①

a) $f(x,y,z) = 4xy^3 - e^z y^4 + y\ln(x) + x\ln(y)$

$\frac{\partial f}{\partial x} = \frac{d}{}(4xy^3 - e^z y^4 + y\ln(x) + x\ln(y))$

Differentiating with respect to x

$\frac{d}{dx}(4xy^3) = 4y^3$

$\frac{d}{dx}(e^z y^4) = 0$

$\frac{d}{dx}(y\ln(x)) = \frac{y}{x}$

$\frac{d}{dx}(x\ln(y)) = \ln(y)$

$\frac{df}{dx} = 4y^3 + \frac{y}{x} + \ln(y)$

Differentiating with respect to y

$\frac{d}{dy}(4xy^3) = 12xy^2$ $\quad \frac{d}{dy}(\cancel{-}e^z y^4) = -4e^z y^3$ $\quad \frac{d}{dy}(y(\ln(x))) = \ln(x)$

$\frac{d}{dy}(x(\ln(y))) = \frac{x}{y}$

$\frac{df}{dy} = 12xy^2 - 4e^z y^3 + \ln(x) + \frac{x}{y}$

Differentiating with respect to z

$\frac{d}{dz}(-e^z y^4) = -e^z y^4$

b) $f(x,y,z) = \sin(x+2y^2) - e^{4x-y+2z}$

$\frac{d}{dx}(\sin(x+2y^2)) = \cos(x+2y^2)(x+2y^2)' =$
$= \sin(x+2y^2) \cdot 1$

$\frac{d}{dx}(-e^{4x-y+2z}) = -(4+1)e^{4x-y+2z}$

$\frac{df}{dx} = \cos(x+2y^2) - (4+1)e^{4x-y+2z}$

$\frac{d}{dy}(\sin(x+2y^2)) = 4y\cos(x+2y^2)$

$\frac{d}{dx}(-e^{4x-y+2z}) = e^{4x-y+2z}$

$\frac{df}{dy} = 4y\cos(x+2y^2) + e^{4x-y+2z}$

$$\frac{df}{dz} = 0 - 2e^{4x-y+2z} = -2e^{4x-y+2z}$$

$$\frac{d}{dz}(\sin(x+2y^2)) = 0$$

$$\frac{d}{dz}(-e^{4x-y+2z}) = -2e^{4x-y+2z}$$

c) $f(x,y,z) = y\sqrt{x^2+yz}$

~~$\frac{df}{dx} = y\sqrt{x^2+y}$~~

$$\frac{df}{dx} = (y)'\sqrt{x^2+yz} + y(\sqrt{x^2+yz})' = 0 + y\cdot\frac{1\cdot(x^2+yz)'}{2\sqrt{x^2+yz}} = \frac{2xy}{2\sqrt{x^2+yz}} = \frac{xy}{\sqrt{x^2+yz}}$$

$$\frac{df}{dx} = \frac{xy}{\sqrt{x^2+yz}}$$

$$\frac{df}{dy} = (y)'\sqrt{x^2+yz} + y(\sqrt{x^2+yz})' = \sqrt{x^2+yz} + \frac{y(x^2+yz)'}{2\sqrt{x^2+yz}} = \sqrt{x^2+yz} + \frac{yz}{2\sqrt{x^2+yz}} = \frac{2(x^2+yz)+yz}{2\sqrt{x^2+yz}} = \frac{2x^2+2yz+yz}{2\sqrt{x^2+yz}} = \frac{2x^2+3yz}{2\sqrt{x^2+yz}}$$

$$\frac{df}{dy} = \frac{2x^2+3yz}{2\sqrt{x^2+yz}}$$

$$\frac{df}{dz} = (y)'\sqrt{x^2+yz} + y(\sqrt{x^2+yz})' = 0 + \frac{y(x^2+yz)'}{2\sqrt{x^2+yz}} = \frac{y^2}{2\sqrt{x^2+yz}}$$

$$\frac{df}{dz} = \frac{y^2}{2\sqrt{x^2+yz}}$$

d) $f(x,y,z) = \frac{x^2}{y^2+1} - \frac{y^2}{x^3+z^2}$

$$\frac{df}{dx} = \left(\frac{x^2}{y^2+1}\right)' - \left(\frac{y^2}{x^3+z^2}\right)' = \frac{2x}{y^2+1} + \frac{y^2\cdot 3x^2}{(x^3+z^2)^2}$$

$$\frac{df}{dx}\left(\frac{x^2}{y^2+1}\right)' = \frac{(x^2)'(y^2+1) - x^2(y^2+1)'}{(y^2+1)^2} = \frac{2x(y^2+1) - x^2\cdot 0}{(y^2+1)^2} = \frac{2x(y^2+1)}{(y^2+1)^2} = \frac{2x}{y^2+1}$$

$$\frac{df}{dx}\left(\frac{y^2}{x^3+z^2}\right)' = \frac{(y^2)'(x^3+z^2) - y^2(x^3+z^2)'}{(x^3+z^2)^2} = \frac{0 - y^2\cdot 3x^2}{(x^3+z^2)^2} = -\frac{y^2\cdot 3x^2}{(x^3+z^2)^2}$$

$$\frac{\partial f}{\partial y} = \left(\frac{x^2}{y^2+1}\right)' + \left(\frac{-y^2}{x^3+z^2}\right)' =$$

$$= \frac{-2x^2y}{(y^2+1)^2} - \frac{2y}{(x^3+z^2)}$$

$$\frac{\partial f}{\partial y}\left(\frac{x^2}{y^2+1}\right) = \frac{(x^2)'(y^2+1) - x^2(y^2+1)'}{(y^2+1)^2} =$$

$$= \frac{0 - x^2 \cdot 2y}{(y^2+1)^2} = \frac{-2x^2y}{(y^2+1)^2}$$

$$\frac{\partial f}{\partial y}\left(\frac{-y^2}{x^3+z^2}\right) = \frac{(-y^2)'(x^3+z^2) - (-y^2)(x^3+z^2)'}{(x^3+z^2)^2}$$

$$= \frac{-2y(x^3+z^2) - 0}{(x^3+z^2)^2} = \frac{-2y\cancel{(x^3+z^2)}}{(x^3+z^2)^{\cancel{2}}}$$

$$= \frac{-2y}{(x^3+z^2)}$$

$$\frac{\partial f}{\partial z} = \left(\frac{x^2}{y^2+1}\right)' + \left(\frac{-y^2}{x^3+z^2}\right)' =$$

$$= \frac{2y^2z}{(x^3+z^2)^2}$$

$$\frac{\partial f}{\partial z}\left(\frac{x^2}{y^2+1}\right) = 0$$

$$\frac{\partial f}{\partial z}\left(\frac{-y^2}{x^3+z^2}\right) = \cancel{(-y^2)'(x^3+z^2)} -$$

$$= \frac{(-y^2)'(x^3+z^2) - (-y^2)(x^3+z^2)'}{(x^3+z^2)^2} =$$

$$= \frac{0 + y^2(2z)}{(x^3+z^2)^2} = \frac{2y^2z}{(x^3+z^2)^2}$$

(2)

a) $f(x,y,z) = \ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{6}\right)$

$$\boxed{\frac{\partial f}{\partial x} = -\frac{1}{x} - \frac{1}{x+y} + \frac{1}{x}}$$

$$\frac{d}{dx}\left(\ln\left(\frac{x}{6}\right)\right) = \left(\ln\left(\frac{x}{6}\right)\right)' = \frac{1}{\frac{x}{6}}\left(\frac{x}{6}\right)' =$$

$$= \frac{6}{x}\left(\frac{(x)'\cdot 6 - x(6)'}{36}\right) = \frac{6}{x}\cdot\frac{6}{36} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\ln\left(\frac{1}{x+y}\right)\right) = \left(\ln\left(\frac{1}{x+y}\right)\right)' = \frac{1}{\frac{1}{x+y}}\left(\frac{1}{x+y}\right)' =$$

$$= (x+y)\left(\frac{(1)'(x+y) - 1(x+y)'}{(x+y)^2}\right) =$$

$$= (x+y)\cdot\frac{(-1)}{(x+y)^2} = \frac{-1}{x+y}$$

$$\frac{d}{dx}\left(\ln\left(\frac{y}{x}\right)\right) = \left(\ln\left(\frac{y}{x}\right)\right)' = \frac{1}{\frac{y}{x}}\left(\frac{y}{x}\right)' =$$

$$= \frac{x}{y}\left[\frac{(y)'(x) - y(x)'}{x^2}\right] =$$

$$= \frac{x}{y}\left[\frac{0-y}{x^2}\right] = \frac{-xy}{yx^2} = \frac{-1}{x}$$

$$\boxed{\frac{df}{dy} = \frac{1}{y} - \frac{1}{x+y}}$$

$$\frac{d}{dy}\left(\ln\left(\frac{y}{x}\right)\right) = \left(\ln\left(\frac{y}{x}\right)\right)' =$$

$$= \frac{1}{\frac{y}{x}}\left(\frac{y}{x}\right)' = \frac{x}{y}\left(\frac{\overset{1}{(y)'}x - y\overset{0}{(x)'}}{x^2}\right) =$$

$$= \frac{x}{y}\cdot\frac{x}{x^2} = \frac{1}{y}$$

$$\frac{d}{dy}\left(\ln\left(\frac{1}{x+y}\right)\right) = \left(\ln\left(\frac{1}{x+y}\right)\right)' = \frac{1}{\frac{1}{x+y}}\left(\frac{1}{x+y}\right)' =$$

$$= (x+y)\left(\frac{\overset{0}{(1)'}(x+y) - (1)\overset{1}{(x+y)'}}{(x+y)^2}\right) =$$

$$= \frac{(x+y)(-1)}{(x+y)^2} = -\frac{1}{x+y}$$

$$\frac{d}{dy}\left(\ln\left(\frac{x}{6}\right)\right) = 0$$

$$\frac{df}{dz} = 0$$

$$\Rightarrow \quad \frac{df}{dx} = \underbrace{-\frac{1}{x} - \frac{1}{x+y} + \frac{1}{x}}_{(x+y)x} =$$

$$= \frac{-x-\cancel{y}-x+x+\cancel{y}}{(x+y)x} = -\frac{\cancel{x}}{(x+y)\cancel{x}} = -\frac{1}{x+y}$$

$$\nabla f = \left(-\frac{1}{x+y};\ \frac{1}{y} - \frac{1}{x+y};\ 0\right)$$

b) $f(x,y,z) = \sin\left(\frac{z}{z^2+x}\right) - \frac{6x^2+y}{y^2-z^2}$

$$\frac{df}{dx} = -\cos\left(\frac{z}{z^2+x}\right)\cdot\frac{z}{(z^2+x)^2} - \frac{12x}{(y^2-z^2)}$$

$$\frac{d}{dx}\left(\sin\left(\frac{z}{z^2+x}\right)\right) = \cos\left(\frac{z}{z^2+x}\right)\cdot\left(\frac{z}{z^2+x}\right)' =$$

$$= \cos\left(\frac{z}{z^2+x}\right)\left(\frac{\overset{0}{(z)'}(z^2+x) - z\overset{1}{(z^2+x)'}}{(z^2+x)^2}\right) =$$

$$= \cos\left(\frac{z}{z^2+x}\right)\cdot\frac{-z}{(z^2+x)^2} =$$

$$\frac{d}{dx}\left(\frac{-6x^2+y}{y^2-z^2}\right) = \frac{(-6x^2+y)'(y^2-z^2) - (-6x^2+y)\overset{0}{(y^2-z^2)'}}{(y^2-z^2)^2}$$

$$= \frac{-12x\cancel{(y^2-z^2)}}{(y^2-z^2)^2} = \frac{-12x}{(y^2-z^2)}$$

$$\frac{df}{dy} = \frac{-12x^2y - y^2 - z^2}{(y^2-z^2)^2}$$

$$\frac{d}{dy}\left(\sin\left(\frac{z}{z^2+x}\right)\right) = 0$$

$$\frac{d}{dy}\left(\frac{6x^2+y}{y^2-z^2}\right) = \frac{(6x^2+y)'(y^2-z^2)-(6x^2+y)(y^2-z^2)'}{(y^2-z^2)^2} =$$

$$= \frac{(y^2-z^2)-(6x^2+y)\cdot 2y}{(y^2-z^2)^2} =$$

$$= \frac{y^2-z^2-12x^2y-2y^2}{(y^2-z^2)^2} = \frac{-y^2-12x^2y-z^2}{(y^2-z^2)^2}$$

$$\frac{df}{dz} = \cos\left(\frac{z}{z^2+x}\right)\left(\frac{x-z^2}{(z^2+x)^2}\right) - \frac{12x^2z+2yz}{y^2-z^2}$$

$$\frac{d}{dz}\left(\sin\left(\frac{z}{z^2+x}\right)\right) = \left(\sin\left(\frac{z}{z^2+x}\right)\right)' =$$

$$= \cos\left(\frac{z}{z^2+x}\right)\cdot\left(\frac{z}{z^2+x}\right)' =$$

$$= \cos\left(\frac{z}{z^2+x}\right)\left(\frac{(z)'(z^2+x)-z(z^2+x)'}{(z^2+x)^2}\right)$$

$$= \cos\left(\frac{z}{z^2+x}\right)\left(\frac{z^2+x-2z^2}{(z^2+x)^2}\right) =$$

$$= \cos\left(\frac{z}{z^2+x}\right)\left(\frac{x-z^2}{(z^2+x)^2}\right)$$

$$\frac{d}{dz}\left(\frac{6x^2+y}{y^2-z^2}\right) = \frac{(6x^2+y)'(y^2-z^2)-(6x^2+y)(y^2-z^2)'}{(y^2-z^2)^2} =$$

$$= \frac{-(6x^2+y)(-2z)}{y^2-z^2} = \frac{-12x^2z-2yz}{y^2-z^2} =$$

$$\nabla f = \left(-\cos\left(\frac{z}{z^2+x}\right)\cdot\frac{z}{(z^2+x)^2} - \frac{12x}{(y^2-z^2)};\ -\frac{12x^2y-y^2-z^2}{(y^2-z^2)^2};\ \frac{-12x^2z+2yz}{y^2-z^2} + \cos\left(\frac{z}{z^2+x}\right)\left(\frac{x-z^2}{(z^2+x)^2}\right) \right)$$

c) $f(x,y,z) = x2^{y+z} + y3^{x+z} + z5^{x+y}$

$$\frac{df}{dx} = 2^{y+z} + y.3^{x+z}.\ln(3) + z.5^{x+y}.\ln(5)$$

$$\frac{df}{dy} = x.2^{y+z}.\ln(2) + 3^{x+z} + z.5^{x+y}.\ln(5)$$

$$\frac{df}{dz} = x.2^{y+z}.\ln(2) + y.3^{x+z}.\ln(3) + 5^{x+y}$$

$$\nabla f = \left(2^{y+z} + y.3^{x+z}.\ln(3) + z5^{x+y}(\ln(5));\ x2^{y+z}.\ln(2) + 3^{x+z} + z.5^{x+y}.\ln(5),\ x.2^{y+z}.\ln(2) + y3^{x+z}.\ln(3) + 5^{x+y}\right)$$

d) $f(x,y,z) = x\ln(y+z) + y(\ln(x+z)) + z\ln(x+y)$

$$\frac{d}{dx}(x\ln(y+z)) = \cancel{x(\ln(y+z))'\frac{1}{y+z}(y+z)'} =$$

$$= (x)'\ln(y+z) + x(\ln(y+z))' = \ln(y+z) + x.0 = \ln(y+z)$$

$$\frac{d}{dx}(y(\ln(x+z))) = (\overset{0}{y})'(\ln(x+z)) + y(\ln(x+z))' = 0 + y.\frac{1}{x+z}(x+z)' =$$

$$= \frac{y}{x+z}$$

$$\frac{d}{dx}(z(\ln(x+y))) = (\overset{0}{z})'(\ln(x+y)) + z(\ln(x+y))' = z.\frac{1}{x+y}.(x+y)' = \frac{z}{x+y}$$

$$\frac{df}{dx} = \ln(y+z) + \frac{y}{x+z} + \frac{z}{x+y}$$

$$\frac{d}{dy}(x\ln(y+z)) = (\overset{0}{x})'(\ln(y+z)) + x(\ln(y+z))' = 0 + \frac{x}{y+z}.(y+z)' =$$

$$= \frac{x}{y+z}$$

$$\frac{d}{dy}(y\ln(x+z)) = (y)'(\ln(x+z)) + y(\overset{0}{\ln(x+z)})' = \ln(x+z) + 0 = \ln(x+z)$$

$$\frac{d}{dy}(z\ln(x+y)) = (\overset{0}{z})'(\ln(x+y)) + z.(\ln(x+y))' = 0 + \frac{z}{x+y}.(x+y)' = \frac{z}{x+y}$$

$$\frac{df}{dy} = \frac{x}{y+z} + \ln(x+z) + \frac{z}{x+y}$$

$$\frac{d}{dz}(x\ln(y+z)) = \overset{0}{(x)'}\ln(y+z) + (x)(\ln(y+z))' =$$
$$= 0 + \frac{x}{y+z}\cdot(\overset{0}{y}+\overset{1}{z})' = \frac{x}{y+z}$$

$$\frac{d}{dz}(y\ln(x+z)) = \overset{0}{(y)'}\ln(x+z) + y.(\ln(x+z))' = 0 + \frac{y}{x+z}(\overset{0}{x}+\overset{1}{z})' =$$
$$= \frac{y}{x+z}$$

$$\frac{d}{dz}(z\ln(x+y)) = \overset{1}{(z)'}\ln(x+y) + z.(\overset{0}{\ln(x+y)})' = \ln(x+y)$$

$$\frac{df}{dz} = \frac{x}{y+z} + \frac{y}{x+z} + \ln(x+y)$$

$$\nabla f = \left(\ln(y+z) + \frac{y}{x+z} + \frac{z}{x+y};\ \frac{x}{y+z} + \ln(x+z) + \frac{z}{x+y};\ \frac{x}{y+z} + \frac{y}{x+z} + \ln(x+y)\right)$$

(5) $f(x,y) = \sin(x+y) + y\cos x$ at point $\left(\frac{\pi}{3}; \frac{\pi}{6}\right)$ with respect to vector $\left(\frac{3}{5}; \frac{4}{5}\right)$

$$\frac{df}{dx} = [\sin(x+y)]' + [y\cos x]' =$$
$$= \cos(x+y)(\overset{1}{x}+\overset{0}{y})' + (y)'\cos x + y.(\cos x)' =$$
$$= \cos(x+y) + 0 + y.(-\sin x) = \cos(x+y) - y.\sin x$$

$$\frac{df}{dy} = (\sin(x+y))' + (y\cos x)' = \cos(x+y)(\overset{0}{x}+\overset{1}{y})' + (y)'\cos x + y(\overset{0}{\cos x})' =$$
$$= \cos(x+y) + \cos x$$

$$\nabla f_{(x,y)} = (\cos(x+y) - y.\sin x;\ \cos(x+y) + \cos x)$$

$$\nabla f\left(\frac{\pi}{3}; \frac{\pi}{6}\right) = \left(\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \frac{\pi}{6}.\sin\frac{\pi}{3};\ \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + \cos\frac{\pi}{3}\right)$$

$$\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \cos\left(\frac{6\pi+3\pi}{18}\right) = \cos\frac{9\pi}{18} = \cos\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\nabla f\left(\frac{\pi}{3};\frac{\pi}{6}\right) = \left(\overset{0}{\overset{\parallel}{\cos\frac{\pi}{2}}} - \frac{\pi}{6}\cdot\sin\frac{\pi}{3};\ \underset{0}{\underset{\parallel}{\cos\frac{\pi}{2}}} + \cos\frac{\pi}{3}\right)$$

$$\nabla f\left(\frac{\pi}{3};\frac{\pi}{6}\right) = \left(-\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2};\ \frac{1}{2}\right) \qquad \Delta f\left(\frac{\pi}{3};\frac{\pi}{6}\right) = \left(\frac{-\sqrt{3}\pi}{2};\ \frac{1}{2}\right)$$

$$\nabla_v f = \nabla f\left(\frac{\pi}{3};\frac{\pi}{6}\right)\circ v \qquad v\left(\frac{3}{5};\frac{4}{5}\right)$$

$$\cancel{\nabla_v f = \nabla f(} \quad \nabla_v f = \left(\frac{-\sqrt{3}\pi}{2};\ \frac{1}{2}\right)\circ\left(\frac{3}{5};\ \frac{4}{5}\right)$$

$$\cancel{\nabla_v f} \quad \nabla_v f = \left(\frac{-\sqrt{3}\pi}{2}\cdot\frac{3}{5}\right) + \frac{1}{2}\cdot\frac{4}{5} =$$

$$= -\frac{3\sqrt{3}\pi}{10} + \frac{\overset{3}{2}}{5} = \frac{4-3\sqrt{3}\pi}{10}$$

$$10$$

$$\nabla_v f = \frac{4-3\sqrt{3}\pi}{10}$$