$F[y]$ на $C^2[0,1]$: $y(1)=0$

$$F[y] = \int_0^1 dx\,((y')^2 - 2xy)$$

$$\Delta F[y] = \Delta \int_0^1 dx\,((y')^2 - 2xy) =$$

$$= \int_0^1 dx\,((y' + (\delta y)')^2 - 2x(y+\delta y)) - \int_0^1 dx\,((y')^2 - 2xy) =$$

$$= \int_0^1 dx\,\left\{ \cancel{(y')^2} + 2y'\,\delta y' + (\delta y')^2 \cancel{-2xy} - 2x\,\delta y \cancel{-(y')^2} \cancel{+2xy} \right\} =$$

$$= \int_0^1 dx\,(2y'(\delta y)' + (\delta y')^2 - 2x\,\delta y)$$

$$\delta F[y] = \int_0^1 dx\,(2y'(\delta y)' - 2x\,\delta y) =$$

$$= \int_0^1 2y'(\delta y)'\,dx - \int_0^1 2x\,\delta y\,dx =$$

$$= \int_0^1 \overbrace{(2y'\delta y)'}^{2y'(\delta y)' + 2y''\delta y} dx - \int_0^1 2y''\,\delta y\,dx - \int_0^1 2x\,\delta y\,dx =$$

$$-2\int_0^1 \delta y\,dx\,(y'' + x)$$

$$= -2\int_0^1 dx\,(y'' + x)\,\delta y + 2y'\delta y\Big|_0^1 = 0$$

$$y(1) = 0 \Rightarrow \delta y\big|_{x=1} = 0$$

$$y'' + x = 0$$

$$y'' = -x$$

$$y' = -\frac{x^2}{2} + C_1$$

$$y = -\int \frac{x^2}{2}\,dx + C_1 x = -\frac{1}{2}\cdot\frac{x^3}{3} + C_1 x + C_0$$

$$y(x) = -\frac{x^3}{6} + C_1 x + C_0 \quad (*)$$

Граничное условие в т. $x = 0$

$$2y'\big|_{x=0} = 0 \qquad (\delta y \text{ - произв. в т. } x = 0)$$

$$y(1) = -\frac{1}{6} + C_1 + C_0 = 0 \qquad \Rightarrow C_0 = \frac{1}{6}.$$

$$y'(0) = -\frac{x^2}{2} + C_1\Big|_{x=0} = +C_1 = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow \boxed{y(x) = -\frac{x^3}{6} + \frac{1}{6}}$$