Recurrence Relation :-

It is an equation that defines a equation recursively.

1) Substitution Method :-

(i) Forward Substitution :- (NOT USED FREQUENTLY) ∵ of Guess work

Uses the initial condition in the initial term & value for the next is generated.

Eg:- $T(n) = T(n-1) + n$

$T(0) = 0$ — Initial Condition

⇒ Let $n=1$ then, (If 1 no. then its sum is 1)

$T(1) = T(1-1) + 1$

$= T(0) + 1$

$= 0 + 1$

$= 1$, $\boxed{T(1) = 1}$ — (i)

$n=2$, then (If 2 no. then its sum is $1+2=3$)

$T(2) = T(2-1)+2$

$= T(1)+2$

$= 1+2$

$= 3$, $\boxed{T(2) = 3}$ — (ii)

$n=3$, then (if 3 No. then sum is $1+2+3 \rightarrow 6$)

$T(3) = T(2)+3$

$= 3+3$

$= 6$, $\boxed{T(2) = 6}$ — (iii)

(If n no.s than $(1+2+3 \cdots n)$ is the sum)

Then, we write as $\boxed{T(n) = \frac{n(n+1)}{2}}$ } GUESSED from (i), (ii) & (iii)

$$T(n) = \frac{n^2}{2} + \frac{n}{2}$$

$$T(n) = O(n^2)$$ Ans.

(ii) <u>Backward Substitution</u> :- In this backward values are substituted recursively.

Eg:- $T(n) = \begin{cases} 0 & , n = 0 \\ T(n-1) + n & , n > 0 \end{cases}$

$\Rightarrow \quad T(n) = \underbrace{T(n-1)}_{} + n$ —— ①

$T(n) = T(n-1) + n$

$T(n-1) = T(n-2) + (n-1)$

$T(n-2) = T(n-3) + (n-2)$

$T(n) = [T(n-2) + (n-1)] + n$

$= \underbrace{T(n-2)}_{} + (n-1) + n$ —— ②

$= [T(n-3) + (n-2)] + (n-1) + n$

$= T(n-3) + (n-2) + (n-1) + n$ —— ③

⋮

$T(n) = T(n-k) + (n-(k-1)) + (n-(k-2)) + \cdots (n-1) + n$ —— ④

Assume $n-k = 0$

$n = k$

$= T(0) + (n-(n-1)) + (n-(n-2)) + \cdots (n-1) + n$

$= 0 + (1) + 2 + \cdots (n-1) + n$

$= \dfrac{n(n+1)}{2} = \dfrac{n^2+n}{2}$

$= O(n^2)$ Ans

Example 2 :-

$$\boxed{\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right)+C \\ T(1) &= 1 \end{aligned}}$$

$\Rightarrow T(n) = 2\underbrace{T\left(\frac{n}{2}\right)}+C$

$= 2\left(2T\left(\frac{n}{4}\right)+C\right)+C$

$= 4T\left(\frac{n}{4}\right)+2C+C$

$= 4\underbrace{T\left(\frac{n}{4}\right)}+3C \xrightarrow[\text{In general}]{} 2^2T\left(\frac{n}{2^2}\right)+(2^2-1)C$ — (1)

$= 4\left(2T\left(\frac{n}{2\times 4}\right)+C\right)+3C$

$= 8T\left(\frac{n}{8}\right)+4C+3C$

$= 8T\left(\frac{n}{8}\right)+7C \xrightarrow[\text{In general}]{} 2^3T\left(\frac{n}{2^3}\right)+(2^3-1)C$ — (2)

In general,

$T(n) = 2^kT\left(\frac{n}{2^k}\right)+(2^k-1)C$ — (3)

Put $2^k = n$ in (3)

$T(n) = nT\left(\frac{\cancel{n}}{\cancel{n}}\right)+(n-1).C$

$= n\,T(1)+(n-1).C$

$= n(1)+(n-1)C$

$= n+(n-1)C \Rightarrow \boxed{T(n) = O(n)}$ Ans.

Example 3:- $\boxed{\begin{array}{c} T(n) = T\left(\frac{n}{3}\right) + c \\ T(1) = 1 \end{array}}$

Sol^n $\Rightarrow$ $T(n) = \underbrace{T\left(\frac{n}{3}\right)} + c$ ... I

$= \left(T\left(\frac{n}{3\times 3}\right) + c\right) + c$

$= \underbrace{T\left(\frac{n}{9}\right)} + 2c$

$= \left(T\left(\frac{n}{3\times 9}\right) + c\right) + 2c$

$= T\left(\frac{n}{27}\right) + 3c$

* imp.

Relation b/w log & exponential

$b^x = y$

$x = \log_b y$

In general,

$= T\left(\frac{n}{3^k}\right) + kc$

let $3^k = n \Rightarrow [\, k = \log_3 n \,]$

$= T\left(\frac{\cancel{n}}{\cancel{n}}\right) + \log_3 n \cdot c$

$= T(1) + \log_3 n \cdot c$

$= 1 + \log_3 n \cdot c$

$\boxed{T(n) = c \log_3 n + 1}$ Ans.

$T(n) = O(\log_3 n)$ Ans.

Q ⇒ Solve the equation by substitution method.

$$T(n) = 2T\left(\frac{n}{2}\right) + n \quad \text{———} \quad (A)$$

When have to show that it is asymptotically bound by $O(n \log n)$.

Sol.ⁿ ⇒

$$T(n) = O(n \log n)$$

To prove $T(n) \leq C * n \log n$ → Basic def.ⁿ of Big-oh

$$\text{———} (1)$$

let eq. (1) is true for $n = \frac{n}{2}$

By using Induction ⇒

$$T\left(\frac{n}{2}\right) \leq C \frac{n}{2} \log \frac{n}{2}$$

Put in (A)

$$T(n) \leq \cancel{2} C \frac{n}{\cancel{2}} \log \frac{n}{2} + n$$

$$T(n) \leq C n \log \frac{n}{2} + n$$

$$T(n) = Cn \log \frac{n}{2} + n$$

{ ∵ Just to solve out ∴ we take only = }

$$T(n) = Cn \log n - Cn \log 2 + n$$

$$= Cn \log n - \underbrace{(Cn \log 2 - n)}$$

⇓

This value is Zero
∵ $n \neq -ve$
$\log n \neq -ve$

So, we conclude

$$T(n) \leq Cn \log n$$

$$T(n) = O(n \log n) \quad \text{True.}$$

OR

Solve this equation by Substitution method :-

$$T(n) = \begin{cases} 1 & ; n = 1 \\ 2T\left(\frac{n}{2}\right) + n & ; n > 1 \end{cases}$$

$$T(n) = 2\underbrace{T\left(\frac{n}{2}\right)} + n \quad \text{--- (1)}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{2^2}\right) + \frac{n}{2}$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{2^3}\right) + \frac{n}{4}$$

$$= 2\left[2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right] + n$$

$$= 4T\left(\frac{n}{4}\right) + \frac{\cancel{2}n}{\cancel{2}} + n$$

$$= 4\underbrace{T\left(\frac{n}{4}\right)} + 2n \quad \text{--- (2)}$$

$$= 4\left[2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right] + 2n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + \frac{\cancel{4}n}{\cancel{2^2}} + 2n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n \quad \text{--- (3)}$$

$$\vdots$$

$$T(n) = 2^k \underbrace{T\left(\frac{n}{2^k}\right)} + kn$$

let $\boxed{\begin{aligned} T\left(\frac{n}{2^k}\right) &= T(1) \\ \therefore \frac{n}{2^k} &= 1 \\ n &= 2^k \\ k &= \log n \end{aligned}}$

$$T(n) = 2^k \, \underline{T(1)} + k\underline{n}$$

$$= n \times 1 + n \log n$$

$$= n + \underbrace{n \log n}_{\text{Higher term}}$$

$$\therefore O(n \log n) \text{ Ans}$$

Example:- Solve using Substitution method.

$$T(n) = \begin{cases} 1 & , n=0 \\ T(n-1) + \log n & , n>0 \end{cases}$$

$\Rightarrow$ $T(n) = T(n-1) + \log n$ — ①

$= [T(n-2) + \log(n-1)] + \log n$

$= T(n-2) + \log(n-1) + \log n$ — ②

$= [T(n-3) + \log(n-2)] + \log(n-1) + \log n$

$= T(n-3) + \log(n-2) + \log(n-1) + \log n$ — ③

for K times

$= T(n-K) + \log 1 + \log 2 \text{ - - - - - } \log(n-1) + \log n$

$\because$ $n-K=0$

$n=K$

$= T(0) + \log n!$

$\Rightarrow$ $1 + \log n!$

$= O(n \log n)$ Ans.

Example:-

$$T(n) = \begin{cases} 1 & , n=1 \\ T(n/2)+1 & , n>1 \end{cases}$$

$\Rightarrow$ $T(n) = \underbrace{T\left(\frac{n}{2}\right)}+1$ —— (1)

$$T(n) = T\left(\frac{n}{2}\right)+1$$
$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{2^2}\right)+1$$
$$\vdots$$

$= \left[T\left(\frac{n}{2^2}\right)+1\right]+1$

$= \underbrace{T\left(\frac{n}{2^2}\right)}+2$ —— (2)

$= \left[T\left(\frac{n}{2^3}\right)+1\right]+2$

$= T\left(\frac{n}{2^3}\right)+3$ —— (3)

$\vdots$

$= T\left(\frac{n}{2^k}\right)+k$ —— (4)

Let $\frac{n}{2^k} = 1$

$n = 2^k$

$k = \log n$

$= T(1) + \log n$

$= 1 + \log n$

$= O(\log n)$ Ans