2. Перевірити, що форма ω є повним диференціалом і обчислити $\int\limits_A^B \omega$.

Якщо довести, що $\omega = dg$, $g \in C^1(\Gamma)$, то за формулою Ньютона-Лейбніца $\int\limits_A^B \omega = \int\limits_A^B dg = g(u(b)) - g(u(a)) = g(B) - g(A)$.

1) $\omega = (x_2 dx_1 - x_1 dx_2)\, x_1^{-2}$, $A = (2, 1)$, $B = (1, 2)$.

Якщо $\omega = dg$, то $\begin{cases} \frac{\partial g}{\partial x_1} = \frac{x_2}{x_1^2} \\ \frac{\partial g}{\partial x_2} = -\frac{1}{x_1}. \end{cases}$ Розв'яжемо цю систему.

З першого рівняння $g = \int \frac{x_2}{x_1^2} dx_1 = -\frac{x_2}{x_1} + C(x_2)$. Підставимо в друге $-\frac{1}{x_1} + C'(x_2) = -\frac{1}{x_1}$. Тоді $C'(x_2) = 0 \Rightarrow C(x_2) = C_1 \Rightarrow g = -\frac{x_2}{x_1} + C_1$, $\omega = dg$.

$\int\limits_A^B \omega = g(B) - g(A) = -\frac{2}{1} + \frac{1}{2} = -\frac{3}{2}$.

2) $\omega = (x_1^4 + 4x_1x_2^3)\,dx_1 + (6x_1^2x_2^2 - 5x_2^4)\,dx_2$, $A = (-2, -1)$, $B = (3, 0)$

$\begin{cases} \frac{\partial g}{\partial x_1} = x_1^4 + 4x_1x_2^3 \\ \frac{\partial g}{\partial x_2} = 6x_1^2x_2^2 - 5x_2^4 \end{cases}$ $\Rightarrow g = \int (x_1^4 + 4x_1x_2^3)\,dx_1 = \frac{x_1^5}{5} + 2x_1^2x_2^3 + C(x_2)$. Підставимо в друг

$6x_1^2x_2^2 + C'(x_2) = 6x_1^2x_2^2 - 5x_2^4 \Rightarrow C(x_2) = \int(-5x_2^4)\,dx_2 = -x_2^5 + C_1 =$

$\Rightarrow g = \frac{x_1^5}{5} + 2x_1^2x_2^3 - x_2^5 + C_1$

$\int\limits_A^B \omega = g(B) - g(A) = \left(\frac{243}{5} + 0 - 0\right) - \left(-\frac{32}{5} - 8 + 1\right) = 62$

4) $\omega = x_2x_3\,dx_1 + x_3x_1\,dx_2 + x_1x_2\,dx_3$, $A = (1, 2, 3)$, $B = (6, 1, 1)$

$\begin{cases} \frac{\partial g}{\partial x_1} = x_2x_3 \\ \frac{\partial g}{\partial x_2} = x_3x_1 \\ \frac{\partial g}{\partial x_3} = x_1x_2 \end{cases}$

$\Rightarrow g = \int x_2x_3\,dx_1 = x_1x_2x_3 + C(x_2, x_3)$. Підставимо в друге:

$x_1x_3 + \frac{\partial C}{\partial x_2} = x_1x_3 \Rightarrow C = \int 0\,dx_2 = C_1(x_3) \Rightarrow g = x_1x_2x_3 + C_1(x_3)$

$x_1x_2 + C_1'(x_3) = x_1x_2 \Rightarrow C_1(x_3) = C_2 \Rightarrow g = x_1x_2x_3 + C_2$

$\int\limits_A^B \omega = g(B) - g(A) = 6 \cdot 1 \cdot 1 - 1 \cdot 2 \cdot 3 = 0$

3. Знайти функцію, що має такий диференціал в A:

4) $dz = (x_1^2 - 2x_2x_3)dx_1 + (x_2^2 - 2x_1x_3)dx_2 + (x_3^2 - 2x_1x_2)dx_3$, $A = \mathbb{R}^3$

$$\begin{cases} \frac{\partial z}{\partial x_1} = x_1^2 - 2x_2x_3 \\ \frac{\partial z}{\partial x_2} = x_2^2 - 2x_1x_3 \\ \frac{\partial z}{\partial x_3} = x_3^2 - 2x_1x_2 \end{cases}$$

$\Rightarrow z = \int (x_1^2 - 2x_2x_3)dx_1 = \frac{x_1^3}{3} - 2x_1x_2x_3 + C(x_2, x_3)$. В друге:

$-2x_1x_3 + \frac{\partial C}{\partial x_2} = x_2^2 - 2x_1x_3 \Rightarrow C = \int x_2^2 dx_2 = \frac{x_2^3}{3} + C_1(x_3) \Rightarrow$

$\Rightarrow z = \frac{x_1^3}{3} - 2x_1x_2x_3 + \frac{x_2^3}{3} + C_1(x_3)$. В третє

$-2x_1x_2 + C_1'(x_3) = x_3^2 - 2x_1x_2 \Rightarrow C_1 = \int x_3^2 dx_3 = \frac{x_3^3}{3} + C_2 \Rightarrow$

$z = \frac{x_1^3}{3} - 2x_1x_2x_3 + \frac{x_2^3}{3} + \frac{x_3^3}{3} + C_2$

4. Знайти роботу сили земного тяжіння по переміщенню матер. точки маси m з т. (x_1, x_2, x_3) в т. (y_1, y_2, y_3). Вісь Ox_3 направлена вгору.

За модулем $F = mg$, векторно $\vec{F} = (0, 0, -mg)$. (локально)

Тоді $\omega = F_1dx_1 + F_2dx_2 + F_3dx_3 = -mg\,dx_3$,

$$\begin{cases} \frac{\partial z}{\partial x_1} = 0 \\ \frac{\partial z}{\partial x_2} = 0 \\ \frac{\partial z}{\partial x_3} = -mg \end{cases} \Rightarrow z = -mgx_3 + C$$

$A = \int_A^B \omega = z(B) - z(A) = -mgy_3 + mgx_3 = mg(x_3 - y_3)$

A18

1. Обчислити площу поверхні

1) $x_3 = x_1x_2$, $x_1^2 + x_2^2 \le 1$

$S = \int_S 1\,d\sigma = \int_T \sqrt{A^2+B^2+C^2}\,dx_1dx_2 = \left| T = \{(x_1,x_2) \mid x_1^2 + x_2^2 \le 1\} \right|$

$= \int_T \sqrt{1 + x_2^2 + x_1^2}\,dx_1dx_2 = \left| \begin{matrix} \text{полярні} \\ \text{координати} \end{matrix} \right| =$

$= \int_0^{2\pi} \left(\int_0^1 \sqrt{1+r^2}\, r\,dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^1 \frac{1}{2}(1+r^2)^{1/2} d(1+r^2) \right) d\varphi =$

$= \int_0^{2\pi} \frac{(1+r^2)^{3/2}}{3} \Big|_0^1 d\varphi = \int_0^{2\pi} \left(\frac{2\sqrt{2}}{3} - \frac{1}{3} \right) d\varphi = \frac{2\pi}{3}(2\sqrt{2} - 1)$

$A = \frac{\partial(u_2, u_3)}{\partial(t, s)} = \begin{vmatrix} \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial s} \\ \frac{\partial u_3}{\partial t} & \frac{\partial u_3}{\partial s} \end{vmatrix}$

$B = \frac{\partial(u_3, u_1)}{\partial(t, s)} = \begin{vmatrix} \frac{\partial u_3}{\partial t} & \frac{\partial u_3}{\partial s} \\ \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial s} \end{vmatrix}$

$C = \frac{\partial(u_1, u_2)}{\partial(t, s)} = \begin{vmatrix} \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial s} \\ \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial s} \end{vmatrix}$

Якщо $\begin{cases} x_1 = t \\ x_2 = s \\ x_3 = f(t, s) \end{cases}$, то

$\sqrt{A^2 + B^2 + C^2} = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2}$

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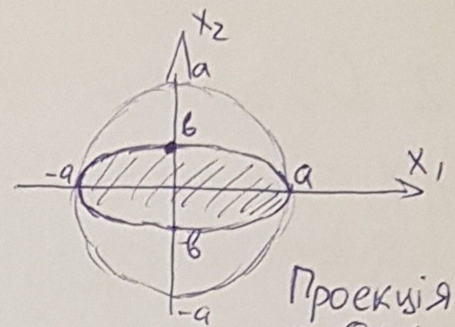
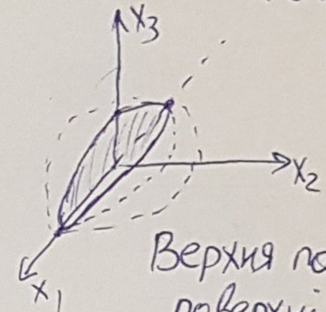


Проекція на Ox_1x_2

1\. 2) $x_1^2+x_2^2+x_3^2=a^2$, $\quad \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\le 1$, $0<b<a$

$$T=\left\{(x_1,x_2)\,\middle|\,\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\le 1\right\}$$

Параметри – x_1, x_2; $\quad x_3=\sqrt{a^2-x_1^2-x_2^2}$



Верхня половина поверхні

$$S=2\iint_S 1\,d\sigma=2\iint_T\sqrt{A^2+B^2+C^2}\,dx_1dx_2=$$

$$=2\iint_T\sqrt{1+\left(\frac{-x_1}{\sqrt{a^2-x_1^2-x_2^2}}\right)^2+\left(\frac{-x_2}{\sqrt{a^2-x_1^2-x_2^2}}\right)^2}\,dx_1dx_2=$$

$$=2\iint_T\frac{a}{\sqrt{a^2-x_1^2-x_2^2}}\,dx_1dx_2=\left|\begin{array}{lll}\text{узаг. полярні коорд.} & & \\ x_1=ar\cos\varphi & r\in[0,1] & \\ x_2=br\sin\varphi & \varphi\in[0,2\pi] & J=abr\end{array}\right|=$$

$$=2\int_0^{2\pi}\left(\int_0^1\frac{a\cdot abr}{\sqrt{a^2-a^2r^2\cos^2\varphi-b^2r^2\sin^2\varphi}}\,dr\right)d\varphi=2a^2b\int_0^{2\pi}\left(\int_0^1\frac{d(a^2-a^2r^2\cos^2\varphi-b^2r^2\sin^2\varphi)}{2\sqrt{a^2-a^2r^2\cos^2\varphi-b^2r^2\sin^2\varphi}}\cdot\right.$$

$$\left.\cdot\frac{1}{(-a^2\cos^2\varphi-b^2\sin^2\varphi)}\right)d\varphi=2a^2b\int_0^{2\pi}\frac{1}{(-a^2\cos^2\varphi-b^2\sin^2\varphi)}\cdot\sqrt{a^2-a^2r^2\cos^2\varphi-b^2r^2\sin^2\varphi}\,\Big|_{r=0}^{1}\,d\varphi=$$

$$=2a^2b\int_0^{2\pi}\frac{\sqrt{a^2\sin^2\varphi-b^2\sin^2\varphi}-a}{-a^2\cos^2\varphi-b^2\sin^2\varphi}\,d\varphi=2a^2b\left(\sqrt{a^2-b^2}\cdot 2\int_0^{\pi}\frac{\sin\varphi\,d\varphi}{-a^2\cos^2\varphi-b^2\sin^2\varphi}-a\cdot 2\int_0^{\pi}\frac{d\varphi}{-a^2\cos^2\varphi-b^2\sin^2\varphi}\right.$$

$$=2a^2b\left(-2\sqrt{a^2-b^2}\int_0^{\pi}\frac{d(\cos\varphi)}{b^2+(a^2-b^2)\cos^2\varphi}-2a\int_0^{\pi}\frac{d(\operatorname{ctg}\varphi)}{b^2+a^2\operatorname{ctg}^2\varphi}\right)=\left|\int\frac{dx}{a^2+bx^2}=\frac{1}{ab}\operatorname{arctg}\frac{bx}{a}+C\right|$$

$$=2a^2b\left(-2\sqrt{a^2-b^2}\cdot\frac{1}{b\sqrt{a^2-b^2}}\operatorname{arctg}\left(\frac{\sqrt{a^2-b^2}}{b}\cos\varphi\right)\Big|_0^{\pi}-2a\cdot\frac{1}{ab}\operatorname{arctg}\left(\frac{a}{b}\operatorname{ctg}\varphi\right)\Big|_0^{\pi}\right)=$$

$$=2a^2b\left(-\frac{2}{b}\left(\operatorname{arctg}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)-\operatorname{arctg}\left(\frac{\sqrt{a^2-b^2}}{b}\right)\right)-\frac{2}{b}\left(\operatorname{arctg}(-\infty)-\operatorname{arctg}(+\infty)\right)\right)=$$

$$=4a^2\operatorname{arctg}\frac{\sqrt{a^2-b^2}}{b}+4\pi a^2.$$

2. Обчислити площу поверхні:

1) $\begin{cases} x_1=(b+a\cos\psi)\cos\varphi \\ x_2=(b+a\cos\psi)\sin\varphi \\ x_3=a\sin\psi \end{cases}$ $\quad \varphi\in[\varphi_1,\varphi_2]$, $\psi\in[\psi_1,\psi_2]$, $(0<a\le b)$ $\quad$ Весь тор: $\varphi\in[0,2\pi]$, $\psi\in[0,2\pi]$

(ЧАСТИНА ТОРА)

$$A=\frac{\partial(u_2,u_3)}{\partial(\varphi,\psi)}=\begin{vmatrix}(b+a\cos\psi)\cos\varphi & -a\sin\psi\sin\varphi\\ 0 & a\cos\psi\end{vmatrix}=a\cos\psi\cos\varphi(b+a\cos\psi)$$

$$B=\frac{\partial(u_3,u_1)}{\partial(\varphi,\psi)}=\begin{vmatrix}0 & a\cos\psi\\ -(b+a\cos\psi)\sin\varphi & -a\sin\psi\cos\varphi\end{vmatrix}=a\cos\psi\sin\varphi(b+a\cos\psi)$$

$$C=\frac{\partial(u_1,u_2)}{\partial(\varphi,\psi)}=\begin{vmatrix}-(b+a\cos\psi)\sin\varphi & -a\sin\psi\cos\varphi\\ (b+a\cos\psi)\cos\varphi & -a\sin\psi\sin\varphi\end{vmatrix}=a(b+a\cos\psi)(\sin\psi\sin^2\varphi+\sin\psi\cos^2\varphi)$$
$$=a(b+a\cos\psi)\sin\psi$$

$$S=\int_S 1d\sigma=\int_T\sqrt{A^2+B^2+C^2}\,d\varphi d\psi=\int_{\varphi_1}^{\varphi_2}\Big(\int_{\psi_1}^{\psi_2}a(b+a\cos\psi)\underbrace{\sqrt{\cos^2\psi\cos^2\varphi+\cos^2\psi\sin^2\varphi+\sin^2\psi}}_{1}\,d\psi\Big)d\varphi$$

$\{T=[\varphi_1,\varphi_2]\times[\psi_1,\psi_2]\}$

$$=\int_{\varphi_1}^{\varphi_2}a(b\psi+a\sin\psi)\Big|_{\psi=\psi_1}^{\psi_2}d\varphi=\int_{\varphi_1}^{\varphi_2}a(b(\psi_2-\psi_1)+a(\sin\psi_2-\sin\psi_1))d\varphi=$$

$$=a(\varphi_2-\varphi_1)(b(\psi_2-\psi_1)+a(\sin\psi_2-\sin\psi_1)).$$

$$S_{\text{всього тора}}=a\cdot2\pi(b\cdot2\pi+0)=4\pi^2ab$$

2) $x_1=r\cos\varphi$, $x_2=r\sin\varphi$, $x_3=\varphi$; $0\le r\le1$, $0\le\varphi\le2\pi$

$$A=\frac{\partial(u_2,u_3)}{\partial(r,\varphi)}=\begin{vmatrix}\sin\varphi & r\cos\varphi\\ 0 & 1\end{vmatrix}=\sin\varphi \qquad B=\frac{\partial(u_3,u_1)}{\partial(r,\varphi)}=\begin{vmatrix}0 & 1\\ \cos\varphi & -r\sin\varphi\end{vmatrix}=-\cos\varphi$$

$$C=\frac{\partial(u_1,u_2)}{\partial(r,\varphi)}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$S=\int_S 1d\sigma=\int_T\sqrt{A^2+B^2+C^2}\,dr\,d\varphi=\int_0^1\Big(\int_0^{2\pi}\sqrt{\sin^2\varphi+\cos^2\varphi+r^2}\,d\varphi\Big)dr=$$

$$=\int_0^1 2\pi\sqrt{1+r^2}\,dr=2\pi\Big(\frac{r}{2}\sqrt{1+r^2}+\frac12\ln|r+\sqrt{1+r^2}|\Big)\Big|_0^1=2\pi\Big(\frac{\sqrt2}{2}+\frac12\ln(1+\sqrt2)\Big)$$