Домашнее задание №7. Механика Стружиной Ксении

№2 $\{q,p\}=1$, т.к. канонически сопряженные, то $\{q,q\}=\{p,p\}=0$

$Q=-p,\quad P=q+Ap^2,\quad A-\text{const}$

а) $\{Q,P\}=\{-p,q+Ap^2\}=-\{p,q\}=\{q,p\}=1$

$\{P,P\}=\{q+Ap^2,q+Ap^2\}=\{q,q\}+A\{q,p^2\}+A\{p^2,q\}+A^2\{p^2,p^2\}=$

$=0+\underbrace{2A\{q,p\}-2A\{q,p\}}_{0}+4A^2\underbrace{\{p,p\}}_{0}=0$

$\{Q,Q\}=\{-p,-p\}=0$

б) $F_1(q,Q)\Rightarrow p=\dfrac{\partial F_1}{\partial q},\quad P=-\dfrac{\partial F_1}{\partial Q}$

$Q=-p,\Rightarrow p=-Q \longrightarrow -Q=\dfrac{\partial F_1}{\partial q}$, Значит, $F_1=-Qq+f(Q,t)$

$P=q+Ap^2=q+AQ^2$

$q+AQ^2=-\dfrac{\partial F_1}{\partial Q}\Rightarrow F_1=-Qq+f(Q,t)\Rightarrow$

$q+AQ^2=-\left(-q+\dfrac{\partial f(Q,t)}{\partial Q}\right)=q-\dfrac{\partial f(Q,t)}{\partial Q}=q+AQ^2$

$\dfrac{\partial f(Q,t)}{\partial Q}=-AQ^2\Rightarrow f(Q,t)=-\dfrac{AQ^3}{3}+g(t)$

Значит, $F_1(q,Q)=-Qq-\dfrac{AQ^3}{3}+C$, т.к. F_1 не зависит от t, $\Rightarrow$ g тоже.

в) $F_2(q,P)\Rightarrow p=\dfrac{\partial F_2}{\partial q},\quad Q=\dfrac{\partial F_2}{\partial P}$

$P=q+Ap^2\Rightarrow p=\sqrt{\dfrac{P-q}{A}}$ и $Q=-p=-\sqrt{\dfrac{P-q}{A}}$

$p=\dfrac{\partial F_2}{\partial q}=\sqrt{\dfrac{P-q}{A}}\Rightarrow F_2=\int\sqrt{\dfrac{P-q}{A}}\,dq=\dfrac{1}{\sqrt{A}}\int\sqrt{P-q}\,dq=\dfrac{1}{\sqrt{A}}\cdot\dfrac{-1}{(1+\frac{1}{2})}(P-q)^{3/2}+f(P)$

$=-\dfrac{(P-q)^{3/2}}{\sqrt{A}}\cdot\dfrac{2}{3}+f(P)$

$-\sqrt{\dfrac{P-q}{A}}=Q=\dfrac{\partial F_2}{\partial P}=-\dfrac{2}{3\sqrt{A}}\cdot\dfrac{3}{2}(P-q)^{1/2}+\dfrac{\partial f(P)}{\partial P}=$

$=-\sqrt{\dfrac{P-q}{A}}+\dfrac{\partial f(P)}{\partial P}=-\sqrt{\dfrac{P-q}{A}}\Rightarrow f(P)=\text{const}$

$F_2(q,P)=-\dfrac{(P-q)^{3/2}}{\sqrt{A}}\cdot\dfrac{2}{3}+C,\quad C-\text{const}$

Значит, $\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} q_0 \cos\omega t + \frac{p_0}{m\omega}\sin\omega t \\ p_0\cos\omega t - q_0 m\omega \sin(\omega t) \end{pmatrix}$

$\{q(t), p(t)\} = \{q_0\cos\omega t + \frac{p_0}{m\omega}\sin(t\omega),\ p_0\cos\omega t - q_0 m\omega\sin(\omega t)\} =$

$= \{q_0, p_0\}\cos^2\omega t - \underbrace{\{q_0, q_0\}}_{0}\cdot C + \underbrace{\{p_0, p_0\}}_{0} D - \{p_0, q_0\}\sin^2(\omega t) =$

$= \{q_0, p_0\}(\cos^2\omega t + \sin^2\omega t) = 1$

б) $F_1(q_0, p(t), t)$ $\quad (q_0, p_0) \to (q(t), p(t))$

Из пункта а)

$q(t) = q_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t \Rightarrow \frac{q - q_0\cos\omega t}{\sin\omega t} m\omega = p_0$ (1)

$p(t) = p_0\cos\omega t - q_0 m\omega \sin\omega t \Rightarrow$

$p(t) = \frac{\cos\omega t}{\sin\omega t}\cdot m\omega(q - q_0\cos\omega t) - q_0 m\omega\sin\omega t =$

$= ctg(\omega t) m\omega (q - q_0\cos\omega t) - q_0 m\omega\sin\omega t$

$\cancel{F_1(q_0, q, t)} \Rightarrow p_0 = \frac{\partial F_1}{\partial q_0} \overset{(1)}{\Rightarrow} F_1(q_0, q, t) = \frac{q_0 m\omega q}{\sin\omega t} - \frac{q_0^2}{2} ctg(\omega t) m\omega + f(q,t)$

Т.к. $p = -\frac{\partial F_1}{\partial q}$, то $\cancel{\cos}\, ctg(\omega t) m\omega(q - q_0\cos\omega t) + q_0 m\omega\sin\omega t =$

$= \frac{q_0 m\omega}{\sin\omega t} + \frac{\partial f(q,t)}{\partial q} \Rightarrow$

$\frac{\partial f(q,t)}{\partial q} = -m\omega q\, ctg(\omega t) + \underbrace{q_0 \overbrace{\frac{\cos^2\omega t}{\sin\omega t} m\omega + \frac{q_0 m\omega \sin^2\omega t}{\sin\omega t}}^{1} - \frac{q_0 m\omega}{\sin\omega t}}_{0}$

$f(q,t) = -\frac{q^2}{2} m\omega\, ctg(\omega t) + c$, c — const

Значит, $F_1(q_0, q, t) = \frac{q_0 m\omega q}{\sin(\omega t)} - \frac{q_0^2\, ctg(\omega t) m\omega}{2} - \frac{q^2}{2} m\omega\, ctg(\omega t) + c$

б) $F_1(q,Q)$ $\qquad p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}$

$P = \ln\left(\frac{p}{2q}\right), \quad Q = \frac{pq}{2} \Rightarrow p = \frac{2Q}{q} \Rightarrow P = \ln\left(\frac{2Q}{2q^2}\right) = \ln\left(\frac{Q}{q^2}\right)$

Тоді $\frac{2Q}{q} = p = \frac{\partial F_1}{\partial q} \Rightarrow F_1(q,Q) = 2Q\ln q + f(Q)$

$\ln\left(\frac{Q}{q^2}\right) = P = -\frac{\partial F_1}{\partial Q} = -2\ln q + -\frac{\partial f(Q)}{\partial Q} = \ln Q - 2\ln q$

$\frac{\partial f(Q)}{\partial Q} = -\ln Q \Rightarrow f(Q) = -\int \ln Q \, dQ = -Q\ln Q + Q + c$, c-const

Значить, $\underbrace{F_1(q,Q) = 2Q\ln q - Q\ln Q + Q + c}$

№1. $H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$

а) $\{q_0, p_0\} = 1$

$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q$

$A = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{pmatrix} \qquad \chi_A = \lambda^2 - \frac{1}{m}(-m\omega^2) = \lambda^2 + \omega^2 \Rightarrow \lambda_\pm = \pm i\omega$

$\lambda_+ = i\omega \qquad u_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, тоді $Au_1 = \begin{pmatrix} \frac{b}{m} \\ -am\omega^2 \end{pmatrix} = \begin{pmatrix} i\omega a \\ i\omega b \end{pmatrix} \Rightarrow u_1 = \begin{pmatrix} 1 \\ im\omega \end{pmatrix}$

$\cancel{\begin{matrix} \frac{b}{m} = i\omega a \\ -am\omega^2 = i\omega b \end{matrix} \Rightarrow \frac{a}{b} = \frac{b}{am\omega^2} \Rightarrow \frac{a^2}{b^2} = \frac{1}{m^2\omega^2} \quad \frac{a}{b} = \frac{1}{im\omega}}$

$\frac{b}{m} = i\omega a \Rightarrow \frac{b}{a} = i\omega m \qquad b = i\omega m, \ a = 1$ - ok

$\lambda_- = -i\omega \quad u_2 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad Au_2 = \begin{pmatrix} \frac{b}{m} \\ -m\omega^2 a \end{pmatrix} = \begin{pmatrix} -i\omega a \\ -i\omega b \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} 1 \\ -im\omega \end{pmatrix}$

$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \sum c_i u_i e^{\lambda_i t} = \begin{pmatrix} c_1 \cdot 1 \cdot e^{i\omega t} + c_2 \cdot 1 \cdot e^{-i\omega t} \\ im\omega \cdot c_1 \cdot e^{i\omega t} + c_2 \cdot (-im\omega) \cdot e^{-i\omega t} \end{pmatrix} =$

$= \begin{pmatrix} c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ im\omega(c_1 e^{i\omega t} - c_2 e^{-i\omega t}) \end{pmatrix} = \begin{pmatrix} \tilde{c}_1 \cos\omega t + \tilde{c}_2 \sin\omega t \\ m\omega(\tilde{c}_2 \cos\omega t - \tilde{c}_1 \sin\omega t) \end{pmatrix}$

$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \tilde{c}_1 \\ m\omega \tilde{c}_2 \end{pmatrix} \qquad \begin{matrix} \tilde{c}_1 = q_0 \\ \tilde{c}_2 = \frac{p_0}{m\omega} \end{matrix}$

б) $F_2(q_0, p(t), t)$

$q(t) = q_0 \cos\omega t + \frac{p_0}{m\omega}\sin\omega t$, $p(t) = p_0\cos\omega t - q_0 m\omega \sin\omega t$

$q = q_0\cos\omega t + \frac{\sin\omega t}{m\omega}\left(\frac{p + q_0 m\omega\sin\omega t}{\cos\omega t}\right) \Leftarrow p_0 = \frac{p + q_0 m\omega \sin\omega t}{\cos\omega t}$

$'' \; q_0\cos\omega t + \frac{tg\,\omega t\; p}{m\omega} + \cancel{q_0 m\omega \sin\omega t}\; tg\,\omega t\, q_0 \sin\omega t$

$p_0 = \frac{\partial F_2}{\partial q_0} = \frac{p + q_0 m\omega\sin\omega t}{\cos\omega t} \Rightarrow$

$F_2(q_0, p, t) = \frac{q_0 p}{\cos\omega t} + \frac{q_0^2}{2} m\omega\, tg\,\omega t + f(p,t)$

$q = \frac{\partial F_2}{\partial p} = q_0\cos\omega t + tg\,\omega t\left(\frac{p}{m\omega} + q_0\sin\omega t\right) =$

$= \frac{q_0}{\cos\omega t} + \frac{\partial f(p,t)}{\partial p} \Rightarrow$

$\frac{\partial f}{\partial p}(p,t) = q_0\cos\omega t + tg\,\omega t\left(\frac{p}{m\omega} + q_0 \sin\omega t\right) - \frac{q_0}{\cos\omega t} =$

$= q_0\cos\omega t + \frac{tg(\omega t)\, p}{m\omega} + \frac{q_0}{\cos\omega t}(\sin^2\omega t - 1) =$

$= q_0 \cos\omega t + \frac{p\, tg\,\omega t}{m\omega} - q_0\cos\omega t = \frac{p\, tg\,\omega t}{m\omega} \Rightarrow$

$f(p,t) = \frac{p^2}{2}\frac{tg\,\omega t}{m\omega} + c$, c – const

Значит, $F_2(q_0, p, t) = \frac{q_0 p}{\cos\omega t} + \frac{q_0^2}{2} m\omega\, tg\,\omega t + \frac{p^2\, tg\,\omega t}{2m\omega} + c$

№3 а) $F = -\frac{\partial U}{\partial x}$ $\quad U = -Fx$

$T_{кин} = \frac{m\dot{x}^2}{2}$ $\quad L = T_{кин} - U = \frac{m\dot{x}^2}{2} + Fx$

$H = \dot{x}p - L$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$

$H = p\cdot\frac{p}{m} - \frac{m}{2}\cdot\left(\frac{p}{m}\right)^2 - Fx = \frac{p^2}{m} - \frac{p^2}{2m} - Fx = \frac{p^2}{2m} - Fx$

б) $Q = -p,\ P = x + Ap^2$ $\quad x = P - AQ^2,\ p = -Q$

$\tilde{H}(Q, P, t) = H(x(Q,P), p(Q,P), t) =$

$= \tilde{H}(P - AQ^2, -Q, t) = \frac{Q^2}{2m} - F(P - AQ^2) = -FP$ при

$\frac{Q^2}{2m} = -FAQ^2 \Rightarrow A = -\frac{1}{2mF}$

в) $A = -\frac{1}{2mF}$

$\begin{cases} \dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \\ \dot{Q} = \frac{\partial \tilde{H}}{\partial P} = -F \end{cases}$ Значит, $P(t) = P(0) = const - P_0$

$Q(t) = -Ft + Q(0) = -Ft + Q_0,\ Q_0 - const$

Итого, выполняя обратное преобразование, получим:

$p(t) = -Q(t) = Ft - Q_0$

$x(t) = P - AQ^2 = P_0 + \frac{1}{2mF}\cdot(-Ft + Q_0)^2 = P_0 + \frac{(Ft - Q_0)^2}{2mF}$

№4 $F_2(q, P) = q^2 e^P$

а) $Q = Q(q,p),\ P = P(q,p)$

Производящая функция второго рода: $p = \frac{\partial F_2}{\partial q},\ Q = \frac{\partial F_2}{\partial P}$

$p = 2qe^P,\ Q = q^2 e^P$

$e^P = \frac{p}{2q} \Rightarrow Q = q^2\cdot\frac{p}{2q} = \frac{pq}{2}$

$\Downarrow$

$P = \ln\left(\frac{p}{2q}\right)$

Ответ: $P = \ln\frac{p}{2q},\ Q = \frac{pq}{2}$