(1) $$S[y] = 2y^2(\pi) + \int_0^{\pi} dx\left((y'(x))^2 - y^2(x) + 3y(x)\cos 2x\right)$$
$$y(x) \in C^2[0,\pi],\ y(0) = 0$$

$$\Delta S[y] = S[y+\delta y] - S[y] =$$

$$= 2(y+\delta y)^2(\pi) - 2y^2(\pi) + \int_0^{\pi} dx\Big(\big(y'(x) + (\delta y)'(x)\big)^2 - (y+\delta y)^2(x) +$$

$$+ 3\big(\cancel{y(x)} + \delta y(x)\big)\cos 2x - (y'(x))^2 + y^2(x) - \cancel{3y(x)\cos 2x}\Big) =$$

$$= 4y\delta y(\pi) + 2\delta y^2(\pi) + \int_0^{\pi} dx\Big(2y'(x)\delta y'(x) + (\delta y'(x))^2 -$$

$$- 2y\delta y(x) - (\delta y(x))^2 + 3\delta y(x)\cos 2x\Big)$$

$$\delta S[y] = 4y(\pi)\delta y(\pi) + \int_0^{\pi} dx\big(2y'(x)\delta y'(x) - 2y(x)\delta y(x) +$$

$$+ 3\delta y(x)\cos 2x\big)$$

$$\int_0^{\pi} 2y'(x)\delta y'(x)dx = \int_0^{\pi} 2y'(x)\,d\,\delta y(x) = 2y'(x)\delta y(x)\Big|_0^{\pi} -$$

$$- \int_0^{\pi} \delta y(x)\,d\,2y'(x) = 2y'(x)\delta y(x)\Big|_0^{\pi} - 2\int_0^{\pi} y''(x)\delta y(x)\,dx$$

$$\delta S[y] = 4y(\pi)\delta y(\pi) + \int_0^{\pi} dx\big(3\cos 2x - 2y(x) - 2y''(x)\big)\delta y(x) +$$

$$+ 2y'(x)\delta y(x)\Big|_0^{\pi} =$$

$$= \big(4y(\pi) + 2y'(\pi)\big)\delta y(\pi) - 2y'(0)\delta y(0) + \int_0^{\pi} dx\big(3\cos 2x - 2y - 2y''\big)\delta y$$

$$2y'' + 2y - 3\cos 2x = 0$$

$$\Downarrow$$

$$\boxed{y(x) = C_2 \sin x + C_1 \cos x - \frac{1}{2}\cos(2x)}$$

$$y(0) = C_1 - \frac{1}{2} = 0 \Rightarrow \boxed{C_1 = \frac{1}{2}}$$

$\delta y\big|_{x=\pi}$ – произвольна $\Rightarrow (4y + 2y')\big|_{x=\pi} = 0$

$$(2y + y')\big|_{x=\pi} = 0$$

$$y'(x) = C_2 \cos x - C_1 \sin x + \frac{1}{2}\cdot 2 \sin 2x$$

$$y'(\pi) = -C_2$$

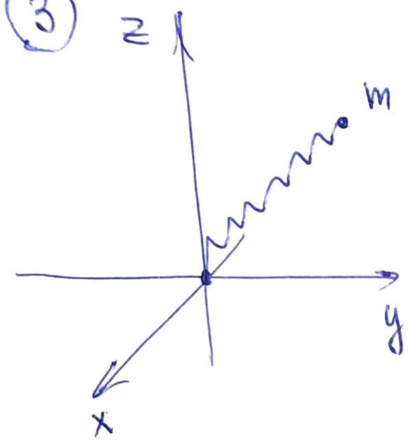
$$y(\pi) = -C_1 - \frac{1}{2}$$

$$\Rightarrow 2y(\pi) + y'(\pi) = -2C_1 - 1 - C_2 = 0$$

$$C_2 = -2C_1 - 1 = -2\cdot\frac{1}{2} - 1 = -2 \Rightarrow \boxed{C_2 = -2}$$

Ответ: $y(x) = -2\sin x + \frac{1}{2}\cos x - \frac{1}{2}\cos(2x)$

③



$$z = \frac{1}{2(x^2+y^2)}$$

Введем цилиндрические коорд. ρ, z, φ

$$\rho = \sqrt{x^2+y^2} \Rightarrow z = \frac{1}{2\rho^2} \Rightarrow \rho^2 = \frac{1}{2z}$$

$$l = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{1}{2z}+z^2}$$

$$L = T_{кин} - U$$

$$T_{кин} = \frac{m}{2}\left(\dot\rho^2+\dot z^2+\rho^2\dot\varphi^2\right) = \frac{m}{2}\left(\frac{\dot z^2}{8z^3}+\dot z^2+\frac{\dot\varphi^2}{2z}\right)$$

$$U = \frac{k}{2}\left(\frac{1}{2z}+z^2\right)$$

б) $\frac{\partial L}{\partial \varphi}=0 \Rightarrow \frac{\partial L}{\partial\dot\varphi} = const$ (З.С.И)

$\frac{\partial L}{\partial t}=0 \Rightarrow$ Выполняется З.С.Э.: $E = T+U = const$

а) $$\boxed{L = T - U = \frac{m}{2}\left(\frac{\dot z^2}{8z^3}+\dot z+\frac{\dot\varphi^2}{2z}\right) - \frac{k}{2}\left(\frac{1}{2z}+z^2\right)}$$

Ур-ие Эйлера-Лагранжа

$$L_\varphi = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = 0 \Rightarrow \frac{d}{dt}\left(\frac{2\dot\varphi m}{4z}\right)=0 \Rightarrow y = \overset{const}{\overset{\shortparallel}{\frac{\dot\varphi m}{2z}}}$$

$$L_z = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot z}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow$$

$$\frac{d}{dt}\left(\frac{2m\dot z}{16z^3} + \frac{2m\dot z}{2}\right) - \left(-\frac{3}{16} m z^2 \cdot \frac{1}{z^4} - \frac{m\dot\varphi^2}{4z^2} + \frac{k}{4z^2} - \frac{\cancel{2}zk}{\cancel{2}}\right)$$

в) При $z = const = z_0 \Rightarrow \dot z = 0$

$$\cancel{W_z\big|_{z=z_0} = \frac{m\dot\varphi^2}{4z_0} - \frac{k}{4z_0} - \frac{kz_0}{2}}$$

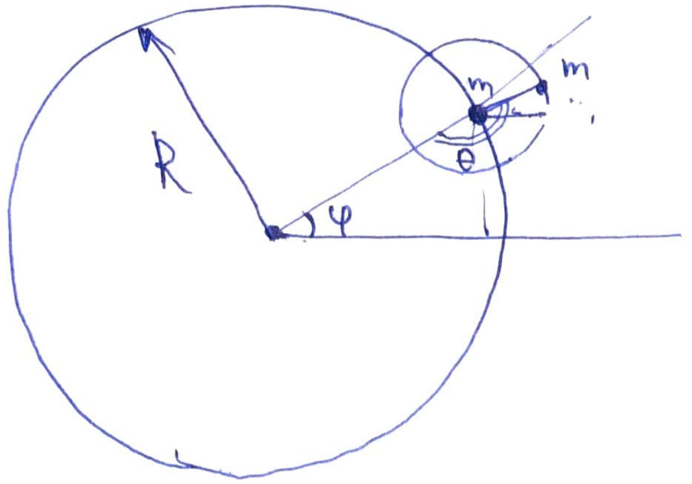
$$W_z\big|_{z=z_0} = \frac{m\dot\varphi^2}{4z_0^2} - \frac{k}{4z_0^2} + z_0 k = 0 \Rightarrow z_0^3 = \frac{m\dot\varphi^2 - k}{4k}$$

$$\Downarrow$$

$z \neq 0$

$m\dot\varphi \neq k$

4



Для первой частицы

$$x_1 = R\cos\varphi \qquad \dot{x}_1 = -R\sin\varphi\,\dot{\varphi}$$

$$y_1 = R\sin\varphi \qquad \dot{y}_1 = R\cos\varphi\,\dot{\varphi}$$

Для второй

$$x_2 = R\cos\varphi + \ell\cos(\theta+\varphi-\pi) = R\cos\varphi - \ell\cos(\theta+\varphi)$$

$$y_2 = R\sin\varphi + \ell\sin(\theta+\varphi-\pi) = R\sin\varphi - \ell\sin(\theta+\varphi)$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) =$$

$$= \frac{m}{2}\left(R^2\dot{\varphi}^2\right) + \frac{m}{2}\Big(\big(-R\sin\varphi\,\dot{\varphi} + \ell\sin(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\big)^2 +$$

$$+ \big(R\cos\varphi\,\dot{\varphi} - \ell\cos(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\big)^2 =$$

$$= \frac{m}{2}\left(R^2\dot{\varphi}^2\right) + \frac{m}{2}\Big(R^2\dot{\varphi}^2 + \ell^2(\dot{\theta}+\dot{\varphi})^2 \cancel{-2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell(}$$

$$-2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell\left(\sin\varphi\sin(\theta+\varphi) + \cos\varphi\cos(\theta+\varphi)\right)\Big)$$

$U = 0$

Значит, $L = T = \frac{m}{2}(R^2\dot{\varphi}^2) + \frac{m}{2}(R^2\dot{\varphi}^2 + \ell^2(\dot{\theta}+\dot{\varphi})^2 -$

$- 2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell\cos(\theta))$

д) $\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполняется З.С.Э: $E = T + U = T = const$

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = const$

$\frac{\partial L}{\partial \dot{\varphi}} = \underbrace{2mR^2\dot{\varphi} + 2\ell^2\dot{\varphi} + 2\dot{\theta}\ell^2 - 2(2\dot{\varphi}+\dot{\theta})R\ell\cos\theta}_{const}$

$\cancel{\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = const}$

$\cancel{\frac{\partial L}{\partial \dot{\theta}} = 2\ell^2\dot{\theta} + 2\dot{\varphi}\ell^2 - 2\dot{\varphi}R\ell}$