№ 1.

$$S[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x)\right) dx$$

$y(x) \in C^\infty[0,1] \qquad y'(0)=0 \qquad y(1)=-3$

$$S[y+\delta y] - S[y] = \int_0^1 \left(((y+\delta y)'')^2 + 5((y+\delta y)')^2 + 4(y+\delta y)^2\right) dx -$$

$$- \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x)\right) dx = \int_0^1 \left((\delta y'')^2 + 5(\delta y')^2 + 4(\delta y)^2 + \right.$$

$$\left. + 2y''\delta y'' + 10 y'\delta y' + 8y\delta y\right) dx$$

$$\Rightarrow \delta S[y] = \int_0^1 (2y''\delta y'' + 10y'\delta y' + 8y\delta y)\, dx =$$

$$= \int_0^1 2y'' d(\delta y') + \int_0^1 10 y' d(\delta y) + \int_0^1 8y\,\delta y\, dx =$$

$$= 2y''\cdot\delta y'\Big|_0^1 - 2\int_0^1 \delta y' y''' dx + 10y'\cdot\delta y\Big|_0^1 - 10\int_0^1 \delta y\cdot y'' dx + \int_0^1 8y\delta y dx =$$

$$= 2y''\cdot\delta y'\Big|_0^1 - 2\int_0^1 y''' d\delta y + 10y'\cdot\delta y\Big|_0^1 - 10\int_0^1 \delta y\cdot y'' dx + 8\int_0^1 y\delta y dx =$$

$$= 2y''\cdot\delta y'\Big|_0^1 - 2\cdot y'''\cdot\delta y\Big|_0^1 + 2\int_0^1 \delta y\cdot y'''' dy + 10y'\cdot\delta y\Big|_0^1 - 10\int_0^1 \delta y\cdot y'' dx + 8\int_0^1 y\delta y dx =$$

$$= \int_0^1 (2y'''' - 10y'' + 8y)\,\delta y\, dx + 2y''\cdot\delta y'\Big|_0^1 + (10y' - 2y''')\delta y\Big|_0^1$$

$$y'''' - 5y'' + 4y = 0$$

~~учтено~~ $y(1) = -3$ ~~[illegible]~~

~~$y'(1)=y''(1)=0$~~

т.к по усл. $y'(0)=0$

$y(0)$ не зафиксирован $\Rightarrow \delta(y(0))$ любой $\Rightarrow 10\cancel{y'}(0) - 2y'''(0) = 0$

$y'(0)=0 \Rightarrow \delta y'(0) = 0 \Rightarrow y''$ в т.0 прим. $\forall$ конечн. знач.

$y'''(0) = 0.$

$y'(1)$ не зафиксирован $\Rightarrow \delta y'(1)$ любой $\Rightarrow \underline{y''(1) = 0}$

$y'''' - 5y'' + 4y = 0 \qquad t^4 - 5t^2 + 4 = 0$

$(t^2-1)(t^2-4) = 0$

$t = \pm 1, \pm 2$

$$y(x) = c_1\cdot e^{-2x} + c_2\cdot e^{-x} + c_3\cdot e^{x} + c_4\cdot e^{2x}$$

учитывая: $y'(0)=0$, $y(1)=-3$, $y'''(0)=0$, $y''(1)=0$:

$$\begin{cases} (1)\quad y'(0) = -2c_1 - c_2 + c_3 + 2c_4 = 0 \\ (2)\quad y(1) = c_1\cdot e^{-2} + c_2\cdot e^{-1} + c_3\cdot e + c_4\cdot e^2 = -3 \\ (3)\quad y'''(0) = -8c_1 - c_2 + c_3 + 8c_4 = 0 \\ (4)\quad y''(1) = 4c_1\cdot e^{-2} + c_2\cdot e^{-1} + c_3\cdot e + 4\cdot c_4 e^2 = 0 \end{cases}$$

$(1) - (3):\ 6c_1 - 6c_4 = 0 \Leftrightarrow c_1 = c_4$ ~~[illegible]~~

т.к. $c_1 = c_4$, то у (1): $c_2 = c_3$

$$-\begin{cases} c_1(e^2+e^{-2}) + c_2(e+e^{-1}) = -3 \\ 4c_1(e^2+e^{-2}) + c_2(e+e^{-1}) = 0 \end{cases}$$

$$3c_1(e^2+e^{-2}) = 3$$

$$c_4 = c_1 = \frac{1}{e^2+e^{-2}}$$

$$4 + c_2(e+e^{-1}) = 0$$

$$c_3 = c_2 = -\frac{4}{e+e^{-1}}$$

$$\Rightarrow \overset{\text{экстремаль}}{y(x)} = \frac{e^{-2x}}{e^2+e^{-2}} - \frac{4e^{-x}}{e+e^{-1}} - \frac{4e^{x}}{e+e^{-1}} + \frac{e^{2x}}{e^2+e^{-2}}$$

б) $F[y(x)] = S[y(x)] + 6y'(1) =$ $\qquad y'(0)=0$

$$= \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x)) + 6y''(x)\right) dx$$

$$F[y+\delta y] - F[y] = \int_0^1 \left((\delta y'')^2 + 5(\delta y')^2 + 4(\delta y)^2 + 6\delta y'' + 2y''\delta y'' + 10y'\delta y' + 8y\delta y\right) dx$$

$$\Rightarrow \delta F[y] = \int_0^1 (6\delta y'' + 2y''\delta y'' + 10y'\delta y' + 8y\delta y)\,dx =$$

$$= \underbrace{\int_0^1 (2y'''' - 10y'' + 8y)\delta y\, dx + 2y''\delta y'\Big|_0^1 + (10y' - 2y''')\delta y\Big|_0^1} + 6\int_0^1 d\delta y' =$$

$$= \underbrace{\qquad\qquad\qquad\qquad} + 6\delta y'\Big|_0^1 =$$

$$= \int_0^1 (2y'''' - 10y'' + 8y)\delta y\, dx + (2y''+6)\delta y'\Big|_0^1 + (10y' - 2y''')\delta y\Big|_0^1$$

$$y'''' - 5y'' + 4y = 0$$

$y'(0)=0 \quad y(1) = -3$

$y(0)$ не зафиксирован $\Rightarrow \delta(y(0))$ любой $\Rightarrow 10\cancel{y'(0)} - 2y'''(0) = 0$

$y'(1)$ не зафиксирован $\Rightarrow \delta(y'(1))$ — любой $\qquad \boxed{y'''(0)=0}$

$\Downarrow$

$2y''(1) + 6 = 0 \Leftrightarrow \underline{y''(1) = -3}$

аналог. п.а): $y(x) = C_1 \cdot e^{-2x} + C_2 \cdot e^{-x} + C_3 \cdot e^{x} + C_4 \cdot e^{2x}$

$y'(0) = -2C_1 - C_2 + C_3 + 2C_4 = 0$

$y(1) = C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + C_4 \cdot e^2 = -3$

$y'''(0) = -8C_1 - C_2 + C_3 + 8C_4 = 0$

$y''(1) = 4C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + 4C_4 \cdot e^2 = -3$

$(1) - (3)$: $6C_1 - 6C_4 = 0 \qquad C_1 = C_4$

Из (1): $C_2 = C_3$

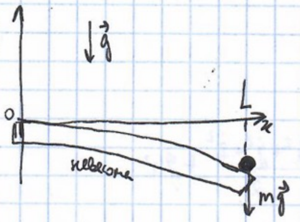
$$-\begin{cases} C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = -3 \\ 4C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = -3 \end{cases}$$

$3C_1(e^2 + e^{-2}) = 0$

$C_1 = C_4 = 0$

$C_2 = C_3 = -\frac{3}{e + e^{-1}} \qquad y(x) = -\frac{3}{e+e^{-1}} \cdot e^{-x} - \frac{3}{e+e^{-1}} \cdot e^{x}$

№2.



Кинет. эн.: 0

$L = U$

$y(0)=y'(0)=0$

$y(L),\ y'(L) - \forall$

$\forall\, \delta y'(L)$:

$$\frac{\partial}{\partial y''}\left(\frac{K(y'')^2}{2}+Lmgy'\right)\Big|_{x=L}=0 \iff y''(L)=0$$

$\forall\, \delta y(L)$:

$$\left(\frac{\partial}{\partial y'}-\frac{d}{dx}\frac{\partial}{\partial y''}\right)\left(\frac{K(y'')^2}{2}+Lmgy'\right)\Big|_{x=L}=0 \iff y'''(L)=\frac{Lmg}{K}$$

$$\delta U_{тяж} = mgy(L)\,dx \qquad \delta U_{упр} = \frac{K(y'')^2}{2}dx$$

$$U[y(x)] = \int_0^L\left(\frac{K(y'')^2}{2}+mgy'(L)\right)dx = \int_0^L\left(\frac{K(y'')^2}{2}+Lmgy'(x)\right)dx$$

длина → L, пер.

Из лекции: $$\delta U[\delta y(x)] = \int_0^L \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\cdot\frac{\partial L}{\partial \dot y}+\frac{d^2}{dx^2}\cdot\frac{\partial L}{\partial \ddot y}\right)\delta y(x)\,dx +$$

$$+ \frac{\partial L}{\partial \ddot y}\cdot\delta\dot y(x)\Big|_0^L + \left(\frac{\partial L}{\partial \dot y}-\frac{d}{dx}\cdot\frac{\partial L}{\partial \ddot y}\right)\delta y(x)\Big|_0^L =$$

$$= \int_0^L Ky^{(IV)}(x)\,\delta y(x)\,dx + Ky''(x)\,\delta y'(x)\Big|_0^L + \left(Lmg - Ky'''(x)\right)\delta y(x)\Big|_0^L$$

$y(0)=0,\ y'(0)=0 \quad y''(L)=0 \quad y'''(L)=\frac{mgL}{K}$

$K\cdot y^{(IV)}(x)=0$

$y(x)=C_1+C_2x+C_3x^2+C_4x^3$

$y(0)=C_1=0$

$y'(0)=C_2=0$

$y''(L)=2C_3+6C_4\cdot L=0$

$y'''(L)=6C_4=\frac{mgL}{K}$ $\qquad C_4=\frac{mgL}{6K}$

$C_3=-3C_4\cdot L=-\frac{mgL^2}{2K}$

$$y(x)=-\frac{mgL^2}{2K}x^2+\frac{mgL}{6K}x^3$$

№3.

$m = 1$

$T = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} \quad U = mgh = gz$

$L = T - U = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполн. ЗСЭ $E = T + U = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} + gz$

$f(\vec{r}) = R^2 - \vec{r}^{\,2} = 0$

$\mathcal{L} = L + \lambda(R^2 - x^2 - y^2 - z^2) = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz + \lambda(R^2 - x^2 - y^2 - z^2)$

Уравнения Э-Л:

$\mathcal{L}_x = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\right) - \frac{\partial}{\partial x} = \frac{d}{dt}(\dot{x}) + 2\lambda x = \ddot{x} + 2\lambda x = 0$

$\mathcal{L}_y = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\right) - \frac{\partial}{\partial y} = \ddot{y} + 2\lambda y = 0$

$\mathcal{L}_z = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{z}}\right) - \frac{\partial}{\partial z} = \ddot{z} + g + 2\lambda z = 0$

$\frac{d^2}{dt^2} f(\vec{r}) = (R^2 - x^2 - y^2 - z^2)'' = -2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \underbrace{x\ddot{x} + y\ddot{y} + z\ddot{z}}) = 0$

$\cancel{\left(0 = \mathcal{L}_x + \mathcal{L}_y + \mathcal{L}_z = \ddot{x} + \ddot{y} + \ddot{z} - 2\lambda(x+y+z)\right)}$

$x \cdot \mathcal{L}_x + y\mathcal{L}_y + z \cdot \mathcal{L}_z = x\ddot{x} + 2\lambda x^2 + y\ddot{y} + 2\lambda y^2 + z\ddot{z} + gz + 2\lambda z^2 = 0$

$-\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + 2\lambda(x^2+y^2+z^2) + gz = 0$

$-\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + 2\lambda R^2 + gz = 0$

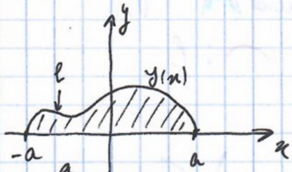
$-2E + 3gz + 2\lambda R^2 = 0$

$\lambda = \frac{+2E - 3gz}{2R^2}$

$N = \lambda\left(\frac{\partial f(\vec{r})}{\partial x}, \frac{\partial f(\vec{r})}{\partial y}, \frac{\partial f(\vec{r})}{\partial z}\right) = -2\lambda(x, y, z) = \frac{-2E + 3gz}{R^2}(x, y, z)$

№4.

$l > 2a$



$$l = \int_{-a}^{a} \sqrt{1+y'^2}\,dx$$

$$J[y(x)] = \int_{-a}^{a} y(x)\,dx + \lambda\left(\int_{-a}^{a}\sqrt{1+y'^2}\,dx - l\right) = \int_{-a}^{a}\left(y(x) + \lambda\sqrt{1+y'^2} - \frac{\lambda l}{2a}\right)dx$$

$$J[y(x)] = \int_{-a}^{a} L\,dx$$

$$L = y + \lambda\sqrt{1+y'^2} - \frac{\lambda l}{2a}$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) - 1 = 0$$

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c \qquad \frac{\lambda^2 y'^2}{1+y'^2} = (x+c)^2$$

$$\lambda^2 \cdot y'^2 = (1+y'^2)(x+c)^2 = (x+c)^2 + y'^2(x+c)^2$$

$$y'^2(\lambda^2 - (x+c)^2) = (x+c)^2$$

$$y'^2 = \frac{(x+c)^2}{\lambda^2-(x+c)^2} \qquad y' = \pm\frac{x+c}{\sqrt{\lambda^2-(x+c)^2}} \qquad y = \tilde{c} \pm \sqrt{\lambda^2-(x+c)^2}$$

т.к. $\left(\tilde{c} \pm \sqrt{\lambda^2-(x+c)^2}\right)' = \dfrac{-2x - 2c}{2\sqrt{\lambda^2-(x+c)^2}}$

$y(a) = y(-a) = 0$

т.е. чётн. $\Rightarrow c = 0$

$y = \tilde{c} \pm \sqrt{\lambda^2-(x+c)^2}$

$$\Rightarrow y = \tilde{c} \pm \sqrt{\lambda^2 - x^2}$$

$$y(-a) = y(a) = \tilde{c} \pm \sqrt{\lambda^2 - a^2} = 0$$

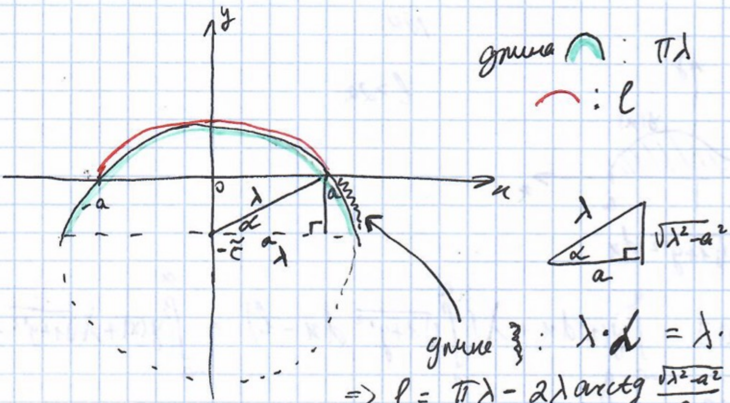
$$\Rightarrow \tilde{c} = \mp\sqrt{\lambda^2 - a^2}$$

отсюда $(*)$ $y(x) = \mp\sqrt{\lambda^2-a^2} \pm \sqrt{\lambda^2-x^2}$

$$\Rightarrow (y \pm \tilde{\tilde{c}})^2 = \lambda^2 - x^2 \qquad (y \pm \tilde{\tilde{c}})^2 + x^2 = \lambda^2$$

$\Rightarrow$ Когда макс. площадь, тогда окр-ть с центром $(0, \pm\tilde{\tilde{c}})$ и радиусом λ.

б)



длина: $\lambda\cdot\alpha = \lambda\cdot \operatorname{arctg}\frac{\sqrt{\lambda^2-a^2}}{a}$

$\Rightarrow l = \pi\lambda - 2\lambda \operatorname{arctg}\frac{\sqrt{\lambda^2-a^2}}{a}$

в) $l = \frac{\pi a}{\sqrt{2}}$

$\pi\lambda - 2\lambda \operatorname{arctg}\frac{\sqrt{\lambda^2-a^2}}{a} = \frac{\pi a}{\sqrt{2}}$

$2\lambda \operatorname{arctg}\frac{\sqrt{\lambda^2-a^2}}{a} = \pi\lambda - \frac{\pi a}{\sqrt{2}}$

$\frac{\sqrt{\lambda^2-a^2}}{a} = \operatorname{tg}\left(\frac{\pi\lambda - \frac{\pi a}{\sqrt{2}}}{2\lambda}\right)$

Решение: $\lambda = a\sqrt{2}$: $1 = \frac{\sqrt{2a^2-a^2}}{a} = \operatorname{tg}\left(\frac{a\sqrt{2}\pi - \frac{\pi a}{\sqrt{2}}}{2a\sqrt{2}}\right) = \operatorname{tg}\left(\frac{\pi}{4}\right) = 1$

$\overset{(*)}{\Rightarrow} y_{(x)} = \mp\sqrt{2a^2-a^2} \pm \sqrt{2a^2-x^2} = \mp a \pm \sqrt{2a^2-x^2}$