6. SOLVING TRANSCENDENT EQUATIONS

Objectives of the paper:

- Learning a way of studying transcendent equations, in order to solve them using the Matlab environment,
- Recapitulation ofcertain properties of elementary mathematical functions,
- Fixing knowledge about solvingtranscendent equations, both numerically and symbolically, using the Matlab programming environment, by studying some examples and solving some problems.

It is recommended to go through Annex M6 before studying paragraphs 6.1 and 6.2.

6.1. Elements regarding the solving of transcendent equations in Matlab

Having the transcendent equation:

`f(x) = 0` ($f:I\subseteq \boldsymbol{R} \rightarrow \boldsymbol{R}$)

Solving this equation in Matlab requires following the two stages specified in the annex M6.

Separation (location of solutions)

In the first step, solutions are located either near their approximate values or in subintervals that each contains one solution. The grapho-analytical method is used to locate solutions. For this purpose, the equation is rewritten as:

`g(x) = h(x)`

If the interval I is not known a priori, then it is determined depending on the definition domains and images of the functions g andh, as well as on the basis of their monotony, periodicity and continuity properties.Next, the g andh functions are then graphically represented over the interval I.

From the graph one can read the intersection points of the graphs of the two functions with the mouse using the Matlabg input function, with one of the syntaxes:

```
[x,y] = ginput
[x,y] = ginput(n)
```

where:

- x is a vector that containsthe abscissae of the "read" points on the graph
- yis a vector that contains the ordinates of the "read" points on the graph
- nis athe numberof"read" points on the graph.

In the first case, the "reading" of the points ends by pressing the ENTER key. In the second case, the termination of the "reading" process of the points is done automatically after the "reading" of the nth point.

Values contained by the vector x represent the approximate values from which the calculation of solutions is started with a predetermined precision.

It is very important that the "reading" of the graph of a point of intersection is done as accurately as possible, in order to minimize the error of the solution - which represents the abscissa of the intersection point - and the calculation time of the solution.

The decomposition of g(x) = h(x) is not unique. As a result, the localization problem can be performed again on a case-by-case basis.

Calculation of solutions with a predetermined precision

For calculating the solutions, it is necessary to define the function f beforehand, in a function file.

The calculation of each solution of the transcendent equation is done using the Matlab function *fzero*. It implements a numerical computation method that is based on a combination of bisection and secant methods and a method of function interpolation.

Depending on the intended purpose, the following syntaxes of the function `fzero` can be used:

- if the intention is only to determine the solution, one of the following syntaxes can be used:

```
x = fzero(file_name,x0)
x = fzero(file_name,x0,options)
```

- if besides determining the solution, the evaluation of the function f for the found solution is also of interest, the function should be used with the syntax:

```
[x,fval]= fzero(file_name,x0)
[x,fval]= fzero(file_name,x0,options)
```

- if the reason for stopping the execution of the numeric method is of interest, one of the following syntaxes should be used:

```
[x,fval,exitflag]= fzero(file_name,x0)
[x,fval,exitflag]= fzero(file_name,x0,options)
```

- if certain additional data is of interest, such as the number of performed iterations, one of the following syntaxes should be used:

```
[x,fval,exitflag,output]= fzero(file_name,x0)
[x,fval,exitflag,output]= fzero(file_name,x0,options)
```

where:

- *`file_name`* represents a string which contains the name of the function file in which the function *f* was defined;
- *`x0`* represents the approximate value of the solution sought, or a vector containing the ends of the subinterval in which the solution was located, `a` and`b`, such that *f(a)f(b)<0*;
- *`options`* represents a structure that contains optimization options for calculating the solution; is an optional argument; optimization options can be changed using the Matlab function *`optimset`*; the *`optimset`* function receives an even number of arguments, in the form of pairs of *option parameter name–option value*; e.g., setting the solution's precision, stored in the Matlab parameter *`TolX`*, can be done by the Matlab command:

```
options=optimset('TolX',precision_value)
```

- *x* representsthe solution calculated with a predetermined precision (either the default precision or one established through the function *optimset*);
- *fval* represents the value of the function f for the calculatedsolution x;
- *exitflag* represents a control value, which is strictly positive if a solution has been found, or strictly negative, otherwise;
- *output* represents a structure that contains the following information: the number of iterations (*iterations*), number of performed evaluations (*funcCount*), as well as the algorithms used to determine the solution (*algorithm*).

Matlab can also be used to solve transcendent equations with parameters. For this purpose, the toolbox *Symbolic Math* is used,which was briefly presented in the paragraph 4.1.

6.2. Examples

Example 6.1: Determine a solution around the value -0.5 with a precision of 10^{-10} for the transcendent equation:

$$\sin(x)+\left(\frac{x}{3}-0.5\right)^3+1=0$$

At the same time also determine the number of performed iterations.

Solution: Because the value at which the search for the solution is started is specified, it is no longer necessary to go through a solution localization stage; we can go directly to the solution's calculation stage. For this purpose, the leftmost member of the equation is first defined in a function file (for example, eqtrans1.m):

```
function f=eqtrans1(x)
f=sin(x)+(x/3-0.5).^3+1;
```

After defining the left most member, the following Matlab program sequence is executed (for example, script file):

```
% setting calculation precision
options=optimset('Tolx',10^(-10));
% calculating the solution
[x,fval,exitflag,output]=fzero('eqtrans1',-0.5,options);
% solution
x

% number of iterations
iter=output.iterations
```

After the sequence above is executed:

```
x =-0.6704
iter=      4
```

Obtained solution is x=-0.6704 after 4 performed iterations.

Comment: To access a member of the structure `options` the following construct is used `options.member_name`.

<u>Example 6.2</u>: Find an approximate solution from the range [–0.5,0.5] for the transcendent equation:

$$(x-1)^4 - tg(x) = 0$$

<u>Solution:</u>

1. The solution separation stage to determine whether there is a single solution within the required range is performed. In this sense, the function on the left-hand side of the equation is graphically represented on the required interval and it is determined how many intersection points of the graph with the Ox axis exist ($g(x)=h(x)$, with $g(x)=f(x)$ and $h(x)=0$):

```
>> x=-0.5:0.1:0.5;
>>plot(x,(x-1).^4-tan(x),x,zeros(size(x)))
```

The graph is shown in the figure 6.1.

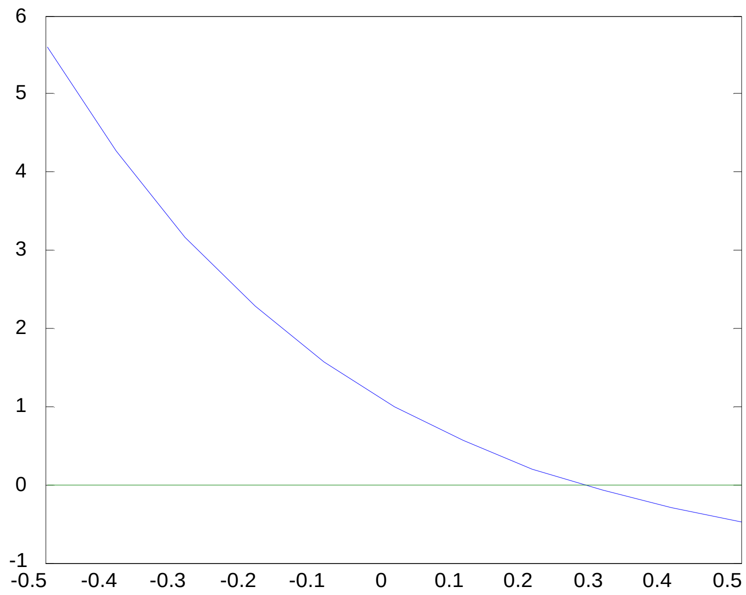

Fig.6.1. *Graph of the function in the left side of the equation in example 6.2.*

It is noted that there is only one equation solution within this range.

2. To determine the solution, define the function in the left-hand side of the equation in a function file:

```
function f=eqtrans2(x)
f= (x-1).^4-tan(x);
```

In order to determine whether it is necessary to "read" of the graph an approximate value of the solution, or if the requested interval can be used to search for it, it should be verified whether the function in the left-hand side of the equation has different signs for the two ends of the interval:

```
>>eqtrans2(-0.5)
  ans =5.6088
>>eqtrans2(0.5)
  ans =-0.4838
```

It can be noted that the function has different signs for the two ends of the interval. Therefore, it is possible to proceed directly to the determination of the solution by using the command:

```
>>x=fzero('eqtrans2',[-0.5,0.5])
```

The obtained solution is:

```
  x =0.2728
```

Example 6.3: Solve the transcendent equation:

$$\cos(\pi x + 0.5) - 2^x + 2.5 = 0$$

Solution.:

1. To locate the solutions of the equation, the equation is rewritten as a form:

$$2^x = \cos(\pi x + 0.5) + 2.5$$

In order to graphically represent the two functions, therepresentation intervals, both on the Ox and on the Oy axis, must be determined.These are deduced from the properties of the two functions. Thus, the function in the left-hand member is the exponential function with the base 2 and has the properties: it is definedon**R**, it is strictly increasing, continuous, the image of the function is the interval $(0,\infty)$. The function in the right member is defined on **R**, is periodic with a period of 2, continuous, limited andthe image of the function is the interval [1.5,3.5].

From these properties we deduce that the two graphs of the functions have at least one intersection point, having the ordinate in the interval [1.5,3.5]. The limits of the graphic representation on the Oy axis must include this interval, and the limits on the Ox axis must include an interval as small as possible $[a,b]$ with$2^a \le 1.5$ and $2^b \ge 3.5$ (for example, a=0 and$b$=2).

The graphs are generated using the following Matlab sequence:

```
% graphical representation interval
x=0:0.05:2;

% graphical representation of the two functions
plot(x,2.^x,'b',x,cos(pi*x+0.5)+2.5,'r:')

% setting the representation limits
axis([0 2 1 4])
```

The graphs are shown in figure 6.2.

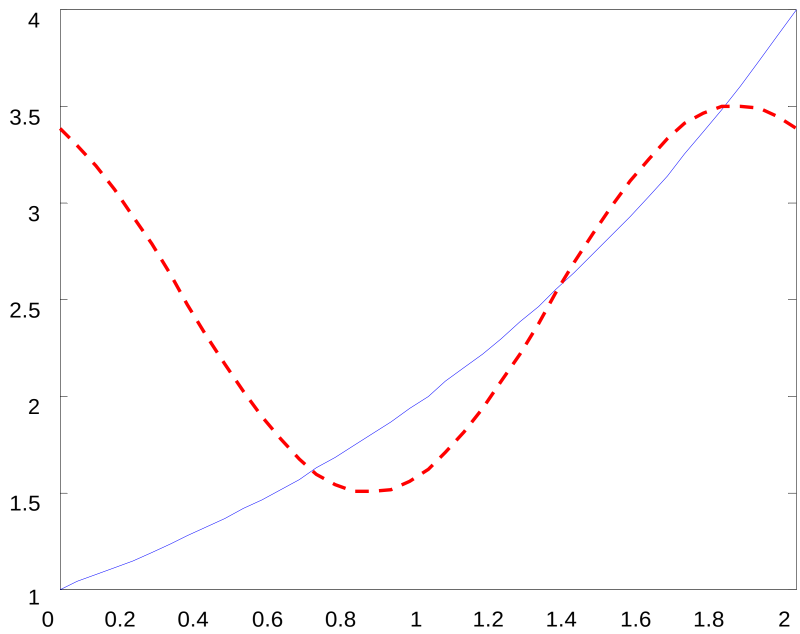

Fig.6.2. *Graphs for the functions in example 6.3.*

It can be seen from the graph that there are three intersection points of the graphs of the two functions. The intersection point coordinates can be "read" from the graph using the Matlab function*ginput*:

```
[x0,y0]=ginput
```

or

```
[x0,y0]=ginput(3)
```

After reading the coordinates of the points with the mouse on the graph (the first reading mode requires pressing the ENTER key after reading the third point), we obtain:

```
x0 =     0.6889     1.3618     1.7995
y0 =     1.6360     2.5658     3.4868
```

2. Before calculating the solutions, the right member of the initial equation is defined (under the form f(x)=0) in a function file:

```
function f=eqtrans3(x)
f=cos(pi*x+0.5)-2.^x+2.5;
```

Solutions are determinedby employing the Matlab function*fzero*3 times, for each solution sending as start value the corresponding x0(i) value:

```
sol1=fzero('eqtrans3',x0(1))
sol2=fzero('eqtrans3',x0(2))
sol3=fzero('eqtrans3',x0(3))
```

The obtained solutions are:

```
sol1 =     0.6888
sol2 =     1.3650
sol3 =     1.8047
```

Example 6.4: Solve the transcendent equation:

$a\cos(u-v)+b\sin(u+v)=0$, with the unknown u.

Solution: Because in the case of symbolic calculation Matlab returns the solution in the most favorable case, prior to solving the equation symbolically it is necessary to identify the compatibility situations.

Discussions:

(I) if a=0 and b=0, equation becomes 0=0, so the solution is $u\in\mathbf{R}$;

(II) if $a\neq 0$ and b=0, equation becomes $a\cos(u-v)=0$, which has an infinity of solutions; these are obtained with the formula:

$$u_k=v+(2\cdot k+)\frac{\pi}{2},\ k\in\mathbf{R}$$

(III) if a=0 and $b\neq 0$, equation becomes $b\sin(u+v)=0$, which has an infinity of solutions, obtained with the formula:

$$u_k=-v+k\cdot\pi,\ k\in\boldsymbol{Z};$$

(IV) if $a\neq 0$ and $b\neq 0$, the following Matlab sequence is executed, and the discussion will continue depending on the result returned by Matlab:

```
% defining the symbolic objects
syms a b u v;
% defining the left member of the equation written under the form
% f(x)=0
f=a*cos(u-v)+b*sin(u+v);
% solving the equation in the most favorable case
solu=simplify(solve(f,u))
```

The following expression is obtained:

```
solu = v-atan((a+b*sin(2*v))/cos(2*v)/b)
```

The discussion continues according to the values of the expression denominator obtained:

(IV.1) if $\cos(2\cdot v)\neq 0$, then the solutions of the equation are:

$$u_k=v-arctg\left(\frac{a+b\cdot\sin(2\cdot v)}{b\cdot\cos(2\cdot v)}\right)+k\cdot\pi,\ k\in\boldsymbol{Z}$$

(IV.2) if $\cos(2\cdot v)=0$, we distinguish the following cases, depending on the signs of the sinus and cosine of the parameter v, cases that can be handled by Matlab code:

```
ff=expand(f);

% (a) sin(v)=sqrt(2)/2, cos(v)=sqrt(2)/2
ffa=subs(ff,sin(v),sqrt(2)/2);
ffa=subs(ffa,cos(v),sqrt(2)/2);
solu_a=solve(ffa,u)

% (b) sin(v)=sqrt(2)/2, cos(v)=-sqrt(2)/2
ffb=subs(ff,sin(v),sqrt(2)/2);
ffb=subs(ffb,cos(v),-sqrt(2)/2);
solu_b=solve(ffb,u)

% (c) sin(v)=-sqrt(2)/2, cos(v)=-sqrt(2)/2
ffc=subs(ff,sin(v),-sqrt(2)/2);
ffc=subs(ffc,cos(v),-sqrt(2)/2);
solu_c=solve(ffc,u)

% (d) sin(v)=-sqrt(2)/2, cos(v)=sqrt(2)/2
ffd=subs(ff,sin(v),-sqrt(2)/2);
ffd=subs(ffd,cos(v),sqrt(2)/2);
solu_d=solve(ffd,u)
```

Performing the Matlab sequence above the following is obtained:

```
solu_a = -1/4*pi
solu_b = 1/4*pi
solu_c = -1/4*pi
solu_d = 1/4*pi
```

Therefore, these solutions have been obtained:

$$u_k = \frac{\pi}{4} + k \cdot \pi,\ k \in \boldsymbol{Z},$$

respectively,

$$u_k = -\frac{\pi}{4} + k \cdot \pi,\ k \in \boldsymbol{Z}.$$

6.3. Problems to solve

P6.1. Solve the transcendent equations:

a) $x^2 - 3 = \sin(x) + \sqrt{|x|}$ (The solution will be determined in the vicinity of the value x_0=1.)

b) $e^{-x^3} = \ln(1 - x + \frac{x^2}{3})$ (The solution will be determined in the vicinity of x_0=-3, with the precision of 10^{-6}. Also, display the number of performed iterations.)

c) $2^{-\sin(x)} + 4 - x\ln(x) = 0$ (The solution will be determined in the interval [3.1,5]. It will be checked beforehand if a solution has been separated inside this interval.)

d) $(x-3)^2 + 5 - \cos(|x|) = 0$ (The solution will be determined in an interval of choice.)

P6.2. How many solutions has the equation below? Determine two solutions of distinct absolute values:

$$e^{\cos(x)} = \sin(x) + 1.$$

P6.3. Solve the equation with the unknown x:

$$2b\cos^2(x) + 2a\sin(x)\cos(x) = p\sqrt{a^2 + b^2} + b, \quad a \neq 0$$

Hint: To simplify symbolic expressions, you can use Matlab commands: `simple`, `simplify`.

ANNEX M6. ELEMENTS REGARDING THE SOLVING OF TRANSCENDENT EQUATIONS

M6.1. Transcendentequations

It is called a ***transcendent equation (with the unknown x)*** an equation

$f(x) = 0$ ($f : I \subseteq \mathbf{R} \to \mathbf{R}$)

that is not an algebraic equation.

Solving the above equation is equivalent to determining the zeros of the $f$function.

A transcendent equation can have a finite number of solutions, an infinity ofsolutions or no solution.

Solving transcendent equations using numerical methods involves the following two steps:

I. ***separating (localizing)*** the solutions, i.e decomposing the search interval$I$ into a subinterval partition so that each subinterval contains at most a solution;

II. **calculation of solutions with a predetermined precision**, usually starting from their approximate values.

The first step can be done by various methods, for example, by using ***the grapho-analytical method***. This method is based on the rewriting of the transcendent equation in the form $g(x) = h(x)$, g, h:$I \to \mathbf{R}$, the graphical representation of the two functions$g$ and$h$ and the use of their properties of monotony, periodicity and continuity. The solutions of the transcendent equation are the abscissae of the intersection points of the graphs of the functions$g$ and$h$. There are no "recipes" for decomposition$f(x)=g(x)-h(x)$; the decomposition is based on the ability and experience of the user. "Experience" means, among other things, knowledge of the aspects of paragraph M6.2 below.

M6.2. Some properties of elementary functions

Table M6.1. synthesizes certain properties of elementary functions.

Table M6.1. ***Properties of elementary functions.***

Function name	Function expression(f(x))	Domain of definition	Image	Properties
identity function	f(x)=x	***R***	***R***	○strictly increasing ○continuous ○the graph is the first bisector
affine function	f(x)=a·x+b, a,b∈***R,*** a≠0	***R***	***R***	○strictly increasing, if a>0; strictly decreasing, if a<0 ○continuous ○the graph is a straight line
second-degree polynomial function	$f(x)=a\cdot x^2+b\cdot x+c$, a,b,c∈***R,*** a≠0	***R***	$\left[-\frac{\Delta}{4\cdot a},\infty\right)$, if a>0 / $\left(-\infty,-\frac{\Delta}{4\cdot a}\right]$, if a<0, $\Delta=b^2-4\cdot a\cdot c$	○is not monotonous, except on portions ○continuous ○graph is a parabola with thevertex $\left(-\frac{b}{2\cdot a},-\frac{\Delta}{4\cdot a}\right)$
third-degree polynomial function	$f(x)=x^3$	***R***	***R***	○strictly increasing ○continuous
radical function	$f(x)=\sqrt[n]{x}$ or $g(x)=\sqrt[2\cdot n-1]{x}$, $n\in\boldsymbol{N}$, $n\geq 2$	[0,∞) for f, ***R*** for g	[0,∞) for f, ***R*** for g	○strictly increasing ○continuous
power function	$f(x)=x^a$, a∈***R******	(0,∞)	(0,∞)	○strictly increasing, ifa>0; strictly decreasing, if a<0 ○continuous
exponential function	$f(x)=a^x$, a∈***R***, a>0, a≠1	***R***	(0,∞)	○strictly increasing, if a>1; strictly decreasing, if a<1 ○continuous
logarithmic function	$f(x)=\log_a(x)$, a∈***R***, a>0, a≠1	(0,∞)	***R***	○strictly increasing, if a>1; strictly decreasing, if a<1 ○continuous
sinefunction	f(x)=sin(x)	***R***	[-1,1]	○periodic (period 2π) ○odd ○continuous
cosine function	f(x)=cos(x)	***R***	[-1,1]	○periodic (period 2π) ○even ○continuous

Table M6.1. ***Properties of elementary functions - continuation***

Function name	Function expression(f(x))	Domain of definition	Image	Properties
tangent function	f(x)=tg(x)	***R*** · *D*, $D=\left\{(2k+1)\frac{\pi}{2} \middle| k \in Z\right\}$	***R***	○periodic (period π) ○ is not monotonous on the domain of definition; is strictly increasing on each maximum subinterval ○ is notcontinuouson the domain of definition; is continuouson each maximum subinterval
cotangent function	f(x)=ctg(x)	***R*** · *D*, $D=\left\{k \cdot \pi \middle| k \in Z\right\}$	***R***	○periodic (period π) ○ is not monotonous on the domain of definition;is strictly decreasing on each maximum subinterval ○ is not continuous on the domain of definition; is continuous on each maximum subinterval
arcsine function	f(x)=arcsin(x)	[-1,1]	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	○strictly increasing ○continuous
arccosine function	f(x)=arccos(x)	[-1,1]	[0,π]	○strictly decreasing ○continuous
arctangent function	f(x)=arctg(x)	***R***	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	○strictly increasing ○continuous
arccotangent function	f(x)=arcctg(x)	***R***	(0,π)	○strictly decreasing ○continuous
absolute value function	f(x)=|x|	***R***	[0,∞)	○is notmonotonous ○continuous