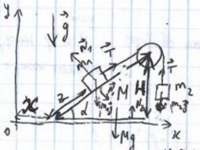
N1



Число степеней свободы: ... $5-3=2$

$\frac{H}{\sin\alpha}=L$

ℓ - длина нити

коорд x у m_2: $x+\sqrt{L^2-H^2}$ $\quad (x+\sqrt{L^2-H^2})^{\cdot}=\dot x$

коорд y у m_2: $H-\ell+L-z$ $\quad (H-\ell+L-z)^{\cdot}=-\dot z$

$$T=\frac{M\dot x^2}{2}+m_1\frac{\left((\dot x+\dot z\cos\alpha)^2+(\dot z\cdot\sin\alpha)^2\right)}{2}+\frac{m_2}{2}\cdot(\dot x^2+\dot z^2)$$

$x\leftrightarrow X$ - ц.м. центр масс по оси x:

$$X=\frac{Mx+m_1(x+z\cos\alpha)+m_2(x+L\cos\alpha)}{M+m_1+m_2}$$

$$\dot X=\frac{M\dot x+m_1(\dot x+\dot z\cos\alpha)+m_2\dot x}{M+m_1+m_2}=\frac{\dot x(M+m_1+m_2)+m_1\cdot\dot z\cos\alpha}{M+m_1+m_2}=\dot x+\frac{m_1\cdot\dot z\cdot\cos\alpha}{M+m_1+m_2}$$

$$\dot x=\dot X-\frac{m_1\dot z\cos\alpha}{M+m_1+m_2}$$

$$\dot X^2=\dot x^2+\frac{2m_1\dot z\cdot\cos\alpha\,\dot x}{M+m_1+m_2}+\frac{m_1^2\cdot\dot z^2\cdot\cos^2\alpha}{(M+m_1+m_2)^2}$$

$$\dot x^2=\dot X^2-\frac{2m_1\dot z\cos\alpha\cdot\dot X}{M+m_1+m_2}+\frac{m_1^2\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}$$

$$T=\frac{M}{2}\cdot\left(\dot X^2-\frac{2m_1\dot z\cos\alpha\dot X}{M+m_1+m_2}+\frac{m_1^2\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}\right)+$$
$$+\frac{m_1}{2}\left(\left(\dot X-\frac{m_1\cdot\dot z\cos\alpha}{M+m_1+m_2}+\dot z\cos\alpha\right)^2+(\dot z\cdot\sin\alpha)^2\right)+$$
$$+\frac{m_2}{2}\left(\dot X^2-\frac{2m_1\dot z\cos\alpha\dot X}{M+m_1+m_2}+\frac{m_1^2\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}+\dot z^2\right)=$$

$$=\frac{M+m_1+m_2}{2}\cdot\dot X^2+\frac{m_1}{2}\left(\cancel{\frac{2M\dot z\cos\alpha\dot X}{M+m_1+m_2}}+\frac{Mm_1\cdot\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}+\frac{m_1^2\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}+\dot z^2\cos^2\alpha-\right.$$
$$-\cancel{2\dot X\frac{m_1\dot z\cos\alpha}{M+m_1+m_2}}-2\frac{m_1\dot z^2\cos^2\alpha}{M+m_1+m_2}+\cancel{2\dot X\dot z\cos\alpha}+\dot z^2\sin^2\alpha-\cancel{\frac{2m_2\dot z\cos\alpha\dot X}{M+m_1+m_2}}+\left.\frac{m_1^2m_2\dot z^2\cos^2\alpha}{(M+m_1+m_2)^2}\right)+$$
$$+\frac{m_2}{2}\cdot\dot z^2=\frac{M+m_1+m_2}{2}\cdot\dot X^2+\frac{m_1}{2}\cdot\left(\frac{M+m_2+m_1\sin^2\alpha}{M+m_1+m_2}\right)\cdot\dot z^2+\frac{m_2}{2}\dot z^2$$

$$U=m_2g\left(H-\ell+\frac{H}{\sin\alpha}-z\right)+m_1g\cdot z\cdot\sin\alpha$$

$$L=T-U=\frac{M+m_1+m_2}{2}\cdot\dot X^2+\dot z^2\left(\frac{m_1}{2}\cdot\frac{M+m_2+m_1\cdot\sin^2\alpha}{M+m_1+m_2}+\frac{m_2}{2}\right)-m_2g\left(H-\ell+\frac{H}{\sin\alpha}-z\right)-m_1g\,z\cdot\sin\alpha$$

Ур. Эйлера-Лагранжа:

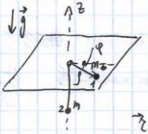
$$L_z=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot z}\right)-\frac{\partial L}{\partial z}=\frac{d}{dt}\left(2\dot z\cdot\left(\frac{m_1}{2}\cdot\frac{M+m_2+m_1\cdot\sin^2\alpha}{M+m_1+m_2}+\frac{m_2}{2}\right)\right)-m_2g+m_1g\cdot\sin\alpha=$$
$$=2\ddot z\left(\frac{m_1}{2}\cdot\frac{M+m_2+m_1\cdot\sin^2\alpha}{M+m_1+m_2}+\frac{m_2}{2}\right)-m_2g+m_1g\sin\alpha=0$$

$$L_X=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot X}\right)-\underbrace{\frac{\partial L}{\partial X}}_{=0}=\frac{d}{dt}\left(2\dot X\cdot\frac{M+m_1+m_2}{2}\right)=\ddot X(M+m_1+m_2)=0$$

$\frac{\partial L}{\partial X}=0\Rightarrow$ выполн ЗСИ: $\dot X(M+m_1+m_2)=const$

$\frac{\partial L}{\partial t}=0\Rightarrow$ выполн ЗСЭ: $E=T+U$

№2.



$3-1=2$ степени свободы ; ρ и φ координаты (полярные коорд. в гор. пл-сти)

длина нити: l

$$\vec{v} = \dot{\rho}\cdot\vec{e}_\rho + \rho\cdot\dot{\varphi}\cdot\vec{e}_\varphi \qquad ((l-\rho)^{\cdot})^2 = \dot{\rho}^2$$

$$T = \frac{m\cdot(\dot{\rho}^2+\rho^2\cdot\dot{\varphi}^2)}{2} + \frac{m\dot{\rho}^2}{2}$$

$$U = -mg(l-\rho)$$

лагранжиан $L = T - U = \frac{m(\dot{\rho}^2+\rho^2\cdot\dot{\varphi}^2)}{2} + \frac{m\dot{\rho}^2}{2} + mg(l-\rho)$

Ур. Эйл.-Лагр.: $L_\rho = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\rho}}\right) - \frac{\partial L}{\partial\rho} = \frac{d}{dt}(m\cdot\dot{\rho}+m\cdot\dot{\rho}) - m\cdot\rho\cdot\dot{\varphi}^2 + mg =$

$= 2m\ddot{\rho} - m\rho\dot{\varphi}^2 + mg = 0$

$\frac{\partial L}{\partial\varphi} = 0 \Rightarrow$ выполн. ЗСИ : $m\rho^2\dot{\varphi} = const$

$L_\varphi = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) - \frac{\partial L}{\partial\varphi} = \frac{d}{dt}(m\cdot\rho^2\cdot\dot{\varphi}) = m((\rho^2)^{\cdot}\cdot\dot{\varphi} + \rho^2\cdot\ddot{\varphi}) = 0$

$m(2\rho\cdot\dot{\rho}\cdot\dot{\varphi} + \rho^2\cdot\ddot{\varphi}) = 0$

Стационарные по координате частицы 2 решения.

у 2ой частицы коорд. $l-\rho$ ⇒ стац. реш. $\rho = const = \rho_0$

$\Rightarrow L_\rho = -m\rho_0\cdot\dot{\varphi}^2 + mg = 0 \qquad \rho_0\cdot\dot{\varphi}^2 = g \qquad \dot{\varphi}^2 = \frac{g}{\rho_0}$

L_φ: $m\rho_0^2\cdot\dot{\varphi} = const = C \Rightarrow \dot{\varphi} = \frac{C}{m\rho_0^2}$

⇒ стационарные траектории при условии $\rho = const$, $\dot{\varphi} = const$

$\frac{\partial L}{\partial t} = 0$ т.к. лагранжиан не зависит явно от t

⇒ выполн. ЗСЭ: $E = T + U$.

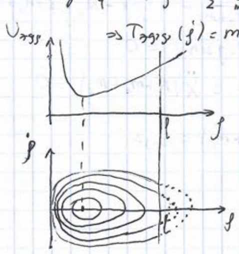
$$\frac{m(\dot{\rho}^2+\rho^2\cdot\dot{\varphi}^2)}{2} + \frac{m\dot{\rho}^2}{2} - mg(l-\rho) = const$$

$$m\dot{\rho}^2 + \frac{m\rho^2\dot{\varphi}^2}{2} + mg\rho = const$$

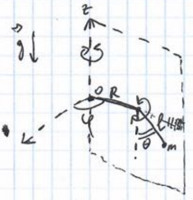
Подст. $\dot{\varphi}$: $m\dot{\rho}^2 + \frac{m}{2}\cdot\frac{C^2}{m^2\cdot\rho^2} + mg\rho = const$

$\Rightarrow T_{эфф}(\dot{\rho}) = m\dot{\rho}^2$, $U_{эфф}(\rho) = \frac{C^2}{2m\rho^2} + mg\rho$

$\rho \in (0, l]$



N3.



число степеней свободы: 2

Обобщ. коорд: $\varphi \in [0, 2\pi)$, $\theta \in [-\pi, \pi)$

$x = (R + l\cdot\sin\theta)\cdot\cos\varphi$ $\qquad \dot x = l\cdot\cos\theta\cdot\dot\theta\cdot\cos\varphi - (R+l\sin\theta)\cdot\sin\varphi\cdot\dot\varphi$

$y = (R + l\cdot\sin\theta)\cdot\sin\varphi$ $\qquad \dot y = l\cos\theta\cdot\dot\theta\sin\varphi + (R+l\sin\theta)\cdot\cos\varphi\cdot\dot\varphi$

$z = -l\cos\theta$ $\qquad \dot z = l\sin\theta\cdot\dot\theta$

$U = -mg\cdot l\cdot\cos\theta \qquad T = \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2) =$

$= \frac{m}{2}\big(l^2\cdot\cos^2\theta\cdot\dot\theta^2\cdot\cos^2\varphi + (R+l\sin\theta)^2\cdot\sin^2\varphi\cdot\dot\varphi^2 - \cancel{2l\cos\theta\cdot\dot\theta\cdot\cos\varphi\cdot(R+l\sin\theta)\cdot\sin\varphi\cdot\dot\varphi} +$

$+ l^2\cos^2\theta\cdot\dot\theta^2\sin^2\varphi + (R+l\sin\theta)^2\cdot\cos^2\varphi\cdot\dot\varphi^2 + \cancel{2l\cos\theta\cdot\dot\theta\cdot(R+l\sin\theta)\cdot\cos\varphi\cdot\dot\varphi\sin\varphi} + l^2\sin^2\theta\cdot\dot\theta^2\big) =$

$= \frac{m}{2}\left(l^2\cdot\dot\theta^2 + (R+l\cdot\sin\theta)^2\cdot\dot\varphi^2\right)$

$L = T - U = \frac{m}{2}\left(l^2\cdot\dot\theta^2 + (R+l\sin\theta)^2\cdot\dot\varphi^2\right) + mgl\cdot\cos\theta = \frac{m}{2}\left(l^2\dot\theta^2 + R^2\dot\varphi^2 + 2Rl\sin\theta\,\dot\varphi^2 + l^2\sin^2\theta\,\dot\varphi^2\right) + mgl\cos\theta$

$L_\varphi = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \underbrace{\frac{\partial L}{\partial\varphi}}_{=0} = \frac{d}{dt}\left(2\dot\varphi(R+l\sin\theta)^2\cdot\frac{m}{2}\right) = 0$

$\frac{\partial L}{\partial\varphi} = 0 \Rightarrow$ выполн. ЗСИ: $m\dot\varphi(R+l\sin\theta)^2 = const = C \qquad \dot\varphi = \frac{C}{m(R+l\sin\theta)^2}$

$L_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(ml^2\cdot\dot\theta\right) - mRl\cos\theta\cdot\dot\varphi^2 - \frac{m}{2}\cdot l^2\cdot 2\sin\theta\cdot\cos\theta\cdot\dot\varphi^2 + mgl\cdot\sin\theta = 0$

$ml^2\ddot\theta - mR\cdot l\cdot\cos\theta\cdot\dot\varphi^2 - ml^2\sin\theta\cdot\cos\theta\cdot\dot\varphi^2 + mgl\sin\theta = 0$

Стационарные по θ решения, $\theta(t) = \theta_0 = const$

$R\cdot\cos\theta_0\cdot\dot\varphi^2 + l\sin\theta_0\cdot\cos\theta_0\cdot\dot\varphi^2 - g\sin\theta_0 = 0$

$\dot\varphi^2 = \frac{g\sin\theta_0}{R\cos\theta_0 + l\cos\theta_0\cdot\sin\theta_0}$

$D = l^2\sin^2\theta_0\cos^2\theta_0 + 4Rg\cos\theta_0\cdot\sin\theta_0$

если $\sin\theta_0\cdot\cos\theta_0(l^2\sin\theta_0\cdot\cos\theta_0 + 4Rg) > 0$, то 2 реш.; < 0 0 реш.; $= 0$ 1 реш.

число реш. зависит от угла

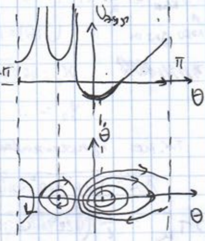
$\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполн. ЗСЭ: $E = T + U = \frac{m}{2}\cdot l^2\cdot\dot\theta^2 + \frac{m}{2}\cdot(R+l\sin\theta)^2\cdot\dot\varphi^2 - mgl\cdot\cos\theta =$

подст. $\dot\varphi$

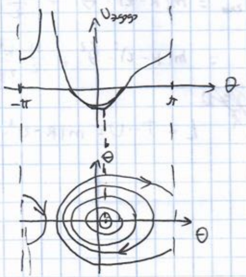
$= \frac{m}{2}l^2\cdot\dot\theta^2 + \frac{m}{2}(R+l\sin\theta)^2\cdot\frac{C^2}{m^2\cdot(R+l\sin\theta)^4} - mgl\cdot\cos\theta = const$

$\underbrace{\frac{m}{2}l^2\cdot\dot\theta^2}_{T_{эфф}(\dot\theta)} + \underbrace{\frac{C^2}{2m\cdot(R+l\sin\theta)^2} - mgl\cdot\cos\theta}_{U_{эфф}(\theta)} = const$

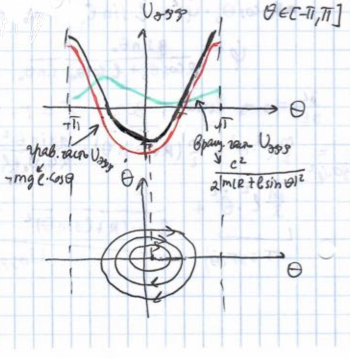
$R < l$


$U_{эфф}$
$-\pi$
π
θ
$\dot{\theta}$
θ

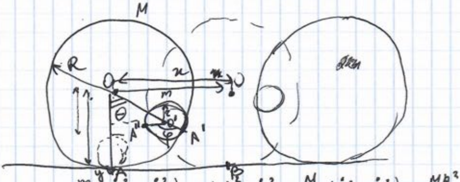
$R = l$


$U_{эфф}$
$-\pi$
π
θ
$\dot{\theta}$
θ

$R > l$


$U_{эфф}$
$\theta \in [-\pi, \pi]$
θ
$-\pi$
π
грав. часть $U_{эфф}$
$-mgl \cdot \cos\theta$
вращ. часть $U_{эфф}$
$\frac{c^2}{2[m(R + l\sin\theta)]^2}$
$\dot{\theta}$
θ

№4.



2 степени свободы: θ, x (на картинке)
сдвиг по Ox

$$T_{кин} = \frac{m}{2}(\dot{x}_{O'}^2 + \dot{y}_{O'}^2) + \frac{mr^2}{2}\cdot\dot{\psi}^2 + \frac{M}{2}(\dot{x}_0^2+\dot{y}_0^2) + \frac{MR^2}{2}\cdot\dot{\tilde{\varphi}}$$

$x_{O'} = (R-r)\cdot\sin\theta + x$ $\quad \dot{x}_{O'} = (R-r)\cdot\cos\theta\cdot\dot{\theta} + \dot{x}$ $\quad x_0 = x$ $\quad AB = x$

$y_{O'} = (R-r)\cdot\cos\theta$ $\quad \dot{y}_{O'} = -(R-r)\cdot\sin\theta\cdot\dot{\theta}$ $\quad y_0 = 0$ $\quad R\tilde{\varphi} = x$

Т.к. нет проскальзывания, то длины дуг равны:

$R(\theta+\tilde{\varphi}) = r\varphi$; $\varphi = \frac{R}{r}(\theta+\tilde{\varphi})$

$\psi = \varphi - (\theta+\tilde{\varphi})$ вращение труб m $\quad \theta+\tilde{\varphi}$ - вращение O' относительно O против ч. стрелки

$\dot{\psi} = \dot{\varphi} - (\dot{\theta}+\dot{\tilde{\varphi}}) = \frac{R}{r}(\dot{\theta}+\dot{\tilde{\varphi}}) - \dot{\theta} - \frac{\dot{x}}{R}$

по малым углам

$$T_{кин} = \frac{m}{2}(\dot{x}^2 + 2(R-r)\cdot\cos\theta\cdot\dot{\theta}\cdot\dot{x} + (R-r)^2\cdot\cos^2\theta\cdot\dot{\theta}^2) +$$
$$+ \frac{mr^2}{2}\cdot\frac{1}{R^2r^2}(R\dot{x}-\dot{x}r)^2 + \frac{M}{2}\dot{x}^2 + \frac{MR^2}{2}\cdot\frac{\dot{x}^2}{R^2} =$$
$$= \frac{m}{2}(\dot{x}^2 + 2(R-r)\cos\theta\cdot\dot{\theta}\cdot\dot{x} + (R-r)^2\cos^2\theta\cdot\dot{\theta}^2) + M\dot{x}^2 + \frac{m\dot{x}^2(R-r)^2}{2R^2}$$

$$U = mg(R-r)\cdot(1-\cos\theta)$$

$$L = T - U = \frac{m}{2}(\dot{x}^2 + 2(R-r)\cdot\cos\theta\cdot\dot{\theta}\cdot\dot{x} + (R-r)^2\cdot\cos^2\theta\cdot\dot{\theta}^2) + M\dot{x}^2 + \frac{m\dot{x}^2(R-r)^2}{2R^2} -$$
$$- mg(R-r)\cdot(1-\cos\theta)$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполн ЗСЭ : $E = T + U$

$$L_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}(m(R-r)^2\cdot\dot{\theta} + m\dot{x}(R-r)\cdot\cos\theta) + \sin\theta\cdot m\dot{x}\dot{\theta}(R-r) + mg(R-r)\cdot\sin\theta = 0$$

$$m(R-r)^2\cdot\ddot{\theta} + m\ddot{x}(R-r)\cos\theta - m\dot{x}(R-r)\cdot\sin\theta\cdot\dot{\theta} + m\dot{x}\dot{\theta}\cdot\sin\theta(R-r) + mg(R-r)\cdot\sin\theta = 0$$

$(R-r)\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \Rightarrow (R-r)\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0$

$$L_x = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \underbrace{\left(\frac{\partial L}{\partial x}\right)}_{=0} = 0$$

т.к. $\frac{\partial L}{\partial x} = 0$ выполн. ЗСИ : $(2M+m)\cdot\dot{x} + \frac{m(R-r)^2}{R^2}\cdot\dot{x} + m\dot{\theta}(R-r)\cdot\cos\theta = const$

для малых θ: $\quad \ddot{x} = 0$

$\ddot{\theta} + \frac{g\theta}{R-r} = 0$ это ур-ие гармонических колебаний с угловой частотой $\omega = \sqrt{\frac{g}{R-r}}$