(1)

a) $\vec{F} = f_1\vec{r} + f_2\vec{p} + f_3[\vec{r}\times\vec{p}]$

f_i – скалярные ф-ции от $\vec{r}^2, \vec{p}^2, (\vec{r}\cdot\vec{p})$.

$$\{\vec{F}, (\vec{M}\cdot\vec{n})\} = \{f_1\vec{r}, (\vec{M}\cdot\vec{n})\} + \{f_2\vec{p}, (\vec{M}\cdot\vec{n})\} + \{f_3[\vec{r}\times\vec{p}], (\vec{M}\cdot\vec{n})\}$$

Посчитаем $\{f_1\vec{r}, (\vec{M}\cdot\vec{n})\}$.

$$\{f_1\vec{r}, (\vec{M}\cdot\vec{n})\} = f_1\{\vec{r}, (\vec{M},\vec{n})\} + \vec{r}\{f_1, (\vec{M}\cdot\vec{n})\}$$

- $\{r_l, \varepsilon_{ijk} r_j p_k n_i\} = \varepsilon_{ijk} n_i \{r_l, r_j p_k\} \overset{k=l}{=} \varepsilon_{ijk} n_i r_j =$
$= \varepsilon_{lij} n_i r_j = [\vec{n}\times\vec{r}]_l \Rightarrow \{\vec{r}, (\vec{M}\cdot\vec{n})\} = [\vec{n}\times\vec{r}]$.

- $\{f_1, (\vec{M}\cdot\vec{n})\} = \sum\limits_{i=1}^{3}\{r_i, (\vec{M}\cdot\vec{n})\}\dfrac{\partial f_1}{\partial r_i} + \sum\limits_{i=1}^{3}\{p_i, (\vec{M}\cdot\vec{n})\}\dfrac{\partial f_1}{\partial p_i}$.

$$\frac{\partial f_1}{\partial r_i} = 2r_i\frac{\partial f_1}{\partial \vec{r}^2} + p_i\frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}$$

$$\frac{\partial f_1}{\partial p_i} = 2p_i\frac{\partial f_1}{\partial \vec{p}^2} + r_i\frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}$$

$\{p_l, \varepsilon_{ijk} r_j p_k n_i\} =$
$= \varepsilon_{ijk} n_i p_k \{p_l, r_j\} \overset{j=l}{=}$
$= -\varepsilon_{ijk} n_i p_k = +\varepsilon_{jik} n_i p_k =$
$= [\vec{n}\times\vec{p}]_l$

Следовательно,

$$\{f_1, (\vec{M}\cdot\vec{n})\} = \sum_{i=1}^{3}\Big(2\frac{\partial f_1}{\partial\vec{r}^2} r_i[\vec{n}\times\vec{r}]_i + \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} p_i[\vec{n}\times\vec{r}]_i +$$
$$+ 2\frac{\partial f_1}{\partial\vec{p}^2} p_i[\vec{n}\times\vec{p}]_i + \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} r_i[\vec{n}\times\vec{p}]_i\Big) =$$
$$= 2\frac{\partial f_1}{\partial\vec{r}^2}\underbrace{(\vec{r}, [\vec{n}\times\vec{r}])}_{0} + \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\vec{p}\cdot(\vec{n}\times\vec{r}) + 2\frac{\partial f_1}{\partial\vec{p}^2}\underbrace{\vec{p}\cdot(\vec{n}\times\vec{p})}_{\vec{n}(\vec{p}\times\vec{p})=0} +$$
$$+ \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\vec{r}\cdot(\vec{n}\times\vec{p}) =$$
$$= \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\big(\vec{r}(\vec{n}\times\vec{p}) + \vec{p}(\vec{n}\times\vec{r})\big) = \frac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\big(\vec{p}(\vec{r}\times\vec{n}) + \vec{p}(\vec{n}\times\vec{r})\big) = 0$$

Значит, аналогично

$$\{f_2, (\vec{M}\cdot\vec{n})\} = \{f_3, (\vec{M}\cdot\vec{n})\} = 0.$$

Тогда

$$\{\vec{F}, (\vec{M}\cdot\vec{n})\} = f_1\{\vec{r}, (\vec{M}\cdot\vec{n})\} + f_2\{\vec{p}, (\vec{M}\cdot\vec{n})\} + f_3\{[\vec{r}\times\vec{p}], (\vec{M}\cdot\vec{n})\},$$

где $\{\vec{r}, (\vec{M}\cdot\vec{n})\} = [\vec{n}\times\vec{r}]$,

$$\{\vec{p}, (\vec{M}\cdot\vec{n})\} = [\vec{n}\times\vec{p}].$$

$$\{[\vec{r}\times\vec{p}], (\vec{M}\cdot\vec{n})\} = \{\vec{M}, (\vec{M}\cdot\vec{n})\}$$

$$\{M_i, M_j n_j\} = n_j\{M_i, M_j\} = \varepsilon_{ijk} n_j M_k = [\vec{n}\times\vec{M}]_i \Rightarrow$$

$$\Rightarrow \{[\vec{r}\times\vec{p}], (\vec{M}\cdot\vec{n})\} = [\vec{n}\times\vec{M}].$$

Таким образом,

$$\boxed{\{\vec{F}, (\vec{M}\cdot\vec{n})\} = f_1[\vec{n}\times\vec{r}] + f_2[\vec{n}\times\vec{p}] + f_3[\vec{n}\times\vec{M}],}$$

б) $\{\vec{F}, \vec{M}^2\} = \{f_1\vec{r}, \vec{M}^2\} + \{f_2\vec{p}, \vec{M}^2\} + \{f_3[\vec{r}\times\vec{p}], \vec{M}^2\}$

Посчитаем $\{f_1\vec{r}, \vec{M}^2\}$.

$\{f_1\vec{r}, \vec{M}^2\} = f_1\{\vec{r}, \vec{M}^2\} + \vec{r}\{f_1, \vec{M}^2\}$

• $\{r_l, \vec{M}^2\} = \{r_l, M_k^2\} = 2M_k\{r_l, M_k\} =$

$= 2M_k\{r_l, \varepsilon_{ijk} r_i p_j\} \underset{l=j}{=} 2M_k r_i \varepsilon_{ilk} = 2\varepsilon_{lki} M_k r_i =$

$= 2[\vec{M}\times\vec{r}]_l \Rightarrow \{\vec{r}, \vec{M}^2\} = 2[\vec{M}\times\vec{r}]$

• $\{f_1, \vec{M}^2\} = \{r_l, \vec{M}^2\}\dfrac{\partial f_1}{\partial r_l} + \{p_l, \vec{M}^2\}\dfrac{\partial f_1}{\partial p_l} =$

$= 2[\vec{M}\times\vec{r}]_l\left(2r_l\dfrac{\partial f_1}{\partial \vec{r}^2} + p_l\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\right) + 2[\vec{M}\times\vec{p}]\left(2p_l\dfrac{\partial f_1}{\partial \vec{p}^2} + r_l\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\right) =$

$= 4\underbrace{(\vec{r}\cdot[\vec{M}\times\vec{r}])}_{=0}\dfrac{\partial f_1}{\partial \vec{r}^2} + 2(\vec{p}\cdot[\vec{M}\times\vec{r}])\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} +$

$+ 4\underbrace{(\vec{p}\cdot[\vec{M}\times\vec{p}])}_{=0}\dfrac{\partial f_1}{\partial \vec{p}^2} + 2(\vec{r}\cdot[\vec{M}\times\vec{p}])\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} =$

$M\,p\,r \quad p\,r\,M$

$= 2\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\left(\vec{p}\cdot[\vec{M}\times\vec{r}] + \underbrace{\vec{r}[\vec{M}\times\vec{p}]}_{=\vec{p}[\vec{r}\times\vec{M}]}\right) = 0$

Значит, аналогично

$\{f_2, \vec{M}^2\} = \{f_3, \vec{M}^2\} = 0.$

Тогда

$\{\vec{F}, \vec{M}^2\} = f_1\{\vec{r}, \vec{M}^2\} + f_2\{\vec{p}, \vec{M}^2\} + f_3\{[\vec{r}\times\vec{p}], \vec{M}^2\},$

где $\{\vec{r}, \vec{M}^2\} = 2[\vec{M}\times\vec{r}]$

- $\{p_\ell, M^2\} = 2M_k\{p_\ell, M_k\} = 2M_k\{p_\ell, \varepsilon_{ijk} r_i p_j\} =$

$= 2M_k\, \varepsilon_{ijk}\, p_j\, (-1) = 2\,\varepsilon_{\ell k j} M_k p_j = 2[\vec{M}\times\vec{p}\,]_\ell \Rightarrow$

$i=\ell$

$\Rightarrow \{\vec{p}, \vec{M}^2\} = 2[\vec{M}\times\vec{p}\,].$

- $\{[\vec{r}\times\vec{p}\,], \vec{M}^2\} = \{\vec{M}, \vec{M}^2\} = 2\vec{M}\{\vec{M},\vec{M}\} = 0$

Таким образом,

$$\boxed{\{\vec{F}, \vec{M}^2\} = 2f_1[\vec{M}\times\vec{r}\,] + 2f_2[\vec{M}\times\vec{p}\,]}$$

② $L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}$, $\dot{x} \equiv \frac{dx}{dt}$

а) $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$

Преобр. Лежандра

$$L(q,\dot{q},t) \longmapsto \left[\dot{q}_\alpha p_\alpha - L(q,\dot{q},t)\right]\Big|_{\dot{q}_\alpha = f_\alpha(q,p,t)} = H(q,p,t)$$

$$\boxed{H(x,p,t) = \frac{p^2}{m} - \frac{m}{2}\cdot\frac{p^2}{m^2} + \frac{m\omega^2x^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2}}$$

б) $\alpha = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right)$

$$\alpha\bar{\alpha} = \frac{m\omega}{2}\left(x+i\frac{p}{m\omega}\right)\left(x-i\frac{p}{m\omega}\right) = \frac{m\omega}{2}\left(x^2+\frac{p^2}{m^2\omega^2}\right) =$$

$$= \frac{m\omega x^2}{2} + \frac{p^2}{2m\omega} = \frac{1}{\omega}\left(\frac{m\omega^2x^2}{2} + \frac{p^2}{2m}\right)$$

$$\Rightarrow \boxed{H(x,p,t) = \omega\alpha\bar{\alpha}.}$$

в) $\{\alpha,\bar{\alpha}\} = \frac{m\omega}{2}\left\{x+i\frac{p}{m\omega}, x-i\frac{p}{m\omega}\right\} =$

$$= \frac{m\omega}{2}\left(\overset{0}{\{x,x\}} - \frac{i}{m\omega}\overset{1}{\{x,p\}} + \frac{i}{m\omega}\overset{-1}{\{p,x\}} + \frac{1}{m\omega}\overset{0}{\{p,p\}}\right) =$$

$$= \frac{m\omega}{2}\left(\frac{-2i}{m\omega}\right) = -i \Rightarrow \boxed{\{\alpha,\bar{\alpha}\} = -i}$$

$$\{\alpha, H\} = \{\alpha, \omega\alpha\bar{\alpha}\} = \omega\{\alpha,\alpha\bar{\alpha}\} = \omega\alpha\{\alpha,\bar{\alpha}\} = -i\omega\alpha$$

$$\Rightarrow \boxed{\{\alpha, H\} = -i\omega\alpha}$$

2) $\alpha = \sqrt{\frac{m\omega}{2}} \left(x + i \frac{p}{m\omega} \right) \in C^{\infty}(M)$

$$\frac{d\alpha}{dt} = \{\alpha, H\} + \underset{\substack{\parallel \\ 0}}{\frac{\partial \alpha}{\partial t}} = -i\omega\alpha$$

$$\int \frac{d\alpha}{\alpha} = -\int i\omega\, dt \Rightarrow \ln\alpha = -i\omega t + c \Rightarrow \alpha(t) = \alpha_0 e^{-i\omega t}$$

$$\Rightarrow \boxed{\alpha(t) = \alpha_0 e^{-i\omega t}, \ \alpha_0 > 0}$$

③ $$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x})$$

$$B = |\vec{B}| = \text{const}, \quad c = \text{const}$$

a) $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2c}y \Rightarrow \dot{x} = \frac{p_x + \frac{qB}{2c}y}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2c}x \Rightarrow \dot{y} = \frac{p_y - \frac{qB}{2c}x}{m}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$H = \left(\frac{p_x}{m} + \frac{qB}{2mc}y\right)p_x + \left(\frac{p_y}{m} - \frac{qB}{2mc}x\right)p_y + \frac{p_z^2}{m} -$$
$$- \frac{m}{2}\cdot\frac{1}{m^2}\left(\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2 + p_z^2\right) -$$
$$- \frac{qB}{2cm}\left(p_y x - \frac{qB}{2c}x^2 - y p_x - \frac{qB}{2c}y^2\right) =$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + p_x y\frac{qB}{2mc} - \cancel{p_y x\frac{qB}{2mc}} -$$
$$- \frac{1}{2m}\cdot\left(\frac{qB}{2c}y\right)^2 - \cancel{p_x y\frac{qB}{2cm}} + \cancel{p_y x\frac{qB}{2mc}} - \frac{1}{2m}\left(\frac{qB}{2c}x\right)^2 -$$
$$- p_y x\frac{qB}{2mc} + \frac{1}{m}\left(\frac{qB}{2c}x\right)^2 + p_x y\frac{qB}{2mc} + \frac{1}{m}\left(\frac{qB}{2c}y\right)^2 =$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2m}\cdot\left(\frac{qB}{2c}\right)^2(y^2 + x^2) +$$
$$+ \frac{qB}{2mc}(p_x y - p_y x) \Rightarrow$$

$$\Rightarrow \boxed{H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{q^2B^2}{8mc^2}(x^2 + y^2) + \frac{qB}{2mc}(p_x y - p_y x)}$$

б) $x(0) = y(0) = z(0) = 0$

$p_x(0) = p_z(0) = p$

$p_y(0) = 0$

Канонические ур-ия Гамильтона:

$$\begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \end{cases}$$

(1) $\dot{x} = \dfrac{\partial H}{\partial p_x} = \dfrac{p_x}{m} + \dfrac{qB}{2mc} y$

(2) $\dot{y} = \dfrac{\partial H}{\partial p_y} = \dfrac{p_y}{m} - \dfrac{qB}{2mc} x$

(3) $\dot{z} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m}$

(4) $\dot{p}_x = -\dfrac{\partial H}{\partial x} = -\dfrac{q^2 B^2}{4mc^2} x + \dfrac{qB}{2mc} p_y$

(5) $\dot{p}_y = -\dfrac{\partial H}{\partial y} = -\dfrac{q^2 B^2}{4mc^2} y - \dfrac{qB}{2mc} p_x$

(6) $\dot{p}_z = -\dfrac{\partial H}{\partial z} = 0 \Rightarrow p_z(t) = const$ + н.у. $\Rightarrow \underline{p_z(t) = p}$

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(3) $\Rightarrow \dot{z} = \dfrac{p}{m} \underset{+\text{н.у.}}{\Rightarrow} \underline{z(t) = \dfrac{p}{m} t}$

Введем обозначение: $k = \dfrac{qB}{2mc}$

Тогда $$\begin{cases} \dot{x} = p_x m^{-1} + ky \\ \dot{y} = p_y m^{-1} - kx \\ \dot{p}_x = -mk^2 x + kp_y \\ \dot{p}_y = -mk^2 y - kp_x \end{cases}$$

$$A=\begin{pmatrix} 0 & k & m^{-1} & 0 \\ -k & 0 & 0 & m^{-1} \\ -mk^2 & 0 & 0 & k \\ 0 & -mk^2 & -k & 0 \end{pmatrix}$$

$$\chi_A(\lambda)=\begin{vmatrix} -\lambda & k & m^{-1} & 0 \\ -k & -\lambda & 0 & m^{-1} \\ -mk^2 & 0 & -\lambda & k \\ 0 & -mk^2 & -k & -\lambda \end{vmatrix} = \lambda^2 k^2\left(\frac{\lambda^2}{k^2}+4\right).$$

Собственные значения: $\lambda_1=0,\ \lambda_2=2ik,\ \lambda_3=-2ik.$

$\boxed{\lambda_1=0}$ Пусть $u=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$Au=\begin{pmatrix} kb+m^{-1}c \\ -ka+m^{-1}d \\ -mk^2a+kd \\ -mk^2b-kc \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

Собственные векторы:

$$u_{11}=\begin{pmatrix} 0 \\ \frac{-1}{mk} \\ 1 \\ 0 \end{pmatrix},\quad u_{12}=\begin{pmatrix} \frac{1}{mk} \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\boxed{\lambda_2=2ik}$

$$Au=\begin{pmatrix} kb+m^{-1}c \\ -ka+m^{-1}d \\ -mk^2a+kd \\ -mk^2b-kc \end{pmatrix}=\begin{pmatrix} 2ika \\ 2ikb \\ 2ikc \\ 2ikd \end{pmatrix} \Rightarrow$$

Собственный вектор:

$$u_2=\begin{pmatrix} -i \\ 1 \\ mk \\ imk \end{pmatrix}=\begin{pmatrix} -1/mk \\ -i/mk \\ -i \\ 1 \end{pmatrix}$$

$\boxed{\lambda_3=-2ik}$

$$Au=\begin{pmatrix} kb+m^{-1}c \\ -ka+m^{-1}d \\ -mk^2a+kd \\ -mk^2b-kc \end{pmatrix}=\begin{pmatrix} -2ika \\ -2ikb \\ -2ikc \\ -2ikd \end{pmatrix} \Rightarrow$$

Собственный вектор

$$u_3=\begin{pmatrix} -1/mk \\ i/mk \\ i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ -\frac{1}{mk} \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{mk} \\ 0 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} -\frac{1}{mk} \\ -\frac{i}{mk} \\ -i \\ 1 \end{pmatrix} e^{2ikt} + C_4 \begin{pmatrix} -\frac{1}{mk} \\ \frac{i}{mk} \\ i \\ 1 \end{pmatrix} e^{-2ikt}$$

Начальные усл-ия

$$\begin{cases} x(0) = \frac{C_2}{mk} - \frac{C_3}{mk} - \frac{C_4}{mk} = 0 & (1) \\ y(0) = -\frac{C_1}{mk} - \frac{iC_3}{mk} + \frac{iC_4}{mk} = 0 & (2) \\ p_x(0) = C_1 - iC_3 + iC_4 = p & (3) \\ p_y(0) = C_2 + C_3 + C_4 = 0 & (4) \end{cases} \iff \begin{cases} C_1 = \frac{p}{2} \\ C_2 = 0 \\ C_3 = \frac{ip}{4} \\ C_4 = \frac{-ip}{4} \end{cases}$$

$(1), (4) \Rightarrow C_2 = 0, \quad C_3 = -C_4$

$(2) \Rightarrow -C_1 - iC_3 - iC_3 = 0 \Rightarrow C_1 = -2iC_3$

$(3) \Rightarrow -2iC_3 - iC_3 - iC_3 = p \Rightarrow C_3 = \frac{-p}{4i} = \frac{ip}{4}$

Таким образом,

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \frac{p}{2mk} \sin(2kt) \\ \frac{p}{2mk} (-1 + \cos 2kt) \\ \frac{p}{2} (1 + \cos 2kt) \\ -\frac{p}{2} \sin(2kt) \end{pmatrix} = \begin{cases} \frac{pc}{qB} \sin\left(\frac{qB}{mc} t\right) \\ \frac{pc}{qB} \left(-1 + \cos\left(\frac{qB}{mc} t\right)\right) \\ \frac{p}{2} \left(1 + \cos\left(\frac{qB}{mc} t\right)\right) \\ -\frac{p}{2} \sin\left(\frac{qB}{mc} t\right) \end{cases}$$

То есть

$$\boxed{\begin{aligned} x(t) &= \frac{pc}{qB}\sin\left(\frac{qB}{mc}t\right) \\ y(t) &= \frac{pc}{qB}\left(\cos\left(\frac{qB}{mc}t\right)-1\right) \\ z(t) &= \frac{p}{m}t \\ p_x(t) &= \frac{p}{2}\left(\cos\left(\frac{qB}{mc}t\right)+1\right) \\ p_y(t) &= -\frac{p}{2}\sin\left(\frac{qB}{mc}t\right) \\ p_z(t) &= p \end{aligned}}$$

в) $\vec{v} = \left(\frac{p_x}{m} + ky,\ \frac{p_y}{m} - kx,\ \frac{p_z}{m}\right) = (v_1, v_2, v_3)$

$$\{v_1, v_2\} = \underbrace{\left\{\frac{p_x}{m}, \frac{p_y}{m}\right\}}_{0} - \left\{\frac{p_x}{m}, kx\right\} + \left\{ky, \frac{p_y}{m}\right\} - \underbrace{\{ky, kx\}}_{0} =$$

$$= -\frac{k}{m}\underbrace{\{p_x, x\}}_{-1} + \frac{k}{m}\underbrace{\{y, p_y\}}_{1} = \frac{2k}{m} = \frac{qB}{m^2c}$$

$$\{v_1, v_3\} = \left\{\frac{p_x}{m}, \frac{p_z}{m}\right\} + \frac{k}{m}\{y, p_z\} = 0$$

$$\{v_2, v_3\} = \left\{\frac{p_y}{m}, \frac{p_z}{m}\right\} - \frac{k}{m}\{x, p_z\} = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\{v_1, v_2\} = -\{v_2, v_1\} = \frac{qB}{m^2c} \\ &\{v_1, v_3\} = \{v_3, v_1\} = \{v_2, v_3\} = \{v_3, v_1\} = 0 \end{aligned}}$$

④ $$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m\omega^2}{2}(x^2 + y^2)$$

а) $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \Rightarrow \dot{y} = \frac{p_y}{m}$$

$$H = \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{m}{2}\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2}\right) + \frac{m\omega^2}{2}(x^2 + y^2) =$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2) \Rightarrow$$

$$\Rightarrow \boxed{H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)}$$

б) • $$J_1 = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{m\omega^2}{2}(x^2 - y^2)$$

$$\frac{dJ_1}{dt} = \{J_1, H\} + \frac{\partial J_1}{\partial t} = \{J_1, H\}$$

$$\{J_1, H\} = \left\{ \frac{p_x^2 - p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 - y^2),\ \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2) \right\}$$

Обозначим:

$$\alpha = \frac{p_x^2}{2m} + \frac{m\omega^2}{2}x^2, \quad \beta = \frac{p_y^2}{2m} + \frac{m\omega^2}{2}y^2.$$

$$\{J_1, H\} = \{\alpha - \beta, \alpha + \beta\} = \{\alpha, \beta\} - \{\beta, \alpha\} = 2\{\alpha, \beta\}$$

$$\{\alpha, \beta\} = \left\{ \frac{p_x^2}{2m} + \frac{m\omega^2}{2}x^2, \frac{p_y^2}{2m} + \frac{m\omega^2}{2}y^2 \right\} = 0,$$ т.к. не 0 дают скобки вида $\{q_\alpha, p_\alpha\}$ $\{p_\alpha, q_\alpha\}$

$$\Rightarrow \frac{dJ_1}{dt} = 0 \Rightarrow J_1(t) = \text{const}$$

- $J_2 = \frac{1}{m} p_x p_y + m\omega^2 x y$

$$H + J_2 = \frac{p_x^2 + 2p_x p_y + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + 2xy + y^2) =$$

$$= \frac{(p_x + p_y)^2}{2m} + \frac{m\omega^2}{2}(x+y)^2$$

$$\{J_2, H\} = \{H + J_2, H\} = \left\{\frac{(p_x+p_y)^2}{2m} + \frac{m\omega^2}{2}(x+y)^2, \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)\right\} =$$

$$= \frac{\cancel{m}\omega^2}{4\cancel{m}}\{(p_x+p_y)^2, x^2+y^2\} + \frac{\omega^2}{4}\{(x+y)^2, p_x^2+p_y^2\} =$$

$$= \frac{\omega^2}{2}(p_x+p_y)\{p_x+p_y, x^2+y^2\} + \frac{\omega^2}{2}(x+y)\{x+y, p_x^2+p_y^2\} =$$

$$= \frac{\omega^2}{2}(p_x+p_y)\Big(2x\overset{-1}{\overset{\shortparallel}{\{p_x, x\}}} + 2y\overset{-1}{\overset{\shortparallel}{\{p_y, y\}}}\Big) +$$

$$+ \frac{\omega^2}{2}(x+y)\Big(2p_x\underset{1}{\underset{\shortparallel}{\{x, p_x\}}} + 2p_y\underset{1}{\underset{\shortparallel}{\{y, p_y\}}}\Big) =$$

$$= \frac{\omega^2}{2}\Big((x+y)(2p_x+2p_y) - (p_x+p_y)(2x+2y)\Big) = 0$$

$$\frac{dJ_2}{dt} = \overset{=0}{\{J_2, H\}} + \overset{=0}{\frac{\partial J_2}{\partial t}} = 0 \Rightarrow J_2(t) = \text{const}$$

- $J_3 = \omega(x p_y - y p_x)$

$$H + J_3 = \frac{p_y^2}{2m} + \omega x p_y + \frac{m\omega^2}{2}x^2 + \frac{p_x^2}{2m} - \omega y p_x + \frac{m\omega^2}{2}y^2 =$$

$$= \left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\,\omega x\right)^2 + \left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\,\omega y\right)^2$$

$$\{J_3, H\} = \{J_3, H + J_3\} = \omega\left\{x p_y, \left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\,\omega x\right)^2 + \left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\,\omega y\right)^2\right\} -$$

$$- \omega\left\{y p_x, \left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\,\omega x\right)^2 + \left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\,\omega y\right)^2\right\}$$

$$= \underline{\underline{\omega}} x \{\underline{p_y}, \frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\} \cdot 2\left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\right) +$$

$$+ \underline{\underline{\omega}} p_y \{\underline{x}, \frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\} \, 2\left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\right) -$$

$$- \left(\underline{\underline{\omega}} y \{\underline{p_x}, \frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\omega x\} + \underline{\underline{\omega}} p_x \{y, \frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\omega x\}\right) 2 \cdot \left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\omega x\right)$$

$$= 2\omega\left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\right)\left(-\sqrt{\frac{m}{2}}\omega(-1)x + p_y\frac{1}{\sqrt{2m}}1\right) -$$

$$- 2\omega\left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\omega x\right)\left(y\sqrt{\frac{m}{2}}\omega(-1) + p_x\frac{1}{\sqrt{2m}} \cdot 1\right) =$$

$$= 2\omega\left(\left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\right)\left(\sqrt{\frac{m}{2}}\omega x + \frac{p_y}{\sqrt{2m}}\right) - \left(\frac{p_y}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\omega x\right)\left(\frac{p_x}{\sqrt{2m}} - \sqrt{\frac{m}{2}}\omega y\right)\right) =$$

$$= 0 \quad \Rightarrow \{J_3, H\} = 0$$

$$\frac{dJ_3}{dt} = \underset{0}{\underset{\shortparallel}{\{J_3, H\}}} + \underset{0}{\underset{\shortparallel}{\frac{\partial J_3}{\partial t}}} = 0 \quad \Rightarrow J_3(t) = \text{const}.$$

в) $\mathcal{L} = \text{span}(J_1, J_2, J_3)$.

Док-ть: $\forall\ l_1, l_2 \in \mathcal{L} \Rightarrow \{l_1, l_2\} \in \mathcal{L}$.

- $\{\alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3, \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3\} =$

$= \{J_1, J_2\}(\alpha_1\beta_2 - \alpha_2\beta_1) + \{J_1, J_3\}(\alpha_1\beta_3 - \alpha_3\beta_1) +$

$+ \{J_2, J_3\}(\alpha_2\beta_3 - \alpha_3\beta_2) \Rightarrow$ достаточно док-ть, что $\{J_i, J_j\} \in \mathcal{L}$.

- $\{J_1, J_2\} = \{J_1 + H, J_2\} = \{\frac{p_x^2}{m} + m\omega^2 x^2, \frac{1}{m}p_x p_y + m\omega^2 xy\} =$

$= \frac{1}{m} 2p_x \{p_x, xy\} m\omega^2 + m\omega^2 2x \{x, p_x p_y\} \frac{1}{m} =$

$= 2p_x \omega^2 y(-1) + 2\omega^2 x p_y = 2\omega(\omega(xp_y - yp_x)) = 2\omega J_3 \in L$

• $\{J_2, J_3\} = \{J_2 + H, J_3\} = \{\frac{(p_x+p_y)^2}{2m} + \frac{m\omega^2}{2}(x+y)^2, xp_y - yp_x\} \cdot \omega =$

$= \omega \frac{1}{2m} \cdot 2(p_x+p_y)\{\underline{p_x} + \underline{p_y}, \underline{x p_y} - \underline{y p_x}\} +$

$+ \frac{m\omega^3}{2} \cdot 2(x+y)\{\underline{x} + \underline{y}, \underline{x} p_y - y\underline{p_x}\} =$

$= \frac{\omega}{m}(p_x+p_y)(-p_y+p_x) + m\omega^3(x+y)(-y+x) =$

$= 2\omega\left(\frac{1}{2m}(p_x^2 - p_y^2) + \frac{m\omega^2}{2}(x^2-y^2)\right) = 2\omega J_1 \in L$

• $\{J_1, J_3\} = \{J_1 + H, J_3\} = \{\frac{p_x^2}{m} + m\omega^2 x^2, xp_y - yp_x\}\omega =$

$= \frac{\omega}{m} 2p_x \{\underline{p_x}, \underline{x}p_y\} + m\omega^3 \cdot 2x\{\underline{x}, -y\underline{p_x}\} =$

$= \frac{2\omega}{m} p_x p_y(-1) + 2m\omega^3 x(-y) = -2\omega\left(\frac{1}{m}p_x p_y + m\omega^2 xy\right) =$

$= -2\omega J_2 \in L.$

Значит, L инвариантно отн. $\{\ ,\ \}$.

⑤ $H = \frac{\vec{M}^2}{2I} - \gamma \vec{M}\cdot\vec{B}$

$\vec{M} = (M_1, M_2, M_3)$, $\vec{B} = (b_1, b_2, b_3) = \overrightarrow{const}$

$I, \gamma > 0$ – константы

1) $\frac{dM_i}{dt} = \{M_i, H\}$

$\{M_i, H\} = \frac{1}{2I}\{M_i, \vec{M}^2\} - \gamma\{M_i, \vec{M}\cdot\vec{B}\}$

• $\{M_i, M_j^2\} = 2M_j\{M_i, M_j\} = 2M_j \varepsilon_{ijk} M_k =$

$= 2\varepsilon_{ijk} M_j M_k = 2[\vec{M}\times\vec{M}]_i = 0$

• $\{M_i, M_j b_j\} = b_j\{M_i, M_j\} = b_j \varepsilon_{ijk} M_k = \varepsilon_{ijk} b_j M_k = [\vec{B}\times\vec{M}]_i$

$\Rightarrow \{M_i, H\} = -\gamma[\vec{B}\times\vec{M}]_i \Rightarrow \boxed{\frac{dM_i}{dt} = -\gamma[\vec{B}\times\vec{M}]_i}$

2) $\vec{B} = (0, 0, B)$

$[\vec{B}\times\vec{M}] = (-BM_2, BM_1, 0)$

$$\begin{cases} \dot{M}_1 = \gamma B M_2 & (1) \\ \dot{M}_2 = -\gamma B M_1 & (2) \\ \dot{M}_3 = 0 & (3) \end{cases}$$

$(3) \Rightarrow M_3(t) = M_3 = const$

$(1),(2) \Rightarrow \begin{cases} \dot{M}_1 = \gamma B M_2 \\ \dot{M}_2 = -\gamma B M_1 \end{cases}$; $A = \begin{pmatrix} 0 & \gamma B \\ -\gamma B & 0 \end{pmatrix}$

$$\chi_A(\lambda) = \lambda^2 + \gamma B \Rightarrow \quad \begin{array}{l}\lambda_1 = i\gamma B \\ \lambda_2 = -i\gamma B\end{array}$$

$\underline{\lambda_1 = i\gamma B}$, $\quad v = \begin{pmatrix} a \\ b \end{pmatrix}$

$$A v_1 = \gamma B \begin{pmatrix} b \\ -a \end{pmatrix} = \gamma B \begin{pmatrix} ai \\ bi \end{pmatrix} \Rightarrow \begin{cases} b = ai \\ -a = bi \end{cases} \Rightarrow v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\underline{\lambda_2 = -i\gamma B}$,

$$A v_2 = \gamma B \begin{pmatrix} b \\ -a \end{pmatrix} = \gamma B \begin{pmatrix} -ai \\ -bi \end{pmatrix} \to \begin{cases} b = -ai \\ a = bi \end{cases} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = c_i v_i e^{\lambda_i t} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\gamma B t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\gamma B t} =$$

$$= \begin{pmatrix} c_1 e^{i\gamma B t} + c_2 e^{-i\gamma B t} \\ c_1 i e^{i\gamma B t} - c_2 i e^{-i\gamma B t} \end{pmatrix} = \begin{pmatrix} c_1' \cos\gamma B t + c_2' \sin\gamma B t \\ c_2' \cos\gamma B t - c_1' \sin\gamma B t \end{pmatrix}$$

Ombem:

$$\boxed{\begin{array}{l} M_1(t) = c_1 \cos(\gamma B t) + c_2 \sin(\gamma B t) \\ M_2(t) = c_2 \cos(\gamma B t) - c_1 \sin(\gamma B t) \\ M_3(t) = M_3 = \text{const.} \end{array}}$$