Exercise 1.

Solution. For any $b \in B$, we have

$$j(b - pj(b)) = jb - jp(jb) = 0$$

by the assumption, hence $b - pj(b) \in \ker j = \operatorname{im} i$. Since i is injective, this allows us to define $q(b) := i^{-1}(b - pj(b))$. Then we have

$$qi(a) = i^{-1}(i(a) - pji(a)) = i^{-1}(i(a)) = a$$

(since $ji = 0$) for all $a \in A$, which shows that $qi = \mathrm{id}_A$. Also, for any $b \in B$, we have

$$(iq + pj)(b) = i(i^{-1}(b - pj(b))) + pj(b) = b,$$

which shows that $iq + pj = \mathrm{id}_B$ and we are done. $\square$

Exercise 2.

Solution.

(1) Let a, b, c be as in the formula and let $[c'] = [c] \in H_n(C_*)$. Then there exists some $d \in C_{n+1}$ such that $c' = c + \partial d$ and since g is surjective, there must also exist some $b' \in B_{n+1}$ such that $g(b') = d$. Then we have

$$g(b + \partial b') = g(b) + g\partial(b') = c + \partial g(b') = c + \partial d = c'$$

and

$$f(a) = \partial b = \partial b + \partial\partial b' = \partial(b + \partial b'),$$

which implies $\partial_*[c'] = [a] = \partial_*[c]$.

(2) If $[c] \in \operatorname{im} g_*$ then there is some $b \in \ker Z_n(B)$ such that $g(b) = c$, so that

$$\partial[c] = [f^{-1}(\partial b)] = [f^{-1}(0)] = [0],$$

which shows that $\operatorname{im} g_* \subseteq \ker \partial_*$.
Conversely, suppose that $\partial_*[c] = 0$, i.e. we have $b \in B_n, a \in A_{n-1}$ such that $g(b) = c, f(a) = \partial b$ and $[a] = 0$. Then there must exist $a' \in A_n$ such that $a = \partial a'$, which implies $\partial b = f\partial a' = \partial f(a')$, hence $\partial(b - f(a')) = 0$ and $b - f(a') \in Z_n(B)$. Also $g(b - f(a')) = g(b) = c$, which shows that $g_*[b - f(a')] = [c]$. Thus $\operatorname{im} g_* \supseteq \ker \partial_*$.

(3) Let $a \in Z_n(a)$. Then $f_*[a] = 0$ iff $f(a) = \partial b$ for some $b \in B_{n+1}$ iff $[a] = \partial_*(g_*[b])$, which shows that $\ker f_* = \operatorname{im} \partial_*$ (the last equivalence holds because $\partial g(b) = g\partial(b) = gf(a) = 0$, hence $g(b) \in Z_{n+1}(C)$).

$\square$