\left[\sup_t |h'(s,t,iy)|^b\right] < \infty, \tag{3.1}$$

In particular, there exists a $\theta \in (0,1)$ such that for all y small enough,

$$\sup_{0 \leqslant s \leq t \leq T} |h'(s,t,iy)| \leqslant y^{-\theta}. \tag{3.2}$$

Remark 1. *We remark that in contrast to the moment estimate established in [1], the estimate* (3.1) *has supremum w.r.t. t inside the expectation. This improvement over the estimate of [1] is an artifact of our new method. We note that this uniform moment estimate therefore imply the unform decay estimate (3.2). The similare decay estimate of [1] is not uniform w.r.t t.*

Remark 2. *Semimartingales satisfying conditions* 1) *and* 2) *are indeed weakly* 1/2*-Hölder. This is not obvious and it does not follow directly from the fact that martingales are time-changed Brownian motions. But one can check that indeed this is true, fir example using the GRR inequality. Therefore, the proof of the estimate* (3.2) *from* (3.1) *is similar to the proof in [1].*

We now prove Proposition 3.1

Proof. We use the formula

$$\log|f_t'(iy+U_t)| = \left\{\int_0^t \frac{2\left(X_r^2 - Y_r^2\right)}{\left(X_r^2+Y_r^2\right)^2} dr\right\}. \tag{3.3}$$

The key idea behind obtaining a moment estimate on $|h'(s,t,iy)|$ is to use the Itô-Tanaka trick. More precisely, we want to realize the integral on the right hand side of (3.3) as the quadratic variation term in the Itô-formula and then write it in terms of a stochastic integral. That way, we have the martingale techniques at our disposal to obtain the required moment estimates. To implement this strategy, we apply the Itô formula to $\log(X_t^2+Y_t^2)$. It will follow using (2.5) and some straightforward computations that

$$\begin{aligned}\log\left(X_t^2+Y_t^2\right) = \log y^2 + \int_s^t \frac{2X_r}{X_r^2+Y_r^2} dU_r - \int_s^t \frac{2X_r^2}{X_r^2+Y_r^2} d[U]_r \\ - \int_s^t \frac{4(X_r^2-Y_r^2)}{(X_r^2+Y_r^2)^2} dr + \int_s^t \frac{1}{X_r^2+Y_r^2} d[U]_r.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}2\log|h_t'(s,t,iy)| &= \int_s^t \frac{2X_r}{X_r^2+Y_r^2} dU_r - \int_s^t \frac{2X_r^2}{\left(X_r^2+Y_r^2\right)^2} d[U]_r - \log\left(1+\frac{X_t^2}{Y_t^2}\right) \\ &\quad - 2\log\left(\frac{Y_r}{y}\right) + \int_s^t \frac{1}{X_r^2+Y_r^2} d[U]_r \\ &\le \int_s^t \frac{2X_r}{X_r^2+y_r^2} dU_r - \int_s^t \frac{2X_r^2}{\left(X_r^2+Y_r^2\right)^2} d[U]_r \\ &\quad - \log\left(1+\frac{X_r^2}{Y_r^2}\right) - \left(2-\frac{\kappa}{2}\right)\log\left(\frac{y_t}{y}\right),\end{aligned} \tag{3.4}$$

where we have used $\frac{d[U]_t}{dt} \le \kappa$ and $\log\frac{Y_t}{y} = \int_s^t \frac{2}{X_r^2+Y_r^2} dr$ in the last inequality. To appreciate the crux of the argument, let us first note that since $\kappa < 4$

$$\log\left(1+\frac{X_t^2}{Y_t^2}\right) + \left(2-\frac{\kappa}{2}\right)\log\frac{Y_t}{y} \geqslant 0.$$

This implies

$$|h'(s,t,iy)|^2 \le \exp\left\{\int_s^t \frac{2X_r}{X_r^2+Y_r^2} dU_r - \int_s^t \frac{2X_r^2}{X_r^2+Y_r^2} d[U]_r\right\}. \tag{3.5}$$

If U is a semimartingale, (i.e. when $A \equiv 0$) note the that the right hand-side is an exponential local martingale of the local martingale $\int_s^t \frac{2X_r}{X_r^2+Y_r^2} dU_r$. Since positive local martingales, are supermartingales, this implies that $\mathbb{E}\left[|h'(s,t,iy)|^2\right] \leqslant 1$ which already gives the second moment estimate. To get a higher bound estimate. To get higher moment estimate and to also handle the A term. we have to do some additional work as follows, which is in the same vein as explained above. Write N_t for the local martingale $N_t = \int_s^t \frac{2X_r}{X_r^2+Y_r^2} dM_r$. Note that $[U] = [M]$ and $[N]_t = \int_s^t \frac{4X_r^2}{(X_r^2+Y_r^2)^2} d[U]_r$. Then,

$$\int_s^t \frac{2X_r}{X_r^2+Y_r^2} dU_r = N_t + \int_s^t \frac{2X_r}{X_r^2+Y_r^2} \dot{A}_r dr \le N_t + \frac{\delta^2}{2} \int_s^t \frac{X_r^2}{(X_r^2+Y_r^2)^2} dr + \frac{2}{\delta^2} \int_s^t \dot{A}_r^2 dr.$$

Furthermore, note that

$$\begin{aligned}\int_s^t \frac{X_r^2}{(X_r^2+Y_r^2)^2} dr &= \frac{1}{4} \int_s^t \frac{2}{X_r^2+Y_r^2} dr + \frac{1}{4} \int_s^t \frac{2\left(X_r^2-Y_r^2\right)}{(X_r^2+Y_r^2)^2} dr \\ &= \frac{1}{4} \log\left(\frac{Y_t}{y}\right) + \frac{1}{4} \log|h'(s,t,iy)|. \end{aligned} \tag{3.6}$$

Therefore,

$$\int_s^t \frac{2X_r}{(X_r+Y_r)^2} dU_r \le N_t + \frac{\delta^2}{8} \log\left(\frac{Y_t}{y}\right) + \frac{\delta^2}{8} \log|h_t'(s,t,iy)| + \frac{2}{\delta^2} \int_s^t \dot{A_r}^2 dr.$$

Plugging this estimate in the (3.4) gives us

$$\begin{aligned} 2\log|h'(s,t,iy)| \le N_t &- \frac{[N]_t}{2} - \log\left(1 + \frac{X_t^2}{Y_t^2}\right) \\ &+ \left(\frac{\delta^2}{8} + \frac{\kappa}{2} - 2\right) \log\left(\frac{Y_t}{y}\right) \\ &+ \frac{\delta^2}{8} \log|h'(s,t,iy)| + \frac{2}{\delta^2} \int_s^t \dot{A_r}^2 dr. \end{aligned}$$

Using $\log\left(1+\frac{X_t^2}{Y_t^2}\right) \geqslant 0$, we get that

$$\left(2 - \frac{\delta^2}{8}\right) \log|h'(s,t,iy)| \leqslant N_t - \frac{[N]_t}{2} + \left(\frac{\delta^2}{8} + \frac{\kappa}{2} - 2\right) \log\left(\frac{Y_t}{y}\right) + \frac{2}{\delta^2} \int_s^t \dot{A}_r^2 dr. \tag{3.7}$$

Next, we obtain a lower bound on $\log|h'(s,t,iy)|$. To this send, note that using conditions 1) and 2)

$$\begin{aligned} [N]_t = \int_0^t \frac{4X_r^2}{(X_r^2+Y_r^2)^2} d[U]_r &\leqslant 4\kappa \int_s^t \frac{X_r^2}{(X_r^2+Y_1^2)^2} dr \\ &= \kappa \log \frac{Y_t}{y} + \kappa \log|h'(s,t,iy)|. \end{aligned} \tag{3.8}$$

Therefore,

$$\kappa \log|h'(s,t,iy)| \geqslant [N]_t - \kappa \log \frac{Y_t}{y}. \tag{3.9}$$

Next, we pick constants $\lambda, \mu > 0$ and multiply the (3.7) by λ and the lower bound by μ. Then, taking the difference of the (3.7) and (4.2), we get

$$\begin{aligned}
&\left(\lambda\left(2-\frac{\delta^2}{8}\right)-\mu\kappa\right)\log|h'(s,t,iy)| \\
&\leq \lambda N_t - \frac{\lambda}{2}[N]_t + \lambda\left(\frac{\delta^2}{8}+\frac{\kappa}{2}-2\right)\log\left(\frac{Y_t}{y}\right) \\
&+ \frac{2\lambda}{\delta^2}\int_s^t \dot{A}_r^2 dr - \mu[N]_t + \kappa\mu\log\left(\frac{Y_t}{y}\right) \\
&= \lambda N_t - \left(\frac{\lambda}{2}+\mu\right)[N]_t + \frac{2\lambda}{\delta^2}\int_s^t \dot{A}_r^2 dr + + \left\{\kappa\mu + \lambda\left(\frac{\delta^2}{8}+\frac{\kappa}{2}-2\right)\right\}\log\left(\frac{Y_t}{y}\right).
\end{aligned}$$

We want to choose λ and μ such that

$$\kappa\mu + \lambda\left(\frac{\delta^2}{8}+\frac{\kappa}{2}-2\right) < 0.$$

Therefore, since $\log\left(\frac{Y_t}{y}\right) \geqslant 0$, we get

$$\left\{\lambda\left(2-\frac{\delta^2}{8}\right)-\mu\kappa\right\}\log\mu'(s,t,iy)\ |\leq \lambda N_t - \left(\frac{\lambda}{2}+\mu\right)[N]_t + \frac{2\lambda}{\delta^2}\int_s^t \dot{A}_r^2 dr. \tag{3.10}$$

Note that the purpose behind multiplying (3.7) by λ and (4.2) by μ is to only drop a certain portion of $\log\left(\frac{Y_t}{y}\right)$ and not any larger portion of it as we did while obtaining the (3.5). Therefore, using (3.10), we obtain using Hölder inequality that

$$\begin{aligned}
\mathbb{E}\left[|h'(s,t,iy)|^{\left(2-\frac{\delta^2}{8}\right)-\mu\kappa}\right] &\leq \mathbb{E}\left[\exp\left\{\lambda N_t - \left(\frac{\lambda}{2}+\mu\right)[N]_t\right\}\exp\left\{\frac{2\lambda}{\delta^2}\int_s^t \dot{A}_r^2 dr\right\}\right] \\
&\leq \mathbb{E}\left[\exp\left\{\lambda p N_t - p\left(\frac{\lambda}{2}+\mu\right)[N]_t\right\}\right]^{1/p}\mathbb{E}\left[\exp\left\{\frac{2\lambda q}{\delta^2}\int_s^t \dot{A}_r^2 dr\right\}\right]^{1/q},
\end{aligned}$$

where $p > 1$ are such that $\frac{1}{p}+\frac{1}{q}=1$. We would also require

$$p\left(\frac{\lambda}{2}+\mu\right) = \frac{\lambda^2 p^2}{2},$$

so that $\exp\left\{pN_t - p\left(\frac{\lambda}{2}+\mu\right)[N]_t\right\}$ is a exponential local martingale of $\lambda p N_t$. Again, since positive local martingales are supermartingales, we obtain that

$$\mathbb{E}\left[\exp\left\{\lambda p N_t - p\left(\frac{\lambda}{2}+\mu\right)[N]_t\right\}\right] \leqslant 1.$$

Therefore, using conditions 1) and 2),

$$\mathbb{E}\left[|h'(s,t,iy)|^{\lambda\left(2-\frac{\delta^2}{8}\right)-\mu\kappa}\right] \leq \mathbb{E}\left[\frac{2\lambda q}{\delta^2}\int_0^T \dot{A}_r^2 dr\right] < \infty.$$

It only remains to check that one can choose λ, μ such that

$$\lambda\left(2-\frac{\delta^2}{8}\right)-\mu\kappa > 2.$$

More precisely, we claim that if $\kappa \in [0,4)$, then there exists constants $\delta > 0$, $p > 1$, $\lambda > 0$, $\mu > 0$ such that

$$\begin{aligned}
&1)\ \kappa\mu + \lambda\left(\tfrac{\delta^2}{8} + \tfrac{\kappa}{2} - 2\right) \leqslant 0,\\
&2)\ p\left(\tfrac{\lambda}{2} + \mu\right) = \tfrac{\lambda^2 p^2}{2},\\
&3)\ \lambda\left(2 - \tfrac{\delta^2}{8}\right) - \mu\kappa > 2.
\end{aligned}$$

This can be easily checked using some simple algebra. Therefore, taking $b = \lambda\left(2 - \frac{\delta^2}{8}\right) - \mu\kappa$, we obtain

$$\mathbb{E}\left[|h'(s,t,iy)|^b\right] \leqslant \mathbb{E}\left[\exp\left\{\frac{2\lambda q}{\delta^2}\int_0^T A_r^2 dr\right\}\right]^{1/q}.$$

To obtain the moment bound for $|\sup_t |h_t'(s,t,iy)|$, we again use the (3.10) to similarly obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_t |h'(s,t,iy)|^{\lambda\left(2-\frac{\delta^2}{8}-\mu\kappa\right)}\right]\\
&\leq \mathbb{E}\left[\sup_t \exp\left\{\lambda p N_t - p\left(\frac{\lambda}{2}+\mu\right)[N]_t\right\}\right]^{1/p}\mathbb{E}\left[\exp\left\{\frac{2\lambda q}{\delta^2}\int_0^T \dot{A}_r^2 dr\right\}\right]^{1/q}. \qquad (3.11)
\end{aligned}$$

Note that $\sup_t \exp\left\{\lambda p N_t - p\left(\frac{\lambda}{2}+\mu\right)[N]_t\right\} = \exp\left\{\sup\left(\lambda p N_t - \frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2}[\lambda p N]_t\right)\right\}$, and since $\lambda p N_t$ is a local martingale, it is equal in distribution to a time change of Brownian motion by either Dambis-Dubins-Schwarz martingale Embedding Theorem. Therefore, we have thatsup$_t\left(\lambda p N_t - \frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2}[\lambda p N]_t\right)$ is dominated by $\sup_{t\geqslant 0}\left(B_t - \frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2}t\right)$. Note that the quantity is just stochastically dominated because we only have equality in distribution, but this is enough for our conclusion. Also, it is a well-known result that

$$\sup_t\left(B_t - \frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2}t\right)$$

is an exponential random variable with parameter $2\frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2}$. Therefore, if we require $2\frac{p\left(\frac{\lambda}{2}+\mu\right)}{\lambda^2 p^2} > 1$ then the right-hand side of (3.11) is finite. It is again a simple exercise that such parameters μ, λ, δ, p can be chosen such that

$$b = \lambda\left(2 - \frac{\delta^2}{8}\right) - \mu\kappa > 2.$$

This completes the proof of the (3.1). The estimate (3.2) follows from (3.1) using a standard argument, see [1] for more details.

□

□

Remark 3. *The purpose behind developing a technique to prove the existence of γ which is not based on exact computations is precisely to handle driving functions such as $|B_t|$, as asked by Avelio-Sepulveda. For drivers such as $|B_t|$ is unlikely to find exact martingales observables to obtain the moment estimate of $|f_t'(iy)|$. Our method bypasses this exact computations and therefore provides alternative route which as the potential of handling $|B_t|$ as well.*

4 Simpleness of the trace

We now prove the simpleness of the curve γ. The proof of the simpleness in the case of a general semimartingale satisfying conditions 1) and 2) is more involved compared to the Brownian motion because of the reasons mentioned in the Introduction. Therefore, we fist prove some preparatory results which will be important in the proof. It will be convenient to extend the definition of U_t for $t \in [0, \infty)$, by defining $U_t = U_T$, $\forall t \geq T$. We will need to consider the stochastic flow $h(s,t,x)$ started in $x > 0$ defined by (**??**). Clearly, the solution $h(s,t,x)$ is well-defined for $t < T(s,x)$, where

$$T(s,x) = \inf\{t \geqslant s \mid \quad h(s,t,x) = 0\}, \tag{4.1}$$

With the convention that $\inf(\phi) = +\infty$. It is a priori not clear that $T(s,x) < \infty$ a.s., but we claim that for semimartingales U satisfying conditions 1) and 2), this is indeed true. Furthermore, we also claim that $T(s,x)$ has larger than one moments, and that it is of order $O(x^2)$, at least in expectation.

Proposition 4.1. *If U is a semimartingale satisfying conditions* 1) *and* 2) *then there exists a $p > 1$ depending only on κ, α such that*

$$\sup_{s,x} \mathbb{E}\left[\left|\frac{T(s,x) - s}{x^2}\right|^p\right] < \infty \tag{4.2}$$

Proof. Writing $h_t(x)$ for $h(s,t,x)$, we start by applying Itô formula to $\log(h_t(x))$. It follows that

$$\begin{aligned}
\log h_t(x) &= \log(x) + \int_s^t \frac{1}{h_r(x)} dU_r - \int_s^t \frac{2}{h_r(x)^2} dr - \frac{1}{2} \int_s^t \frac{1}{h_r(x)^2} d[U]_r \\
&= \log(x) + \int_s^t \frac{1}{h_r(x)} dM_r + \int_s^t \frac{1}{h_r(x)} \dot{A}_r dr - \int_s^t \frac{2}{h_r(x)^2} dr - \frac{1}{2} \int_s^t \frac{1}{h_r(x)^2} d[U]_r \\
&\leq \log(x) + \int_s^t \frac{1}{h_r(x)} dM_r - \frac{1}{2} \int_s^t \frac{1}{h_r(x)^2} d[M]_r \\
&+ \left(\frac{\delta^2}{2} - 2\right) \int_1^t \frac{1}{h_r(x)^2} dr + \frac{1}{2\delta^2} \int_s^t \dot{A}_r^2 dr,
\end{aligned}$$

where we have used that

$$\frac{\dot{A}_n}{h_r(x)} \leqslant \frac{\delta^2}{2} \frac{1}{h_r(x)^2} + \frac{1}{2\delta^2} \dot{A}_r^2,$$

for δ small enough, in the last line. Next, for $\eta > 0$ small enough, we write

$$\left(\frac{\delta^2}{2} - 2\right) \int_s^t \frac{1}{h_r(x)^2} dr = \left(\frac{\delta^2}{2} - 2 + \eta\right) \int_s^t \frac{1}{h_r(x)^2} dr - \eta \int_s^t \frac{1}{h_r(x)^2} dr. \tag{4.3}$$

Using that $\frac{d[M]_t}{dt} \leq \kappa$, note that

$$\left(\frac{\delta^2}{2} - 2 + \eta\right) \int_s^t \frac{1}{h_r(x)^2} dr \leqslant \frac{\left(\frac{\delta^2}{2} - 2 + \eta\right)}{\kappa} \int_s^t \frac{1}{h_r(x)^2} d[M]_r. \tag{4.4}$$

Therefore, we obtain that for all $t < T(s,x)$,

$$\log h_t(x) \leqslant \log(x) + \int_s^t \frac{1}{h_r(x)} dM_r - \left\{ \frac{1}{2} - \frac{\left(\frac{\delta^2}{2} - 2 + \eta\right)}{\kappa} \right\} \int_s^t \frac{1}{h_r(x)^2} d[M]_r$$
$$+ \frac{1}{2\delta^2} \int_s^t \dot{A}_r^2 dr - \int_s^t \frac{\eta}{h_r(x)^2} dr. \tag{4.5}$$

Now, let

$$T_\varepsilon(s,x) = \inf\{t \geqslant s \mid \quad h(s,t,x) = \varepsilon\}.$$

Then, for $t \leq T_\varepsilon(s,x)$ it follows from (4.5) that

$$\log h_t(x) \leqslant \log(x) + \Theta - \int_s^t \frac{\eta}{h_r(x)^2} dr, \tag{4.6}$$

where $\Theta = \sup_{t \leq T_\varepsilon(s,x)} \left\{ \int_s^t \frac{1}{h_r(x)} dM_r - \left\{ \frac{1}{2} - \frac{\left(\frac{\delta^2}{2} - 2 + \eta\right)}{\kappa} \right\} \int_s^t \frac{1}{h_r(x)^2} d[M]_r \right\} + \frac{1}{2\delta^2} \int_0^T \dot{A}_r^2 dr$. Therefore, using Gronwall inequality Lemma 2.3, we obtain that

$$h_t(x) \leqslant \sqrt{x^2 e^{2\Theta} - 2\eta(t-s)}. \tag{4.7}$$

Therefore, it follows that

$$T_\varepsilon(s,x) - s \leq \frac{1}{2\eta} \left(x^2 e^{2\Theta} - \varepsilon^2 \right), \tag{4.8}$$

which implies that

$$\mathbb{E}\left[|T_\varepsilon(s,x) - s|^p \right] \leq C(\eta,p) \left(\varepsilon^{2p} + \mathbb{E}\left[x^{2p} e^{2p\Theta} \right] \right). \tag{4.9}$$

In order to estimate $\mathbb{E}\left[e^{2p\Theta} \right]$ we use the same argument as in Proposition 3.1, i.e. we dominate the $\sup_{t \leq T_\epsilon(s,x)}(\cdots)$ by an exponential random variable. Again, using that $\kappa < 4$ it can be easily checked that for some $p > 1$, $\mathbb{E}\left[e^{2p\Theta} \right]$ is bounded. Then, letting $\epsilon \to 0+$ in (4.9), using Monotone Convergence Theorem, we obtain that

$$\mathbb{E}\left[|T(s,x) - s|^p \right] \leqslant c x^{2p} \tag{4.10}$$

which completes the proof.

□

We will sometimes need to specify the dependence of $T^{s,x}$ on the driving function U, and we will write $T^{s,x}(U)$ to do so. We will also need to consider $T^{s,x}(M)$ i.e. when U is simply a martingale and $A \equiv 0$. Besides the upper-bound provided by Proposition 4.1, we will also need a lower bound on $\mathbb{E}[T^{s,x}(M)]$. More generally, note that for any stopping time τ such that $\tau < \infty$, a.s., by Optimal Sampling Theorem, the process $\{M_t - M_\tau\}_{t \geqslant \tau}$ is a local martingale, and it clearly satisfies conditions 1) and 2). Therefore, one can consider the random variable $T^{\tau,x}(M)$ for any such stopping time τ. We further claim the following.

Lemma 4.2. *Let τ be a stopping time such that $\tau < \infty$ a.s. Then,*

$$\mathbb{E}\left[T^{\tau,x}(M) - \tau \mid \mathcal{F}_\tau \right] \geqslant \frac{x^2}{4}. \tag{4.11}$$

Proof. Let $l_t = l_t(x)$ denote the solution to the equation

$$dl_t = dM_t - \frac{2}{l_t}dt, \;\; l_\tau = x, t \geqslant \tau. \tag{4.12}$$

Clearly, since $l_\tau > 0$ for $\tau < T^{\tau,x}(M)$,

$$l_t \leqslant x + M_t - M_\tau \quad \forall t < T^{\tau_1 x}(M). \tag{4.13}$$

Next, consider the process l_t^2. By Itô-formula,

$$l_t^2 = x^2 + \int_\tau^t 2l_r dM_r + [M]_t - [M]_\tau - 4(t-\tau). \tag{4.14}$$

Note using optional stopping theorem that condition on $\mathcal{F}_\tau$, the process $\int_\tau^\cdot l_r dM_r$ is a local martingale. We claim that is in fact a true martingale. To this end, we observe that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_\tau^t l_r dM_r\right)^2\right] &= \mathbb{E}\left[\int_\tau^t l_r^2 d[M]_r\right] \\
&\leq \kappa \mathbb{E}\left[\int_\tau^t l_r^2 dr\right] \\
&\leqslant \kappa \mathbb{E}\left[\int_\tau^t (x + M_r - M_\tau)^2 dr\right],
\end{aligned}$$

where we have used that $\frac{d[M]_t}{dt} < \kappa$ and the (4.13) in the last inequality. Further note that

$$\begin{aligned}
\mathbb{E}\left[(x + M_r - M_\tau)^2\right] &\leqslant 2x^2 + 2\mathbb{E}\left[(M_r - M_\tau)^2\right] \\
&= 2x^2 + 2\mathbb{E}\left[[\mathrm{M}]_r - [\mathrm{M}]_\tau\right] \\
&\leq 2x^2 + 2\kappa\mathbb{E}[r - \tau] \\
&\leq 2x^2 + 2\kappa r.
\end{aligned}$$

Since the second moment of $\int_\tau^t l_r dM_r$ is bounded for all bounded t, this verifies that indeed is a true martingale. Now, note that $l_T\tau, x_{(M)} = 0$. Therefore, using (4.14)

$$\begin{aligned}
x^2 + \int_\tau^{T^{\tau,x}(M)} l_r dM_r &= 4\left(T^{\tau,x}(M) - \tau\right) - \left([M]_t - [M)_\tau\right) \\
&\leq 4\left(T^{\tau,x}(M) - \tau\right).
\end{aligned}$$

Also, since $\int_\tau^\cdot l_r dM_r$ is a martingale w.r.t. to $\mathbb{P}(\cdot \mid \mathcal{F}_\tau)$ and since $\mathbb{E}\left[T^{\tau,x}(M)\right] < \infty$ using Proposition 4.1

$$\mathbb{E}\left[\int_\tau^{T^{\tau,x}(M)} l_r dM_r | \mathcal{F}_\tau\right] = 0.$$

Therefore,

$$\mathbb{E}\left[T^{\tau,x}(M) - \tau | \mathcal{F}_\tau\right] \geqslant \frac{x^2}{4}.$$

□

We now finally state the following proposition which is the key towards establishing the simpleness of the curve γ.

Proposition 4.3. *If U is a semimartingale satisfying conditions* 1) *and* 2) *, then*

$$\mathbb{P}\left[\lim_{x\to 0+} T^{s,x}(U) = 0 \quad \text{for all } s \geqslant 0\right] = 1.$$

Assuming the above proposition, we can easily establish the simpleness of the curve γ via a similar argument as in [5]. It now remains to prove Proposition 4.3. To this end, we first note that

$$U_t - U_s = M_t - M_s + A_t - A_s \leqslant M_t - M_s + \int_s^t \left|\dot{A}_r\right| dr.$$

Therefore, if $\tilde{U}_t = M_t + \int_0^t \left|\dot{A}_r\right| dr$, then $U_t - U_s \leqslant \tilde{U}_t - \tilde{U}_s$. This in turn implies using Lemma 2.3 that

$$T^{s,x}(U) \leqslant T^{s,x}(\tilde{U}).$$

Therefore, it suffices to prove that simultaneously for all $s \geq 0$,

$$\lim_{x\to 0+} T^{s,x}(\tilde{U}) = 0. \tag{4.15}$$

The advantage from switching from U to $\tilde{U}$ is that $\tilde{U}$ has monotonic increasing bounded variation part. We will crucially make use of this observation. Also, since $\tilde{U}_t$ is constant for $t > T$ it easily follows that $T^{s,x}(\tilde{U}) = s + \frac{x^2}{4}$ for $s > T$ and the (4.15) is trivially true. To prove the (4.15) for $s \leq T$, we first note that using Lemma 2.3 that for all $x_1 \geq x_2$, we have $h(s,t,x_1) \geqslant h(s,t,x_2)$. Therefore, $T^{s,x_1}(\tilde{U}) \geqslant T^{s,x_2}(\tilde{U})$. This in turn, implies that $T^{s,0+} - s := \lim_{x\to 0+} T^{s,x} - s$ exists. The core part of the argument is to prove that this limit is indeed zero for all $s \in [0,T]$. To do so, we make an observation that using the flow property of $h(\cdot,\cdot,\cdot)$ for any $s \leqslant u \leqslant T^{s,x}$,

$$T^{s,x} = T^{u,h(s,u,x)} \geqslant T^{u,0+}. \tag{4.16}$$

Therefore, if $T^{s,x} - s$ is small, then it implies that $T^{0,x} - u$ is small for all $s < u < T^{s,x}$. We utilize this observation to cover the interval $[0,T]$ using the intervals of the form $[s, T^{s,x}]$. More precisely, let $x_n = 2^{-n}$. Then, for each n, define recursively

$$s_0(n) = 0 \quad s_1(n) = T^{0,x_n},$$

$$s_{k+1}(n) = T^{s_k(n),x_n}.$$

We run this recursion enough number of times (say K_n times) so that $s_{k+1}(n)$ crosses T. Then, the interval $[0,T]$ is covered by union of intervals $[s_k(n), s_{k+1}(n)]$. To get an estimate of the size of K_n, we require a lower bound on the increments $s_{k+1}(n) - s_k(n)$. To this end, we resort to Lemma 4.2 as follows. Define a sequence m_k by $m_0 = 0$ and $m_k = s_k - \sum_{i=1}^k \mathbb{E}\left[s_i - s_{i-1} | \mathcal{F}_{s_{i-1}}\right]$. Then, since $\mathbb{E}[s_k] < \infty$ by Proposition 4.1, it easily follows that m_k is a discrete martingale. Furthermore, note that $\tilde{U}_t - \tilde{U}_s = M_t - M_s + \int_s^t \left|\dot{A}_r\right| dr \geqslant M_t - M_s$. Therefore, by Lemma 2.3

$$s_i - s_{i-1} = T^{s_{i-1},x_n}(\tilde{U}) - s_{i-1} \geqslant T^{s_{i-1},x_n}(M) - s_{i-1}.$$

Therefore, using Lemma 4.2,

$$\mathbb{E}\left[s_i - s_{i-1} \mid \mathcal{F}_{s_{i-1}}\right] \geqslant \frac{x_n^2}{4},$$

and it follows that

$$s_k \geqslant m_k + k\frac{x_n^2}{4}.$$

Since martingale m_k is of mean zero, this suggests to choose K_n such that $s_{K_n} > T$ it suffices to take $K_n = \left\lceil \frac{16T}{x_n^2} \right\rceil$. To make it precise, we claim that for $K_n = \left\lceil \frac{16T}{x_n^2} \right\rceil$,

$$m_{K_n} \xrightarrow{\mathbb{P}} 0, \text{ as } n \to \infty.$$

To prove it, note that since m_k is a martingale, using Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[m_{K_n}^p\right] \leq C_p \mathbb{E}\left[\left(\sum_{i=1}^{K_n} (m_i - m_{i-1})^2\right)^{p/2}\right].$$

We choose $p \in (1,2)$ so that the Proposition 4.1 is valid. Therefore,

$$\mathbb{E}\left[m_{K_n}^p\right] \leqslant C_p \sum_{i=1}^{K_n} \mathbb{E}\left[|m_i - m_{i-1}|^p\right].$$

Also, using Proposition 4.1

$$\begin{aligned}
\mathbb{E}\left[(m_i - m_{i-1})^p\right] &\leqslant C_p \mathbb{E}\left[|s_i - s_{i-1}|^p\right] \\
&= C_p \mathbb{E}\left[|T^{s_{i-1},x_n} - s_{i-1}|^p\right] \\
&\leqslant C x_n^{2p}.
\end{aligned}$$

This implies that $\mathbb{E}\left[m_{K_n}^p\right] \leqslant C x_n^{2p} K_n \to 0$, as $n \to \infty$. This in particular implies that $m_{K_n} \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Since convergence in probability implies a.s. convergence on a subsequence, we conclude that a.s. for infinitely many n, we have $m_{K_n} > -T$. In particular,

$$s_{K_n} \geqslant m_{K_n} + K_n \frac{{x_n}^2}{4} > T,$$

which implies that almost surely for infinitely many n, $[0,T]$ is covered by $\bigcup_{k=0}^{K_n-1} [s_k, s_{k+1}]$. Next, we now establish that each of the increment $s_{k+1} - s_k$ for $0 \leq k \leq K_n - 1$ are uniformly small. To this end, let $p > 1$ be from Proposition 4.1 and consider the event

$$E_n = \{ \text{ For some } 0 \leq k \leq K_n - 1,\ s_{k+1} - s_k > 2^{-n\frac{p-1}{p}} \}.$$

Then, using Proposition 4.1,

$$\begin{aligned}
\mathbb{P}\left[E_n\right] &\leqslant K_n \mathbb{P}\left[s_{k+1} - s_k > 2^{-n\frac{p-1}{p}}\right] \\
&\leq K_n \frac{E\left[|s_{k+1} - s_k|^p\right]}{2^{-n(p-1)}} \\
&\leq C \frac{1}{x_n^2} \frac{x_n^{2p}}{2^{-n(p-1)}} \\
&= C 2^{-n(p-1)}.
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \mathbb{P}[E_n] < \infty$ and Borell-Cantelli Lemma implies that almost surely, for all n large enough and for all k with $0 \leq k \leq K_n - 1$, $s_{k+1}(n) - s_k(n) \leq 2^{-n(p-1)/p}$. Finally, for any $u \in [0, T]$, $u \in [s_k(n), s_{k+1}(n)]$ for some $0 \leq k \leq K_n - 1$, for infinitely many n. Therefore, by (4.16)

$$\begin{aligned} T^{u,0+} - u &\leqslant T^{s_k(n),x_n} - u \\ &\leqslant T^{s_k(n),x_n} - s_k(n) \\ &= s_{k+1}(n) - s_k(n) \\ &\leq 2^{-n\frac{(p-1)}{p}}. \end{aligned}$$

Taking $n \to \infty$, we set $T^{u,0+} - u = 0$ and this finishes the proof of Proposition 4.3.

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