

GDSC: Algorithms & Data Structures

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1. Preamble

1.1 Credits

The further work is mostly based on the **Algorithms & Data Structures** course held by Professor **Sergey Kopelevich** in **Higher School of Economics, Saint Petersburg, Applied Mathematics & Computer Science Bachelor's Program**, 1-3 semesters, 2021-2022.

The materials:

1. [SPb HSE, 1st-2nd semesters, Fall 2021/22, Algorithms Lectures Abstract](#)
2. [SPb HSE, 3rd semester, Fall 2021/22, Algorithms Lectures Abstract](#)

Many thanks to Professor **Sergey Kopelevich**!

2. Big-O Notations

2.1 Definitions of $O, o, \Omega, \omega, \Theta$

2.2 Asymptotic types: linear, quadratic, polylog, exponential

For now refer to **GDSC Competitive Programming Abstract (topics 1-5), Fall 2023**.

3. Basic data structures

3.1 Arrays, doubly linked list, singly linked list

3.2 `std::vector` and how it works internally

3.3 stack, queue, deque

3.4 Keeping minimum for $O(1)$: min-stack, min-queue implementation

For now refer to **GDSC Competitive Programming Abstract (topics 1-5), Fall 2023**.

4. Master Theorem

4.1 Master Theorem

Algorithms that are written in a recursive manner oftentimes utilize *divide-and-conquer* technique which implies division of the task into smaller subtasks that are processed by further recursive calls of the algorithm; once the subtask is small enough it is considered as a base case and processed manually. Some examples include **Merge sort algorithm**, **Binary search tree traversal**, etc.

For such algorithms we need to define their asymptotics. **Master Theorem** is a generalized method that yields asymptotically tight bounds for divide and conquer algorithms [wiki].

Theorem 4.1. Master Theorem

Consider the following recurrence relation: $T(n) = a \cdot T(\frac{n}{b}) + f(n)$ where $f(n) = n^c$ for constants $a > 0, b > 1, c \geq 0$; let $k = \log_b n$ be the recursion depth. Then the following holds:

$$\begin{cases} T(n) = \Theta(a^k) = \Theta(n^{\log_b a}), & a > b^c \\ T(n) = \Theta(f(n)) = \Theta(n^c), & a < b^c \\ T(n) = \Theta(k \cdot f(n)) = \Theta(n^c \cdot \log n), & a = b^c \end{cases} \quad (1)$$

Proof.

$$T(n) = f(n) + a \cdot T(\frac{n}{b}) = f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \dots + a^k f(\frac{n}{b^k}) \quad | \quad f(n) = n^c$$

$$T(n) = n^c + a \cdot (\frac{n}{b})^c + a^2 \cdot (\frac{n}{b^2})^c + \dots + a^k \cdot (\frac{n}{b^k})^c$$

$$T(b) = n^c \cdot (1 + \frac{a}{b^c} + (\frac{a}{b^c})^2 + \dots + (\frac{a}{b^c})^k)$$

Let $q = \frac{a}{b^c}$ and $S(q) = 1 + q + \dots + q^k$:

1. If $q = 1$: $S(q) = 1 + 1 + \dots + 1 = k + 1 = \log_b n + 1 \implies T(n) = \Theta(f(n) \cdot \log n)$.
2. If $q < 1$: $S(q)$ is a geometric progression, thus it is equal to $S(q) = \frac{1 - q^{k+1}}{1 - q} = \text{const} = \Theta(1) \implies T(n) = \Theta(f(n))$.
3. If $q > 1$: $S(q) = q^k + \frac{q^k - 1}{q - 1} = \Theta(q^k) \implies T(n) = \Theta(a^k \cdot (\frac{n}{b^k})^c) = \Theta(a^k)$.

Note: $f(n)$ could be $O(n^c)$; it does not violate the proof ($f(n) = O(n^c) = C \cdot n^c$).

□

4.2 Generalized Master Theorem

Theorem 4.2. Generalized Master Theorem

In the case of $f(n) = n^c \cdot \log_d n$ Master Theorem still holds:

$$T(n) = a \cdot T(\frac{n}{b}) + n^c \cdot \log_d n, \quad a > 0, \quad b > 1, \quad c \geq 0, \quad d \geq 0.$$

$$\begin{cases} T(n) = \Theta(a^k) = \Theta(n^{\log_b a}), & a > b^c \\ T(n) = \Theta(f(n)) = \Theta(n^c \cdot \log^d n), & a < b^c \\ T(n) = \Theta(k \cdot f(n)) = \Theta(n^c \cdot \log^{d+1} n), & a = b^c \end{cases} \quad (2)$$

4.3 Algorithm for recurrence relations

There are also recurrence relations with the following form:

$$T(n) = a_0 \cdot T(n - p_0) + a_1 \cdot T(n - p_1) + \dots + a_k \cdot T(n - p_k) \quad a_i, p_i > 0, \sum p_i > 1$$

There exists an algorithm of how to find asymptotics for such relations:

Theorem 4.3. Algorithm for recurrence relations

Given $T(n)$ of the above form with the above constants, then the following holds:

$$T(n) = \Theta(\alpha^n), \text{ such that } \alpha > 1 \text{ and it is the **only root** of the equation: } \alpha^n = a_0 \cdot \alpha^{n-p_0} + \dots + a_k \cdot \alpha^{n-p_k}$$

Example. Use of Master Theorem

$$1. T(n) = 4 \cdot T\left(\frac{n}{2}\right) + 20 \cdot n^{\frac{3}{2}}$$

$$a = 4, b = 2, c = \frac{3}{2}, f(n) = 20 \cdot n^{\frac{3}{2}} \implies a = 4 > b^c = \sqrt{8} \implies T(n) = \Theta(n^{\log_b a}) = \Theta(n^2).$$

$$2. \text{ Merge sort algorithm recurrence relation: } T(n) = 2 \cdot T\left(\frac{n}{2}\right) + C \cdot n^1$$

$$a = 2, b = 2, c = 1 \implies a = 4 = b^c = 2^1 \implies T(n) = \Theta(n^1 \cdot \log n)$$

Example. Use of Algorithm for recurrence relations

$$1. T(n) = T(n - 1) + 6 \cdot T(n - 2)$$

$$T(n) = \Theta(\alpha), \text{ notice that } \alpha = 3 \text{ satisfies the equation: } 3^n = 3^{n-1} + 6 \cdot 3^{n-2}.$$

$$2. T(n) = T(n - 1) + T(n - 2) + T(n - 3)$$

$$1 = \alpha^{-1} + \alpha^{-2} + \alpha^{-3} \implies \alpha \approx 1.839$$

5. Amortized Analysis

5.1 Definition and general perception of the concept

Remember that we were talking about `std::vector` we said that its `push_back` operation works for an average of $\Theta(1)$. It was due to presence of 2 **distinct states**:

1. vector has enough capacity to fit the next pushed element.
2. vector does not have enough capacity and has to make itself twice bigger (i.e. reallocating a memory buffer of size $2 \cdot N$).

There are definitely more complicated scenarios where number of such *interesting states* is much greater, thus we need a unified approach of how to define this average, or **amortized**, time for an operation.

Definition 5.1. The amortized analysis

The amortized analysis is an approach that allows to determine an average running time (time complexity) of operations o_1, o_2, \dots, o_k in a sequence S over that sequence S .

There are several methods that are referred to as amortized analysis ([wiki](#)). We are going to discuss **Potential method**.

5.2 Amortized analysis: Potential method

Definition 5.2. Potential method

Introduce a **potential function** called $\Phi : \mathbf{S} \rightarrow \mathbf{R}_0^+$ where \mathbf{S} is a set of states of the considered data structure and $\mathbf{R}_0^+ = [0, +\infty)$, i.e. the potential function maps states of the data structure to some non-negative values. As an important edge case for the initial state S_{init} : $\Phi(S_{init}) = 0$.

Let o_i be an individual operation within some sequence of operations named Q . Let S_{i-1} be the state of the considered data structure before the execution of the operation o_i and S_i be the state after the execution of o_i . Let Φ be a chosen potential function, then the amortized time for an operation o_i is defined as follows:

$$T_a(o_i) = T_r(o_i) + (\Phi(S_i) - \Phi(S_{i-1})) \text{ where:}$$

1. $T_a(o)$ - amortized time of the operation.
2. $T_r(o)$ - real/actual time spent on the operation.

Theorem 5.1. Potential method yields an upper bound

The amortized time of a sequence of operations always yields an **upper bound** of the the real/actual time for the considered sequence of operations, i.e.:

$\forall O = o_1, o_2, \dots, o_n$ be a sequence of operations, define:

1. $T_a(O) = \sum_{i=1}^n T_a(o_i)$ - amortized time of the sequence O
2. $T_r(O) = \sum_{i=1}^n T_r(o_i)$ - real time of the sequence O

Then: $T_r(O) \leq T_a(O)$

Proof. As definition for $T_a(O)$ suggests:

$$\begin{aligned} T_a(O) &= \sum_{i=1}^n (T_r(o_i) + \Phi(S_i) - \Phi(S_{i-1})) = T_r(O) + \Phi(S_n) - \Phi(S_0) \\ T_a(O) &= T_r(O) + \underbrace{\Phi(S_n)}_{\geq 0} - \underbrace{\Phi(S_0)}_{=0} \geq T_r(O) \end{aligned}$$

□

Note: Generally speaking, Φ could be any function that satisfies the constraints but the idea is to find such Φ that would represent the closest upper bound of the real/actual time for the considered sequence of operations.

Example. Amortized analysis for `std::vector::push_back` operation

1. Analyse `push_back` operation that expands the vector:

Let $\Phi = 2 \cdot s - N$ where the s is the actual size of a vector and N is its capacity. Remember that the expansion occurs once $s == N$.

Let's consider a `push_back` operation that doubles the size of the vector:

1. $T_r = s + 1 = N + 1$ - because we need to copy all the existing elements in a new buffer + place a new element; here $s = N$ because the extension could only happen once $s = N$.

2. $\Phi_0 = 2 \cdot s - N \geq 0$ - value of the potential function before the operation.

3. $\Phi_1 = 2 \cdot (s + 1) - 2N \geq 0$ - value of the potential function after the operation.

Thus, we have:

$$T_a = T_r + \Phi_1 - \Phi_0 = N + 1 + (2s + 2 - 2N - 2s + N) = N + 1 + (2 - N) = N - N + 3 = \Theta(1).$$

2. Analyse `push_back` operation that does not expand the vector:

Notice that in this case $\Delta\Phi = \Phi_1 - \Phi_0 = \text{const}$ and $T_r = \text{const}$, thus $T_a = \text{const} = \Theta(1)$.

For more examples check the [wiki](#) page.

6. Binary Search

6.1 Definition, use cases

6.1.1 Simplest version

Given a sorted array of size n . We need to find an element x in this array for $O(\log n)$:

```

1 | int find(int l, int r, int x) { // [l,r]
2 |     // a is sorted in the ascending order
3 |     while(l <= r) {
4 |         int m = (l + r) / 2;
5 |         if (a[m] == x) return m;
6 |         else if (a[m] < x) l = m + 1;
7 |         else r = m - 1;
8 |     }
9 |     return -1; // i.e., not found
10| }

```

The time complexity is indeed $O(\log n)$ since on each iteration the $|r - l|$ decreases its value in a half ($|r - l| \rightarrow \frac{|r-l|}{2}$).

The problem of such an implementation is that if there are multiple occurrences of x in the array, the algorithm will return **some index** in a range $[l_{ans}, r_{ans})$ where $\forall i = l_{ans}, \dots, r_{ans} - 1 : a[i] == x$, although we would like to get either of both sides of the range, i.e. either l_{ans} or r_{ans} .

6.1.2 Lower bound / Upper bound

The above problem leads to the following implementations of the binary search algorithm that searches for the forementioned indicies:

1. $\min i : a_i \geq x$, i.e. l_{ans} :

```

1 | int lower_bound(int l, int r, int x) {
2 |     while (r - l > 1) {
3 |         int m = (l + r) / 2;
4 |         if (a[m] < x) l = m; // Notice that we keep the invariant: a[l] < x
5 |         else r = m; // => a[r] >= x
6 |     }
7 |     // (a[l] < m) && (a[r] >= m) => r is a minimum index
8 |     return r;
9 | }

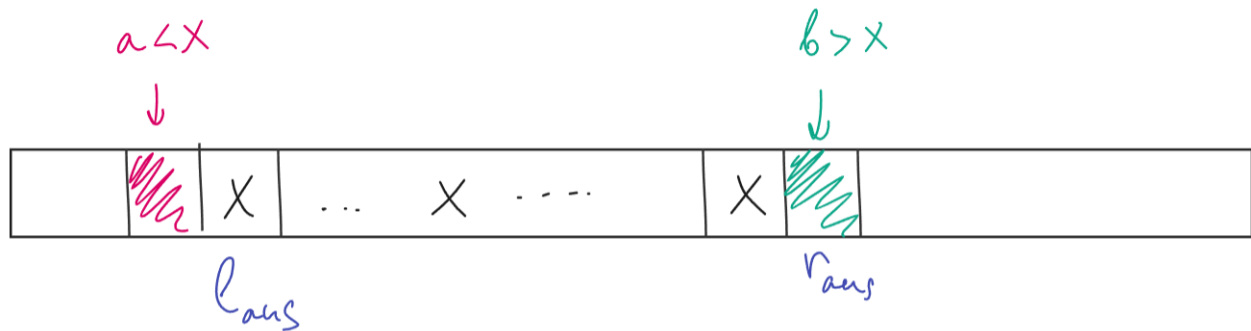
```

2. $\min i : a_i > x$, i.e. r_{ans} :

```

1 | int upper_bound(int l, int r, int x) {
2 |     while (r - l > 1) {
3 |         int m = (l + r) / 2;
4 |         if (a[m] > x) r = m;
5 |         else l = m;
6 |     }
7 |     // Now a[r] > x => l is the last index for which a[l] <= x
8 |     return r;
9 | }

```



6.1.3 STL implementation

In C++ language we are already provided with the template functions that do the same thing:

```

1  /* returns iterator (for int* it is a pointer) */
2  int i = std::lower_bound(a, a + n, x) - a;
3  int i = std::upper_bound(a, a + n, x) - a;
4
5  // For other containers (e.g., std::vector, std::set) that define begin()/end()
   operations use the following:
6  std::lower_bound(
7      std::begin(container),
8      std::end(container),
9      element
10 ) // e.g., for std::vector<T> the return type is std::vector<T>::iterator, i.e.
    pointer to the element

```

6.2 Use of a predicate

We could abstract the implementation even further via searching for a predicate. Predicate is such a function $f : S \rightarrow \{0, 1\}$. Then let's consider the predicate $f(i) = \text{if}(a_i < x) \text{ then } 0 \text{ else } 1$ - in this case the binary search will find such indices l and r that satisfy the following:

1. $l + 1 = r$
2. $f(l) = 0$ (i.e. $a_l < x$)
3. $f(r) = 1$ (i.e. $a_r \geq x$)

```

1  int binary_search(int l, int r, int x) {
2      while (r - l > 1) {
3          int m = (l + r) / 2;
4          if (f(m)) r = m; // invariant: a[r] >= x
5          else l = m; // invariant a[l] < x
6      }
7      return r;
8  }
9
10 // If you want to parameterize predicate as well:
11 // #1: provide a 4th argument as a pointer to a function
12 // (see: https://www.cprogramming.com/tutorial/function-pointers.html)
13 int binary_search(int l, int r, int x, bool(*f)(int)) {
14     ...
15 }
16
17 // #2: template parameter
18 template<typename /* or class */ Func>

```

```

19 | int binary_search(int l, int r, int x, Func f) {
20 |     ...
21 | }
22 | // possible call:
23 | binary_search(l, r, x, []() { /* predicate impl */ });

```

6.3 Correctness

You might already noticed that the functions (arrays are also akin functions, i.e. $a : \{0, 1, \dots, n-1\} : \mathbb{N}$) over which we apply the binary search algorithm are all monotonic functions, i.e. they comply to $x < y : f(x) < f(y)$ (strict monotonically increasing functions; the rest are alike).

Lemma. *The binary search algorithm over some range $[x, y]$ is correct iff the considered function f is monotonic over $[x, y]$.*

6.4 Binary search for functions over \mathbb{R}

It is possible to use the binary search with real numbers as well. Let's consider a problem of *finding square root of x* :

Problem statement: Given $x \in \mathbb{R}$. Find a value $y \in \mathbb{R}$ so that $y^2 == x$ (for us it is $|y^2 - x| < \varepsilon$):

```

1 | double my_sqrt(double x /* never use 'float' */) {
2 |     double l = 0.0;
3 |     double r = x + 1;
4 |
5 |     const double EPS = 1e-9; // 10^(-9)
6 |
7 |     while(r - l > EPS) {
8 |         double m = (r + l) / 2;
9 |         // Preserve invariant: l^2 <= x, r^2 > x
10 |        if (m*m > x) r = m;
11 |        else l = m;
12 |    }
13 |
14 |    return (l + r) / 2;
15 | }

```

In the above notice: if $0 < y < 1$ then $y^2 < y$, and if $y \geq 1$ then $y^2 \geq 1$. Thus, for $0 < x < 1$ the corresponding y is greater than x , i.e. we select $r = x + 1$.

Example. Root of a polynomial $P(x)$

Given a polynomial $P(x)$ of **an odd degree**, i.e. $\deg P = 2k + 1$ with the coefficient for x^{2k+1} be equal to 1. There exists a root $x_0 \in \mathbb{R}$, and we need to find it.

Solution:

We could do that using binary search with any precision ε . First, we need to find points l and r , such that: $P(l) < 0$ and $P(r) > 0$ (e.g., $l = -\infty$, $r = +\infty$ - **MAX_INT** and **MIN_INT** in C++):

```

1 | for (l = -1; P(l) >= 0; l *= 2);
2 | for (r = 1; P(r) <= 0; r *= 2);

```

And finally the root search:

```

1 | while (r - l > EPS) {
2 |     double m = (l + r) / 2;
3 |     if (P(m) < 0) l = m; // P(l) < 0
4 |     else r = m; // P(r) >= 0

```



```
5 ||     }  
6 ||     return (l + r) / 2;
```

Actually we need exactly $k := \frac{r-l}{\varepsilon}$ iterations, thus the while-loop could be changed by the for-loop.

7. Ternary Search

7.1 Definition, use cases

Imagine that we need to find a maximum of a function $f(x)$ in the interval $[l, r]$ but now the function is no longer monotonic. Is there a fast way (i.e. as fast as logarithmic asymptotic as in the binary search algorithm) of finding a point x_0 such that $f(x_0) \rightarrow \max$?

The answer is yes if our function satisfies some constraint.

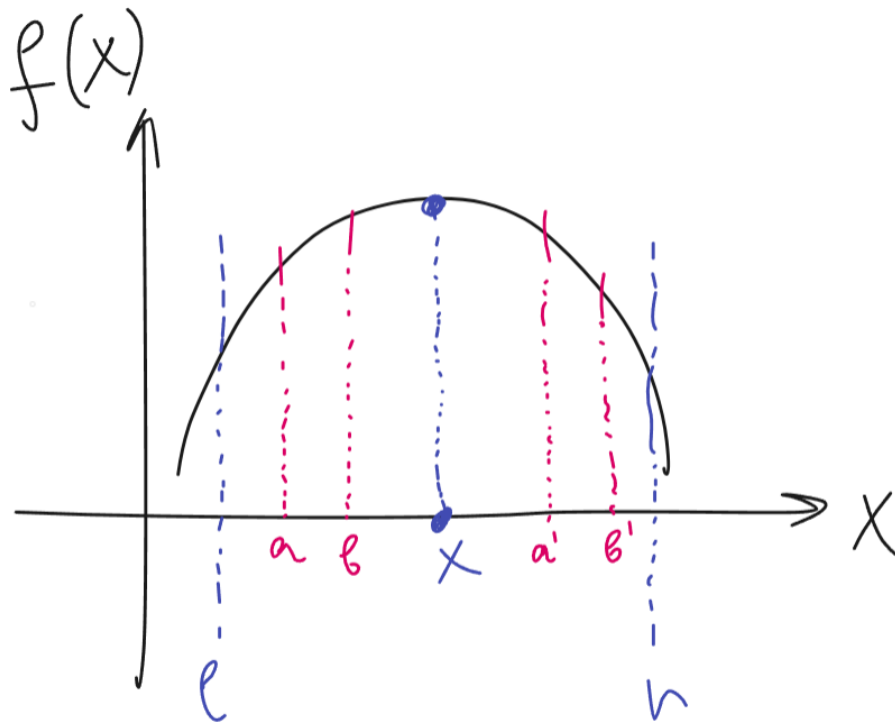
Definition 7.1. Ternary search algorithm

Given a function $f(x)$ and a range $[l, r]$ in which the function is [unimodal](#) and convex, and we need to find maximum in the range (maximum for convex functions, minimum for concave functions).

Since the function is unimodal over $[l, r]$ the following holds:

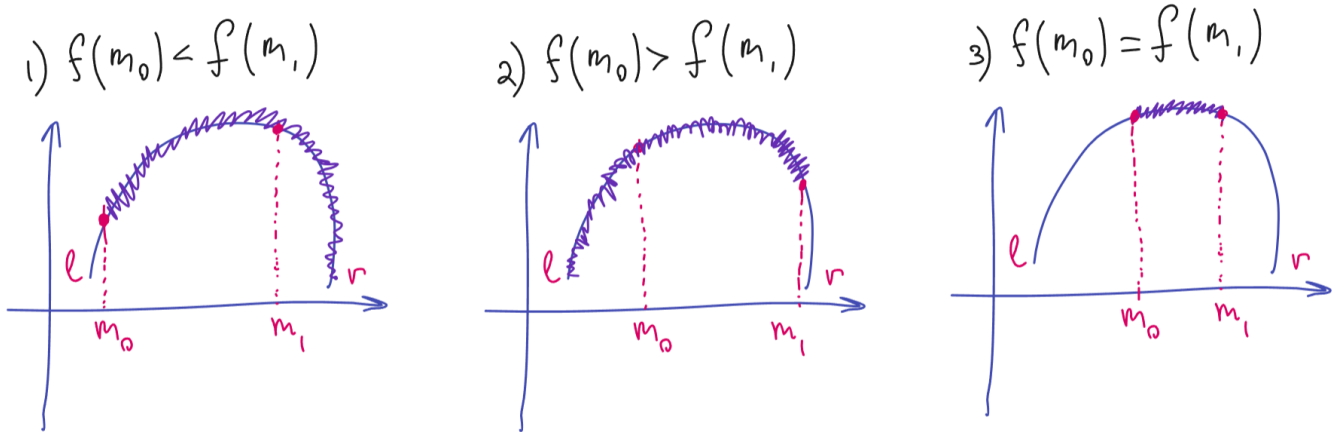
1. $\forall a, b : l \leq a < b \leq x_0 : f(a) < f(b)$
2. $\forall a, b : x_0 \leq a < b \leq r : f(a) > f(b)$

Where x_0 is a point where $f(x) = \max$



For the algorithm let's consider two points m_0 and m_1 such that $l < m_0 < m_1 < r$, there are several cases:

1. $f(m_0) < f(m_1)$, then the maximum of the function f cannot be located in the interval $[l, m_0]$ since there is an interval $[m_0, r]$ on which there are points where function takes greater values.
2. $f(m_0) > f(m_1)$, then the maximum of the function f cannot be located in the interval $[m_1, r]$ since the interval $[l, m_1]$ contains points in which the function takes greater values.
3. $f(m_0) = f(m_1)$, in this case the maximum will be located in the interval $[m_0, m_1]$ but for the implementation simplify this condition branch can be merged with any of the above.



As for the choice of the points m_0 and m_1 we need to select something that would truncate the remaining searching area in a manner that would give us a logarithmic asymptotic. Let's make m_0 be the first 3rd of the interval and m_1 be the second 3rd of the interval:

$$1. m_0 = l + \frac{r-l}{3}$$

$$2. m_1 = l + 2 \cdot \frac{r-l}{3} = \frac{3l+2r-2l}{3} = \frac{2r+l}{3} = \frac{3r-r+l}{3} = r - \frac{r-l}{3}$$

Notice that according to the above statement the searching interval becomes $\frac{2}{3}$ of the initial interval, thus:

$$T(n) = T\left(\frac{2}{3}n\right) + 1$$

According to **Master Theorem**:

$$a = 1, b = \frac{2}{3}, c = 0, f(n) = n^c = 1 \implies a = b^c \implies \Theta(n^c \cdot \log n) = \Theta(\log n) \implies$$

$$T(n) = T\left(\frac{2}{3}n\right) + 1 = \Theta(\log n)$$

7.2 Implementation

Here is the implementation for a function which is an array:

```

1  int ternary_search(vector<int>& arr) {
2      // 'arr' is a unimodal function N -> Z
3      int l = 0;
4      int r = arr.size();
5
6      while(r - l > 2 /* not 1 because m0 = l, m1 = r possible */) {
7          int m0 = l + (r - l) / 3;
8          int m1 = r - (r - l) / 3;
9
10         if (arr[m0] < arr[m1]) {
11             l = m0;
12         }
13         else {
14             r = m1;
15         }
16     }
17
18     int ans = l;
19
20     if (r < arr.size() && arr[r] > arr[ans]) {
21         ans = r;
22     }
23
24     return ans;

```

25 || }

Here is the implementation for a function f over floating-point numbers:

```
1 | int ternary_search(double l, double r, double (*f)(double)) {
2 |     while (r - l > EPS) {
3 |         double m0 = l + (r - l) / 3;
4 |         double m1 = r - (r - l) / 3;
5 |
6 |         if (f(m0) < f(m1)) {
7 |             l = m0;
8 |         }
9 |         else {
10 |             r = m1;
11 |         }
12 |     }
13 |
14 |     return (l + r) / 2;
15 | }
```