(1) $$S[y] = 2y^2(\pi) + \int_0^{\pi} dx \left((y'(x))^2 - y^2(x) + 3y(x)\cos 2x\right)$$

$$y(x) \in C^2[0,\pi],\ y(0) = 0$$

$$\Delta S[y] = S[y+\delta y] - S[y] =$$

$$= 2(y+\delta y)^2(\pi) - 2y^2(\pi) + \int_0^{\pi} dx \Big((y'(x) + (\delta y)'(x))^2 - (y+\delta y)^2(x) +$$

$$+ 3(\cancel{y(x)} + \delta y(x))\cos 2x - (y'(x))^2 + y^2(x) \cancel{- 3y(x)\cos 2x}\Big) =$$

$$= 4y\delta y(\pi) + 2\delta y^2(\pi) + \int_0^{\pi} dx \Big(2y'(x)\delta y'(x) + (\delta y'(x))^2 -$$

$$- 2y\delta y(x) - (\delta y(x))^2 + 3\delta y(x)\cos 2x\Big)$$

$$\delta S[y] = 4y(\pi)\delta y(\pi) + \int_0^{\pi} dx \Big(2y'(x)\delta y'(x) - 2y(x)\delta y(x) +$$

$$+ 3\delta y(x)\cos 2x\Big)$$

$$\int_0^{\pi} 2y'(x)\delta y'(x)dx = \int_0^{\pi} 2y'(x)\, d\,\delta y(x) = 2y'(x)\delta y(x)\Big|_0^{\pi} -$$

$$- \int_0^{\pi} \delta y(x)\, d\,2y'(x) = 2y'(x)\delta y(x)\Big|_0^{\pi} - 2\int_0^{\pi} y''(x)\delta y(x)\,dx$$

$$\delta S[y] = 4y(\pi)\delta y(\pi) + \int_0^{\pi} dx \left(3\cos 2x - 2y(x) - 2y''(x)\right)\delta y(x) +$$

$$+ 2y'(x)\delta y(x)\Big|_0^{\pi} =$$

$$= (4y(\pi) + 2y'(\pi))\delta y(\pi) - 2y'(0)\delta y(0) + \int_0^{\pi} dx \left(3\cos 2x - 2y - 2y''\right)\delta y$$

$$2y'' + 2y - 3\cos 2x = 0$$

$$\Downarrow$$

$$\boxed{y(x) = C_2 \sin x + C_1 \cos x - \frac{1}{2}\cos(2x)}$$

$$y(0) = C_1 - \frac{1}{2} = 0 \Rightarrow \boxed{C_1 = \frac{1}{2}}$$

$\delta y|_{x=\pi}$ – произвольна $\Rightarrow (4y + 2y')|_{x=\pi} = 0$

$$(2y + y')|_{x=\pi} = 0$$

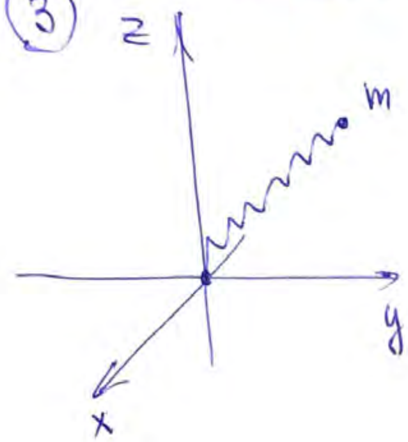
$$y'(x) = C_2 \cos x - C_1 \sin x + \frac{1}{2}\cdot 2\sin 2x$$

$$\left.\begin{aligned} y'(\pi) &= -C_2 \\ y(\pi) &= -C_1 - \frac{1}{2} \end{aligned}\right| \Rightarrow 2y(\pi) + y'(\pi) = -2C_1 - 1 - C_2 = 0$$

$$C_2 = -2C_1 - 1 = -2\cdot\frac{1}{2} - 1 = -2 \Rightarrow \boxed{C_2 = -2}$$

<u>Ответ:</u> $y(x) = -2\sin x + \frac{1}{2}\cos x - \frac{1}{2}\cos(2x)$

③



$$z = \frac{1}{2(x^2+y^2)}$$

Введем цилиндрические коорд. ρ, z, φ

$$\rho = \sqrt{x^2+y^2} \Rightarrow z = \frac{1}{2\rho^2} \Rightarrow \rho^2 = \frac{1}{2z}$$

$$l = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{1}{2z}+z^2}$$

$$L = T_{кин} - U$$

$$T_{кин} = \frac{m}{2}\left(\dot\rho^2+\dot z^2+\rho^2\dot\varphi^2\right) = \frac{m}{2}\left(\frac{\dot z^2}{8z^3}+\dot z^2+\frac{\dot\varphi^2}{2z}\right)$$

$$U = \frac{k}{2}\left(\frac{1}{2z}+z^2\right)$$

б) $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot\varphi} = const$ (З.С.И)

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ Выполняется З.С.Э. : $E = T + U = const$

~~Ур-ие Эйлера - Лагранжа~~

а) $$\boxed{L = T - U = \frac{m}{2}\left(\frac{\dot z^2}{8z^3}+\dot z+\frac{\dot\varphi^2}{2z}\right) - \frac{k}{2}\left(\frac{1}{2z}+z^2\right)}$$

Ур-ие Эйлера - Лагранжа

$$L_\varphi = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = 0 \Rightarrow \frac{d}{dt}\left(\frac{2\dot\varphi m}{4z}\right) = 0 \Rightarrow J = \frac{\overset{\text{const}}{\overset{\parallel}{\dot\varphi m}}}{2z}$$

$$L_z = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot z}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow$$

$$\frac{d}{dt}\left(\frac{2m\dot z}{16z^3}+\frac{2m\dot z}{2}\right)-\left(-\frac{3}{16}\,mz^2\cdot\frac{1}{z^4}-\frac{m\dot\varphi^2}{4z^2}+\frac{k}{4z^2}-\frac{\cancel{2}zk}{\cancel{2}}\right)$$

в) При $z=\text{const}=z_0 \Rightarrow \dot z=0$

$$\cancel{W_z\Big|_{z=z_0}=\frac{m\dot\varphi^2}{4z_0}-\frac{k}{4z_0}-\frac{kz_0}{2}}$$

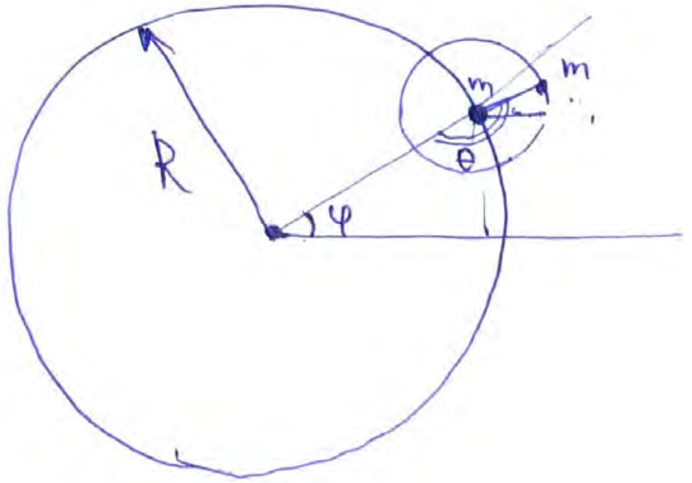
$$W_z\Big|_{z=z_0}=\frac{m\dot\varphi^2}{4z_0^2}-\frac{k}{4z_0^2}+z_0k=0 \Rightarrow z_0^3=\frac{m\dot\varphi^2-k}{4k}$$

$$\Downarrow$$

$z\neq 0$

$m\dot\varphi\neq k$

(4)



Для первой частицы

$$x_1 = R\cos\varphi \qquad \dot{x}_1 = -R\sin\varphi\,\dot{\varphi}$$

$$y_1 = R\sin\varphi \qquad \dot{y}_1 = R\cos\varphi\,\dot{\varphi}$$

Для второй

$$x_2 = R\cos\varphi + \ell\cos(\theta+\varphi-\pi) = R\cos\varphi - \ell\cos(\theta+\varphi)$$

$$y_2 = R\sin\varphi + \ell\sin(\theta+\varphi-\pi) = R\sin\varphi - \ell\sin(\theta+\varphi)$$

$$T = \frac{m}{2}(\dot{x}_1^2+\dot{y}_1^2) + \frac{m}{2}(\dot{x}_2^2+\dot{y}_2^2) =$$

$$= \frac{m}{2}(R^2\dot{\varphi}^2) + \frac{m}{2}\Big(\big(-R\sin\varphi\,\dot{\varphi} + \ell\sin(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\big)^2 +$$

$$+ \big(R\cos\varphi\,\dot{\varphi} - \ell\cos(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\big)^2 =$$

$$= \frac{m}{2}(R^2\dot{\varphi}^2) + \frac{m}{2}\Big(R^2\dot{\varphi}^2 + \ell^2(\dot{\theta}+\dot{\varphi})^2 \cancel{-2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell(}$$

$$-2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell\big(\sin\varphi\sin(\theta+\varphi) + \cos\varphi\cos(\theta+\varphi)\big)\Big)$$

$U = 0$

Значит, $L = T = \frac{m}{2}\left(R^2\dot{\varphi}^2\right) + \frac{m}{2}\left(R^2\dot{\varphi}^2 + \ell^2(\dot{\theta}+\dot{\varphi})^2 - \right.$

$\left. - 2\dot{\varphi}(\dot{\theta}+\dot{\varphi})R\ell\cos(\theta)\right)$

б) $\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполняется З.С.Э: $\mathcal{E} = T + U = T = const$

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = const$

$\frac{\partial L}{\partial \dot{\varphi}} = 2mR^2\dot{\varphi} + 2\ell^2\dot{\varphi} + 2\dot{\theta}\ell^2 - 2(2\dot{\varphi}+\dot{\theta})R\ell\underbrace{\cos\theta}_{const}$

$\cancel{\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = const}$

$\cancel{\frac{\partial L}{\partial \dot{\theta}} = 2\ell^2\dot{\theta} + 2\dot{\varphi}\ell^2 - 2\dot{\varphi}R\ell}$