The Golden Digits National Contest

1st Edition, February 2024

Problem 1. Determine all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$$f\left(\frac{y}{f(x)}\right) + x = f(xy) + f(f(x)),$$

for any positive real numbers x and y.

Pavel Ciurea

Solution. Rewrite the condition in the statement using the map $y \mapsto y/x$. It then becomes

$$f\left(\frac{y}{xf(x)}\right) - f(y) = f(f(x)) - x. \tag{†}$$

Let $\lambda_x = 1/(xf(x))$. It follows inductively that for any positive integer N we have

$$-\frac{f(y)}{N} < \frac{f\left(y\lambda_x^N\right) - f(y)}{N} = f(f(x)) - x = \frac{f(y) - f\left(y\lambda_x^{-N}\right)}{N} < \frac{f(y)}{N}.$$

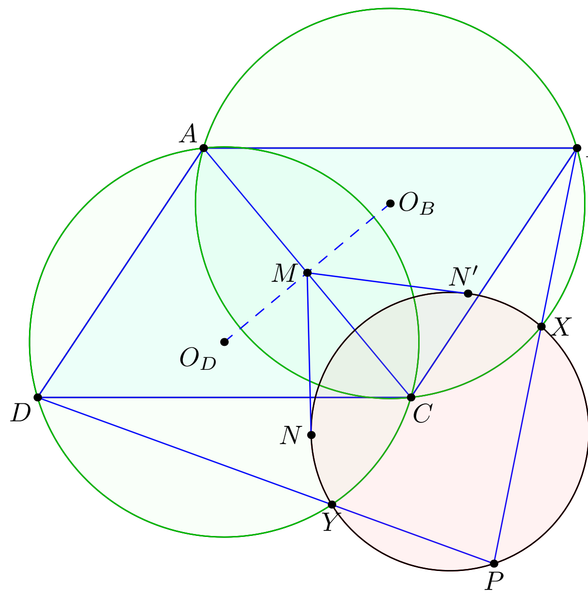
Taking N to be large enough, it follows that $f(f(x)) = x$ so, in particular, f is injective. Recall the equation (†). These properties yield $xf(x) = 1$ for every positive real number x.

To conclude, the only function which satisfies the condition is $f(x) = 1/x$, which works trivially.

Problem 2. Let $ABCD$ be a parallelogram and P a point in the plane. The line BP intersects the circumcircle of ABC again at X and the line DP intersects the circumcircle of DAC again at Y. Let M be the midpoint of AC. The point N lies on the circumcircle of PXY so that MN is a tangent to this circle. Prove that MN and AM have the same length.

David-Andrei Anghel

Solution 1. Let ω be the circumcircle of PXY and ω_B, ω_D the circumcircles of ABC and ADC respectively, with centers O_B, O_D and equal radius R. Also, consider the linear function $f(Q) = AQ^2 - \text{Pow}_\omega(Q)$. We wish to show that $f(M) = 0$, i.e. $f(B) + f(D) = 0$.



Observe that

$$\begin{aligned} f(B) &= AB^2 - BX \cdot BP \\ &= AB^2 - (BP - PX) \cdot BP \\ &= AB^2 - BP^2 + PX \cdot PB \\ &= AB^2 - BP^2 + PO_B^2 - R^2, \end{aligned}$$

where the last equality is obtained from the power of P with respect to ω_B. Therefore, we have $f(B) + f(D) = AB^2 + AD^2 - BP^2 - DP^2 + PO_B^2 + PO_D^2 - 2R^2$. Using the median formula

$$4PM^2 = 2(BP^2 + DP^2) - BD^2 = 2(PO_B^2 + PO_D^2) - O_BO_D^2,$$

which yields

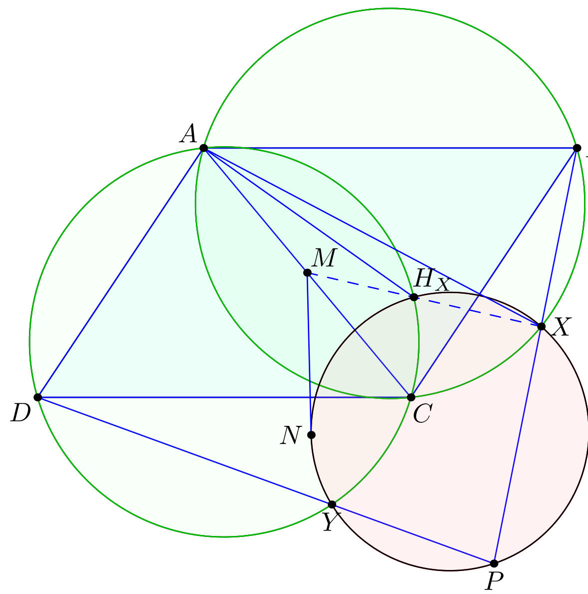
$$f(B) + f(D) = AB^2 + AD^2 + \frac{O_BO_D^2 - BD^2}{2} - 2R^2.$$

Finally, we have $O_BO_D^2 = 4O_BM^2 = 4R^2 - AC^2$, and $2(AB^2 + AD^2) = AC^2 + BD^2$, by the parallelogram formula. Combining these two, we get

$$f(B) + f(D) = AB^2 + AD^2 + \frac{4R^2 - AC^2 - BD^2}{2} - 2R^2 = 0,$$

exactly what we wanted to prove.

Solution 2. Consider H_X, the X-Humpty point in the triangle XAC. It is well-known that H_X lies on the median XM, as well as on the circle ω_D (since ω_B and ω_D are symmetric with respect to AC).



A quick angle chase gives

$$\begin{aligned}\measuredangle H_XYD = \measuredangle H_XAD &= \measuredangle H_XAC + \measuredangle CAD \\ &= \measuredangle H_XXA + \measuredangle ACB \\ &= \measuredangle H_XXA + \measuredangle AXB \\ &= \measuredangle H_XXB,\end{aligned}$$

or equivalently, $\measuredangle H_XYP = \measuredangle H_XXP$, implying that $H_X \in \omega$. But then,

$$MN^2 = \operatorname{Pow}_\omega(M) = MH_X \cdot MX = MA^2,$$

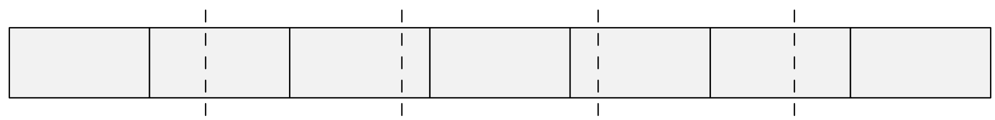
where the last equality is yet another well-known property. This concludes the solution.

Problem. There are m identical rectangular chocolate bars and n people. Each chocolate bar may be cut into two (possibly unequal) pieces at most once. For which m and n is it possible to split the chocolate evenly among all the people?

D. Bugaenko and N. Konstantinov

Solution. It is possible if and only if $m \geqslant n$ or $m/n = 1 - 1/k$ for some integer $k \geqslant 2$.

We begin by showing that these values do work. Let ℓ be the length of the chocolate bars. If $m \geqslant n$ then place the chocolate bars in a line, forming a rectangle of length $m\ell$. Now, simply perform vertical cuts which split this large rectangle into n equal sections.



Because $m \geqslant n$ the distance between any two cuts is $m\ell/n \geqslant \ell$ hence each chocolate bar is cut at most once. Now, assume that $m/n = 1 - 1/k$ for some integer $k \geqslant 2$. Observe that in this case, $k-1$ divides m. Split the chocolate bars into groups of size $k-1$.

Like before, combine the chocolate bars in each group to form a rectangle of length $\ell(k-1)$ and perform vertical cuts which split each such rectangle into k equal sections. Thus, we divide the chocolate evenly into $km/(k-1) = n$ portions.

Note that we cut each chocolate bar at most once. Every chocolate piece created in this manner has length $t \cdot \ell/k$ for some integer $t \geqslant 1$. Thus, if a chocolate bar would be cut at least twice, two of these cuts would be at a distance of at most $(k-2)\ell/k < (k-1)\ell/k$, absurd.

Assume that the condition holds for some m and n with $m < n$. We may also assume, for the sake of simplicity, that each chocolate bar has area one. Then, each person should receive a total area of chocolate equal to $m/n < 1$ so each chocolate bar must be cut.

Suppose that after the cuts, the resulting chocolate pieces have areas $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2m}$. Of course, for each i, the pieces of areas a_i and a_{2m+1-i} make up a chocolate bar. We claim that a_i/a_1 is always an integer. We proceed inductively; of course, this holds for $i = 1$.

Before continuing, observe that $a_{2m} = m/n$. If $a_{2m} > m/n$ then this piece is too large by itself. If $a_{2m} < m/n$, then any other piece of chocolate contributes with an area of at least a_1, which is too much, as $a_{2m} + a_1 = 1 > m/n$. Either way, the chocolate cannot be split evenly.

Now, assume that for $i = 1, \ldots, k$ it is true that a_i/a_1 is an integer. Note that since $a_{k+1} + a_{2m-k} = 1 > m/n$ then a_{2m-k} adds up to m/n with some $a_{i_1}, \ldots, a_{i_l} < a_{k+1}$. Hence, the indices $i_1, \ldots, i_l$ are at most k so due to the induction, $(a_{i_1} + \cdots + a_{i_l})/a_1$ is an integer.

Hence, we may write $a_{2m-k} + t \cdot a_1 = m/n$. Observe further that $m/n = a_{2m} = 1 - a_1$ hence we may conclude that $(t+1) \cdot a_1 = 1 - a_{2m-k} = a_{k+1}$. Therefore, a_{k+1}/a_1 is an integer too, as desired. This completes the induction.

Now, we may write $m/n = a_{2m} = (t-1) \cdot a_1$ and $1 = a_{2m} + a_1 = t \cdot a_1$ for some integer $t \geqslant 2$ to infer that $m/n = 1 - 1/t$, as we wanted to prove. To conclude, the only m and n that work are the ones described above.