Типові задачі-2

1. а) $$\int_1^{+\infty} \frac{\sin(x-1)}{2x-2}\,dx = \left| \begin{matrix} x-1=t & dx=dt \\ t_1=0 & t_2=+\infty \end{matrix} \right| = \int_0^{+\infty} \frac{\sin t}{2t}\,dt = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

б) $$\int_{-\infty}^{+\infty} e^{-x^2+x}\,dx = \int_{-\infty} e^{-(x-\frac{1}{2})^2+\frac{1}{4}}\,dx = \left| \begin{matrix} x-\frac{1}{2}=t \\ dx=dt \end{matrix} \right| = \int_{-\infty}^{+\infty} e^{-t^2+\frac{1}{4}}\,dt = e^{\frac{1}{4}}\cdot 2\int_0^{+\infty} e^{-t^2}\,dt =$$
$$= \sqrt[4]{e}\cdot 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt[4]{e}\cdot\sqrt{\pi}.$$

в) $$\int_0^{+\infty} \frac{\cos 3x-\cos 4x}{x}\,dx = \left| \begin{matrix} \text{Інтеграл Фрулані} \\ f(x)=\cos x - \text{неперервна} \\ \int_1^{+\infty}\frac{\cos x}{x}\,dx - \text{збіж. за Діріхле} \end{matrix} \right| = -\cos 0\cdot\ln\frac{3}{4} = \ln\frac{4}{3}$$

г) $$\int_0^{+\infty} \frac{e^{-x^2}-e^{-2x^2}}{x^2}\,dx = \left| \begin{matrix} u=e^{-x^2}-e^{-2x^2} & dv=\frac{dx}{x^2} \\ du=(-2xe^{-x^2}+4xe^{-2x^2})dx & v=-\frac{1}{x} \end{matrix} \right| = \left.\frac{e^{-2x^2}-e^{-x^2}}{x}\right|_0^{+\infty} + \int_0^{+\infty}(-2e^{-x^2}+4e^{-2x^2})\,dx =$$

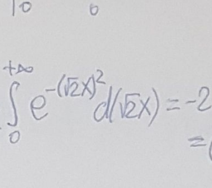
$$= \left\{ \begin{matrix} \frac{e^{-2x^2}-e^{-x^2}}{x} \xrightarrow[x\to\infty]{} \frac{0-0}{\infty} = 0 \\ \lim\limits_{x\to 0}\frac{e^{-2x^2}-e^{-x^2}}{x} = \left|\frac{0}{0}\right| = \lim\limits_{x\to 0}\frac{-4xe^{-2x^2}+2xe^{-x^2}}{1} = 0 \end{matrix} \right\} = -2\int_0^{+\infty} e^{-x^2}\,dx + \frac{4}{\sqrt{2}}\int_0^{+\infty} e^{-(\sqrt{2}x)^2}\,d(\sqrt{2}x) = -2\cdot\frac{\sqrt{\pi}}{2} + 2\sqrt{2}\cdot\frac{\sqrt{\pi}}{2}$$
$$= (\sqrt{2}-1)\sqrt{\pi}.$$

Правило Лопіталя

2. а) $$\int_0^{+\infty} \frac{x^7}{1+x^{13}}\,dx = \left| \begin{matrix} x^{13}=t & x=t^{1/13} \\ dx=\frac{1}{13}t^{-12/13} & t_1=0 \quad t_2=+\infty \end{matrix} \right| = \int_0^{+\infty} \frac{t^{7/13}}{1+t}\cdot\frac{1}{13}t^{-12/13}\,dt = \frac{1}{13}\int_0^{+\infty}\frac{t^{-5/13}}{1+t}\,dt =$$
$$= \frac{1}{13}B\left(\frac{8}{13},\frac{5}{13}\right) = \frac{1}{13}\,\frac{\Gamma(\frac{8}{13})\Gamma(\frac{5}{13})}{\Gamma(1)} = \frac{1}{13}\cdot\frac{\pi}{\sin\frac{8\pi}{13}}.$$

б) $$\int_0^{+\infty} t^{\alpha+2}e^{-2t}\ln t\,dt = \left(\int_0^{+\infty} t^{\alpha+2}e^{-2t}\,dt\right)'_\alpha = \left| \begin{matrix} 2t=s \\ dt=\frac{ds}{2} \end{matrix} \right| = \left(\int_0^{+\infty}\left(\frac{s}{2}\right)^{\alpha+2}e^{-s}\,\frac{ds}{2}\right)'_\alpha =$$
$$= \left(2^{-\alpha-3}\cdot\int_0^{+\infty} s^{\alpha+2}e^{-s}\,ds\right)'_\alpha = \left(2^{-\alpha-3}\cdot\Gamma(\alpha+3)\right)'_\alpha = 2^{-\alpha-3}\ln 2\cdot\Gamma(\alpha+3) + 2^{-\alpha-3}\Gamma'(\alpha+3),\ \alpha>-3$$

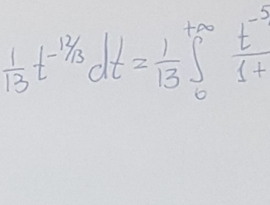
3. а) $$\int_0^1\left(\int_1^{x_1+1} f(x_1,x_2)\,dx_2\right)dx_1 = \int_1^2\left(\int_{x_2-1}^1 f(x_1,x_2)\,dx_1\right)dx_2$$



б) $$\int_0^1\left(\int_{x_1^2}^{2x_1^2} f(x_1,x_2)\,dx_2\right)dx_1 = \int_0^1\left(\int_{\sqrt{\frac{x_2}{2}}}^{\sqrt{x_2}} f(x_1,x_2)\,dx_1\right)dx_2 +$$
$$+\int_1^2\left(\int_{\sqrt{\frac{x_2}{2}}}^1 f(x_1,x_2)\,dx_1\right)dx_2$$

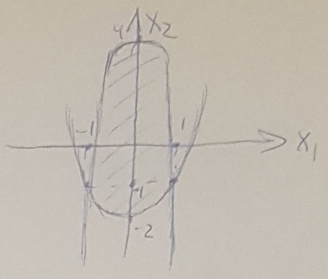
$x_2=x_1^2 \Rightarrow x_1=\sqrt{x_2}$

$x_2=2x_1^2 \Rightarrow x_1=\sqrt{\frac{x_2}{2}}$



4. A: $x_2 = x_1^2 - 2,\ x_2 = 4 - 5x_1^4$

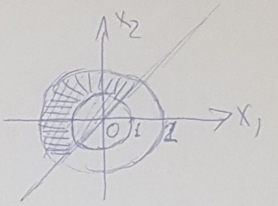
$$\begin{cases} x_2 = x_1^2 - 2 \\ x_2 = 4 - 5x_1^4 \end{cases} \qquad \begin{aligned} & x_1^2 - 2 = 4 - 5x_1^4 \\ & 5x_1^4 + x_1^2 - 6 = 0 \\ & x_1^2 = 1;\ -\tfrac{6}{5} \\ & x_1 = \pm 1 \quad x_2 = -1 \end{aligned}$$



$$\iint_A x_1 x_2\, dx_1 dx_2 = \int_{-1}^{1}\left(\int_{x_1^2-2}^{4-5x_1^4} x_1 x_2\, dx_2\right) dx_1 =$$

$$= \int_{-1}^{1} x_1 \cdot \frac{x_2^2}{2}\Big|_{x_2 = x_1^2-2}^{x_2 = 4-5x_1^4} dx_1 = \int_{-1}^{1} \frac{x_1}{2}\left((4-5x_1^4)^2 - (x_1^2-2)^2\right) dx_1 = 0$$ (непарна ф-ція, симетричний проміжок)

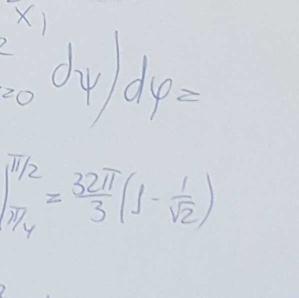
5. A: $1 \le x_1^2 + x_2^2 \le 4,\ x_2 \ge x_1$; $p = x_2^2$



$$m = \iint_A p\, dx_1 dx_2 = \left| \begin{aligned} x_1 &= r\cos\varphi \\ x_2 &= r\sin\varphi \end{aligned} \right| = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(\int_1^2 (r\sin\varphi)^2 \cdot r\, dr\right) d\varphi =$$

$$= \int_{\pi/4}^{5\pi/4} \sin^2\varphi \cdot \frac{r^4}{4}\Big|_1^2 d\varphi = \frac{15}{4}\int_{\pi/4}^{5\pi/4} \frac{1-\cos 2\varphi}{2} d\varphi = \frac{15}{4}\left(\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right)\Big|_{\pi/4}^{5\pi/4} = \frac{15}{4}\left(\frac{5\pi}{8} - \frac{1}{4} - \frac{\pi}{8} + \frac{1}{4}\right) = \frac{15\pi}{8}$$

6. $x_1^2 + x_2^2 + x_3^2 = 4,\ x_3^2 \ge x_1^2 + x_2^2.$



$$V = \iiint_A 1\, dx_1 dx_2 dx_3 = \left\{ \begin{matrix} \text{сферичні} \\ \text{координати} \end{matrix} \right\} =$$

$$= 2\int_0^{2\pi}\left(\int_{\pi/4}^{\pi/2}\left(\int_0^2 r^2 \cos\psi\, dr\right) d\psi\right) d\varphi = 2\int_0^{2\pi}\left(\int_{\pi/4}^{\pi/2} \frac{r^3}{3}\cos\psi\Big|_{r=0}^{2} d\psi\right) d\varphi =$$

$$= 2\int_0^{2\pi}\left(\int_{\pi/4}^{\pi/2} \frac{8}{3}\cos\psi\, d\psi\right) d\varphi = 4\pi \int_{\pi/4}^{\pi/2} \frac{8}{3}\cos\psi\, d\psi = \frac{32\pi}{3}\sin\psi\Big|_{\pi/4}^{\pi/2} = \frac{32\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

7. $\vec{F}(x_1, x_2) = (x_1^2, x_2^2)$ Межа півкола $x_1^2 + x_2^2 = 1$ при $x_2 \ge 0$ (проти год. стр)

$$\begin{cases} x_1 = \cos t \\ x_2 = \sin t \end{cases},\ t \in [0, \pi] \qquad A = \int_\Gamma F_1 dx_1 + F_2 dx_2 = \int_\Gamma x_1^2 dx_1 + x_2^2 dx_2 =$$

$$= \int_0^\pi \left(\cos^2 t \cdot (-\sin t) + \sin^2 t \cdot \cos t\right) dt = \int_0^\pi \cos^2 t\, d(\cos t) + \int_0^\pi \sin^2 t\, d(\sin t) = \left(\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3}\right)\Big|_0^\pi = -\frac{2}{3}$$

8. $\int\limits_{\Gamma} x_1^2 dx_2 - x_2 dx_1 \ominus \quad \Gamma = \{(t^2, t^3) \mid t \in [0,2]\}$, t - зростає

$$\ominus \int\limits_0^2 (t^4 \cdot 3t^2 - t^3 \cdot 2t)\,dt = \int\limits_0^2 (3t^6 - 2t^4)\,dt = \frac{3}{7}t^7 - \frac{2}{5}t^5\Big|_0^2 = \frac{3}{7}\cdot 128 - \frac{2}{5}\cdot 32 = \frac{15\cdot 128 - 7\cdot 64}{35} = \frac{1472}{35}$$

9. $x_2 = \sqrt{x_1 + 1}$, $x_1 \in [0,1] \quad \rho(x_1, x_2) = x_2^2$ Нехай x_1 - параметр

$$m = \int\limits_{\Gamma} \rho\, dl = \int\limits_0^1 (\sqrt{x_1+1})^2 \cdot \sqrt{1^2 + \left(\frac{1}{2\sqrt{x_1+1}}\right)^2}\,dx_1 = \int\limits_0^1 (x_1+1)\sqrt{\frac{4x_1+5}{4x_1+4}}\,dx_1 =$$

$$= \frac{1}{2}\int\limits_0^1 \sqrt{4x_1^2 + 9x_1 + 5}\,dx_1 = \frac{2}{2}\int\limits_0^1 \sqrt{4(x_1^2 + \frac{9}{4}x_1 + \frac{5}{4})}\,dx_1 = \int\limits_0^1 \sqrt{(x_1 + \frac{9}{8})^2 - \frac{1}{64}}\,dx_1 =$$

$$= \frac{x_1 + \frac{9}{8}}{2}\sqrt{(x_1+\frac{9}{8})^2 - \frac{1}{64}} - \frac{\frac{1}{64}}{2}\ln\left((x_1 + \frac{9}{8}) + \sqrt{(x_1+\frac{9}{8})^2 - \frac{1}{64}}\right)\Big|_0^1 = \frac{17}{16}\sqrt{18} - \frac{1}{128}\ln\left(\frac{17}{8} + \sqrt{18}\right) - \frac{9}{16}\sqrt{5} + \frac{1}{128}\ln\left(\frac{9}{8} + \sqrt{5}\right)$$

10. $\int\limits_{\Gamma} x_1 x_2\, dl \ominus \quad \Gamma = \{(2t+1, 3t+2) \mid t \in [0,2]\}$

$$\ominus \int\limits_0^2 (2t+1)(3t+2)\sqrt{2^2 + 3^2}\,dt = \sqrt{13}\cdot\int\limits_0^2 (6t^2 + 7t + 2)\,dt = \sqrt{13}\left(2t^3 + \frac{7t^2}{2} + 2t\right)\Big|_0^2 = \sqrt{13}(16 + 14 + 4) = 34\sqrt{13}$$

11. $S = \{(3t_1, \cos t_2, \sin t_2, 4t_2) \mid 0 \le t_1 \le 1,\ 0 \le t_2 \le 2\pi\}$

$$S = \int\limits_S 1\, d\sigma \ominus$$

$$A = \frac{\partial(u_2, u_3)}{\partial(t_1, t_2)} = \begin{vmatrix} 0 & \cos t_2 \\ 0 & 4 \end{vmatrix} = 0$$

$$B = \frac{\partial(u_3, u_1)}{\partial(t_1, t_2)} = \begin{vmatrix} 0 & 4 \\ 3\cos t_2 & -3t_1 \sin t_2 \end{vmatrix} = -\cancel{3}12\cos t_2$$

$$C = \frac{\partial(u_1, u_2)}{\partial(t_1, t_2)} = \begin{vmatrix} 3\cos t_2 & -3t_1 \sin t_2 \\ 0 & \cos t_2 \end{vmatrix} = 3\cos^2 t_2$$

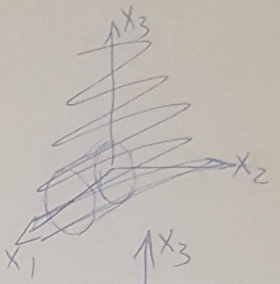
$$\ominus \int\limits_0^1 \left(\int\limits_0^{2\pi} \sqrt{0^2 + (-12\cos t_2)^2 + (3\cos^2 t_2)^2}\,dt_2\right)dt_1 = \int\limits_0^{2\pi} \sqrt{144\cos^2 t_2 + 9\cos^4 t_2}\,dt_2 =$$

$$\left\{\begin{matrix} \cos^2 - \text{парна}, \\ \pi - \text{періодична} \end{matrix}\right\} = 4\int\limits_0^{\pi/2} 3\cos t_2 \cdot \sqrt{16\cos^2 t_2 + 1}\,dt_2 = 12\int\limits_0^{\pi/2} \sqrt{17 - 16\sin^2 t_2}\,d(\sin t_2) = 48\int\limits_0^{\pi/2} \sqrt{\frac{17}{16} - \sin^2 t_2}\,d(\sin t_2) =$$

$$= 48\left(\frac{\frac{17}{16}}{2}\arcsin\frac{\sin t_2}{\sqrt{\frac{17}{16}}} + \frac{\sin t_2}{4}\sqrt{\frac{17}{16} - \sin^2 t_2}\right)\Big|_0^{\pi/2} = \frac{51}{2}\left(\arcsin\frac{4}{\sqrt{17}} + \frac{1}{16}\right.$$

12. $S: x_1^2+x_3^2=1,\ x_2\in[0,1]$

(зовн. бік)



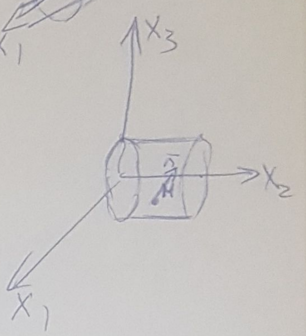
$$\begin{cases} x_1=\cos t \\ x_3=\sin t \\ x_2=s \end{cases}$$

$t\in[0,2\pi]$
$s\in[0,1]$

$$A=\frac{\partial(u_2,u_3)}{\partial(t,s)}=\begin{vmatrix} 0 & 1 \\ \cos t & 0 \end{vmatrix}=-\cos t$$

$$B=\frac{\partial(u_3,u_1)}{\partial(t,s)}=\begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix}=0$$

$$C=\frac{\partial(u_1,u_2)}{\partial(t,s)}=\begin{vmatrix} -\sin t & 0 \\ 0 & 1 \end{vmatrix}=-\sin t$$



При $t=0, s=\frac{1}{2}$: $M(1,\frac{1}{2},0)$ $\bar{n}=(-1,0,0)$

З кінця $\bar{n}$ видно внутр. бік.

Орієнтація від'ємна.

$$\int_S x_1\,dx_2\wedge dx_3+x_2\,dx_3\wedge dx_1+x_3\,dx_1\wedge dx_2=-\int_0^{2\pi}\left(\int_0^1\left(\cos t\cdot(-\cos t)+s\cdot 0+\sin t\cdot(-\sin t)\right)ds\right)dt$$

$$=-\int_0^{2\pi}\left(\int_0^1 -1\,ds\right)dt=\int_0^{2\pi}dt=2\pi.$$