1. $a>0,\ b>0,\quad \frac{e^{-ax}-e^{-bx}}{x}=\int_a^b e^{-\alpha x}\,d\alpha,\ x>0$ (бо $\int e^{-\alpha x}d\alpha=-\frac{e^{-\alpha x}}{x}+C$)

$$\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}dx=\int_0^{+\infty}\left(\int_a^b e^{-\alpha x}d\alpha\right)dx=\int_a^b\left(\int_0^{+\infty}e^{-\alpha x}dx\right)d\alpha=\int_a^b -\frac{e^{-\alpha x}}{\alpha}\Big|_{x=0}^{+\infty}d\alpha=$$

$$=\int_a^b\frac{1}{\alpha}d\alpha=\ln\alpha\Big|_a^b=\ln b-\ln a=\ln\frac{b}{a}$$ (Використано теорему про інтегрування невласного інтеграла по параметру)

II спосіб: Интеграл Фрулані. $f(x)=e^{-x}$, $f\in C^1([0,+\infty))$, $f(+\infty)=0$

$$\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}dx=(f(+\infty)-f(0))\ln\frac{a}{b}=-\ln\frac{a}{b}=\ln\frac{b}{a}.$$

2. $a,b>0$ $\quad\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}=-\cos 0\cdot\ln\frac{a}{b}=\ln\frac{b}{a}$,

як інтеграл Фруллані, бо $f(x)=\cos x$, $f\in R([0,1])$, $f\in C(\{0\})$,
$\int_1^{+\infty}\frac{\cos x}{x}dx$ – збіжний за ознакою Діріхле

3. $\int_0^{+\infty}\frac{\sin^2\alpha x}{x^2}dx$, $\alpha\in\mathbb{R}$

a) $I(\alpha)=\int_0^{+\infty}\frac{\sin^2\alpha x}{x^2}dx$ – рівномірно збіжний на $[a,b]\subset(0,+\infty)$ за ознакою Вейерштрасса: $\left|\frac{\sin^2\alpha x}{x^2}\right|\le\frac{1}{x^2},\ x\ge1$, $\int_1^{+\infty}\frac{dx}{x^2}$ - збіж.

$\left|\frac{\sin^2\alpha x}{x^2}\right|\le\frac{\alpha^2x^2}{x^2}=\alpha^2\le b^2$, $\int_0^1 b^2dx$ - збіж.

$I'(\alpha)=\int_0^{+\infty}\frac{\sin2\alpha x}{x}dx$ – рівномірно збіжний на $[a,b]\subset(0,+\infty)$ за ознакою Діріхле: 1. $\left|\int_0^A\sin 2\alpha x\,dx\right|=\left|\frac{1-\cos2\alpha x}{2\alpha}\right|\le\frac{1}{\alpha}\le\frac{1}{a}=C$

$$I'(\alpha)=\int_0^{+\infty}\frac{\sin2\alpha x}{x}dx=\left|\begin{matrix}2\alpha x=t\\x=\frac{t}{2\alpha}\\dx=\frac{dt}{2\alpha}\end{matrix}\right|=\int_0^{+\infty}\frac{\sin t}{t}dt=\frac{\pi}{2},\ \alpha>0$$

2. $\frac{1}{x}\searrow$

3. $\frac{1}{x}\xrightarrow[x\to+\infty]{}0$

$I(\alpha)=\frac{\pi}{2}\alpha+C$, $\alpha\ge0$ (за теоремою про неперервність)

$I(0)=0\Rightarrow C=0\Rightarrow I(\alpha)=\frac{\pi}{2}\alpha,\ \alpha\ge0$.

I – парна, тому $I(\alpha)=\frac{\pi}{2}|\alpha|$, $\alpha\in\mathbb{R}$.

3. б) $I(\alpha)=\int\limits_0^{+\infty} \frac{\sin^2\alpha x}{x^2}dx = \left| \begin{matrix} u=\sin^2\alpha x & dv=\frac{dx}{x^2} \\ du=2\sin\alpha x\cos\alpha x\cdot\alpha\, dx=\alpha\cdot\sin 2\alpha x\, dx \\ v=-\frac{1}{x} \end{matrix} \right| = -\frac{\sin^2\alpha x}{x}\Big|_0^{+\infty} + \alpha\int\limits_0^{+\infty}\frac{\sin 2\alpha x}{x}dx$

$= \left| \begin{matrix} \frac{\sin^2\alpha x}{x} \sim \frac{(\alpha x)^2}{x}=\alpha^2 x \to 0,\ x\to 0 \\ \frac{\text{обмеж.}}{\infty}=0,\ x\to\infty \end{matrix} \right| = \begin{cases} \left|2\alpha x=t\right| = \alpha\int\limits_0^{+\infty}\frac{\sin t}{t}dt=\frac{\pi\alpha}{2},\ \alpha>0 \\ \left|-2\alpha x=t\right| = -\alpha\int\limits_0^{+\infty}\frac{\sin t}{t}dt=-\frac{\pi\alpha}{2},\ \alpha<0 \end{cases} \Rightarrow I(\alpha)=\frac{\pi|\alpha|}{2}$

4. 1) $\int\limits_0^{+\infty}\frac{1-\cos\alpha x}{x^2}dx = \left| \begin{matrix} u=1-\cos\alpha x & dv=\frac{dx}{x^2} \\ du=\alpha\sin\alpha x\, dx & v=-\frac{1}{x} \end{matrix} \right| = \frac{\cos\alpha x-1}{x}\Big|_0^{+\infty} + \int\limits_0^{+\infty}\frac{\alpha\sin\alpha x}{x}dx =$

$= \left| \frac{\cos\alpha x-1}{x} = -\frac{2\sin^2\frac{\alpha x}{2}}{x} \sim \frac{-2\left(\frac{\alpha x}{2}\right)^2}{x} = -\frac{\alpha^2 x}{4} \to 0,\ x\to 0 \right| = \alpha\int\limits_0^{+\infty}\frac{\sin\alpha x}{x}dx = \frac{\pi|\alpha|}{2},\ \alpha\in\mathbb{R}$

2) $\int\limits_0^{+\infty}\frac{1-e^{-\alpha x^2}}{x^2}dx = \left| \begin{matrix} u=1-e^{-\alpha x^2} & dv=\frac{dx}{x^2} \\ du=2\alpha x e^{-\alpha x^2}dx & v=-\frac{1}{x} \end{matrix} \right| = \frac{e^{-\alpha x^2}-1}{x}\Big|_0^{+\infty} + 2\alpha\int\limits_0^{+\infty} e^{-\alpha x^2}dx =$

$(\alpha>0)$

$= \left| \begin{matrix} \frac{e^{-\alpha x^2}-1}{x} \sim \frac{-\alpha x^2}{x} = -\alpha x \to 0,\ x\to 0 \\ \frac{\text{обмеж.}}{\infty}=0,\ x\to\infty \end{matrix} \right| \left| \begin{matrix} \sqrt{\alpha}\,x=t \\ \sqrt{\alpha}\, dx=dt \end{matrix} \right| = 2\sqrt{\alpha}\int\limits_0^{+\infty} e^{-t^2}dt = 2\sqrt{\alpha}\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi\alpha}$

3) $\int\limits_{-\infty}^{+\infty} e^{-2x^2+10x+3}dx = \int\limits_{-\infty}^{+\infty} e^{-2(x^2-5x-\frac{3}{2})}dx = \int\limits_{-\infty}^{+\infty} e^{-2\left(\left(x-\frac{5}{2}\right)^2-\frac{31}{4}\right)}dx =$

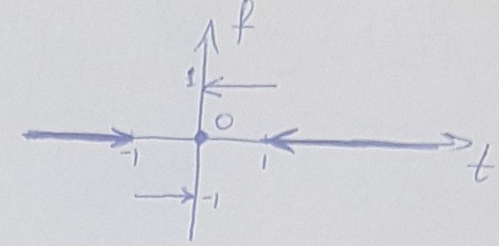
$= \int\limits_{-\infty}^{+\infty} e^{\frac{31}{2}}\cdot e^{-\left(\sqrt{2}\left(x-\frac{5}{2}\right)\right)^2}dx = \left| \begin{matrix} \sqrt{2}\left(x-\frac{5}{2}\right)=t \\ \sqrt{2}\,dx=dt \end{matrix} \right| = e^{\frac{31}{2}}\cdot\frac{1}{\sqrt{2}}\int\limits_{-\infty}^{+\infty} e^{-t^2}dt = e^{\frac{31}{2}}\cdot\frac{1}{\sqrt{2}}\cdot 2\int\limits_0^{+\infty} e^{-t^2}dt = \sqrt{\frac{e^{31}\pi}{2}}$

(ПАРНІСТЬ)

4) $\alpha>0,\ \beta>0:\ \int\limits_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x^2}dx =$

$= \int\limits_0^{+\infty}\frac{1-e^{-\beta x^2}}{x^2}dx - \int\limits_0^{+\infty}\frac{1-e^{-\alpha x^2}}{x^2}dx = \sqrt{\pi\beta}-\sqrt{\pi\alpha}$

5. 1) $f(t)=\begin{cases} \operatorname{sign} t, & |t|\le 1 \\ 0, & |t|>1 \end{cases}$



Коефіцієнти Фур'є:

$$a(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos\lambda t\,dt=\frac{1}{\pi}\left(\int_{-1}^{0}(-\cos\lambda t)dt+\int_{0}^{1}\cos\lambda t\,dt\right)=\frac{1}{\pi}\left(-\frac{\sin\lambda t}{\lambda}\Big|_{t=-1}^{0}+\frac{\sin\lambda t}{\lambda}\Big|_{t=0}^{1}\right)=$$

$$=\frac{1}{\pi}\left(-\frac{1}{\lambda}+\frac{1}{\lambda}\right)=0,\ \lambda\in R$$ (також випливає з непарності f)

$$b(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\lambda t\,dt=\frac{1}{\pi}\left(\int_{-1}^{0}(-\sin\lambda t)dt+\int_{0}^{1}\sin\lambda t\,dt\right)=\frac{1}{\pi}\left(\frac{\cos\lambda t}{\lambda}\Big|_{t=-1}^{0}-\frac{\cos\lambda t}{\lambda}\Big|_{t=0}^{1}\right)$$

$$=\frac{1}{\pi}\left(\frac{1-\cos\lambda}{\lambda}-\frac{\cos\lambda-1}{\lambda}\right)=\frac{2(1-\cos\lambda)}{\pi\lambda},\ \lambda\neq 0;\quad b(0)=0.$$

$$I(t)=\int_{0}^{+\infty}(a(\lambda)\cos\lambda t+b(\lambda)\sin\lambda t)d\lambda=\int_{0}^{+\infty}\frac{2(1-\cos\lambda)}{\pi\lambda}\sin\lambda t\,d\lambda$$

За критеріями збіжності: 1) при $t\neq -1;0;1$ $\exists f'(t)\Rightarrow I(t)=f(t)$

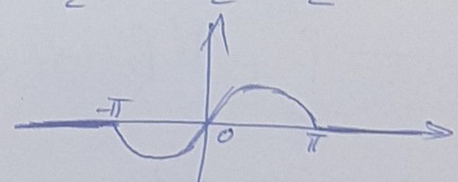
2) при $t=-1;0;1$ – розрив I роду і існують однобічні похідні.

$$I(0)=\frac{f(0-)+f(0+)}{2}=\frac{(-1)+1}{2}=0$$

$$I(1)=\frac{f(1-)+f(1+)}{2}=\frac{1+0}{2}=\frac{1}{2}$$

$$I(-1)=\frac{f(-1-)+f(-1+)}{2}=\frac{0+(-1)}{2}=-\frac{1}{2}$$

2) $f(t)=\begin{cases} \sin t, & |t|\le\pi \\ 0, & |t|>\pi \end{cases}$



f – непарна $\Rightarrow a(\lambda)=0,\ \lambda\in R$

$$b(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\lambda t\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin t\cdot\sin\lambda t\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(\lambda-1)t-\cos(\lambda+1)t}{2}dt=$$

$$=\frac{1}{2\pi}\left(\frac{\sin(\lambda-1)t}{\lambda-1}-\frac{\sin(\lambda+1)t}{\lambda+1}\right)\Big|_{t=-\pi}^{\pi}=\frac{1}{2\pi}\left(\frac{\sin(\lambda-1)\pi}{\lambda-1}-\frac{\sin(\lambda+1)\pi}{\lambda+1}\right)=\frac{1}{2\pi}\left(-\frac{\sin\pi\lambda}{\lambda-1}+\frac{\sin\pi\lambda}{\lambda+1}\right)=$$

$$=\frac{1}{\pi}\frac{\sin\pi\lambda}{1-\lambda^2},\ \lambda\neq\pm 1$$

$$I(t)=\int_{0}^{+\infty}\frac{1}{\pi}\frac{\sin\pi\lambda}{1-\lambda^2}\sin\lambda t\,d\lambda,\ t\in R$$

f – неперервна і має однобічні похідні на $R\Rightarrow I(t)=f(t),\ t\in R$

6. 1) $f(t)=e^{-\frac{(t-a)^2}{2\sigma^2}}, t\in\mathbb{R}, (a\in\mathbb{R}, \sigma>0)$

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\lambda t}\cdot e^{-\frac{(t-a)^2}{2\sigma^2}}dt$$

$$\hat{f}'(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\lambda t}\cdot it\, e^{-\frac{(t-a)^2}{2\sigma^2}}dt=ia\hat{f}(\lambda)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\lambda t}\cdot i(t-a)e^{-\frac{(t-a)^2}{2\sigma^2}}dt=$$

$$=ia\hat{f}(\lambda)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\lambda t}\cdot i\cdot\left(e^{-\frac{(t-a)^2}{2\sigma^2}}\right)'\cdot(-\sigma^2)dt=$$

$$=ia\hat{f}(\lambda)+\frac{1}{\sqrt{2\pi}}\left(-i\sigma^2 e^{i\lambda t}\cdot e^{-\frac{(t-a)^2}{2\sigma^2}}\Big|_{-\infty}^{+\infty}-\sigma^2\lambda\int_{-\infty}^{+\infty} e^{i\lambda t}\cdot e^{-\frac{(t-a)^2}{2\sigma^2}}dt\right)=(ia-\sigma^2\lambda)\hat{f}(\lambda)$$

$$(\ln\hat{f}(\lambda))'=ia-\sigma^2\lambda \Rightarrow \ln\hat{f}(\lambda)=\ln C+\frac{(ia-\sigma^2\lambda)^2}{-2\sigma^2}\Rightarrow \hat{f}(\lambda)=Ce^{-\frac{(ia-\sigma^2\lambda)^2}{2\sigma^2}}$$

$$\hat{f}(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(t-a)^2}{2\sigma^2}}dt=\left|\begin{matrix}\frac{t-a}{\sqrt{2}\sigma}=s\\ dt=\sqrt{2}\sigma ds\end{matrix}\right|=\frac{\sigma}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-s^2}ds=\frac{\sigma}{\sqrt{\pi}}\cdot\sqrt{\pi}=\sigma$$

$$\Rightarrow C\cdot e^{\frac{a^2}{2\sigma^2}}=\sigma\Rightarrow C=\sigma e^{-\frac{a^2}{2\sigma^2}}\Rightarrow \hat{f}(\lambda)=\sigma e^{\frac{-a^2-(ia-\sigma^2\lambda)^2}{2\sigma^2}}$$

2) $f(t)=\begin{cases}1, & |t|\le a\\ 0, & |t|>a\end{cases}, a\ge 0$

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\lambda t}f(t)dt=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{i\lambda t}dt=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{i\lambda t}}{i\lambda}\Big|_{-a=t}^{a=t}=$$

$$=\frac{1}{\sqrt{2\pi}}\frac{e^{i\lambda a}-e^{-i\lambda a}}{i\lambda}=\frac{1}{\sqrt{2\pi}}\frac{\cos(\lambda a)+i\sin(\lambda a)-\cos(\lambda a)+i\sin(\lambda a)}{i\lambda}=$$

$$=\frac{2\sin(\lambda a)}{\sqrt{2\pi}\cdot\lambda}, \lambda\neq 0$$

A7

1. $\int_0^{+\infty} x^{2n} e^{-x^2} dx = \left| \begin{array}{ll} x^2 = t & x = \sqrt{t} \\ dx = \frac{dt}{2\sqrt{t}} & \\ t_1 = 0 & t_2 = +\infty \end{array} \right| = \int_0^{+\infty} t^n e^{-t} \cdot \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^{+\infty} t^{n-\frac{1}{2}} e^{-t} dt = \frac{1}{2} \Gamma\left(n+\frac{1}{2}\right) =$

$(n \in \mathbb{N})$

$= \frac{1}{2} \cdot \left(n-\frac{1}{2}\right) \Gamma\left(n-\frac{1}{2}\right) = \frac{1}{2}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) = \ldots = \frac{1}{2}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \cdot \ldots \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) =$

$= \frac{1}{2} \cdot \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdot \ldots \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}}$

$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$

$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

$\boxed{\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0}$

2. 1) $I(\alpha) = \int_0^{+\infty} e^{-x^\alpha} dx = \left| \begin{array}{lll} x^\alpha = t & x = t^{\frac{1}{\alpha}} & t_1 = 0 \\ dx = \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt & & t_2 = +\infty \end{array} \right| = \int_0^{+\infty} e^{-t} \cdot \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt = \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$

$(\alpha > 0)$

2) $\lim\limits_{\alpha \to +\infty} I(\alpha) = \lim\limits_{\alpha \to +\infty} \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) = \lim\limits_{\alpha \to +\infty} \Gamma\left(1+\frac{1}{\alpha}\right) = \Gamma(1) = 1$

3. 1) $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \left| \begin{array}{ll} \sin^2 x = t & \sin^5 x = t^{5/2} \\ dt = 2\sin x \cos x \, dx & \cos^3 x = (1-\sin^2 x)^{\frac{3}{2}} = (1-t)^{3/2} \\ t_1 = \sin^2 0 = 0 & t_2 = \sin^2 \frac{\pi}{2} = 1 \end{array} \right| =$

$= \frac{1}{2}\int_0^1 t^{5/2}(1-t)^{3/2} dt = \frac{1}{2} B\left(\frac{7}{2}, \frac{5}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)} =$

$= \frac{1}{2} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{5!} = \frac{5 \cdot 3 \cdot 3 \cdot \pi}{2^6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{3\pi}{2^9} = \frac{3\pi}{512}$

$\boxed{B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx}$

$\boxed{B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \alpha > 0, \ \beta > 0}$

$\boxed{\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}}$

2) $\int_0^1 \sqrt{x-x^2} dx = \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} dx = B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{\left(\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)\right)^2}{2!} = \frac{\pi}{8}$

3) $\int_0^1 \frac{dx}{\sqrt[\alpha]{1-x^\alpha}} = \left| \begin{array}{lll} x^\alpha = t & x = t^{1/\alpha} & t_1 = 0 \\ dx = \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt & & t_2 = 1 \end{array} \right| = \int_0^1 (1-t)^{-\frac{1}{\alpha}} \cdot \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt = \frac{1}{\alpha} B\left(1-\frac{1}{\alpha}, \frac{1}{\alpha}\right) =$

$(\alpha > 1)$

$= \frac{1}{\alpha} \frac{\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma(1)} = \frac{1}{\alpha} \cdot \frac{\pi}{\sin \pi/\alpha} = \frac{\frac{\pi}{\alpha}}{\sin\frac{\pi}{\alpha}}$

$\boxed{\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}, \quad \alpha \neq 0; \pm 1; \pm 2; \ldots}$

A7

4. 1) $\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x^\beta}dx = \left| \begin{matrix} t=x^\beta & x=t^{\frac{1}{\beta}} & t_1=0 \\ dx=\frac{1}{\beta}t^{\frac{1}{\beta}-1}dt & & t_2=+\infty \end{matrix} \right| =$ $(\beta>0)$

$$\boxed{\int_0^{+\infty} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}dx = B(\alpha,\beta),\quad \alpha>0,\beta>0}$$

$= \int_0^{+\infty} \frac{t^{\frac{\alpha-1}{\beta}}}{1+t}\cdot\frac{1}{\beta}t^{\frac{1}{\beta}-1}dt = \frac{1}{\beta}\int_0^{+\infty}\frac{t^{\frac{\alpha}{\beta}-1}}{1+t}dt = \left| \begin{matrix} \alpha_1-1=\frac{\alpha}{\beta}-1 \Rightarrow \alpha_1=\frac{\alpha}{\beta} \\ \alpha_1+\beta_1=1 \Rightarrow \beta_1=1-\frac{\alpha}{\beta} \end{matrix} \right| =$

$= \frac{1}{\beta}B\left(\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\right) = \frac{1}{\beta}\frac{\Gamma(\frac{\alpha}{\beta})\Gamma(1-\frac{\alpha}{\beta})}{\Gamma(1)} = \frac{1}{\beta}\cdot\frac{\pi}{\sin\frac{\pi\alpha}{\beta}}$, збіжний при $\begin{cases} \frac{\alpha}{\beta}>0 \\ 1-\frac{\alpha}{\beta}>0 \\ \beta>0 \end{cases}$ $\begin{cases} \alpha>0 \\ \beta>0 \\ \beta>\alpha \end{cases} \Leftrightarrow \alpha\in(0,\beta),\ \beta>0.$

2) $\int_0^{+\infty}\frac{x^{\alpha-1}}{(1+x)^\beta}dx = \left| \begin{matrix} \alpha_1-1=\alpha-1 \Rightarrow \alpha_1=\alpha \\ \alpha_1+\beta_1=\beta \Rightarrow \beta_1=\beta-\alpha \end{matrix} \right| = B(\alpha,\beta-\alpha) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}$, збіжний при $\begin{cases} \alpha>0 \\ \beta-\alpha>0 \\ \beta>0 \end{cases} \Leftrightarrow$ $(\beta>0)$

$\alpha\in(0,\beta),\ \beta>0.$

5. 1) $\int_0^{+\infty}\frac{x^{\alpha-1}\ln x}{1+x}dx = \left(\int_0^{+\infty}\frac{x^{\alpha-1}}{1+x}dx\right)'_\alpha \overset{4\,1)}{=} \left(\frac{\pi}{\sin\pi\alpha}\right)'_\alpha = -\frac{\pi^2\cos\pi\alpha}{\sin^2\pi\alpha}$, $\alpha\in(0,1)$

$(\alpha\in(0,1))$

2) $\int_0^{+\infty}x^\alpha e^{-ax}\ln x\,dx = \left(\int_0^{+\infty}x^\alpha e^{-ax}dx\right)'_\alpha = \left| \begin{matrix} ax=t & t_1=0 \\ dx=\frac{dt}{a} & t_2=+\infty \end{matrix} \right| = \left(\int_0^{+\infty}\frac{t^\alpha}{a^\alpha}e^{-t}\frac{dt}{a}\right)'_\alpha =$

$(a>0,\ \alpha>-1)$

$= \left(\frac{\Gamma(\alpha+1)}{a^{\alpha+1}}\right)'_\alpha = \frac{\Gamma'(\alpha+1)a^{\alpha+1}-\Gamma(\alpha+1)a^{\alpha+1}\ln a}{a^{2\alpha+2}} = \frac{\Gamma'(\alpha+1)-\Gamma(\alpha+1)}{a^{\alpha+1}}$

6. Довести рівність $\int_0^1\frac{dx}{\sqrt{1-x^4}}\cdot\int_0^1\frac{x^2dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$

$\int_0^1\frac{dx}{\sqrt{1-x^4}} = \left| \begin{matrix} x^4=t & x=t^{\frac14} & t_1=0 \\ dx=\frac14 t^{-\frac34}dt & & t_2=1 \end{matrix} \right| = \int_0^1\frac{\frac14 t^{-\frac34}dt}{\sqrt{1-t}} = \frac14\int_0^1 t^{-\frac34}(1-t)^{-\frac12}dt = \frac14 B\left(\frac14,\frac12\right) = \frac14\frac{\Gamma(\frac14)\Gamma(\frac12)}{\Gamma(\frac34)}$

$\int_0^1\frac{x^2dx}{\sqrt{1-x^4}} = \left| \begin{matrix} x^4=t & x=t^{\frac14} & t_1=0 \\ dx=\frac14 t^{-\frac34}dt & & t_2=1 \end{matrix} \right| = \int_0^1\frac{\sqrt{t}\cdot\frac14 t^{-\frac34}dt}{\sqrt{1-t}} = \frac14\int_0^1 t^{-\frac14}(1-t)^{-\frac12}dt = \frac14 B\left(\frac34,\frac12\right) = \frac14\frac{\Gamma(\frac34)\Gamma(\frac12)}{\Gamma(\frac54)} = \frac14\cdot\frac{\Gamma(\frac34)\cdot\Gamma(\frac12)}{\frac14\Gamma(\frac14)}$

Враховуючи, що $(\Gamma(\frac12))^2=\pi$, добуток рівний $\frac{\pi}{4}$.

7. Відомо, що $\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)}\int_0^{+\infty}t^{\beta-1}e^{-xt}dt$, $x>0$, $\beta>0$ (при заміні $s=xt$)

$\int_0^{+\infty}\frac{\cos\alpha x}{x^\beta}dx = \int_0^{+\infty}\frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty}t^{\beta-1}e^{-xt}\cos\alpha x\,dt\right)dx = \frac{1}{\Gamma(\beta)}\int_0^{+\infty}\left(\int_0^{+\infty}t^{\beta-1}e^{-xt}\cos\alpha x\,dx\right)dt \ominus$

$(\alpha>0,\ 0<\beta<1)$

$I = \int_0^{+\infty}e^{-xt}\cos\alpha x\,dx = \left| \begin{matrix} u=e^{-xt} & dv=\cos\alpha x\,dx \\ du=-te^{-xt}dx & v=\frac{\sin\alpha x}{\alpha} \end{matrix} \right| = \frac{e^{-xt}\sin\alpha x}{\alpha}\Big|_{x=0}^{+\infty} + \frac{t}{\alpha}\int_0^{+\infty}e^{-xt}\sin\alpha x\,dx = \left| \begin{matrix} u=e^{-xt} & dv=\sin\alpha x\,dx \\ du=-te^{-xt} & v=-\frac{\cos\alpha x}{\alpha} \end{matrix} \right| =$

$(\alpha>0,\ t>0)$ $\{\sin 0=0,\ e^{-\infty}=0\}$

$= \frac{t}{\alpha}\left(-\frac{e^{-xt}\cos\alpha x}{\alpha}\Big|_{x=0}^{+\infty} - \frac{t}{\alpha}\int_0^{+\infty}e^{-xt}\cos\alpha x\,dx\right) = \frac{t}{\alpha^2} - \frac{t^2}{\alpha^2}I \Rightarrow \left(1+\frac{t^2}{\alpha^2}\right)I = \frac{t}{\alpha^2} \Rightarrow I = \frac{t}{t^2+\alpha^2}$

$\ominus \frac{1}{\Gamma(\beta)}\int_0^{+\infty}\frac{t^\beta dt}{t^2+\alpha^2} = \cancel{\frac{1}{2\Gamma(\beta)}\int_0^{+\infty}d(t^2+\alpha^2)} \left| \begin{matrix} \frac{t^2}{\alpha^2}=s & t=\alpha\sqrt{s} & s_1=0 \\ dt=\frac{\alpha}{2\sqrt{s}}ds & t^2+\alpha^2=\alpha^2(1+s) & s_2=+\infty \end{matrix} \right| = \frac{1}{\Gamma(\beta)}\int_0^{+\infty}\frac{\alpha^\beta s^{\beta/2}\cdot\frac{\alpha}{2\sqrt{s}}ds}{\alpha^2(1+s)} = \frac{\alpha^{\beta-1}}{2\Gamma(\beta)}B\left(\frac{\beta+1}{2},\frac{1-\beta}{2}\right)$

$= \frac{\alpha^{\beta-1}}{2\Gamma(\beta)}\frac{\pi}{\sin\pi(\frac{\beta+1}{2})}$