Домашнее задание №5. Механика.

Стружинской Ксении

№1. $S[y(x)] = \int_0^1 ((y''(x))^2 + 5(y'(x))^2 + 4y^2(x))\,dx$

$y(x) \in C^\infty[0,1]$; $y'(0)=0$, $y(1)=-3$

$\delta S = \delta\int_0^1((y''(x))^2+5(y'(x))^2+4y^2(x))\,dx = \int_0^1(\delta(y''(x))^2+\delta 5(y'(x))^2+$

$+\delta 4y^2(x))\,dx \overset{*}{=} \int_0^1(2y''\delta(y'')+10y'\delta(y')+8y\delta y)\,dx =$

$= \int_0^1(2y''(\delta y)''+10y'(\delta y)'+8y\delta y)\,dx \;\textcircled{=}$

(*) заменим $y(x)$ на y для удобства

$\int_0^1(2y''(\delta y)'')\,dx = 2y''(\delta y)'\big|_0^1 - \int_0^1(2y'')'(\delta y)'\,dx$

$\int_0^1(2y'')'(\delta y)'\,dx = (2y'')'\delta y\big|_0^1 - \int_0^1(2y'')''\delta y\,dx$

$\int_0^1 10y'(\delta y)'\,dx = 10y'\delta y\big|_0^1 - \int_0^1(10y')'\delta y\,dx = 10y'\delta y\big|_0^1 - 10\int_0^1 y''\delta y\,dx$

$\textcircled{=}\; 2y''(\delta y)'\big|_0^1 - 2y'''\delta y\big|_0^1 + \int_0^1\underbrace{(2y'''' - 10y''+8y)}\delta y\,dx + 10y'\delta y\big|_0^1$

Т.е. $2y''''-10y''+8y=0$ $\quad y''''-5y''+4y=0$

Т.к. $y(0)$ не фиксирован условием, то $\delta(y(0))$ любой

$10y'(0) - 2y'''(0) = 0$, но $y'(0)=0 \Rightarrow \boxed{y'''(0)=0}$

$y'(1)$ не фиксирован условием, $\Rightarrow \delta(y'(1))$ любой, $\boxed{y''(1)=0}$

$y''''-5y''+4y=0$ $\quad$ хар мн-н: $t^4-5t^2+4=0$

$(t^2-1)(t^2-4)=0$

$t=\pm1;\ t=\pm2$

Значит, $y(x) = C_1e^x + C_2e^{-x} + C_3e^{2x} + C_4e^{-2x}$

Воспользуемся ограничениями из условия и прямоугольников:

$$\begin{cases} C_1e + C_2e^{-1} + C_3e^2 + C_4e^{-2} = -3 \\ C_1 - C_2 + 2C_3 - 2C_4 = 0 \\ C_1 - C_2 + 8C_3 - 8C_4 = 0 \\ C_1e + C_2e^{-1} + 4e^2C_3 + 4C_4e^{-2} = 0 \end{cases} \Rightarrow C_3 = C_4 \rightarrow C_1 = C_2$$

$$\begin{cases} C_1(e+e^{-1}) + C_3(e^2+e^{-2}) = -3 \\ C_1(e+e^{-1}) + C_3(4e^2+4e^{-2}) = 0 \end{cases} \quad C_3 = \frac{3}{4e^2+4e^{-2}-e^2-e^{-2}} = \frac{1}{e^2+e^{-2}}$$

$C_1(e+e^{-1}) + 1 = -3 \Rightarrow C_1 = -\frac{4}{e+e^{-1}}$

Значит, экстремаль $\boxed{y(x) = \frac{e^{2x}+e^{-2x}}{e^2+e^{-2}} - \frac{4(e^x+e^{-x})}{e+e^{-1}}}$

1б) $F[y(x)] = S[y(x)] + 6y'(1)$, $y'(0)=0$, $y(1) = -3 \Rightarrow$

$S[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x)\right)dx$,

$F[y(x)] = \int_0^1 \left((y'')^2 + 5(y')^2 + 4y^2 + 6y''\right)dx$, т.к. $y'(0)=0$

Тогда $\delta F[y(x)] = \int_0^1 dx\left(\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{\partial L}{\partial y''}\delta y''\right) =$

$= \int_0^1 \left(8y\,\delta y + 10y'\delta y' + 6\delta y'' + 2y''\delta y''\right)dx$ Аналогично с 1а):

$= \int_0^1 \left(2y'''' - 10y'' + 8y\right)\delta y\,dx + (2y'' + 6)\delta y'\Big|_0^1 + (10y' - 2y''')\delta y\Big|_0^1$, т.к.

$6\int_0^1 \delta y'' = 6\delta y'\Big|_0^1 - \underbrace{\int_0^1 (6)'\delta y'\,dx}_{0} = 6\delta y'\Big|_0^1$

$2y'''' - 10y'' + 8y = 0$

$y = C_1e^x + C_2e^{-x} + C_3e^{2x} + C_4e^{-2x}$

При этом $\underline{y'(0)=0}$, $\underline{y(1)=-3}$,

$y(0)$ не фиксирован $\Rightarrow 10y'(0) - 2y'''(0) = 0 \Rightarrow \underline{y'''(0)=0}$ (т.к. $y'(0)=0$)

$y'(1)$ не фиксирован $\Rightarrow 2y''(1) + 6 = 0 \Rightarrow \underline{y''(1) = -3}$

Составим систему на основании четырех условий:

$$\begin{cases} C_1e + C_2e^{-1} + C_3e^2 + C_4e^{-2} = -3 \\ C_1 - C_2 + 2C_3 - 2C_4 = 0 \\ C_1e + C_2e^{-1} + 4e^2C_3 + 4C_4e^{-2} = -3 \\ C_1 - C_2 + 8C_3 - 8C_4 = 0 \end{cases} \Rightarrow C_3 = C_4 \Rightarrow C_1 = C_2$$

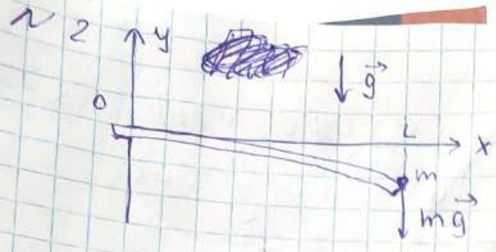
$C_1(e+e^{-1}) + C_3(e^2+e^{-2}) = -3$

$C_1(e+e^{-1}) + 4C_3(e^2+e^{-2}) = -3$ $\Rightarrow C_3(e^2+e^{-2}) = 0$ $C_3 = 0 = C_4$

$C_1 = C_2 = -\frac{3}{e+e^{-1}}$

$y_{экстр}(x) = -\frac{3}{e+e^{-1}}\left(e^{-x} + e^x\right)$

№ 2



$\frac{k(y'')^2}{2}$

Т.к. начало балки закреплено, то $y(0) = y'(0) = 0$, а конец балки свободный $\Rightarrow y(L)$ и $y'(L)$ любые.

Из лекций/семинаров для балки-балкона с доп. у усл:

$\delta U_{тяж} = mg\,y(L)\,dx \qquad \delta U_{упр} = \frac{k(y'')^2}{2}dx$

$$U[y(x)] = \int_0^L \left(\frac{k(y'')^2}{2} + mg\,y(L)\right)dx = \int_0^L \left(\frac{k(y'')^2}{2} + Lmg\,y'(x)\right)dx$$

$$\delta U[\delta y(x)] = \int_0^L \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial y''}\right)\delta y(x)\,dx +$$

$$+ \frac{\partial \mathcal{L}}{\partial y''}\delta y'(x)\Big|_0^L + \left(\frac{\partial \mathcal{L}}{\partial y'} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y''}\right)\delta y(x)\Big|_0^L = \int_0^L k\,y''''(x)\,\delta y(x)\,dx +$$

$$+ k\,y''(x)\,\delta y'(x)\Big|_0^L + (Lmg - k\,y'''(x))\,\delta y(x)\Big|_0^L$$

Т.к. $\delta y'(L)$ любое, то

$$\frac{\partial}{\partial y''}\left(\frac{k(y'')^2}{2} + Lmgy'\right)\Big|_{x=L} = 0 \Rightarrow y''(L) = 0$$

Т.к. $\delta y(L)$ любое, то

$$\left(\frac{\partial}{\partial y'} - \frac{d}{dx}\frac{\partial}{\partial y''}\right)\left(\frac{k(y'')^2}{2} + Lmgy'\right)\Big|_{L=x} = 0 \Rightarrow y'''(L) = \frac{Lmg}{k}$$

Используя подчеркнутое соотношение, получим:

$k\,y''''(x) = 0$

$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

$$\begin{cases} C_1 = 0 & y(0) \\ C_2 = 0 & y'(0) \\ 2C_3 + 6LC_4 = 0 & y''(L) \Rightarrow C_3 = -\frac{mgL^2}{2k} \\ 6C_4 = \frac{Lmg}{k} & y'''(L) \Rightarrow C_4 = \frac{Lmg}{6k} \end{cases}$$

Значит, $y(x) = -\frac{mgL^2}{2k}x^2 + \frac{mgL}{6k}x^3$, — конфигурация балки из принципа наименьшего действия

№3

$$T = \frac{\dot x^2+\dot y^2+\dot z^2}{2}$$

$$U = mgh = zg$$

$$L = T - U = \frac{\dot x^2+\dot y^2+\dot z^2}{2} - zg$$

L явно не зависит от времени, $\frac{\partial L}{\partial t} = 0 \Rightarrow$ выполнен ЗСЭ

$$\underline{E = T + U = \frac{\dot x^2+\dot y^2+\dot z^2}{2} + zg = const}$$

$f(\vec r) = R^2 - \vec r^2 = 0$ связь

$$\mathcal{L} = L + \lambda(R^2-x^2-y^2-z^2) = \frac{\dot x^2+\dot y^2+\dot z^2}{2} - zg + \lambda(R^2-x^2-y^2-z^2)$$

Уравнения Эйлера-Лагранжа:

$$\mathcal{L}_x = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(\dot x) + 2\lambda x = \ddot x + 2\lambda x = 0$$

$$\mathcal{L}_y = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot y}\right) - \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}(\dot y) + 2\lambda y = \ddot y + 2\lambda y = 0$$

$$\mathcal{L}_z = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot z}\right) - \frac{\partial \mathcal{L}}{\partial z} = \ddot z + 2\lambda z + g = 0$$

$$0 = \mathcal{L}_x + \mathcal{L}_y + \mathcal{L}_z = \ddot x + \ddot y + \ddot z + 2\lambda x + 2\lambda y + 2\lambda z -$$

$$\frac{d^2 f(\vec r)}{dt^2} = \frac{d^2}{dt^2}(R^2-x^2-y^2-z^2) = -2(x\ddot x + y\ddot y + z\ddot z + \dot x^2 + \dot y^2 + \dot z^2) = 0$$

$$x\ddot x + y\ddot y + z\ddot z = -(\dot x^2+\dot y^2+\dot z^2) \quad + gz$$

$$x\mathcal{L}_x + y\mathcal{L}_y + z\mathcal{L}_z = x(\ddot x + 2\lambda x) + y(\ddot y + 2\lambda y) + z(\ddot z + 2\lambda z + g) = 0 =$$

$$= -(\dot x^2+\dot y^2+\dot z^2) + 2\lambda(x^2+y^2+z^2) + gz =$$

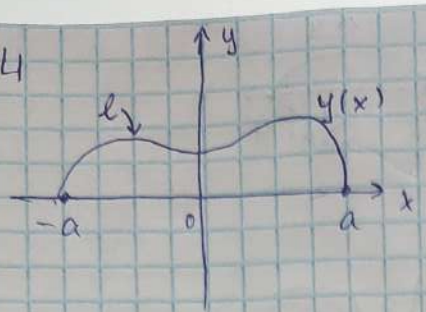
$$= -\dot x^2 - \dot y^2 - \dot z^2 + 2\lambda + gz = 0$$

$$-2E + 2\lambda R^2 + 3gz = 0 \quad (\text{т.к. } 2E = \dot x^2+\dot y^2+\dot z^2 + 2gz)$$

Значит, $\lambda = \frac{2E - 3gz}{2R^2}$

Тогда $N = \lambda\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = -2\lambda(x,y,z) = \frac{+3gz - 2E}{R^2}(x,y,z)$

№4



$l > 2a$

$y(a) = y(-a) = 0$

$y(x) \in C^2[-a, a]$

$\int\limits_{-a}^{a} \sqrt{1+(y')^2}\,dx = l = const$

Нам необходимо найти такую $y(x)$, что $\int\limits_{-a}^{a} y\,dx$ - наибольш

$$J(y) = \int\limits_{-a}^{a} y\,dx + \lambda\left(\int\limits_{-a}^{a}\sqrt{1+(y')^2}\,dx - l\right) = \int\limits_{-a}^{a}\underbrace{\left(y + \lambda\sqrt{1+(y')^2} - \frac{\lambda l}{2a}\right)}_{L(x,y,y')}dx$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$\lambda\left(\frac{y''}{(1+(y')^2)^{3/2}} - \frac{1}{\lambda}\right) = 0$$ – условие экстремальности $y(x)$

т.е. $\frac{y''}{(1+(y')^2)^{3/2}} - \frac{1}{\lambda} = 0$

т.к. $\frac{\partial L}{\partial x} = 0$, то выполнено „ЗСЭ"

$$„E" = y'\frac{\partial L}{\partial y'} + \cancel{\lambda'\frac{\partial L}{\partial \lambda'}}^{0} - L = \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} - y + \frac{\lambda l}{2a} - \lambda\sqrt{1+(y')^2} = const$$

тогда $\frac{\lambda(y')^2}{\sqrt{1+(y')^2}} - y - \lambda\sqrt{1+(y')^2} = const = c$

$$\lambda\left(\frac{\overbrace{(y')^2 - (1+(y')^2)}^{-1}}{\sqrt{1+(y')^2}}\right) = y + c$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{\lambda^2}{(y+c)^2} - 1}$$

$\Downarrow$

$\frac{dx}{dy} = \sqrt{\frac{(y+c)^2}{\lambda^2 - (y+c)^2}}$, тогда $x + c_1 = \int \frac{y+c}{\sqrt{\lambda^2-(y+c)^2}}\,dy$

Если $y + c = \lambda\sin\varphi$, то $x + c_1 = \lambda\cos\varphi \Rightarrow$

$(y+c)^2 + (x+c_1)^2 = \lambda^2$

$y(a) = y(-a) = 0 \Rightarrow (a+c_1)^2 + c^2 = (-a+c_1)^2 + c^2 = \lambda^2$

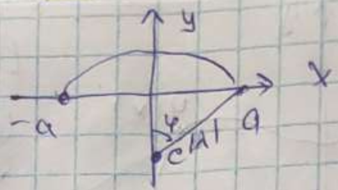
$2ac_1 = -2ac_1$

$c_1 = 0$ т.к. $2a \neq 0$

$c \geq 0$, $\Rightarrow 2a < l \leq a\pi$

$x^2 + (y+c)^2 = \lambda^2 \Rightarrow y = \sqrt{\lambda^2 - x^2} - c$, $c \geq 0$

б) $\ell = 2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda|$



в) При $\ell = \frac{\pi a}{\sqrt{2}}$

$$\frac{\pi a}{\sqrt{2}} = 2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda| \Rightarrow |\lambda| = a\sqrt{2}$$

и $y(x) = \sqrt{2a^2 - x^2} - a$