

**Remark.** All sets are assumed to be topological spaces and all maps are assumed to be continuous unless stated otherwise. The symbol ' $\simeq$ ' will denote that two topological spaces are homeomorphic. The closed unit interval will be denoted by  $I$  or  $J$ .

**Exercise 1.** Prove that being homotopic is an equivalence relation (on the set of continuous maps between topological spaces).

*Solution.* Let  $f, g, k : X \rightarrow Y$  be such that  $f \sim g$ ,  $g \sim k$ , i.e. there exist maps  $h, h' : X \times I \rightarrow Y$  such that  $h(x, 0) = f(x)$ ,  $h(x, 1) = g(x)$ ,  $h'(x, 0) = g(x)$ ,  $h'(x, 1) = k(x)$ .

- Reflexivity: the map  $H_1 : X \times I \rightarrow Y$  defined by  $H_1(x, t) := f(x)$  for all  $t \in I$  is a homotopy between  $f$  and itself.
- Symmetry: the map  $H_2 : X \times I \rightarrow Y$  defined by  $H_2(x, t) := h(x, 1 - t)$  for all  $t \in I$  is a homotopy between  $g$  and  $f$ .
- Transitivity: the map  $H_3 : X \times I \rightarrow Y$  defined by  $H_3(x, t) := \begin{cases} h(x, 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(x, 2t) & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$  is a homotopy between  $f$  and  $k$ .

□

**Exercise 2.** Let  $\simeq$  be an equivalence relation on a topological space  $X$ . Prove that the map  $f : X/\simeq \rightarrow Y$  is continuous iff  $f \circ p : X \rightarrow Y$  is continuous, where  $p : X \rightarrow X/\simeq$  is the canonical quotient projection.

*Solution.* The direction " $\Rightarrow$ " follows from the facts that  $p$  is continuous (in fact, the quotient topology is the final topology with respect to  $p$ ) and the composition of continuous functions is again continuous. For " $\Leftarrow$ ", let  $U \subseteq Y$  be open. Then

$$p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U)$$

is open by continuity of  $f \circ p$ , so  $f^{-1}(U)$  must also be open by the definition of quotient topology and we are done. □

**Exercise 3.** Show that  $D^n/S^{n-1} = S^n$  using the map  $f : D^n \rightarrow S^n$  given by

$$f(x_1, \dots, x_n) = (2\sqrt{1 - \|x\|^2}, 2\|x\|^2 - 1)$$

.

*Solution.* It's easy to see that  $f$  is continuous. Moreover, its restriction to the interior of  $D^n$  gives a bijection to  $S^n \setminus \{(0, \dots, 0, 1)\}$  (the inverse function is given by  $(\mathbf{y}, z) \mapsto \frac{1}{\sqrt{1-z}}\mathbf{y}$ ) and we have  $f(S^{n-1}) = \{(0, \dots, 0, 1)\}$ , so we can define  $f' : D^n/S^{n-1} \rightarrow S^n$  by  $f'([x]) = f(x)$ . Then  $f'$  is a bijection, and by the previous exercise it is continuous. Finally, both  $D^n/S^{n-1}$  and  $S^n$  are compact (Hausdorff) spaces (since both  $D^n$  and  $S^n$  are closed bounded subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ , respectively, and  $S^{n-1} \subseteq D^n$  is closed), so  $f'$  must be a homeomorphism (a general fact for continuous bijections between compact spaces). □

**Exercise 4.** Let  $f : X \rightarrow Y$  and  $M_f = X \times I \cup_{j \times 1} Y$ . Moreover, let  $\iota_X : X \rightarrow M_f$  be given by  $x \mapsto (x, 0)$ ,  $\iota_Y : Y \rightarrow M_f$  be given by  $y \mapsto [y]$  and  $r : M_f \rightarrow Y$  be given by  $r(y) = y$ ,  $r(x, t) = f(x)$ . Show that

- i)  $Y$  is a deformation retract of  $M_f$ ,
- ii)  $r \circ \iota_X = f$ ,
- iii)  $\iota_Y \circ f \sim \iota_X$ .

*Solution.*

- i) Geometrically, the deformation retraction is realized by pushing  $X$  along  $I$  towards  $Y$ .
- ii) We have  $r \circ \iota_X(x) = r(x, 0) = f(x)$  for all  $x \in X$ .
- iii) The required homotopy  $h : X \times J \rightarrow M_f$  is given by  $h(x, s) = [(x, s)]$ .

□

**Exercise 5.** Show that the pair  $(M_f, X)$  has the homotopic extension property (HEP), i.e.  $\iota_X$  is a cofibration.

*Solution.* Let  $g : I \times J \rightarrow \{0\} \times J \cup I \times \{0\}$  be any continuous map such that  $g(0, s) = (0, s)$  and  $g(1, s) = (1, 0)$ . Then the map  $r : M_f \times J \rightarrow X \times \{0\} \times J \cup M_f \times \{0\}$  defined by  $r(x, t, s) = (x, g(t, s))$  and  $r(y, s) = (y, 0)$  is the required retraction. □

**Exercise 6.** The smash product between two based spaces is defined by

$$(C, c_0) \wedge (D, d_0) := (C \times D) / (C \times \{d_0\} \cup \{c_0\} \times D).$$

Show that  $X/A \wedge Y/B = (X \times Y) / (X \times B \cup A \times Y)$ .

*Solution.* Let  $p_1 : X \times Y \rightarrow X/A \times Y/B$  be given by  $p_1(x, y) = ([x], [y])$  and  $p_2 : X/A \times Y/B \rightarrow X/A \wedge Y/B$  be given by  $p_2([x], [y]) = ([x], [y])$ . Then the composition  $p_2 \circ p_1$  is continuous and factors through  $(X \times Y) / (X \times B \cup A \times Y)$ , which implies that the canonical bijection between  $(X \times Y) / (X \times B \cup A \times Y)$  and  $X/A \wedge Y/B$  is continuous (using exercise 2). Using the definition of quotient topology several times, it can be shown that this bijection is also open, hence a homeomorphism. □

**Exercise 7.** Let  $A = \{\frac{1}{n} \cup \{0\}\} \subseteq \mathbb{R}$ . Show that  $(I, A)$  does not have the HEP, i.e. the inclusion  $A \hookrightarrow I$  is not a cofibration.

*Solution.* If  $A \times J \cup I \times \{0\}$  was a retract of  $I \times J$ , the retraction would have to preserve connected subsets. But  $A \times J \cup I \times \{0\}$  is locally connected while  $I \times J$  is not, a contradiction. □