EM Algorithm

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1 Assigment 1 - Estimation of the shape without rotation

The first assignment was to implement the EM algorithm to estimate the shape s of an object without rotation, by finding the maximum likelihood of the parameters η_0 and η_1 of the Bernoulli distribution (η_0 for background and η_1 for the forground).

The EM algorithm is an iterative algorithm that alternate between the E-step and the M-step.

The goal is to maximize the probability :

Proof that all we need form the data is the average image $\psi = \frac{1}{m}\sum_{x\in T^m} x$

$$\frac{1}{m}\sum_{x\in T^m} p(x;s,\eta) -> \max_{s,\eta} \tag{1}$$

$$\log p(x;s,\eta) = \langle x, \eta(s)\rangle - n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1}) \tag{2}$$

Subsituting (2) in (1) :

$$\begin{aligned}\frac{1}{m}\sum_{x\in T^m}&(\langle x, \eta(s)\rangle - n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1}))\\ &= \langle \psi, \eta(s)\rangle - n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1})\end{aligned} \tag{3}$$

E-step

In the E-step, we compute the posterior probability of the hidden variable s given the data ψ and the current estimate of the parameters η. To begin, we set the initial value of the η : $\eta_0 = 0$ $\eta_1 = 1$. We return the estimated value of the hidden variable s and move on with the M-step.

M-step

In this step, we compute the new estimate of the parameters η given the data ψ and the current estimate of the hidden variable s.

$$\eta_0 = \frac{1}{n_0}\sum \psi(1-s)$$

$$\eta_1 = \frac{1}{n_1}\sum \psi s$$

Where n_0 is the number of pixels in the background and n_1 is the number of pixels in the foreground.

We return the new estimate of the parameters η and move on with the E-step.

Results

The algorithm converge and find the right shape of the object (fig. 1) with $\eta_0 = 0.5673$ and $\eta_1 = 0.3436$.

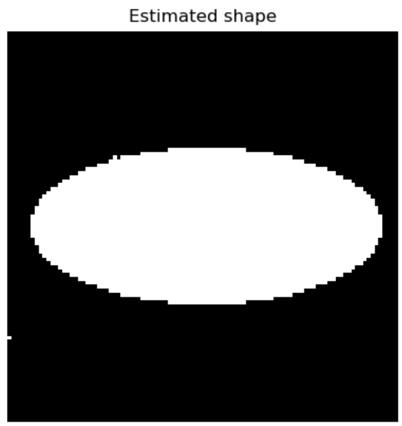

Figure 1: Shape of assignment 1

2 Assigment 2 - Estimation of the shape with rotation

The second assignment was to implement the EM algorithm to estimate the shape s of an object with rotation. There would be 4 possible rotation : 0, 90, 180 and 270 degrees.

Again, we will use the EM algorithm to estimate the shape of the object and the rotation.

E-step

The probability of $\alpha_x(r) = p(r|x; s, \eta)$
The posterior probabilities of $\alpha_x(r)$ can be computed by softmax function:

$$\alpha_x(r) = softmax_r(\langle x, \eta(T_r s)\rangle + log\pi_r)$$

Proof :

$$\begin{aligned}
\alpha_x(r) &= \frac{p(x|r; s, \eta)p(r)}{\sum p(r|x; s, \eta)p(r)} \\
&= \frac{e^{\langle x,\eta(T_r s)\rangle - n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1})}\pi_r}{\sum e^{\langle x,\eta(T_r s)\rangle - n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1})}\pi_r} \\
&= \frac{e^{\langle x,\eta(T_r s)\rangle + \log\pi_r} e^{-n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1})}}{\sum e^{\langle x,\eta(T_r s)\rangle + log\pi_r} e^{-n_{s=0}\log(1+e^{\eta_0}) - n_{s=1}\log(1+e^{\eta_1})}} \\
&= \frac{e^{\langle x,\eta(T_r s)\rangle + \log\pi_r}}{\sum e^{\langle x,\eta(T_r s)\rangle + \log\pi_r}} \\
&= softmax_r(\langle x, \eta(T_r s)\rangle + \log\pi_r)
\end{aligned}$$

M-step

In this step, we compute the new estimate of the parameters η and π given the data ψ and the current estimate of the hidden variable s and r.

The maximiser for the pose probabilities π_r can be found in closed form using the Lagrange multiplier method:

Lagrangian :

$$\mathcal{L} = \frac{1}{m} \sum_{x \in T^m} \sum_{r \in R} \alpha_x(r) + \lambda \left(\sum_{r \in R} \pi_r - 1 \right)$$

We take the derivative of the Lagrangian with respect to π_r and λ and set them to zero.

$$\frac{\partial \mathcal{L}}{\partial \pi_r} = \frac{1}{m} \sum_{x \in T^m} \frac{\alpha_x(r)}{\pi_r} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{r \in R} \pi_r - 1 = 0$$

If we solve the equations above, we get :

$$\pi_r = \frac{1}{m} \sum_{x \in T^m} \alpha_x(r)$$

$$\lambda = \sum_{r \in R} \alpha_x(r) = 1$$

Now we can fix π_r and maximize the parameters η and the shape s, by using the same method as in the first assignment, using *shape_mle*().

Proof that all you need from the data is the array :

$$\psi = \frac{1}{m} \sum_{x \in T^m} \sum_{r \in R} \alpha_x(r) T_r^T x$$

$$\begin{aligned}
&\frac{1}{m} \sum_{x \in T^m} \sum_{r} \alpha_x(r) (\langle T_r^T x, \eta(s) \rangle - n_{s=0} \log(1 + e^{\eta_0}) - n_{s=1} \log(1 + e^{\eta_1})) \\
= &\left\langle \frac{1}{m} \sum_{x \in T^m} \sum_{r} \alpha_x(r) T_r^T x, \eta(s) \right\rangle - n_{s=0} \log(1 + e^{\eta_0}) - n_{s=1} \log(1 + e^{\eta_1}) \\
&= \langle \psi, \eta(s) \rangle - n_{s=0} \log(1 + e^{\eta_0}) - n_{s=1} \log(1 + e^{\eta_1})
\end{aligned}$$

Results

The algorithm converge and find the right shape of the object (fig. 2) with $\eta = [0.4516,\ 0.5502]$ and $\pi = [0.3\ 0.3\ 0.2\ 0.2]$.

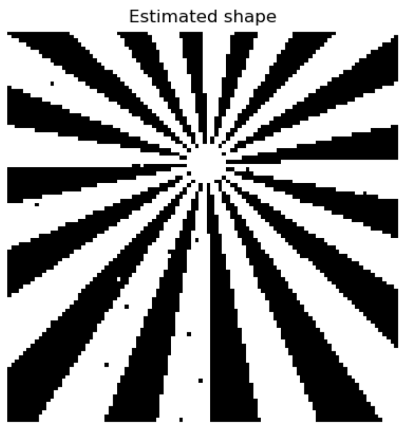

Figure 2: Shape of assignment 2