

# The Golden Digits National Contest

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**Problem 1.** Let  $n \geq 2$  be an integer. Prove that for any positive real numbers  $a_1, a_2, \dots, a_n$ ,

$$\frac{1}{2\sqrt{2}} \sum_{i=1}^n 2^i a_i^2 \geq \sum_{1 \leq i < j \leq n} a_i a_j.$$

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**Solution.** We prove the inequality by inducting on  $n$ . For  $n = 2$ , the base case, the inequality is equivalent to  $a_1^2 + 2a_2^2 \geq \sqrt{2}a_1a_2$  which is true by the am-gm inequality. Now, assume that the inequality is true for some integer  $n \geq 2$ . We will prove it also holds for  $n + 1$ .

By applying the inequality for the  $n$  terms  $a_2, \dots, a_{n+1}$  it follows that

$$\frac{1}{2\sqrt{2}} \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \geq \sum_{2 \leq i < j \leq n+1} a_i a_j.$$

Therefore, in order to prove the general inequality for  $n + 1$ , it suffices to show that

$$\frac{1}{2\sqrt{2}} \left( 2a_1^2 + \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \right) \geq a_1(a_2 + \dots + a_{n+1}). \quad (\dagger)$$

One may easily observe that  $2 \geq 2^0 + 2^{-1} + \dots + 2^{-(n-1)}$ . Inequality  $(\dagger)$  then follows as such:

$$\frac{1}{2\sqrt{2}} \left( 2a_1^2 + \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \right) \geq \sum_{i=2}^{n+1} \frac{2^{-(i-2)} a_1^2 + 2^{i-1} a_i^2}{2\sqrt{2}} \geq \sum_{i=2}^{n+1} a_1 a_i,$$

by applying the am-gm inequality on each term.

**Remark 1.** More generally, with the same approach, one can prove that

$$y \sum_{i=1}^n x^i a_i^2 \geq \sum_{1 \leq i < j \leq n} a_i a_j,$$

provided that  $x > 1$  and  $2y(x-1) \geq 1/\sqrt{x}$ . In the problem above, we used  $x = 2$  and  $y = 1/2\sqrt{2}$ .

**Remark 2.** We present another possible approach. The inequality is equivalent to

$$\sum_{i=1}^n \left( 1 + \frac{2^i}{\sqrt{2}} \right) a_i^2 \geq \left( \sum_{i=1}^n a_i \right)^2,$$

which is true by the Cauchy-Schwarz inequality, because

$$\sum_{i=1}^n \frac{1}{1 + 2^i/\sqrt{2}} < \sum_{i=1}^{\infty} \frac{1}{1 + 2^i/\sqrt{2}} \approx 0.9927 < 1.$$

**Problem 2.** Let  $n$  be a positive integer. Consider an infinite checkered board. A set  $S$  of cells is *connected* if one may get from any cell in  $S$  to any other cell in  $S$  by only traversing edge-adjacent cells in  $S$ . Find the largest integer  $k_n$  with the following property: in any connected set with  $n$  cells, one can find  $k_n$  disjoint dominoes.

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**Solution.** The answer is  $k_n = \lfloor (n+2)/4 \rfloor$ . Firstly, we show that we can always find  $\lfloor (n+2)/4 \rfloor$  disjoint dominoes in a connected set with  $n$  cells. To do so, we will reinterpret the problem using a graph. Consider an arbitrary connected set  $S$  with  $n$  cells.

Construct a graph  $G$ , where each vertex represents a cell in our connected set and two vertices are connected if and only if their corresponding cells share an edge. Naturally, finding  $k_n$  disjoint dominoes is equivalent to finding  $k_n$  disjoint edges.

Because any cell is adjacent to four other cells of the board, for any vertex  $v \in G$  we have  $\deg v \leq 4$ . Consequently, it suffices to prove the following graph-theoretic claim:

**Claim.** In any connected graph  $G$  with  $n$  vertices and maximal degree  $\Delta \leq 4$ , there exist at least  $k_n = \lfloor (n+2)/4 \rfloor$  disjoint edges.

*Proof.* We will prove this by strong induction on  $n$ . The cases  $n \leq 5$  are trivially true. For  $n \geq 6$ , assume that the claim holds for  $1, 2, \dots, n-1$ . We will show that it also holds for  $n$ .

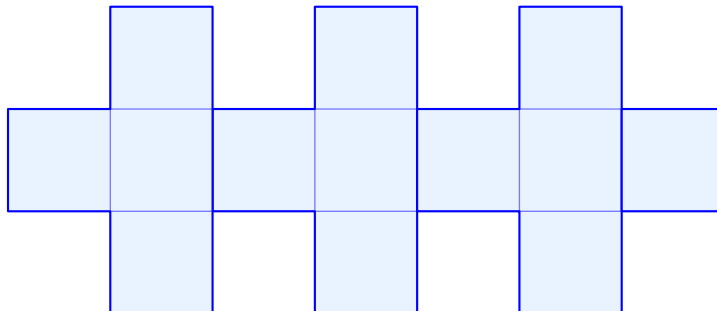
This graph is connected, so we may consider a spanning tree  $T$  of  $G$ . Consider an arbitrary vertex  $v_0$  as the root of  $T$ . As  $|T| \geq 6$  then there exists at least one vertex  $v$  so that  $\text{dist}(v_0, v) \geq 2$ . Hence, let  $\max_{v \in T} \text{dist}(v_0, v) = d \geq 2$ .

Consider a vertex  $u$  so that  $\text{dist}(v_0, u) = d$ . Let  $w$  be the parent of  $u$  and  $L_w$  be the set of leaves emerging from  $w$ . As  $\text{dist}(v_0, w) = d-1 \geq 1$ , then  $w$  also has a parent. Since  $\deg w \leq 4$  then  $|L_w| \leq 3$ . Consider the edge  $uw$  and remove  $L_w \cup \{w\}$  from  $T$ .

We thus form a tree  $T'$  with  $|T| - 1 - |L_w| \geq n - 4$  vertices. Hence, by the inductive hypothesis, in this tree exclusively we may find  $\lfloor (|T'| + 2)/4 \rfloor \geq \lfloor (n+2)/4 \rfloor - 1$  disjoint edges. By also counting  $uw$ , we get the desired  $\lfloor (n+2)/4 \rfloor$  disjoint edges in  $T$ , finishing the proof.  $\square$

Now, it suffices to provide an example for which no more than  $k_n = \lfloor (n+2)/4 \rfloor$  dominoes can be selected. Evidently, it suffices to provide an example for  $n = 4\ell + 1$ , for which  $k_n = \ell$ , because for  $n = 4\ell + \varepsilon$  with  $\varepsilon \in \{0, -1, -2\}$ , it suffices to remove some cells.

For  $n = 4\ell + 1$ , consider a sequence of  $\ell$  T-tetrominoes, followed by a single cell, as seen below.



**Problem 3.** Let  $p$  be a prime number and  $\mathcal{A}$  be a finite set of integers, with at least  $p^k$  elements. Denote by  $N_{\text{even}}$  the number of subsets of  $\mathcal{A}$  with even cardinality and sum of elements divisible by  $p^k$ . Define  $N_{\text{odd}}$  similarly. Prove that  $N_{\text{even}} \equiv N_{\text{odd}} \pmod{p}$ .

**Solution 1.** Let  $\mathcal{A} = \{a_1, \dots, a_n\}$  with  $n \geq p^k$ . We will encode each subset of  $\mathcal{A}$  as an  $n$ -digit code  $(\chi_1, \dots, \chi_n)$  where  $\chi_i = 1$  if  $a_i$  belongs to the subset and  $\chi_i = 0$  otherwise. Evidently, the sum of elements of the subset is  $\chi_1 a_1 + \dots + \chi_n a_n$ .

We will construct an  $n$ -variable polynomial  $f(x_1, \dots, x_n)$  with integer coefficients, whose degree does not exceed  $p^k - 1$  and which satisfies

$$f(\chi_1, \dots, \chi_n) \equiv \begin{cases} 1 \pmod{p} & \text{if } p^k \mid \chi_1 a_1 + \dots + \chi_n a_n; \\ 0 \pmod{p} & \text{otherwise} \end{cases} \quad (\dagger)$$

for any  $\chi_1, \dots, \chi_n \in \{0, 1\}$ . To exhibit a construction, we will consider a sequence of monomials. The term  $x_1$  is written  $a_1$  times, followed by  $x_2$  written  $a_2$  times and so on, ending with  $x_n$  written  $a_n$  times and the constant 1 written  $p^k - 1$  times:

$$\underbrace{(x_1, \dots, x_1)}_{a_1 \text{ times}}, \underbrace{(x_2, \dots, x_2)}_{a_2 \text{ times}}, \dots, \underbrace{(x_n, \dots, x_n)}_{a_n \text{ times}}, \underbrace{(1, \dots, 1)}_{p^k - 1 \text{ times}},$$

for a total of  $N := a_1 + \dots + a_n + p^k - 1$  monomials. Then, for each  $1 \leq i \leq N$ , let  $g_i(x_1, \dots, x_n)$  be the  $i$ -th term of this monomial sequence. Consider every  $(p^k - 1)$ -tuple of polynomials of the form  $g_i$ , take their product and sum everything in order to get

$$f(x_1, \dots, x_n) := \sum g_{i_1}(x_1, \dots, x_n) \cdots g_{i_{p^k-1}}(x_1, \dots, x_n),$$

for every  $1 \leq i_1 < \dots < i_{p^k-1} \leq N$ . Evidently, this polynomial has degree  $d \leq p^k - 1$ . Moreover, given any  $\chi_1, \dots, \chi_n \in \{0, 1\}$ , then there are precisely  $\chi_1 a_1 + \dots + \chi_n a_n + p^k - 1$  ones among the numbers  $g_i(\chi_1, \dots, \chi_n)$  so

$$f(\chi_1, \dots, \chi_n) \equiv \binom{\chi_1 a_1 + \dots + \chi_n a_n + p^k - 1}{p^k - 1} \pmod{p},$$

which satisfies the congruence  $(\dagger)$  due to Lucas' theorem. This construction is satisfactory.

Let  $C_{u_1, \dots, u_n}$  be the coefficient of the term  $x_1^{u_1} \cdots x_n^{u_n}$  in the expansion of  $f$ . For the sake of brevity, in what follows, any instance of  $\chi_i$  will denote a number in  $\{0, 1\}$ . The main observation is

$$N_{\text{even}} - N_{\text{odd}} \equiv \sum_{\chi_1, \dots, \chi_n} (-1)^{\chi_1 + \dots + \chi_n} f(\chi_1, \dots, \chi_n) \pmod{p}.$$

We will denote  $\chi_1 + \dots + \chi_n$  by  $\Sigma \chi_i$ . Observe that the right-hand side is equal to

$$\begin{aligned} \sum_{\chi_1, \dots, \chi_n} (-1)^{\Sigma \chi_i} \sum_{u_1, \dots, u_n} C_{u_1, \dots, u_n} \chi_1^{u_1} \cdots \chi_n^{u_n} &= \sum_{u_1, \dots, u_n} C_{u_1, \dots, u_n} \sum_{\chi_1, \dots, \chi_n} (-1)^{\Sigma \chi_i} \chi_1^{u_1} \cdots \chi_n^{u_n} \\ &= \sum_{u_1, \dots, u_n} C_{u_1, \dots, u_n} \prod_{i=1}^n \left( \sum_{\chi_i \in \{0, 1\}} (-1)^{\chi_i} \chi_i^{u_i} \right). \end{aligned}$$

Because  $\deg f < p^k \leq n$  then among every  $u_1, \dots, u_n$  there exists at least one index  $i$  such that  $u_i = 0$ , hence the right-hand side is equal to zero, as desired.