Exercise 1.

Solution. By cutting the sphere with two handles X into two congruent parts such that each contains one half of each handle, we obtain the decomposition $X = A \cup B$, where $A \simeq B \simeq S^1 \vee S^1$ and $A \cap B \simeq S^1 \sqcup S^1 \sqcup S^1$. Since

$$H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

and $H_n(\bigsqcup_{\alpha \in J} X_\alpha) = \bigoplus_{\alpha \in J} H_n(X_\alpha)$ (and the fact that A and B are connected), Mayer-Vietoris yields

$$\begin{aligned} \cdots \to & 0 \to 0 \to H_2(X) \xrightarrow{\partial_*} \\ \to & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} H_1(X) \xrightarrow{\partial_*} \\ \to & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0. \end{aligned}$$

Now, $H_0(X) = \mathbb{Z}$ since X is clearly connected and $H_n(\mathbb{R}P^2) = 0$ for $n \geq 3$ (since all the other omitted terms of the long exact sequence above are zero). Next, a careful examination of the inclusions $A \cap B \to A, A \cap B \to B$ reveals that f is given by $f(1,0,0) = (1,0,1,0), f(0,1,0) = (0,1,0,1), f(0,0,1) = (1,1,1,1)$, hence $\operatorname{im} f \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_2(X) = \ker f \cong \mathbb{Z}$. Another geometric examination of the inclusions shows that $h(a,b,c) = (a+b+c, a+b+c)$ for all $a, b, c \in \mathbb{Z}$, hence $\operatorname{im} \partial_* = \ker h \cong \mathbb{Z} \oplus \mathbb{Z}$. Together with the fact that

$$\ker \partial_* = \operatorname{im} g \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \ker g = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \operatorname{im} f \cong \mathbb{Z} \oplus \mathbb{Z},$$

this gives us (by the standard decomposition of the long exact sequence) a short exact sequence

$$0 \to \ker \partial_* = \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to \operatorname{im} \partial_* = \mathbb{Z} \oplus \mathbb{Z} \to 0$$

which splits, since $\mathbb{Z} \oplus \mathbb{Z}$ is a free (hence projective) $\mathbb{Z}$-module. Therefore

$$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

□

Exercise 2.

Solution. We have $\mathbb{R}P^2 = D^2 \cup_i M$, where M is the Möbius band and $i : S^1 \to M$ is the inclusion to the boundary of M. In fact, S^1 is a deformation retract of M, hence

$$H_n(M) = H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

Using Mayer-Vietoris with $A = D^2$ and $B = M$ and noting that $D^2 \cap M = S^1$ and D^2 is contractible, we have the following long exact sequence:

$$\begin{aligned}\cdots \to & H_2(S^1) = 0 \to H_2(D^2) \oplus H_2(M) = 0 \to H_2(\mathbb{R}P^2) \xrightarrow{\partial_*} \\ \to & H_1(S^1) = \mathbb{Z} \xrightarrow{f} H_1(D^2) \oplus H_1(M) = \mathbb{Z} \to H_1(\mathbb{R}P^2) \xrightarrow{\partial_*} \\ \to & H_0(S^1) = \mathbb{Z} \xrightarrow{g} H_0(D^2) \oplus H_0(M) = \mathbb{Z} \oplus \mathbb{Z} \to H_0(\mathbb{R}P^2) \to 0.\end{aligned}$$

First off, $H_0(\mathbb{R}P^2) = \mathbb{Z}$ since $\mathbb{R}P^2$ is connected (it can be realized as a quotient of S^2 and $H_n(\mathbb{R}P^2) = 0$ for $n \geq 3$ (since all the other omitted terms of the long exact sequence above are zero). Next, the map f is nonzero, hence injective (in fact, it is just multiplication by 2, since i is a degree 2 covering map), since $j \circ i \sim \mathrm{id}_{S^1}$ for some $j : M \to S^1$ and for the same reason g is injective (in fact, it is the diagonal map $g(a) = (a, a)$). Thus we have $0 = \ker f = \operatorname{im} \partial_*$, hence $H_2(\mathbb{R}P^2) = \ker \partial_* = \operatorname{im} 0 = 0$, and there is a short exact sequence $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to H_1(\mathbb{R}P^2) \to 0$. It follows that $H_1(\mathbb{R}P^2)$ is the cokernel of $2\times$, so that $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$. □