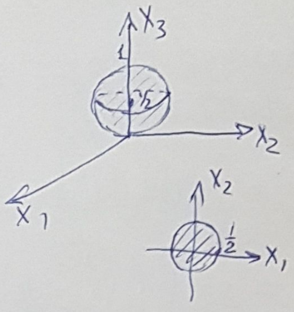
A11

$$1.\quad 1)\ \int_C \sqrt{x_1^2+x_2^2+x_3^2}\,dx_1\,dx_2\,dx_3\ ,\quad C:\ x_1^2+x_2^2+x_3^2=x_3$$



Сфер. координати:

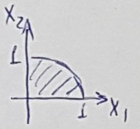
$$x_1=r\cos\varphi\cos\psi \qquad \varphi\in[0,2\pi]$$
$$x_2=r\sin\varphi\cos\psi \qquad \psi\in\left[0,\tfrac{\pi}{2}\right]$$
$$x_3=r\sin\psi \qquad J=r^2\cos\psi$$

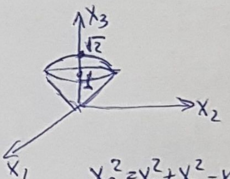
$\underline{x_1^2+x_2^2+x_3^2=r^2} \Rightarrow\quad r^2=r\sin\psi,\ r=\sin\psi$ – рівняння сфери $\Rightarrow r\in[0,\sin\psi]$

$$I=\int_0^{2\pi}\left(\int_0^{\pi/2}\left(\int_0^{\sin\psi} r\cdot r^2\cos\psi\,dr\right)d\psi\right)d\varphi=\int_0^{2\pi}\left(\int_0^{\pi/2}\frac{r^4}{4}\cos\psi\Big|_{r=0}^{\sin\psi}d\psi\right)d\varphi=\int_0^{2\pi}\left(\int_0^{\pi/2}\frac{\sin^4\psi\cos\psi}{4}d\psi\right)d\varphi=$$

$$=\frac{1}{4}\int_0^{2\pi}\left(\int_0^{\pi/2}\sin^4\psi\,d\sin\psi\right)d\varphi=\frac{1}{4}\int_0^{2\pi}\frac{\sin^5\psi}{5}\Big|_0^{\pi/2}d\varphi=\frac{1}{4}\int_0^{2\pi}\frac{1}{5}d\varphi=\frac{1}{4}\cdot\frac{1}{5}\cdot 2\pi=\frac{\pi}{10}$$

$$2)\ \int_0^1\left(\int_0^{\sqrt{1-x_1^2}}\left(\int_{\sqrt{x_1^2+x_2^2}}^{\sqrt{2-x_1^2-x_2^2}} x_3^2\,dx_3\right)dx_2\right)dx_1 \ominus$$





$x_3^2=x_1^2+x_2^2$ – конус

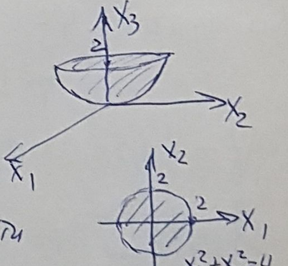
$x_1^2+x_2^2+x_3^2=2$ – сфера

$$\varphi\in\left[0,\tfrac{\pi}{2}\right],\ \psi\in\left[\tfrac{\pi}{4},\tfrac{\pi}{2}\right],\ r\in[0,\sqrt{2}]$$

$$\ominus\int_0^{\pi/2}\left(\int_{\pi/4}^{\pi/2}\left(\int_0^{\sqrt{2}} r^2\sin^2\psi\cdot r^2\cos\psi\,dr\right)d\psi\right)d\varphi=\int_0^{\pi/2}\left(\int_{\pi/4}^{\pi/2}\frac{r^5}{5}\sin^2\psi\cos\psi\Big|_{r=0}^{\sqrt{2}}d\psi\right)d\varphi=$$

$$=\frac{4\sqrt{2}}{5}\int_0^{\pi/2}\left(\int_{\pi/4}^{\pi/2}\sin^2\psi\cos\psi\,d\psi\right)d\varphi=\frac{4\sqrt{2}}{5}\int_0^{\pi/2}\frac{\sin^3\psi}{3}\Big|_{\psi=\pi/4}^{\pi/2}d\varphi=\frac{4\sqrt{2}}{5}\int_0^{\pi/2}\frac{1-\frac{1}{2\sqrt{2}}}{3}d\varphi=\frac{4\sqrt{2}-2}{15}\cdot\frac{\pi}{2}=\frac{2\sqrt{2}-1}{15}\pi$$

$$2.\ \int_C (x_1^2+x_2^2)\,dx_1\,dx_2\,dx_3 \ominus,\quad C:\ x_1^2+x_2^2=2x_3,\ x_3=2$$



$$\varphi\in[0,2\pi],\ r\in[0,2]\quad h\in\left[\tfrac{r^2}{2},2\right]$$

$\underline{x_1^2+x_2^2=r^2}\qquad J=r$

$$x_1=r\cos\varphi$$
$$x_2=r\sin\varphi$$
$$x_3=h$$

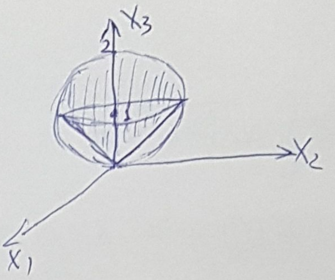
~~Сферичні~~

Циліндричні координати

$$\ominus\int_0^{2\pi}\left(\int_0^2\left(\int_{r^2/2}^2 r^2\cdot r\,dh\right)dr\right)d\varphi=2\pi\int_0^2\left(\int_{r^2/2}^2 r^3\,dh\right)dr=2\pi\int_0^2 r^3\left(2-\frac{r^2}{2}\right)dr=$$

$$=2\pi\int_0^2\left(2r^3-\frac{r^5}{2}\right)dr=2\pi\left(\frac{r^4}{2}-\frac{r^6}{12}\right)\Big|_0^2=2\pi\left(8-\frac{16}{3}\right)=\frac{16\pi}{3}$$

3. $x_1^2+x_2^2 \le x_3^2$, $x_1^2+x_2^2+x_3^2=2x_3$



Циліндр. коорд.: $r \le h$, $r^2+(h-1)^2=1 \Rightarrow$
$\varphi\in[0,2\pi]$, $r\in[0,1]$ $\quad h = 1+\sqrt{1-r^2}$

$$V = \int_A 1\,dx_1dx_2dx_3 = \int_0^{2\pi}\left(\int_0^1\left(\int_r^{1+\sqrt{1-r^2}} r\,dh\right)dr\right)d\varphi =$$

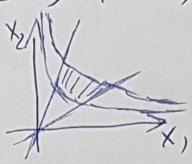
$$= 2\pi\int_0^1\left(\int_r^{1+\sqrt{1-r^2}} r\,dh\right)dr = 2\pi\int_0^1 r(1+\sqrt{1-r^2}-r)\,dr = 2\pi\left(\left(\frac{r^2}{2}-\frac{r^3}{3}\right)\Big|_0^1 - \frac{1}{2}\int_0^1 (1-r^2)^{1/2}d(1-r^2)\right) =$$

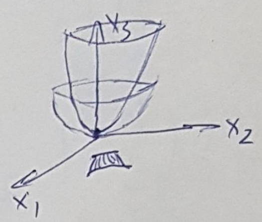
$$= 2\pi\left(\frac{1}{2}-\frac{1}{3}-\frac{(1-r^2)^{3/2}}{3}\Big|_0^1\right) = 2\pi\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}\right) = \pi$$

4. $x_3 = x_1^2+x_2^2$, $x_3 = 2(x_1^2+x_2^2)$, $x_1x_2=1$, $x_1x_2=2$, $x_1=2x_2$, $2x_1=x_2$, $x_1\ge 0$, $x_2 \ge 0$

$x_1x_2 = y_1$ $\quad x_1 = \sqrt{\frac{y_1}{y_2}}$
$\frac{x_2}{x_1} = y_2$ $\quad x_2 = \sqrt{y_1y_2}$
$x_3 = y_3$

$$J = \begin{vmatrix} \frac{1}{2\sqrt{y_1y_2}} & -\frac{\sqrt{y_1}}{2\sqrt{y_2^3}} & 0 \\ \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2y_2}$$



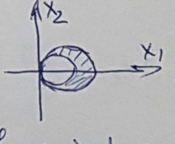


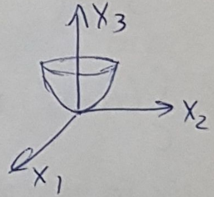
$y_1\in[1,2]$ $\quad y_2\in[\frac{1}{2},2]$ $\quad x_3\in\left[\frac{y_1}{y_2}+y_1y_2,\ \frac{2y_1}{y_2}+2y_1y_2\right]$

$$V = \int_A 1\,dx_1dx_2dx_3 = \int_1^2\left(\int_{1/2}^2\left(\int_{\frac{y_1}{y_2}+y_1y_2}^{\frac{2y_1}{y_2}+2y_1y_2}\frac{1}{2y_2}dy_3\right)dy_2\right)dy_1 = \int_1^2\left(\int_{1/2}^2\left(\frac{y_1}{2y_2^2}+\frac{y_1}{2}\right)dy_2\right)dy_1 =$$

$$= \int_1^2\left(-\frac{y_1}{2y_2}+\frac{y_1y_2}{2}\right)\Big|_{y_2=\frac{1}{2}}^2 dy_1 = \int_1^2\left(-\frac{y_1}{4}+y_1+y_1-\frac{y_1}{4}\right)dy_1 = \int_1^2\frac{3}{2}y_1\,dy_1 = \frac{3y_1^2}{4}\Big|_1^2 = \frac{9}{4}$$

5. 1) $x_3 = x_1^2+x_2^2$, $x_1^2+x_2^2 = x_1$, $x_1^2+x_2^2 = 2x_1$, $x_3=0$





Циліндричні: $\varphi\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\begin{matrix} r^2 = r\cos\varphi \\ r^2 = 2r\cos\varphi\end{matrix} \Rightarrow r\in[\cos\varphi, 2\cos\varphi]$

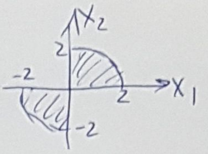
$h\in[0,r^2]$ $\quad J = r$

$$V = \int_A 1\,dx_1dx_2dx_3 = \int_{-\pi/2}^{\pi/2}\left(\int_{\cos\varphi}^{2\cos\varphi}\left(\int_0^{r^2} r\,dh\right)dr\right)d\varphi = \int_{-\pi/2}^{\pi/2}\left(\int_{\cos\varphi}^{2\cos\varphi} r^3dr\right)d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2}\frac{r^4}{4}\Big|_{r=\cos\varphi}^{r=2\cos\varphi}d\varphi = \frac{15}{4}\int_{-\pi/2}^{\pi/2}\cos^4\varphi\,d\varphi = \frac{15}{4}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\varphi}{2}\right)^2d\varphi = \frac{15}{4}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{4}+\cos 2\varphi+\frac{1}{8}+\frac{\cos 4\varphi}{8}\right)d\varphi =$$

$$= \frac{15}{4}\left(\frac{3}{8}\varphi+\frac{\sin 2\varphi}{2}+\frac{\sin 4\varphi}{32}\right)\Big|_{\varphi=-\frac{\pi}{2}}^{\pi/2} = \frac{45}{32}\pi$$

5. 2) $x_3^2=x_1x_2$, $\quad x_1^2+x_2^2=4$



Циліндричні: $\varphi\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$, $r\in[0,2]$

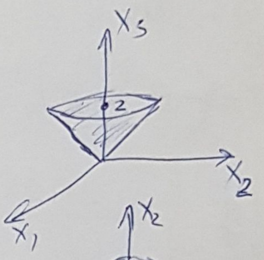
$h\in[-r\sqrt{\sin\varphi\cos\varphi},\ r\sqrt{\sin\varphi\cos\varphi}]$

$$V=\int_A 1\,dx_1dx_2dx_3=2\int_0^{\pi/2}\left(\int_0^2\left(\int_{-r\sqrt{\sin\varphi\cos\varphi}}^{r\sqrt{\sin\varphi\cos\varphi}} r\,dh\right)dr\right)d\varphi=2\int_0^{\pi/2}\left(\int_0^2 2r^2\sqrt{\sin\varphi\cos\varphi}\,dr\right)d\varphi=$$

$$=4\int_0^{\pi/2}\sqrt{\sin\varphi\cos\varphi}\cdot\frac{r^3}{3}\Big|_0^2 d\varphi=\frac{32}{3}\int_0^{\pi/2}\sqrt{\sin\varphi\cos\varphi}\,d\varphi=\left|\sin^2\varphi=t\right|=\frac{16}{3}\int_0^1\frac{dt}{\sqrt[4]{t}\sqrt[4]{1-t}}=$$

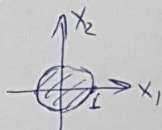
$$=\frac{16}{3}B\left(\frac34,\frac34\right)=\frac{16}{3}\cdot\frac{\Gamma(\frac34)\Gamma(\frac34)}{\Gamma(\frac32)}=\frac{16}{3}\frac{(\Gamma(\frac34))^2}{\frac12\cdot\sqrt{\pi}}=\frac{32}{3\sqrt{\pi}}\left(\Gamma\left(\frac34\right)\right)^2$$

~~3)~~ 6. $4x_1^2+4x_2^2=x_3^2$, $x_3=2$ Центр ваги-?



$V=\frac13 hS=\frac13\cdot2\cdot\pi\cdot1^2=\frac{2\pi}{3}$. Вважаємо, що $\rho=1$.

Внаслідок симетрії відносно Ox_1x_3, Ox_2x_3:



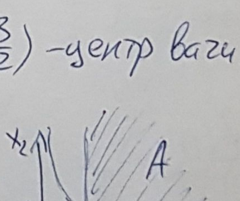
$x_{1ц}=0$, $x_{2ц}=0$.

$x_{3ц}=\frac{1}{V}\int_A x_3\,dx_1dx_2dx_3 =$ Циліндр. коорд. $\varphi\in[0,2\pi]$, $r\in[0,1]$, $h\in[2r,2]$

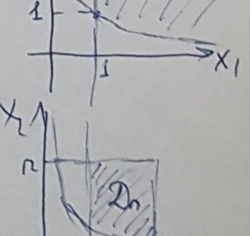
$$=\frac{3}{2\pi}\int_0^{2\pi}\left(\int_0^1\left(\int_{2r}^2 h\cdot r\,dh\right)dr\right)d\varphi=3\int_0^1\left(\int_{2r}^2 hr\,dh\right)dr=3\int_0^1\frac{h^2r}{2}\Big|_{h=2r}^2 dr=$$

$$=3\int_0^1(2r-2r^3)dr=3\left(r^2-\frac{r^4}{2}\right)\Big|_0^1=\frac32$$ Отже, $(0,0,\frac32)$ - центр ваги.

7. $\int_A\frac{dx_1dx_2}{x_1^\alpha x_2^\beta} =$ $\quad A=\{(x_1,x_2)\mid x_1x_2\geq1,\ x_1\geq1\}$



$$=\lim_{n\to\infty}\int_{D_n}\frac{dx_1dx_2}{x_1^\alpha x_2^\beta}=\lim_{n\to\infty}\int_1^n\left(\int_{\frac{1}{x_1}}^n\frac{dx_2}{x_1^\alpha x_2^\beta}\right)dx_1=$$

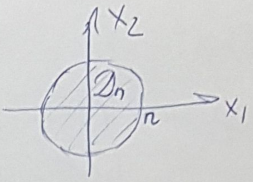


$$=\lim_{n\to\infty}\int_1^n\frac{1}{x_1^\alpha}\cdot\frac{n^{-\beta+1}-(\frac{1}{x_1})^{-\beta+1}}{-\beta+1}dx_1=\lim_{n\to\infty}\frac{1}{-\beta+1}\left(\frac{1}{n^{\beta-1}}\cdot\frac{x_1^{-\alpha+1}}{-\alpha+1}\Big|_1^n-\frac{x_1^{\beta-\alpha}}{\beta-\alpha}\Big|_1^n\right)=$$

$$=\lim_{n\to\infty}\frac{1}{1-\beta}\left(\frac{1}{n^{\alpha+\beta-2}(1-\alpha)}+\frac{1}{(\alpha-1)n^{\beta-1}}-\frac{1}{\beta-\alpha}\frac{1}{n^{\alpha-\beta}}+\frac{1}{\beta-\alpha}\right)$$ - скінченна, якщо

$\alpha+\beta-2>0$, $\beta-1>0$, $\alpha-\beta>0 \iff \underline{\alpha>\beta>1}$

8. $\int\limits_{\mathbb{R}^2} e^{-x_1^2-x_2^2}\cos(x_1^2+x_2^2)\,dx_1dx_2$



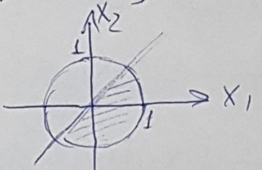
$=\lim\limits_{n\to\infty}\int\limits_{D_n} e^{-x_1^2-x_2^2}\cos(x_1^2+x_2^2)\,dx_1dx_2=$

$=\lim\limits_{n\to\infty}\int\limits_0^{2\pi}\left(\int\limits_0^n r e^{-r^2}\cos(r^2)\,dr\right)d\varphi=\lim\limits_{n\to\infty}\pi\int\limits_0^n e^{-r^2}\cos(r^2)\,d(r^2)=\pi\int\limits_0^{+\infty}e^{-t}\cos t\,dt=$

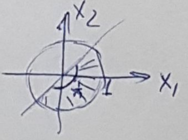
$=\left|\begin{matrix}u=e^{-t} & dv=\cos t\,dt\\ du=-e^{-t}dt & v=\sin t\end{matrix}\right|=\pi\left(e^{-t}\sin t\Big|_0^{+\infty}+\int\limits_0^{+\infty}e^{-t}\sin t\,dt\right)=\left|\begin{matrix}u=e^{-t} & dv=\sin t\,dt\\ du=-e^{-t}dt & v=-\cos t\end{matrix}\right|$

$=\pi\left(-e^{-t}\cos t\Big|_0^{+\infty}-\int\limits_0^{+\infty}e^{-t}\cos t\,dt\right)=\pi-I\Rightarrow I=\frac{\pi}{2}$

9. $\int\limits_A \frac{dx_1dx_2}{(x_1^2+x_2^2)^{1/3}} = \quad A=\{(x_1,x_2)\mid x_1^2+x_2^2\le 1,\ x_1\ge x_2\}$



$=\lim\limits_{n\to\infty}\int\limits_{D_n}\frac{dx_1dx_2}{(x_1^2+x_2^2)^{1/3}}=$



$=\lim\limits_{n\to\infty}\int\limits_{-\frac{3\pi}{4}}^{\pi/4}\left(\int\limits_{\frac{1}{n}}^{1}\cancel{dx}\,\frac{r}{r^{2/3}}\,dr\right)d\varphi=$

$=\lim\limits_{n\to\infty}\int\limits_{-\frac{3\pi}{4}}^{\pi/4}\frac{3}{4}r^{4/3}\Big|_{r=\frac{1}{n}}^{1}d\varphi=\lim\limits_{n\to\infty}\frac{3\pi}{4}\left(1-\frac{1}{n^{4/3}}\right)=\frac{3\pi}{4}.$