Algebraic Structures

- Defxn
- elementary properties of algebraic Structures
- Semi group
- Monoid
- Group
- homomorphism
- isomorphism and automorphism
- Subgroups
- normal subgroups, cyclic groups.

- **Algebraic Structure:** →

→ if there exists a system such that it consists of a non-empty set and one or more operations on that set, then that system is called an algebraic system. It is generally denoted by $(A, op_1, op_2 \dots op_n)$ where A is a non-empty set and $op_1, op_2 \dots op_n$ are operations on A.

An algebraic system is also called an algebraic structure because the operations on Set A define structure on the elements of A.

(2) Binary operation

Consider a non-empty set A and a function such that $f: A \times A \rightarrow A$, then f is called binary operation on A whose domain is the set of ordered pairs of elements of A.

If $*$ is a binary operation on A, then it may be written as $a * b$.

- A binary operation can be denoted by any of the symbols $+, -, *, \oplus, \Delta, \boxed{\nabla}, \vee, \wedge$ etc.

- TABLES OF OPERATION

Consider a non-empty finite set $A = \{a_1, a_2, a_3 \ldots a_n\}$

A binary operation $*$ on A can be described by means of table as shown below:

$*$	a_1	a_2	a_3		a_n
a_1	$a_1 * a_1$	$a_1 * a_2$			
a_2	$a_2 * a_1$	$a_2 * a_2$			
a_3			$a_3 * a_3$		
a_n					$a_n * a_n$

Q:→ Consider the set $A = \{1, 2, 3\}$ and a binary operation $*$ on the set A defined by $a * b = 2a + 2b$. Represent operation $*$ as a table on A. (2)

Ans:→ The table of operations is shown as below:

$*$	1	2	3
1	4	6	8
2	6	8	10
3	8	10	12

- Properties of Binary Operations:→
- Closure Property:→ Consider a non-empty set A and a binary operation $*$ on A. Then A is closed under the operation $*$, if $a * b \in A$, where a and b are elements of A.

eg. the operation of addition on the set of integers is a closed operation i.e. if $a, b \in Z$, then $a+b \in Z \ \forall a, b \in Z$.

Ans:→ Consider the set $A = \{-1, 0, 1\}$. Determine whether A is closed under (i) addition 2) multiplication.

(1) The sum of the elements is $(-1)+(-1) = -2$ and $1+1=2$ does not belong to A. Hence A is not closed under ~~multi~~ addition.

(2) The multiplication of every 2 elements of the set are

$-1 * 0 = 0$; $\quad -1 * 1 = -1$; $\quad -1 * -1 = 1$

$0 * -1 = 0$; $\quad 0 * 1 = 0$; $\quad 0 * 0 = 0$

$1 * -1 = -1$; $\quad 1 * 0 = 0$; $\quad 1 * 1 = 1$

Since each multiplication belongs to A, hence A is closed under multiplication.

Q2. Consider the set $A = \{1, 3, 5, 7, 9 \ldots\}$, the set of odd +ve integers. Determine whether A is closed under (i) addition (2) multiplication

Ans:→ (i) The set A is not closed under addition because the addition of 2 odd numbers produces an even number which does not belong to A.

(ii) The set A is closed under the operation multiplication because the multiplication of 2 odd nos produces an odd no. So, for every $a, b \in A$ we have $a * b \in A$.

(2) Associative property :- Consider a non-empty set A and a binary operation $*$ on A. Then, the operation $*$ on A is associative, if for everyone $a, b, c \in A$, we have $(a * b) * c = a * (b * c)$ (3)

Q:- Consider the binary operation $*$ on $\mathbb{Q}$, the set of rational numbers defined by

$$a * b = a + b - ab \quad \forall\ a, b \in \mathbb{Q}$$

Determine whether $*$ is associative.

(b) Consider the binary operation $*$ on the set N of +ve integers defined by

$$a * b = a^b$$

Determine whether $*$ is associative?

Ans:- (a) $(a * b) * c = (a + b - ab) * c$

$a * b$

Put the value

$$= \underbrace{(a + b - ab)}_{a} + \underbrace{c}_{b} - \frac{(a + b - ab)c}{ab}$$

$$= a + b - ab + c - ca - bc + abc$$

$$= a + b + c - ab - ac - bc + abc$$

Similarly, we have

$$a * (b * c) = (a + b + c - ab - ac - bc + abc)$$

$$\therefore (a * b) * c = a * (b * c)$$

Hence $*$ is associative.

(b) $*$ will be associative if

$$a * (b*c) = (a*b)*c \quad \forall a,b,c \in N.$$

Take $a=2$, $b=2$, $c=3$ and consider

$$a * (b*c) = 2 * (2*3)$$
$$= 2 * 2^3$$
$$= 2*8 = 2^8 = 256$$

$$(a*b)*c = (2*2)*3$$
$$= 2^2 * 3$$
$$= 4*3$$
$$= 4^3 = 64$$

Hence $a*(b*c) \neq (a*b)*c$

~~Hence~~ $\therefore$ $*$ is non-associative.

- Commutative Property :→

Consider a non-empty set A and a binary operation $*$ on A. $*$ is commutative if for every $a,b \in A$, we have $a*b = b*a$

Q:→ (a) Consider the binary operation $*$ on Q, the set of rational numbers, defined by

$$a*b = a^2+b^2 \quad \forall a,b \in Q$$

Determine whether $*$ is commutative? (4)

(b) Consider $S = \{a, b, c, d\}$ and $*$ be a binary operation on S defined as shown in the following table.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	a	b
c	c	b	a	a
d	d	a	a	a

Determine (i) whether $*$ is associative?
(ii) whether $*$ is commutative?

Sol'ⁿ (a) Let us assume some elements $a, b \in Q$, then by definition

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$

Hence $*$ is commutative.

(b) (i) Let $a, b, c \in S$ and consider

$$b * (c * c) = b * a = b$$

and $(b * c) * c = a * c = c$

$$b * (c * c) \neq (b * c) * c$$

Thus, $*$ is non-associative.

(ii) $b * c = a$ and $c * b = b$

$$\Rightarrow b * c \neq c * b$$

$\therefore$ $*$ is non-commutative.

Ans - (2) Let $a, b, c \in \mathbb{Q}$, then by defxn, we have

$$(a * b) * c = \left(\frac{ab}{2}\right) * c$$

$$= \frac{\frac{ab}{2} \cdot c}{2} = \frac{abc}{4}.$$

Similarly, $a * (b * c) = a * \left(\frac{bc}{2}\right)$

$$= \frac{\frac{abc}{2}}{2} = \frac{abc}{4}$$

$\therefore$ $a * (b * c) = a * (b * c)$

Hence $*$ is associative.

Q.⇒ Consider the binary operation $*$ and $\mathbb{Q}$, set of rational no's defined by $a * b = \frac{ab}{2}$ $\forall a, b \in \mathbb{Q}$

Determine whether $*$ is (i) associative (ii) Commutative

(i) Let $a, b \in \mathbb{Q}$, then we have

$$a * b = \frac{ab}{2} = b * a = \frac{ba}{2}$$

$$= b * a$$

Hence $*$ is commutative

Q:-> Show that $(R, +)$ is a group.

Closure law :-> $a+b \in R \quad \forall\ a, b \in R$

$\because$ Sum of 2 real nos is a real number. So, closure law holds.

Associative law :-> $a+(b+c) = (a+b)+c$

$\therefore$ A.L. holds

Identity :-> $a+0 = a = 0+a$ for all $a \in R$

$\therefore$ 0 is the identity element.

Inverse :-> $a * a' = a' * a = e \rightarrow$ 0

$a+(-a)=0$

$a+(-a) = (-a)+a$

$= 0$

$\therefore$ $(-a)$ is the inverse of element.

$\therefore$ $(R, +)$ is a group.

Q:-> Show that $(Z_6, +_6)$ is a group.

$Z_6 = \{0, 1, 2, 3, 4, 5\}$

$5 +_6 4 = 3$

$5 +_6 5 = 4$

$\frac{10}{6}$ Remainder

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Closure :- Since all entries in the table are elements of Z_6, so closure law holds.

Associative :- $a +_6 (b +_6 c) = (a +_6 b) +_6 c$
$\forall\ a, b, c \in Z_6$

Identity Element :- 0 is the identity element

Inverse element :- The inverse is that when combined with operation gives us identity

Inverse of 0 is 0.
" " 1 is 5
" " 2 is 4
" " 3 is 3
" " 4 is 2
" " 5 is 1

So $(Z_6, +_6)$ is a group.

Q:⇒ Show that $G = \{1, 2, 3, 4, 5\}$ is not a group under addition modulo 6.

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

$\because 0 \notin G$

$\therefore$ closure law does not hold

$\Rightarrow (G, +_6)$ is not a group

Q:⇒ Show that $G = \{1, 2, 3, 4, 5\}$ is not a group under multiplication modulo 6.

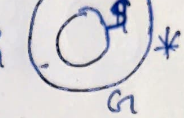
$\times_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

$\because 0 \notin G$

$\therefore$ closure law does not hold

$\Rightarrow (G, \times_6)$ is not a group.

Subgroup :-> Let us consider a group $(G, *)$. Also, let $S \subseteq G$; then $(S, *)$ is called a subgroup if it satisfies the following conditions : (1)



(1) The operation $*$ is closed operation on S.

(2) The operation $*$ is an associative operation

(3) As e is an identity element belonged to G. It must belong to the set S i.e. The identity element of $(G, *)$ must belongs to $(S, *)$.

(4) For every element $a \in S$, a^{-1} also belongs to S.

Q – Let $(I, +)$ be a group, where I is the set of all integers and $(+)$ is an addition operation. Determine whether the following subsets of G are subgroups of G.

(a) The set $(G_1, +)$ of all odd integers

(b) The set $(G_2, +)$ of all +ve integers.

Ans :-> (a) The set G_1 of all odd integers is not a subgroup of G. It does not satisfy the closure property, since addition of 2 odd integers is always even.

(b) **Closure property :->** The set G_2 is closed under the operation $+$, since addition of 2 even integers is always even.

(c) **Associative property :->** The operation $+$ is associative since $(a+b)+c = a+(b+c)$ for every $a, b, c \in G_2$.

④ Identity :→ The element 0 is the identity element. Hence $0 \in G_2$.

⑤ Inverse :- The inverse of every element $a \in G_2$ is $-a \notin G_2$. Hence, the inverse of every element does not exists.

Since the system $(G_2, +)$ does not satisfy all the conditions of subgroups. Hence $(G_2, +)$ is not a subgroup of $(I, +)$

Abelian Group :→ Let us consider an algebraic system $(G, *)$ where $*$ is the binary operxn on G. Then the system $(G, *)$ is said to be an abelian group if it satisfies all the properties of the group plus an additional property.

(i) The operation $*$ is commutative

i.e. $a * b = b * a \quad \forall a, b \in G$.

... an algebraic system $(G, *)$ where G is the set of all non-0 real numbers (2)
and $*$ is a binary operation defined by $a * b = \frac{ab}{4}$. Show that $(G, *)$ is an abelian group.

Ans:→ **Closure property:→** The set G is closed under the operation $*$. Since $a * b = \frac{ab}{4}$ is a real number. Hence belongs to G.

Associative:– $(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{(ab)c}{16}$

$= \frac{abc}{16}$

Similarly $a * (b * c) = a * \left(\frac{bc}{4}\right)$

$= \frac{a(bc)}{16} = \frac{abc}{16}$

Identity:– To find the identity element, let us assume that e is a +ve real no. Then for $a \in G$.

$e * a = a \Rightarrow \frac{ea}{4} = a$

or $e = 4$

Similarly, $a * e = a$

$\frac{ae}{4} = a$ or $e = 4$

Thus, the identity of an element in G is 4.

Inverse:- Let us assume that $a \in G$. If $a^{-1} \in G$ is an inverse of a then $a * a^{-1} = 4$

$\Rightarrow \frac{aa^{-1}}{4} = 4$ or $a^{-1} = \frac{16}{a}$

Similarly, $a^{-1} * a = 4$ gives

$\frac{a^{-1}a}{4} = 4$ or $a^{-1} = \frac{16}{a}$

Thus, the inverse of an element a in G is $\frac{16}{a}$.

Commutative:-> The operation $*$ on G is commutative

Since $a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$

Thus, the algebraic system $(G, *)$ is closed, associative, has identity element has inverse and commutative. Hence the system $(G, *)$ is an abelian group.

~~Q-> Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and~~

Q-> Let (G, o) be a group. Show that if (G, o) is an Abelian group then $(a o b)^2 = a^2 o b^2$ for all a and b in G.

Soln:-> Let us assume that G is an Abelian group

$(a o b)^2 =$

$(a o b) o (a o b) = a o (b o a) o b$ [$\because o$ is associative]

$= a o (a o b) o b = (a o a) o (b o b) = a^2 o b^2$

Hence $(a o b)^2 = a^2 o b^2$