Лекция 5

Числа Гурвица

Уравнение транспозиции

$$H^\circ(u, p_1, p_2, \ldots) = \sum_{m=0}^{\infty} \sum_{\mu} h^\circ_{m,\mu}\, p_{\mu_1} p_{\mu_2} \cdots \frac{u^m}{m!}$$

ряд для несвязных чисел Гурвица

$$H(u, p_1, p_2, \ldots) = \sum_{m=0}^{\infty} \sum_{\mu} h_{m,\mu}\, p_{\mu_1} p_{\mu_2} \cdots \frac{u^m}{m!}$$

– для связных чисел

Теорема

$$\frac{\partial H^\circ}{\partial u} = \underbrace{\frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j)\, p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right)}_{=: W} H^\circ$$

Обозначение $H^\circ(u, p_1, p_2, \ldots) = \sum_{m=0}^{\infty} H^\circ_m(p_1, p_2, \ldots) \frac{u^m}{m!}$

Следствие $W H^\circ_m(p_1, p_2, \ldots) = H^\circ_{m+1}(p_1, p_2, \ldots)$

▷ $\frac{\partial H^\circ}{\partial u} = W H^\circ = W\left(H^\circ_0(p_1, \ldots)\right) \frac{u^1}{1!} + W H_2(p_1, \ldots) \frac{u^2}{2!} + \ldots$

$\|$ W не действует на u!

$$\frac{\partial}{\partial u}\left(H_0^\circ(p_1,p_2,\dots)+H_1^\circ(p_1,p_2,\dots)\frac{u^1}{1!}+H_2(p_1,p_2,\dots)\frac{u^2}{2!}+\dots\right)=$$

$$=H_1^\circ(p_1,p_2,\dots)+H_2^\circ(p_1,p_2,\dots)\frac{u^1}{1!}+H_3^\circ(p_1,p_2,\dots)\frac{u^2}{2!}+\dots$$

Т.К. У НАС $\frac{\partial}{\partial u}\left(\frac{u^n}{n!}\right)=\frac{u^{n-1}}{(n-1)!}$ И ВСЕ КОЭФ-ТЫ СДВИГАЮТСЯ

<u>ПРИМЕР</u> $$W=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}\left((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}+ij\,p_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\right)$$

$m=2$

$$H_0^\circ(p_1,p_2,\dots)=e^{p_1}=1+p_1+\frac{p_1^2}{2!}+\dots$$

0 ТРАНСПОЗИЦИЙ $\quad \nu=1^n \quad h^\circ_{0,1^n}=\frac{1}{n!}$

$$H_1^\circ(p_1,p_2,\dots)=W\,H_0^\circ(p_1,p_2,\dots)$$

$n=3$

$$W=\frac{1}{2}\left(2p_1^2\frac{\partial}{\partial p_2}+p_2\frac{\partial^2}{\partial p_1^2}\right)+2\cdot\frac{1}{2}\left(3p_1p_2\frac{\partial}{\partial p_3}+1\cdot 2p_3\frac{\partial^2}{\partial p_1\partial p_2}\right)+\dots$$

$i=j=1$ $\quad n=2$

$n=1 \to$ НИЧЕГО НЕТ, Т.К. НЕТ p_0

$$H_1^\circ(p_1,p_2,\dots)=\frac{1}{2}p_2\left(\frac{\partial^2}{\partial p_1^2}e^{p_1}\right)=\frac{1}{2}p_2e^{p_1}=$$

$$=\frac{1}{2}p_2+\frac{1}{2}p_1p_2+\dots$$

$m=1$

$$\frac{h^\circ_{1,2^1}}{2!} = \frac{1}{2}$$

$$H^\circ_1(p_1,p_2,\dots) = \sum_{\nu} h^\circ_{1,\nu}\, p_{\nu_1} p_{\nu_2}\dots$$

Доказательство теоремы

$$\frac{\partial H^\circ}{\partial u} = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}\left((i+j)\, p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right) H^\circ$$

$$H^\circ = \sum_{m=0}^{\infty}\sum_{\nu} h^\circ_{m,\nu}\, p_{\nu_1} p_{\nu_2}\dots \frac{u^m}{m!}$$

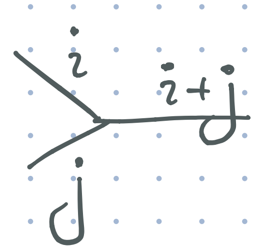
$$\frac{\partial H^\circ}{\partial u} = \sum_{m=0}^{\infty}\sum_{\nu} h^\circ_{m+1,\nu}\, p_{\nu_1}, p_{\nu_2},\dots \frac{u^m}{m!}$$

Хотим в правой части при $\frac{u^m}{m!}$ вместо $h^\circ_{m,\nu}$ получить $h^\circ_{m+1,\nu}$

$$\sigma = \tau_m \circ \dots \circ \tau_1$$

$$\tau_{m+1}\circ\sigma = \tau_{m+1}\circ\tau_m\circ\dots\circ\tau_1$$

Если циклы i, j склеились, то



$$p_i p_j \longrightarrow p_{i+j} \longrightarrow p_i p_j$$

— // — i, j разрезаются

$$H^\circ = \exp(H) \qquad H^\circ := H^\circ(p_1, p_2, \dots)$$
$$H := H(u, p_1, p_2, \dots)$$

$$\frac{\partial \exp(H)}{\partial u} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j} \frac{\partial}{\partial p_i \partial p_j} \right) \exp(H)$$

$$||$$

1) $\exp(H) \dfrac{\partial H}{\partial u}$

2) $\dfrac{\partial \exp(H)}{\partial p_k} = \exp(H) \dfrac{\partial H}{\partial p_k}$

3) $\dfrac{\partial^2}{\partial p_i \partial p_j} \exp(H) = \left(\dfrac{\partial^2 H}{\partial p_i \partial p_j} + \dfrac{\partial H}{\partial p_i} \cdot \dfrac{\partial H}{\partial p_j} \right) \exp(H)$

$$\frac{\partial H^\circ}{\partial u} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j) p_i p_j \frac{\partial H}{\partial p_{i+j}} + ij\, p_{i+j} \left(\frac{\partial^2 H}{\partial p_i \partial p_j} + \right.\right.$$
$$\left.\left. + \frac{\partial H}{\partial p_i} \cdot \frac{\partial H}{\partial p_j} \right)\right)$$

Формула Гурвица (1893)

$$h_{n-2+\ell(\nu),\nu} = \frac{(n-2+\ell(\nu))!}{|\mathrm{Aut}\,\nu|} n^{\ell(\nu)-3} \prod_{i=1}^{\ell(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}$$

$\ell(\nu)$ — число циклов в ν

$\nu = \nu_1, \nu_2, \dots, \nu_{\ell(\nu)}$

например, $\nu = 2,2,1,1 \Rightarrow |\mathrm{Aut}\,\nu| = 2!\,2!$

$n-2+\ell(\nu)$ — min число транспозиций, чтобы получить связное число

$n-1+\ell(\nu)-1$