A3

1. 1) $f(x,y)=x^4+2y^4-4x^2y^2+\ln(x+y^2)+x\sin(x+y),\ x>-y^2$

$$\frac{\partial f}{\partial x}=4x^3-8xy^2+\frac{1}{x+y^2}+\sin(x+y)+x\cos(x+y)$$

$$\frac{\partial f}{\partial y}=8y^3-8x^2y+\frac{2y}{x+y^2}+x\cos(x+y)$$

2) $f(x,y)=xy+\frac{x}{y}+\operatorname{arctg}\frac{y}{x}+\left(\frac{x}{y}\right)^y,\ xy>0$

$$\frac{\partial f}{\partial x}=y+\frac{1}{y}-\frac{\frac{y}{x^2}}{1+\frac{y^2}{x^2}}+\frac{yx^{y-1}}{y^y}=y+\frac{1}{y}-\frac{y}{x^2+y^2}+\left(\frac{x}{y}\right)^{y-1}$$

$$\frac{\partial f}{\partial y}=x-\frac{x}{y^2}+\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}+e^{y(\ln x-\ln y)}(\ln x-\ln y-1)$$

$$\left(\frac{x}{y}\right)^y=e^{y(\ln x-\ln y)}$$

3) $f(x,y)=\frac{x}{\sqrt{x^2+y^2}}+x^y+\arcsin\frac{1}{1+x^2y^2},\ x>0$

$$\frac{\partial f}{\partial x}=\frac{\sqrt{x^2+y^2}-x\cdot\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}+yx^{y-1}-\frac{\frac{2xy^2}{(1+x^2y^2)^2}}{\sqrt{1-\frac{1}{(1+x^2y^2)^2}}}$$

$$\frac{\partial f}{\partial y}=\frac{-x\cdot\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2}+x^y\ln x-\frac{\frac{2yx^2}{(1+x^2y^2)^2}}{\sqrt{1-\frac{1}{(1+x^2y^2)^2}}}$$

2. 1) $f(x,y,z)=x^3\sin(yz)+(x^2+y^2)e^{x+z}$

$$\frac{\partial f}{\partial x}=3x^2\sin(yz)+2xe^{x+z}+(x^2+y^2)e^{x+z}=3x^2\sin(yz)+(2x+x^2+y^2)e^{x+z}$$

$$\frac{\partial^2 f}{\partial x^2}=6x\sin(yz)+(2+2x)e^{x+z}+(2x+x^2+y^2)e^{x+z}$$

$$\frac{\partial f}{\partial y}=x^3z\cos(yz)+2ye^{x+z}$$

$$\frac{\partial^2 f}{\partial y\partial z}=x^3\cos(yz)-x^3yz\sin(yz)+2ye^{x+z}$$

2. 2) $f(x,y,z) = x\ln(yz) + xyz\,e^{x+y+z} = x\ln(yz) + xe^x \cdot ye^y \cdot ze^z$, $x,y,z>0$

$$\frac{\partial f}{\partial x} = \ln(yz) + (x+1)e^x \cdot ye^y \cdot ze^z$$

$$\frac{\partial^2 f}{\partial x^2} = (x+2)e^x \cdot ye^y \cdot ze^z$$

$$\frac{\partial^2 f}{\partial x^2 \partial y} = (x+2)e^x \cdot (y+1)e^y \cdot ze^z$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y} + (x+1)e^x \cdot (y+1)e^y \cdot ze^z$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = (x+1)e^x \cdot (y+1)e^y \cdot (z+1)e^z$$

$$\dots$$

$$\frac{\partial^6 f}{\partial x \partial^2 y \partial^3 z} = (x+1)e^x \cdot (y+2)e^y \cdot (z+3)e^z$$

3. $f(x,y) = x^2 - y^5$ т. $\overline{x}^0 = (1,1)$ $\qquad \overline{a} = (2,3)$

$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -5y^4 \qquad \overrightarrow{grad}\, f = (2x, -5y^4)$

$\overrightarrow{grad}\, f(\overline{x}^0) = (2,-5)$

$$f'_{\overline{a}}(\overline{x}^0) = \frac{\partial f}{\partial x}(\overline{x}^0)\cdot a_1 + \frac{\partial f}{\partial y}(\overline{x}^0)\cdot a_2 = 2\cdot 2 + (-5)\cdot 3 = -11$$

4. 1) $f(x,y) = \sqrt[3]{x^3+y^3}$, $(x,y)\in\mathbb{R}^2$, $\overline{x}^0 = (0,0)$.

$\frac{\partial f}{\partial x} = \frac{3x^2}{3\sqrt[3]{(x^3+y^3)^2}}$ – незастосовна в т. $\overline{x}^0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x\to 0}\frac{f(0+\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt[3]{(\Delta x)^3} - 0}{\Delta x} = 1 = L_1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y\to 0}\frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{\sqrt[3]{(\Delta y)^3} - 0}{\Delta y} = 1 = L_2$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - L_1\cdot\Delta x - L_2\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\sqrt[3]{(\Delta x)^3+(\Delta y)^3} - \Delta x - \Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \nexists$$

Озн. Гейне: $\Delta x = \frac{1}{n}$, $\Delta y = \frac{1}{n}$: $\frac{\sqrt[3]{\frac{1}{n^3}+\frac{1}{n^3}} - \frac{1}{n} - \frac{1}{n}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = \frac{\sqrt[3]{2}-2}{\sqrt{2}}$; $\Delta x = \frac{1}{n}$, $\Delta y = 0$: $\frac{\sqrt[3]{\frac{1}{n^3}+0} - \frac{1}{n} - 0}{\sqrt{\frac{1}{n^2}+0}} = 0$

Ф-ція не є диференційовною в т. $\overline{x}^0$

A3

4. 2) $f(x,y)=\begin{cases} e^{-\frac{1}{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$ в т. $(0,0)$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{-\frac{1}{\Delta x^2}}}{\Delta x}=\left|\frac{1}{\Delta x}=t\right|=\lim_{t\to\infty}\frac{t}{e^{t^2}}=0$$

$\frac{\partial f}{\partial y}(0,0)=0$ (аналогічно)

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(0+\Delta x,0+\Delta y)-f(0,0)-L_1\Delta x-L_2\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{e^{-\frac{1}{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}=\left|\sqrt{\Delta x^2+\Delta y^2}=t\right|=$$

$$=\lim_{t\to\infty}\frac{t}{e^{t^2}}=0$$

Отже, f - диференційовна в т. $(0,0)$ і

$df((0,0),(\Delta x,\Delta y))=L_1\cdot\Delta x+L_2\Delta y=0.$

3) $f(x,y)=|x-1|+|y-2|$, $(x,y)\in\mathbb{R}^2$ т. $(1,2)$

$$\frac{\partial f}{\partial x}(1,2)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x,2)-f(1,2)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|-0}{\Delta x}\ \nexists$$

За Гейне: $\Delta x=\frac{1}{n}$: $\frac{|\frac{1}{n}|}{\frac{1}{n}}=1\to 1$

$\Delta x=-\frac{1}{n}$: $\frac{|-\frac{1}{n}|}{-\frac{1}{n}}=-1\to -1$

Отже, f не є диференційовною в т. $(1,2)$

4) $f(x,y)=tg\frac{x^2}{y}$, $y\neq 0$, $\left|\frac{x^2}{y}\right|<\frac{\pi}{2}$, $\overline{x}^0=\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

$\frac{\partial f}{\partial x}=\frac{\frac{2x}{y}}{\cos^2\frac{x^2}{y}}$, $\frac{\partial f}{\partial y}=\frac{-\frac{x^2}{y^2}}{\cos^2\frac{x^2}{y}}$ – існують і неперервні в околі т. $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

За теоремою f - диференційовна в т. $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

$L_1=\frac{\partial f}{\partial x}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=4$ $\quad L_2=\frac{\partial f}{\partial y}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=-2$

$df\left(\left(\frac{\pi}{4},\frac{\pi}{4}\right),(\Delta x,\Delta y)\right)=4\Delta x-2\Delta y$

5. 1) $f(x,y,z) = \frac{x^4}{12} - \frac{x^3(y+z)}{6} + \frac{x^2yz}{2} + g(\underbrace{y-x}_{u}, \underbrace{z-x}_{v})$ ($g: \mathbb{R}^2 \to \mathbb{R}$ – диференційовна)

$$\frac{\partial f}{\partial x} = \frac{x^3}{3} - \frac{x^2(y+z)}{2} + xyz + \frac{\partial g}{\partial u}\cdot(-1) + \frac{\partial g}{\partial v}\cdot(-1)$$

$$\frac{\partial f}{\partial y} = -\frac{x^3}{6} + \frac{x^2 z}{2} + \frac{\partial g}{\partial u}\cdot 1 + \frac{\partial g}{\partial v}\cdot 0$$

$$\frac{\partial f}{\partial z} = -\frac{x^3}{6} + \frac{x^2 y}{2} + \frac{\partial g}{\partial u}\cdot 0 + \frac{\partial g}{\partial v}\cdot 1$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = xyz$$

2) $f(x,y,z) = \frac{xy}{z}\ln x + x g(\underbrace{\tfrac{y}{x}}_{u}, \underbrace{\tfrac{z}{x}}_{v})$, $x>0, z \neq 0$

$$\frac{\partial f}{\partial x} = \frac{y}{z}\ln x + \frac{y}{z} + g + x\cdot\left(\frac{\partial g}{\partial u}\cdot\left(-\frac{y}{x^2}\right) + \frac{\partial g}{\partial v}\cdot\left(-\frac{z}{x^2}\right)\right)$$

$$\frac{\partial f}{\partial y} = \frac{x}{z}\ln x + x\cdot\left(\frac{\partial g}{\partial u}\cdot\frac{1}{x} + \frac{\partial g}{\partial v}\cdot 0\right)$$

$$\frac{\partial f}{\partial z} = -\frac{xy}{z^2}\ln x + x\cdot\left(\frac{\partial g}{\partial u}\cdot 0 + \frac{\partial g}{\partial v}\cdot\frac{1}{x}\right)$$

$$x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = \underline{\frac{xy}{z}\ln x} + \underline{\underline{\frac{xy}{z}}} + \underline{xg} \underbrace{- y\cdot\frac{\partial g}{\partial u} - z\frac{\partial g}{\partial v} + y\frac{\partial g}{\partial u} + z\frac{\partial g}{\partial v}} =$$

$$= f + \frac{xy}{z}$$

6. 1) $f(x,y,z) = g(\underbrace{xy}_{u}, \underbrace{\tfrac{y}{z}}_{v})$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\cdot y \qquad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\cdot x + \frac{\partial g}{\partial v}\cdot\frac{1}{z} \qquad \frac{\partial f}{\partial z} = \frac{\partial g}{\partial v}\cdot\left(-\frac{y}{z^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 g}{\partial u^2}\cdot y\right)\cdot y \qquad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial g}{\partial u} + y\cdot\left(\frac{\partial^2 g}{\partial u^2}\cdot x + \frac{\partial^2 g}{\partial u\partial v}\cdot\frac{1}{z}\right) \qquad \frac{\partial^2 f}{\partial x\partial z} = y\cdot\frac{\partial^2 f}{\partial u\partial v}\cdot\left(-\frac{y}{z^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = x\cdot\left(\frac{\partial^2 g}{\partial u^2}\cdot x + \frac{\partial^2 g}{\partial u\partial v}\cdot\frac{1}{z}\right) + \frac{1}{z}\cdot\left(\frac{\partial^2 g}{\partial v\partial u}\cdot x + \frac{\partial^2 g}{\partial v^2}\cdot\frac{1}{z}\right)$$

$$\frac{\partial^2 f}{\partial z\partial y} = -\frac{1}{z^2}\cdot\frac{\partial g}{\partial v} - \frac{y}{z^2}\cdot\left(\frac{\partial^2 g}{\partial v\partial u}\cdot x + \frac{\partial^2 g}{\partial v^2}\cdot\frac{1}{z}\right) \qquad \frac{\partial^2 f}{\partial z^2} = \frac{2y}{z^3}\cdot\frac{\partial g}{\partial v} - \frac{y}{z^2}\cdot\frac{\partial^2 g}{\partial v^2}\cdot\left(-\frac{y}{z^2}\right)$$

A3

7. $z^3 - 3xyz = a^3, \ (x,y) \in R^2 \qquad z = z(x,y)$

по x: $3z^2 \cdot \dfrac{\partial z}{\partial x} - 3yz - 3xy \cdot \dfrac{\partial z}{\partial x} = 0 \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{3yz}{3z^2 - 3xy} = \dfrac{yz}{z^2 - xy}$

по y: $3z^2 \cdot \dfrac{\partial z}{\partial y} - 3xz - 3xy \cdot \dfrac{\partial z}{\partial y} = 0 \Rightarrow \dfrac{\partial z}{\partial y} = \dfrac{3xz}{3z^2 - 3xy} = \dfrac{xz}{z^2 - xy}$