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N1.

$\vec{F}: R^3 \times R^3 \to R^3$

$\vec{F} = f_1 \vec{r} + f_2 \vec{p} + f_3 [\vec{r} \times \vec{p}]$

$\{r_i, p_j\} = \delta_{ij}$

f_i – скалярные ф-ции от $\vec{r}^2, \vec{p}^2, (\vec{r}, \vec{p})$

а) $\{\vec{F}, (\vec{M} \cdot \vec{n})\}$

$\vec{M} = [\vec{r} \times \vec{p}] = \varepsilon_{kij} r_i p_j$

$(M, n) = \varepsilon_{ijk} r_j p_k n_i$

$\{f_1 \vec{r} + f_2 \vec{p} + f_3 [\vec{r} \times \vec{p}], (\vec{M} \cdot \vec{n})\}$

① $\{f_1 \vec{r}, (\vec{M} \cdot \vec{n})\} = f_1 \{\vec{r}, (\vec{M} \cdot \vec{n})\} + \vec{r} \{f_1, (\vec{M} \cdot \vec{n})\}$

$f_1 \{r_l, \varepsilon_{ijk} r_j p_k n_i\} + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} \{\vec{r}^2, (\vec{M} \cdot \vec{n})\} + \frac{\partial f_1}{\partial \vec{p}^2} \{\vec{p}^2, (\vec{M} \cdot \vec{n})\} + \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \{(\vec{r} \cdot \vec{p}), (\vec{M} \cdot \vec{n})\} \right) =$

$= f_1 \varepsilon_{ijk} r_j \underbrace{\{r_l, p_k\}}_{\delta_{lk}} n_i + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} \{r_l^2, \varepsilon_{ijk} r_j p_k n_i\} + \right.$

$+ \frac{\partial f_1}{\partial \vec{p}^2} \{p_l^2, \varepsilon_{ijk} r_j p_k n_i\} + \left. \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \{r_l p_l, \varepsilon_{ijk} r_j p_k n_i\} \right) =$

$= f_1 \varepsilon_{ijl} r_j n_i + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} \cdot 2 r_l \{r_l, \varepsilon_{ijk} r_j p_k n_i\} + \frac{\partial f_1}{\partial \vec{p}^2} \cdot 2 p_l \cdot \{p_l, \varepsilon_{ijk} r_j p_k n_i\} + \right.$

$+ \left. \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \cdot \left(r_l \{p_l, \varepsilon_{ijk} r_j p_k n_i\} + p_l \{r_l, \varepsilon_{ijk} r_j p_k n_i\} \right) \right) =$

$= -f_1 \varepsilon_{lji} r_j n_i + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} \cdot 2 r_l \cdot \overset{-\varepsilon_{lji} r_j n_i}{\varepsilon_{ijl} r_j n_i} + \frac{\partial f_1}{\partial \vec{p}^2} \cdot 2 p_l \cdot (-\overset{-\varepsilon_{lki} p_k n_i}{\varepsilon_{ilk} p_k n_i}) + \right.$

$+ \left. \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \left(r_l \cdot (-\overset{-\varepsilon_{lki} p_k n_i}{\varepsilon_{ilk} p_k n_i}) + p_l \cdot \overset{-\varepsilon_{lji} r_j n_i}{\varepsilon_{ijl} r_j n_i} \right) \right) =$

$= -f_1 \cdot [\vec{r} \times \vec{n}]_l + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} \cdot 2 r_l (-[\vec{r} \times \vec{n}]_l) + \frac{\partial f_1}{\partial \vec{p}^2} \cdot 2 p_l (-[\vec{p} \times \vec{n}]_l) + \right.$

$+ \left. \frac{\partial f}{\partial (\vec{r} \cdot \vec{p})} \cdot \left(r_l (-[\vec{p} \times \vec{n}]_l) + p_l \cdot (-[\vec{r} \times \vec{n}]_l) \right) \right)$

$\Rightarrow \{f_1 \vec{r}, (\vec{M} \cdot \vec{n})\} = -f_1 [\vec{r} \times \vec{n}] + \vec{r} \left(\frac{\partial f_1}{\partial \vec{r}^2} (-2) \cdot \overset{0}{(\vec{r}, [\vec{r} \times \vec{n}])} + \right.$

$+ \left. \frac{\partial f_1}{\partial \vec{p}^2} \cdot (-2) \cdot \overset{0}{(\vec{p}, [\vec{p} \times \vec{n}])} + \frac{\partial f}{\partial (\vec{r} \cdot \vec{p})} \underbrace{\left(-(\vec{r}, [\vec{p} \times \vec{n}]) - (\vec{p}, [\vec{r} \times \vec{n}]) \right)}_{0} \right) = -f_1 [\vec{r} \times \vec{n}]$

т.к. $a[b \times c] = b[c \times a] = c[a \times b]$

② $\{f_2 \vec{p}, (\vec{M} \cdot \vec{n})\} = f_2 \{\vec{p}, (\vec{M} \cdot \vec{n})\} + \vec{p} \{f_2, (\vec{M} \cdot \vec{n})\}$

$-f_2 \underset{-\varepsilon_{lki}}{\varepsilon_{ilk}} p_k n_i + \vec{p} \left[\frac{\partial f_2}{\partial \vec{p}^2} \cdot 2 p_l \varepsilon_{ilk} p_k n_i + \frac{\partial f_2}{\partial \vec{r}^2} \cdot 2 r_l \cdot \varepsilon_{ijl} r_j n_i + \frac{\partial f_2}{\partial (\vec{r} \cdot \vec{p})} \left(r_l \cdot (-\varepsilon_{lki} p_k n_i) + p_l \cdot \varepsilon_{lji} r_j n_i \right) \right] =$

$= -f_2 \cdot [\vec{p} \times \vec{n}]_l + \vec{p} \cdot \left[\frac{\partial f_2}{\partial \vec{p}^2} 2 p_l (-[\vec{p} \times \vec{n}]_l) + \frac{\partial f_2}{\partial \vec{r}^2} \cdot 2 r_l (-[\vec{r} \times \vec{n}]_l) + \right.$

$+ \left. \frac{\partial f}{\partial (\vec{r} \cdot \vec{p})} \cdot \left(r_l \cdot (-[\vec{p} \times n]_l) + p_l (-[\vec{r} \times \vec{n}]_l) \right) \right]$

$\Rightarrow \{f_2 \vec{p}, (\vec{M} \cdot \vec{n})\} = -f_2 [\vec{p} \times \vec{n}] + \vec{p} \left[\frac{\partial f_2}{\partial \vec{r}^2} (-2) \overset{0}{(\vec{r}, [\vec{r} \times \vec{n}])} + \right.$

$+ \left. \frac{\partial f_1}{\partial \vec{p}^2} \cdot (-2) \cdot \underset{0}{(\vec{p}, [\vec{p} \times \vec{n}])} + \frac{\partial f}{\partial (\vec{r} \cdot \vec{p})} \cdot \underbrace{\left(-(\vec{r}, [\vec{p} \times \vec{n}]) - (\vec{p}, [\vec{r} \times \vec{n}]) \right)}_{0} \right] = -f_2 [\vec{p} \times \vec{n}]$

(3) $\{f_3 \vec{M}, (\vec{M}\cdot\vec{n})\} = f_3 \{M, (\vec{M}\cdot\vec{n})\} + M\{f_3, (\vec{M}\cdot\vec{n})\} =$

$$f_3 \{M_e, M_k\} \frac{\partial(\vec{M}\cdot\vec{n})}{\partial M_k} + \vec{M}\cdot\left(\frac{\partial f_3}{\partial \vec{r}^2}\{\vec{r}^2, (\vec{M}\cdot\vec{n})\} + \frac{\partial f_3}{\partial \vec{p}^2}\{\vec{p}^2, (\vec{M}\cdot\vec{n})\} + \frac{\partial f_3}{\partial(\vec{r}\cdot\vec{p})}\cdot\{(\vec{r}\cdot\vec{p}), (\vec{M}\cdot\vec{n})\}\right).$$

$$f_3 \{M_l, M_k\} \frac{\partial(\vec{M}\cdot\vec{n})}{\partial M_k} = f_3 \cdot \varepsilon_{lki} M_i n_k = -f_3 \varepsilon_{lik} M_i \cdot n_k = -f_3 [M \times n]_l$$

$$\Rightarrow f_3 \{M, (\vec{M}\cdot\vec{n})\} = -f_3 \cdot [M\times n].$$

Ответ $\{F, (\vec{M}\cdot\vec{n})\} = -f_1[\vec{r}\times\vec{n}] - f_3[\vec{M}\times\vec{n}] - f_2[\vec{p}\times\vec{n}]$

б) $\{\vec F, \vec M^2\} = \{f_1\vec r + f_2\vec p + f_3[\vec r\times\vec p], \vec M^2\}$

① $\{f_1\vec r, \vec M^2\} = f_1\{\vec r,\vec M^2\} + \vec r\{f_1,\vec M^2\} =$

$$= f_1\cdot\{\vec r,\vec M^2\} + \vec r\cdot\left(\frac{\partial f_1}{\partial \vec r^2}\{\vec r^2,\vec M^2\} + \frac{\partial f_1}{\partial \vec p^2}\{\vec p^2,\vec M^2\} + \frac{\partial f_1}{\partial(\vec r\cdot\vec p)}\{(\vec r\cdot\vec p),\vec M^2\}\right) =$$

$$= f_1\cdot \underbrace{2M_i\{r_l, \varepsilon_{ijk}r_jp_k\}}_{\{r_l,M_i\}\cdot 2M_i} + \vec r\cdot 2M_i\cdot\left(\frac{\partial f_1}{\partial \vec r^2}\{r_l^2,\varepsilon_{ijk}r_jp_k\} + \frac{\partial f_1}{\partial\vec p^2}\{p_l^2,\varepsilon_{ijk}r_jp_k\} + \frac{\partial f_1}{\partial(\vec r\cdot\vec p)}\{r_lp_l,\varepsilon_{ijk}r_jp_k\}\right) =$$

$$= f_1\cdot 2M_i\varepsilon_{ijk}r_j\{r_l,p_k\} + \vec r\cdot 2M_i\left(\frac{\partial f_1}{\partial\vec r^2}2r_l\{r_l,\varepsilon_{ijk}r_jp_k\} + \frac{\partial f_1}{\partial \vec p^2}2p_l\{p_l,\varepsilon_{ijk}r_jp_k\} + \frac{\partial f_1}{\partial(\vec r\cdot\vec p)}\cdot(r_l\{p_l,\varepsilon_{ijk}r_jp_k\} + p_l\{r_l,\varepsilon_{ijk}r_jp_k\})\right) =$$

$$= f_1\cdot\underbrace{2M_i\varepsilon_{ijl}r_j}_{2\varepsilon_{lij}M_ir_j} + \vec r\cdot 2\left(\frac{\partial f_1}{\partial\vec r^2}\cdot 2r_l\overset{\varepsilon_{lij}}{\overset{\|}{\varepsilon_{ijl}}}M_ir_j - \frac{\partial f_1}{\partial\vec p^2}2p_l\underset{\underset{-\varepsilon_{lik}}{\|}}{\varepsilon_{ilk}}M_ip_k + \frac{\partial f_1}{\partial(\vec r\cdot\vec p)}\cdot(-r_l\overset{-\varepsilon_{lik}}{\overset{\|}{\varepsilon_{ilk}}}M_ip_k + p_l\overset{\varepsilon_{lij}}{\overset{\|}{\varepsilon_{ijl}}}M_ir_j)\right)$$

$$\{f_1\vec r,\vec M^2\} = 2f_1\cdot[\vec M\times\vec r] + \vec r\cdot\left(4\frac{\partial f_1}{\partial \vec r^2}\cancel{(\vec r,[\vec M\times\vec r])} + 4\frac{\partial f_1}{\partial\vec p^2}\cancel{(\vec p,[\vec M\times\vec p])} + 2\frac{\partial f_1}{\partial(\vec r\cdot\vec p)}\underbrace{((\vec r,[\vec M\times\vec p]) + (\vec p,[\vec M\times\vec r]))}_{0}\right) = 2f_1[\vec M\times\vec r]$$

(т.к. $a\cdot[b\times c] = b\cdot[c\times a] = c\cdot[a\times b]$)

② $$\{f_2\vec p,\vec M^2\} = 2f_2\cdot[\vec M\times\vec p] + \vec p\left(4\frac{\partial f_2}{\partial\vec r^2}\cancel{(\vec r,[\vec M\times\vec r])} + 4\frac{\partial f_2}{\partial\vec p^2}\cancel{(\vec p,[\vec M\times\vec p])} + 2\frac{\partial f_2}{\partial(\vec r\cdot\vec p)}\cdot\underbrace{((\vec r,[\vec M\times\vec p]) + (\vec p,[\vec M\times\vec r]))}_{0}\right) = 2f_2[\vec M\times\vec p]$$

③ $\{f_3\vec M,\vec M^2\} = f_3\{\vec M,\vec M^2\} + \vec M\{f_3,\vec M^2\} =$

$$= f_3\cdot\{\vec M,\vec M^2\} + \vec M\cdot\left(\frac{\partial f_3}{\partial\vec r^2}\cdot\{\vec r^2,\vec M^2\} + \frac{\partial f_3}{\partial(\vec r\vec p)}\{(\vec r\vec p),\vec M^2\} + \frac{\partial f_3}{\partial\vec p^2}\{\vec p^2,\vec M^2\}\right)$$

$$f_3\underbrace{2\cdot\varepsilon_{ijk}M_kM_j}_{\{M_i,M_j\}\cdot 2M_j} + \vec M\left(4\frac{\partial f_3}{\partial\vec r^2}(\vec r,[\vec M\times\vec r]) + 4\frac{\partial f_3}{\partial\vec p^2}(\vec p,[\vec M\times\vec p]) + 2\frac{\partial f_3}{\partial(\vec r\cdot\vec p)}\cdot((\vec r,[\vec M\times\vec p]) + (\vec p,[\vec M\times\vec r]))\right) =$$

$= -2f_3\varepsilon_{ikj}M_kM_j$

$$= -2f_3\cancel{[\vec M\times\vec M]_i} + \vec M\left(4\frac{\partial f_3}{\partial\vec r^2}\cancel{(\vec r,[\vec M\times\vec r])} + 4\frac{\partial f_3}{\partial\vec p^2}\cancel{(\vec p,[\vec M\times\vec p])} + 2\frac{\partial f_3}{\partial(\vec r\cdot\vec p)}\cdot\underbrace{((\vec r,[\vec M\times\vec p]) + (\vec p,[\vec M\times\vec r]))}_{0}\right)$$

$\Rightarrow \{f_3\vec M,\vec M^2\} = 0$

~~Обобщим за~~ $\cancel{A_i = 4\frac{\partial f_i}{\partial \vec r^2}(\vec r,[\vec M\times\vec r]) + 4\frac{\partial f_i}{\partial\vec p^2}(\vec p,[\vec M\times\vec p]) + 2\frac{\partial f_i}{\partial(\vec r\cdot\vec p)}((\vec r,[\vec M\times\vec p]) + (\vec p,[\vec M\times\vec r]))}$, $\cancel{i=1,2,3}$

Тогда ответ: $2f_1[\vec M\times\vec r] + 2f_2[\vec M\times\vec p]$

N2.

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} \qquad \dot{x} \equiv \frac{dx}{dt}$$

а) $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \dot{x} = \frac{p}{m}$

$$H = p\cdot\dot{x} - L = \frac{p^2}{m} - L = \frac{p^2}{m} - \frac{m\cdot\frac{p^2}{m^2}}{2} + \frac{m\omega^2 x^2}{2} = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} =$$

$$= \boxed{\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}}$$

б) $a = \sqrt{\frac{m\omega}{2}}\left(x + i\cdot\frac{p}{m\omega}\right)$

$\bar{a} = \sqrt{\frac{m\omega}{2}}\left(x - i\cdot\frac{p}{m\omega}\right)$

$a\cdot\bar{a} = \frac{m\omega}{2}\left(x^2 + \frac{p^2}{m^2\omega^2}\right) = \frac{m\omega x^2}{2} + \frac{p^2}{2m\omega}$

$H = \boxed{a\cdot\bar{a}\cdot\omega} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$

в) $\{a,\bar{a}\} = \frac{\partial a}{\partial x}\{x,\bar{a}\} + \frac{\partial a}{\partial p}\{p,\bar{a}\} = \sqrt{\frac{m\omega}{2}}\underbrace{\{x,x\}}_{0}\frac{\partial\bar{a}}{\partial x} + \frac{i}{\sqrt{2m\omega}}\underbrace{\{p,x\}}_{-1}\cdot\frac{\partial\bar{a}}{\partial x} +$

$+ \sqrt{\frac{m\omega}{2}}\underbrace{\{x,p\}}_{1}\frac{\partial\bar{a}}{\partial p} + \frac{i}{\sqrt{2m\omega}}\underbrace{\{p,p\}}_{0}\frac{\partial\bar{a}}{\partial p} = -\frac{i}{\sqrt{2m\omega}}\cdot\sqrt{\frac{m\omega}{2}} + \sqrt{\frac{m\omega}{2}}\cdot\left(-\frac{i}{\sqrt{2m\omega}}\right) =$

$= -\frac{i}{2} - \frac{i}{2} = \boxed{-i}$

$\{a,H\} = \{a, a\bar{a}\omega\} = \{a,a\bar{a}\}\cdot\omega = \left(\bar{a}\underbrace{\{a,a\}}_{0} + a\underbrace{\{a,\bar{a}\}}_{-i}\right)\cdot\omega =$

$= \boxed{-ai\omega}$

г) $\begin{cases} \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \\ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \end{cases}$ $\qquad H = a\cdot\bar{a}\cdot\omega$

$\dot{a} = \sqrt{\frac{m\omega}{2}}\left(\dot{x} + i\frac{\dot{p}}{m\omega}\right) = \sqrt{\frac{m\omega}{2}}\cdot\left(\underbrace{\frac{\partial H}{\partial a}}_{\bar{a}\omega}\cdot\underbrace{\frac{\partial a}{\partial p}}_{=\frac{i}{\sqrt{2m\omega}}} - i\cdot\frac{\partial H}{\partial a}\cdot\frac{\partial a}{\partial x}\cdot\frac{1}{m\omega}\right) +$

$+ \sqrt{\frac{m\omega}{2}}\cdot\left(\frac{\partial H}{\partial\bar{a}}\cdot\frac{\partial\bar{a}}{\partial p} - i\cdot\frac{\partial H}{\partial\bar{a}}\cdot\frac{\partial\bar{a}}{\partial x}\cdot\frac{1}{m\omega}\right) =$

$\dot{a} = \{a,H\}$

$= \sqrt{\frac{m\omega}{2}}\left(\bar{a}\omega\cdot\frac{1}{\sqrt{2m\omega}} - \frac{i}{m\omega}\cdot\bar{a}\cdot\omega\cdot\sqrt{\frac{m\omega}{2}}\right) +$

$+ \sqrt{\frac{m\omega}{2}}\cdot\left(a\omega\cdot\left(-\frac{i}{\sqrt{2m\omega}}\right) - \frac{i}{m\omega}a\omega\cdot\sqrt{\frac{m\omega}{2}}\right) =$

$= \cancel{\frac{i\bar{a}\cdot\omega}{2}} - \cancel{\frac{i\bar{a}\cdot\omega}{2}} - \frac{i\cdot a\omega}{2} - \frac{i a\omega}{2} = -ai\omega$

$\Rightarrow \dot{a} = -ai\omega$

$\boxed{a = Ce^{-i\omega t}}$

N3.

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{qB}{2c}(x\dot{y}-y\dot{x})$$

а) $p_1 = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2c}\cdot y \qquad \dot{x} = \frac{p_1 + \frac{qB}{2c}\cdot y}{m}$

$p_2 = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2c}\cdot x \qquad \dot{y} = \frac{p_2 - \frac{qB}{2c}\cdot x}{m}$

$p_3 = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \dot{z} = \frac{p_3}{m}$

$$H = p_1\dot{x}+p_2\dot{y}+p_3\dot{z} - \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{qB}{2c}(x\dot{y}-y\dot{x}) =$$

$$= \frac{p_1^2+\frac{p_1 qB}{2c}\cdot y}{m} + \frac{p_2^2 - \frac{p_2 qB}{2c}\cdot x}{m} + \frac{p_3^2}{m} - \frac{1}{2m}\Big(p_1^2 + p_1\cdot\frac{qB}{c}\cdot y + \frac{q^2B^2}{4c^2}\cdot y^2 +$$

$$+p_2^2 - p_2\cdot\frac{qB}{c}\cdot x + \frac{q^2B^2}{4c^2}\cdot x^2 + p_3^2\Big) - \frac{qB}{2c}\left(\frac{p_2 x - \frac{qB}{2c}\cdot x^2}{m} - \frac{p_1 y + \frac{qB}{2c}y^2}{m}\right) =$$

$$= \frac{1}{m}\Big(p_1^2 + \cancel{\frac{qBy}{2c}}\cdot p_1 + p_2^2 - \cancel{\frac{qBx}{2c}}\cdot p_2 + p_3^2 - \frac{p_1^2}{2} - \frac{qBy}{\cancel{2c}}\cdot p_1 - \frac{q^2B^2y^2}{8c^2} -$$

$$-\frac{p_2^2}{2} + \cancel{\frac{qBx}{2c}}\cdot p_2 - \frac{q^2B^2x^2}{8c^2} - \frac{p_3^2}{2} - \frac{qBx}{2c}\cdot p_2 + \frac{q^2B^2x^2}{4c^2} + \frac{qBy}{2c}\cdot p_1 + \frac{q^2B^2}{4c^2}\cdot y^2\Big) =$$

$$= \frac{1}{m}\left(\frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{p_3^2}{2} + \frac{q^2B^2y^2}{8c^2} + \frac{q^2B^2x^2}{8c^2} - \frac{qBxp_2}{2c} + \frac{qByp_1}{2c}\right).$$

б)

$\dot{p}_1 = -\frac{\partial H}{\partial x} = \left(-\frac{q^2B^2x}{4c^2} + \frac{qBp_2}{2c}\right)\cdot\frac{1}{m}$

$\dot{p}_2 = -\frac{\partial H}{\partial y} = \left(-\frac{q^2B^2y}{4c^2} - \frac{qBp_1}{2c}\right)\cdot\frac{1}{m}$

$\dot{p}_3 = -\frac{\partial H}{\partial z} = 0 \qquad p_3 = const$, но т.к. $p_3(0) = p$, то $p_3(t) = p$

$\dot{x} = \frac{\partial H}{\partial p_1} = \frac{1}{m}\left(p_1 + \frac{qBy}{2c}\right)$

$\dot{y} = \frac{\partial H}{\partial p_2} = \frac{1}{m}\left(p_2 - \frac{qBx}{2c}\right)$

$\dot{z} = \frac{\partial H}{\partial p_3} = \frac{1}{m}\cdot p_3 = \frac{p}{m}$

$x(0) = y(0) = z(0) = 0$

$p_1(0) = p_3(0) = p$

$p_2(0) = 0$

т.к $z(0)=0$, то $z = \frac{p}{m}t$

Пусть $A = \begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix}$ Обознач: $N = \frac{q\beta}{2cm}$

$$\dot{A} = \underbrace{\begin{pmatrix} 0 & N & \frac{1}{m} & 0 \\ -N & 0 & 0 & \frac{1}{m} \\ -mN^2 & 0 & 0 & N \\ 0 & -mN^2 & -N & 0 \end{pmatrix}}_{B} \cdot A$$

$$-N \cdot \begin{vmatrix} -N & 0 & \frac{1}{m} \\ -mN^2 & 0 & N \\ 0 & -N & 0 \end{vmatrix} = N^3 - N^3 = 0$$

$$\frac{1}{m} \cdot \begin{vmatrix} -N & 0 & \frac{1}{m} \\ -mN^2 & 0 & N \\ 0 & -mN^2 & 0 \end{vmatrix} = mN^4 - mN^4 = 0$$

$$\chi_B(\lambda) = \lambda^4 + (2N^2 + N^2 + N^2)\lambda^2 = \lambda^4 + 4N^2\lambda^2 = \lambda^2(\lambda^2 + 4N^2)$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 2iN \quad \lambda_3 = -2iN$$

$\lambda_1 = 0$:

$$Bu = \lambda_1 u \qquad u := \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\begin{pmatrix} bN + \frac{c}{m} \\ -aN + \frac{d}{m} \\ -amN^2 + Nd \\ -bmN^2 - Nc \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

собств. векторы: $\begin{pmatrix} \frac{1}{mN} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{mN} \\ 1 \\ 0 \end{pmatrix}.$

$\lambda_2 = 2Ni$: $Bu = \lambda_2 u$

$$\begin{pmatrix} bN + \frac{c}{m} \\ -aN + \frac{d}{m} \\ -amN^2 + Nd \\ -bmN^2 - Nc \end{pmatrix} = \begin{pmatrix} 2Ni\,a \\ 2Ni \cdot b \\ 2Ni \cdot c \\ 2Ni \cdot d \end{pmatrix}$$

собств. вектор: $\begin{pmatrix} -i \\ 1 \\ mN \\ mNi \end{pmatrix}$

$\lambda_3 = -2Ni$: $Bu = \lambda_3 u$

$$\begin{pmatrix} bN + \frac{c}{m} \\ -aN + \frac{d}{m} \\ -amN^2 + Nd \\ -bmN^2 - Nc \end{pmatrix} = \begin{pmatrix} -2Ni\,a \\ -2Ni\,b \\ -2Ni\,c \\ -2Ni\,d \end{pmatrix}$$

собств. вектор: $\begin{pmatrix} i \\ +1 \\ +mN \\ -mNi \end{pmatrix}$

$$A = \begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix} = c_0 \cdot \begin{pmatrix} \frac{1}{mN} \\ 0 \\ 0 \\ 1 \end{pmatrix} e^0 + c_1 \cdot \begin{pmatrix} 0 \\ -\frac{1}{mN} \\ 1 \\ 0 \end{pmatrix} e^0 + c_2 \cdot \begin{pmatrix} -i \\ 1 \\ mN \\ mNi \end{pmatrix} e^{2Nit} + c_3 \cdot \begin{pmatrix} i \\ 1 \\ mN \\ -mNi \end{pmatrix} \cdot e^{-2Nit}$$

$$x(0) = C_0 \cdot \frac{1}{mN} - C_2 \cdot i + C_3 i = 0$$

$$y(0) = -C_1 \cdot \frac{1}{mN} + C_2 + C_3 = 0$$

$$p_1(0) = C_1 + C_2 \cdot mN + C_3 \cdot mN = p$$

$$p_2(0) = C_0 + C_2 \cdot mNi - C_3 \cdot mNi = 0$$

$$\cancel{i(C_2+C_3) = C_0 \cdot \frac{1}{mN}}$$

$$\cancel{C_2+C_3 = C_1 \cdot \frac{1}{mN}}$$

$$\cancel{\Rightarrow \frac{C_0}{mNi} = \frac{C_1}{mN}} \qquad C_0 = iC_1$$

$$\cancel{mN(C_2+C_3) = C_1}$$

$$\cancel{C_2+C_3 = \frac{C_1}{mN} = \frac{C_1}{mN} \Rightarrow C_1 = 0}$$

$$\cancel{\Rightarrow C_2 = -C_3}$$

$$C_0 \cdot \frac{1}{mN} = i(C_2 - C_3)$$

$$C_0 = mNi(C_3 - C_2)$$

$$mNi(C_2 - C_3) = mNi(C_3 - C_2)$$

$$2C_2 = 2C_3 \qquad \boxed{C_2 = C_3} \rightarrow$$

$$C_1 + 2C_2 mN = p$$

$$\boxed{C_0 = 0}$$

$$\cancel{+\begin{cases} -C_1 \cdot \frac{1}{mN} + C_3 mN = 0 \\ C_1 + C_3 \cdot mN = p \end{cases}}$$

$$\cancel{2C_3 mN = p}$$

$$\cancel{\boxed{C_3 = \frac{p}{2mN}}}$$

$$\cancel{-\frac{C_1}{mN} + \frac{p}{2mN} = 0}$$

$$\cancel{-2C_1 + p = 0 \Leftrightarrow \boxed{C_1 = \frac{p}{2}}}$$

$$\cancel{\boxed{C_0} = mNi\left(\frac{p}{2mN} - 0\right) = \boxed{\frac{pi}{2}}}$$

$$-\frac{C_1}{mN} + 2C_2 = 0$$

$$-C_1 + 2C_2 mN = 0$$

$$4C_2 mN = p$$

$$\boxed{C_2 = \frac{p}{4mN} = C_3}$$

$$\boxed{C_1 = \frac{p}{2}}$$

$$\begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{pi}{4mN} \cdot e^{2Nit} + \frac{pi}{4mN} \cdot e^{-2Nit} \\ \frac{p}{4mN} \cdot e^{2Nit} + \frac{p}{4mN} \cdot e^{-2Nit} - \frac{p}{2mN} \\ \frac{p}{4} \cdot e^{2Nit} + \frac{p}{4} \cdot e^{-2Nit} + \frac{p}{2} \\ \frac{pi}{4} \cdot e^{2Nit} - \frac{pi}{4} \cdot e^{-2Nit} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{p}{2mN} \cdot \sin(2Nt) \\ \frac{p}{2mN} \cdot \cos(2Nt) - \frac{p}{2mN} \\ \frac{p}{2} \cdot \cos(2Nt) + \frac{p}{2} \\ -\frac{p}{2} \sin(2Nt) \end{pmatrix} = \begin{pmatrix} \frac{pc}{qB} \sin\left(\frac{qB}{mc} t\right) \\ \frac{pc}{qB} \left(\cos\left(\frac{qB}{mc} t\right) - 1\right) \\ \frac{p}{2} \left(\cos\left(\frac{qB}{mc} t\right) + 1\right) \\ -\frac{p}{2} \cdot \sin\left(\frac{qB}{mc} t\right) \end{pmatrix}$$

b) $\vec{v} = (v_1, v_2, v_3) = (\dot{x}, \dot{y}, \dot{z})$

$$\{\dot{x}, \dot{y}\} = \frac{1}{m^2} \left\{ p_1 + \frac{q\beta y}{2c}, p_2 - \frac{q\beta x}{2c} \right\} =$$

$$= \frac{1}{m^2} \cdot \frac{q\beta}{2c} \underbrace{\{y, p_2\}}_{=1} - \frac{1}{m^2} \cdot \frac{q\beta}{2c} \underbrace{\{p_1, x\}}_{=-1} = \frac{2}{m^2} \cdot \frac{q\beta}{2c} = \frac{q\beta}{c \cdot m^2}$$

$$\{\dot{y}, \dot{z}\} = \frac{1}{m^2} \left\{ p_2 - \frac{q\beta x}{2c}, p_3 \right\} = \frac{1}{m^2} \overbrace{\{p_2, p_3\}}^{=0} - \frac{q\beta x}{2cm^2} \{x, p_3\} = 0$$

$$\{\dot{x}, \dot{z}\} = \frac{1}{m^2} \left\{ p_1 + \frac{q\beta y}{2c}, p_3 \right\} = 0$$

№4.

$$L = \frac{m}{2}(\dot x^2+\dot y^2) - \frac{m\omega^2}{2}(x^2+y^2)$$

Пусть $p_x = p_1$
$p_y = p_2$

а) $p_1 = \frac{\partial L}{\partial \dot x} = m\dot x \qquad p_2 = \frac{\partial L}{\partial \dot y} = m\dot y$

$\dot x = \frac{p_1}{m} \qquad \dot y = \frac{p_2}{m}$

$$H = p_1\dot x + p_2\dot y - \frac{m}{2}(\dot x^2+\dot y^2) + \frac{m\omega^2}{2}(x^2+y^2) =$$
$$= \frac{p_1^2}{m} + \frac{p_2^2}{m} - \frac{m}{2}\left(\frac{p_1^2}{m^2}+\frac{p_2^2}{m^2}\right) + \frac{m\omega^2}{2}(x^2+y^2) =$$
$$= \frac{p_1^2}{m} + \frac{p_2^2}{m} - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)$$

б) $J_1 = \frac{1}{2m}(p_1^2 - p_2^2) + \frac{m\omega^2}{2}(x^2-y^2)$

$J_2 = \frac{1}{m}p_1p_2 + m\omega^2 xy$

$J_3 = \omega(xp_2 - yp_1)$

$\dot p_1 = -\frac{\partial H}{\partial x} = -m\omega^2 x \qquad \dot p_2 = -\frac{\partial H}{\partial y} = -m\omega^2 y$

$\dot x = \frac{\partial H}{\partial p_1} = \frac{p_1}{m} \qquad \dot y = \frac{\partial H}{\partial p_2} = \frac{p_2}{m}$

интеграл движения должен удовл.: $\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\cdot\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}\right) = 0$

$$0 = \frac{dJ_1}{dt} = \frac{\partial J_1}{\partial x}\cdot\dot x + \frac{\partial J_1}{\partial p_1}\cdot\dot p_1 + \frac{\partial J_1}{\partial y}\cdot\dot y + \frac{\partial J_1}{\partial p_2}\cdot\dot p_2 =$$
$$= m\omega^2 x\cdot\dot x + \frac{p_1}{m}\cdot\dot p_1 - m\omega^2 y\dot y - \frac{p_2}{2m}\dot p_2 =$$
$$= m\omega^2\cdot x\cdot\frac{p_1}{m} + \frac{p_1}{m}\cdot(-m\omega^2 x) - m\omega^2 y\cdot\frac{p_2}{m} - \frac{p_2}{2m}\cdot(-m\omega^2 y) = 0$$

$\Rightarrow J_1$ – интеграл движения

$$0 = \frac{dJ_2}{dt} = \frac{\partial J_2}{\partial x}\cdot\dot x + \frac{\partial J_2}{\partial p_1}\cdot\dot p_1 + \frac{\partial J_2}{\partial y}\cdot\dot y + \frac{\partial J_2}{\partial p_2}\cdot\dot p_2 =$$
$$= m\omega^2 y\cdot\frac{p_1}{m} + \frac{p_2}{m}\cdot(-m\omega^2 x) + m\omega^2 x\cdot\frac{p_2}{m} + \frac{p_1}{m}(-m\omega^2 y) = 0$$

$\Rightarrow J_2$ – интеграл движения

$$0 = \frac{dJ_3}{dt} = \frac{\partial J_3}{\partial x}\cdot\dot x + \frac{\partial J_3}{\partial p_1}\cdot\dot p_1 + \frac{\partial J_3}{\partial y}\cdot\dot y + \frac{\partial J_3}{\partial p_2}\cdot\dot p_2 =$$
$$= p_2\omega\cdot\frac{p_1}{m} + (-\omega y)\cdot(-m\omega^2 x) + (-\omega p_1)\cdot\frac{p_2}{m} + \omega x\cdot(-m\omega^2 y) = 0$$

$\Rightarrow J_3$ – интеграл движения

в) лин. оболочка $\mathcal{L}$, порождённая всевозможными лин. комбинациями ф-ций J_i, инвариантна относ. скобки Пуассона.

Достаточно показать, что $\forall i,j$ $\{J_i, J_j\} \in \mathcal{L}$

$$\{J_1, J_2\} = \left\{ \frac{1}{2m}(p_1^2 - p_2^2) + \frac{m\omega^2}{2}(x^2 - y^2), \frac{1}{m} p_1 p_2 + m\omega^2 xy \right\} =$$

$$= \frac{m\omega^2 y}{2m}\{p_1^2, x\} - \frac{m\omega^2 x}{2m}\{p_2^2, y\} + \frac{m\omega^2}{2}\cdot\frac{p_2}{m}\{x^2, p_1\} - \frac{m\omega^2}{2}\cdot\frac{p_1}{m}\{y^2, p_2\} =$$

$$= \frac{\omega^2 y}{2}\cdot 2p_1\{p_1, x\} - \frac{\omega^2 x}{2}\cdot 2p_2\{p_2, y\} + \frac{\omega^2 p_2}{2}\, 2x\{x, p_1\} - \frac{\omega^2 p_1}{2}\, 2y\{y, p_2\} =$$

$$= -\omega^2 y p_1 + \omega^2 x p_2 + \omega^2 x p_2 - \omega^2 y p_1 = -2\omega^2 y p_1 + 2\omega^2 x p_2 =$$

$$= 2\omega^2(xp_2 - yp_1) = 2\omega J_3 \Rightarrow \{J_1, J_2\} \in \mathcal{L}$$

$$\{J_1, J_3\} = \left\{ \frac{1}{2m}(p_1^2 - p_2^2) + \frac{m\omega^2}{2}(x^2 - y^2), \omega(xp_2 - yp_1) \right\} =$$

$$= \frac{\omega p_2}{2m}\{p_1^2, x\} + \frac{p_1\omega}{2m}\{p_2^2, y\} - \frac{m\omega^3 y}{2}\{x^2, p_1\} - \frac{m\omega^3 x}{2}\{y^2, p_2\} =$$

$$= \frac{\omega p_1 p_2}{m}\{p_1, x\} + \frac{\omega p_1 p_2}{m}\{p_2, y\} - m\omega^3 xy\{x, p_1\} - m\omega^3 xy\{y, p_2\} =$$

$$= -\frac{\omega p_1 p_2}{m} - \frac{\omega p_1 p_2}{m} - m\omega^3 xy - m\omega^3 xy = -\omega\left(\frac{2p_1p_2}{m} + 2m\omega^2 xy\right) =$$

$$= -2\omega J_2 \Rightarrow \{J_1, J_3\} \in \mathcal{L}$$

$$\{J_2, J_3\} = \left\{ \frac{1}{m} p_1 p_2 + m\omega^2 xy, \omega(xp_2 - yp_1) \right\} =$$

$$= \frac{\omega p_2^2}{m}\{p_1, x\} - \frac{\omega p_1^2}{m}\{p_2, y\} - m\omega^3 y^2\{x, p_1\} + m\omega^3 x^2\{y, p_2\} =$$

$$= -\frac{\omega p_2^2}{m} + \frac{\omega p_1^2}{m} - m\omega^3 y^2 + m\omega^3 x^2 =$$

$$= \omega\left(\frac{p_1^2}{m} - \frac{p_2^2}{m} - m\omega^2(y^2 - x^2)\right) = 2\omega J_1 \Rightarrow \{J_2, J_3\} \in \mathcal{L}$$

$\{J_1, J_1\} = \{J_2, J_2\} = \{J_3, J_3\} = 0$, т.к. скобка Пуассона кососимметрична и билинейна

N5.

$$H = \frac{\vec{M}^2}{2I} - \gamma \vec{M}\vec{B}$$

$$\vec{M} = (M_1, M_2, M_3) \qquad \vec{B} = (0, 0, B)$$

$$H = \frac{M_1^2 + M_2^2 + M_3^2}{2I} - \gamma(M_1 B_1 + M_2 B_2 + M_3 B_3)$$

$$\{M_i, M_i\} = 0$$

$$\{M_i, M_j\} = \{\varepsilon_{ijk} r_j p_k, \varepsilon_{jki} r_k p_i\} = \varepsilon_{ijk} \varepsilon_{jki} \overset{-1}{\{p_k, r_k\}} r_j p_i + \varepsilon_{ikj} \{r_k, p_k\} p_j \varepsilon_{jik} r_i =$$

$$= -r_j p_i + r_i p_j = \varepsilon_{kij} r_i p_j \cdot \varepsilon_{ijk} = \varepsilon_{ijk} M_k$$

$$\frac{dM_i}{dt} = \{M_i, H\} = \{M_i, \frac{M_1^2 + M_2^2 + M_3^2}{2I} - \gamma(M_1 B_1 + M_2 B_2 + M_3 B_3)\} =$$

$$= \sum_{j \neq i} \frac{\{M_i, M_j\} \cdot 2M_j}{2I} - \gamma\{M_i, M_j\} \cdot B_j = \sum_{j \neq i} \frac{2\varepsilon_{ijk} M_k M_j}{2I} - \gamma \varepsilon_{ijk} M_k B_j =$$

$$= \sum_{j \neq i} -\frac{\varepsilon_{ikj} M_k M_j}{I} - \gamma \varepsilon_{ijk} M_k B_j = \sum_{j \neq i} \frac{-\overset{}{\underset{0}{[\vec{M} \times \vec{M}]_i}}}{I} - \gamma \varepsilon_{ijk} M_k B_j =$$

$$= \gamma \varepsilon_{ikj} M_k B_j = \gamma [\vec{M} \times \vec{B}]_i$$

$$\dot{M}_1 = \gamma(M_2 B_3 - M_3 B_2)$$

$$\dot{M}_2 = \gamma(M_3 B_1 - M_1 B_3)$$

$$\dot{M}_3 = \gamma(M_1 B_2 - M_2 B_1)$$

$$\vec{B} = (0, 0, B): \quad \begin{cases} \dot{M}_1 = \gamma M_2 B \\ \dot{M}_2 = -\gamma M_1 B \\ \dot{M}_3 = 0 \end{cases} \qquad M_3 = const$$

Пусть $M_{12} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$: $\dot{M}_{12} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} M_{12} B$

$$M_{12} = \exp\left(\begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} Bt\right) \cdot M_{12}(0) = \begin{pmatrix} \cos\gamma Bt & \sin\gamma Bt \\ -\sin\gamma Bt & \cos\gamma Bt \end{pmatrix} \cdot M_{12}(0)$$

$$M_1 = M_1(0) \cdot \cos(\gamma Bt) + M_2(0) \cdot \sin(\gamma Bt)$$

$$M_2 = -M_1(0) \cdot \sin(\gamma Bt) + M_2(0) \cdot \cos(\gamma Bt)$$

$$M_3 = const$$