Task

$$\prod_{n\geq 1}\left(1+tq^{2n-1}\right)=\sum_{k\geq 0}\frac{t^k q^{k^2}}{(1-q^2)(1-q^4)\dots(1-q^{2k})}$$

Solution

Notice that the coefficient of t^k on the left side depends on the k brackets from which we take tq^m. Enumerate this brackets by odd numbers, where $2n'+1$ connected with $(1+tq^{2n'+1})$.
Let i is the minimal degree of coefficient q^i of t^k and i is the sum of odd numbers from 1 to $2k-1$.
Then consider some other coefficient q^n and we can choose any bracket instead of $2k-1$. In other words, we can increase $2k-1$ by $0, 2, 4, 6, \dots$. A generating function of this bracket is $q^{2k-1}(1+q^2+q^4+\dots)=\frac{q^{2k-1}}{1-q^2}$.

Then consider next bracket $2k-3$, smallest number by which we can increase the number of this bracket is 4, since we cannot increase $2k-3$ by 2. A generating function of this bracket is $q^{2k-3}(1+q^4+\dots)=\frac{q^{2k-3}}{1-q^4}$.
Similarly for all other brackets.

$$\frac{q}{1-q^{2k}}\cdot\ldots\cdot\frac{q^{2k-1}}{1-q^2}=\frac{q^{k^2}}{(1-q^2)\cdot\ldots\cdot(1-q^{2k})}$$

Which proves the equation.