Exercise 3.

Solution. Let u, v be any two distinct points of X, so that $(u, v) \in (X \times X) \setminus \Delta$. Since X is Hausdorff, there exist open disjoint sets $U, V \subseteq X$ such that $u \in U, v \in V$. Then $(u, v) \in U \times V \subseteq (X \times X) \setminus \Delta$. Since $U \times V$ is open in the product topology (it belongs to its basis) and $(u, v) \in (X \times X) \setminus \Delta$ were chosen arbitrarily, this means that $(X \times X) \setminus \Delta$ is open, hence Δ is closed. $\square$

Exercise 4.

Solution. Consider the map $f : X \to X \times X$ given by $f(x) = (x, r(x))$ (which is clearly continuous) and let $\Delta_A := \{(a, a) \in A \times A\}$. Since $r(a) = a$ for $a \in A$ by the definition of retraction, it follows that $f(x) \in \Delta_A$ if and only if $x \in A$. Also A is Hausdorff (it is a subset of a Hausdorff space X), so Δ_A is closed by the previous exercise, hence $A = f^{-1}(\Delta_A)$ is closed as well. $\square$

Exercise 5.

Solution. Since $A \hookrightarrow X$ is a cofibration, there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$. Using the previous exercise, this implies that $X \times \{0\} \cup A \times I$ is closed in $X \times I$ (because $X \times I$ is Hausdorff, as a product of two Hasudorff spaces). Therefore $X \times I \setminus (X \times \{0\} \cup A \times I)$ is open, and since the canonical projection $p : X \times I \to X$ is open, the image

$$p(X \times I \setminus (X \times \{0\} \cup A \times I)) = X \setminus A$$

must be open as well. Thus A is closed. $\square$