

a)

$$\begin{cases} m\ddot{x} = -k_1x - k_2(x-y) \\ 2m\ddot{y} = k_2(x-y) - k_3(y-z) \\ m\ddot{z} = k_3(y-z) - k_4z \end{cases}$$

б)  $\ddot{X} = -AX$

$$A = \frac{1}{m} \begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} 5 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -2 & 5-\lambda \end{vmatrix} = (5-\lambda)((2-\lambda)(5-\lambda)-2) + 2((\lambda-5)) =$$

$$= (5-\lambda)((2-\lambda)(5-\lambda)-4) =$$

$$= -(\lambda-5)(10-7\lambda+\lambda^2-4) = -(\lambda-5)(\lambda^2-7\lambda+6) = -(\lambda-5)(\lambda-1)(\lambda-6)$$

$$\omega_1^2 = \frac{5k}{m} \rightarrow \omega_1 = \sqrt{\frac{5k}{m}}$$

$$\omega_2^2 = \frac{k}{m} \rightarrow \omega_2 = \sqrt{\frac{k}{m}} \quad - \text{нормальные частоты}$$

$$\omega_3^2 = \frac{6k}{m} \rightarrow \omega_3 = \sqrt{\frac{6k}{m}}$$

$\lambda=5$

$$\begin{pmatrix} 0 & -2 & 0 \\ -1 & -3 & -1 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{cases} x_1 + 3x_2 + x_3 = 0 \\ x_2 = 0 \end{cases} \quad \psi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{\lambda=1} \quad \begin{pmatrix} 4 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & -4 \\ -1 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = x_2 - x_3 \\ x_2 = 2x_3 \end{cases} \quad \psi_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

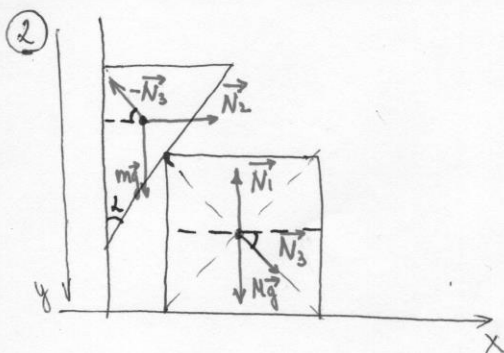
$$\underline{\lambda=5} \quad \begin{pmatrix} -1 & -2 & 0 \\ -1 & -4 & -1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & -1 \\ +1 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} 2x_2 = -x_3 \\ x_1 = -4x_2 - x_3 \end{cases} \Rightarrow \psi_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Ответ в пункте б):

Нормальные частоты:  $\sqrt{\frac{5k}{m}}$ ,  $\sqrt{\frac{k}{m}}$ ,  $\sqrt{\frac{6k}{m}}$ .

Нормальные моды:  $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ .



a) # степеней свободы = 2 - 1 = 1.

$$\text{б) } \begin{cases} M\ddot{x} = N_3 \cos \alpha \\ 0 = N_2 - N_3 \cos \alpha \\ 0 = N_1 - N_3 \sin \alpha - Mg \\ m\ddot{y} = mg - N_3 \sin \alpha \\ \ddot{x} = \ddot{y} \tan \alpha \end{cases}$$

б)  $N_2 = N_3 \cos \alpha$

$$\ddot{x} = \frac{N_3 \cos \alpha}{M}$$

$$N_1 = N_3 \sin \alpha + Mg$$

$$\ddot{y} = g - \frac{N_3}{m} \sin \alpha$$

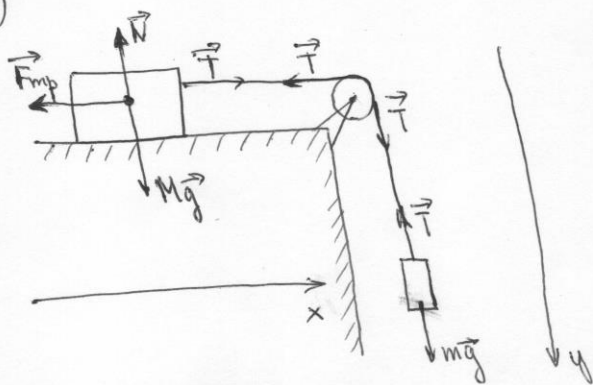
$$\frac{N_3 \cos \alpha}{M} = g \tan \alpha - \frac{N_3}{m} \sin \alpha \tan \alpha$$

$$N_3 \left( \frac{\cos \alpha}{M} + \frac{\sin \alpha \tan \alpha}{m} \right) = g \tan \alpha \Rightarrow N_3 = \frac{(g \tan \alpha) M m}{m \cos \alpha + M \sin \alpha \tan \alpha}$$

$$\ddot{x} = \frac{mg \sin \alpha}{m \cos \alpha + M \sin \alpha \tan \alpha}$$

$$\ddot{y} = \frac{mg \cos \alpha + Mg \sin \alpha \tan \alpha - Mg \tan \alpha \sin \alpha}{m \cos \alpha + M \sin \alpha \tan \alpha} = \frac{mg \cos \alpha}{m \cos \alpha + M \sin \alpha \tan \alpha}$$

3



\* Заметим, что если  $|kN| > |T|$ , то купчик M не будет двигаться. В этом случае  $T = mg$ ,  $N = Mg$   
 $kMg > mg$   
 $\Rightarrow \boxed{k > \frac{m}{M}}$

a) # степеней свободы  $= 2 - 1 = 1$

$$\delta) \begin{cases} m\ddot{y} = mg - T \\ M\ddot{x} = T - kN \\ N = Mg \\ \ddot{x} = \ddot{y} \end{cases}$$

$$\begin{aligned} \delta) \quad \ddot{y} &= g - \frac{T}{m} \\ \ddot{x} &= \frac{T}{M} - \frac{kMg}{M} \end{aligned}$$

$$g - \frac{T}{m} = \frac{T}{M} - kg$$

$$T\left(\frac{1}{m} + \frac{1}{M}\right) = g(1+k)$$

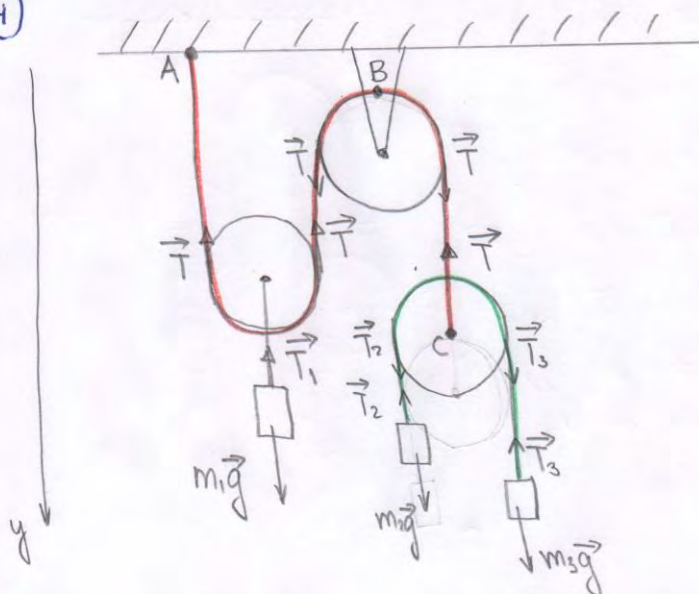
$$T = \frac{mMg(1+k)}{m+M}$$

$$\ddot{x} = \ddot{y} = g - \frac{Mg(1+k)}{m+M} = \frac{mg + Mg - Mg - Mgk}{m+M} =$$

$$= \boxed{\frac{m - Mk}{m+M} g}$$



4



$$\begin{aligned}
 1. \quad & m_1 \ddot{x}_1 = m_1 g - T_1 \\
 & m_2 \ddot{x}_2 = m_2 g - T_2 \\
 & m_3 \ddot{x}_3 = m_3 g - T_3 \\
 & T_2 = T_3 = T/2 \\
 & T_1 = 2T
 \end{aligned}$$

2. Пусть в момент времени  $t=0$  длина участка АВ красной нити равнялась  $l_{AB}$ , длина участка ВС —  $l_{BC}$ .

Если бы блок С был неподвижен:  $\ddot{x}_2 + \ddot{x}_3 = 0$

$$\begin{cases}
 x_2 = \tilde{x}_2 + \Delta l_{BC} \Rightarrow \tilde{x}_2 = x_2 - \Delta l_{BC} \\
 x_3 = \tilde{x}_3 + \Delta l_{BC} \Rightarrow x_3 = -x_2 + \Delta l_{BC} \cdot 2 \Rightarrow \Delta l_{BC} = \frac{x_2 + x_3}{2} \\
 \ddot{x}_2 + \ddot{x}_3 = 0
 \end{cases}$$

Для красной нити имеем:  $l_{AB} + 2x_1 + l_{BC} + \frac{x_2 + x_3}{2} = \text{const} \Rightarrow$

$$\Rightarrow 4\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$$

$$3. \quad \ddot{x}_1 = g - \frac{2T}{m_1}, \quad \ddot{x}_2 = g - \frac{T}{2m_2}, \quad \ddot{x}_3 = g - \frac{T}{2m_3}$$

$$4g - \frac{8T}{m_1} + g - \frac{T}{2m_2} + g - \frac{T}{2m_3} = 0$$

$$T \left( \frac{8}{m_1} + \frac{1}{2m_2} + \frac{1}{2m_3} \right) = 6g \Rightarrow T = \frac{12g \cdot m_1 m_2 m_3}{16 m_2 m_3 + m_1 m_3 + m_1 m_2}$$

$$\ddot{x}_1 = \frac{16 m_2 m_3 g + m_1 m_3 g + m_1 m_2 g - 24 m_2 m_3 g}{16 m_2 m_3 + m_1 m_3 + m_1 m_2} =$$

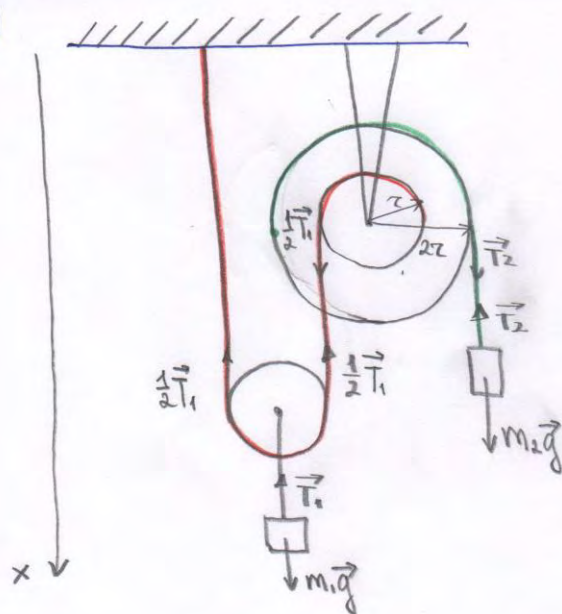
$$= \frac{m_1 m_3 + m_1 m_2 - 8 m_2 m_3}{16 m_2 m_3 + m_1 m_3 + m_1 m_2} g$$

$$\ddot{x}_2 = \frac{16 m_2 m_3 + m_1 m_2 - 5 m_1 m_3}{16 m_2 m_3 + m_1 m_3 + m_2 m_3} g$$

$$\ddot{x}_3 = \frac{16 m_2 m_3 + m_1 m_3 - 5 m_1 m_2}{16 m_2 m_3 + m_1 m_3 + m_2 m_3} g$$

• Число степеней свободы =  $3 - 1 = 2$

5



$$\begin{aligned} m_1 \ddot{x}_1 &= m_1 g - T_1 \\ m_2 \ddot{x}_2 &= m_2 g - T_2 \\ T_2 &= \frac{1}{2} \cdot \frac{1}{2} T_1 = \frac{1}{4} T_1 \end{aligned}$$

по часовой стрелке

Пусть катушка повернулась на угол  $\varphi$ . Тогда

$$\Delta l_{\text{сп}} = x_2 = 2r\varphi$$

$$\Delta l_{\text{кр}} = -r\varphi = \frac{-x_2}{2} \Rightarrow x_1 = \frac{\Delta l_{\text{кр}}}{2} = \frac{-x_2}{4} \Rightarrow 4x_1 = -x_2$$

$$\Rightarrow 4\ddot{x}_1 = -\ddot{x}_2$$

• Число степеней свободы:  $2 - 1 = 1$

$$\ddot{x}_1 = g - \frac{T_1}{m_1} = g - \frac{4T_2}{m_1}$$

$$\ddot{x}_2 = g - \frac{T_2}{m_2}$$

$$4g - \frac{16T_2}{m_1} = -g + \frac{T_2}{m_2} \Rightarrow 5g = T_2 \left( \frac{16}{m_1} + \frac{1}{m_2} \right)$$

$$\Rightarrow T_2 = \frac{5m_1 m_2 g}{16m_2 + m_1}$$

$$\ddot{x}_1 = g - \frac{4 \cdot 5m_2 g}{16m_2 + m_1} = \frac{16m_2 g + m_1 g - 20m_2 g}{16m_2 + m_1} = \frac{m_1 - 4m_2}{16m_2 + m_1} g$$

$$\ddot{x}_2 = g - \frac{5m_1 g}{16m_2 + m_1} = \frac{16m_2 + m_1 - 5m_1}{16m_2 + m_1} g = \frac{4(4m_2 - m_1)}{16m_2 + m_1} g$$



⑥

$$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi) = (\vec{e}_x, \vec{e}_y, \vec{e}_z) \begin{pmatrix} \sin\theta \cos\varphi & \cos\theta \cos\varphi & -\sin\varphi \\ \sin\theta \sin\varphi & \cos\theta \sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \tan\theta = (x^2 + y^2)^{1/2}/z, \quad \tan\varphi = y/x$$

$$\vec{e}_r = \vec{e}_x \sin\theta \cos\varphi + \vec{e}_y \sin\theta \sin\varphi + \vec{e}_z \cos\theta$$

$$\begin{aligned} \dot{\vec{e}}_r &= \vec{e}_x (\cos\theta \cos\varphi \cdot \dot{\theta} - \sin\theta \sin\varphi \cdot \dot{\varphi}) + \\ &+ \vec{e}_y (\cos\theta \sin\varphi \cdot \dot{\theta} + \sin\theta \cos\varphi \cdot \dot{\varphi}) + \\ &+ \vec{e}_z \cdot (-\sin\theta) \dot{\theta} = \end{aligned}$$

$$\begin{aligned} &= \dot{\theta} (\vec{e}_x \cos\theta \cos\varphi + \vec{e}_y \cos\theta \sin\varphi - \vec{e}_z \sin\theta) \\ &+ \dot{\varphi} (-\vec{e}_x \sin\theta \sin\varphi + \vec{e}_y \sin\theta \cos\varphi) = \end{aligned}$$

$$= \dot{\theta} \vec{e}_\theta + \dot{\varphi} \sin\theta \vec{e}_\varphi \Rightarrow \boxed{\dot{\vec{e}}_r = \dot{\theta} \vec{e}_\theta + \dot{\varphi} \sin\theta \vec{e}_\varphi}$$

$$\vec{e}_\theta = \vec{e}_x \cos\theta \cos\varphi + \vec{e}_y \cos\theta \sin\varphi - \vec{e}_z \sin\theta$$

$$\begin{aligned} \dot{\vec{e}}_\theta &= \vec{e}_x (-\sin\theta \cos\varphi \dot{\theta} - \cos\theta \sin\varphi \dot{\varphi}) + \vec{e}_y (-\sin\theta \sin\varphi \dot{\theta} + \cos\theta \cos\varphi \dot{\varphi}) - \\ &- \vec{e}_z \cos\theta \dot{\theta} = -\dot{\theta} (\vec{e}_x \sin\theta \cos\varphi + \vec{e}_y \sin\theta \sin\varphi + \vec{e}_z \cos\theta) + \end{aligned}$$

$$+ \dot{\varphi} \cos\theta (-\vec{e}_x \sin\varphi + \vec{e}_y \cos\varphi) = -\dot{\theta} \vec{e}_r + \dot{\varphi} \cos\theta \vec{e}_\varphi \Rightarrow$$

$$\Rightarrow \boxed{\dot{\vec{e}}_\theta = -\dot{\theta} \vec{e}_r + \dot{\varphi} \cos\theta \vec{e}_\varphi}$$

$$\vec{e}_\varphi = -\vec{e}_x \sin\varphi + \vec{e}_y \cos\varphi$$

$$\dot{\vec{e}}_\varphi = -\vec{e}_x \cos\varphi \dot{\varphi} - \vec{e}_y \sin\varphi \dot{\varphi} = -\dot{\varphi} (\cos\varphi \vec{e}_x + \sin\varphi \vec{e}_y -$$

$$+ \sin^2\theta \sin\varphi \vec{e}_y + \sin\theta \cos\theta \vec{e}_z + \cos^2\theta \cos\varphi \vec{e}_x + \cos^2\varphi \sin\varphi \vec{e}_y -$$

$$- \sin\theta \cos\theta \vec{e}_z) = -\dot{\varphi} (\cos\varphi + \sin\varphi) =$$

$$= -\dot{\varphi} (\sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta) \Rightarrow \boxed{\dot{\vec{e}}_\varphi = -\dot{\varphi} (\sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta)}$$

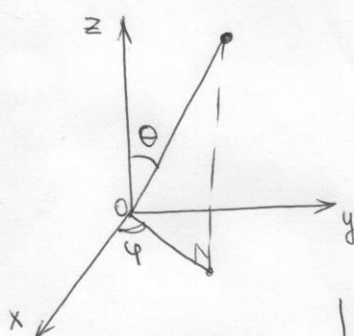


$$\vec{r} = r \vec{e}_r$$

$$\dot{\vec{r}} = \dot{r} \vec{e}_r + r \dot{\vec{e}}_r = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta + r \dot{\varphi} \sin \theta \vec{e}_\varphi$$

$$\begin{aligned} \ddot{\vec{r}} &= \ddot{r} \vec{e}_r + \dot{r} (\dot{\theta} \vec{e}_\theta + \dot{\varphi} \sin \theta \vec{e}_\varphi) + \dot{r} \dot{\theta} \vec{e}_\theta + r \ddot{\theta} \vec{e}_\theta + \\ &+ r \dot{\theta} (-\dot{\theta} \vec{e}_r + \dot{\varphi} \cos \theta \vec{e}_\varphi) + \dot{r} \dot{\varphi} \sin \theta \vec{e}_\varphi + r \dot{\varphi} \sin \theta \dot{\varphi} \vec{e}_\varphi + \\ &+ r \dot{\varphi} \cos \theta \dot{\theta} \vec{e}_\varphi + r \dot{\varphi} \sin \theta (-\dot{\varphi}) (\sin \theta \vec{e}_r + \cos \theta \vec{e}_\theta) = \end{aligned}$$

$$\begin{aligned} &= \vec{e}_r (\ddot{r} - r(\dot{\theta})^2 - r(\dot{\varphi})^2 \sin^2 \theta) + \\ &+ \vec{e}_\theta (2\dot{r}\dot{\theta} + r\ddot{\theta} - r(\dot{\varphi})^2 \sin \theta \cos \theta) + \\ &+ \vec{e}_\varphi (\dot{r}\dot{\varphi} \sin \theta + r\dot{\theta}\dot{\varphi} \cos \theta + \dot{r}\dot{\varphi} \sin \theta + r\ddot{\varphi} \sin \theta + r\dot{\varphi}\dot{\theta} \cos \theta) \end{aligned}$$



$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad \begin{aligned} \theta &\in [0, \pi] \\ \varphi &\in [0, 2\pi) \end{aligned}$$

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| = \begin{vmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} =$$

$$= r^2 \sin \theta \begin{vmatrix} \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi \\ \sin \theta \sin \varphi & \cos \theta \sin \varphi & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{vmatrix} =$$

$$= r^2 \sin \theta \left[ -\sin \varphi (-\sin^2 \theta \sin \varphi - \cos^2 \theta \sin \varphi) - \cos \varphi (-\sin^2 \theta \cos \varphi - \cos^2 \theta \cos \varphi) \right]$$

$$= r^2 \sin \theta (\sin^2 \varphi + \cos^2 \varphi) = r^2 \sin \theta = 0 \Leftrightarrow \begin{aligned} r &= 0 \\ \sin \theta &= 0 \\ \theta &\in [0, \pi] \end{aligned} \Leftrightarrow \begin{aligned} r &= 0 \\ \theta &= 0 \\ \theta &= \pi \end{aligned}$$

$\Rightarrow$  переход к сферическим координатам невыполним на  $\mathbb{R}^3 \setminus \{0_z\} \subset \mathbb{R}^3$ .