Task

$$\prod_{n\geq 1}\left(1+tq^{2n-1}\right)=\sum_{k\geq 0}\frac{t^kq^{k^2}}{(1-q^2)\,(1-q^4)\ldots(1-q^{2k})}$$

Solution

Notice that the coefficient of t^k on the left side depends on the k brackets from which we take tq^m. Enumerate this brackets by odd numbers, where $2n'+1$ connected with $(1+tq^{2n'+1})$.
Let i is the minimal degree of coefficient q^i of t^k and i is the sum of odd numbers from 1 to $2k-1$.
Then consider some other coefficient q^n and we can choose any bracket instead of $2k-1$. In other words, we can increase $2k-1$ by $0,2,4,6,\ldots$. A generating function of this bracket is $q^{2k-1}(1+q^2+q^4+\ldots)=\frac{q^{2k-1}}{1-q^2}$.

Then consider next bracket $2k-3$, smallest number by which we can increase the number of this bracket is 4, since we cannot increase $2k-3$ by 2. A generating function of this bracket is $q^{2k-3}(1+q^4+\ldots)=\frac{q^{2k-3}}{1-q^4}$.
Similarly for all other brackets.

$$\frac{q}{1-q^{2k}}\cdot\ldots\cdot\frac{q^{2k-1}}{1-q^2}=\frac{q^{k^2}}{(1-q^2)\cdot\ldots\cdot(1-q^{2k})}$$

Which proves the equation.

Task
Let f_n be the Fibonacci sequence $f_1 = 1,\ f_2 = 2$
Prove that each positive integer admits a unique representation in a form
$a_1 f_1 + a_2 f_2 + \ldots + a_n f_n + \ldots$ such that

- each a_i is either 0 or 1
- there are finitely many numbers a_i equal to 1
- no two consequent numbers a_i are equal to 1

Solution
We'll proof this statement by induction
If we can represent $1, \ldots, n$ as sum of Fibonacci numbers, we also can represent $n+1$ Basis:

$$1 = f_1$$
$$2 = f_2$$
$$3 = f_3$$
$$4 = f_1 + f_3$$

Step:
If $n+1$ is a Fibonacci number, it is itself representation.
Suppose that $n+1$ isn't a Fibonacci number, then

$$\exists i \in \mathbb{Z} : F_i < n+1 < F_{i+1}$$
$$n + 1 - F_i = m$$
$$m < F_i$$

Then we can represent $n+1$ as representation of m plus F_i

Then proof the uniqueness
$n \in \mathbb{N}$ and n has 2 representations: A_1, A_2
Let $A_1' = A_1/A_2,\ A_2' = A_2/A_1$, so $A_1' \cap A_2' = \varnothing$ and $A_1',\ A_2'$ represent the same number.
Suppose the the largest element of A_1' larger then the largest element of A_2', but it means that A_1' represent larger number, cause sum of elements of A_2' less then the largest element of A_1' (else A_2' should have this number in it's representation). It follows that there cannot be 2 different representations.

Task

Let $D(n,m)$ be the number of ways to move from the point $(0,0)$ to the point (n,m) moving each time one step upwards or one step rightwards. Give a closed form for the generating function

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D(n,m)x^n y^m =$$
$$= 1 + x + y + x^2 + y^2 + 2xy + \dots$$

Solution

It is known that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{n}{m} x^{n-m} y^m =$$
$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m\right) =$$
$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right) =$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(x+y)^n =$$
$$\sum_{n=0}^{\infty}\binom{n}{0}x^n + \sum_{n=0}^{\infty}\binom{n+1}{1}x^n y + \sum_{n=0}^{\infty}\binom{n+2}{2}x^n y^2 + \dots =$$
$$\sum_{n=0}^{\infty}\binom{n}{0}x^n + \sum_{n=1}^{\infty}\binom{n}{1}x^{n-1} y + \sum_{n=2}^{\infty}\binom{n}{2}x^{n-2} y^2 + \dots =$$
$$1 + (x+y) + (x+y)^2 + \dots =$$
$$1 + x + y + x^2 + y^2 + 2xy + \dots$$

Task (Another q-binomial theorem) The variables x and y. do not commute, but they satisfy the following relation: $yx = qxy$. If a is any expression containing x, y or q, than $qa = aq$. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

For example,

$$(x+y)^2 = x^2 + (1+q)xy + y^2$$

Solution

When we multiply $(x+y)^n$, each term is a string of length n which contains x and y. If there are k letters y in string, we can define it as $x^{n-k}y^k$ and each change from one letter to another contributing a factor of q. Coefficient of this term is q^a, where a is the number of changes. Let $\alpha(n,k)$ be the sum of these coefficients over all strings with k letters y. Then $\alpha(n,k)$ is a polynomial in q, and

$$(x+y)^n = \sum_{k=0}^{n} \alpha(n,k) x^{n-k} y^k$$

We identify the coefficients $\alpha(n,k)$ with Gaussian coefficients.

$$(x+y)^n = (x+y)^{n-1}(x+y) = \left(\sum_{k=0}^{n-1} C(n-1,k) x^{n-1-k} y^k \right) (x+y)$$

Multiplying by x, we have to change this x over all k letters y to reach the required form, giving a factor of q^k. Multiplying by y, no changes are required. So we have

$$\alpha(n,k) = q^k \alpha(n-1,k) + \alpha(n-1,k-1)$$

Thus the coefficients $\alpha(n,k)$ satisfy the same recurrence and initial conditions as the Gaussian coefficients, and so are equal to them.