Discrete Mathematics

Er. Meenakshi Rana

→ It is a kind of mathematics in which we will study about discrete values.

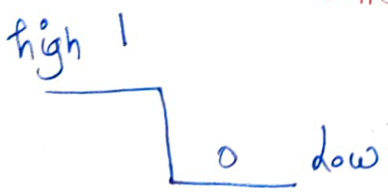
No. of students in your class? → Particular value 70 or 72

Height of students in your class?
↘ Continuous value, vary from student to student.

Today is friday → True, false
Yes, no
1, 0

In computer sience, we call
Digital → Discrete
Analog → Continuous

→ Why? we study
→ : Logic making and problem-solving capabilities.
In CS: → Better understanding of digital computers.



• **Propositions**

→ A proposition is a statement that is either true or false, but not both.

Decterative: this is apple
Imperative: eat this apple
Interrogative: is this apple?

Today is friday
Washington is the capital of America

True, false
Truth value of preposition

Notation :-
Preposition can be denoted by any letter
p, q, r, s, t

p: Today is friday
q: Washington is the capital of America.

- Propositional logic :

The area of logic that deals with prepositions.

" Elephants are bigger than mice."

Is this a statement ? — Yes

Is this a proposition? — Yes

What is the truth value of the proposition? — True

(2) NEGATION :->

→ is opposite to proposition

p: Today is friday
$\neg p$: Today is not friday
or It is not the case that today is friday.

...ive or (XOR)
...plication (if - then)
Biconditional (if and only if)

Conjuction (Binary operator)

p : "Today is friday"
q : "It is raining today".
$p \wedge q$: "Today is friday and it is raining today".

→ The proposition is true when this condition happens.

→ Let p and q be propositions. The conjuction of p and q, denoted by $p \wedge q$, is the proposition "p and q" which is true when both p and q are true and is false otherwise.

Truth Table :→

The truth Table for the conjuction of 2 propositions

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (Binary operator)

→ <u>inclusive or</u>: The disjunction is true when atleast of the 2 propositions is true.

"Students who have taken calculus or computer science can take this class."

⇒ one cond. shud be true or both shud be true.

exclusive or :⇒ The disjunction is true only when one of the proposition is true.

" Ice cream or pudding will be served after lunch."

→ Let p and q be propositions. The <u>disjunction</u> of p and q, denoted by $p \vee q$, is the proposition "p or q" which is false when both p and q are false and is true otherwise.

Truth Table

The truth Table for disjunction of 2 propositions → Inclusive or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The truth Table for the Exclusive or (XOR) of 2 Propositions

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

it is not necessary that every time, have to convert a positive statement to [ne]gative statement. We can also negate a negative statement to positive statement. ②

→ Defination of Negation

→ Let p be a proposition. The negation of p, denoted by $\neg p$ is the statement "It is not the case that p."

→ Truth Table :→

Method to show the relationship of propositions

Truth Table for the Negation

p	$\neg p$
T	F
F	T

" 520 < 111 "

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

"$y > 5$"

Is this a statement? yes

Is this a proposition? no (we don't know the value of y)

Truth value?

Depends on the value of y, but this value is not specified.

Ex. "What time is it?"

Is this a statement? no

Is this a proposition? no

Condition:→

* A proposition has to be a statement.

- Predicate (दावा करना)

Property of Subject

→ Expression that contain variables

He goes to school

first part — Subject

Second Part — Predicate — Property of subject

$x > 5$

P(9) True

P(2) false

It is not proposition

$7 > 5$ → True

$2 > 5$ ✗ false

versal Quantifier

(x) is true for all values of x in the universe of discourse.

↓

for all

$\forall$ $\quad \forall x (Px)$

Existential Quantifier

There exists a value of x in the universe of discourse such that $P(x)$ is true.

↓

$\exists$ $\quad \exists x P(x)$

Q- What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the universe of discourse consists of positive integers not exceeding 4?

$0^2 < 10$ $\quad 1^2 < 10$ $\quad 2^2 < 10$ $\quad 3^2 < 10$

$0 < 10$ ✓ $\quad 1 < 10$ ✓ $\quad 4 < 10$ ✓ $\quad 9 < 10$

$4^2 \not< 10$

$16 \not< 10$

false ←

"$y > 5$"

Is this a statement? yes

Is this a proposition? no (we don't know the value of y)

Truth value?

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Subject

Second Part
Predicate
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$4^2 \not< 10$

$\cancel{16 < 10}$

false ←

Q- What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and universe of discourse consists of +ve integers not exceeding 4?

True

$0^2 < 10$

$0 < 10$

---

logical operators (connectives)

p: Today is friday

q: It is raining today

Today is friday and It is raining today

logical operators / Connectives

Compound Proposition

p: Today is friday

q: Today is Monday.

Today is friday or Today is Monday.

→ They are used to form new compound propositions from 2 or more existing propositions. The logical operators are also called as

Connectives

Types

→ Conjuction (AND)

→ Disjunction (OR)

e of duality :→

Two formulas A_1 and A_2 are said to be dual of each other if either one can be obtained from the either by replacing $\wedge$ (AND) by $\vee$ (OR) and $\vee$(OR) by $\wedge$ (AND). Also if the formula contains T(True) or F (false), then we replace T by F and F by T to obtain the dual.

Note → 1:→ The 2 connectives $\wedge$ and $\vee$ are called dual of each other

2. Like AND and OR $\uparrow$ (NAND) and $\downarrow$ (NOR) are dual of each other.

Example:→ Determine the dual of each of the following:

(a) $p \wedge (q \wedge r)$

$= p \vee (q \vee r)$

(b) $\neg p \vee \neg q$

$= \neg p \wedge \neg q$

(c) $(p \wedge \neg q) \vee (\neg p \wedge q)$

$= (p \vee \neg q) \wedge (\neg p \vee q)$

~~(d) $(p \downarrow q) \downarrow (p \downarrow q)$~~

(d) $(p \uparrow q) \uparrow (p \uparrow q)$

$= (p \downarrow q) \downarrow (p \downarrow q)$

(c) $((\neg p \vee q) \wedge (q \wedge \neg s)) \vee (p \vee F)$, here F means false

$= ((\neg p \wedge q) \vee (q \vee \neg s)) \wedge (p \wedge T)$

• **Functionally complete set of connectives :->**

→ Connectives are $\wedge$, $\vee$, $\sim$, $\rightarrow$ and $\leftrightarrow$

→ A set of connectives is called functionally complete if every formula can be expressed in terms of equivalent formula containing the connectives from this set.

Q:-> Write an equivalent formula for $P \wedge (R \Leftrightarrow S) \vee (S \Leftrightarrow P)$ which does not involve biconditional.

Ans:-> We know that $P \Leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$ — (1)

Apply eqn (1) to this :->

$P \wedge (R \Leftrightarrow S) \vee (S \Leftrightarrow P)$ which does not involve biconditional.

$= [P \wedge ((R \rightarrow S) \wedge (S \rightarrow R)) \vee (S \rightarrow P) \wedge (P \rightarrow S))]$

ers

Proposition can be true or false

Predicate can not be true or false until value is not assigned to variable.

→ Predicate becomes Proposition if value is assigned to variable.

→ Propositional fxn $P(x)$

$P(x) : x > 3$, then what is truth value of $P(4)$

$4 > 3$

true

Quantifiers :→ $x < 5$

2nd way :→ all values less than 5, x will be true. (values not exceeding 4)

Universe of discourse (ie. range)

→ A quantifier is an operator used to create a proposition from a propositional function. An other way if we do not assign values to $p(x)$

→ Well formed formula

Propositional logic uses a symbolic "language" represent the logical structure of a compound proposition.

Like any language, this language also has grammatical rules for putting symbols together in the right way.

Any expression that obeys the syntactic rules of propositional logic is called a well formed formula.

Rules of Propositional logic:→

- Any capital letter by itself is a WFF.
- Any WFF can be prefixed with "¬" (Negation) The result will be a WFF.
- if A and B are WFF, then $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$ are WFFs.

WFF	non-WFF
$\neg(P \wedge Q)$	$\neg P \wedge Q$
$\neg(P \vee Q)$	
$P \rightarrow (P \vee Q)$	$(P \rightarrow Q) \rightarrow (\ \wedge Q)$
$P \rightarrow Q \rightarrow R$	$(P \rightarrow Q$
	$(P \wedge Q) \rightarrow Q$

Normal form

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Let $A(P_1, P_2, P_3 \cdots P_n)$ be a statement formula then the construction of truth table may not be practical always.

So, we consider alternate procedure known as reduction to normal form.

(1) Disconjuction Normal form :⇒

A statement form which consist of disjunction b/w conjuction (and) is called DNF.

eg. (1) $(p \wedge q) \vee r$

(2) $(p \wedge \neg q) \overset{or}{\vee} (\neg p \wedge r) \overset{or}{\vee} (r \wedge \neg q)$

Q.:→ Obtain the DNF of form $(p \to q) \wedge (\neg p \wedge q)$

Sol:⇒ We know that

$$p \to q \Leftrightarrow \neg p \vee q$$

So $(\neg p \vee q) \wedge (\neg p \wedge q)$

apply distributive law

$$(\neg p \wedge \neg p \wedge q) \overset{or}{\vee} (q \wedge \neg p \wedge q)$$

$$= (\neg p \wedge q) \vee (q \wedge \neg p)$$

disjunction

• Conjuction Normal form

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→ A statement which consist of conjuction b/w disjunction is called CNF
→ conjuction b/w 2 disjunction

eg ÷ ① $p \wedge q$ (and) ② $(\neg p \vee q) \wedge (\neg p \vee r)$ (and)

Q:→ Obtain the CNF of the form
$(p \wedge q) \vee (\neg p \wedge q \wedge r)$

Solution: $(p \wedge q) \vee (\neg p \wedge q \wedge r)$

using distributive law
$(p \vee (\neg p \wedge q \wedge r)) \wedge (q \vee (\neg p \wedge q \wedge r))$

$= [(p \vee \neg p) \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \neg p) \wedge (q \vee q) \wedge (q \vee r)]$

$\Rightarrow (p \vee q) \wedge (p \vee r) \wedge [(q \vee \neg p) \wedge q \wedge (q \vee r)]$ ans)

Q→ ①. Obtain DNF of $p \vee (\neg p \rightarrow (q \vee (q \rightarrow \neg r))$

Solution:- $p \vee (\neg p \rightarrow (q \vee (\neg q \vee \neg r)))$

$= p \vee (p \vee (q \vee (\neg q \vee \neg r))$

$\Rightarrow p \vee (p \vee (q \vee (\neg q \vee q \vee \neg r)))$

$= p \vee (p \vee (q \vee \neg r))$

$= p \vee (p \vee q \vee \neg r)$

$= p \vee p \vee p \vee q \vee p \vee \neg r$

$= p \vee q \vee \neg r$

$p \vee (\neg p \to (q \vee (q \to \neg r))$

$p \vee (\neg p \to (q \vee (\neg q \vee \neg r))$

$= p \vee (\neg p \to (\underbrace{q \vee \neg q}_{1} \vee q \vee \neg r))$

$= p \vee (\neg p \to (q \vee \neg r)) \to$ $\quad \neg(p \to q)$ $\quad \neg p \vee q$

$= p \vee (\neg(\neg p \vee (q \vee \neg r))$

$= \quad p \vee (p \vee (q \vee \neg r)$

$= p \vee (p \vee q \vee p \vee \neg r)$ idempoted law

$= \quad p \vee (p \vee q \vee \neg r)$

$= \quad p \vee p \vee p \vee q \vee p \vee \neg r$

$p \vee q \vee \neg r$

disjunction

Q-2. Obtain CNF of $(p \to q) \wedge (q \vee (p \wedge r))$ and determine whether or not its tautology.

Sol: We know that $p \to q \Leftrightarrow \neg p \vee q$

$= (\neg p \vee q) \wedge (q \vee (p \wedge r))$

$= (\neg p \vee q) \wedge (q \vee p) \wedge (q \vee r)$

conjuction

Con:

Truth Table

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p	q	r	$\neg p$	$\neg p \vee q$	$q \vee p$	$q \vee r$	$\neg(p \vee q)$ $(q \vee p)$ $(q \vee r)$
T	T	T	F	T			
T	T	F	F	T			
T	F	T	F	F			
T	T	F	F	T			
F	F	T	T	T			
F	T	F	T	T			
F	F	T	T	T			
F	F	F	T	T			

3) find CNF of $p \wedge (p \to q)$

$$(p \to q = \neg p \vee q)$$

$$= p \wedge (\neg p \vee q)$$

$$= (p \wedge \neg p) \vee (p \wedge q)$$

$$= 1 \vee (p \wedge q)$$

$$= \underline{p \wedge q} \to \text{and}$$

do we study Predicate in DM? (8)

↓

to deal with variables

Predicate is also called as Propositional fxn.

Denote : P (Variables)

What is the truth value of P(a) and P(2)?

→ Discussed above.

Note :→ if a value is assigned to the variable x, then $P(x)$ becomes a proposition.

Q:→ Consider the propositional function $Q(x, y, z)$ defined by :

$x + y = z$

What is the truth value of $Q(2,3,5)$?

→ $2+3=5$

$5=5$

True

What is the truth value of $Q(0,1,2)$?

→ $0+1=2$

$1=2$ → false

Rules of Inference for propositional logic

Rules of Inference :→ are the templates for constructing valid arguments.
(1)

↓

deriving conclusions from evidences.

Types of Inference Rules:

/ Elementary valid Argument form

(1) Modus Ponenes or Rule of detachment

$$\begin{array}{ll} \overset{T}{p} \rightarrow \overset{T}{q} & \therefore T \checkmark \\ p & T \checkmark \\ \hline \therefore q \ T & \end{array}$$

if p is true

valid

if p is True q has to be true.

Conclusion must be true

p	q	p → q
T	T	T ✓
T	F	F
F	T	T ✓
F	F	T ✓

OR $[(p \rightarrow q) \wedge p] \rightarrow q$

argument

(2.) Modus Tollens :→

$$\begin{array}{ll} \overset{F}{p} \rightarrow \overset{F}{q} & T \\ \neg q & T \\ \hline \therefore \neg p & \cancel{F} \ T \\ \hline \end{array}$$

OR $[p \rightarrow q) \wedge \neg q] \rightarrow \neg p$

Hence it is valid argument

③ Hypothetical syllogism

$$\begin{array}{l} \quad\;\; T \quad\;\; T \rightarrow \text{has to be true} \\ \quad\;\; p \rightarrow q \quad T \\ T\; q \rightarrow r \quad F/T \quad T \\ \hline T\; p \rightarrow r \quad T \quad (F\;T) \end{array}$$

we will try to make value false. Lets see if we can do this.

Let make $p \rightarrow r$ false.

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

④ Disjunctive syllogism:

$$\begin{array}{l} F\; p \vee q^{T} \quad T \\ \sim p \quad\; T \\ \hline \therefore q \quad T \end{array}$$

or

$$[p \vee q] \wedge \neg p \rightarrow q$$

⑤ Addition.

$$\begin{array}{l} p \;\; T \\ \hline \therefore p \vee q \\ \;\; T \quad T = T \end{array}$$

or $p \rightarrow (p \vee q)$

⑥ Simplification

$$\begin{array}{l} \overset{T}{p} \wedge \overset{T}{q} \; T \\ \hline p_T \end{array}$$

or

$$\begin{array}{l} \overset{T}{p} \wedge \overset{T}{q} \\ \hline q \; T \end{array}$$

$(p \wedge q) \to p$ or $(p \wedge q) \to q$

(2)

(7) Conjunction

$p \quad T$

$q \quad T$

$\therefore \; p \wedge q = T$

$\quad T \quad T$

Hence it is valid argument

$[(p) \wedge (q)] \to (p \wedge q)$

(8) Resolution :-

let's take

$p \vee q \quad T$ (p: T/F, q: F)

$\neg p \vee r \quad T$ (p: F/T, r: T)

$q \vee r = T$ (q: F, r: T)

or $[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$

(9) Constructive dilemma

$(p \to q) \wedge (r \to s)$

$p \vee r$

$q \vee s$

(10) Destructive dilemma

$(p \to q) \wedge (r \to s)$

$\neg q \vee \neg s$

$\neg p \vee \neg r$

Q:→ If Ram works hard, then he will get a job
Ram works hard. ∴, he will get a job.
check validity.

Sol'→ p : Ram works hard

q : He will get a job

Premises: $p \to q$, p Conclusion: q

1. $p \to q$ Premise

2. p Premise

3. q Modus ponens using 1 and 2

The argument is valid.

Q2. If a man is a bachelor, he is unhappy.
If a man is unhappy, he dies young.
∴ bachelors die young. check
validity.

Sol'→ p: A Man is a bachelor

r: A man dies young

q: ~~the~~ A man is unhappy

Premises: $p \to q$, $q \to r$ Conclusion: $p \to r$

1. $p \to q$ Premise
2. $q \to r$ Premise
3. $p \to r$ Hypothetical syllogism using 1 and 2

$\to$ Argument is valid.

- If this no is divisible by 6, then it is divisible by 3. This no is not divisible by 3. Therefore, this number is not divisible by 6.

Sol:-> p: This no. is divisible by 6.
q: It is divisible by 3.

Premises: $p \to q$, $\neg q$ Conclusion: $\neg p$

1. $p \to q$ Premise
2. $\neg q$ Premise
3. $\neg p$ Modus Tollens using 1 and 2

$\to$ Argument is valid

Q> Either Ram is not guilty (p) or Shyam is telling the truth. Shyam is not telling the truth (q). ∴ Ram is not guilty

Sol:- p: Ram is not guilty

q: Shyam is telling the truth

Premises: $p \vee q$, $\neg q$ Conclusion: p

1. $p \vee q$ Premise
2. $\neg q$ Premise
3. p disjunctive syllogism using 1 & 2

→ Argument is valid.