Контрольная работа № 3 Струихиной Ксении

№ 1 $S[y(x)] = \int_0^1 \left(\underbrace{2(y')^2 + \frac{y^2}{2} + e^x(2y' - y)}_{\mathcal{H}} \right) dx$

$\delta S[y(x)] = \int_0^1 \left(\frac{\partial \mathcal{H}}{\partial y} \delta y + \frac{\partial \mathcal{H}}{\partial y'} \delta y' \right) dx = \int_0^1 \left(y\delta y - e^x \delta y + 4y'\delta y' + 2e^x \delta y' \right)$

$-4y'' + y - 3e^x$

$= \int_0^1 \left((y - e^x)\delta y) + (4y' + 2e^x)\delta y' \right) dx = \int_0^1 (y - e^x - 4y'' - 2e^x)\delta y\, dx + (4y' + 2e^x)\delta y \Big|_0^1$

$\int_0^1 (4y' + 2e^x)\delta y'\, dx = (4y' + 2e^x)\delta y \Big|_0^1 - \int_0^1 (4y'' + 2e^x)\delta y\, dx$

$4y'' - y = -3e^x \quad 4t^2 - 1 = 0 \quad t = \pm\frac{1}{2}$

$y = C_1 e^{-\frac{x}{2}} + C_2 e^{\frac{x}{2}} - e^x$

$y(1) = \sqrt{e} - e$

Т.к. $y(0)$ произвольное, то $\delta(y(0))$ ~~то~~ принимает любое значение, $\Rightarrow 4y'(0) + 2 = 0 \Rightarrow y'(0) = -\frac{1}{2}$

$$\begin{cases} \sqrt{e} - e = C_1 e^{-\frac{1}{2}} + C_2\sqrt{e} - e & y(1) = \sqrt{e} - e \\ -\frac{1}{2}C_1 + \frac{1}{2}C_2 - 1 = -\frac{1}{2} & y'(0) = -\frac{1}{2} \end{cases}$$

$C_2 = -1 + 2 + C_1 = C_1 + 1$

$\cancel{\sqrt{e} - e = C_1\frac{1}{\sqrt{e}} + C_1\sqrt{e} - \frac{1}{3}\sqrt{e} - \frac{e}{3}}$ $\quad \sqrt{e} = C_1\frac{1}{\sqrt{e}} + (C_1 + 1)\sqrt{e}$

$\cancel{\sqrt{e}\left(1 - C_1 + \frac{1}{3} - \frac{C_1}{e}\right) = \frac{2}{3}e}$ $\quad C_1\left(\frac{1}{\sqrt{e}} + \sqrt{e}\right) = 0 \quad \underline{C_1 = 0}$

$\cancel{C_1 = \frac{\sqrt{e} - \frac{2}{3}e + \frac{1}{3}\sqrt{e}}{\frac{1}{\sqrt{e}} + \sqrt{e}} = \frac{3e - 2e\sqrt{e} + e}{3(1+e)} = \frac{4e - 2e\sqrt{e}}{3 + 3e}}$ $\quad \underline{C_2 = 1}$

$\cancel{C_2 = \frac{4e - 2e\sqrt{e} - 1 - e}{3 + 3e} = \frac{3e - 2e\sqrt{e} - 1}{3 + 3e}}$

Ответ: $\cancel{\frac{4e - 2e\sqrt{e}}{3 + 3e} e^{-\frac{x}{2}} + \frac{3e - 2e\sqrt{e} - 1}{3 + 3e} e^{\frac{x}{2}} - \frac{e^x}{3}}$ $\quad e^{\frac{x}{2}} - e^x = y_{\text{экстр}}$

№ 2 $S[y(x)] = \int_0^{\pi/2} \underbrace{\left((y'')^2 - y^2 + 8y''e^{x-\pi/2}\right)}_{L} dx$

$\delta S[y(x)] = \int_0^{\pi/2} \left(\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{\partial L}{\partial y''}\delta y''\right) dx =$

$= \int_0^{\pi/2} \left(-2y\delta y + \delta y''(2y''+8e^{x-\pi/2})\right) dx =$

$\int_0^{\pi/2}(2y''+8e^{x-\pi/2})\delta y'' dx = (2y''+8e^{x-\pi/2})\delta y'\Big|_0^{\pi/2} - \int (2y'''+8e^{x-\pi/2})\delta y' dx =$

$= (2y''+8e^{x-\pi/2})\delta y'\Big|_0^{\pi/2} - (2y'''+8e^{x-\pi/2})\delta y\Big|_0^{\pi/2} + \int_0^{\pi/2}(2y''''+8e^{x-\pi/2})\delta y\, dx$

$= \int_0^{\pi/2} dx\, \delta y(-2y+2y''''+8e^{x-\pi/2}) + (2y''+8e^{x-\pi/2})\delta y'\Big|_0^{\pi/2} - (2y'''+8e^{x-\pi/2})\delta y\Big|_0^{\pi/2}$

$y'''' - y = -4e^{x-\pi/2}$ $t^4 - 1 = 0$, $t = \pm 1$, $t = \pm i$

$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x - xe^{x-\pi/2}$

$y'(0) = 0$ $y'(\pi/2) = -1$ $y(\pi/2) = e^{-\pi/2} - \pi/2$

$y(0)$ произвольное $\Rightarrow$ $\delta y(0)$ любое $\Rightarrow$ $2y'''(0) + 8e^{-\pi/2} = 0$

$2y'''(0) = -4e^{-\pi/2}$

$\boxed{(-xe^{x-\pi/2})' = -xe^{x-\pi/2} - e^{x-\pi/2}}$

$$\begin{cases} C_1 - C_2 - C_3\sin 0 + C_4\cos 0 = e^{-\pi/2} \\ C_1e^{\pi/2} - C_2e^{-\pi/2} - C_3\sin\pi/2 + C_4\cos\pi/2 - \pi/2 - 1 = -1 \\ C_1e^{\pi/2} + C_2e^{-\pi/2} + C_3\cos\pi/2 + C_4\sin\pi/2 - \pi/2 = e^{-\pi/2} - \pi/2 \\ C_1 - C_2 + C_3\cdot 0 - C_4\cos 0 - (e^{0-\pi/2}\cdot 0 + 3e^{0-\pi/2}) = -4e^{-\pi/2} \end{cases}$$

$$\begin{cases} C_1 - C_2 + C_4 = e^{-\pi/2} \\ C_1e^{\pi/2} - C_2e^{-\pi/2} - C_3 = \pi/2 \\ C_1e^{\pi/2} + C_2e^{-\pi/2} + C_4 = e^{-\pi/2} \\ C_1 - C_2 - C_4 = -e^{-\pi/2} \end{cases}$$

$\to C_4 = e^{-\pi/2}$ и $C_1 = C_2$

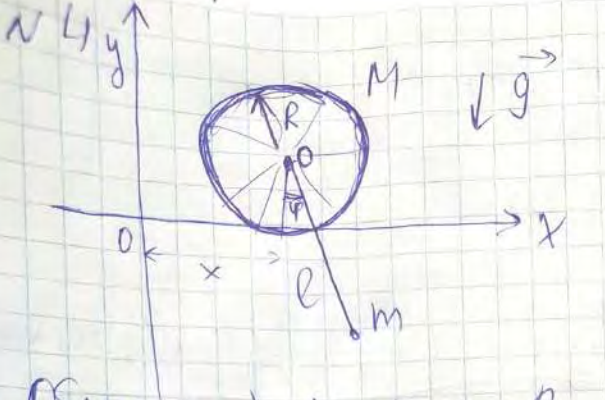
$$\begin{cases} C_1(e^{\pi/2} - e^{-\pi/2}) - C_3 = \pi/2 \\ C_1(e^{\pi/2} + e^{-\pi/2}) = 0 \end{cases}$$

$C_1 = C_2 = 0$ $C_3 = -\pi/2$

$\cancel{\begin{cases} C_1e^{\pi} + C_2 = \frac{1}{2} \\ C_1 - C_2 = \frac{e^{-\pi/2}}{2} \end{cases}}$ $\cancel{\begin{cases} C_1e^{\pi} + C_2 = \frac{1}{2} \\ C_1 = C_2 + \frac{e^{-\pi/2}}{2} \end{cases}}$

$\cancel{C_1(1+e^{\pi}) = \frac{1-e^{-\pi/2}}{2}}$ $\cancel{C_1 = \frac{1-e^{-\pi/2}}{2(1+e^{\pi})}}$ $\cancel{C_2 = \frac{1}{2} - \frac{e^{\pi} - e^{-\pi/2}}{2(1+e^{\pi})}}$

№ 4



$y_m = -l\cos\varphi + R$

$x_m = x_0 + l\sin\varphi$

$U = +mg(l\cos\varphi - R) \sim mgl\cos\varphi$ т.к. $mgR = const$

$\cancel{\dot{y}_m = l\sin\varphi}$

$\dot{y}_m = l\sin\varphi\cdot\dot\varphi$

Обруч (ун) $x_0 = x$, $y_0 = R$

$T_{кин} = \frac{M}{2}\dot{x}^2 + E_{вр.} + \frac{m}{2}(\dot{x}_m^2 + \dot{y}_m^2)$

$E_{вр} = MR^2 \cdot \frac{\omega^2}{2}$ $\quad\theta$ - угол поворота обруча, тогда $\theta R = x$

$\dot{x} = R\dot\theta$

$E_{вр} = MR^2 \cdot \left(\frac{\dot{x}}{R}\right)^2 \cdot \frac{1}{2} = M\dot{x}^2$ $\quad \omega = \dot\theta = \frac{\dot{x}}{R}$

$T_{кин} = M\dot{x}^2 + \frac{m}{2}(\dot{x}^2 + l^2\sin^2\varphi\,\dot\varphi^2 + 2\dot{x}l\dot\varphi\cos\varphi + \dot\varphi^2\sin^2\varphi\, l^2 + l^2\cos^2\varphi$

$L = T_{кин} - U = M\dot{x}^2 + \frac{m}{2}(\dot{x}^2 + l^2\dot\varphi^2 + 2\dot{x}l\dot\varphi\cos\varphi) + mgl\cos\varphi$

const

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполнен ЗСЭ: $E = T + U = const$

$\cancel{L_\varphi = 0 \Rightarrow L_{\dot\varphi} = 0}$

$L_x = 0 \Rightarrow$ ЗСИ $\quad L_{\dot{x}} = const$

$L_{\dot{x}} = 2M\dot{x} + m\dot{x} + 2l\dot\varphi\cos\varphi = const$

$$C_3 = \frac{e^{\pi/2}(1-e^{-\pi/2})}{2(1+e^{\pi})} - e^{\pi/2}\left(\frac{1+e^{\pi}-e^{\pi}+e^{\pi/2}}{2(1+e^{\pi})}\right) - \pi/2 =$$

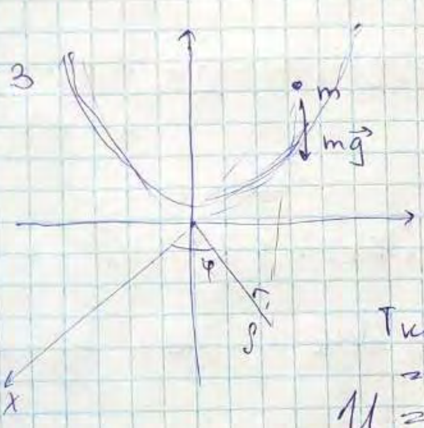
$$= \left(\frac{e^{\pi/2}-1-e^{-\pi/2}-1}{2(1+e^{\pi})}\right) - \pi/2$$

$$y_{экстр} = \frac{1-e^{-\pi/2}}{2(1+e^{\pi})}e^{x} + \frac{1+e^{\pi/2}}{2(1+e^{\pi})}e^{-x} + \left(\frac{e^{\pi/2}-e^{-\pi/2}}{2(1+e^{\pi})} - \pi/2\right)\cos x +$$

$$+ \sin x\left(\frac{e^{-\pi/2}}{2}\right) - x e^{x-\pi/2}$$

$$\boxed{y_{экстр} = -\pi/2\cos x + e^{-\pi/2}\sin x - x e^{x-\pi/2}}$$

№ 3



$z^2 = x^2 + y^2 + a^2 \quad z > 0$ Введем цил. координаты: ρ, z, φ

$\rho = \sqrt{x^2+y^2} \quad z^2 = \rho^2 + a^2$

$T_{кин} = \frac{m}{2}(\dot\rho^2 + \dot z^2 + \rho^2\dot\varphi^2)$

$\rho^2 = z^2 - a^2$

$\dot z^2 = \dot\rho^2 \quad \rho^2 = z^2 - a^2$

$\rho = \sqrt{z^2-a^2}$

$\dot\rho = \frac{1}{2}\frac{2z\cdot\dot z}{\sqrt{z^2-a^2}}$

$$\boxed{\dot\rho^2 = \frac{(z\cdot\dot z)^2}{z^2-a^2}}$$

$T_{кин} = \frac{m}{2}\left(2\dot z^2 + (z^2-a^2)\dot\varphi^2\right) = \frac{m}{2}\left(\dot z^2 + (z^2-a^2)\cdot\dot\varphi^2 + \frac{(z\cdot\dot z)^2}{z^2-a^2}\right)$

$U = mgz$

$L = T - U = \frac{m}{2}\left(2\dot z^2 + (z^2-a^2)\cdot\dot\varphi^2\right) - mgz$

$\frac{m}{2}\left(\frac{(z\cdot\dot z)^2}{z^2-a^2} + \dot z^2 + (z^2-a^2)\cdot\dot\varphi^2\right) - mgz$

$\frac{\partial L}{\partial\varphi} = 0 \Rightarrow$ ЗСИ

$L_\varphi = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \quad \frac{d}{dt}\left(m(z^2-a^2)\right) = 0 \Rightarrow J = m(z^2-a^2) = const$

$mz^2 = const$

$z^2 = const$

При $z = const$ выполнено

$L_z = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot z}\right) - \frac{\partial L}{\partial z} = 0$

$\frac{d}{dt}(2m\dot z) - mz\dot\varphi^2 = 0 \quad 2m\ddot z - mz\dot\varphi^2 = 0 \quad \ddot z = \frac{z\dot\varphi^2}{2}$

Если $z = const$ $\dot z = 0 \Rightarrow \dot\varphi = 0$

$\frac{\partial\mathcal{L}}{\partial t} = 0 \Rightarrow$ выполнен ЗСЭ:

$E = T + U = \frac{m}{2}\left(2\dot z^2 + (z^2-a^2)\dot\varphi^2\right) + mgz = const$

$L_z = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot z}\right) - \frac{\partial\mathcal{L}}{\partial z} = 0 = \frac{d}{dt}\left(m\dot z + \frac{m\dot z z^2}{z^2-a^2}\right) + mg - \frac{m\dot z^2 z}{2(z^2-a^2)}$

$-\left(-\frac{2a^2z}{(z^2-a^2)^2}\frac{\dot z^2 m}{2}\right) = \left(m\dot z + \frac{m\dot z z^2}{z^2-a^2}\right)^{\cdot} + mg + \frac{a^2 z\cdot\dot z^2 m}{2(z^2-a^2)^2} = 0$

$z = const$

$\dot z = 0 \Rightarrow mg = 0$

$\frac{\partial\mathcal{L}}{\partial t} = 0 \Rightarrow$ выполнен ЗСЭ: $E = T + U = \frac{m}{2}\left(\frac{(z\dot z)^2}{z^2-a^2} + \dot z^2 + (z^2-a^2)\dot\varphi^2\right) + mgz = const$