Notes on the algorithms used in artemide ver.3.

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I. CONVOLUTION AND GRIDS FOR DISTRIBUTIONS WITH TWIST-2 INPUT

The convolution has the general form

$$[C \otimes f](x, b, \mu) = \sum_f \int_x^1 \frac{dy}{y} C_{f,f'}\left(\frac{x}{y}, b, \mu\right) f_{f'}(y, \mu). \tag{1.1}$$

Here the μ is internal parameter specified in the model. Furthermore, In the grid I save the function multiplied by x I get

$$\begin{aligned} x[C \otimes f](x, b, \mu) &= \sum_f \int_x^1 dy \frac{x}{y} C_{f,f'}\left(\frac{x}{y}, b, \mu\right) f_{f'}(y, \mu) \\ &= \sum_f \int_x^1 dy \frac{x}{y} C_{f,f'}(y, b, \mu) f_{f'}\left(\frac{x}{y}, \mu\right) \\ &= \sum_f \int_x^1 dy C_{f,f'}(y, b,) F_{f'}\left(\frac{x}{y}, \mu\right), \end{aligned} \tag{1.2}$$

where $F(x) = xf(x)$ is the function stored in LHAPDF grids.

One needs a grid for this convolution in (x,b).

For it I use the algorithm from QCDnum. The coefficient function is presented as

$$C_{f,f'}(x, b, \mu) = \sum_{n=0} \sum_{k=0}^{n} a_s^n \mathbf{L}^k C_{f,f'}^{(n,k)}(x). \tag{1.3}$$

The PDF is interpolated over the logarithmical grid

$$\text{X-grid}: \quad [B_x, N_x], \quad x_i = 10^{-B_x + \Delta_x i}, \text{ with } \Delta_x = \frac{B_x}{N_x}, \ i = 0, ..., N_x. \tag{1.4}$$

Such, grid span logarithmically $x \in [10^{-B}, 1]$. Over this grid I use qubic Lagrange interpolation

$$F(x) = \sum_{i=0}^{N_x - 1} \theta(x_i < x < x_{i+1}) \sum_{r=-1,0,1,2} P_{i,r}(x) F(x_{i+r}), \tag{1.5}$$

where P is the Lagrange polynomial over the logarithmic grid

$$P_{i,r}(x) = \prod_{\substack{l=-1,0,1,2 \\ l \neq r}} \frac{\log_{10} x - \log_{10} x_{i+l}}{\log_{10} x_{i+k} - \log_{10} x_{i+l}} = \prod_{\substack{l=-1,0,1,2 \\ l \neq r}} \left(\frac{\log_{10} \frac{x}{x_{i+k}}}{\Delta(k-l)} + 1 \right), \tag{1.6}$$

The end segments are approximated by square polynomials, i.e. if $i = 0$ $l \in 0, 1, 2$ [+if $k < 0 \to 0$] if $i = N_x - 1$ $l \in -1, 0, 1$ [+if $k > 1 \to 0$].

Now the function f can be represented as

$$F(x) = \sum_{i=0}^{N_x} F(x_i) W_i(x), \tag{1.7}$$

where

$$W_i(x) = \sum_{k=-2,-1,0,1} \theta(x_{i+k} < x < x_{i+k+1}) P_{i+k,k}(x). \tag{1.8}$$

- the function W has finite support $x_{i-2} < x < x_{i+2}$ and looks like a symmetric bump , with max at $x = x_i$
- For $x = x_i$ (node) one has $W_i(x_j) = \delta_{ij}$

Therefore, the convolution can be presented as

$$[C \otimes f](x, b, \mu) = \sum_{f'} \sum_{n=0} \sum_{k=0}^{n} \sum_{i=0}^{N_x-1} a_s^n \mathbf{L}^k F_{f'}(x_i, \mu) \mathfrak{T}_{ff'}^{(n,k)}(x, x_i), \tag{1.9}$$

where

$$\mathfrak{T}_{ff'}^{(n,k)}(x, x_i) = \int_x^1 dy C_{ff'}^{(n,k)}(y) W_i\left(\frac{x}{y}\right). \tag{1.10}$$

For the $x = x_j$ $\in$grid one has

$$[C \otimes f](x_i, b, \mu) = \sum_{f'} \sum_{n=0} \sum_{k=0}^{n} \sum_{j=0}^{N_x-1} a_s^n \mathbf{L}^k \mathfrak{T}_{ff';ij}^{(n,k)} f_{f'}(x_j, \mu), \tag{1.11}$$

where

$$\mathfrak{T}_{ff',ij}^{(n,k)} = \int_{x_i}^1 dy\ C_{ff'}^{(n,k)}(y) W_j\left(\frac{x_i}{y}\right). \tag{1.12}$$

Note that since the function W has restrictired support one has

$$\mathfrak{T}_{ff',ij}^{(n,k)} = \int_{\max(x_i/x_{j+2}, x_i)}^{\min(1, x_i/x_{j-2})} dy C_{ff'}^{(n,k)}(y) W_j\left(\frac{x_i}{y}\right). \tag{1.13}$$

The matrix has the following form

$$\begin{pmatrix}
\mathfrak{T}_{00} & \mathfrak{T}_{01} & \mathfrak{T}_{02} & \dots & \mathfrak{T}_{0(N-3)} & \mathfrak{T}_{0(N-2)} & \mathfrak{T}_{0(N-1)} & \mathfrak{T}_{0N} \\
\color{red}{\mathfrak{T}_{10}} & \color{red}{\mathfrak{T}_{11}} & \color{red}{\mathfrak{T}_{12}} & \dots & \color{red}{\mathfrak{T}_{1(N-3)}} & \mathfrak{T}_{1(N-2)} & \mathfrak{T}_{1(N-1)} & \mathfrak{T}_{1N} \\
0 & \color{red}{\mathfrak{T}_{10}} & \color{red}{\mathfrak{T}_{11}} & \dots & \color{red}{\mathfrak{T}_{1(N-4)}} & \mathfrak{T}_{2(N-2)} & \mathfrak{T}_{2(N-1)} & \mathfrak{T}_{2N} \\
0 & 0 & \color{red}{\mathfrak{T}_{10}} & \dots & \color{red}{\mathfrak{T}_{1(N-5)}} & \mathfrak{T}_{3(N-2)} & \mathfrak{T}_{3(N-1)} & \mathfrak{T}_{3N} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & \color{red}{\mathfrak{T}_{10}} & \mathfrak{T}_{(N-2)(N-2)} & \mathfrak{T}_{(N-2)(N-1)} & \mathfrak{T}_{(N-2)N} \\
0 & 0 & 0 & \dots & 0 & \mathfrak{T}_{(N-1)(N-2)} & \mathfrak{T}_{(N-1)(N-1)} & \mathfrak{T}_{(N-1)N} \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \mathfrak{T}_{NN}
\end{pmatrix}, \tag{1.14}$$

where by red I show the part which has a pattern.

The coefficient function has three parts

$$C(x) = C_\delta(x) + C_S(x) + C_R(x), \tag{1.15}$$

where

$$C_\delta(x) = c_\delta \delta(1-x), \qquad C_S(x) = c_S[g(x)]_+, \tag{1.16}$$

and C_R is a regular part. Clearly the contribution of C_δ to $\mathfrak{T}$ is

$$c_\delta \delta_{ij} \to \mathfrak{T}_{ff',ij}^{(n,k)} \tag{1.17}$$

The regular part contributes

$$\int_{x_i/x_{j+2}}^{\min(1, x_i/x_{j-2})} dy C_R(y) W_j\left(\frac{x_i}{y}\right) \to \mathfrak{T}_{ff',ij}^{(n,k)} \tag{1.18}$$

Finally, the plus-part contributes

$$\begin{aligned}
&\int_{x_i/x_{j+2}}^{\min(1,x_i/x_{j-2})} dy C_S(y) W_j\left(\frac{x_i}{y}\right) = \int_{x_i/x_{j+2}}^{\min(1,x_i/x_{j-2})} dy c_S [g(y)]_+ W_j\left(\frac{x_i}{y}\right) \\
&= \int_{x_i/x_{j+2}}^{\min(1,x_i/x_{j-2})} dy c_S g(y)_+ (W_j\left(\frac{x_i}{y}\right) - W_j(x_i)) - c_S \int_0^{x_i/x_{j+2}} dy g(y) W_j(x_i) \\
&= \int_{x_i/x_{j+2}}^{\min(1,x_i/x_{j-2})} dy c_S g(y)_+ (W_j\left(\frac{x_i}{y}\right) - \delta_{ij}) - c_S \int_0^{x_i/x_{j+2}} dy g(y) \delta_{ij} \to \mathfrak{T}^{(n,k)}_{ff',ij},
\end{aligned} \tag{1.19}$$

where it is used that $W_j(x_i) = \delta_{ij}$.