

N 1.

$$S[y(n)] = \int_0^1 ((y''(n))^2 + 5(y'(n))^2 + 4y^2(n)) dx$$

$$y(x) \in C^\infty[0,1] \quad y'(0)=0 \quad y(1)=-3$$

$$S[y+\delta y] - S[y] = \int_0^1 ((y+\delta y)''^2 + 5(y+\delta y)'^2 + 4(y+\delta y)^2) dx - \int_0^1 ((y''(n))^2 + 5(y'(n))^2 + 4y^2(n)) dx = \int_0^1 ((\delta y'')^2 + 5(\delta y')^2 + 4(\delta y)^2 + 2y''\delta y'' + 10y'\delta y' + 8y\delta y) dx$$

$$\Rightarrow \delta S[y] = \int_0^1 (2y''\delta y'' + 10y'\delta y' + 8y\delta y) dx =$$

$$= \int_0^1 2y'' d(\delta y') + \int_0^1 10y' d(\delta y) + \int_0^1 8y \delta y dx =$$

$$= 2y'' \cdot \delta y' \Big|_0^1 - 2 \int_0^1 y'' \delta y' dx + 10y' \cdot \delta y \Big|_0^1 - 10 \int_0^1 y' \delta y dx + \int_0^1 8y \delta y dx =$$

$$= 2y'' \cdot \delta y' \Big|_0^1 - 2 \int_0^1 y'' \delta y' dx + 10y' \cdot \delta y \Big|_0^1 - 10 \int_0^1 y' \delta y dx + 8 \int_0^1 y \delta y dx =$$

$$= 2y'' \cdot \delta y' \Big|_0^1 - 2y'' \cdot \delta y \Big|_0^1 + 2 \int_0^1 y'' \delta y dx + 10y' \cdot \delta y \Big|_0^1 - 10 \int_0^1 y' \delta y dx + 8 \int_0^1 y \delta y dx =$$

$$= \int_0^1 (2y'' - 10y' + 8y) \delta y dx + 2y'' \cdot \delta y' \Big|_0^1 + (10y' - 2y'') \delta y \Big|_0^1$$

$$y''' - 5y'' + 4y = 0$$

~~$y(1) = -3$~~ ~~$y'(1) = 0$~~ ~~$y''(1) = 0$~~ ~~$y'''(1) = 0$~~

~~$y'(1) = 0$~~ ~~$y''(1) = 0$~~ ~~$y'''(1) = 0$~~

т.к. $y'(0)=0$

$y'(0)$ не задан $\Rightarrow \delta(y(0))$ произв. $\Rightarrow 10y'(0) - 2y''(0) = 0$

$y'(0)=0 \Rightarrow \delta y'(0)=0 \Rightarrow y''$ в т.о критич. & констант.

$y'''(0)=0$

$y'(1)$ не задан $\Rightarrow \delta y'(1)$ произв. $\Rightarrow y''(1)=0$

$$y''' - 5y'' + 4y = 0$$

$$t^3 - 5t^2 + 4 = 0$$

$$(t^2 - 1)(t^2 - 4) = 0$$

$$t = \pm 1, \pm 2$$

$$y(x) = C_1 \cdot e^{-2x} + C_2 \cdot e^{-x} + C_3 \cdot e^x + C_4 \cdot e^{2x}$$

функции: $y'(0)=0, y(1)=-3, y'''(0)=0, y''(1)=0$:

(1) $y'(0) = -2C_1 - C_2 + C_3 + 2C_4 = 0$

(2) $y(1) = C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + C_4 \cdot e^2 = -3$

(3) $y'''(0) = -8C_1 - C_2 + C_3 + 8C_4 = 0$

(4) $y''(1) = 4C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + 4C_4 \cdot e^2 = 0$

(1) - (3): $6C_1 - 6C_4 = 0 \Rightarrow C_1 = C_4$

$$7. \text{K. } C_1 = C_4, \text{ pg 14): } C_2 = C_3$$

$$- \begin{cases} C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = -3 \\ 4C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = 0 \end{cases}$$

$$3C_1(e^2 + e^{-2}) = 3$$

$$C_4 = C_1 = \frac{1}{e^2 + e^{-2}}$$

$$4 + C_2(e + e^{-1}) = 0$$

$$C_3 = C_2 = -\frac{4}{e + e^{-1}}$$

$$\Rightarrow \overset{\text{Kostensatz}}{y(x)} = \frac{e^{-2x}}{e^2 + e^{-2}} - \frac{4e^{-x}}{e + e^{-1}} - \frac{4e^x}{e + e^{-1}} + \frac{e^{2x}}{e^2 + e^{-2}}$$

$$\delta) F[y(x)] = S[y(x)] + 6y'(1) =$$

$$= \int_0^1 ((y''(x))^2 + 5(y'(x))^2 + 4y^2(x) + 6y''(x)) dx \quad y'(0) = 0$$

$$F[y + \delta y] - F[y] = \int_0^1 ((\delta y'')^2 + 5(\delta y')^2 + 4(\delta y)^2 + 6\delta y'' + 2y''\delta y' + 10y'\delta y' + 8y\delta y) dx$$

$$\Rightarrow \delta F[y] = \int_0^1 (6\delta y'' + 2y''\delta y' + 10y'\delta y' + 8y\delta y) dx =$$

$$= \int_0^1 (2y'''' - 10y'' + 8y)\delta y dx + 2y''\delta y'|_0^1 + (10y' - 2y''')\delta y|_0^1 + 6\int_0^1 \delta y' =$$

$$\delta F[y] = \int_0^1 (2y'''' - 10y'' + 8y)\delta y dx + 6\delta y'|_0^1 =$$

$$= \int_0^1 (2y'''' - 10y'' + 8y)\delta y dx + (2y'' + 6)\delta y'|_0^1 + (10y' - 2y''')\delta y|_0^1$$

$$y'''' - 5y'' + 4y = 0$$

$$y'(0) = 0 \quad y(1) = -3$$

$$y(0) \text{ не задан } \Rightarrow \delta(y(0)) \text{ не задан } \Rightarrow 10y'(0) - 2y'''(0) = 0$$

$$y'(1) \text{ не задан } \Rightarrow \delta(y'(1)) \text{ не задан}$$

$$y'''(0) = 0$$

$$\Downarrow$$

$$2y''(1) + 6 = 0 \Leftrightarrow y''(1) = -3$$

аналогично: $y(x) = C_1 \cdot e^{-2x} + C_2 \cdot e^{-x} + C_3 \cdot e^x + C_4 \cdot e^{2x}$

$$y'(0) = -2C_1 - C_2 + C_3 + 2C_4 = 0$$

$$y(1) = C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + C_4 \cdot e^2 = -3$$

$$y'''(0) = -8C_1 - C_2 + C_3 + 8C_4 = 0$$

$$y''(1) = 4C_1 \cdot e^{-2} + C_2 \cdot e^{-1} + C_3 \cdot e + 4C_4 \cdot e^2 = -3$$

$$(1) - (3): 6C_1 - 6C_4 = 0 \quad C_1 = C_4$$

$$\text{из (1): } C_2 = C_3$$

$$- \begin{cases} C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = -3 \\ 4C_1(e^2 + e^{-2}) + C_2(e + e^{-1}) = -3 \end{cases}$$

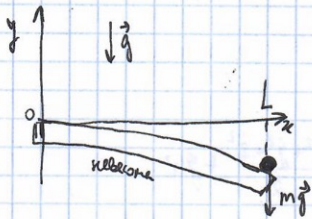
$$3C_1(e^2 + e^{-2}) = 0$$

$$C_1 = C_4 = 0$$

$$C_2 = C_3 = -\frac{3}{e + e^{-1}}$$

$$y(x) = -\frac{3}{e + e^{-1}} \cdot e^{-x} - \frac{3}{e + e^{-1}} \cdot e^x$$

N2.



Kinemat. AH: 0

$$L = U$$

$$y(0) = y'(0) = 0$$

$$y(L) = y'(L) = 0$$

$$\delta y'(L):$$

$$\frac{\partial}{\partial y''} \left(\frac{k(y'')^2}{2} + mgy' \right) \Big|_{x=L} = 0 \Leftrightarrow y''(L) = 0$$

$$\delta y(L):$$

$$\left(\frac{\partial}{\partial y'} - \frac{d}{dx} \frac{\partial}{\partial y''} \right) \left(\frac{k(y'')^2}{2} + mgy' \right) \Big|_{x=L} = 0 \Leftrightarrow y'''(L) = \frac{Lmg}{k}$$

$$\delta U_{\text{grav}} = mgy(L) \delta y$$

$$\delta U_{\text{spring}} = \frac{k(y'')^2}{2} \delta y$$

$$U[y(x)] = \int_0^L \left(\frac{k(y'')^2}{2} + mgy' \right) dx = \int_0^L \left(\frac{k(y'')^2}{2} + Lmgy'(x) \right) dx$$

$$\delta U_{\text{gesamt}} = \delta U[y(x)] = \int_0^L \left(\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial L}{\partial y''} \right) \delta y(x) dx +$$

$$+ \frac{\partial L}{\partial y'} \cdot \delta y'(x) \Big|_0^L + \left(\frac{\partial L}{\partial y''} - \frac{d}{dx} \frac{\partial L}{\partial y'''} \right) \delta y''(x) \Big|_0^L$$

$$= \int_0^L k y''(x) \delta y(x) dx + k y''(L) \delta y'(L) + (Lmg - k y'''(L)) \delta y(L)$$

$$y(0) = 0, y'(0) = 0, y''(L) = 0, y'''(L) = \frac{mgL}{k}$$

$$k \cdot y''(x) = 0$$

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$y(0) = c_1 = 0$$

$$y'(0) = c_2 = 0$$

$$y''(L) = 2c_3 + 6c_4 \cdot L = 0$$

$$y'''(L) = 6c_4 = \frac{mgL}{k}$$

$$c_4 = \frac{mgL}{6k}$$

$$c_3 = -\frac{3c_4 \cdot L}{L} = -\frac{mgL^2}{2k}$$

$$y(x) = -\frac{mgL^2}{2k} x^2 + \frac{mgL}{6k} x^3$$

N3.

$$m=1$$

$$T = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} \quad V = mgh = gz$$

$$L = T - V = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - gz$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \text{konstant.} \quad \text{S.C.} \quad E = T + V = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} + gz$$

$$f(\vec{r}) = R^2 - \vec{r}^2 = 0$$

$$\mathcal{L} = L + \lambda (R^2 - x^2 - y^2 - z^2) = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - gz + \lambda (R^2 - x^2 - y^2 - z^2)$$

Yakovlev 7-11:

$$\mathcal{L}_x = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} \right) - \frac{\partial}{\partial x} = \frac{d}{dt} (\dot{x}) + 2\lambda x = \ddot{x} + 2\lambda x = 0$$

$$\mathcal{L}_y = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{y}} \right) - \frac{\partial}{\partial y} = \ddot{y} + 2\lambda y = 0$$

$$\mathcal{L}_z = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{z}} \right) - \frac{\partial}{\partial z} = \ddot{z} + g + 2\lambda z = 0$$

$$\frac{d^2}{dt^2} f(\vec{r}) = (R^2 - x^2 - y^2 - z^2)'' = -2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + x\ddot{x} + y\ddot{y} + z\ddot{z}) = 0$$

$$(\mathcal{L}_x + \mathcal{L}_y + \mathcal{L}_z = \ddot{x} + \ddot{y} + \ddot{z} - 2\lambda(x+y+z))$$

$$x \cdot \mathcal{L}_x + y \cdot \mathcal{L}_y + z \cdot \mathcal{L}_z = x\ddot{x} + 2\lambda x^2 + y\ddot{y} + 2\lambda y^2 + z\ddot{z} + gz + 2\lambda z^2 = 0$$

$$-\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + 2\lambda(x^2 + y^2 + z^2) + gz = 0$$

$$-\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + 2\lambda R^2 + gz = 0$$

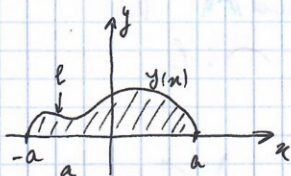
$$-2E + 3gz + 2\lambda R^2 = 0$$

$$\lambda = \frac{-2E + 3gz}{2R^2}$$

$$N = \lambda \left(\frac{\partial f(\vec{r})}{\partial x}, \frac{\partial f(\vec{r})}{\partial y}, \frac{\partial f(\vec{r})}{\partial z} \right) = -2\lambda (x, y, z) = \frac{-2E + 3gz}{R^2} (x, y, z)$$

N4.

$$l > 2a$$



$$l = \int_{-a}^a \sqrt{1+y'^2} dx$$

$$S[y(x)] = \int_{-a}^a y(x) dx + \lambda \left(\int_{-a}^a \sqrt{1+y'^2} dx - l \right) = \int_{-a}^a \left(y(x) + \lambda \sqrt{1+y'^2} - \frac{\lambda l}{2a} \right) dx$$

$$S[y(x)] = \int_{-a}^a L dx$$

$$L = y + \lambda \sqrt{1+y'^2} - \frac{\lambda l}{2a}$$

$$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dx} \left(\frac{\lambda y'}{\sqrt{1+y'^2}} \right) - 1 = 0$$

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c$$

$$\frac{\lambda^2 y'^2}{1+y'^2} = (x+c)^2$$

$$\lambda^2 \cdot y'^2 = (1+y'^2)(x+c)^2 = (x+c)^2 + y'^2(x+c)^2$$

$$y'^2(\lambda^2 - (x+c)^2) = (x+c)^2$$

$$y'^2 = \frac{(x+c)^2}{\lambda^2 - (x+c)^2}$$

$$y' = \pm \frac{x+c}{\sqrt{\lambda^2 - (x+c)^2}}$$

$$y = \tilde{c} \pm \sqrt{\lambda^2 - (x+c)^2}$$

$$y(a) = y(-a) = 0$$

$$\text{T.e. } c=0$$

$$y = \tilde{c} \pm \sqrt{\lambda^2 - x^2}$$

$$\Rightarrow y = \tilde{c} \pm \sqrt{\lambda^2 - x^2}$$

$$y(-a) = y(a) = \tilde{c} \pm \sqrt{\lambda^2 - a^2} = 0$$

$$\Rightarrow \tilde{c} = \mp \sqrt{\lambda^2 - a^2}$$

$$\text{отсюда } y(x) = \mp \sqrt{\lambda^2 - a^2} \pm \sqrt{\lambda^2 - x^2}$$

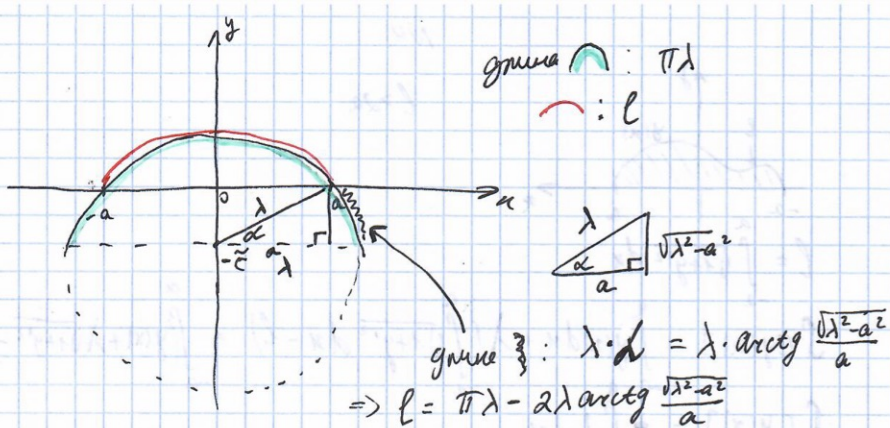
$$\Rightarrow (y \pm \tilde{c})^2 = \lambda^2 - x^2$$

$$(y \pm \tilde{c})^2 + x^2 = \lambda^2$$

$$\begin{aligned} \text{т.к. } (\tilde{c} \pm \sqrt{\lambda^2 - (x+c)^2})' &= \\ &= \frac{-2x \pm 2c}{2\sqrt{\lambda^2 - (x+c)^2}} \end{aligned}$$

→ Когда мысленно, тогда окр-ль с центром $(0, \pm \tilde{c})$ и радиусом λ .

8)



b) $l = \frac{\pi a}{\sqrt{2}}$

$$\pi\lambda - 2\lambda \arctg \frac{\sqrt{\lambda^2 - a^2}}{a} = \frac{\pi a}{\sqrt{2}}$$

$$2\lambda \arctg \frac{\sqrt{\lambda^2 - a^2}}{a} = \pi\lambda - \frac{\pi a}{\sqrt{2}}$$

$$\frac{\sqrt{\lambda^2 - a^2}}{a} = \tg \left(\frac{\pi\lambda - \frac{\pi a}{\sqrt{2}}}{2\lambda} \right)$$

Решение: $\lambda = a\sqrt{2} : 1 = \frac{\sqrt{2a^2 - a^2}}{a} = \tg \left(\frac{a\sqrt{2}\pi - \frac{\pi a}{\sqrt{2}}}{2a\sqrt{2}} \right) = \tg \left(\frac{\pi}{4} \right) = 1$

$\Rightarrow y_{\text{мнм}} = \pm \sqrt{2a^2 - a^2} \pm \sqrt{2a^2 - x^2} = \pm a \pm \sqrt{2a^2 - x^2}$