Мозговой Владислав

№1

$$S[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x)\right)dx$$

$$y(x) \in C^{\infty}[0,1] \qquad y'(0) = 0 \qquad y(1) = -3$$

$$\delta S[\delta y(x)] = \int_0^1 \left(8y(x) - \frac{d}{dx}(10y'(x)) + \frac{d^2}{dx^2}(2y''(x))\right)\delta y(x)\,dx + 2y''(x)\delta y'(x)\Big|_0^1$$

$$+ (10y'(x) - 2y'''(x))\delta y(x)\Big|_0^1$$

$$= \int_0^1 (2y^{(4)}(x) - 10y''(x) + 8y(x))\delta y(x)dx + 2y''(x)\,\delta y'(x)\Big|_0^1 + (10y'(x) - 2y^{(3)}(x))\delta y(x)\Big|_0^1$$

экстремаль:

$$y^{(4)}(x) - 5y''(x) + 4y(x) = 0$$

граничные условия:

$$y(1) = -3 \qquad y'(0) = 0$$

$$\left(\frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''}\right)\Big|_{x=0} = 0 \qquad \frac{\partial L}{\partial y''}\Big|_{x=1} = 0$$

$$\lambda^4 - 5\lambda^2 + 4 = (\lambda^2 - 1)(\lambda^2 - 4)$$

$$\Rightarrow y(x) = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x} \qquad y(1) = c_1e + \frac{c_2}{e} + c_3e^2 + \frac{c_4}{e^2} = -3$$

$$y'(x) = c_1e^x - c_2e^{-x} + 2c_3e^{2x} - 2c_4e^{-2x} \qquad y'(0) = c_1 - c_2 + 2c_3 - 2c_4 = 0$$

$$\left(\frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''}\right)\Big|_{x=0} = 10y'(x) - 2y'''(x)\Big|_{x=0} = 0 \Rightarrow y'''(x)\Big|_{x=0} = 0$$

$$y''(x) = c_1e^x + c_2e^{-x} + 4c_3e^{2x} + 4c_4e^{-2x}$$

$$y'''(x) = c_1e^x - c_2e^{-x} + 8c_3e^{2x} - 8c_4e^{-2x}$$

$$y'''(0) = c_1 - c_2 + 8c_3 - 8c_4 = 0$$

$$\frac{\partial L}{\partial y''}\Big|_{x=1} = 2y''(1) = 0 \Leftrightarrow y''(1) = 0$$

$$y''(1) = c_1e + \frac{c_2}{e} + 4c_3e^2 + \frac{4c_4}{e^2} = 0$$

$$\Rightarrow C_1 = C_2 = -\frac{4}{e+e^{-1}} \qquad C_3 = C_4 = \frac{1}{e^2+e^{-2}}$$

$$\Rightarrow y(x) = \frac{e^{2x}+e^{-2x}}{e^2+e^{-2\cancel{x}}} - \frac{4(e^{x}+e^{-x})}{e+e^{-1}}$$

б) $F[y(x)] = S[y(x)] + 6y'(1)$

$$y'(0)=0 \Rightarrow F[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x) + 6y''(x)\right)dx$$

$$\delta F[\delta y(x)] = \int_0^1 \left(2y^{(4)}(x) - 10y''(x) + 8y(x)\right)\delta y(x)dx + (2y''(x)+6)\delta y'(x)\Big|_0^1 + (10y'(x) - 2y'''(x))\delta y(x)\Big|_0^1$$

Экстремаль удовлетворяет уравнению $\lambda^4 - 5\lambda^2 + 4$

с граничными условиями

$$y(1) = -3 \qquad y'(0) = 0$$

$$10y'(0) - 2y'''(0) = 0 \Rightarrow y'''(0) = 0$$

$$2y''(1) + 6 = 0 \Rightarrow y''(1) = -3$$

$$y''(1) = C_1 e + \frac{C_2}{e} + 4C_3 e^2 + \frac{4C_4}{e^2} = -3$$

$$\Rightarrow C_1 = C_2 = -\frac{3}{e+e^{-1}} \qquad C_3 = C_4 = 0$$

$$y(x) = -\frac{3(e^{x}+e^{-x})}{e+e^{-1}}$$

№2

$$\Phi[y(x)] = 2y(\alpha) + \int_0^1 (y^2+(y')^2)dx = 2y(\alpha) + \int_0^\alpha (y^2+(y')^2)dx + \int_\alpha^1 (y^2+(y')^2)dx$$

$$\Delta\Phi = 2\delta y(\alpha) + \int_0^1 2y\delta y + (\delta y)^2 + 2y'\delta y' + (\delta y')^2 dx \ominus$$

$$* \int_0^\alpha 2y'\delta y' dx = \int_0^\alpha 2y' d\delta y = 2y'\delta y\Big|_0^\alpha - \int_0^\alpha \delta y\, 2y'' dx$$

$$\int_\alpha^1 2y'\delta y' dx = \int_\alpha^1 2y' d\delta y = 2y'\delta y\Big|_\alpha^1 - \int_\alpha^1 \delta y\, 2y'' dx$$

$2y'\delta y\Big|_0^\alpha = 0$ при $x=0$ т.к. δy фиксировано

$2y'\delta y\Big|_\alpha^1 = 0$ при $x=1$ т.к. δy фиксировано

$\Rightarrow$ в точке α $\exists$ производная $y'(\alpha)$

$$\Delta\Phi \ominus 2\delta y(\alpha) + \int_0^1 2y\delta y + (\delta y)^2 + (\delta y')^2 - \delta y\, 2(y)'' dx$$

б) $\delta\Phi_{y_0}[\delta y_0] = 2\delta y(\alpha) + \int_0^1 (2y - 2y'')\delta y\, dx$

$\varphi_{y_0}[\delta y] = \int_0^1 (\delta y)^2 - (\delta y')^2 dx$ удовлетворяет $\lim\limits_{\|\delta y\|\to 0} \dfrac{\varphi_{y_0}[\delta y(x)]}{\|\delta y(x)\|} = 0$

$$\begin{cases} 2y - 2y'' = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

$y = y'' \Rightarrow$ хар. многочлен $t^2 = 1 \Rightarrow t = \pm 1$

$y(x) = c_1 e^x + c_2 e^{-x}$

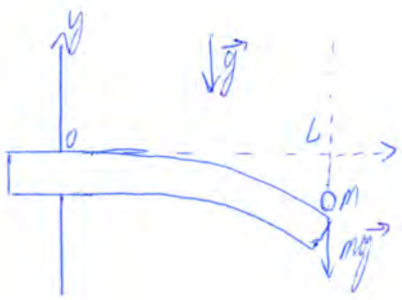
$$\begin{cases} y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \\ y(1) = c_1 e + c_2 e^{-1} = 0 \end{cases}$$

$c_1(e - e^{-1}) = 0$

$c_1 = c_2 = 0$

$y(x) = 0$

№3



Кинетической энергии у балки нет, поэтому балка принимает положение, в котором потенциальная энергия в экстремуме

$$\delta U_{тяж} = mg\, y(L)$$

$$\delta U_{упр} = \frac{k}{2}(y'')^2 dx$$

$$U[y(x)] = \int_0^L \left(mg\, y(L) + \frac{k}{2}(y''(x))^2 dx\right) = \int_0^L \left(mg\, y'(x) + \frac{k}{2}(y''(x))^2\right) dx$$

$$\delta U[\delta y(x)] = \int_0 k y^{(4)}(x)\,\delta y(x)\,dx + k y''\,\delta y'(x)\Big|_0^L + (mg - k y''')\,\delta y(x)\Big|_0^L$$

Экстремаль $\delta U[\delta y(x)]$ удовлетворяет $\cancel{k} y^{(4)}(x) = 0$

$y(0)=0 \quad y'(0)=0 \quad y''(L)=0 \quad y'''(L)=\frac{mg}{k}$

$$k\, y''(x)\Big|_{x=L} = 0 \Rightarrow y''(L) = 0$$

$$mg - k y'''(x)\Big|_{x=L} = 0 \Rightarrow y''(L) = 0$$

$\cancel{mg - k y'''(x)|_{x=L} = 0} \Rightarrow \quad y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$

$$y(0) = c_4 = 0$$

$$y'(0) = 3c_1 x^2 + 2c_2 x + c_3\Big|_{x=0} = c_3 = 0$$

$$y''(L) = 6c_1 x + 2c_2\Big|_{x=L} = 6c_1 L + 2c_2 = 0$$

$$y'''(L) = 6c_1\Big|_{x=L} = 6c_1 = \frac{mg}{k}$$

$$c_1 = \frac{mg}{6k} \quad c_2 = -\frac{mgL}{2k} \quad c_3 = c_4 = 0$$

$$y(x) = \frac{mg}{6k}x^3 - \frac{mgL}{2k}x^2$$

№1

$$T_{кин} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} \qquad V = gz$$

Общая энергия $E = T_{кин} + V = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} + gz$

$$L = T_{кин} - V = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполняется З.С.Э.

$\mathcal{L} = L - \lambda f(\bar{r})$, где $f(\bar{r}) = R^2 - \bar{r}^2 = 0$

$$\Rightarrow \mathcal{L} = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz + \lambda(x^2+y^2+z^2-R^2)$$

$\mathcal{L}_x: \ \ddot{x} - 2\lambda x = 0$

$\mathcal{L}_y: \ \ddot{y} - 2\lambda y = 0$

$\mathcal{L}_z: \ \ddot{z} - 2\lambda z + g = 0$

Найдем λ

$$x\mathcal{L}_x + y\mathcal{L}_y + z\mathcal{L}_z = x\ddot{x} + y\ddot{y} + z\ddot{z} - 2\lambda\overbrace{(x^2+y^2+z^2)}^{R^2} + gz = 0$$

$$2(x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \ddot{f}(\bar{r}) = 0$$

$$\Rightarrow x\ddot{x} + y\ddot{y} + z\ddot{z} = -(\dot{x}^2+\dot{y}^2+\dot{z}^2) = -2E + 2gz$$

$$2\lambda R^2 = gz - 2E + 2gz \Rightarrow \lambda = \frac{3gz - 2E}{2R^2}$$

$$\bar{N} = \left(\lambda\frac{\partial f(\bar{r})}{\partial x}, \lambda\frac{\partial f(\bar{r})}{\partial y}, \frac{\partial f(\bar{r})}{\partial z}\right) = 2\lambda(x,y,z) = \frac{3gz-2E}{R^2}(x,y,z)$$

$$\Rightarrow \bar{N} = \frac{3gz - 2E}{R^2}(x,y,z)$$

№5

$y(x) \in C^2[-a,a]$

$I(y) = \int\limits_{-a}^{a} \sqrt{1+y'^2}\,dx = l = const$

$l > 2a$

$y(a) = y(-a) = 0$

а)

$$J(y) = \int\limits_{-a}^{a} y\,dx + \lambda\left(\int\limits_{-a}^{a}\sqrt{1+y'^2}\,dx - l\right) = \int\limits_{-a}^{a}\left(y + \lambda\sqrt{1+y'^2} - \frac{\lambda l}{2a}\right)dx$$

если $J(y)$ имеет экстремум в $y(x)$, то

$$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) - \frac{\partial h}{\partial y} = 0, \text{ т.е. } \frac{y''}{(1+y'^2)^{\frac{3}{2}}} - \frac{1}{\lambda} = 0$$

$$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{d}{dx}\left(\lambda\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\lambda y''}{(1+y'^2)^{\frac{3}{2}}} \qquad \frac{\partial h}{\partial y} = 1$$

$\frac{\partial h}{\partial x} = 0 \Rightarrow$ выполнен ЗСЭ

$$y'\frac{\partial h}{\partial y'} + x'\frac{\partial h}{\partial x'} - h = \frac{\lambda y'^2}{\sqrt{1+y'^2}} - y - \lambda\sqrt{1+y'^2} + \frac{\lambda l}{2a} = const$$

Обозначим $\frac{\lambda y'^2}{\sqrt{1+y'^2}} - y - \lambda\sqrt{1+y'^2} = c_1 = const$

$$\frac{\lambda y'^2}{\sqrt{1+y'^2}} - \lambda\sqrt{1+y'^2} = c_1 + y$$

$$-\lambda = (y+c_1)\sqrt{1+y'^2}$$

$$y' = \sqrt{\frac{\lambda^2}{(y+c_1)^2} - 1}$$

т.е. $\frac{dx}{dy}=\sqrt{\frac{(y+c_1)^2}{\lambda^2-(y+c_1)^2}} \Rightarrow x+c_2=\int\frac{y+c_1}{\sqrt{\lambda^2-(y+c_1)^2}}dy$

пусть $y+c_1=\lambda\sin\varphi$, тогда $x+c_2=\lambda\cos\varphi$

следовательно $\left.\begin{array}{l} y+c_1=\lambda\sin\varphi \\ x+c_2=\lambda\cos\varphi \end{array}\right| \Rightarrow (x+c_2)^2+(y+c_1)^2=\lambda^2$

$y(a)=y(-a)=0 \Rightarrow (a+c_2)^2+c_1^2=(-a+c_2)^2+c_1^2=\lambda^2 \Rightarrow c_2=0$

Так как $y(x)$ является функцией, то $c_1\geq 0 \Rightarrow 2a<l\leq\pi a$

$x^2+(y+c_1)^2=\lambda^2$, $c_1\geq 0$, т.е.

$y=\sqrt{\lambda^2-x^2}-c_1$, $c_1\geq 0$

б) $l=2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda|$

в) при $l=\frac{\pi a}{\sqrt{2}}$

$\frac{\pi a}{\sqrt{2}}=2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda| \Rightarrow |\lambda|=a\sqrt{2}$

$y(x)=\sqrt{2a^2-x^2}-a$