СЕМИНАР ПО МАТАНУ

$\boxed{1}$ $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ НАЙТИ $\int$ ПО ВНУТРЕННОСТИ

$$\int\limits_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\leq 1} dx\,dy =$$

ЗАМЕНА $\begin{cases} x_1=\frac{x}{a} \\ y_1=\frac{y}{b} \end{cases}$

$x_1^2+y_1^2=1$

$$=\int\limits_{x_1^2+y_1^2\leq 1} |J|\,dx_1\,dy_1 =$$

$$\frac{1}{J}=\left|\frac{\partial(x_1,y_1)}{\partial(x,y)}\right|=\begin{vmatrix}\frac{1}{a} & 0 \\ 0 & \frac{1}{b}\end{vmatrix}=\frac{1}{ab} \Rightarrow |J|=ab$$

или $J=abr$

$r>0$ $\varphi\in[0,2\pi)$

$$=ab\underbrace{\int\limits_{x_1^2+y_1^2\leq 1} dx_1\,dy_1}_{S_{\text{круга}}}=\pi ab$$

2 Фигура ограничена кривой

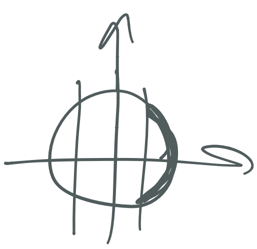
$$A := \begin{cases} (x^2+y^2)^2 = 2a^2(x^2-y^2) \\ x^2+y^2 \geq a^2 \end{cases}$$

и

$$\begin{cases} x = r\sin\varphi \\ y = r\cos\varphi \end{cases} \Rightarrow \begin{cases} r^4 = 2a^2r^2\cos 2\varphi \\ r \geq a \end{cases}$$

$r>0,\ \varphi\in[0,2\pi)$

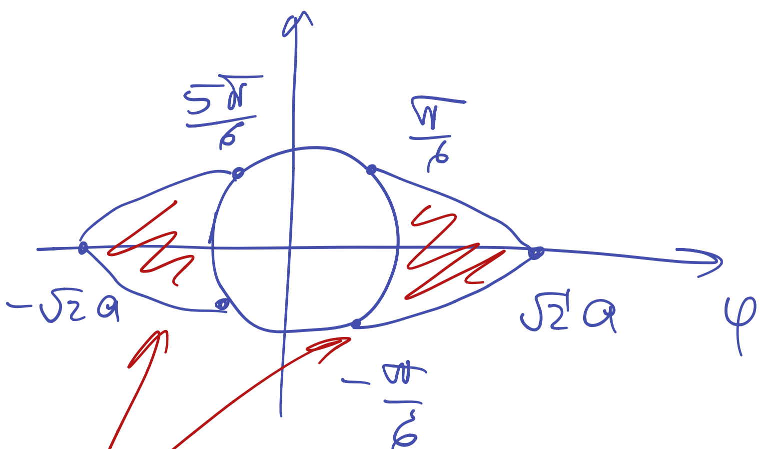
$$\Downarrow$$



$$\begin{cases} r = \sqrt{2}a\sqrt{\cos 2\varphi} \\ r \geq a \end{cases}$$

$$\Downarrow$$

$$\cos 2\varphi \geq \frac{1}{2} \Rightarrow \varphi \in \left[-\frac{\pi}{6}; \frac{\pi}{6}\right]$$



т.к. 2 симметричных куска

$$S = \int_A dx\,dy = \iint\limits_{a\leq r\leq\sqrt{2}a\sqrt{\cos 2\varphi}} r\,dr\,d\varphi = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\int_a^{a\sqrt{2}\sqrt{\cos 2\varphi}} r\,dr\right)d\varphi =$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2a^2\cos 2\varphi - a^2}{2}\, d\varphi = a^2\left(\sqrt{3} - \frac{\pi}{3}\right)$$

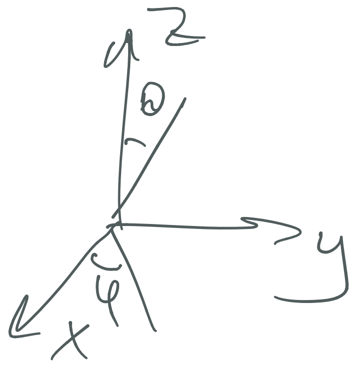
$\boxed{13}$

$$\iiint\limits_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1} \exp\left(\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2}\right) dx\,dy\,dz =$$

$$\begin{cases} x_1 = \dfrac{x}{a} \\ y_1 = \dfrac{y}{b} \\ z_1 = \dfrac{z}{c} \end{cases} \qquad |J| = abc$$

$$= \iiint\limits_{x_1^2+y_1^2+z_1^2\le 1} \exp\left(\sqrt{x_1^2+y_1^2+z_1^2}\right) abc\, dx_1\, dy_1\, dz_1 =$$

$$\begin{cases} x_1 = r\sin\theta\cos\varphi \\ y_1 = r\sin\theta\sin\varphi \\ z_1 = r\cos\theta \end{cases} \qquad \begin{matrix} \theta \in [0;\pi] \\ \varphi \in [0, 2\pi) \end{matrix}$$



сферич. коорд-ты.

$$|J| = r^2 \sin\theta$$

$$= abc \int\limits_{r\le 1} \exp(r)\, r^2 \sin\theta \, dr\, d\theta\, d\varphi =$$

$$= abc \int_0^1 e^r r^2 dr \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{=2} \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} =$$

$$= 4\pi abc \underbrace{\int_0^1 e^r r^2 dr}_{=e} = 4\pi e abc$$

$\boxed{4}$ $\begin{cases} x^4+y^4+z^4=1 \\ x\geq 0,\ y\geq 0,\ z\geq 0 \end{cases}$

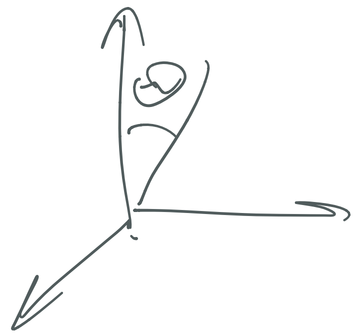
$\begin{cases} x_1 = x^2 \\ y_1 = y^2 \\ z_1 = z^2 \end{cases}$ $\qquad |J| = \dfrac{1}{8xyz}$

аналогично

$\boxed{5}$ $\displaystyle\int_{x\in\mathbb{R}^n:\,|x|\le 1} \frac{1}{|x|^p}\,dx$ $\displaystyle\int_{x\in\mathbb{R}^n:\,|x|\ge 1} \frac{1}{|x|^p}\,dx$

$$\sum_{n=1}^{\infty} \mu\left(\left\{x:\ \frac{1}{|x|^p}\ge n\right\}\right)<\infty$$

СФЕРИЧ. КООРДИНАТЫ



$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ \vdots \\ x_{n-1} = r\sin\theta_1\ldots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n = r\sin\theta_1\ldots\sin\theta_{n-2}\sin\theta_{n-1} \end{cases}$$

$\theta_1,\ldots,\theta_{n-2}\in[0;\pi]$

$\theta_{n-1}\in[0;2\pi)$

$$|J| = r^{n-1}\cdot F(\theta_1,\ldots,\theta_{n-1})$$

$$\int_{x \in \mathbb{R}^n : |x| \leq 1} \frac{1}{|x|^p} \, dx =$$

$$= \int_{0 \leq r \leq 1} \frac{1}{r^p} r^{n-1} F(\theta_1, \ldots, \theta_{n-1}) \, dr \, d\theta_1 \ldots d\theta_{n-1} =$$

$$= \underbrace{\int_0^1 r^{n-1-p} \, dr}_{< \infty} \; \underbrace{\int_{\theta_i} F(\ldots) \, d\theta_1 \ldots d\theta_{n-1}}_{< \infty}$$

$n-1-p > -1 \Rightarrow p < n$