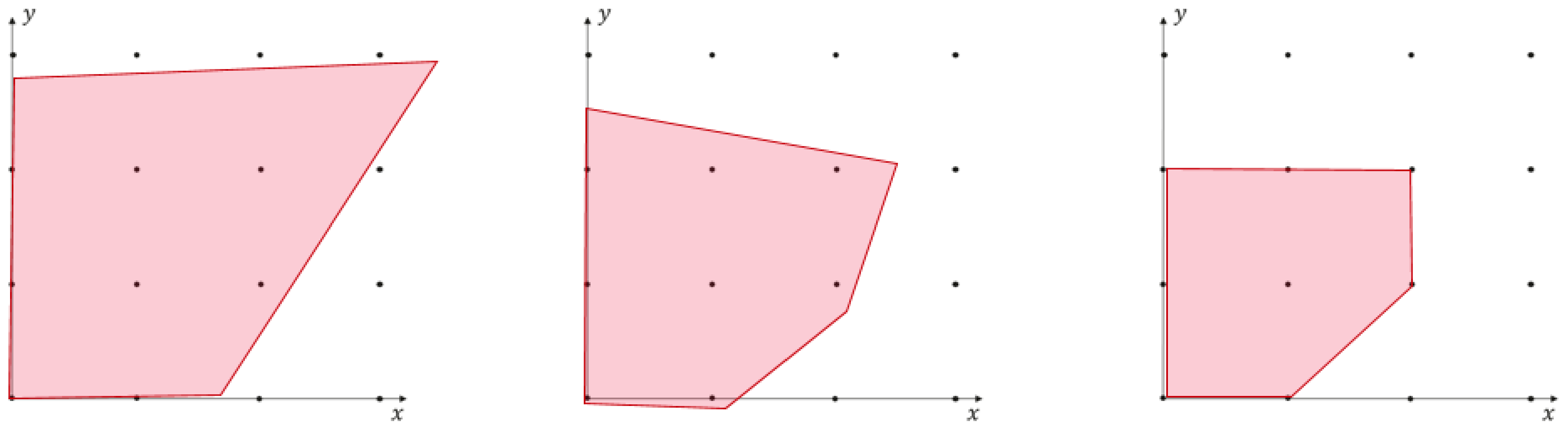
Discrete Optimization and Integer Programming

Introduction in Integer Programming



Outline

- Introduction in MILP
  - Definitions
  - MILP vs LP
- Modeling with MILP
  - Discrete variables
  - Binary variables

Scope

Modeling with discrete variables and linear constraints
- some tips
- many examples

Solving (mixed) integer linear programs with branch-and-bound methods
- preprocessing
- Relaxations
- branching rules

Tightening bounds with cutting-planes
- basic theory of polyhedra
- specific classes of valid inequalities

Splitting up large-scale problems with LP decomposition techniques
- column-generation
- Lagrangian relaxation
- Bender’s decomposition

Definitions

Mixed integer linear program
A mathematical model of an optimization problem where:

- the objective is to minimize or maximize a linear expression
- all constraints are linear equalities and inequalities
- some variables have integer or binary values

Subtypes

- Mixed Integer Linear Program (**MILP**): some variables have integer values
- Integer Linear Program (ILP or **IP**): all variables have integer values
- Binary Integer Linear Program (BILP or **BIP**): all variables have binary values

Standard form

$$\min \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} h_k y_k$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{p} g_{ik} y_k \leq b_i \ \forall i \in \{1, \dots, m\}$$
$$x_j \in \mathbb{Z}_+ \ \forall j \in \{1, \dots, n\}$$
$$y_k \in \mathbb{R}_+ \ \forall k \in \{1, \dots, p\}$$

Problem instance is defined by

$$c \in \mathbb{Q}^n, h \in \mathbb{Q}^p, a \in \mathbb{Q}^{m \times n}, g \in \mathbb{Q}^{m \times p}$$

Vectorial form

$$\max\{ cx + hy \,|\, ax + gy \leq b, x \in \mathbb{Z}_+^n \ y \in \mathbb{R}_+^p \}$$

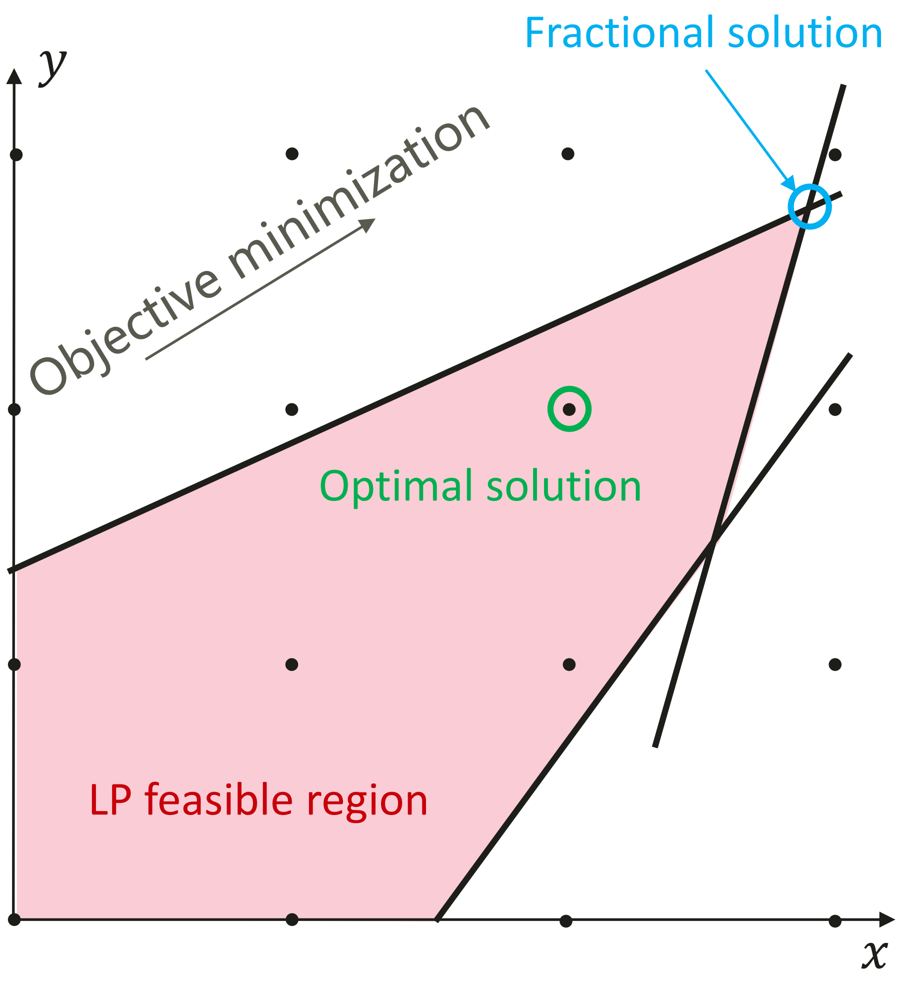
Definitions

$$\min\{\, cx + hy \mid ax + gy \le b, x \in \mathbb{Z}_+^n\ y \in \mathbb{R}_+^p \}$$

Solutions

- Solution: $\{(x, y) \mid x \in \mathbb{Z}_+^n\ y \in \mathbb{R}_+^p\}$
- Feasible solution: $\{(x, y) \mid ax + gy \le b, x \in \mathbb{Z}_+^n\ y \in \mathbb{R}_+^p\}$
- Optimal solution: $\arg\min\{\, cx + hy \mid ax + gy \le b, x \in \mathbb{Z}_+^n\ y \in \mathbb{R}_+^p\}$
- Fractional solution: $\arg\min\{\, cx + hy \mid ax + gy \le b, \boldsymbol{x} \in \mathbb{R}_+^{\boldsymbol{n}}\ y \in \mathbb{R}_+^p\}$
- (Global) optimum: $\min\{cx + hy \mid ax + gy \le b, x \in \mathbb{Z}_+^n\ y \in \mathbb{R}_+^p\}$

Solve MILP = find optimal solution.



MILP vs LP

Linear programming

$LP \in P$

- the ellipsoid algorithm is polynomial-time
- the simplex algorithm is practical (even if non-polynomial)

Mixed-integer linear programming

MILP is NP − complete in the strong sense

- no known (pseudo)polynomial-time algorithm for solving MILP
- no known general practical algorithm for large-scaled MILP

Idea: solve LP and round variables

Why not solve the LP relaxation and round the solutions to the closest integer ?

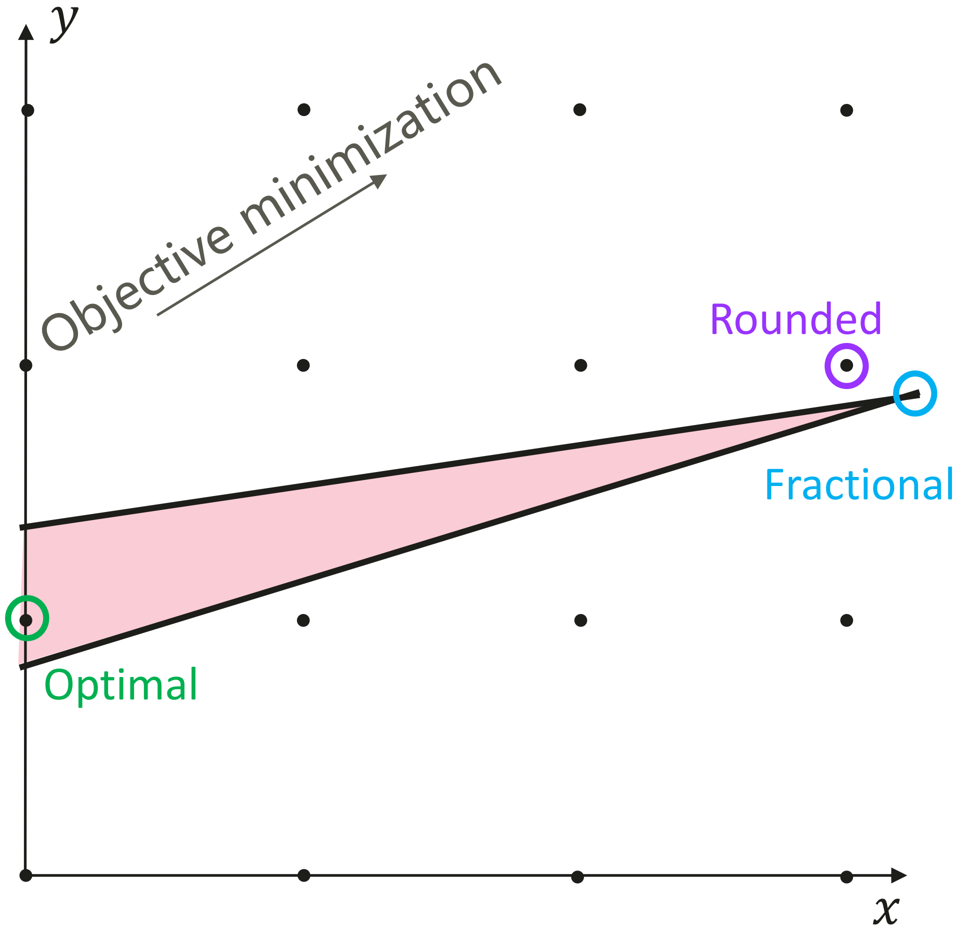
- the optimum value of the LP relaxation is far from that of the IP

MILP vs LP

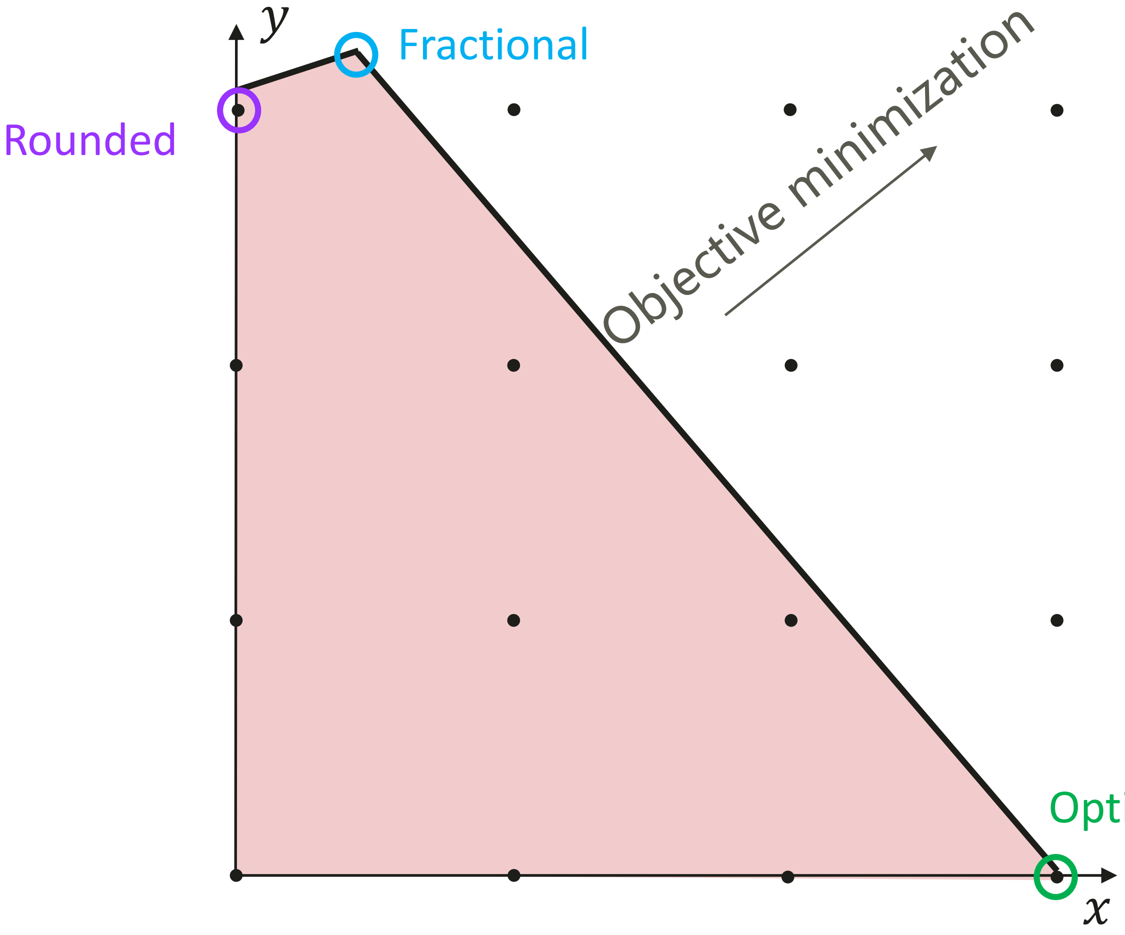
Idea: solve LP and round variables

Why not solve the LP relaxation and round the solutions to the closest integer ?

Rounded solution is not necessarily feasible



Rounding may results feasible but far from optimal



MILP vs LP

Idea: solve LP and round variables
Why not solve the LP relaxation and round the solutions to the closest integer ?

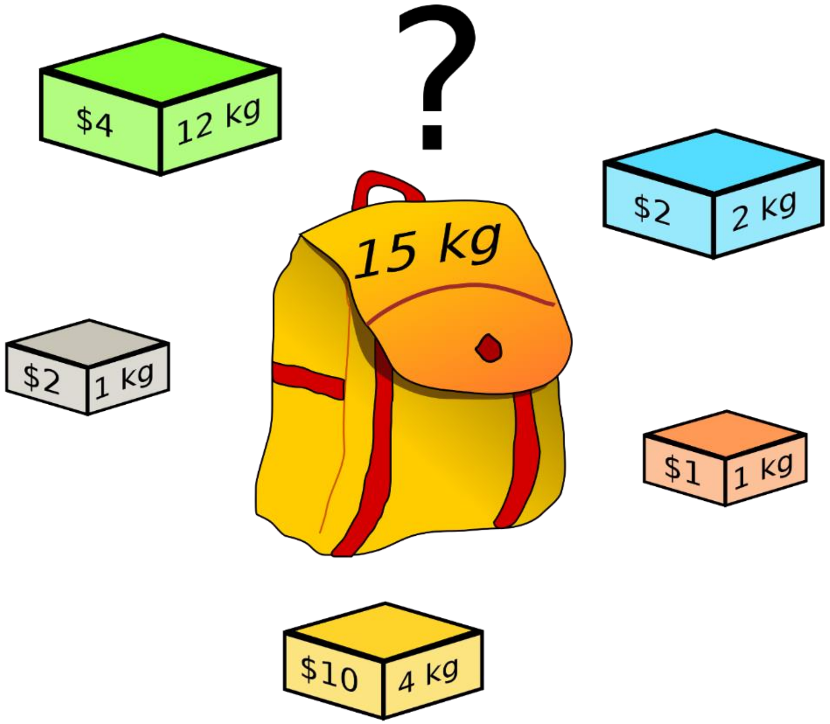
Even worse for BIP: (0.5, 0.5, . . .) can be a fractional solution but gives no information

Good news
Solving the associated LP relaxation results a lower bound on the optimal solution of MILP (for minimization statement)

Exercise
- Describe MILP with optimal fractional solution
- Solve and draw optimal and fractional solutions

$$\max x + 0.64y$$
s.t.
$$50x + 31y \leq 250$$
$$3x - 2y \geq -4$$
$$x, y \in \mathbb{Z}_+$$

0.5 means put the item in the knapsack or not?



How to solve MILP

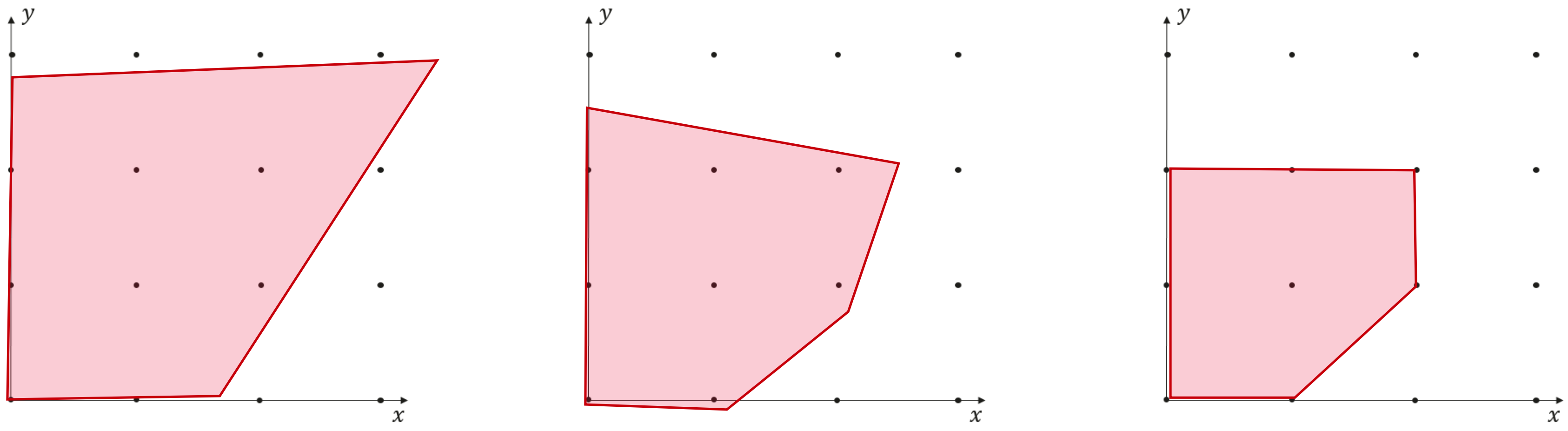
Use LP-based generic algorithms

- Pure LP (if the fractional solution is integer)
- Heuristics (find sub-optimal solutions)
- Cutting-planes method (to shrink the LP polyhedron)
- Branch-and-bound (intelligent enumeration of the solutions)
- Decomposition (separation of the subproblems)

These methods will be discussed in details in the later lectures

Modeling MILP

- a typical MILP can have many formulations
- not all formulations are created equal
- we need to find the good formulations



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Modeling with discrete variables

Combinatorial problems

In many optimization problems, some physical entities are indivisible and one need to find:

- count: the number of elements in a finite discrete set
- selection: one (best) element out of a finite discrete set
- order: a permutation of a list of elements
- schedule: a timed permutation
- graph: a substructure in a given graph

Modeling

- Map discrete variables to non-negative integers
- Model feasibility and optimality conditions with linear functions

Let’s look at some examples

Modeling with discrete variables

Scheduling

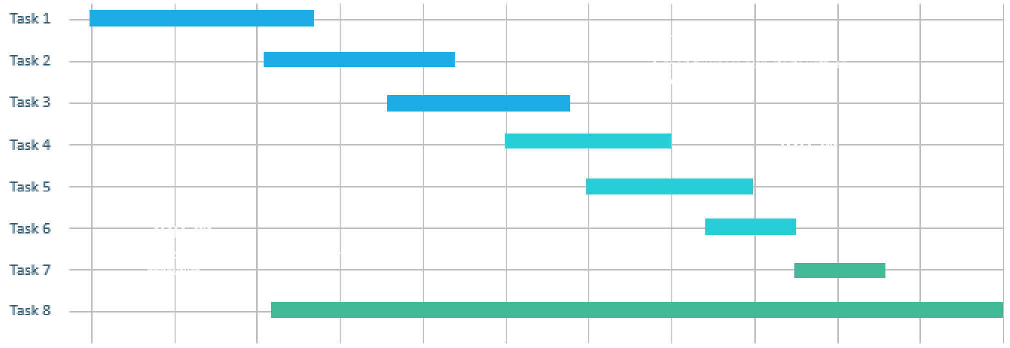
- Decide when to commit resources between a variety of possible tasks
- Tasks are partially ordered and associated to shared resources
- Order and date their execution
- Production process (jobs/machines), computing (processes/processors), project management

Discrete variables

- Assign a task to a time period (ex: starting time of a job)
- Count the load of a resource at each time
- Model the relative order of two tasks

Single machine scheduling

Project scheduling



Modeling with discrete variables

Timetabling

- Decide how to commit resources between a variety of possible tasks
- Tasks are associated to time periods
- Coordinate the resources to execute a task
- Assign a valid sequence of tasks to each resource
- Transport (flight/airplane), workplace (shift/nurse), school (class/teacher/room)

Discrete variables

- Assign a task to a resource (ex: shift of a nurse)
- Model the in/compatibility of 2 resources at same time
- Model a sequence of tasks
- Assign a sequence of tasks to a resource

School timetable

		Monday	Tuesday	Wednesday	Thursday	Friday
8:10am		Arrive	Arrive	Arrive	Arrive	Arrive
8:30am	Registration	Registration	Registration	Registration	Registration	Registration
8.35am	MATHS	MATHS	MATHS	MATHS	MATHS	MATHS
9.00am-9.15am	Assembly	Assembly	Pilates/Yoga	Tai Chi	Hymn Practice	Celebration Assembly
9.15 – 9.35am	Lesson 1	CLL	Read Write Inc Z	Maths	SWIMMING	Maths
9.35-10.15am	Lesson 2		Mathematics		SWIMMING	
10:15-10:30am	Snacks	Snacks	Snacks	Snacks	SWIMMING	Snacks
10.30-10.50am	Break	Break time	Break time	Break time	SWIMMING	Break time
10.50-11.30	Lesson 3	Maths	Mathematics	10:50 RWI PSED	CLL	CLL CLL/Library
11.30-12.10	Lesson 4		MUSIC	PE	EAD	
12.10-12.20	Storytime	Storytime	Storytime	Storytime	Storytime	Storytime
12.20-1.30pm	Lunch	Lunch Break time/sleep	Lunch Break time/sleep	Lunch Break time/sleep	Lunch Break time/sleep	Lunch Break time/sleep
1.30-1.35	Registration	Registration	Registration	Registration	Registration	Registration
1.35-1.55pm	Lesson 5	Literacy	MATHS	MUSIC	UTW	Literacy
1.55-2.35pm	Lesson 6	CLL Z				UTW
2.35-2.40	Optional Break	Break	Break	Break	Break	Break
2.40-3.20pm	Lesson 7	PE	CLL	UTW	EAD	Mathematics
3.20-3.30am	Storytime	Storytime	Storytime	Storytime	Storytime	Storytime
3.30-4.00pm	After-School Club	Let's do Chinese	Let's Get Sticky	Let's sing	Let's Get Moving	Let's read

Modeling with discrete variables

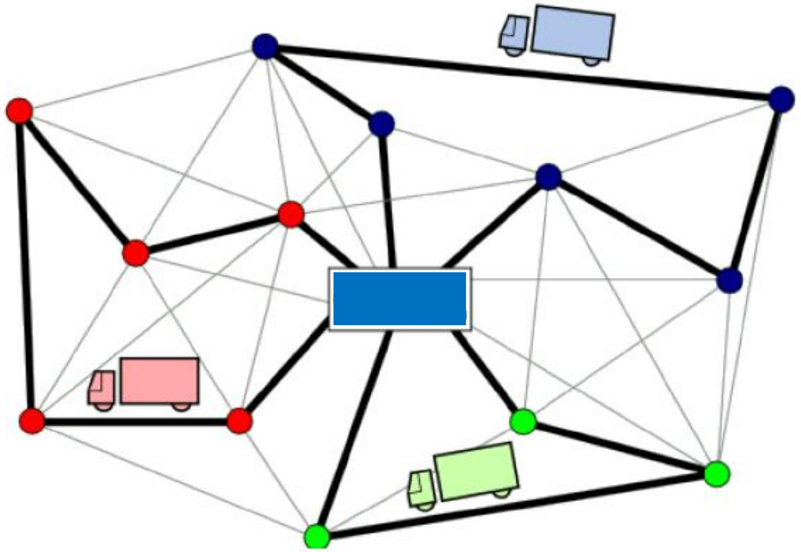
Routing

- Selecting paths in a network along which to send network traffic
- Different travelling measures: cost, load, distance, time
- Assign vehicles to paths
- Capacity, precedence between nodes, time windows, etc.

Discrete variables

- Assign a node to a vehicle
- Model the relative order of 2 nodes in a path
- Model a path
- Count the cost or weight of a path
- Assign a path to a vehicle

Vehicle routing problem



Modeling with discrete variables

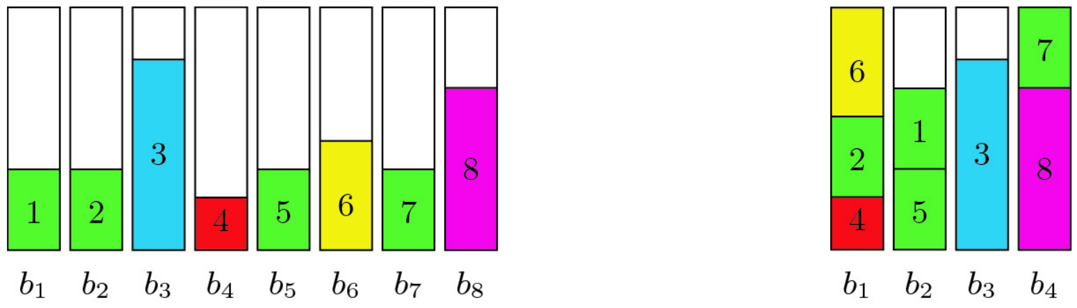
Packing and geometric placement

- Assign items to containers and place them in 1D, 2D or 3D without overlapping
- Different packing measures: profit, size, weight
- Minimize the number of containers or the total gap, maximize the profit of the packed items
- Some items are incompatible or must be at a given distance the lengths on some dimension can be cumulated
- Manufacturing (cut of paper, steel), transport and stock, location and sizing, puzzles.

Discrete variables

- Assign an item to a container
- Assign a container to a set of items
- Count the load of a container
- Assign an item to a coordinate

Bin packing problem



A feasible solution, with 8 bins

An optimal solution, with 4 bins

Modeling with binary variables

Boolean condition as a 0-1 variable

- Decision: is item j selected?
- Assignment: is item j assigned to value i?
- Value indicator: is variable x positive (or $x \geq \alpha$)?
- Condition indicator: does constraint c hold?

Logical/numeric condition as a linear combination of 0-1 variables

- Disjunction (exclusive disjunction): (either) δ_1 or δ_2
- Dependency: if δ_1 then δ_2
- Exclusive alternative: exactly 1 out of n
- Counter or bound: exactly/at least/at most k out of n

Non-linear value functions with 0-1 variables

- Set-up value: $f(x) =$ either $a(x)$ or $a(x) + b$
- Discrete values: $f(x) = f_i$ if $x = i$
- Piecewise linear: $f(x)$ linear on $[a_i - 1, a_i]$

Modeling with binary variables

Modeling yes/no decision

Example: Integer Knapsack problem

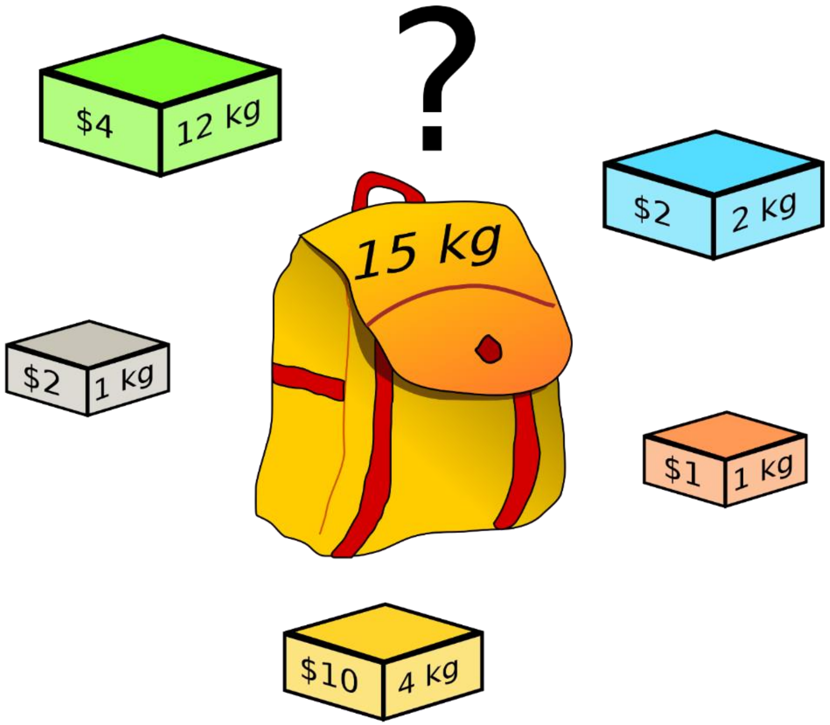
Given n items with associated values c_j and weights w_j and a knapsack of capacity K. Find a subset of items of maximum value to pack (the total weight does not exceed K).

Question to be answered

Is item j selected?

Modeling

0-1 decision variable: $x_j = 1$ iff j is selected.



Exercise

Describe MILP of Integer Knapsack Problem

Modeling with binary variables

Modeling multiple choice (1 out of n)

Example: Minimal cost assignment

- set of workers $\boldsymbol{W}$
- set of tasks $\boldsymbol{N}$, $|\mathbf{W}| = |\mathbf{N}|$
- cost function $c: W \times N \to \mathbb{Z}$

Assign worker on each task with minimal cost.

Question to be answered

Is worker i assigned on task j?

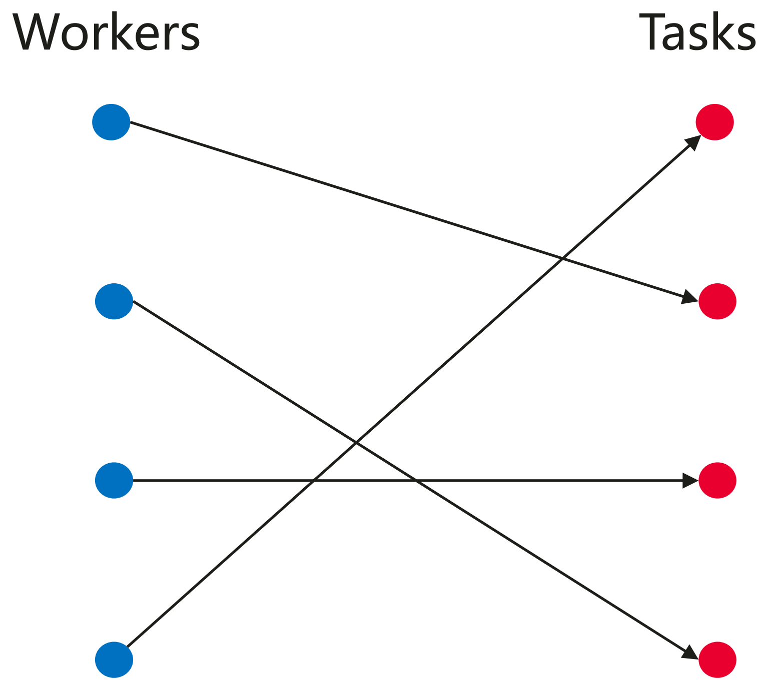
Modeling

- 0-1 decision variable: $x_{ij} = 1$ iff i assigned on j
- Exactly one assignment per worker

$$\forall i \in W: x_{ij} = 1 \iff x_{ik} = 0, \forall\, k \neq j \quad \Rightarrow \quad \boxed{\sum_{j \in N} x_{ij} = 1.}$$

- Exactly one assignment per task

$$\forall j \in N:\ x_{ij} = 1 \iff x_{kj} = 0, \forall\, k \neq i \quad \Rightarrow \quad \boxed{\sum_{i \in W} x_{ij} = 1.}$$



Exercise

Prove that for this problem fractional solution equals to optimal solution.

Modeling with binary variables

Modeling if-then dependency

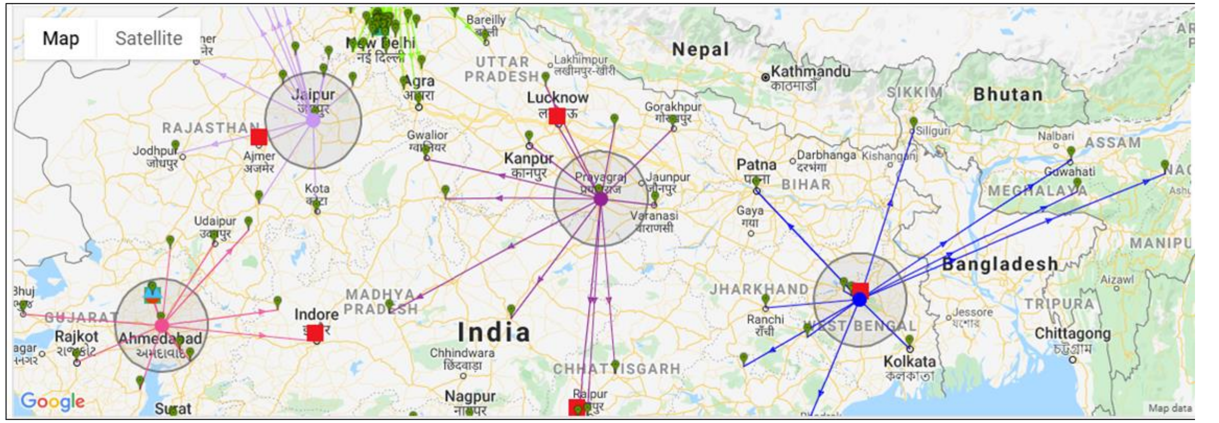
Example: Uncapacitated Facility Location Problem

Given n potential facility locations and m customers to serve from one facility, associated to costs c_j for opening facility j and d_{ji} for serving customer i from facility j. Find a subset of locations to open facilities that minimizes the cost (opening and service).

Questions to be answered

- Is facility j opened?
- Is customer i served from facility j?

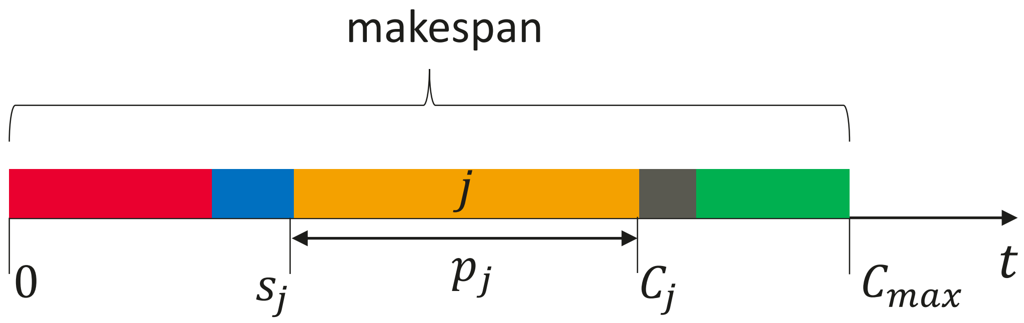
Modeling



Exercise
Describe MILP variables and constraints of uncapacitated facility location problem

Modeling with binary variables

Modeling exclusive disjunction (either-or)



Example: Scheduling Problem $1|r_j, d_j|C_{max}$
Find a minimal makespan schedule of n tasks with durations $\pi \in \mathbb{Z}_+$ on single machine without preemptions and simultaneous execution of multiple tasks. Each task j should be started not earlier than its *release time* r_j and be completed not later than its deadline d_j.

Questions to be answered

- Starting time of task j?

Modeling

- Integer variable $s_j \in [r_j, d_j - p_j + 1]$
- Non simultaneous processing:
  - 0-1 indicator variable $x_{ij} = 1$ iff $s_j - s_i \geq p_i$
  - Linearized constraint (let $M = \sum_{j=1}^{n} p_j$) ⟶ $\boxed{s_j - s_i \geq M\,(x_{ij} - 1) + p_i}$
- Either i precedes j or j precedes i
  - Constraint: $x_{ij} = 1 \Leftrightarrow x_{ji} = 0$ ⟶ $\boxed{x_{ij} + x_{ji} = 1}$

Exercise
Let $s_j = 0\ \forall j \in \{1, \dots, n\}$ .State the problem without using integer start time variables.