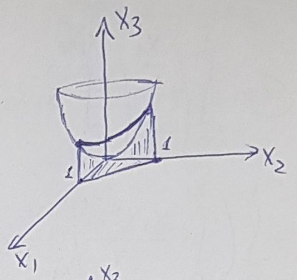
A10

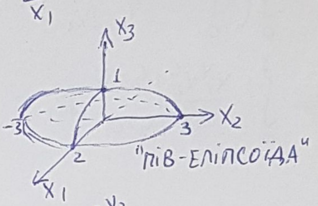
1. Зобразити тіло з заданим об'ємом:

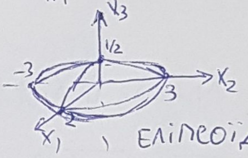
1) $\int_0^1\left(\int_0^{1-x_1}(x_1^2+x_2^2)dx_2\right)dx_1 = \int_0^1\left(\int_0^{1-x_1}\left(\int_0^{x_1^2+x_2^2}1\,dx_3\right)dx_2\right)dx_1$



2) $\int_A\sqrt{1-\frac{x_1^2}{4}-\frac{x_2^2}{9}}\,dx_1dx_2 = \quad A=\left\{(x_1,x_2)\,\middle|\,\frac{x_1^2}{4}+\frac{x_2^2}{9}\le 1\right\}$

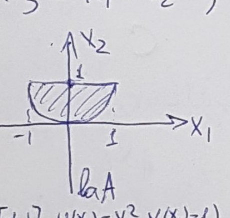
$= \int_A\left(\int_0^{\sqrt{1-\frac{x_1^2}{4}-\frac{x_2^2}{9}}}1\,dx_3\right)dx_2\Big)dx_1 = \int_A\left(\int_{-\frac{1}{2}\sqrt{1-\frac{x_1^2}{4}-\frac{x_2^2}{9}}}^{\frac{1}{2}\sqrt{1-\frac{x_1^2}{4}-\frac{x_2^2}{9}}}1\,dx_3\right)dx_1dx_2$



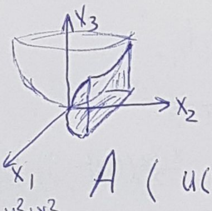


ЕЛІПСОЇД

2. $x_3=x_1^2+x_2^2$, $x_2=x_1^2$, $x_2=1$, $x_3=0$



ваА



(основа $[-1,1]$, $u(x_1)=x_1^2$, $v(x_1)=1$)

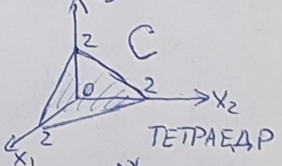
A ($u(x_1,x_2)=0$, $v(x_1,x_2)=x_1^2+x_2^2$)

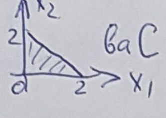
$V=\int_A 1\,dx_1dx_2dx_3 = \int_{-1}^1\left(\int_{x_1^2}^1\left(\int_0^{x_1^2+x_2^2}1\,dx_3\right)dx_2\right)dx_1 = \int_{-1}^1\left(\int_{x_1^2}^1(x_1^2+x_2^2)dx_2\right)dx_1 =$

$=\int_{-1}^1\left(x_1^2x_2+\frac{x_2^3}{3}\right)\Bigg|_{x_2=x_1^2}^{x_2=1}dx_1 = \int_{-1}^1\left(x_1^2+\frac{1}{3}-x_1^4-\frac{x_1^6}{3}\right)dx_1 = \left(\frac{x_1^3}{3}+\frac{x_1}{3}-\frac{x_1^5}{5}-\frac{x_1^7}{21}\right)\Bigg|_{-1}^1 = \frac{2}{3}+\frac{2}{3}-\frac{2}{5}-\frac{2}{21} =$

$=\frac{70-42-10}{105}=\frac{18}{105}=\frac{6}{35}.$

3. 1) $\int_C\frac{dx_1dx_2dx_3}{(2+x_1+x_2+x_3)^3} = \quad C:\ x_1+x_2+x_3=2,\ x_i=0, i=1;2;3$



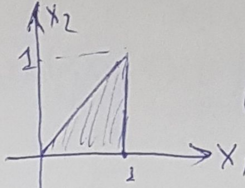


$= \int_0^2\left(\int_0^{2-x_1}\left(\int_0^{2-x_1-x_2}\frac{dx_3}{(2+x_1+x_2+x_3)^3}\right)dx_2\right)dx_1 = \int_0^2\left(\int_0^{2-x_1}\frac{-1}{2(2+x_1+x_2+x_3)^2}\Bigg|_{x_3=0}^{x_3=2-x_1-x_2}dx_2\right)dx_1 =$

$=\int_0^2\left(\int_0^{2-x_1}\left(\frac{1}{2(2+x_1+x_2)^2}-\frac{1}{32}\right)dx_2\right)dx_1 = \int_0^2\left(-\frac{1}{2(2+x_1+x_2)}-\frac{x_2}{32}\right)\Bigg|_{x_2=0}^{x_2=2-x_1}dx_1 =$

$=\int_0^2\left(\frac{1}{2(2+x_1)}-\frac{1}{8}-\frac{2-x_1}{32}\right)dx_1 = \left(\frac{1}{2}\ln|2+x_1|-\frac{x_1}{8}-\frac{x_1}{16}+\frac{x_1^2}{64}\right)\Bigg|_0^2 = \frac{1}{2}\ln 4-\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{2}\ln 2 = \frac{1}{2}\ln 2-\frac{5}{16}.$

3. 2) $\int\limits_C x_1x_2^2x_3^3\,dx_1dx_2dx_3 \ominus$ C: $x_3=x_1x_2$, $x_2=x_1$, $x_1=1$, $x_3=0$, $x_1\geq x_2$, $x_i\geq 0$, $i=1,2,3$

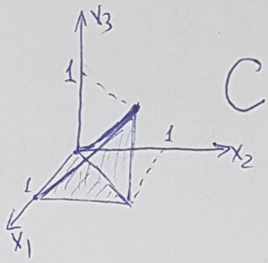


Ba C

$$\ominus \int_0^1\left(\int_0^{x_1}\left(\int_0^{x_1x_2} x_1x_2^2x_3^3\,dx_3\right)dx_2\right)dx_1 =$$

$$=\int_0^1\left(\int_0^{x_1}\frac{x_1x_2^2x_3^4}{4}\Big|_{x_3=0}^{x_1x_2}dx_2\right)dx_1=\int_0^1\left(\int_0^{x_1}\frac{x_1^4x_2^5}{4}dx_2\right)dx_1=$$

$$=\int_0^1\frac{x_1^4x_2^6}{24}\Big|_{x_2=0}^{x_2=x_1}dx_1=\int_0^1\frac{x_1^{10}}{24}dx_1=\frac{x_1^{11}}{254}\Big|_0^1=\frac{1}{254}$$



C

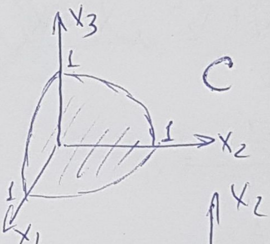
3) $\int\limits_C x_1x_2x_3\,dx_1dx_2dx_3 \ominus$ C: $x_1^2+x_2^2+x_3^2=1$, $x_i\geq 0$, $i=1,2,3$

$$\ominus\int_0^1\left(\int_0^{\sqrt{1-x_1^2}}\left(\int_0^{\sqrt{1-x_1^2-x_2^2}}x_1x_2x_3\,dx_3\right)dx_2\right)dx_1=$$

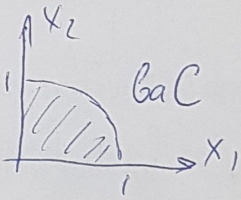
$$=\int_0^1\left(\int_0^{\sqrt{1-x_1^2}}x_1x_2\cdot\frac{x_3^2}{2}\Big|_{x_3=0}^{\sqrt{1-x_1^2-x_2^2}}dx_2\right)dx_1=\int_0^1\left(\int_0^{\sqrt{1-x_1^2}}\frac{x_1x_2}{2}(1-x_1^2-x_2^2)dx_2\right)dx_1=$$

$$=\int_0^1\left(\frac{x_1x_2^2}{4}(1-x_1^2)-\frac{x_1x_2^4}{8}\right)\Big|_{x_2=0}^{\sqrt{1-x_1^2}}dx_1=\int_0^1\left(\frac{x_1(1-x_1^2)^2}{4}-\frac{x_1(1-x_1^2)^2}{8}\right)dx_1=$$

$$=\frac18\cdot\left(-\frac12\right)\int_0^1(1-x_1^2)^2\,d(1-x_1^2)=-\frac{1}{16}\cdot\frac{(1-x_1^2)^3}{3}\Big|_0^1=\frac{1}{48}.$$

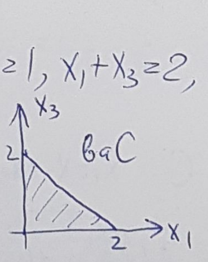


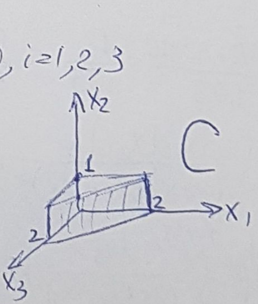
C



Ba C

4) $\int\limits_C x_1\,dx_1dx_2dx_3 \ominus$ C: $x_2=1$, $x_1+x_3=2$, $x_i=0$, $i=1,2,3$





C

$$\ominus\int_0^2\left(\int_0^{2-x_1}\left(\int_0^2 x_1\,dx_2\right)dx_3\right)dx_1=$$

$$=\int_0^2\left(\int_0^{2-x_1}x_1x_2\Big|_{x_2=0}^{2}dx_3\right)dx_1=\int_0^2\left(\int_0^{2-x_1}2x_1\,dx_3\right)dx_1=$$

$$=\int_0^2 2x_1x_3\Big|_{x_3=0}^{2-x_1}dx_1=\int_0^2 2x_1(2-x_1)\,dx_1=\left(2x_1^2-\frac23x_1^3\right)\Big|_0^2=8-\frac{16}{3}=\frac83$$

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3. 5) $\int\limits_C (1+3x_1x_2x_3 + x_1^2x_2^2x_3^2)e^{x_1x_2x_3}\,dx_1dx_2dx_3 =$ $\quad C=[0,1]\times[0,1]\times[0,1]$

$= \int_0^1\left(\int_0^1\left(\int_0^1 (1+3x_1x_2x_3+x_1^2x_2^2x_3^2)e^{x_1x_2x_3}dx_3\right)dx_2\right)dx_1 = \left|\begin{matrix} t=x_1x_2x_3 \\ dt = x_1x_2dx_3 \\ t_1=0 \quad t_2=x_1x_2\end{matrix}\right| =$

$=\int_0^1\left(\int_0^1\left(\int_0^{x_1x_2}(1+3t+t^2)e^t\frac{dt}{x_1x_2}\right)dx_2\right)dx_1 = \int_0^1\left(\int_0^1 \frac{1}{x_1x_2}\cdot(t+t^2)e^t\Big|_{t=0}^{x_1x_2}dx_2\right)dx_1 =$

$\left\{ \int(1+3t+t^2)e^tdt = (1+3t+t^2)e^t - \int(3+2t)e^tdt = (1+3t+t^2)e^t-(3+2t)e^t+\int 2e^tdt = (t+t^2)e^t+C\right\}$

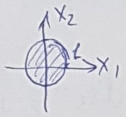
$= \int_0^1\left(\int_0^1(1+x_1x_2)e^{x_1x_2}dx_2\right)dx_1 = \left|\begin{matrix}s=x_1x_2\\ ds=x_1dx_2\\ s_1=0\quad s_2=x_1\end{matrix}\right| = \int_0^1\left(\int_0^{x_1}(1+s)e^s\frac{ds}{x_1}\right)dx_1 = \int_0^1\frac{1}{x_1}\cdot se^s\Big|_{s=0}^{x_1}dx_1 =$

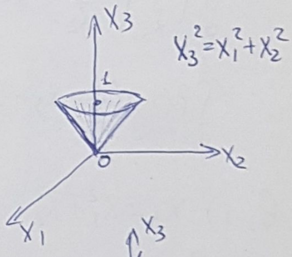
$=\int_0^1 e^{x_1}dx_1 = e^{x_1}\Big|_0^1 = e-1$

$\left\{ \cancel{\int se^s ds = se^s - \int e^s ds =} \right.$
$\left. \int(s+1)e^sds = (s+1)e^s - \int e^sds = se^s+C \right\}$

4. Розставити межі інтегрування різними способами:

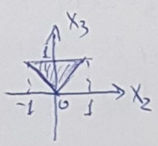
1) $\int_{-1}^1\left(\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\left(\int_{\sqrt{x_1^2+x_2^2}}^1 f(x_1,x_2,x_3)dx_3\right)dx_2\right)dx_1 =$

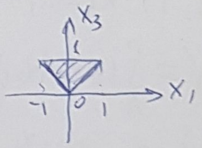




$x_3^2=x_1^2+x_2^2$

$=\int_{-1}^1\left(\int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}}\left(\int_{\sqrt{x_1^2+x_2^2}}^1 f(x_1,x_2,x_3)dx_3\right)dx_1\right)dx_2 =$





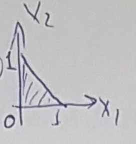
$=\int_{-1}^1\left(\int_{|x_2|}^1\left(\int_{-\sqrt{x_3^2-x_2^2}}^{\sqrt{x_3^2-x_2^2}} f(x_1,x_2,x_3)dx_1\right)dx_3\right)dx_2 =$

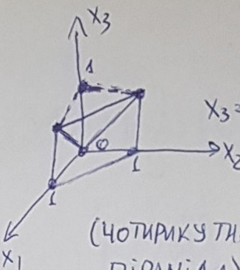
$=\int_0^1\left(\int_{-x_3}^{x_3}\left(\int_{-\sqrt{x_3^2-x_2^2}}^{\sqrt{x_3^2-x_2^2}} f(x_1,x_2,x_3)dx_1\right)dx_2\right)dx_3 =$

$=\int_{-1}^1\left(\int_{|x_1|}^1\left(\int_{-\sqrt{x_3^2-x_1^2}}^{\sqrt{x_3^2-x_1^2}} f(x_1,x_2,x_3)dx_2\right)dx_3\right)dx_1 =$

$=\int_0^1\left(\int_{-x_3}^{x_3}\left(\int_{-\sqrt{x_3^2-x_1^2}}^{\sqrt{x_3^2-x_1^2}} f(x_1,x_2,x_3)dx_2\right)dx_1\right)dx_3$

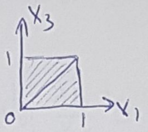
4. 2) $\int_0^1 \left(\int_0^{1-x_1} \left(\int_0^{x_1+x_2} f(x_1,x_2,x_3)\,dx_3 \right) dx_2 \right) dx_1 =$

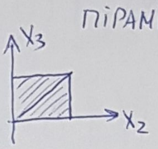




$= \int_0^1 \left(\int_0^{x_1} \left(\int_0^{1-x_1} f(x_1,x_2,x_3)\,dx_2 \right) dx_3 \right) dx_1 + \int_0^1 \left(\int_{x_1}^1 \left(\int_{x_3-x_1}^{1-x_1} f(x_1,x_2,x_3)\,dx_2 \right) dx_3 \right) dx_1 =$

(ЧОТИРИКУТНА ПІРАМІДА)



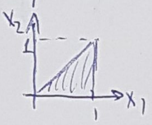


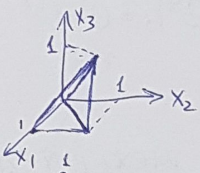
$= \int_0^1 \left(\int_0^{x_2} \left(\int_0^{1-x_2} f(x_1,x_2,x_3)\,dx_1 \right) dx_3 \right) dx_2 + \int_0^1 \left(\int_{x_2}^1 \left(\int_{x_3-x_2}^{1-x_2} f(x_1,x_2,x_3)\,dx_1 \right) dx_3 \right) dx_2$

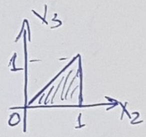
Інші 3 інтеграла аналогічні

5. Замінити інтеграл однократним ($f \in C([0,1])$):

$\int_0^1 \left(\int_0^{x_1} \left(\int_0^{x_2} f(x_3)\,dx_3 \right) dx_2 \right) dx_1 =$



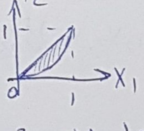


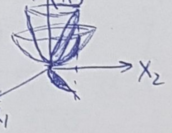


$= \int_0^1 \left(\int_{x_3}^1 \left(\int_{x_2}^1 f(x_3)\,dx_1 \right) dx_2 \right) dx_3 = \int_0^1 \left(\int_{x_3}^1 f(x_3)(1-x_2)\,dx_2 \right) dx_3 = \int_0^1 f(x_3) \left(x_2 - \frac{x_2^2}{2} \right) \Big|_{x_2=x_3}^1 dx_3 =$

$= \int_0^1 f(x_3) \left(\frac{1}{2} - x_3 + \frac{x_3^2}{2} \right) dx_3 = \frac{1}{2} \int_0^1 f(x_3)(x_3-1)^2\,dx_3.$

6. 1) $x_3 = x_1^2 + x_2^2,\ x_3 = 2(x_1^2 + x_2^2),\ x_1 = x_2,\ x_2 = x_1^2$

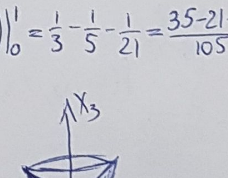




$V = \int_A 1\,dx_1 dx_2 dx_3 = \int_0^1 \left(\int_{x_1^2}^{x_1} \left(\int_{x_1^2+x_2^2}^{2(x_1^2+x_2^2)} dx_3 \right) dx_2 \right) dx_1 = \int_0^1 \left(\int_{x_1^2}^{x_1} (x_1^2 + x_2^2)\,dx_2 \right) dx_1 =$

$= \int_0^1 \left(x_1^2 x_2 + \frac{x_2^3}{3} \right) \Big|_{x_2=x_1^2}^{x_1} dx_1 = \int_0^1 \left(x_1^3 + \frac{x_1^3}{3} - x_1^4 - \frac{x_1^6}{3} \right) dx_1 = \left(\frac{x_1^4}{3} - \frac{x_1^5}{5} - \frac{x_1^7}{21} \right) \Big|_0^1 = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{35-21-5}{105} = \frac{3}{35}$

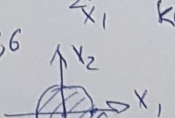
2) $6x_3 = x_1^2 + x_2^2,\ x_3 = \sqrt{x_1^2 + x_2^2}$



$V = \int_{-6}^{6} \left(\int_{-\sqrt{36-x_1^2}}^{\sqrt{36-x_1^2}} \left(\int_{\frac{x_1^2+x_2^2}{6}}^{\sqrt{x_1^2+x_2^2}} 1\,dx_3 \right) dx_2 \right) dx_1 = \left\{ \begin{array}{l} x_3 = \sqrt{x_1^2+x_2^2} \\ 6x_3 = x_1^2 + x_2^2 \end{array} \right. \Rightarrow$

$6x_3 = x_3^2 \Rightarrow x_3 = 0; 6$

конус і параболоїд,



$= \int_{-6}^{6} \left(\int_{-\sqrt{36-x_1^2}}^{\sqrt{36-x_1^2}} \left(\sqrt{x_1^2+x_2^2} - \frac{x_1^2+x_2^2}{6} \right) dx_2 \right) dx_1 = \ldots$

$= \int_0^6 \left(\int_0^{2\pi} \left(\int_{r^2/6}^{r} r\,dh \right) d\varphi \right) dr = \int_0^6 \left(\int_0^{2\pi} \left(r^2 - \frac{r^3}{6} \right) d\varphi \right) dr = 2\pi \int_0^6 \left(r^2 - \frac{r^3}{6} \right) dr = 2\pi \left(\frac{r^3}{3} - \frac{r^4}{24} \right) \Big|_0^6 = 2\pi(72-54) = 36\pi$