Домашнее задание №6. Механика

Струихиной Ксении

№1 $\vec{F}: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$\vec{F} = f_1 \vec{r} + f_2 \vec{p} + f_3 [\vec{r} \times \vec{p}]$ $\quad \{r_i, p_j\} = \delta_{ij}$

$\{\vec{F}, (\vec{M} \cdot \vec{n})\}$, $\vec{M} = [\vec{r} \times \vec{p}]$, $\{\vec{F}, \vec{M}^2\}$

1) $\{\vec{F}, (\vec{M} \cdot \vec{n})\} = \{f_1 \vec{r} + f_2 \vec{p} + f_3 [\vec{r} \times \vec{p}], (\vec{M} \cdot \vec{n})\} =$

$= \{f_1 \vec{r}, (\vec{M} \cdot \vec{n})\} + \{f_2 \vec{p}, (\vec{M} \cdot \vec{n})\} + \{f_3 [\vec{r} \times \vec{p}], (\vec{M} \cdot \vec{n})\}$

$\{f_1 \vec{r}, (\vec{M} \cdot \vec{n})\} = f_1 \{\vec{r}, (\vec{M} \cdot \vec{n})\} + \vec{r} \{f_1, (\vec{M} \cdot \vec{n})\}$

a) $\{f_1, (\vec{M} \cdot \vec{n})\} = \sum\limits_{i=1}^{3} \{r_i, (\vec{M} \cdot \vec{n})\} \frac{\partial f_1}{\partial r_i} + \sum\limits_{i=1}^{3} \{p_i, (\vec{M} \cdot \vec{n})\} \frac{\partial f_1}{\partial p_i}$

$\frac{\partial f_1}{\partial r_i} = 2 \cdot \frac{\partial f_1}{\partial \vec{r}^2} r_i + p_i \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})}$

$\frac{\partial f_1}{\partial p_i} = 2 p_i \frac{\partial f_1}{\partial \vec{p}^2} + r_i \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})}$

т.к. $\{p_l, p_k r_j\} \neq 0$ при $l = j$

$\{p_l, \varepsilon_{ijk} r_j p_k n_i\} = \varepsilon_{ijk} n_i p_k \{p_l, r_j\} = -p_k n_i \varepsilon_{ijk} = \varepsilon_{jik} n_i p_k = [\vec{n} \times \vec{p}]_l$

$\{r_l, p_k\} \neq 0$ при $k = l$

б) $\{r_l, \varepsilon_{ijk} r_j p_k n_i\} = \varepsilon_{ijk} \{r_l, r_j p_k\} n_i = \varepsilon_{ijk} n_i r_j = \varepsilon_{lij} n_i r_j = [\vec{n} \times \vec{r}]_l$

Значит, $\{f_1, (\vec{M} \cdot \vec{n})\} = \sum\limits_{i=1}^{3} \Big(2 \frac{\partial f_1}{\partial \vec{r}^2} r_i [\vec{n} \times \vec{r}]_i + \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} p_i [\vec{n} \times \vec{r}]_i +$

$+ \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} r_i [\vec{n} \times \vec{p}]_i + \frac{2 \partial f_1}{\partial \vec{p}^2} p_i [\vec{n} \times \vec{p}]_i \Big) = 2 \frac{\partial f_1}{\partial \vec{r}^2} (\vec{r}, \overset{0}{[\vec{n} \times \vec{r}]}) +$

$+ \frac{2 \partial f_1}{\partial \vec{p}^2} \vec{p} \cdot \overset{0}{[\vec{n} \times \vec{p}]} + \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \vec{p} (\vec{n} \times \vec{r}) + \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \vec{r} (\vec{n} \times \vec{p}) =$

$\vec{n} [\vec{p} \times \vec{p}] = 0$

$= \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \left(\vec{r} [\vec{n} \times \vec{p}] + \vec{p} [\vec{n} \times \vec{r}] \right) = \frac{\partial f_1}{\partial (\vec{r} \cdot \vec{p})} \overbrace{\left(\vec{p} \cdot [\vec{r} \times \vec{n}] + \vec{p} [\vec{n} \times \vec{r}] \right)}^{0} = 0$

Аналогично, $\{f_2, (\vec{M} \cdot \vec{n})\} = \{f_3, (\vec{M} \cdot \vec{n})\} = 0$

И $\{f_1 \vec{r}, (\vec{M} \cdot \vec{n})\} = f_1 \{\vec{r}, (\vec{M} \cdot \vec{n})\} = f_1 [\vec{n} \times \vec{r}]$

$\{f_2 \vec{p}, (\vec{M} \cdot \vec{n})\} = f_2 \{\vec{p}, (\vec{M} \cdot \vec{n})\} = f_2 [\vec{n} \times \vec{p}]$

$\{[\vec{r} \times \vec{p}], (\vec{M} \cdot \vec{n})\} = \{\vec{M}, (\vec{M} \cdot \vec{n})\}$

$\{M_i, M_j n_j\} = n_j \{M_i, M_j\} = \varepsilon_{ijk} n_j M_k = [\vec{n} \times \vec{M}]_i \Rightarrow$

$\{f_3 [\vec{r} \times \vec{p}], (\vec{M} \cdot \vec{n})\} = f_3 [\vec{n} \times \vec{M}]$

Итого: $\{\vec{F}, (\vec{M} \cdot \vec{n})\} = f_1 [\vec{n} \times \vec{r}] + f_2 [\vec{n} \times \vec{p}] + f_3 [\vec{n} \times \vec{M}]$

2) $\{\vec{F}, \vec{M}^2\} = \{f_1\vec{r}, \vec{M}^2\} + \{f_2\vec{p}, \vec{M}^2\} + \{f_3[\vec{r}\times\vec{p}], \vec{M}^2\}$

$\{f_1\vec{r}, \vec{M}^2\} = f_1\{\vec{r}, \vec{M}^2\} + \vec{r}\{f_1, \vec{M}^2\}$

$\{f_1, \vec{M}^2\} = \{r_k, \vec{M}^2\}\dfrac{\partial f_1}{\partial r_k} + \{p_k, \vec{M}^2\}\dfrac{\partial f_1}{\partial p_k} = 2[\vec{M}\times\vec{r}]_k\left(2r_k\dfrac{\partial f_1}{\partial \vec{r}^2} + p_k\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\right) +$

$+ 2[\vec{M}\times\vec{p}]\left(2p_k\dfrac{\partial f_1}{\partial\vec{p}^2} + r_k\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}\right) = \cancel{4(\vec{r}\cdot[\vec{M}\times\vec{r}])\dfrac{\partial f_1}{\partial\vec{r}^2}} +$

$+ \cancel{4(\vec{p}\cdot[\vec{M}\times\vec{p}])\dfrac{\partial f_1}{\partial\vec{p}^2}} + 2(\vec{p}\cdot[\vec{M}\cdot\vec{r}])\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} + 2(\vec{r}\cdot[\vec{M}\cdot\vec{p}])\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})} =$

$= 2\dfrac{\partial f_1}{\partial(\vec{r}\cdot\vec{p})}(\vec{p}[\vec{M}\times\vec{r}] + \vec{r}[\vec{M}\times\vec{p}]) = 0$

Аналогично, $\{f_3, \vec{M}^2\} = \{f_2, \vec{M}^2\} = 0$

И $\{\vec{F}, \vec{M}^2\} = f_1\{\vec{r}, \vec{M}^2\} + f_2\{\vec{p}, \vec{M}^2\} + f_3\{[\vec{r}\times\vec{p}], \vec{M}^2\}$

$\{r_k, \vec{M}^2\} = \{r_k, M_e^2\} = 2M_e\{r_k, M_e\} = 2M_e\{r_k, \varepsilon_{ije} r_i p_j\} =$

$= 2M_e r_i \varepsilon_{ike} = 2\varepsilon_{kei} M_e r_i = 2[\vec{M}\times\vec{r}]_k \Rightarrow$

$\{\vec{r}, \vec{M}^2\} = 2[\vec{M}\times\vec{r}]$

при $i\neq k$ $\{p_k, r_i\} = 0$

$\{p_k, M^2\} = 2M_e\{p_k, M_e\} = 2M_e\{p_k, \varepsilon_{ije}\underline{r_i}p_j\} =$

$= 2M_e\varepsilon_{ije}p_j(-1) = 2\varepsilon_{kej}M_e p_j = 2[\vec{M}\times\vec{p}]_k \Rightarrow$

$\{\vec{p}, \vec{M}^2\} = 2[\vec{M}\times\vec{p}]$

$\{[\vec{r}\times\vec{p}], \vec{M}^2\} = \{\vec{M}, \vec{M}^2\} = 2\vec{M}\{\vec{M}, \vec{M}\} = 0$

Значит, $\{\vec{F}, \vec{M}^2\} = 2f_1[\vec{M}\times\vec{r}] + 2f_2[\vec{M}\times\vec{p}]$

№2 $L = \dfrac{m\dot{x}^2}{2} - \dfrac{m\omega^2x^2}{2}$, $\dot{x} \equiv \dfrac{dx}{dt}$

а) $p = \dfrac{\partial L}{\partial\dot{x}} = m\dot{x} \Rightarrow \dot{x} = \dfrac{p}{m}$

$H = p\dot{x} - L = \dfrac{p^2}{m} - L = \dfrac{p^2}{m} - \dfrac{m\overbrace{\left(\frac{p}{m}\right)}^{\dot{x}}{}^2}{2} + \dfrac{m\omega^2x^2}{2} = \dfrac{p^2}{2m} + \dfrac{m\omega^2x^2}{2}$

б) $a = \sqrt{\dfrac{m\omega}{2}}\left(x + i\dfrac{p}{m\omega}\right)$, $\bar{a} = \sqrt{\dfrac{m\omega}{2}}\left(x - i\dfrac{p}{m\omega}\right)$

$H = \dfrac{p^2}{2m} + \dfrac{m\omega^2x^2}{2} = \dfrac{m\omega^2}{2}\left(x^2 + \dfrac{p^2}{m^2\omega^2}\right) = \omega\cdot\left(\sqrt{\dfrac{m\omega}{2}}\right)^2\left(x + i\dfrac{p}{m\omega}\right)\left(x - i\dfrac{p}{m\omega}\right) =$

$= \underline{\omega\cdot a\cdot\bar{a}}$

в) $\{a, \bar{a}\} = \frac{m\omega}{2}\{x + \frac{ip}{m\omega}, x - \frac{ip}{m\omega}\} = \frac{m\omega}{2}\left(\ldots\right)$ [illegible]

$+ \frac{i}{m\omega}\{p, x\} - \frac{1}{m^2\omega^2}\{p, p\}\big) = -\frac{2i\, m\omega}{2m\omega} = -i$

Значит, $\{a, \bar{a}\} = -i$

$\{a, H\} = \{a, \omega a \bar{a}\} = \omega\{a, a\bar{a}\} = (\bar{a}\{a, a\} + a\{a, \bar{a}\})\omega = -i\omega a$

г) $a = \sqrt{\frac{m\omega}{2}}\left(x + \frac{ip}{m\omega}\right)$

$\frac{da}{dt} = \{a, H\} + \frac{\partial a}{\partial t} = -i\omega a$

$\int \frac{da}{a} = -\int i\omega dt \Rightarrow \ln a = -i\omega t + c$

$A = [\vec{B}\vec{r}]$ [illegible]

№3 $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x})$

а) $p_1 = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2c}y \Rightarrow \frac{p_1 + \frac{qB}{2c}y}{m} = \dot{x}$

$p_2 = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2c}x \Rightarrow \frac{p_2 - \frac{qB}{2c}x}{m} = \dot{y}$

$p_3 = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \frac{p_3}{m} = \dot{z}$

$H = p\dot{x} - L = p_1\dot{x} + p_2\dot{y} + p_3\dot{z} - \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{qB}{2c}(x\dot{y} - y\dot{x}) =$

$= p_1\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right) + p_2\left(\frac{p_2}{m} - \frac{qBx}{2cm}\right) + \frac{p_3^2}{m}$

$- \frac{m}{2}\left(\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right)^2 + \left(\frac{p_2}{m} - \frac{qBx}{2cm}\right)^2 + \left(\frac{p_3}{m}\right)^2\right) - \frac{qB}{2c}x\left(\frac{p_2}{m} - \frac{qBx}{2cm}\right) +$

$+ \frac{qB}{2c}y\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right) = (p_1^2 + p_2^2 + p_3^2)\frac{1}{m} + \frac{p_1 qBy}{2cm} - \frac{qBxp_2}{2cm}$

$- \frac{m}{2}\left(\frac{p_1^2}{m^2} + \frac{qp_1By}{cm^2} + \frac{q^2B^2y^2}{4c^2m^2} + \frac{p_2^2}{m^2} - \frac{qBxp_2}{cm^2} + \frac{q^2B^2x^2}{4c^2m^2} + \frac{p_3^2}{m^2}\right)$

$- \frac{p_2 qBx}{2c} + \frac{q^2B^2x^2}{4c^2m} + \frac{qByp_1}{2cm} + \frac{q^2B^2y^2}{4c^2m} =$

$= \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{q^2B^2}{8c^2m}(x^2 + y^2) + \frac{qB}{2cm}(p_1 y - p_2 x)$

б) $x(0)=y(0)=z(0)=0,\quad p_x(0)=p_z(0)=p,\quad p_y(0)=0$

$$\dot q_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot p_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\dot x = \frac{\partial H}{\partial p_1} = \frac{p_1}{m} + \frac{qB}{2mc}y \qquad \dot y = \frac{\partial H}{\partial p_2} = \frac{p_2}{m} - \frac{qB}{2mc}x$$

$$\dot z = \frac{\partial H}{\partial p_3} = \frac{p_3}{m} \Rightarrow \underbrace{z(t) = \frac{p}{m}t} \qquad (\text{т.к. } z(0)=0)$$

$$\dot p_1 = -\frac{\partial H}{\partial x} = -\frac{q^2B^2x}{4mc^2} + \frac{qB}{2mc}p_2$$

$$\dot p_2 = -\frac{\partial H}{\partial y} = -\frac{q^2B^2y}{4mc^2} - \frac{qB}{2mc}p_1$$

$$\dot p_3 = -\frac{\partial H}{\partial z} = 0 \Rightarrow p_3(t) = \text{const},\ \text{т.к. } p_3(0)=p, \text{ то } \underline{p_3(t)=p}$$

Обозначим $\frac{qB}{2mc} = s$

Тогда $\dot x = \frac{p_1}{m} + sy$, $\dot y = \frac{p_2}{m} - sx$, $\dot p_1 = -ms^2x + sp_2$

$\dot p_2 = -ms^2y - sp_1$

$$A = \begin{pmatrix} 0 & s & \frac{1}{m} & 0 \\ -s & 0 & 0 & \frac{1}{m} \\ -ms^2 & 0 & 0 & s \\ 0 & -ms^2 & -s & 0 \end{pmatrix} \qquad \chi_A(\lambda) = \begin{vmatrix} -\lambda & s & \frac{1}{m} & 0 \\ -s & -\lambda & 0 & \frac{1}{m} \\ -ms^2 & 0 & -\lambda & s \\ 0 & -ms^2 & -s & -\lambda \end{vmatrix} =$$

$$= \lambda^2 s^2\left(\frac{\lambda^2}{s^2}+4\right)$$

С.З. $\lambda_1 = 0 \quad \lambda_{2,3} = \pm 2is$

1) $\underline{\lambda_1 = 0}$ $\quad u = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$Au = \begin{pmatrix} sb + \frac{c}{m} \\ -sa + \frac{d}{m} \\ -ms^2a + sd \\ -ms^2b - sc \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_{11} = \begin{pmatrix} 0 \\ -\frac{1}{ms} \\ 1 \\ 0 \end{pmatrix} \quad u_{12} = \begin{pmatrix} \frac{1}{ms} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2) $\lambda_2 = 2is$

$$Au = \begin{pmatrix} 2isa \\ 2isb \\ 2isc \\ 2isd \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} -i \\ \frac{1}{ms} \\ ims \end{pmatrix} = \begin{pmatrix} -\frac{1}{ms} \\ -\frac{i}{ms} \\ -i \\ 1 \end{pmatrix}$$

3) $\lambda_3 = -2is$

$$Au = \begin{pmatrix} -2isa \\ -2isb \\ -2isc \\ -2isd \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{1}{ms} \\ \frac{i}{ms} \\ i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ -\frac{1}{ms} \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{ms} \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -\frac{1}{ms} \\ -\frac{i}{ms} \\ -i \\ 1 \end{pmatrix} e^{2ist} + c_4 \begin{pmatrix} -\frac{1}{ms} \\ \frac{i}{ms} \\ i \\ 1 \end{pmatrix} e^{-2ist}$$

Из начальных условий:

$$\begin{cases} 1)\ x(0) = \frac{c_2}{ms} - \frac{c_3}{ms} - \frac{c_4}{ms} = 0 \\ 2)\ y(0) = -\frac{c_1}{ms} - \frac{ic_3}{ms} + \frac{ic_4}{ms} = 0 \\ 3)\ p_1(0) = c_1 - ic_3 + c_4 i = p \\ 4)\ p_2(0) = c_2 + c_3 + c_4 = 0 \end{cases}$$

$ms\cdot 1) + 4) \Rightarrow \underbrace{c_2 = 0}$, $c_3 + c_4 = 0$

$ms\cdot 2) - 3)$ $\cancel{\ldots}$ $-2c_1 = -p \Rightarrow \underbrace{c_1 = \frac{p}{2}}$

$3)$: $\frac{p}{2} - 2ic_3 = p \Rightarrow \underbrace{c_3 = \frac{ip}{4}} \Rightarrow \underbrace{c_4 = -\frac{ip}{4}}$

Значит,
$$\begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{p}{2ms}\sin(2st) \\ \frac{p}{2ms}(\cos(2st) - 1) \\ \frac{p}{2}(1+\cos(2st)) \\ -\frac{p}{2}\sin(2st) \end{pmatrix} = \begin{pmatrix} \frac{pc}{qB}\sin\left(\frac{qB}{mc}t\right) \\ \frac{pc}{qB}\left(\cos\left(\frac{qBt}{mc}\right) - 1\right) \\ \frac{p}{2}\left(1+\cos\left(\frac{qBt}{mc}\right)\right) \\ -\frac{p}{2}\sin\left(\frac{qBt}{mc}\right) \end{pmatrix}$$

и, как выяснили раньше,

$p_3(t) = p$, $z(t) = \frac{pt}{m}$

в) $\vec{v} = (v_1, v_2, v_3) = (\dot{x}, \dot{y}, \dot{z})$

$\vec{v} = \left(\frac{p_1}{m} + sy, \frac{p_2}{m} - sx, \frac{p_3}{m}\right)$

$\{v_1, v_2\} = \underbrace{\{\frac{p_1}{m}, \frac{p_2}{m}\}}_{0} - \{\frac{p_1}{m}, sx\} + \{sy, \frac{p_2}{m}\} - \underbrace{\{sy, sx\}}_{0} =$

$= -\frac{s}{m}\underbrace{\{p_1, x\}}_{-1} + \frac{s}{m}\underbrace{\{y, p_2\}}_{1} = \frac{2s}{m} = \frac{qB}{m^2c}$

$\{v_1, v_3\} = \{\frac{p_1}{m}, \frac{p_3}{m}\} + \frac{s}{m}\{y, p_3\} = 0$

$\{v_2, v_3\} = \{\frac{p_2}{m}, \frac{p_3}{m}\} - \frac{s}{m}\{x, p_3\} = 0$

Значит, $\{v_1, v_2\} = -\{v_2, v_1\} = \frac{qB}{m^2c}$, $\{v_1, v_3\} = \{v_3, v_1\} = \{v_2, v_3\} = \{v_3, v_2\} = 0$

№4 $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m\omega^2}{2}(x^2+y^2)$

а) $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$, аналогично $\dot{y} = \frac{p_y}{m}$

$H = p\dot{x} - L = \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\omega^2}{2}(x^2+y^2) =$

$= \boxed{\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)}$

б) $J_1 = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{m\omega^2}{2}(x^2 - y^2)$

$J_2 = \frac{1}{m}p_x p_y + m\omega^2 xy$

$J_3 = \omega(x p_y - y p_x)$

$\frac{dJ_1}{dt} = \{J_1, H\} + \overset{\nearrow 0}{\frac{\partial J_1}{\partial t}} = \{J_1, H\}$

$\{J_1, H\} = \{\frac{1}{2m}(p_x^2 - p_y^2) + \frac{m\omega^2}{2}(x^2-y^2), \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)\} =$

$= \{(\frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}) - (\frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2}), (\frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}) + (\frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2})\} =$

$= \{\frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}, \frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2}\} - \{\frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2}, \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}\} =$

$= 2\{\frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}, \frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2}\} = 0$

$\frac{dJ_1}{dt} = 0 \Rightarrow J_1(t)$ - const, интеграл движения

$\frac{dJ_2}{dt} = \frac{\partial J_2}{\partial x}\dot{x} + \frac{\partial J_2}{\partial p_1}\dot{p}_1 + \frac{\partial J_2}{\partial y}\dot{y} + \frac{\partial J_2}{\partial p_2}\dot{p}_2 = \frac{m\omega^2 y \cdot p_1}{m} + \frac{p_2}{m}(-m\omega^2 x) +$

$+ \frac{m\omega^2 x \, p_2}{m} + \frac{p_1}{m}(-m\omega^2 y) = 0$

$\frac{dJ_3}{dt} = \frac{\partial J_3}{\partial x}\dot{x} + \frac{\partial J_3}{\partial p_1}\dot{p}_1 + \frac{\partial J_3}{\partial y}\dot{y} + \frac{\partial J_3}{\partial p_2}\dot{p}_2 = \frac{p_2 \omega p_1}{m} + (-\omega y)(-m\omega^2 x) +$

$+ (-\omega p_1)\frac{p_2}{m} + \omega x(-m\omega^2 y) = 0$

в) Покажем, что $\cancel{\{L_\alpha, L_\beta\}}$ $\{J_\alpha, J_\beta\} \in \mathcal{L}$ ($\alpha, \beta = 1, 2$ или 3)

$\{J_1, J_2\} = \{J_1 + H, J_2\} = \{\frac{p_x^2}{m} + m\omega^2 x^2, \frac{p_x p_y}{m} + m\omega^2 xy\} =$

$= \frac{2p_x}{m}\{p_x, xy\}\omega^2 m + \frac{2m\omega^2 x}{m}\{x, p_x p_y\} =$

$= -2p_x\omega^2 y + 2\omega^2 x p_y = 2\omega \cdot \omega(xp_y - yp_x) - 2\omega J_3 \in \mathcal{L}$

$\{J_2, J_3\} = \{J_2 + H, J_3\} = \{\frac{(p_x+p_y)^2}{2m} + \frac{m\omega^2(x+y)^2}{2}, xp_y - yp_x\}\omega =$

$= \frac{2\omega(p_x+p_y)}{2m}\{p_x+p_y, xp_y - yp_x\} + m\omega^2(x+y)\{x+y, xp_y - yp_x\}$

$= 2\omega\left(\frac{p_x^2 - p_y^2}{2m} + \frac{m\omega^2(x^2-y^2)}{2}\right) = 2\omega J_1 \in \mathcal{L}$

$\{J_1, J_3\} = \{J_1 + H, J_3\} = \{\frac{p_x^2}{m} + m\omega^2 x^2, xp_y - yp_x\}\omega =$

$= \frac{\omega}{m} 2p_x\{p_x, p_y x\} + 2mx\omega^3\{x, -yp_x\} = -2\omega\left(\frac{p_x p_y}{m} + m\omega^2 xy\right) =$

$= -2\omega J_2 \in \mathcal{L}$

что и требовалось. (для $\{J_\alpha, J_\alpha\} = 0$ - очевидно)

№5 $H = \frac{\vec{M}^2}{2I} - \gamma\vec{M}\cdot\vec{B}$ $\quad \vec{B} = (0, 0, B)$

Заметим, что $\frac{dM_i}{dt} = \{M_i, H\} = \frac{1}{2I}\{M_i, M_j^2\} - \gamma\{M_i, \vec{M}\cdot\vec{B}\} =$

$= \frac{1}{2I}\left(2M_j\{M_i, M_j\}\right) - \gamma\{M_i, M_j b_j\} = \frac{1}{2I}\cdot 2M_j \varepsilon_{ijk} M_k -$

$-\gamma \underbrace{b_j\{M_i, M_j\}}_{b_j \varepsilon_{ijk} M_k} = \frac{1}{2I}\cdot 2[\vec{M}\times\vec{M}]_i - \gamma\cdot[\vec{B}\cdot\vec{M}]_i = -\gamma[\vec{B}\times\vec{M}]_i$

$\frac{dM_i}{dt} = -\gamma[\vec{B}\times\vec{M}]_i$

$[\vec{B}\times\vec{M}] = (-BM_2, BM_1, 0)$ $\quad \dot{M}_1 = \gamma BM_2, \ \dot{M}_2 = -\gamma BM_1, \ \dot{M}_3 = 0$

$M_3(t) = const$

$A = \begin{pmatrix} 0 & \gamma B \\ -\gamma B & 0 \end{pmatrix}$

$\chi_A = \lambda^2 + \gamma B$

$\lambda_{1,2} = \pm i\gamma B$

$\underline{\lambda_1 = i\gamma B}$

$Au_1 = \gamma B\begin{pmatrix} u_{12} \\ -u_{11} \end{pmatrix} = \gamma B\begin{pmatrix} u_{11} i \\ u_{12} i \end{pmatrix} \Rightarrow u_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda_2 = -i\gamma B$ $\quad Au_2 = \gamma B\begin{pmatrix} u_{22} \\ -u_{21} \end{pmatrix} = \gamma B\begin{pmatrix} -u_{21} i \\ -u_{22} i \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = c_i u_i e^{\lambda_i t} = \begin{pmatrix} c_1 e^{i\gamma Bt} + c_2 e^{-i\gamma Bt} \\ c_1 i e^{i\gamma Bt} - c_2 i e^{-i\gamma Bt} \end{pmatrix} \to$$

$$M_1(t) = c_1 \cos(\gamma Bt) + c_2 \sin(\gamma Bt)$$

$$M_2(t) = c_2 \cos(\gamma Bt) - c_1 \sin(\gamma Bt)$$

$$M_3(t) = \mathrm{const} = M_3$$

в) $\{a, \bar{a}\} = \frac{m\omega}{2}\{x + \frac{ip}{m\omega}, x - i\frac{p}{m\omega}\} = \frac{m\omega}{2}(\underbrace{\{x,x\}}_{=0} - \frac{i}{m\omega}\underbrace{\{x,p\}}_{=1} +$

$+ \frac{i}{m\omega}\overbrace{\{p,x\}}^{-1} + \frac{1}{m^2\omega^2}\overbrace{\{p,p\}}^{0}) = -\frac{2im\omega}{2m\omega} = -i$

Значит, $\{a, \bar{a}\} = \underline{-i}$

$\{a, H\} = \{a, \omega a\bar{a}\} = \omega\{a, a\bar{a}\} = (\bar{a}\overbrace{\{a,a\}}^{0} + a\overbrace{\{a,\bar{a}\}}^{-i})\omega = \underline{-ia\omega}$

г) ~~$a = \sqrt{\frac{m\omega}{2}}(x + \frac{ip}{m\omega})$~~

$\frac{da}{dt} = \{a, H\} + \overbrace{\frac{\partial a}{\partial t}}^{0} = -i\omega a$

$\int \frac{da}{a} = -\int i\omega dt \Rightarrow \ln a = -i\omega t + c$

№3 $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x})$ $B = |\vec{B}|$ c – ск. света

а) $p_1 = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2c}y \Rightarrow \frac{p_1 + \frac{qBy}{2c}}{m} = \dot{x}$

$p_2 = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2c}x \Rightarrow \frac{p_2 - \frac{qBx}{2c}}{m} = \dot{y}$

$p_3 = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \frac{p_3}{m} = \dot{z}$

$H = p\dot{x} - L = p_1\dot{x} + p_2\dot{y} + p_3\dot{z} - \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{qB}{2c}(x\dot{y} - y\dot{x}) =$

$= p_1\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right) + p_2\left(\frac{p_2}{m} - \frac{qBx}{2cm}\right) + \frac{p_3^2}{m} -$

$- \frac{m}{2}\left(\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right)^2 + \left(\frac{p_2}{m} - \frac{qBx}{2cm}\right)^2 + \left(\frac{p_3}{m}\right)^2\right) - \frac{qB}{2c}x\left(\frac{p_2}{m} - \frac{qBx}{2cm}\right) +$

$+ \frac{qB}{2c}y\left(\frac{p_1}{m} + \frac{qBy}{2cm}\right) = (p_1^2 + p_2^2 + p_3^2)\frac{1}{m} + \frac{p_1 qBy}{2cm} - \frac{qBxp_2}{2cm} -$

$- \frac{m}{2}\left(\frac{p_1^2}{m^2} + \frac{qp_1By}{cm^2} + \frac{q^2B^2y^2}{4c^2m^2} + \frac{p_2^2}{m^2} - \frac{qBxp_2}{cm^2} + \frac{q^2B^2x^2}{4c^2m^2} + \frac{p_3^2}{m^2}\right) -$

$- \frac{p_2 qBx}{2c} + \frac{q^2B^2x^2}{4c^2m} + \frac{qByp_1}{2cm} + \frac{q^2B^2y^2}{4c^2m} =$

$= \underline{\frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{q^2B^2}{8c^2m}(x^2 + y^2) + \frac{qB}{2cm}(p_1 y - p_2 x)}$