Task (Another q-binomial theorem) The variables x and y. do not commute, but they satisfy the following relation: $yx = qxy$. If a is any expression containing x, y or q, than $qa = aq$. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

For example,

$$(x+y)^2 = x^2 + (1+q)xy + y^2$$

Solution

When we multiply $(x+y)^n$, each term is a string of length n which contains x and y. If there are k letters y in string, we can define it as $x^{n-k}y^k$ and each change from one letter to another contributing a factor of q. Coefficient of this term is q^a, where a is the number of changes. Let $\alpha(n,k)$ be the sum of these coefficients over all strings with k letters y. Then $\alpha(n,k)$ is a polynomial in q, and

$$(x+y)^n = \sum_{k=0}^{n} \alpha(n,k) x^{n-k} y^k$$

We identify the coefficients $\alpha(n,k)$ with Gaussian coefficients.

$$(x+y)^n = (x+y)^{n-1}(x+y) = \left(\sum_{k=0}^{n-1} C(n-1,k) x^{n-1-k} y^k \right) (x+y)$$

Multiplying by x, we have to change this x over all k letters y to reach the required form, giving a factor of q^k. Multiplying by y, no changes are required. So we have

$$\alpha(n,k) = q^k \alpha(n-1,k) + \alpha(n-1,k-1)$$

Thus the coefficients $\alpha(n,k)$ satisfy the same recurrence and initial conditions as the Gaussian coefficients, and so are equal to them.