Task
Let $D(n,m)$ be the number of ways to move from the point $(0,0)$ to the point (n,m) moving each time one step upwards or one step rightwards. Give a closed form for the generating function

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D(n,m)x^n y^m =$$
$$= 1 + x + y + x^2 + y^2 + 2xy + \dots$$

Solution
It is known that

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n}{m}x^{n-m}y^m =$$
$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m\right) =$$
$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right) =$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(x+y)^n =$$
$$\sum_{n=0}^{\infty}\binom{n}{0}x^n + \sum_{n=0}^{\infty}\binom{n+1}{1}x^n y + \sum_{n=0}^{\infty}\binom{n+2}{2}x^n y^2 + \dots =$$
$$\sum_{n=0}^{\infty}\binom{n}{0}x^n + \sum_{n=1}^{\infty}\binom{n}{1}x^{n-1}y + \sum_{n=2}^{\infty}\binom{n}{2}x^{n-2}y^2 + \dots =$$
$$1 + (x+y) + (x+y)^2 + \dots =$$
$$1 + x + y + x^2 + y^2 + 2xy + \dots$$