Top 3 Funniest Functions for Beginner Analysts

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Preface

In mathematics we often run into various ideas that run counter to our intuitions. This occurs enough in mathematics to motivate an assigned name to these phenomena, Pathological. Hence the motivation for this project was to present an easily digestible introduction to some elementary examples of pathology's that occur in mathematics. To facilitate this discussion we'll begin by revisiting topics that should be familiar to those who've taken calculus.

Section 1

1.1 Continuity and Limits:

If you recall from your very first calculus course, you may have been told that a function f(x) is continuous on its domain if it has no "holes." That early intuition is reflected in the formal definition of what it means for a function to be continuous at a point or on a set.

Definition: A function f : A → B, is continuous at a point $x_0 \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \epsilon$$

If f is continuous at every point in its domain A, then we say that f is continuous on A.

When we say that a function f is continuous at point x_0 in its domain, we are arguing that all x values extremely close to x_0 have outputs also extremely close to $f(x_0)$.
It would then be reasonable to think of $f(x_0)$ as the limit of f(x), for x near x_0. This notion is formalized in the following known as the "$\epsilon - \delta$" definition for functional limits,

Definition: Let f : A → B, and let a be a limit point of the domain A. We say that $\lim_{x \to a}$ f(x) = L provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$ then,

$$|f(x) - L| < \epsilon \tag{1}$$

1.2 Differentiation:

Derivatives should also feel like mostly familiar territory.

Definition: Let f : A → B be a function defined on an interval A. Given $x_0 \in$ A, the derivative of f at x_0 is defined by

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

If g′ exists for all points in A, we say that f is differentiable on A.
Note that for a function to be differentiable at a point x_0 it must be continuous x_0, hence differentiation implies continuity.
But does continuity imply differentiation?

1.3 Sequences of Functions:

Sequences of functions should not feel too unfamiliar, many of you may have already worked with sequences of functions without realizing it. One example of a sequence of functions that you may be familiar with is,

$$f_n(x) = x^n \tag{2}$$

this function in itself is interesting because although each f_n is continuous, the sequence converges to a non-continuous function

$$f(x) = \begin{cases} 0, & x \in (-1, 1) \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$$

1.4 Series of Functions:

If the previous segment felt slightly confusing the following should revitalize familiar notions of sequences of functions for those who explored well into integral calculus.

Taylor Series: Let A denote the closed interval [a,b]. Suppose f is a real function on A, n is a positive integer, f^{n-1} is continuous on A, $f^{(n)}(x)$ exists for every x $\in$ (a, b). Let α, β be distinct points in A, and define

$$P(x) = \sum_{k=0}^{n-1} f^{(k)}(\alpha)\frac{(x-\alpha)^k}{k!}. \tag{3}$$

Then there exists a point t between α and β such that,

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{4}$$

Taylor's theorem shows that f can be approximated by a polynomial of degree n -1, and (4) allowed us to estimate the error.

Taylor Series are extremely nice cases of series of functions. Generally, when g is some sequence of functions, a series of functions is of the form:

$$S_n(x) = \sum_{k=0}^{n} g_k(x)$$

Section 2:

Summary:

Now that we've refreshed your mind with some familiar or long-forgotten mathematical concepts, We shall present three pathological mathematical cases that break common intuitions around continuity and differentiability.

2.1 Dirichlet function:

Consider the real function f defined as,

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{Q}^c. \end{cases}$$

Functions of this form where different values are assumed at the rationals and irrationals are often referred to as Dirichlet functions (named after mathematician Peter Gustav Lejeune Dirichlet) and are nowhere continuous. The intuition behind why this function is nowhere continuous ties back to the earlier notion of continuous functions having no holes. Regardless of the point you are standing on, rational or irrational, by the density of the rationals and irrationals in $\mathbb{R}$, you will always be sitting between two rationals or two irrationals. Hence there are always immediate "holes" to your left and right.

2.2 The Devil's Staircase:

Also known as the Cantor function, the Devil's staircase is a monotonically increasing function from 0 to 1. However, despite going from 0 to 1 on the y-axis, the Cantor function has a derivative of 0 almost everywhere.

The Cantor Function: Let $\mathcal{C}$ denote the Cantor set then the Cantor function c(x) : [0, 1] $\to$ [0, 1] is defined as

$$c(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2^n}, & x \in \mathcal{C} \\ \sup_{y \le x} c(y), & x \in [0,1]/\mathcal{C}, y \in \mathcal{C} \end{cases}$$

The notion of the derivative being zero almost everywhere stems from the fact that the Cantor function only increases on points that belong to the Cantor set, which has a length of zero.

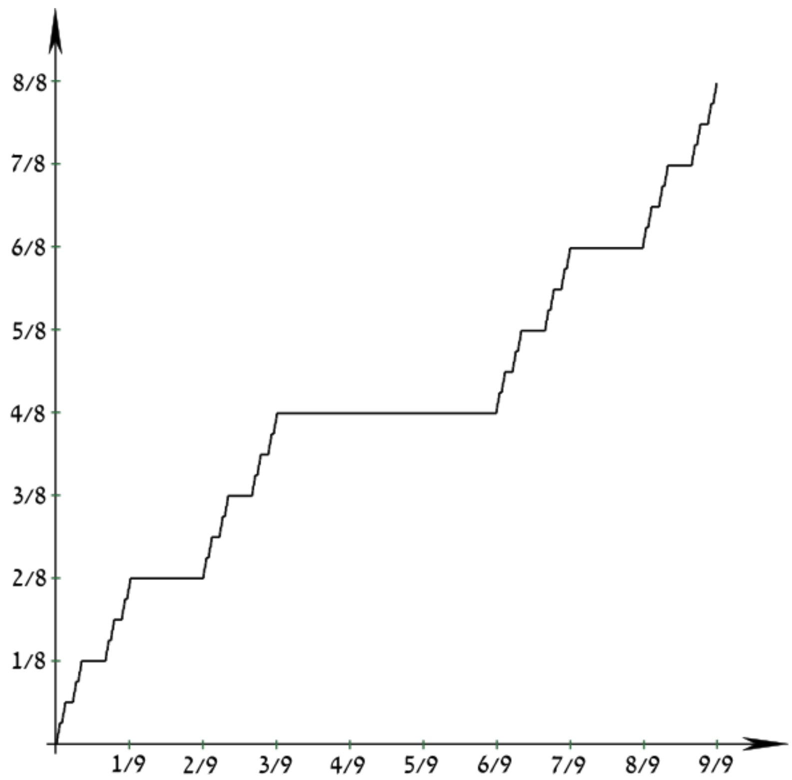

Figure 1: Cantor Function

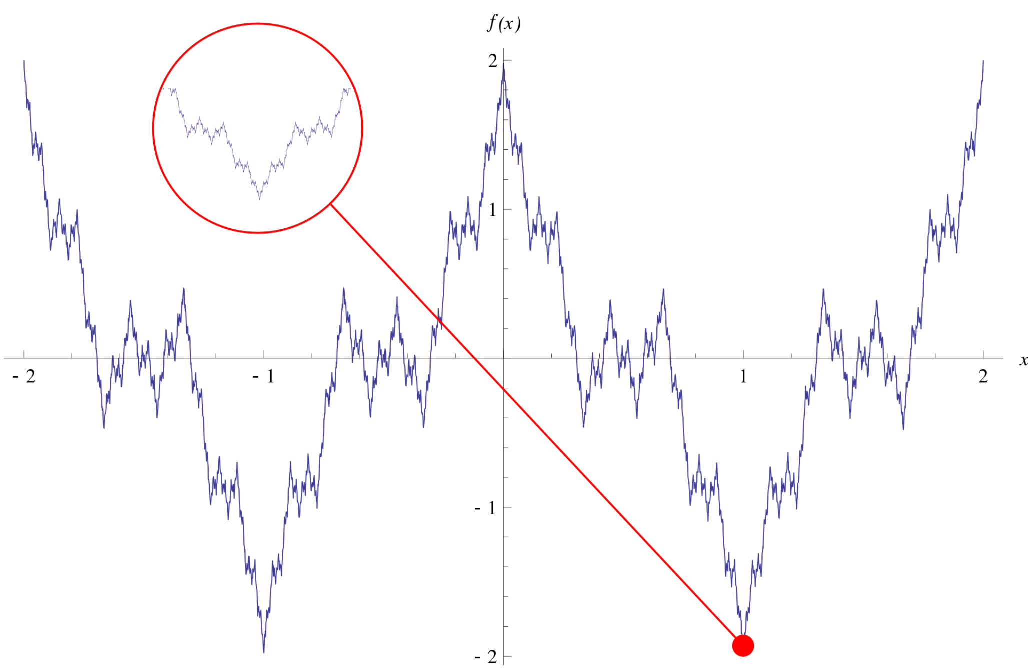

Figure 2: Weierstrass function

2.3 The Weierstrass Function:

> ”I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives”. Charles Hermite; letter to Thomas Joannes Stieltjes about Weierstrass functions,
> Correspondance d’Hermite et de Stieltjes vol.2, p.317-319

We end this with the weirdest of the three functions. First published in 1872 (though discovered sometime before), the Weierstrass function was created with the intent to contradict naive notions that continuity implies differentiation. Hence we present the Weierstrass Function, a function that is continuous everywhere but differential nowhere.

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $a \in (0,1)$ b is odd and $ab > 1 + \frac{3\pi}{2}$

To prove this function converges uniformly is pretty straightforward, since we know that cos is bounded, and a is less than 1, thus this function is bounded by a geometric series. Which we recall $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ for r $\in (0,1)$

Showing this function is not differentiable is a bit more complex, but in essence, one takes advantage of some of the facts around our choice of b.

Upon fixing any x, it is possible to construct sequences that converge to x monotonically from opposing sides, and have their limit definitions converge to opposite signed infinities. Skipping a whole lot of algebra and meticulous sequence construction we eventually get the opposing signed statements:

$|\frac{f(z_m)-f(x_0)}{z_m-x_0}| > (ab)^m(\frac{2}{3} - \frac{\pi}{ab-1}) = \infty$ as $m \longrightarrow \infty$

where the sign of $|\frac{f(z_m)-f(x_0)}{z_m-x_0}|$ is $(-1)^{\alpha_m}$

$|\frac{f(y_m)-f(x_0)}{y_m-x_0}| > (ab)^m(\frac{2}{3} - \frac{\pi}{ab-1}) = \infty$ as $m \longrightarrow \infty$

where the sign of $|\frac{f(y_m)-f(x_0)}{y_m-x_0}|$ is $-(-1)^{\alpha_m}$ Where z_m and y_m are opposite approaching sequences to any given point.

A link to an extensive proof is provided:[(4)] Brent from Berkeley's Project

1 References

(1) Ross KA. Elementary Analysis : The Theory of Calculus. 2nd ed. New York: Springer; 2013. doi:10.1007/978-1-4614-6271-2

(2) Abbott S. Understanding Analysis. Second ed. New York: Springer; 2015. doi:10.1007/978-1-4939-2712-8

(3) Rudin W. Principles of Mathematical Analysis. Third ed. New York: McGraw-Hill; 1976.

(Fig 1) Cantor Function, Wikipedia

(Fig 2) Weierstrass function, Wikipedia