Catalan sequence

Task

Let c_n be the Catalan sequence. Find the limit $\lim\limits_{n\to\infty} \frac{c_{n+1}}{c_n}$

Solution

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

Then

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n+2}\frac{(2n+2)!}{(n+1)!((2n+2)-(n+1))!}}{\frac{1}{n+1}\frac{(2n)!}{n!((2n)-(n))!}} = \lim_{n\to\infty} \frac{\frac{1}{n+2}\frac{(2n+2)!}{(n+1)!(n+1)!}}{\frac{1}{n+1}\frac{(2n)!}{n!n!}} =$$

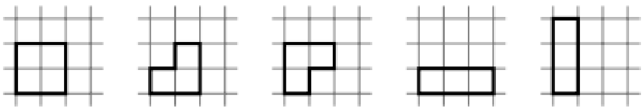
$$\lim_{n\to\infty} \frac{(n+1)\frac{(2n+2)!}{(n+1)!(n+1)!}}{(n+2)\frac{(2n)!}{n!n!}} = \lim_{n\to\infty} \frac{(n+1)(2n+2)!n!n!}{(n+2)(2n)!(n+1)!(n+1)!} =$$

$$\lim_{n\to\infty} \frac{(n+1)(2n+1)(2n+2)}{(n+2)(n+1)(n+1)} = \lim_{n\to\infty} \frac{2(n+1)(2n+1)}{(n+2)(n+1)} =$$

$$\lim_{n\to\infty} \frac{4n+2}{n+2} = \lim_{n\to\infty} \frac{4+\frac{2}{n}}{1+\frac{2}{n}} = 4$$

Task

Prove that the number of pairs of paths of length $2n+2$ sarting at $(0,0)$, ending at the same point, having no other common points and going only up and to the right is equal to c_n.



Solution

Task
Prove the equality:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Solution
We know that

$$\binom{n}{k} = \binom{n}{n-k}$$

Then

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Then we can see that this is the same as choosing $n-k$ objectsfrom the set of power n and k from the other set with the same size. Then, considering the sum of $\binom{n}{k}\binom{n}{n-k}$ for all possible k, we get the number of ways to select n objects from a set of size $2n$.

Task
Let f_n be the Fibonacci sequence. Prove that

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$

Solution
Binet's formula:

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Then

$$f_{n-1}f_{n+1} - f_n^2 =$$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \cdot \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} - \left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)^2 =$$

$$\frac{1}{5}\left(\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right) \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) - \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)^2\right) =$$

$$\frac{1}{2^{2n} \cdot 5}\left(\left((1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}\right) \cdot \left((1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\right) - \left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)^2\right) =$$

$$\frac{1}{2^{2n} \cdot 5}\Big(\left((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1} - (1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1}\right) -$$
$$\left((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - 2(1+\sqrt{5})^n(1-\sqrt{5})^n\right)\Big) =$$

$$\frac{1}{2^{2n} \cdot 5}\Big((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1}$$
$$-(1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1} - (1+\sqrt{5})^{2n} - (1-\sqrt{5})^{2n} + 2(1+\sqrt{5})^n(1-\sqrt{5})^n\Big) =$$

$$\frac{1}{2^{2n} \cdot 5}\left(2(1+\sqrt{5})^n(1-\sqrt{5})^n - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1} - (1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1}\right) =$$

$$\frac{(1-5)^{n-1}}{2^{2n} \cdot 5}\left(2(1-\sqrt{5})(1+\sqrt{5}) - (1-\sqrt{5})^2 - (1+\sqrt{5})^2\right) =$$

$$\frac{(-4)^{n-1}}{4^n \cdot 5}\left(2(1-\sqrt{5})(1+\sqrt{5}) - (6-2\sqrt{5}) - (6+2\sqrt{5})\right) =$$

$$\frac{(-4)^{n-1}}{4^n \cdot 5}\left(-8-6-6\right) =$$

$$\frac{(-4)^{n-1}}{4^n \cdot 5}(-20) = (-1)^n$$