№1

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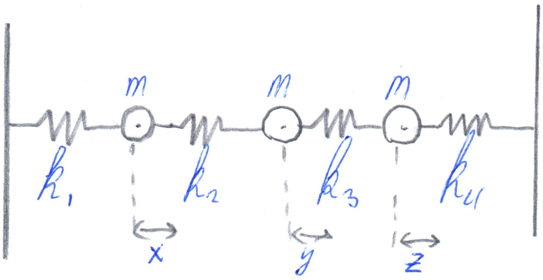


$k_1 = k_4 = 3k$

$k_2 = k_3 = 2k$

а) $$\begin{cases} m\ddot{x} = -k_1 x - k_2(x-y) \\ m\ddot{y} = k_2(x-y) - k_3(y-z) \\ m\ddot{z} = k_3(y-z) - k_4 z \end{cases}$$

б) $$\ddot{X} = -AX$$

$$A = \frac{1}{m}\begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} 5 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = -\lambda^3 + 14\lambda^2 - 57\lambda + 60 = -(\lambda-5)(\lambda^2 - 9\lambda + 12)$$

$$\lambda_1 = 5$$

$$\lambda_{2,3} = \frac{9 \pm \sqrt{33}}{2}$$

$$\omega_i^2 = \frac{\lambda_i k}{m} \Rightarrow \omega_1 = \sqrt{\frac{5k}{m}},\ \omega_2 = \sqrt{\frac{(9+\sqrt{33})k}{2m}},\ \omega_3 = \sqrt{\frac{(9-\sqrt{33})k}{2m}}$$

λ_1) $$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 - x_2 + 2x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \psi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

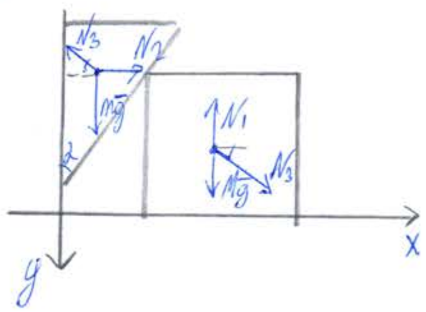
λ_2) $$\begin{pmatrix} 5-\lambda_2 & -2 & 0 \\ -2 & 4-\lambda_2 & -2 \\ 0 & -2 & 5-\lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{\sqrt{33}-1}{4} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + \frac{\sqrt{33}-1}{4}x_3 = 0 \end{cases} \Rightarrow \psi_2 = \begin{pmatrix} +1 \\ \frac{1-\sqrt{33}}{4} \\ 1 \end{pmatrix}$$

λ_3) $$\begin{pmatrix} 5-\lambda_3 & -2 & 0 \\ -2 & 4-\lambda_3 & -2 \\ 0 & -2 & 5-\lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{\sqrt{33}+1}{4} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 - \frac{\sqrt{33}+1}{4}x_3 = 0 \end{cases} \Rightarrow \psi_3 = \begin{pmatrix} +1 \\ \frac{1+\sqrt{33}}{4} \\ 1 \end{pmatrix}$$

Ответы 1б:
$$\omega_1 = \sqrt{\frac{5k}{m}} \qquad \omega_2 = \sqrt{\frac{(9+\sqrt{33})k}{2m}} \qquad \omega_3 = \sqrt{\frac{(9-\sqrt{33})k}{2m}}$$

$$\psi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} +1 \\ \frac{1-\sqrt{33}}{4} \\ 1 \end{pmatrix} \qquad \psi_3 = \begin{pmatrix} +1 \\ \frac{1+\sqrt{33}}{4} \\ 1 \end{pmatrix}$$

№2



а) # степеней свободы $2-1=1$

б) $$\begin{cases} M\ddot{x} = N_3\cos\alpha \\ 0 = N_2 - N_3\cos\alpha \\ 0 = N_1 - N_3\sin\alpha - Mg \\ m\ddot{y} = mg - N_3\sin\alpha \\ \ddot{x} = \ddot{y}\,tg\,\alpha \end{cases}$$

в) $$N_2 = N_3\cos\alpha$$

$$\ddot{x} = \frac{N_3\cos\alpha}{M}$$

$$N_1 = N_3\sin\alpha + Mg$$

$$\ddot{y} = g - \frac{N_3}{m}\sin\alpha$$

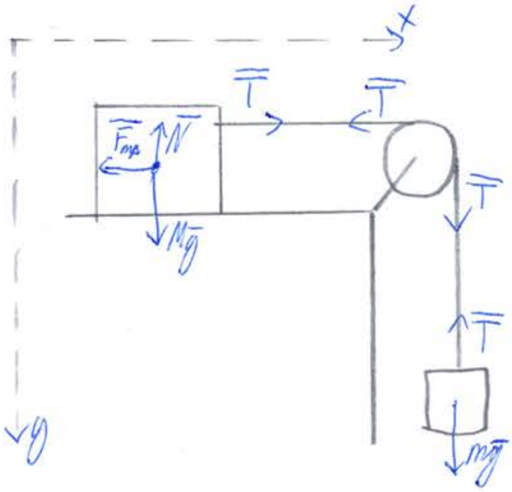
$$\frac{N_3\cos\alpha}{M} = g\,tg\alpha - \frac{N_3}{m}\sin\alpha\,tg\alpha$$

$$N_3\left(\frac{\cos\alpha}{M} + \frac{\sin\alpha\,tg\alpha}{m}\right) = g\,tg\alpha \Rightarrow N_3 = \frac{g\,tg\alpha\,Mm}{m\cos\alpha + M\sin\alpha\,tg\alpha}$$

$$\ddot{x} = \frac{mg\sin\alpha}{m\cos\alpha + M\sin\alpha\,tg\alpha}$$

$$\ddot{y} = \frac{mg\cos\alpha + Mg\sin\alpha\,tg\alpha - Mg\cancel{tg}\sin\alpha\,tg\alpha}{m\cos\alpha + M\sin\alpha\,tg\alpha} = \frac{mg\cos\alpha}{m\cos\alpha + M\sin\alpha\,tg\alpha} = \frac{mg\cos^2\alpha}{m\cos^2\alpha + M\sin^2\alpha}$$

№3



если ~~[illegible]~~ $|\mu N| > |T|$, то ничего не двигается и

$T = mg \quad N = Mg$

$\mu Mg > mg, \ \mu > \frac{m}{M}$

а) # степеней свободы $2 - 1 = 1$

б) $\begin{cases} m\ddot{y} = mg - T \\ M\ddot{x} = T - \mu N \\ N = mg \\ \ddot{x} = \ddot{y} \end{cases}$

в) $\ddot{y} = g - \frac{T}{m}$

$\ddot{x} = \frac{T}{M} - \frac{\mu Mg}{M}$

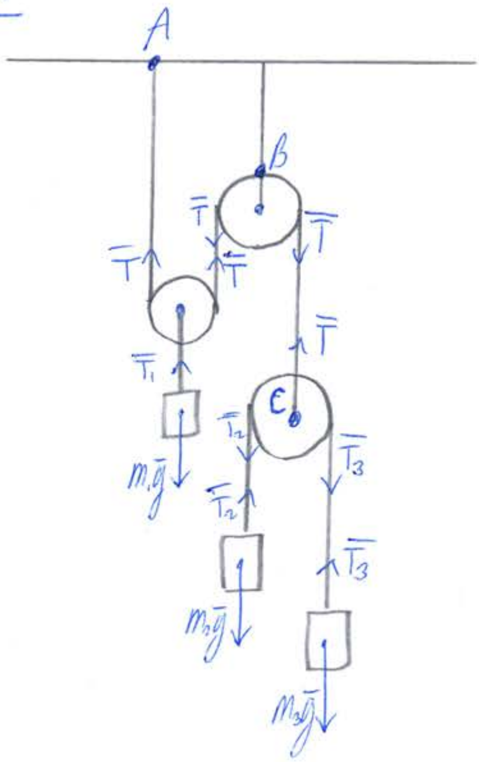
$g - \frac{T}{m} = \frac{T}{M} - \mu g$

$T\left(\frac{1}{m} + \frac{1}{M}\right) = g(1+\mu)$

$T = \frac{mMg(1+\mu)}{m+M}$

$\ddot{x} = \ddot{y} = g - \frac{Mg(1+\mu)}{m+M} = \frac{mg + Mg - Mg - Mg\mu}{m+M} = \frac{m - M\mu}{m+M} g$

№4



$$m_1\ddot{x}_1 = m_1 g - T_1$$
$$m_2\ddot{x}_2 = m_2 g - T_2$$
$$m_3\ddot{x}_3 = m_3 g - T_3$$
$$T_2 = T_3 = \frac{T}{2}$$
$$T_1 = 2T$$

Пусть длина $AB - l_{AB}$, $BC - l_{BC}$

Если блок C неподвижен, то $\ddot{\tilde{x}}_2 + \ddot{\tilde{x}}_3 = 0$

$$\begin{cases} x_2 = \tilde{x}_2 + \Delta l_{BC} \Rightarrow \tilde{x}_2 = x_2 - \Delta l_{BC} \\ x_3 = \tilde{x}_3 + \Delta l_{BC} \Rightarrow x_3 = -x_2 + 2\Delta l_{BC} \Rightarrow \Delta l_{BC} = \frac{x_2 + x_3}{2} \\ \ddot{\tilde{x}}_2 + \ddot{\tilde{x}}_3 = 0 \end{cases}$$

Для нити через верхние 2 блока: $l_{AB} + 2x_1 + l_{BC} + \frac{x_2 + x_3}{2} = const$

$\Rightarrow 4\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$

$$\ddot{x}_1 = g - \frac{2T}{m_1}$$
$$\ddot{x}_2 = g - \frac{T}{2m_2} \quad \Rightarrow \quad 4g - \frac{8T}{m_1} + g - \frac{T}{2m_2} + g - \frac{T}{2m_3} = 0$$
$$\ddot{x}_3 = g - \frac{T}{2m_3}$$

$$\Rightarrow T\left(\frac{8}{m_1} + \frac{1}{2m_2} + \frac{1}{2m_3}\right) = 6g \Rightarrow T = \frac{12\, m_1 m_2 m_3\, g}{16 m_2 m_3 + m_1 m_3 + m_1 m_2}$$

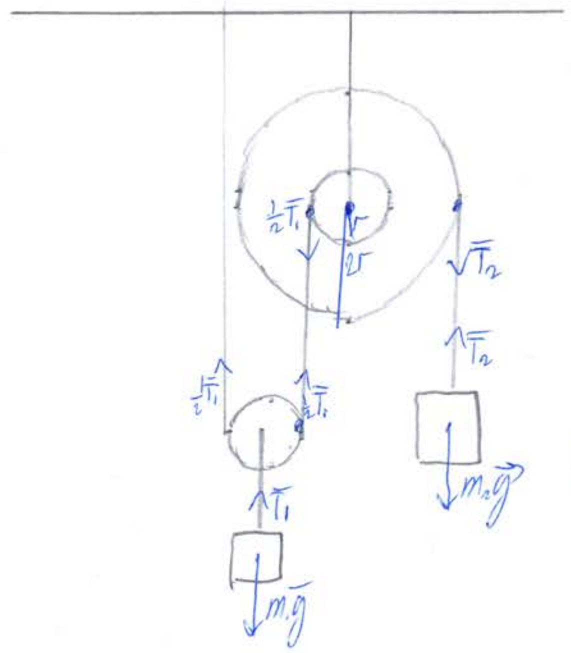
$$\Rightarrow \ddot{x}_1 = \frac{16 m_2 m_3 g + m_1 m_3 g + m_1 m_2 g - 24 m_2 m_3 g}{16 m_2 m_3 + m_1 m_3 + m_1 m_2} = \frac{m_1 m_3 + m_1 m_2 - 8 m_2 m_3}{16 m_2 m_3 + m_1 m_3 + m_1 m_2} g$$

$$\ddot{x}_2 = \frac{16 m_2 m_3 + m_1 m_2 - 5 m_1 m_3}{16 m_2 m_3 + m_1 m_3 + m_1 m_2} g$$

$$\ddot{x}_3 = \frac{16 m_2 m_3 + m_1 m_3 - 5 m_1 m_2}{16 m_2 m_3 + m_1 m_3 + m_1 m_2}$$

степеней свободы $3 - 1 = 2$

№5



$$m_1\ddot{x}_1 = m_1 g - T_1$$
$$m_2\ddot{x}_2 = m_2 g - T_2$$
$$T_2 = \frac{1}{2}\cdot\frac{1}{2}T_1 = \frac{1}{4}T_1$$

\# степеней свободы $2-1=1$

Пусть катушка повернулась на φ по часовой, изменение длины нити через малую катушку Δl_1, через большую Δl_2

$$\Delta l_1 = -r\varphi = -\frac{x_2}{2}$$
$$\Delta l_2 = x_2 = 2r\varphi$$
$$\Rightarrow x_1 = \frac{\Delta l_1}{2} = -\frac{x_2}{4} \Rightarrow 4x_1 = -x_2$$
$$\Rightarrow 4\ddot{x}_1 = -\ddot{x}_2$$
$$\ddot{x}_1 = g - \frac{T_1}{m_1} = g - \frac{4T_2}{m_1}$$
$$\ddot{x}_2 = g - \frac{T_2}{m_2}$$
$$4g - \frac{16T_2}{m_1} = -g + \frac{T_2}{m_2} \Rightarrow 5g = T_2\left(\frac{16}{m_1} + \frac{1}{m_2}\right) \Rightarrow T_2 = \frac{5m_1m_2g}{16m_2+m_1}$$
$$\ddot{x}_1 = g - \frac{20m_2g}{16m_2+m_1} = \frac{m_1-4m_2}{16m_2+m_1}g$$
$$\ddot{x}_2 = g - \frac{5m_1g}{16m_2+m_1} = \frac{4(4m_2-m_1)}{16m_2+m_1}g$$

№6

$$(\bar{e}_r, \bar{e}_\theta, \bar{e}_\varphi) = (\bar{e}_x, \bar{e}_y, \bar{e}_z)\begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \quad tg\,\theta = \frac{(x^2+y^2)^{\frac{1}{2}}}{z} \quad tg\,\varphi = \frac{y}{x}$$

$$\bar{e}_r = \bar{e}_x \sin\theta\cos\varphi + \bar{e}_y \sin\theta\sin\varphi + \bar{e}_z\cos\theta$$

- $$\dot{\bar{e}}_r = \bar{e}_x(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}) + \bar{e}_y(\cos\theta\sin\varphi\,\dot{\theta} + \sin\theta\cos\varphi\,\dot{\varphi}) + \bar{e}_z(-\sin\theta\,\dot{\theta})$$

$$= \dot{\theta}(\bar{e}_x\cos\theta\cos\varphi + \bar{e}_y\cos\varphi\sin\varphi - \bar{e}_z\sin\theta) + \dot{\varphi}(-\bar{e}_x\sin\theta\sin\varphi + \bar{e}_y\sin\theta\cos\varphi)$$

$$= \dot{\theta}\,\bar{e}_\theta + \dot{\varphi}\sin\theta\,\bar{e}_\varphi \Rightarrow \underline{\dot{\bar{e}}_r = \dot{\theta}\,\bar{e}_\theta + \dot{\varphi}\sin\theta\,\bar{e}_\varphi}$$

- $$\bar{e}_\theta = \bar{e}_x\cos\theta\cos\varphi + \bar{e}_y\cos\theta\sin\varphi - \bar{e}_z\sin\theta$$

$$\dot{\bar{e}}_\theta = \bar{e}_x(-\sin\theta\cos\varphi\,\dot{\theta} - \cos\theta\sin\varphi\,\dot{\varphi}) + \bar{e}_y(-\sin\theta\sin\varphi\,\dot{\theta} + \cos\theta\cos\varphi\,\dot{\varphi}) - \bar{e}_z\cos\theta\,\dot{\theta}$$

$$= -\dot{\theta}(\bar{e}_x\sin\theta\cos\varphi + \sin\theta\sin\varphi\,\bar{e}_y + \bar{e}_z\cos\theta) + \dot{\varphi}\cos\theta(-\bar{e}_x\sin\varphi + \bar{e}_y\cos\varphi)$$

$$= -\dot{\theta}\bar{e}_r + \dot{\varphi}\cos\theta\,\bar{e}_r \Rightarrow \underline{\dot{\bar{e}}_\theta = -\dot{\theta}\bar{e}_r + \dot{\varphi}\cos\theta\,\bar{e}_\varphi}$$

- $$\bar{e}_\varphi = -\bar{e}_x\sin\varphi + \bar{e}_\theta\cos\varphi$$

$$\dot{\bar{e}}_\varphi = -\bar{e}_x\cos\varphi\,\dot{\varphi} - \bar{e}_\theta\sin\varphi\,\dot{\varphi} = -\dot{\varphi}(\sin^2\theta\cos\varphi\,\bar{e}_x + \sin^2\theta\sin\varphi\,\bar{e}_y + \sin\theta\cos\theta\,\bar{e}_z + \cos^2\theta\cos\varphi\,\bar{e}_x + \cos^2\varphi\sin\varphi\,\bar{e}_y - \sin\theta\cos\theta\,\bar{e}_r) =$$

$$= -\dot{\varphi}(\cos\varphi + \sin\varphi) = -\dot{\varphi}(\sin\theta\bar{e}_r + \cos\theta\bar{e}_\theta) \Rightarrow \underline{\dot{\bar{e}}_\varphi = -\dot{\varphi}(\sin\theta\bar{e}_r + \cos\theta\bar{e}_\theta)}$$

№6 продолжение

$\bar{r} = r\bar{e}_r$

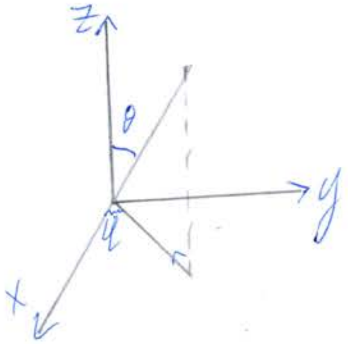
$\dot{\bar{r}} = \dot{r}\bar{e}_r + r\dot{\bar{e}}_r = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta + r\dot{\varphi}\sin\theta\bar{e}_\varphi$

$\ddot{\bar{r}} = \ddot{r}\bar{e}_r + \dot{r}(\dot{\theta}\bar{e}_\theta + \dot{\varphi}\sin\theta\bar{e}_\varphi) + \dot{r}\dot{\theta}\bar{e}_\theta + r\ddot{\theta}\bar{e}_\theta + r\dot{\theta}(-\dot{\theta}\bar{e}_r + \dot{\varphi}\cos\theta\bar{e}_\varphi) + \dot{r}\dot{\varphi}\sin\theta\bar{e}_\varphi$

$+ r\ddot{\varphi}\sin\theta\bar{e}_\varphi + r\dot{\varphi}\cos\theta\,\dot{\theta}\bar{e}_\varphi + r\dot{\varphi}\sin\theta(-\dot{\varphi})(\sin\theta\bar{e}_r + \cos\theta\bar{e}_\theta) =$

$= \bar{e}_r(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta) + \bar{e}_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta) +$

$+ \bar{e}_\varphi(\dot{r}\dot{\varphi}\sin\theta + r\dot{\theta}\dot{\varphi}\cos\theta + \dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + r\dot{\varphi}\dot{\theta}\cos\theta)$



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{matrix} \theta \in [0, \pi] \\ \varphi \in [0, 2\pi) \end{matrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = r^2\sin\theta = 0$$

$\Leftrightarrow \begin{matrix} r = 0 \\ \sin\theta = 0 \\ \theta \in [0, \pi] \end{matrix} \Leftrightarrow \begin{matrix} r = 0 \\ \theta = 0 \\ \theta = \pi \end{matrix}$

$\Rightarrow$ Переход несингулярен на $\mathbb{R}^3 \setminus \{O_z\} \subset \mathbb{R}^3$