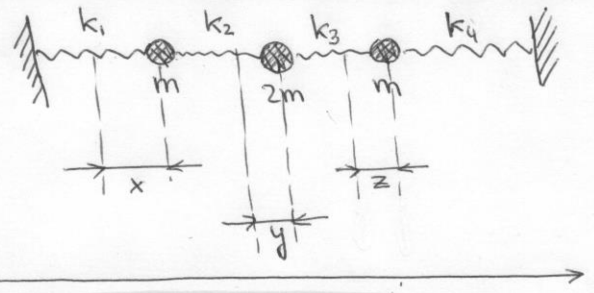
1



$$\boxed{\begin{aligned} k_1 &= k_4 = 3k \\ k_2 &= k_3 = 2k \end{aligned}}$$

а) $$\boxed{\begin{cases} m\ddot{x} = -k_1 x - k_2(x-y) \\ 2m\ddot{y} = k_2(x-y) - k_3(y-z) \\ m\ddot{z} = k_3(y-z) - k_4 z \end{cases}}$$

б) $\ddot{X} = -AX$

$$A = \frac{1}{m}\begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -\frac{k_2}{2} & \frac{k_2+k_3}{2} & -\frac{k_3}{2} \\ 0 & -k_3 & k_3+k_4 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} 5 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -2 & 5-\lambda \end{vmatrix} = (5-\lambda)\left((2-\lambda)(5-\lambda)-2\right) + 2\left((\lambda-5)\right) =$$

$$= (5-\lambda)\left((2-\lambda)(5-\lambda)-4\right) =$$

$$= -(\lambda-5)(10-7\lambda+\lambda^2-4) = -(\lambda-5)(\lambda^2-7\lambda+6) = -(\lambda-5)(\lambda-1)(\lambda-6)$$

$$\omega_1^2 = \frac{5k}{m} \longrightarrow \omega_1 = \sqrt{\frac{5k}{m}}$$

$$\omega_2^2 = \frac{k}{m} \longrightarrow \omega_2 = \sqrt{\frac{k}{m}} \quad \text{– нормальные частоты}$$

$$\omega_3^2 = \frac{6k}{m} \longrightarrow \omega_3 = \sqrt{\frac{6k}{m}}$$

$\underline{\lambda=5}$ $$\begin{pmatrix} 0 & -2 & 0 \\ -1 & -3 & -1 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} x_1 + 3x_2 + x_3 = 0 \\ x_2 = 0 \end{cases} \quad \Psi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{\lambda=1}\quad \begin{array}{c}\\ \times 4\\ \\ \end{array}\begin{pmatrix} 4 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & -4 \\ -1 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = x_2 - x_3 \\ x_2 = 2x_3 \end{cases} \qquad \psi_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underline{\lambda=6}\quad \begin{pmatrix} -1 & -2 & 0 \\ -1 & -4 & -1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 1 \\ +1 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

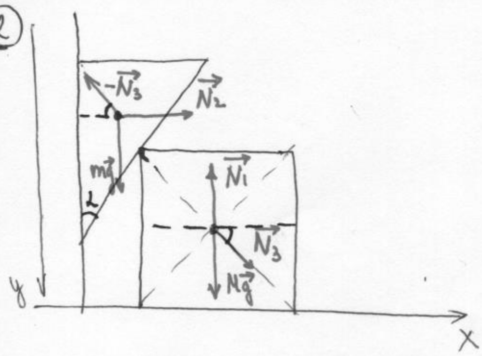
$$\begin{cases} 2x_2 = -x_3 \\ x_1 = -4x_2 - x_3 \end{cases} \Rightarrow \qquad \psi_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Ответ в пункте б):

Нормальные частоты: $\sqrt{\frac{5k}{m}}, \sqrt{\frac{k}{m}}, \sqrt{\frac{6k}{m}}$.

Нормальные моды: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

②



а) # степеней свободы = 2-1 = 1.

б) $$\begin{cases} M\ddot{x} = N_3 \cos\alpha \\ 0 = N_2 - N_3 \cos\alpha \\ 0 = N_1 - N_3 \sin\alpha - Mg \\ m\ddot{y} = mg - N_3 \sin\alpha \\ \ddot{x} = \ddot{y}\, tg\alpha \end{cases}$$

в) $N_2 = N_3 \cos\alpha$

$$\ddot{x} = \frac{N_3 \cos\alpha}{M}$$

$$N_1 = N_3 \sin\alpha + Mg.$$

$$\ddot{y} = g - \frac{N_3}{m} \sin\alpha$$

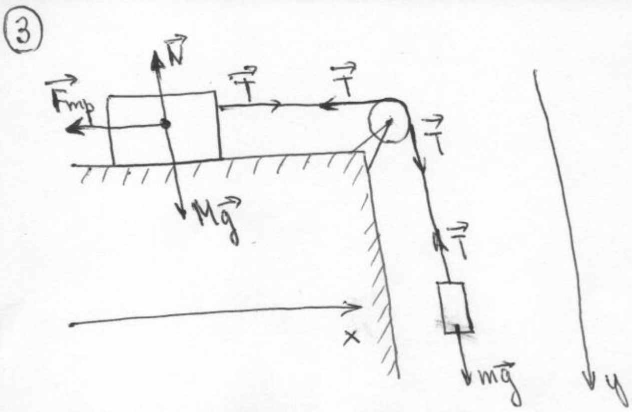
$$\frac{N_3 \cos\alpha}{M} = g\, tg\alpha - \frac{N_3}{m} \sin\alpha\, tg\alpha$$

$$N_3 \left(\frac{\cos\alpha}{M} + \frac{\sin\alpha\, tg\alpha}{m} \right) = g\, tg\alpha \Rightarrow N_3 = \frac{(g\, tg\alpha) Mm}{m\cos\alpha + M\sin\alpha\, tg\alpha}$$

$$\ddot{x} = \frac{mg \sin\alpha}{m\cos\alpha + M\sin\alpha\, tg\alpha}$$

$$\ddot{y} = \frac{mg\cos\alpha + Mg \sin\alpha\, tg\alpha - Mg\, tg\alpha \sin\alpha}{m\cos\alpha + M \sin\alpha\, tg\alpha} = \frac{mg \cos\alpha}{m\cos\alpha + M\sin\alpha\, tg\alpha}$$

③



⊛ Заметим, что если $|kN| > |T|$, то кирпич M не будет двигаться. В этом случае
$T = mg$, $N = Mg$
$kMg > mg$
$\Rightarrow \boxed{k > \frac{m}{M}}$

а) # степеней свободы $= 2 - 1 = 1$

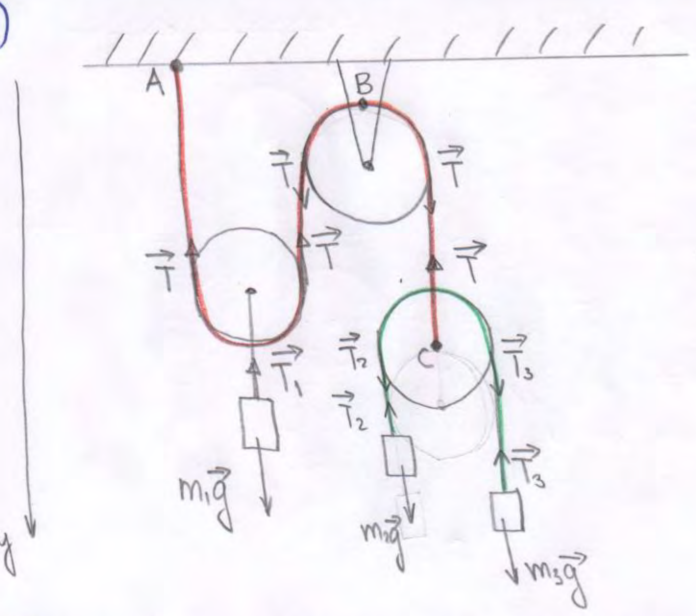
б) $$\begin{cases} m\ddot{y} = mg - T \\ M\ddot{x} = T - kN \\ N = Mg \\ \ddot{x} = \ddot{y} \end{cases}$$

в) $$\ddot{y} = g - \frac{T}{m}$$
$$\ddot{x} = \frac{T}{M} - \frac{kMg}{M}$$

$$g - \frac{T}{m} = \frac{T}{M} - kg$$
$$T\left(\frac{1}{m} + \frac{1}{M}\right) = g(1+k)$$
$$T = \frac{mMg(1+k)}{m+M}$$

$$\ddot{x} = \ddot{y} = g - \frac{Mg(1+k)}{m+M} = \frac{mg \cancel{+Mg} \cancel{-Mg} - Mgk}{m+M} =$$
$$= \boxed{\frac{m - Mk}{m+M}\, g}$$

4



1. $$\boxed{\begin{aligned} &m_1\ddot x_1 = m_1 g - T_1 \\ &m_2\ddot x_2 = m_2 g - T_2 \\ &m_3\ddot x_3 = m_3 g - T_3 \\ &T_2 = T_3 = T/2 \\ &T_1 = 2T \end{aligned}}$$

2. Пусть в момент времени $t=0$ длина участка AB красной нити равнялась l_{AB}, длина участка BC − l_{BC}.

Если бы блок C был неподвижен: $\ddot{\tilde x}_2 + \ddot{\tilde x}_3 = 0$

$$\begin{cases} x_2 = \tilde x_2 + \Delta l_{BC} \Rightarrow \tilde x_2 = x_2 - \Delta l_{BC} \\ x_3 = \tilde x_3 + \Delta l_{BC} \Rightarrow x_3 = -x_2 + \Delta l_{BC}\cdot 2 \Rightarrow \Delta l_{BC} = \dfrac{x_2 + x_3}{2} \\ \ddot{\tilde x}_2 + \ddot{\tilde x}_3 = 0 \end{cases}$$

Для красной нити имеем: $l_{AB} + 2x_1 + l_{BC} + \dfrac{x_2+x_3}{2} = \text{const} \Rightarrow$

$\Rightarrow \boxed{4\ddot x_1 + \ddot x_2 + \ddot x_3 = 0}$

3. $\ddot x_1 = g - \dfrac{2T}{m_1}$, $\ddot x_2 = g - \dfrac{T}{2m_2}$, $\ddot x_3 = g - \dfrac{T}{2m_3}$

$$4g - \frac{8T}{m_1} + g - \frac{T}{2m_2} + g - \frac{T}{2m_3} = 0$$

$$T\left(\overset{2m_3m_2}{\frac{8}{m_1}} + \overset{m_1m_3}{\frac{1}{2m_2}} + \frac{1}{2m_3}\right) = 6g \Rightarrow T = \frac{12g\; m_1m_2m_3}{16m_2m_3 + m_1m_3 + m_1m_2}$$

$$\ddot x_1 = \frac{16m_2m_3 g + m_1m_3 g + m_1m_2 g - 24\, m_2m_3 g}{16m_2m_3 + m_1m_3 + m_2m_3} =$$

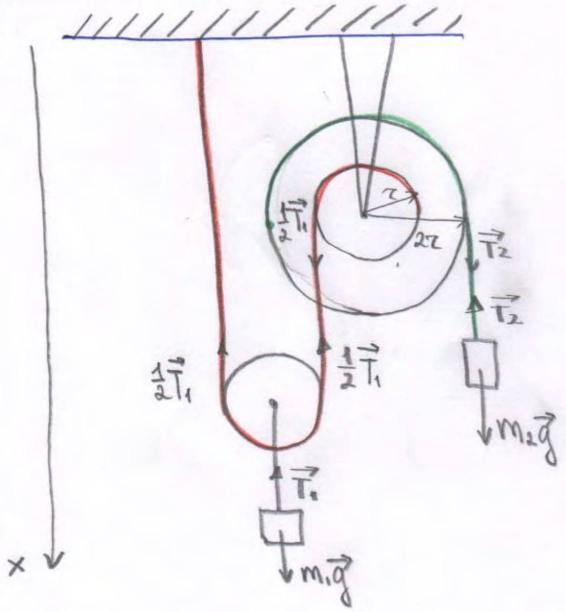
$$= \frac{m_1m_3 + m_1m_2 - 8m_2m_3}{16m_2m_3 + m_1m_3 + m_2m_3}\, g$$

$$\ddot{x}_2 = \frac{16 m_2 m_3 + m_1 m_2 - 5 m_1 m_3}{16 m_2 m_3 + m_1 m_3 + m_2 m_3} g$$

$$\ddot{x}_3 = \frac{16 m_2 m_3 + m_1 m_3 - 5 m_1 m_2}{16 m_2 m_3 + m_1 m_3 + m_2 m_3} g$$

- Число степеней свободы $= 3 - 1 = 2$

5



$$\boxed{\begin{aligned} m_1\ddot{x}_1 &= m_1 g - T_1 \\ m_2\ddot{x}_2 &= m_2 g - T_2 \\ T_2 &= \frac{1}{2}\cdot\frac{1}{2}T_1 = \frac{1}{4}T_1 \end{aligned}}$$

Пусть катушка повернулась на угол φ по часовой стрелке. Тогда

$$\Delta l_{зел} = x_2 = 2r\varphi$$

$$\Delta l_{кр} = -r\varphi = \frac{-x_2}{2} \Rightarrow x_1 = \frac{\Delta l_{кр}}{2} = \frac{-x_2}{4} \Rightarrow 4x_1 = -x_2$$

$$\Rightarrow \boxed{4\ddot{x}_1 = -\ddot{x}_2}$$

- Число степеней свободы: $2-1=1$

$$\ddot{x}_1 = g - \frac{T_1}{m_1} = g - \frac{4T_2}{m_1}$$

$$\ddot{x}_2 = g - \frac{T_2}{m_2}$$

$$4g - \frac{16T_2}{m_1} = -g + \frac{T_2}{m_2} \Rightarrow 5g = T_2\left(\frac{16}{m_1} + \frac{1}{m_2}\right)$$

$$\Rightarrow T_2 = \frac{5m_1m_2 g}{16m_2 + m_1}$$

$$\ddot{x}_1 = g - \frac{4\cdot 5m_2 g}{16m_2 + m_1} = \frac{16m_2 g + m_1 g - 20m_2 g}{16m_2 + m_1} = \frac{m_1 - 4m_2}{16m_2 + m_1}g$$

$$\ddot{x}_2 = g - \frac{5m_1 g}{16m_2 + m_1} = \frac{16m_2 + m_1 - 5m_1}{16m_2 + m_1}g = \frac{4(4m_2 - m_1)}{16m_2 + m_1}g$$

(6)

$$(\vec{e_r},\vec{e_\theta},\vec{e_\varphi}) = (\vec{e_x},\vec{e_y},\vec{e_z})\begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$r = (x^2+y^2+z^2)^{1/2}, \quad tg\,\theta = (x^2+y^2)^{\frac{1}{2}}/z, \quad tg\,\varphi = y/x$$

$$\vec{e_r} = \vec{e_x}\sin\theta\cos\varphi + \vec{e_y}\sin\theta\sin\varphi + \vec{e_z}\cos\theta$$

$$\dot{\vec{e_r}} = \vec{e_x}(\cos\theta\cos\varphi\cdot\dot\theta - \sin\theta\sin\varphi\cdot\dot\varphi) +$$
$$+ \vec{e_y}(\cos\theta\sin\varphi.\dot\theta + \sin\theta\cos\varphi\cdot\dot\varphi) +$$
$$+ \vec{e_z}\cdot(-\sin\theta)\,\dot\theta =$$

$$= \dot\theta(\vec{e_x}\cos\theta\cos\varphi + \vec{e_y}\cos\varphi\sin\varphi - \vec{e_z}\sin\theta)$$
$$+ \dot\varphi(-\vec{e_x}\sin\theta\sin\varphi + \vec{e_y}\sin\theta\cos\varphi) =$$
$$= \dot\theta\,\vec{e_\theta} + \dot\varphi\sin\theta\,\vec{e_\varphi} \Rightarrow \boxed{\dot{\vec{e_r}} = \dot\theta\vec{e_\theta} + \dot\varphi\sin\theta\,\vec{e_\varphi}}$$

$$\vec{e_\theta} = \vec{e_x}\cos\theta\cos\varphi + \vec{e_y}\cos\theta\sin\varphi - \vec{e_z}\sin\theta$$

$$\dot{\vec{e_\theta}} = \vec{e_x}(-\sin\theta\cos\varphi\,\dot\theta - \cos\theta\sin\varphi\,\dot\varphi) + \vec{e_y}(-\sin\theta\sin\varphi\,\dot\theta + \cos\theta\cos\varphi\,\dot\varphi) -$$
$$- \vec{e_z}\cos\theta\,\dot\theta = -\dot\theta(\vec{e_x}\sin\theta\cos\varphi + \sin\theta\sin\varphi\,\vec{e_y} + \vec{e_z}\cos\theta) +$$
$$+ \dot\varphi\cos\theta(-\vec{e_x}\sin\varphi + \vec{e_y}\cos\varphi) = -\dot\theta\vec{e_r} + \dot\varphi\cos\theta\,\vec{e_\varphi} \Rightarrow$$
$$\Rightarrow \boxed{\dot{\vec{e_\theta}} = -\dot\theta\vec{e_r} + \dot\varphi\cos\theta\,\vec{e_\varphi}}$$

$$\vec{e_\varphi} = -\vec{e_x}\sin\varphi + \vec{e_\theta}\cos\varphi$$

$$\dot{\vec{e_\varphi}} = -\vec{e_x}\cos\varphi\,\dot\varphi - \vec{e_\theta}\sin\varphi.\dot\varphi = -\dot\varphi(\sin^2\theta\cos\varphi\,\vec{e_x} +$$
$$+ \sin^2\theta\sin\varphi\,\vec{e_y} + \sin\theta\cos\theta\,\vec{e_z} + \cos^2\theta\cos\varphi\,\vec{e_x} + \cos^2\varphi\sin\varphi\,\vec{e_y} -$$
$$- \sin\theta\cos\theta\,\vec{e_z}) = -\dot\varphi(\cos\varphi + \sin\varphi) =$$
$$= -\dot\varphi(\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}) \Rightarrow \boxed{\dot{\vec{e_\varphi}} = -\dot\varphi(\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta})}$$

$\vec{r} = r\vec{e_r}$

$$\boxed{\dot{\vec{r}} = \dot r\,\vec{e_r} + r\,\dot{\vec{e_r}} = \dot r\,\vec{e_r} + r\dot\theta\,\vec{e_\theta} + r\dot\varphi\sin\theta\,\vec{e_\varphi}}$$

$$\ddot{\vec{r}} = \ddot r\,\vec{e_r} + \dot r\left(\dot\theta\,\vec{e_\theta} + \dot\varphi\sin\theta\,\vec{e_\varphi}\right) + \dot r\dot\theta\,\vec{e_\theta} + r\ddot\theta\,\vec{e_\theta} +$$

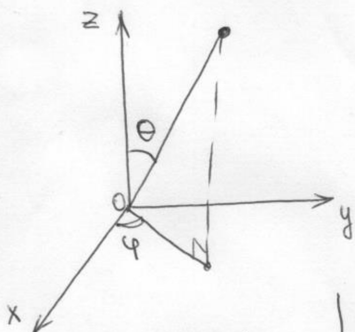
$$+ r\dot\theta\left(-\dot\theta\,\vec{e_r} + \dot\varphi\cos\theta\cdot\vec{e_\varphi}\right) + \dot r\dot\varphi\sin\theta\,\vec{e_\varphi} + r\ddot\varphi\sin\theta\,\vec{e_\varphi} +$$

$$+ r\dot\varphi\cos\theta\,\dot\theta\,\vec{e_\varphi} + r\dot\varphi\sin\theta\,(-\dot\varphi)\left(\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}\right) =$$

$$= \vec{e_r}\left(\ddot r - r(\dot\theta)^2 - r(\dot\varphi)^2\sin^2\theta\right) +$$

$$+ \vec{e_\theta}\left(2\dot r\dot\theta + r\ddot\theta - r(\dot\varphi)^2\sin\theta\cos\theta\right) +$$

$$+ \vec{e_\varphi}\left(\dot r\dot\varphi\sin\theta + r\dot\theta\dot\varphi\cos\theta + \dot r\dot\varphi\sin\theta + r\ddot\varphi\sin\theta + r\dot\varphi\dot\theta\cos\theta\right)$$



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{array}{l} \theta\in[0,\pi] \\ \varphi\in[0,2\pi) \end{array}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = \begin{vmatrix} \sin\theta\cdot\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\sin\theta\begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\sin\theta\left[-\sin\varphi\left(-\sin^2\theta\sin\varphi - \cos^2\theta\sin\varphi\right) - \cos\varphi\left(-\sin^2\theta\cos\varphi - \cos^2\theta\cos\varphi\right)\right]$$

$$= r^2\sin\theta\left(\sin^2\varphi + \cos^2\varphi\right) = r^2\sin\theta = 0 \iff \begin{array}{l} r=0 \\ \sin\theta = 0 \\ \theta\in[0,\pi] \end{array} \iff \begin{array}{l} r=0 \\ \theta=0 \\ \theta=\pi \end{array}$$

$\Rightarrow$ переход к сферическим координатам несингулярен на $\mathbb{R}^3\setminus\{Oz\}\subset\mathbb{R}^3$.