Task
Let f_n be the Fibonacci sequence. Prove that

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$

Solution
Binet's formula:

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Then

$$f_{n-1}f_{n+1} - f_n^2 =$$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \cdot \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} - \left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)^2 =$$

$$\frac{1}{5}\left(\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right) \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) - \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)^2\right) =$$

$$\frac{1}{2^{2n}\cdot 5}\left(\left((1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}\right) \cdot \left((1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\right) - \left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)^2\right) =$$

$$\frac{1}{2^{2n}\cdot 5}\Big(\left((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1} - (1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1}\right) -$$
$$\left((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - 2(1+\sqrt{5})^n(1-\sqrt{5})^n\right)\Big) =$$

$$\frac{1}{2^{2n}\cdot 5}\Big((1+\sqrt{5})^{2n} + (1-\sqrt{5})^{2n} - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1}$$
$$-(1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1} - (1+\sqrt{5})^{2n} - (1-\sqrt{5})^{2n} + 2(1+\sqrt{5})^n(1-\sqrt{5})^n\Big) =$$

$$\frac{1}{2^{2n}\cdot 5}\left(2(1+\sqrt{5})^n(1-\sqrt{5})^n - (1+\sqrt{5})^{n-1}(1-\sqrt{5})^{n+1} - (1+\sqrt{5})^{n+1}(1-\sqrt{5})^{n-1}\right) =$$

$$\frac{(1-5)^{n-1}}{2^{2n}\cdot 5}\left(2(1-\sqrt{5})(1+\sqrt{5}) - (1-\sqrt{5})^2 - (1+\sqrt{5})^2\right) =$$

$$\frac{(-4)^{n-1}}{4^n\cdot 5}\left(2(1-\sqrt{5})(1+\sqrt{5}) - (6-2\sqrt{5}) - (6+2\sqrt{5})\right) =$$

$$\frac{(-4)^{n-1}}{4^n\cdot 5}\Big(-8-6-6\Big) =$$

$$\frac{(-4)^{n-1}}{4^n\cdot 5}(-20) = (-1)^n$$