1. Localization and Spectrum

All rings are assumed to be commutative with identity.

Proposition 1.1. Let A be a ring and S be a multiplicative subset of A. The map $A \to S^{-1}A$ induces a homeomorphism

$$\mathrm{Spec}(S^{-1}R) \to \{\mathfrak{p} \in \mathrm{Spec}\, A : S \cap \mathfrak{p} = \emptyset\}$$

where topology of the right hand side is the subspace topology of $\mathrm{Spec}\, A$. The inverse map is given by $\mathfrak{p} \to S^{-1}\mathfrak{p}$.

Proof. Let $\varphi : A \to S^{-1}A$ be the map sending a to $a/1$. Then we have a continuous map $\mathrm{Spec}\, \varphi : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}\, A$. For simplicity, denote $\mathrm{Spec}\, \varphi$ by h. Let $\mathfrak{p}'$ be a prime ideal in $S^{-1}A$. Then $\varphi^{-1}\mathfrak{p}'$ is a prime ideal in A so that $\varphi^{-1}(\mathfrak{p}') \cap S = \emptyset$. If not, there exists $f \in \varphi^{-1}(\mathfrak{p}') \cap S$. Then $f \in S$ and $f/1 \in \mathfrak{p}'$. Since $f \in S$, $1/f \in S^{-1}A$. This implies that $1/1 \in \mathfrak{p}'$, i.e. $S^{-1}A = \mathfrak{p}'$ which is absurd because $\mathfrak{p}'$ is a prime ideal. Hence $\mathrm{Im}\, h \subset \{\mathfrak{p} \in \mathrm{Spec}\, A : S \cap \mathfrak{p} = \emptyset\}$. Conversely if $\mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}\, A : S \cap \mathfrak{p} = \emptyset\}$, then $\varphi(\mathfrak{p}) = S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}A$. This is because the localization of an integral domain is an integral domain and hence $S^{-1}A/S^{-1}\mathfrak{p} \cong S^{-1}(A/\mathfrak{p})$ is an integral domain. Moreover, $\mathfrak{p} = \varphi^{-1}(S^{-1}\mathfrak{p})$. Therefore $\mathfrak{p} \in \mathrm{Im}\, h$. We find $\mathrm{Im}\, h = \{\mathfrak{p} \in \mathrm{Spec}\, A : S \cap \mathfrak{p} = \emptyset\}$.

Let $h' : \mathrm{Im}\, h \to \mathrm{Spec}(S^{-1}R)$ by $\mathfrak{p} \to S^{-1}\mathfrak{p}$. For $\mathfrak{p} \in \mathrm{Im}\, h$, $h \circ h'(\mathfrak{p}) = h(S^{-1}\mathfrak{p}) = \varphi^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ and for any $\mathfrak{p}'$, $h' \circ h(\mathfrak{p}') = h'(\varphi^{-1}\mathfrak{p}') = S^{-1}(\varphi^{-1}\mathfrak{p}') = \mathfrak{p}'$ by definition. Hence h' is the inverse of h. Now, we only need to show that h is an open mapping.

Let $D(t/s)$ be a standard open subset in $\mathrm{Spec}(S^{-1}A)$. Let us show that $h(D(t/s)) = D(t) \cap \mathrm{Im}\, h$. Suppose $\mathfrak{p} \in D(t) \cap \mathrm{Im}\, h$. Then $\mathfrak{p} \cap S = \emptyset$ and $t \notin \mathfrak{p}$. Then $t/s \notin \mathfrak{p}' = \varphi(\mathfrak{p})$.[1] This shows that $\mathfrak{p}' \in D(t/s)$. In other words, $\mathfrak{p} = h(\mathfrak{p}') \subset h(D(t/s)))$ Therefore $D(t) \cap \mathrm{Im}\, h \subset h(D(t/s))$. Suppose that $\mathfrak{p} \in h(D(t/s))$. Then $\mathfrak{p} \in \mathrm{Im}\, h$ and there is $\mathfrak{p}' \in D(t/s)$ so that $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}')$. Since $\mathfrak{p} \in \mathrm{Im}\, h$, $\mathfrak{p} \cap S = \emptyset$. Since $\mathfrak{p}' \in D(t/s)$, $t/s \notin \mathfrak{p}'$. Now, we want to show $\mathfrak{p} \in D(t)$. Suppose not. $t \in \mathfrak{p}$. Then $t/s \in \mathfrak{p}'$ which leads to the contradiction that $t/s \notin \mathfrak{p}'$. Therefore $t \notin \mathfrak{p}$ and hence $\mathfrak{p} \in D(t)$. We conclude that

$$h(D(t/s)) = D(t) \cap \mathrm{Im}\, h.$$

This shows that h is an open mapping. □

Corollary 1.1. Let A be a ring and $f \in A$. Then we obtain a homeomorphism

$$\mathrm{Spec}\, A_f \to D(f).$$

Proof. Let $\varphi : A \to A_f$ be the localization and $h : \mathrm{Spec}\, A_f \to \mathrm{Spec}\, A$ be its induced map. Then $\mathrm{Im}\, h = \{\mathfrak{p} \in \mathrm{Spec}\, A : \mathfrak{p} \cap S_f = \emptyset\}$, where $S_f = \{f^n : n \geq 0\}$. By definition, $\mathrm{Im}\, h = D(f)$. Using Proposition 1.1, $\mathrm{Spec}\, A_f \to D(f)$ is a homeomorphism.

□

Proposition 1.2. Let A be a ring and I be an ideal. Then the quotient map $A \to A/I$ induces a homeomorphism

$$\mathrm{Spec}(A/I) \to V(I) \subset \mathrm{Spec}\, A.$$

Proof. The bijection

$$\{\text{prime ideals of } A/I\} \longleftrightarrow \{\text{prime ideals of } A \text{ containing } I.\}$$

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[1] If $t/s \in \mathfrak{p}'$, then $t/s = t'/s'$ for $t'/s' \in S^{-1}\mathfrak{p}$. Hence there is $s'' \in S$ so that $s''(ts' - st') = 0$. Since $t' \in \mathfrak{p}$, $s''st' \in \mathfrak{p}$ and $s''s't$ is thus in $\mathfrak{p}$. Since $S \cap \mathfrak{p} = \emptyset$, $s's'' \notin S$. Since $\mathfrak{p}$ is a prime, we obtain $t \in \mathfrak{p}$ which leads to a contradiction that $t \notin \mathfrak{p}$.

implies that the continuous map $h : \mathrm{Spec}(A/I) \to V(I)$ is a bijection. Here h is the induced map of the quotient map.

Let us prove that that map is an open mapping. Claim

$$h(D(s+I)) = D(s) \cap V(I),$$

where s is any representative of $s+I$. Suppose $\mathfrak{p} \in D(s) \cap V(I)$. Then $s \notin \mathfrak{p}$ and $\mathfrak{p}$ contains I. Then $s + I \neq I$. Because $s \notin \mathfrak{p}$, $s + I \notin \mathfrak{p}/I$. Hence $\mathfrak{p}/I \in D(s+I)$, we see that $\mathfrak{p} \in h(D(s+I))$. We find $D(s) \cap V(I) \subset h(D(s))$. Conversely, if $\mathfrak{p} \in h(D(s+I))$, then $s + I \notin \mathfrak{p}/I$. This implies that $s \notin \mathfrak{p}$ and hence $\mathfrak{p} \in D(s)$. We see that $\mathfrak{p} \in D(s) \cap V(I)$. We obtain $h(D(s+I)) \subset D(s) \cap V(I)$.

$\square$