Random Matrix Theory and Multiple SLEs: Local Ergodicity of Dyson Brownian Motion, Universality of General β-Ensembles and their consequences on the simultaneous growth of multiple SLEs; connections with Machine Learning and other projects

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Abstract

We consider simultaneously growing chordal multiple SLE in $\mathbb{H}$, with Dyson Brownian motion as a driver. Using results from Random Matrix Theory (RMT), specifically the fast convergence to 'local equilibrium' of the Dyson Brownian motion, and the Universality of General β-Ensembles, we describe how to obtain geometric and probabilistic information on the multiple SLE. We also give the details of a different project on the multiple SLE curves in the radial case in $\mathbb{D}$. We present a list of future directions, in both the multiple SLE chordal and radial cases.

Contents

1 Introduction to Random Matrix Theory

Random matrices are fundamental objects for the modeling of complex systems. A basic example is the Wigner ensemble. The Wigner matrices H are $N \times N$ symmetric or Hermitian matrices whose entries are random variables that are independent up to the constraint imposed by the symmetry $H = H^*$. Wigner made the observation that the distribution of the distances between consecutive eigenvalues (that is called the gap distribution) follows an universal pattern. Furthermore, he predicted that the universality is not restricted to the Wigner ensemble, but it should hold for any system of sufficient large complexity, described by a large Hamiltonian.

[to be completed]

2 Introduction to SLE Theory

The Loewner equation was introduced by Charles Loewner in 1923 in [29] and it played an important role in the proof of the Bieberbach Conjecture [4] by Louis de Branges in 1985 in [7]. In 2000, Oded Schramm introduced a stochastic version of the Loewner equation in [35]. The stochastic version of the Loewner evolution, i.e. the Schramm-Loewner evolution, SLE_κ, generates a one parameter family of random fractal curves that are proved to describe scaling limits of a number of discrete models that appear in planar Statistical Physics. We refer to [27] for a detailed study of the object and many of its properties.

[To be completed]

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4 Preliminaries

In the last years, there were a number of results on Multiple SLEs. We refer the reader to the sequence of papers [6], [18], [9], [22], [10], [32], [2], [21], [28]. In particular, in [6], [18], [9], [22] there is established a connection between SLE and Random Matrix Theory. Furthermore, in [9], [22] there is a link between the multiple SLE and the Complex Burgers Equation. There is also literature written about the connection between multiple SLEs and Conformal Field Theory (CFT). We refer the reader to [33], as well as [28]. In the paper [25], the authors investigate the pure partition functions of multiple $SLEs$ and prove their existence. In addition in [32] the pure partition functions are shown to be positive and they study the explicit one for $\kappa = 4$. In this paper we create a further link between Random Matrix Theory and Schramm-Loewner Evolution.

An important motivation to study the connection between Random Matrix Theory and SLE Theory comes from the work [6] where ideas in the Conformal Field Theory are linked with the predictions offered by the multiple SLE Theory. Specifically, in [6] is argued that the Dyson process is related by a similarity transformation to the quantum Calogero-Sutherland model.

In our model we consider the simultaneous growth that is related via [6] with the Calogero-Sutherland CFT model. In the papers [32] [25] it is discussed the case when the growth is not simultaneous. The difference between the simultaneous and non-simultaneous case in terms of drivers is discussed in [9]. See also [8] for a discussion on this topic. In this paper, we are interested in the simultaneous growth case (that leads to Dyson Brownian motion as driver) and in the future work we would like to study the non-simultaneous growth.

The main goal of this project is to apply results from Random Matrix Theory in order to obtain geometrical information about multiple SLE curves. The main strategy consists of applying results from the analysis of the Dyson Brownian Motion (DBM) at the level of drivers in the Loewner differential equation (LDE) corresponding to multiple SLE curves and through typical Loewner theory type analysis (including Carathéodory convergence type estimates, Hausdorff distance type estimates) to push the results at the level of the multiple SLE curves/hulls.

5 Multiple Chordal SLEs in the upper half-plane

We consider the Weyl chamber defined as

$$\mathfrak{W}_n = \{\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^N : x_1 < x_2 < .... < x_N\}.$$

Let $B_j(t)$, $t \geqslant 0$, $j = 1, 2, ..., N$ be independent one-dimensional standard Brownian motions.

The Dyson Brownian motion with parameter $\beta > 0$ is a system of stochastic differential equations (SDEs) for the interacting particle system on $\mathbb{R}$ $(X_1^N(t), X_2^N(t), ..., X_N^N(t))$, given by

$$dX_j^N(t) = dB_j(t) + \frac{\beta}{2} \sum_{1 \leqslant k \leqslant N, k \neq j}^{N} \frac{dt}{X_j^N(t) - X_k^N(t)},$$

for $t \in [0, T^{X^N}]$ and $j = 1, 2, ..., N$.

The initial configuration is $\mathbf{X}^N(0) = x^N \in \mathfrak{W}_n$ and $T^{x^n} = \inf\{\mathbf{X}^N(t) \notin \mathfrak{W}_n\}$.

We set as in [22], $\beta = \frac{8}{\kappa}$. We perform the following time-change $V_j^N(t) = X_j^N(\kappa t/N)$ and obtain the system of SDEs $V^N(t) = (V_1^N(t), ...V_N^N(t))$ given by

$$dV_j^N(t) = \sqrt{\kappa} dB_j(t) + \frac{1}{N} \sum_{k=1}^{n} \frac{4}{V_j^N(t) - V_k^N(t)} dt, \qquad t \in [0, \infty).$$

It is well know that for the Dyson Brownian motion when $\beta \geqslant 1$, $\forall x^N \in \mathfrak{W}_N$ we have $T^{x^N} = +\infty$, a.s. Next, we define further for $t \in [0, \infty)$ the multiple SLEs

$$\frac{\partial_t (g_t^N(z))}{\partial t} = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t^N(z) - V_j^N(t)},$$

$t \geqslant 0$, $g_0(z) = z \in \mathbb{H}$.

Each realization of the g_t^N, $t \in [0, +\infty)$ determines a time evolution of the n-tuple $(\gamma_1(t)^N, .., \gamma_2(t)^N)$. In other words, $g_t^N(z)$ is a conformal map from $\mathbb{H} \setminus \cup_{j=1}^N \gamma_k^N(0, t] \to \mathbb{H}$, for $t \in [0, \infty)$, where $\gamma^N(0, t] = \cup_{0 \leqslant s \leqslant t} \gamma_j^N(s), j = 1, 2, ..., N$.

[to be completed]

6 Dyson Brownian Motion

6.1. Preliminaries. Let $\mathcal{M}_n(\beta)$ be the set of all $n \times n$ real ($\beta = 1$), complex ($\beta = 2$) and quaternion matrices ($\beta = 4$), respectively. Let $\mathcal{S}_n(\beta)$ be the set of self-dual elements in $\mathcal{M}_n(\beta)$. Let M_t be an $\mathcal{S}_n(\beta)$ -valued Ornstein-Uhlenbeck process, meaning that M_t satisfies the SDE

$$dM_t = -M_t dt + \frac{\sigma}{\sqrt{\beta n}} d(B_t + B_t^*),$$

where B_t is an $n \times n$ matrices, which elements are standard real complex or quaternion Brownian motions, for $\beta = 1, 2$ and 4 respectively. We consider the following system of n Itô equations:

$$d\lambda_t^i = \frac{2}{\sqrt{n\beta}} dB_t^i - \lambda_i dt + \frac{2}{n} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, \tag{6.1}$$

for $i = 1, 2, ..., n$. The entries $M_t^{i,j}$ $i \leqslant j$ are then independent O-U processes. It is known that the eigenvalues distribution of the previous system is given by (6.1) version of the Dyson Brownian motion. In this case, since the law of the O-U process is absolutely continuous with respect to Brownian motion in every entry, all the previous results concerning the existence of the traces and the other properties are still valid a.s.

We observe that even if connected with classes of matrices for certain values of the parameter β, the dynamics in (6.1) makes sense also for $\beta \geqslant 1$ (that corresponds to $\kappa \leqslant 8$).

In the special cases , when one has the O-U matrix representation, then the analysis of the Dyson Brownian motion, specifically the local ergodicity of the Dyson Brownian motion, gives a tool for proofs of results in Random Matrix Theory. We cover this in more detail in the next section.

We also cover the Local Ergodicity for the Dyson Brownian Motion, for $\beta \geqslant 1$.

6.2. The three steps strategy in the proof of universality. Let us first define Wigner matrices.

Definition 6.1. *A Wigner matrix is a $N \times N$ random Hermitian matrix $H = H^*$ whose entries H_{ij} satisfy the following conditions.*

- *(i) The upper-triangular entries $(H_{ij} : 1 \leqslant i \leqslant j \leqslant N)$ are independent.*
- *(ii) For all i, j, we have $\mathbb{E}(H_{ij}) = 0$ and $\mathbb{E}|H_{ij}|^2 = N^{-1}(1 + O(\delta_{ij}))$*
- *(iii) The random variables $\sqrt{N}H_{ij}$ are bounded in any L^p space, uniformly in N, i, j.*

The fast convergence to local equlibrium appears as one of the steps in the proof of the Universality Conjecture in Random Matrix Theory. We refer the reader to [13] for more details. In short, in [13] the analysis is performed for a Wigner matrix added with a matrix from the Gaussian Orthogonal Ensemble (GOE). The analysis is performed in a 'three-step strategy': Local semicircle law and delocalization of eigenvectors, Universality for the Gaussian Divisible Ensemble (GDE) and Approximation by GDE, or alternatively continuity of the eigenvalues under the matrix O-U process up to small times. Now, we give the details of these steps.

Step 1 can be phrased as 'Local semicircle law and delocalization of eigenvectors' (rigidity of the eigenvalues), and it states that the density of eigenvalues is given by the semicircle law not only as a weak limit macroscopically, but also in a high probability at scales containing N^{ξ} eigenvalues, for all $\xi > 0$, where N is the size of the random matrix. Step 2 is the Universality for Gaussian divisible ensembles (GDE). The GDE are matrices of the form $H_t = e^{-t/2}H_0 + \sqrt{1-e^{-t}}H^{\mathrm{G}}$, where $t > 0$ is a parameter, H_0 is a Wigner matrix and H^{G} is an independent GUE/GOE matrix. The parametrization of H_t is chosen so that H_t can be obtained by an Ornstein-Uhlenbeck (O-U) process starting from H_0. The aim of Step 2 is to prove the bulk universality of H_t for $t = N^{-\tau}$ for the entire range of $0 < \tau < 1$. In Step 3, one has the following continuity of matrix O-U process: In Theorem 15.2 in [13] it is shown that the changes of the local statistics in the bulk under the matrix O-U process flow, up to time scales $t \ll N^{-1/2}$ are negligible.

6.3. Local Ergodicity of the Dyson Brownian Motion. In this subsection we provide the details of the Local Ergodicity of the Dyson Brownian motion.

Let us consider the density of the semicircle law

$$\varrho_{sc}(x) := \frac{1}{2\pi}\sqrt{(4-x^2)_+}.$$

Let us further define

$$n_{sc}(E) := \int_{-\infty}^{E} \varrho_{sc}(x)\mathrm{d}x.$$

We will say that γ_j is the classical location of the j-th eigenvalue

$$\gamma_j = n_{sc}^{-1}(j/N).$$

Let us consider the generator

$$\mathcal{L}_G = \sum_{i=1}^{N} \frac{1}{\beta N}\partial_i^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}\lambda_i + \frac{1}{N}\sum_{j}^{(i)}\frac{1}{\lambda_i - \lambda_j}\right)\partial_i.$$

We consider the Dyson Brownian motion, i.e., dynamics of the eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of H_t that is $H_t = e^{-t/2}H_0 + \sqrt{1-e^{-t}}H^{\mathrm{G}}$, with H_0 Wigner Matrix and H^G an independent matrix from a Gaussian ensemble. We write the distribution of $\boldsymbol{\lambda}$ of H_t at time t as $f_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda})$. We notice that Kolmorogov's forward equation for the evolution of the density f_t takes the form

$$\partial_t f_t = \mathcal{L}_G f_t. \tag{6.2}$$

Let us further consider the following estimate.

A-priori estimate: There exists a $\xi > 0$ such that average rigidity on scale $N^{-1+\xi}$ holds, i.e.,

$$Q = Q_\xi := \sup_{0\leqslant t\leqslant N} \frac{1}{N}\int \sum_{j=1}^{N} (\lambda_j - \gamma_j)^2 f_t(\lambda)\mu_G(\mathrm{d}\lambda) \leqslant CN^{-2+2\xi} \tag{6.3}$$

with a constant C that is uniformly in N.

The main result on the Local Ergodicity of Dyson Brownian Motion states that if the a-priori estimate from before is satisfied, then the local correlation functions of the measure $f_t \mu_G$ are the same as the corresponding ones for the Gaussian measure, $\mu_G = f_\infty \mu_G$, provided that t is larger than $N^{-1+2\xi}$.

When we have the O-U matrix representation, i.e. in the cases $\kappa = 4, \kappa = 8$ that correspond to $\beta = 2$ (GUE) and $\beta = 1$ respectively, the probability distribution of the eigenvalues at the time t, $f_t \mu_G$, is the same as that of the Gaussian divisible matrix

$$H_t = e^{-t/2} H_0 + \left(1 - e^{-t}\right)^{1/2} H^{\mathrm{G}},$$

where H_0 is the initial Wigner matrix and H^{G} is an independent standard GUE (or GOE) matrix.

However, note that the next result is true for any $\beta \geqslant 1$, i.e for $\kappa \leqslant 8$. Also, there is a stronger result ([26]), i.e. fixed energy universality for $\beta = 1$, that we plan to adapt for $\beta = 2$, and use in that form in a future project,

For general $\beta \geqslant 1$, instead of eigenvalues, we think of an interacting particles system $(x_1, ..., x_N)$ on the real line of which dynamics is described via the Dyson BM equations.

In general, the n-point correlation functions of the symmetrized probability measure ν are defined by

$$p^{(n)}_{\nu,N}\left(x_1, x_2, \ldots, x_n\right) := \int_{\mathbb{R}^{N-n}} \nu(\mathbf{x}) \mathrm{d}x_{n+1} \ldots \mathrm{d}x_N, \quad \mathbf{x} = \left(x_1, x_2, \ldots, x_N\right).$$

Remark 6.2. Note that the eigenvalues are unlabeled when one studies the k-point correlations. In the statements of the universality below the eigenvalues are unordered as well.

In particular, when $\nu = f_t \mathrm{d}\mu_G$, one has

$$p^{(n)}_{t,N}\left(x_1, x_2, \ldots, x_n\right) = p^{(n)}_{f_t \mu_G, N}\left(x_1, x_2, \ldots, x_n\right).$$

We also use, as in [13], $p^{(n)}_{G,N}$ for $p^{(n)}_{\mu_G,N}$. When considering the O-U matrix representation case (i.e. $\beta = 1, 2$), one has the following result.

Theorem 6.3 (Theorem 12.4 in [13]). *[Local ergodicity of DBM] Suppose that for some exponent* $\xi \in \left(0, \frac{1}{2}\right)$, *the average rigidity, i.e. the a priori estimate* (6.3), *holds for the solution* f_t *of the* *forward equation* (6.2) *on scale* $N^{-1+\xi}$. *Additionally, suppose that in the bulk the rigidity holds on scale* $N^{-1+\xi}$ *even without averaging, i.e., for any* $\kappa > 0$

$$\sup_{\kappa N \leqslant j \leqslant (1-\kappa)N} |\lambda_j - \gamma_j| \prec N^{-1+\xi}$$

holds for any $t \in \left[N^{-1+2\xi}, N\right]$ *if* $N \geqslant N_0(\xi, \kappa)$ *is large enough. Let* $E \in (-2,2)$ *and* $b = b_N > 0$ *such that* $[E-b, E+b] \subset (-2,2)$. *Then for any integer* $n \geqslant 1$ *and for any compactly supported smooth test function* $O : \mathbb{R}^n \to \mathbb{R}$, *we have, for any* $t \in \left[N^{-1+2\xi}, N\right]$

$$\left| \int_{E-b}^{E+b} \frac{\mathrm{d}E'}{2b} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\alpha} O(\boldsymbol{\alpha}) \left(p_{t,N}^{(n)} - p_{G,N}^{(n)} \right) \left(E' + \frac{\boldsymbol{\alpha}}{N} \right) \right| \leqslant N^{\varepsilon} \left[\frac{N^{-1+\xi}}{b} + \sqrt{\frac{1}{bNt}} \right] \|O\|_{C^1}, \tag{6.4}$$

for any N *sufficiently large,* $N \geqslant N_0(n, \xi, \kappa)$.

In the same manner, the correlation functions of the equilibrium measure μ of the DBM, are denoted by

$$p_{\mu,N}^{(k)}(x_1, x_2, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} \mu(\mathbf{x}) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N$$

Following, [12], we have the following result for $\beta \geqslant 1$.

Theorem 6.4 (Theorem 2.1 of [12]). *Let* $\mathcal{N}_I$ *denote the number of eigenvalues in an interval* $I \subset \mathbb{R}$. *Suppose the initial density* f_0 *satisfies* $S_\mu(f_0) := \int f_0 \log f_0 \mathrm{d}\mu \leqslant CN^m$ *with some fixed exponent* m *independent of* N. *Let* f_t *be the solution of the forward equation* (6.2). *Suppose that the following three assumptions are satisfied for all sufficiently large* N

- *For some* $\mathfrak{a} > 0$ *we have*

$$Q \leqslant N^{-2\mathfrak{a}}.$$

- *There exist constants* $\mathfrak{b} > 0$ *and* $\mathfrak{c} > 0$ *such that*

$$\sup_{t \geqslant 0} \int \mathbf{1} \left\{ \max_{j=1,\ldots N} |x_j - \gamma_j| \geqslant N^{-\mathfrak{b}} \right\} f_t \mathrm{d}\mu \leqslant \exp\left[-N^{\mathfrak{c}}\right].$$

- *For any compact subinterval* $I_0 \subset (-2,2)$, *and for any* $\delta > 0, \sigma > 0$ *and* $n \in \mathbb{N}$ *there is a constant* C_n *depending on* I_0, δ *and* σ *such that for any interval* $I \subset I_0$ *with* $|I| \geqslant N^{-1+\sigma}$ *and for any* $K \geqslant 1$, *we have*

$$\sup_{t \geqslant 0} \int \mathbf{1} \left\{ \mathcal{N}_I \geqslant KN|I| \right\} f_t \mathrm{d}\mu \leqslant C_n K^{-n}.$$

Let $E \in \mathbb{R}$ such that $|E| < 2$ and let $0 < b < 2 - |E|$. There exists a constant $\zeta > 0$, depending only on $\mathfrak{a}$ and $\mathfrak{b}$, such that, for any integer $k \geqslant 1$ and for any compactly supported continuous test function $O : \mathbb{R}^k \to \mathbb{R}$ we have have

$$\begin{aligned} \lim_{N \to \infty} \int_{E-b}^{E+b} \tfrac{\mathrm{d}E'}{2b} \int_{\mathbb{R}^k} \quad & \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_k O\left(\alpha_1, \ldots, \alpha_k\right) \\ & \times \tfrac{1}{\varrho_{sc}(E)^k} \left(p_{t,N}^{(k)} - p_{\mu,N}^{(k)}\right) \left(E' + \tfrac{\alpha_1}{N \varrho_{sc}(E)}, \cdots, E' + \tfrac{\alpha_k}{N \varrho_{sc}(E)}\right) = 0 \end{aligned}$$

for $t = N^{-\zeta}$.

We use these results in the next sections, at the level of drivers of the multiple SLEs. The goal is to translate this result via the multiple Loewener Differential Equation (LDE) to the level of hulls/curves.

7 Results in [23] that need to be adapted to the multiple SLE curves

Lemma 7.1 (Lemma 6.1 in [23]). *For each $\delta > 0$ and $T > 0$ there exists a constant $C(T, \delta)$ such that the following holds. Let $h_k(t, z), k = 1, 2$ be the solutions of backward LDE with the continuous driving terms $(W_k(t))_{t \in [0,T]}, k = 1, 2$, respectively. Then they satisfy*

$$\left|h_1\left(T, z_1\right) - h_2\left(T, z_2\right)\right| \leqslant C(T, \delta)\left(\left\|W_1 - W_2\right\|_{\infty,[0,T]} + \left|z_1 - z_2\right|\right),$$

for any z_1, z_2, such that $\operatorname{Im} z_k > \delta > 0$.

For the proof of the previous lemma, one considers $h_k(t, z), k = 1, 2$ be the solutions of the backward LDE with the continuous driving terms $(W_k(t))_{t \in [0,T]}$, which we also consider to be fixed. Moreover, one writes $\psi(t) = h_1(t, z_1) - h_2(t, z_2)$. Then, one considers

$$\begin{gathered} \partial_t \psi(t) = \zeta(t)(\psi(t) - D(t)), \\ \text{where } \zeta(t) = 2 / \left(\left(h_1\left(t, z_1\right) - W_1(t)\right)\left(h_2\left(t, z_2\right) - W_2(t)\right)\right) \text{ and } D(t) = W_1(t) - W_2(t). \end{gathered}$$

Furthermore, $\partial_t\left(e^{-\int_0^t \zeta(s) \mathrm{d}s} \psi(t)\right) = -\zeta(t) e^{-\int_0^t \zeta(s) \mathrm{d}s} D(t)$ using an integrating factor. Hence

$$\psi(t) = e^{\int_0^t \zeta(s) \mathrm{d}s} \psi(0) - \int_0^t \zeta(u) e^{\int_u^t \zeta(s) \mathrm{d}s} D(u) \mathrm{d}u.$$

In order to obtain the result, on the right hand side it is considered the uniform estimate on $[0, T]$ between the drivers.

In our case, we will consider different topology. Moreover, we will do the analysis for the multiple Loewner differential equation with simultaneous growth, as described in the Preliminaries section.

Similarly, we have

Lemma 7.2. *For each $\delta > 0$ and $T > 0$ there exists a constant $C(T, \delta)$ such that the following holds. Let $g_k(t, z), k = 1, 2$ be the solutions of forward LDE with the continuous driving terms $(W_k(t))_{t\in[0,T]}$, $k = 1, 2,$ respectively*

$$|g_1(T, z_1) - g_2(T, z_2)| \leqslant C(T, \delta)\Big(\|W_1 - W_2\|_{\infty,[0,T]} + |z_1 - z_2|\Big),$$

for any z_1, z_2 such that $\operatorname{Im} g_k(T, z_k) > \delta > 0$.

Proof. The proof is similar to the proof of Lemma 6.1. The only difference is that we replace $\psi(t)$ by $\psi(t) = g_1(t, z_1) - g_2(t, z_1)$ and $\zeta(t)$ by $\zeta(t) = -2/\left(\left(g_1(t, z_1) - W_1(t)\right)\left(g_2(t, z_2) - W_2(t)\right)\right)$. Then I_k is given as and bounded by $I_k = \int_0^t 2 \mid g_k(s, z_k) - W_k(s)|^{-2}\,\mathrm{d}s \leqslant \log\frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(t,z_k)} \leqslant \log\frac{\operatorname{Im} z_k}{\max\left\{\delta, \sqrt{\left((\operatorname{Im} z_k)^2 - 4t\right)^+}\right\}}$, where $a^+ = \max\{a, 0\}$. $\square$

The final result that one gets in this form is the following.

Proposition 7.3 (Proposition 6.2. in [23])**.** *Let K_0 be a hull and $G \subset \mathbb{H}\backslash K_0$ be a compact set. Then there exists a constant $C > 0$ such that if if $g_k, k = 1, 2,$ are two Loewner chains such that $K_k(T) \subset K_0$ for $k = 1, 2,$ then*

$$\|g_1 - g_2\|_{\infty,[0,T]\times G} \leqslant C\,\|W_1 - W_2\|_{\infty,[0,T]}\,.$$

Proof. The claim follows directly from Lemma 6.2 $\square$

In our case, using the type of convergence (local ergodicity or fixed energy) we want to prove with the same type of idea a Caratheodory, Hausdorff convergence at the level of curves/drivers.

8 First two main results

Let us start with the following result in [13], that guarantees that the Dyson Brownian motion has a strong solution and the ordering is preserved for all times.

Theorem 8.1 (Theorem 12.2. of [13])**.** *Let the Weyl chamber be defined as before*

$$\mathfrak{W}_n = \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^N : \lambda_1 < \lambda_2 < .... < \lambda_N\}.$$

Let $\beta \geqslant 1$ and suppose that the initial data satisfy $\boldsymbol{\lambda}(0) \in \mathfrak{W}_n$. Then there exists a unique (strong) solution to Dyson Brownian motion in the space of continuous functions $(\boldsymbol{\lambda}(t))_{t\geqslant 0} \in C\left(\mathbb{R}_+, \bar{\Sigma}_N\right)$. Furthermore, for any $t > 0$ we have $\boldsymbol{\lambda}(t) \in \mathfrak{W}_n$ and $\boldsymbol{\lambda}(t)$ depends continuously on $\boldsymbol{\lambda}(0)$. In particular, if $\boldsymbol{\lambda}(0) \in \mathfrak{W}_n$, i.e., the multiplicity of the initial points is one, then $(\boldsymbol{\lambda}(t))_{t\geqslant 0} \in C\left(\mathbb{R}_+, \mathfrak{W}_n\right)$, i.e., this property is preserved for all times along the evolution.

Even though the results on the fast convergence to local equilibrium of the Dyson Brownian motion are valid for all $\beta \geqslant 1$, i.e. $\kappa \leqslant 8$, we focus on the regime $\kappa \leqslant 4$, in order to work with simple curves, for simplicity. We plan to study the case $8 \geqslant \kappa > 4$ in a future project.

The next results use the statistics of the k-point correlations. This is a way to study the statistics of unlabeled eigenvalues. These statistics can be used to define new observables for the multiple SLE curves such as the probability to find k curves rooted on a real set (using that the k-point correlations can be used to find the probability that a measurable set contains exactly a given number of eigenvalues). Also, one can obtain the probability that there are no eigenvalues in a given region. These quantities can be used to study in the future projects the convergence of the discrete models to the continuum one.

Theorem 8.2. *Let $\beta \geqslant 2$, i.e. $\kappa \leqslant 4$. Let us consider N-multiple SLE curves up to the first hitting time in the upper half-plane. Then, for any choice of initial conditions of Dyson BM, respecting the assumptions of Theorem 6.4, we have that, for $t \geqslant \frac{1}{N^\zeta}$, for some $\zeta > 0$, for any $k \geqslant 1$, the convergence of k-point correlations at the level of drivers, in the sense of* (6.4)*, as $N \to \infty$, gives the convergence of the k-marginals*

$$\mathbb{P}_N^k(g_t^N(z) \in A_N,\ \mathit{for}\, z\ \in G_N) \to \mu^k(g_t(z) \in A,\ \mathit{for}\, z\ \in G_\infty), \tag{8.1}$$

(in a sense to be defined), as $N \to \infty$, where μ is the equilibrium measure for the Dyson BM, $G_N, G_\infty \subset \mathbb{H}$, and A_N, A are Borel sets.

Remark 8.3. Using a similar argument as the one in [17], one can obtain multiple SLE curves, for $\kappa \leqslant 4$, that do not touch each other until their reach the target. Then, in the case of simple curves, we have that the result in the Carathéodory sense gives the result in the Hausdorff sense.

Proof. [Sketch]

8.1. Control at the level of drivers.

- For $t \geqslant N^{-\zeta}$, the control on the drivers is given by the local ergodicity of the Dyson BM for $\beta \geqslant 1$, i.e. $\kappa \leqslant 8$ (see Theorem 6.4).

8.2. From the drivers to the hulls; adaptation to the multiple SLE curves and the work [21] .

- We adapt the results in Section 7 for any hulls to the multiple SLE case, especially Proposition 7.3 by considering different topology at the level of drivers. Section 4.3 in [21], especially after the equation (4.12), in [21] provides ideas for the iterated growth of multiple SLE. The same idea works in the simultaneous growth case.

- One can also use the result in Sec. 4.3 in [21], if assuming the usual Kemppainen-Smirnov crossing conditions (see [24]). In Sec. 4.3 of [21], it is shown that weak convergence of the iterated driving functions implies weak convergence of curves in the sup-norm (and thus also Hausdorff and Carathéodory) topology. The same idea works in the simultaneous growth case.

- In order to achieve the final result, we couple and compare the Dyson BM started from random initial conditions that respect the conditions of Theorem 6.4, and Dyson BM started from the equilibrium measure and then use the multiple Loewner Differential Equation to perform the analysis from the drivers to the hulls.

□

In the special case $\kappa = 4$ using the elements of the proof of the Universality in [13], one obtains the following result. Given the connection with the O-U matrix representation, then one can get also control over the dynamics over time $t \leqslant N^{-\zeta}$, compared with the general case $\beta \geqslant 1$, where the control is obtained in time, only after $t \geqslant N^{-\zeta}$.

Theorem 8.4. *Let $\kappa = 4$. Let us consider N-multiple SLE curves in the upper half-plane to the first hitting time. Then, for any choice of initial conditions of the Dyson BM, respecting the assumptions of Theorem 6.4, we have that, for any $k \geqslant 1$, the convergence of k-point correlations at the level of drivers, in the sense of* (6.4)*, as $N \to \infty$, gives the convergence of the k-marginals*

$$\mathbb{P}_N^k(g_t^N(z) \in A_N,\ for\ z\ \in G_N) \to \mu^k(g_t(z) \in A,\ for\ z\ \in G_\infty), \tag{8.2}$$

(in a sense to be defined), as $N \to \infty$, where μ is the equilibrium measure for the Dyson BM, $G_N, G_\infty \subset \mathbb{H}$, and A_N, A are Borel sets.

Corollary 8.5 (Corollary of Theorem 8.4). *One corollary of the result in [11] gives that the probability to find no eigenvalue in a certain interval that is precisely stated in the paper, after averaging in the interval of size $N^{-1+\delta}$ around a point $u_0 \in (-2, 2)$ is the same as in the GUE case.*

Proof. [Sketch of the proof for the $\kappa = 4$ case]

The proof is similar with the one of Theorem 8.2, with the following difference.

8.3. Control at the level of drivers.

- For $t \geqslant N^{-\zeta}$

  As before, the control on this regime, it is obtained via the Local Ergodicity of the Dyson BM driver (see Theorem 6.4).

- For $t \leqslant N^{-\zeta}$

  Since $\kappa = 4$, i.e. $\beta = 2$, we use the third step in the proof of the Universality in Random Matrix Theory (see Section 6.2). Using the continuity of the matrix O-U process we show that the changes of the local statistics in the bulk under the matrix O-U process flow, up to time scales $t \ll N^{-1/2}$ are negligible (in the sense of Theorem 15.2 in [13]).

8.4. From the drivers to the hulls; adaptation to the multiple SLE curves and the work [21]. The next steps of the proof, i.e. the analysis from the drivers to the hulls and the coupling argument, are done in the same manner as the previous result.

□

Corollary 8.6 (Local Independence on initial conditions of the asymptotic multiple SLE growth model). *Theorem 8.2 can be interpreted as a local independence on initial conditions that respect certain assumptions, for the growth model of N SLE curves, as $N \to \infty$, in time less than $N^{-\zeta}$, for some $\zeta > 0$. These type of results are useful in the context of study of the scaling limits of Random Growth Models (see also [30], [31]).*

9 Updated strategy to prove both main results, Theorem 8.2 and Theorem 8.4

Let $z \in G_N$ compacts, such that $g_t^V(z)$ and $g_t^B(z)$ has no blow-up in finite time. Let us couple the following simultaneous growth multiple Loewner chains

$$dg_t^V(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{2}{g_t^V(z) - V_t^i}dt,$$

$$dg_t^B(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{2}{g_t^B(z) - B_t^i}dt.$$

In the previous formulas, V_t^i are jointly distributed such that the assumptions of the fast relaxation to local equilibrium of Dyson Brownian Motion are satisfied. The second pair of multiple drivers B_t^i are jointly distributed according to the equilibrium measure of the Dyson Brownian motion (i.e. the second dynamics starts directly from the equilibrium).

Equivalently, we write

$$N dg_t^V(z) = \sum_{i=1}^{N} \frac{2}{g_t^V(z) - V_t^i}dt,$$

$$N dg_t^B(z) = \sum_{i=1}^{N} \frac{2}{g_t^B(z) - B_t^i}dt.$$

We have

$$d(g_t^V(z) - V_t^1) + d(g_t^V(z) - V_t^2) + \cdots + d(g_t^V(z) - V_t^N) = \sum_{i=1}^{N}\left[\left(\frac{2}{g_t^V(z) - V_t^i}\right) - dV_t^i\right],$$

and similarly

$$d(g_t^B(z) - B_t^1) + d(g_t^B(z) - B_t^2) + \cdots + d(g_t^B(z) - B_t^N) = \sum_{i=1}^{N}\left[\left(\frac{2}{g_t^B(z) - V_t^i}\right) - dB_t^i\right].$$

We define $\tilde{g}_t^{V_i} := g_t^B(z) - V_t^i, i \in 1, \dots, N$ and $\tilde{g}_t^{B_i} := g_t^B(z) - B_t^i, i \in 1, \dots, N$.

Then, the equations become

$$\sum_{i=1}^{N} \tilde{g}_t^{V_i} = \sum_{i=1}^{N}\left(\int_0^t \frac{2}{\tilde{g}_t^{V_i}}dt - V_t^i\right)$$

and
$$\sum_{i=1}^{N} \tilde{g}_t^{B_i} = \sum_{i=1}^{N} \left(\int_0^t \frac{2}{\tilde{g}_t^{B_i}} dt - B_t^i \right).$$

Let us further consider
$$\sum_{i=1}^{N} \tilde{g}_t^{E',V_i} = \sum_{i=1}^{N} \left(\int_0^t \frac{2}{\tilde{g}_t^{V_i}} dt - \left(E' + \frac{V_t^i}{N\rho_{sc}(E)} \right) \right)$$
and
$$\sum_{i=1}^{N} \tilde{g}_t^{E',B_i} = \sum_{i=1}^{N} \left(\int_0^t \frac{2}{\tilde{g}_t^{B_i}} dt - \left(B_t^i + \frac{E'}{N\rho_{sc}(E)} \right) \right).$$

We are interested in estimating
$$\int_{E-b}^{E+b} \frac{dE'}{2b} \mathbb{E}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{E',V_i} \right) \right] - \int_{E-b}^{E+b} \frac{dE'}{2b} \mathbb{E}_\mu \left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{E',B_i} \right) \right],$$
as $N \to \infty$, for F test functions.

Then, we have
$$\int_{E-b}^{E+b} \frac{dE'}{2b} \mathbb{E}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{E',V_i} \right) \right] - \int_{E-b}^{E+b} \frac{dE'}{2b} \mathbb{E}_\mu \left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{E',B_i} \right) \right]$$

$$= \int_{E-b}^{E+b} \frac{dE'}{2b} \left(\mathbb{E}\left[F\left(\sum_{i=1}^{N} \int_0^t \frac{2}{\tilde{g}_t^{E',V}(z)} dt \right) \right] \right) - \left(\mathbb{E}_\mu \left[F\left(\sum_{i=1}^{N} \int_0^t \frac{2}{\tilde{g}_t^{E',B}(z)} dt \right) \right] \right)$$
$$+ \int_{E-b}^{E+b} \frac{dE'}{2b} \left(\mathbb{E}\left[F\left(\sum_{i=1}^{N} \left(E' + \frac{V_t^i}{N\rho_{sc}(E)} \right) \right) \right] - \mathbb{E}_\mu \left[F\left(\sum_{i=1}^{N} \left(E' + \frac{B_t^i}{N\rho_{sc}(E)} \right) \right) \right] \right).$$

The convergence of the second term on the RHS, as $N \to \infty$ (when considering the drivers written as in the Local Ergodicity result, Theorem 6.4), follows from Theorem 6.4.

For the first term in the RHS, we use the information in the sigma-algebra of the k-point correlations, for any $k \geqslant 1$, to obtain the convergence. In our case, we have the convergence of the k-point correlations in the averaged sense. Note that in [34], the universality of β-ensembles is proved without further averaging. However, in that case there is no dynamics as the system is already in equlibrium. In that case, it is direct to use the information on the sigma-algebra of the k-point correlations in the analysis.

In our case, we check that the averaging of the k-point correlations is enough to give the information needed to obtain the averaging convergence of the integrals, i.e. the first terms in the RHS.

Another strategy, is to consider a Random Matrix Model in which we add a random variable to each entry sampled from a distribution in order to obtain the averaging effect via the displacement of the window of energies, and smooth out the density of the eigenvalues (to make the density of the eigenvalues smoother than in the case of Bernoulli-type entries, for example).

For the general case $\beta \geqslant 2$, in order to obtain the estimate at the level of hulls/curves, after $t \geqslant N^{-\zeta}$, for some $\zeta > 0$, one can study the following form of the multiple Loewner Differential Equation. Let us fix some deterministic time $s > 0$, and let $\tilde{g}_t$ be the solution of the Loewner equation with driving functions $\tilde{V}_t^i = V_{s+t}^i, t \geqslant 0, \forall i \in 1, \cdots N$. We omit in the notation the superscripts in order to not overburden the reader. This solution can be obtained by $g_{s+t} \circ g_s^{-1}$.

$$\partial_t g_{s+t} \circ g_s^{-1}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{2}{g_{s+t} \circ g_s^{-1}(z) - V_{s+t}^i} = \frac{1}{N} \sum_{i=1}^{N} \frac{2}{g_{s+t} \circ g_s^{-1}(z) - \tilde{V}_t^i}$$

and $g_s \circ g_s^{-1}(z) = z$. By the uniqueness of solution of the equation above, we have $\tilde{g}_t(z) = g_{s+t} \circ g_s^{-1}(z)$. We can choose $s = \frac{1}{N^{\zeta}}$, for $\zeta > 0$ as in Theorem 6.4 and study this dynamics.

For $\beta = 2$, using the control on times $t \in [0, N^{-\zeta}]$, there is no need to consider the above chain.

In the case, there is no further averaging (for example, by a stronger result than Theorem 6.4), then the main estimate from this section, will be obtained in a direct manner without the further averaging in the energy windows.

[To be completed]

10 Brownian carousel and the local statistics of the multiple SLE curves

It is known that when β takes the special values $\beta = 1, 2, 4$, the correlation functions can be explicitly expressed. Thus, the analysis of the correlation functions relies heavily on the asymptotic properties of the corresponding orthogonal polynomials.

For non-classical values of β, i.e., $\beta \notin \{1, 2, 4\}$, there is no simple expression of the correlation functions in terms of orthogonal polynomials. In [38] it is proved that in the Gaussian case, i.e., when V is quadratic, the measure describes eigenvalues of tri-diagonal matrices. This idea allowed to establish, between others, the local spacing distributions of the Gaussian β -ensembles in [38].

Specifically, the Brownian carousel studied in [38] gives information about the local statistics of the eigenvalues for the general $\beta \geqslant 1$. We study the consequences of this representation at the level of drivers to the level of multiple SLE curves.

Moreover, in another direction of study, we want to use that in [5], the authors prove that the local spacing distributions are independent of the potential V for certain class of V, for any $\beta \geqslant 1$.

[to be completed]

11 Starting from stationarity

In the previous section, we argued that given a range of random initial conditions, then the Dyson BM reaches in fast time the local equlibruim ('forgetting very fast the initial conditions').

In a different project, we plan to start directly from stationarity the Dyson Brownian motion as a driver of the multiple SLE curves. Computations involving the Dyson BM and the LDE will give new results in this case as well, as the flow of the Dyson BM at the level of drivers will leave the distribution invariant. Then one will get directly the uniform control over time that is needed for the Carathéodory result. [to be completed with the details].

12 General β Ensembles Universality and multiple SLE curves, a third result

In Random Matrix Theory literature there are results concerning the Universality of the general β ensembles (see [5], [34], [3], more references to be added). In this case, there is no relaxation to local equlibruim as in the previous sections, as the drivers are already in equilibrium. However, the universality here is understood in terms of the potential W in the Dyson Brownian Motion dynamics, i.e. we consider

$$d\lambda_t^i = \frac{2}{\sqrt{n\beta}} dB_t^i - \frac{W'(\lambda_t^i)}{2} dt + \frac{2}{n} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, \tag{12.1}$$

for $i = 1, 2, ..., n$.

When $W(x) = \frac{x^2}{2}$, then we have the Dyson Brownoan motion corresponding to the Gaussian ensembles, that we have considered before. The results in the Random Matrix Theory literature discuss the class of potentials W that one can consider in order for the local statistics to be the same with the Gaussian potential $W(x) = \frac{x^2}{2}$.

Adopting the same strategy as in the previous section, we compare multiple Loewner chains with drivers given by the generalized Dyson dynamics (12.1) started from equilibirium, with W assuming the conditions in [34], [3] or [5], and couple the dynamics with the one obtained from the Gaussian potential $W(x) = \frac{x^2}{2}$. Then, one can obtain a convergence result adopting the same strategy as before, with even an easier argument using the results in [34], as there will be no further averaging in the window of E' occuring. Then, using the convergence of the k-point correlations (without averaging) and the sigma-algebra they generate, we obtain also the convergence in the drift part of the multiple Loewner dynamics. Then, having control of both the drift and the drivers, we obtain the result at the level of the multiple SLE.

As a next step, we want to undersand the models corresponding to the classes W of potentials that are covered in [5], [34] and [3]. Also, we want to undestand if there are discrete models of whose drivers can be described with some specific choices of W that are covered in [3], [34] and [5]. One reference for this direction it is Section 5, Theorem 5.1 in [37]. An idea for proof it is that if the convergence to multiple SLE on a certain lattice is already established, then moving from this lattice to another one (in a certain class) is similar to adding a drift in the Dyson BM, that by the Universality of General β Ensembles keeps the same statistics asymptotically as with the Gaussian potential. In Theorem 5.1 of [37] the dynamics of drivers is explicit and depends on the angles of the curve on the lattice. By changing the lattice, one modifies these angles. This can be expressed as a perturbation of a model that already converged to multiple SLE. The current development in this direction consists of checking if some specific perturbations can be covered by the classes of potentials considered in the proof of the Universality of the General β Ensembles.

12.1. Sketch of the result and of the proof in this case.

Theorem 12.1. *Let $\beta \geqslant 2$, i.e. $\kappa \leqslant 4$. Let us consider N-multiple SLE curves up to the first hitting time in the upper half-plane. Then, we have that, for any $k \geqslant 1$, the convergence of k-point correlations at the level of drivers for potentials W respecting the assumptions in [3] or [34], as $N \to \infty$, gives the convergence of the k-marginals*

$$\mathbb{P}^k_W(g^N_t(z) \in A_N, \ for\, z \in G_N) \to \mu^k_{W_G}(g_t(z) \in A, \ for\, z \in G_\infty), \tag{12.2}$$

(in a sense to be defined), as $N \to \infty$, where μ_{W_G} is the measure corresponding to Gaussian potential $W_G(x) = \frac{x^2}{2}$, $G_N, G_\infty \subset \mathbb{H}$, and A_N, A are Borel sets.

Remark 12.2. To obtain an averaged version of the result, the analysis in [5] it is very useful.

Sketch of the proof of 12.1. We couple

$$\sum_{i=1}^{N} \tilde{g}_t^{W,V_i} = \sum_{i=1}^{N} \left(\int_0^t \frac{2}{\tilde{g}_t^{W,V_i}} dt - \left(E + \frac{V_t^i}{N\rho_{sc}(E)} \right) \right)$$

and

$$\sum_{i=1}^{N} \tilde{g}_t^{W_G,B_i} = \sum_{i=1}^{N} \left(\int_0^t \frac{2}{\tilde{g}_t^{W_G,B_i}} dt - \left(E + \frac{B_t^i}{N\rho_{sc}(E)} \right) \right).$$

The drivers of the first chain correspond to the equilibrium measure of the Dyson BM with potential W as in [34] or [3] (see also [5]).

We are interested in estimating

$$\mathbb{E}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{W,V_i} \right) \right] - \mathbb{E}_{\mu_{W_G}}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{W_G,B_i} \right) \right],$$

as $N \to \infty$, for F test functions.

Then, we have

$$\mathbb{E}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{W,V_i} \right) \right] - \mathbb{E}_{\mu_{W_G}}\left[F\left(\sum_{i=1}^{N} \tilde{g}_t^{W_G,B_i} \right) \right]$$

$$= \left(\mathbb{E}\left[F\left(\sum_{i=1}^{N} \int_0^t \frac{2}{\tilde{g}_t^{W,V}(z)} dt \right) \right] \right) - \left(\mathbb{E}_{\mu_{W_G}}\left[F\left(\sum_{i=1}^{N} \int_0^t \frac{2}{\tilde{g}_t^{W_G,B}(z)} dt \right) \right] \right)$$
$$+ \left(\mathbb{E}\left[F\left(\sum_{i=1}^{N} \left(E + \frac{V_t^i}{N\rho_{sc}(E)} \right) \right) \right] - \mathbb{E}_{\mu_{W_G}}\left[F\left(\sum_{i=1}^{N} \left(E + \frac{B_t^i}{N\rho_{sc}(E)} \right) \right) \right] \right).$$

To understand the convergence of the second term on the RHS, as $N \to \infty$, we use [3] or [34]. For the first term in the RHS, we use the information in the sigma-algebra of the k-point correlations, for any $k \geqslant 1$.

[to be completed]

□

12.2. A different statistics: the gaps between the drivers. Applications to the geometry of the multiple SLE curves.

Let $0 \leqslant s \leqslant t$, and define

$$\tilde{\gamma}_s(t) = g_s(\gamma(t)).$$

In the case of one SLE curve, we have the following result.

Lemma 12.3 (Lemma 6.15 of [23]). *Let $W(s),\ \ s\in[0,t]$, be the driver in the Loewner Differential Equation. Then, we have that*

$$\frac{1}{2}\sup_{u\in[s,t]}|W(u)-W(s)|-2\sqrt{|s-t|}\leqslant\sup_{u\in[s,t]}|\operatorname{Re}\tilde{\gamma}_s(u)-W(s)|\leqslant\sup_{u\in[s,t]}|W(u)-W(s)|\ .$$

This lemma gives lower and upper bounds for the width of a box where the curve is observed in a given time interval. A similar estimate can be obtained for multiple curves. This can be used to obtain estimates on the difference between the neighborhood curves. This approach is further combined with the gaps statistics from RMT in order to provide new geometric information on the multiple SLE curves. This idea can be applied, in particular, for the multiple SLE model started from the Gaussian distribution. In the future, the analysis can be extended using the gap universality of the Dyson Brownian motion ([13]) that we discuss in the following.

Let us first consider the a-priori estimate, with the notation before: there exists $\xi>0$ such that

$$Q:=\sup_{0\leqslant t\leqslant N}\frac{1}{N}\int\sum_{j=1}^{N}\left(\lambda_j-\gamma_j\right)^2 f_t(\boldsymbol{\lambda})\mu_G(\,\mathrm{d}\boldsymbol{\lambda})\leqslant CN^{-2+2\xi},$$

with a constant C uniformly in N. We also assume that after time $1/N$ the solution of the equation

$$\partial_t f_t=\mathcal{L}_G f_t,\quad t\geqslant 0$$

satisfies the following entropy (S_μ) bound

$$S_\mu\left(f_{1/N}\right)\leqslant CN^m,$$

for some fixed m. In Lemma 14.6 in [13] it is shown that for $\beta=1,2$ this bound holds.

Theorem 12.4 (Theorem 14.1 of [13] (Gap universality of the Dyson Brownian motion for short time)). *Let $\beta\geqslant 1$ and assume the a-priori estimate and the entropy bound hold. Fix $n\geqslant 1$ and an array of positive integers, $\mathbf{m}=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_+^n$. Let $G:\mathbb{R}^n\to\mathbb{R}$ be a bounded smooth function with compact support and define*

$$\mathcal{G}_{i,\mathbf{m}}(\mathbf{x}):=G\left(N\left(x_i-x_{i+m_1}\right),N\left(x_{i+m_1}-x_{i+m_2}\right),\ldots,N\left(x_{i+m_{n-1}}-x_{i+m_n}\right)\right)$$

Then for any $\xi\in\left(0,\frac{1}{2}\right)$ and any sufficiently small $\varepsilon>0$, independent of N, there exist constants $C,c>0$, depending only on ε and G, such that for any $J\subset\{1,2,\ldots,N-m_n\}$ we have

$$\sup_{t\geqslant N-1+2\xi+\varepsilon}\left|\int\frac{1}{|J|}\sum_{i\in J}\mathcal{G}_{i,\mathbf{m}}(\mathbf{x})\left(f_t\ \mathrm{d}\mu-\mathrm{d}\mu\right)\right|\leqslant CN^{\varepsilon}\sqrt{\frac{N^2Q}{|J|t}}+Ce^{-cN^{\varepsilon}}.$$

We will use the result for the neighboring gaps only, i.e. we will have the case $m_1 = 1$, $m_2 = 2, \ldots, m_n = n$.

[to be completed]

13 Multiple Radial SLEs on the unit disk and the Circular β Ensembles

An important paper in this direction is the work of Cardy in [6]. In [6] it is introduced the framework for this type of analysis but the analysis in the paper leads to a different exponent $\beta = \frac{4}{\kappa}$. However, there is a corrected version published later by Cardy where the factor $\beta = \frac{8}{\kappa}$ is obtained.

13.1. Brief summary on the project on the multiple radial SLE and Circular Dyson Brownian motion with Vivian Healey. In this section, we give a brief description of the work in [17], in order to connect the main result with a theorem on the smallest gap distribution of the Circular Ensembles (see [14]). This joint project with Vivian Healey is detached from the previous independent work.

Let $a = \frac{2}{\kappa}$. We fix positive integer n and let $\boldsymbol{\theta} = \left(\theta^1, \ldots, \theta^n\right)$ with $\theta^1 < \cdots < \theta^n < \theta^1 + \pi$. We further consider $z^j = \exp\left\{2i\theta^j\right\}$ and $\mathbf{z} = \left(z^1, \ldots, z^n\right)$. Then, $z^1, \ldots, z^n$ are n distinct points on the unit circle ordered counterclockwise. Let $\gamma = \left(\gamma^1, \ldots, \gamma^n\right)$ be an n-tuple of curves $\gamma^j : (0, \infty) \to \mathbb{D} \backslash \{0\}$ with $\gamma^j(0+) = z^j$ and $\gamma^j(\infty) = 0$. We write γ_t^j for $\gamma^j[0, t]$ and $\gamma_t = \left(\gamma_t^1, \ldots, \gamma_t^n\right)$.

In the following, we introduce some notations from [17].

- Let D_t^j, D_t be the connected components of $\mathbb{D} \backslash \gamma_t^j, \mathbb{D} \backslash \gamma_t$, respectively, containing the origin. Let $g_t^j : D_t^j \to \mathbb{D}, g_t : D_t \to \mathbb{D}$ be the unique conformal transformations with

$$g_t^j(0) = g_t(0) = 0, \quad (g_t^j)'(0), g_t'(0) > 0$$

- Let T be the first time t such that $\gamma_t^j \cap \gamma_t^k \neq \emptyset$ for some $1 \leqslant j < k \leqslant n$. Define $z_t^j = \exp\left\{2i\theta_t^j\right\}$ by $g_t(\gamma^j(t)) = z_t^j$. Let $\mathbf{z}_t = \left(z_t^1, \ldots, z_t^n\right), \boldsymbol{\theta}_t = \left(\theta_t^1, \ldots, \theta_t^n\right)$. For $\zeta \in \mathbb{H}$, define $h_t(\zeta)$ to be the continuous function of t with $h_0(\zeta) = \zeta$ and

$$g_t(e^{2i\zeta}) = e^{2ih_t(\zeta)}.$$

Note that if $\zeta \in \mathbb{R}$ so that $e^{2i\zeta} \in \partial\mathbb{D}$, we can differentiate with respect to ζ to get

$$|g_t'(e^{2i\zeta})| = h_t'(\zeta).$$

- More generally, if $\mathbf{t} = (t_1, \ldots, t_n)$ is an n-tuple of times, we define $\gamma_{\mathbf{t}}, D_{\mathbf{t}}, g_{\mathbf{t}}$. We let

$$\alpha(\mathbf{t}) = \log g_{\mathbf{t}}'(0).$$

We will say that the curves have the common (capacity a-) parameterization, if for each t

$$\partial_j \alpha(t, t, \ldots, t) = 2a, \quad j = 1, \ldots, n$$

In particular,

$$g_t'(0) = e^{2ant}.$$

The common parametrization terminology in [17] is equivalent with the simultaneous growth terminology in [18] .

Proposition 13.1 (Proposition 3.1 in [17])**.** *[Radial Loewner equation] If γ_t has the common parameterization, then for $t < T$, the functions g_t, h_t satisfy*

$$\dot{g}_t(w) = 2a g_t(w) \sum_{j=1}^{n} \frac{z_t^j + g_t(w)}{z_t^j - g_t(w)}, \quad \dot{h}_t(\zeta) = a \sum_{j=1}^{n} \cot\left(h_t(\zeta) - \theta_t^j\right)$$

If ∂D_t contains an open arc of $\partial\mathbb{D}$ including $w = e^{2i\zeta}$, then

$$\left|g_t'(w)\right| = \exp\left\{-a \int_0^t \sum_{j=1}^{n} \csc^2\left(h_s(\zeta) - \theta_t^j\right) ds\right\}$$

In what follows next, we distinguish between three families of measures (that are being sequentially obtained one from another via tilting with some martingales).

- $\mathbb{P}, \mathbb{E}$ will denote independent SLE_κ with the common parameterization;
- $\mathbb{P}_*, \mathbb{E}_*$ will denote locally independent SLE_κ;
- $\mathcal{P}, \mathcal{E}$ will denote n -radial SLE_κ.

Define

$$F_\alpha(\boldsymbol{\theta}) = \prod_{1 \leqslant j < k \leqslant n} \left|\sin\left(\theta^k - \theta^j\right)\right|^\alpha.$$

Let $\mathbb{P}$ denote the measure on n independent radial SLE_κ curves from θ_0 to 0 with the a -common parameterization.

In Section 3.2, of [17], it is obtained $\mathbb{P}_*$ from $\mathbb{P}$ by tilting by a $\mathbb{P}$ -local martingale M_t. After that, it is obtained $\mathcal{P}$ from $\mathbb{P}_*$ by tilting by a $\mathbb{P}_*$ -local martingale $N_{t,T}$ and then letting $T \to \infty$. Equivalently, it is obtained $\mathcal{P}$ from $\mathbb{P}$ by tilting by $\tilde{N}_{t,T} := M_t N_{t,T}$ and letting $T \to \infty$, where all the martingales described above are explicit in [17].

Let $L_t^j = L_t^j(\gamma_t)$ be the set of loops ℓ with $s(\ell) < s^j(\ell)$ and $s(\ell) \leqslant t$. Define

$$\mathcal{L}_t = I_t \exp\left\{\frac{\mathbf{c}}{2}\sum_{j=1}^{n} m_{\mathbb{D}}\left(L_t^j\right)\right\}$$

Here I_t is the indicator function that $\gamma_t^j \cap \gamma^k = \emptyset$ for $j \neq k$.

Theorem 13.2 (Theorem 3.12. of [17])**.** *Let $0 < \kappa \leqslant 4$. Let $t > 0$ be fixed. For each $T > t$, let $\mu_T = \mu_{T,t}$ denote the measure whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is*

$$\frac{\mathcal{L}_T}{\mathbb{E}^{\boldsymbol{\theta}_0}\left[\mathcal{L}_T\right]}$$

Then as $T \to \infty$, the measure $\mu_{T,t}$ approaches $\mathcal{P}$ with respect to the variation distance. Furthermore, the driving functions $z_t^j = e^{2i\theta_t^j}$ satisfy

$$d\theta_t^j = 2a\sum_{k\neq j}\cot\left(\theta_t^j - \theta_t^k\right)dt + dW_t^j,$$

where W_t^j are independent standard Brownian motions in $\mathcal{P}$.

One of the essential identities in the proof of the above theorem is the following

$$\frac{d\mu_{T,t}/d\mathbb{P}_t}{d\mathcal{P}_t/d\mathbb{P}_t} = 1 + O\left(e^{-u(T-t)}\right).$$

The right-hand side dependes on T but also inside the O notation, there is an N dependence, that we will make explicit. The previous result,it is proved for fixed N, where N is the number of curves. In order to apply results from Random Matrices, we are interested in the asymptotic in N. In our case, in the previous expression, we take T and N simultaneous to $+\infty$, in order to use the next result from Random Matrix Theory (that is a direct consequence of the main result in [14]).

Corollary 13.3 (Corollary 1 of [14]). *Let's denote m_k as the k-th smallest gap, and*

$$\tau_k = n^{(\beta+2)/(\beta+1)} \times (A_\beta/(\beta+1))^{1/(\beta+1)} \, m_k,$$

with A_β an explicit factor in β. Then, for any bounded interval $A \subset \mathbb{R}_+$, we have

$$\lim_{n \to +\infty} \mathbb{P}(\tau_k \in A) = \int_A \frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} dx$$

In particular, the limiting density function for τ_1 is

$$(\beta+1)x^\beta e^{-x^{\beta+1}}.$$

One of the main goals of the project is to use the previous result from Random Matrix Theory literature concerning the distribution of the smallest gap in the case of general Circular Ensembles, to obtain new geometric information on the multiple SLE curves.

Another possibility that we investigate is to start from stationarity and perform computations using the equilibrium measure of the Dyson BM and the simultaneous growth multiple Loewner Differential Equation.

14 Future projects

14.1. Bringing tools from Machine Learning (ML): One curve case. In Machine Learning, approximation of an intractable integration is often achieved by using the unbiased Monte Carlo estimator, but the variances of the estimation are generally very large. Control variates approaches are well-known to reduce the variance of the estimation. This method was successfully applied to specific diffusion processes ([15]) in order to obtain reduced-variance estimations of functionals of them. Diffusion processes appear very often in SLE computations and reduced-variance estimations potentially will bring new insights in the theory. I outline a few possible directions below, many expected to appear as the exploration continues.

- 1)Consider $|h_s'(z)|$, where $h_s(z)$ are time-changed backward SLE maps. Moments of these quantities are studied in order to prove the existence of the SLE trace (see [23]). This quantity can be re-expressed using functionals of the form $\exp(2s - 4\int_0^s \sin^2(\arg(h_u(z))du)$ (see [23]). Obtaining low-variance estimations for this functional for stochastic drivers of Loewner evolution that are 'close' to Brownian Motion (but not Brownian motion itself) will bring new tools to explore random Loewner curves beyond the Brownian and martingale machinery.

- 2) Also in a different direction, even in the Brownian driver case, an interesting object to study is $\int_0^t dt/y_t^2$, where y_t is the imaginary part of the backward Loewner maps. This functional is important in order to understand a fundamental quantity: the law of the SLE tip at a fixed time. We plan to understand expected value, variance and higher moments of this quantity in order to understand it's asymptotic behavior (that will resolve a conjecture on the law of the SLE trace at a fixed time).

- 4) Perform the **Remez algorithm** to obtain the mini-max polynomial that optimally approximates the Brownian driver. This approach combined with the algorithm simulating Loewner curves ([36]) will give an optimal approach on the simulation of the SLE traces problem. Moreover, in general, I want to define and study **a randomized version of Remez algorithm** with the goal to obtain faster the optimal approximation polynomial. The latter project is a general one and is of interest for ML community and beyond.

- Also in a different direction, an interesting unresolved (to the best of my knowledge) problem is a Law of Iterated Logarithm for the SLE curves. A possible first step in this direction is to obtain a low-variance estimate for functionals of the form $E[f(g_t(z)-q(t))]$, where $g_t(z)$ are the forward SLE maps and $q(t)$ is a deterministic function (that describes the possible envelope). The exploration can continue with other functionals that capture the geometry of the curves.

- All these questions have natural extensions in the multiple SLE setting. In order to implement optimal approximations of the multiple SLE curves we foresee the use of the neural networks (given the multi-dimensionality, i.e. number of curves, of the problem).

14.2. Tools from ML: Multiple curves case. In this subsection, we highlight a specific high-dimensional approximation method that we plan to use: the Random Batch Method (RBM). These methods are motivated by the mini-batch methods from Machine Learning. In [19] a Random Batch method (RBM-1) is studied for interacting particle systems with applications to the Dyson Brownian motion ($\beta = 1$). In this paper, it is proved that under suitable conditions on the external potential, interaction force and batch size, that the RBM-1 is converging with a numerical error depending only on the time-step and independent of N (the number of particles). The result gives a Wasserstein norm convergence of the laws of the marginals of the approximation and the true interacting particle dynamics. I plan to extend this method to obtain the result for the marginals of the multiple SLE curves, especially the extreme ones, where the Tracy-Widom distribution as well as the Airy process will appear.

14.3. Further directions to investigate in the Multiple Chordal SLE case.

- 0) Perform the analysis and obtain the corresponding result for $8 \geqslant \kappa \geqslant 4$.

  In this context, we have the following lemma that establishes a connection between the parameters corresponding to SLE duality and the corresponding one in the β ensembles.

  Lemma 14.1. *Let $\kappa, \kappa' \in \mathbb{R}_+$ and let $\beta = \frac{8}{\kappa}$ and $\beta' = \frac{8}{\kappa'}$. Then, the relation $\beta' = \frac{\beta}{4}$ is equivalent with the κ duality in SLE_κ theory, i.e. $\kappa' = 16/\kappa$.*

  Proof. Let $\beta = 8/\kappa$ and $\beta' = 8/\kappa'$. Then, $\beta = 4/\beta'$ is equivalent with $8/\kappa = 4\kappa'/8 = \kappa'/2$. Thus, $\kappa' = 16/\kappa$. □

  We want to use the duality results of SLE Theory to obtain some new information about results in Random Matrix Theory.

- 1) Use the Fixed energy Universality proof in [26] to redo the proof for $\kappa = 4$. Then, the fixed energy result for $\kappa = 4$, it implies the averaging result, and then obtains the main results in this draft in a stronger form.

- 1.b) Provide the same analysis using the universality at the edge (Theorem 18.7 in Section 18.4 of [13]) (note that is not formulated in the averaged sense) and Theorem 17.1 in [12] for the largest eigenvalue. A similar approach will give new results.

- 2) An important model in which one can consider the simultaneous and non-simultaneous growth (more relevant): Consider in the unit disk the $+-$ alternating boundary conditions on half of the circle and $+$ on the other half of the circle. Consider the multiple SLEs in this context and pass to the hydrodynamic limit. Then the outer boundary of the limiting shape behaves like an interface between 0 (all the signs on one side cancel) and + (see also [9]).

- 3) One direction to investigate is the implication of the result for the multiple SLE curves in CFT, using the work in [6].

- 4) Use the extreme gaps result proved in the work [1] along with multiple curves versions of the lemmas in the previous section and obtain the geometric information about the multiple SLE curves (using information on the time dynamics of the smallest and the biggest gap).

- 5) Study the reversibility of the Dyson Brownian motion dynamics and study similar questions in the context of multiple backward SLE (the time reversal of the multiple SLEs).

- 6) Use the β-ensembles universality result in [5] proof, to compare general potentials with the typical Gaussian potential at the level of drivers, and understand the impact that it has on the geometry of the curves/hulls.

- 7) Using the β-Tracy-Widom distribution about the fluctuations of the top eigenvalue, we can obtain some geometric information about the dynamics of the extremal SLE curve (i.e. the curve that grows from the extremal eigenvalue), as well as obtain some comparison between the dynamics of the extremal SLE curve, i.e. the one that grows from the position of the top eigenvalue, and the dynamics of some concrete curve (such as a tilted /straight line) also growing from the position of the top eigenvalue.

- 8) We also plan to use the connection between TASEP large time limit with Tracy-Widom ([20]). In general, there is no link at the moment (that I know about) between the particle systems scaling limits and the SLE curves. We plan to obtain some results that link some the asymptotic behavior of the extreme SLE curve with some results about TASEP. In addition, we plan to establish further the connection between multilevel TASEP, Warren process and multiple SLE (see [16]).

- 9) Connect the development in this direction presented in this draft with the convergence and conformal invariance of discrete models coming from planar Statistical Physics on various lattices ([24], [37]).

14.4. Main directions to be studied in the Multiple Radial SLE case.

- ▸ 1) One direction is to focus on the critical case $\kappa = 4$. We plan use techniques and results about the smallest gap distribution obtained in [1] in the case of the CUE (a particular case of the result in [14], for $\kappa = 4$) and obtain new information on the critical case.

- ▸ 2) We plan to obtain estimates on the geometry of the hulls/curves when one starts from the stationary measure directly.

- ▸ 3) To check another direction: we plan to improve the topology with the estimates on difference between curves as in [36].

References

[1] Gérard Ben Arous and Paul Bourgade. Extreme gaps between eigenvalues of random matrices. *The Annals of Probability*, 41(4):2648–2681, 2013.

[2] Vincent Beffara, Eveliina Peltola, and Hao Wu. On the uniqueness of global multiple SLEs. *arXiv preprint arXiv:1801.07699*, 2018.

[3] Florent Bekerman, Alessio Figalli, and Alice Guionnet. Transport Maps for β-Matrix Models and Universality. *Communications in mathematical physics*, 338(2):589–619, 2015.

[4] Ludwig Bieberbach. Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitzungsberichte Preussische Akademie der Wissenschaften*, 138:940–955, 1916.

[5] Paul Bourgade, László Erdős, and Horng-Tzer Yau. Universality of general β-ensembles. *Duke Mathematical Journal*, 163(6):1127–1190, 2014.

[6] John Cardy. Stochastic Loewner evolution and Dyson's circular ensembles. *Journal of Physics A: Mathematical and General*, 36(24):L379, 2003.

[7] Louis De Branges. A proof of the Bieberbach conjecture. *Acta Mathematica*, 154(1-2):137–152, 1985.

[8] Andrea del Monaco, Ikkei Hotta, and Sebastian Schleißinger. Tightness results for infinite-slit limits of the chordal loewner equation. *Computational Methods and Function Theory*, 18(1):9–33, 2018.

[9] Andrea del Monaco and Sebastian Schleißinger. Multiple SLE and the complex Burgers equation. *Mathematische Nachrichten*, 289(16):2007–2018, 2016.

[10] Julien Dubédat. Commutation relations for Schramm-Loewner evolutions. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 60(12):1792–1847, 2007.

[11] László Erdős, Sandrine Péché, José A Ramírez, Benjamin Schlein, and Horng-Tzer Yau. Bulk universality for wigner matrices. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 63(7):895–925, 2010.

[12] László Erdős, Benjamin Schlein, and Horng-Tzer Yau. Universality of random matrices and local relaxation flow. *Inventiones mathematicae*, 185(1):75–119, 2011.

[13] László Erdos and Horng-Tzer Yau. A dynamical approach to random matrix theory. *Courant Lecture Notes in Mathematics*, 28, 2017.

[14] Renjie Feng and Dongyi Wei. Small gaps of circular beta-ensemble. *arXiv preprint arXiv:1806.01555*, 2018.

[15] Josselin Garnier and Laurent Mertz. A control variate method driven by diffusion approximation. *Communications on Pure and Applied Mathematics*, 2019.

[16] Vadim Gorin and Mykhaylo Shkolnikov. Multilevel Dyson Brownian motions via jack polynomials. *Probability Theory and Related Fields*, 163(3-4):413–463, 2015.

[17] Vivian Olsiewski Healey and Gregory F Lawler. N-sided Radial Schramm-Loewner Evolution. *arXiv preprint arXiv:2002.04128*, 2020.

[18] Ikkei Hotta and Sebastian Schleißinger. Limits of radial multiple SLE and a Burgers-Loewner differential equation. *arXiv preprint arXiv:1904.02556*, 2019.

[19] Shi Jin, Lei Li, and Jian-Guo Liu. Random Batch Methods (RBM) for interacting particle systems. *Journal of Computational Physics*, 400:108877, 2020.

[20] Kurt Johansson. Shape fluctuations and random matrices. *Communications in mathematical physics*, 209(2):437–476, 2000.

[21] Alex Karrila. Multiple SLE type scaling limits: from local to global. *arXiv preprint arXiv:1903.10354*, 2019.

[22] Makoto Katori and Shinji Koshida. Conformal welding problem, flow line problem, and multiple Schramm–Loewner evolution. *arXiv preprint arXiv:1903.09925*, 2019.

[23] Antti Kemppainen. Schramm–Loewner Evolution. In *Schramm–Loewner Evolution*, pages 69–100. Springer, 2017.

[24] Antti Kemppainen, Stanislav Smirnov, et al. Random curves, scaling limits and Loewner evolutions. *The Annals of Probability*, 45(2):698–779, 2017.

[25] Kalle Kytölä and Eveliina Peltola. Pure partition functions of multiple sles. *Communications in Mathematical Physics*, 346(1):237–292, 2016.

[26] Benjamin Landon, Philippe Sosoe, and Horng-Tzer Yau. Fixed energy universality of Dyson Brownian motion. *Advances in Mathematics*, 346:1137–1332, 2019.

[27] Gregory F Lawler. *Conformally invariant processes in the plane.* Number 114. American Mathematical Soc., 2008.

[28] Jonatan Lenells and Fredrik Viklund. Schramm's formula and the Green's function for multiple SLE. *Journal of Statistical Physics*, 176(4):873–931, 2019.

[29] Karl Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Mathematische Annalen*, 89(1-2):103–121, 1923.

[30] James Norris, Vittoria Silvestri, and Amanda Turner. Scaling limits for planar aggregation with subcritical fluctuations. *arXiv preprint arXiv:1902.01376*, 2019.

[31] James Norris and Amanda Turner. Hastings–Levitov aggregation in the small-particle limit. *Communications in Mathematical Physics*, 316(3):809–841, 2012.

[32] Eveliina Peltola and Hao Wu. Global and Local Multiple SLEs for $\kappa \leqslant 4$ and Connection Probabilities for Level Lines of GFF . *Communications in Mathematical Physics*, 366(2):469–536, 2019.

[33] Kazumitsu Sakai. Multiple Schramm–Loewner evolutions for conformal field theories with Lie algebra symmetries. *Nuclear Physics B*, 867(2):429–447, 2013.

[34] M. Scherbina. Change of variables as a method to study general β-models: bulk universality. *Journal of Mathematical Physics*, 2014.

[35] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1):221–288, 2000.

[36] Huy Tran. Convergence of an algorithm simulating Loewner curves. In *Annales Academiae Scientiarum Fennicae. Mathematica*, volume 40, 2015.

[37] Jonathan Tsai. The Loewner driving function of trajectory arcs of quadratic differentials. *Journal of mathematical analysis and applications*, 360(2):561–576, 2009.

[38] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the brownian carousel. *Inventiones mathematicae*, 177(3):463–508, 2009.