Мозговой Владислав

№1

$$S[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x) \right) dx$$

$$y(x) \in C^{\infty}[0,1] \qquad y'(0) = 0 \qquad y(1) = -3$$

$$\delta S[\delta y(x)] = \int_0^1 \left(8y(x) - \frac{d}{dx}(10y'(x)) + \frac{d^2}{dx^2}(2y''(x)) \right) \delta y(x)\, dx + 2y''(x)\delta y'(x)\Big|_0^1$$

$$+ (10y'(x) - 2y'''(x))\delta y(x)\Big|_0^1$$

$$= \int_0^1 (2y^{(4)}(x) - 10y''(x) + 8y(x))\delta y(x)\, dx + 2y''(x)\,\delta y'(x)\Big|_0^1 + (10y'(x) - 2y^{(3)}(x))\delta y(x)\Big|_0^1$$

Экстремаль:

$$y^{(4)}(x) - 5y''(x) + 4y(x) = 0$$

граничные условия:

$$y(1) = -3 \qquad y'(0) = 0$$

$$\left(\frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''} \right)\Big|_{x=0} = 0 \qquad \frac{\partial L}{\partial y''}\Big|_{x=1} = 0$$

$$x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4)$$

$$\Rightarrow y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x} \qquad y(1) = c_1 e + \frac{c_2}{e} + c_3 e^2 + \frac{c_4}{e^2} = -3$$

$$y'(x) = c_1 e^x - c_2 e^{-x} + 2c_3 e^{2x} - 2c_4 e^{-2x} \qquad y'(0) = c_1 - c_2 + 2c_3 - 2c_4 = 0$$

$$\left(\frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''} \right)\Big|_{x=0} = 10y'(x) - 2y'''(x)\Big|_{x=0} = 0 \Rightarrow y'''(x)\Big|_{x=0} = 0$$

$$y''(x) = c_1 e^x + c_2 e^{-x} + 4c_3 e^{2x} + 4c_4 e^{-2x}$$

$$y'''(x) = c_1 e^x - c_2 e^{-x} + 8c_3 e^{2x} - 8c_4 e^{-2x}$$

$$y'''(0) = c_1 - c_2 + 8c_3 - 8c_4 = 0$$

$$\frac{\partial L}{\partial y''}\Big|_{x=1} = 2y''(1) = 0 \Leftrightarrow y''(1) = 0$$

$$y''(1) = c_1 e + \frac{c_2}{e} + 4c_3 e^2 + \frac{4c_4}{e^2} = 0$$

$$\Rightarrow C_1 = C_2 = -\frac{4}{e+e^{-1}} \qquad C_3 = C_4 = \frac{1}{e^2+e^{-2}}$$

$$\Rightarrow y(x) = \frac{e^{2x}+e^{-2x}}{e^2+e^{-2\cancel{x}}} - \frac{4(e^{x}+e^{-x})}{e+e^{-1}}$$

б) $F[y(x)] = S[y(x)] + 6y'(1)$

$$y'(0)=0 \Rightarrow F[y(x)] = \int_0^1 \left((y''(x))^2 + 5(y'(x))^2 + 4y^2(x) + 6y''(x) \right) dx$$

$$\delta F[\delta y(x)] = \int_0^1 \left(2y^{(4)}(x) - 10y''(x) + 8y(x)\right)\delta y(x)\,dx + (2y''(x)+6)\delta y'(x)\Big|_0^1 + (10y'(x) - 2y'''(x))\delta y(x)\Big|_0^1$$

Экстремаль удовлетворяет уравнению $\lambda^4 - 5\lambda^2 + 4$

с граничными условиями

$$y(1) = -3 \qquad y'(0) = 0$$

$$10y'(0) - 2y'''(0) = 0 \Rightarrow y'''(0) = 0$$

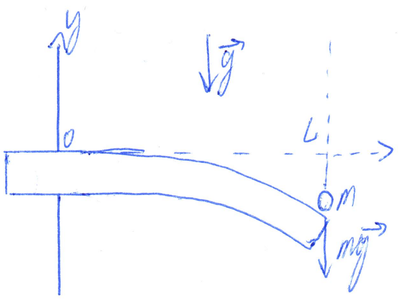
$$2y''(1) + 6 = 0 \Rightarrow y''(1) = -3$$

$$y''(1) = C_1 e + \frac{C_2}{e} + 4C_3 e^2 + \frac{4C_4}{e^2} = -3$$

$$\Rightarrow C_1 = C_2 = -\frac{3}{e+e^{-1}} \qquad C_3 = C_4 = 0$$

$$y(x) = -\frac{3(e^{x}+e^{-x})}{e+e^{-1}}$$

№3



Кинетической энергии у балки нет, поэтому балка принимает положение, в котором потенциальная энергия в экстремуме

$$\delta U_{тяж} = mg\, y(L)$$

$$\delta U_{упр} = \frac{K}{2}(y'')^2 dx$$

$$U[y(x)] = \int_0^L \left(mg\, y(L) + \frac{K}{2}(y''(x))^2 dx\right) = \int_0^L \left(mg\, y'(x) + \frac{K}{2}(y''(x))^2 dx\right.$$

$$\delta U[\delta y(x)] = \int_0^L K y^{(4)}(x)\,\delta y(x)\,dx + K y''\,\delta y'(x)\Big|_0^L + (mg - K y''')\,\delta y(x)\Big|_0^L$$

Экстремаль $\delta U[\delta y(x)]$ удовлетворяет $K y^{(4)}(x) = 0$

$y(0)=0 \quad y'(0)=0 \quad y''(L)=0 \quad y'''(L)=\frac{mg}{K}$

$$K\, y''(x)\Big|_{x=L} = 0 \;\Rightarrow\; y''(L) = 0$$

$$mg - K y'''(x)\Big|_{x=L} = 0 \;\Rightarrow\; y''(L) = 0$$

$\cancel{mg - K y'''(x)\big|_{x=L} = 0} \Rightarrow \quad y(x) = C_1x^3 + C_2x^2 + C_3x + C_4$

$y(0) = C_4 = 0$

$y'(0) = 3C_1x^2 + 2C_2x + C_3\big|_{x=0} = C_3 = 0$

$y''(L) = 6C_1x + 2C_2\big|_{x=L} = 6C_1L + 2C_2 = 0$

$y'''(L) = 6C_1\big|_{x=L} = 6C_1 = \frac{mg}{K}$

$C_1 = \frac{mg}{6K} \quad C_2 = -\frac{mgL}{2K} \quad C_3 = C_4 = 0$

$$y(x) = \frac{mg}{6K}x^3 - \frac{mgL}{2K}x^2$$

№1

$T_{кин} = \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}$ $\qquad V = gz$

Общая энергия $E = T_{кин} + V = \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} + gz$

$L = T_{кин} - V = \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz$

$\dfrac{\partial L}{\partial t} = 0 \Rightarrow$ выполняется ЗС.Э.

$\mathcal{L} = L - \lambda f(\bar{r})$, где $f(\bar{r}) = R^2 - \bar{r}^2 = 0$

$\Rightarrow \mathcal{L} = \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2} - gz + \lambda(x^2+y^2+z^2-R^2)$

$\mathcal{L}_x: \ \ddot{x} - 2\lambda x = 0$

$\mathcal{L}_y: \ \ddot{y} - 2\lambda y = 0$

$\mathcal{L}_z: \ \ddot{z} - 2\lambda z + g = 0$

Найдем λ

$x\mathcal{L}_x + y\mathcal{L}_y + z\mathcal{L}_z = x\ddot{x} + y\ddot{y} + z\ddot{z} - 2\lambda\overbrace{(x^2+y^2+z^2)}^{R^2} + gz = 0$

$2(x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \ddot{f}(\bar{r}) = 0$

$\Rightarrow x\ddot{x} + y\ddot{y} + z\ddot{z} = -(\dot{x}^2+\dot{y}^2+\dot{z}^2) = -2E + 2gz$

$2\lambda R^2 = gz - 2E + 2gz \Rightarrow \lambda = \dfrac{3gz - 2E}{2R^2}$

$\bar{N} = \left(\lambda \dfrac{\partial f(\bar{r})}{\partial x}, \lambda \dfrac{\partial f(\bar{r})}{\partial y}, \dfrac{\partial f(\bar{r})}{\partial z}\right) = 2\lambda(x, y, z) = \dfrac{3gz - 2E}{R^2}(x, y, z)$

$\Rightarrow \bar{N} = \dfrac{3gz - 2E}{R^2}(x, y, z)$

№5

$y(x) \in C^2[-a,a]$

$I(y) = \int\limits_{-a}^{a} \sqrt{1+y'^2}\,dx = l = const$

$l > 2a$

$y(a) = y(-a) = 0$

а)

$J(y) = \int\limits_{-a}^{a} y\,dx + \lambda\left(\int\limits_{-a}^{a}\sqrt{1+y'^2}\,dx - l\right) = \int\limits_{-a}^{a}\left(y + \lambda\sqrt{1+y'^2} - \frac{\lambda l}{2a}\right)dx$

если $J(y)$ имеет экстремум в $y(x)$, то

$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) - \frac{\partial h}{\partial y} = 0$, т.е. $\frac{y''}{(1+y'^2)^{\frac{3}{2}}} - \frac{1}{\lambda} = 0$

$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{d}{dx}\left(\lambda\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\lambda y''}{(1+y'^2)^{\frac{3}{2}}}$ $\qquad \frac{\partial h}{\partial y} = 1$

$\frac{\partial h}{\partial x} = 0 \Rightarrow$ выполнен ЗСЭ

$y'\frac{\partial h}{\partial y'} + x'\frac{\partial h}{\partial x'} - h = \frac{\lambda y'^2}{\sqrt{1+y'^2}} - y - \lambda\sqrt{1+y'^2} + \frac{\lambda l}{2a} = const$

Обозначим $\frac{\lambda y'^2}{\sqrt{1+y'^2}} - 1 - \lambda\sqrt{1+y'^2} = c_1 = const$

$\frac{\lambda y'^2}{\sqrt{1+y'^2}} - \lambda\sqrt{1+y'^2} = c_1 + y$

$-\lambda = (y+c_1)\sqrt{1+y'^2}$

$y' = \sqrt{\frac{\lambda^2}{(y+c_1)^2} - 1}$

т.е. $\frac{dx}{dy}=\sqrt{\frac{(y+c_1)^2}{\lambda^2-(y+c_1)^2}} \Rightarrow x+c_2=\int\frac{y+c_1}{\sqrt{\lambda^2-(y+c_1)^2}}dy$

пусть $y+c_1=\lambda\sin\varphi$, тогда $x+c_2=\lambda\cos\varphi$

следовательно $\left.\begin{array}{l} y+c_1=\lambda\sin\varphi \\ x+c_2=\lambda\cos\varphi \end{array}\right| \Rightarrow (x+c_2)^2+(y+c_1)^2=\lambda^2$

$y(a)=y(-a)=0 \Rightarrow (a+c_2)^2+c_1^2=(-a+c_2)^2+c_1^2=\lambda^2 \Rightarrow c_2=0$

Так как $y(x)$ является функцией, то $c_1\geq 0 \Rightarrow 2a<l\leq\pi a$

$x^2+(y+c_1)^2=\lambda^2$, $c_1\geq 0$, т.е.

$y=\sqrt{\lambda^2-x^2}-c_1$, $c_1\geq 0$

б) $l=2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda|$

в) при $l=\frac{\pi a}{\sqrt{2}}$

$\frac{\pi a}{\sqrt{2}}=2\arcsin\left(\frac{a}{|\lambda|}\right)|\lambda| \Rightarrow |\lambda|=a\sqrt{2}$

$y(x)=\sqrt{2a^2-x^2}-a$