Theorem :- The congruence relation is an equivalence relation.

$15 \bmod 10 \rightarrow 10\overline{)15}$ $\frac{10}{5}$

$X \bmod Y = \frac{X}{Y} = R$

Proof :- **Congruence :-** $a \equiv b \pmod m \rightarrow$ $a \bmod m = b \bmod m$

(equal in every respect)

$\therefore$ $a - b$ is divisible by m.

- **Reflexive :-**

As $a - a = 0$ is divisible by m.

So $a \equiv a \pmod m$ for any integer a

Hence, the congruence relation is reflexive.

- **Symmetric :-**

$a \equiv b \pmod m$

$\Rightarrow$ $a - b$ is divisible by m. $\Rightarrow a - b = mk$

$\Rightarrow$ $-(b - a)$ is divisible by m. $\Rightarrow -(b-a) = mk$

$\Rightarrow$ $b - a$ is divisible by m. $\Rightarrow (b - a) = m(-k)$

$\Rightarrow$ $b \equiv a \pmod m \leftarrow (b-a) = mk_1$

Hence, the congruence relation is symmetric.

• Transitive :→

$a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$\Rightarrow \quad a \equiv c \pmod{m}$

Proof → $a \equiv b \pmod{m}$

$\Rightarrow (a-b)$ is divisible by m

$\Rightarrow \quad a-b = mk_1 \quad \cdots\cdots \text{(i)}$

$b \equiv c \pmod{m}$

$= (b-c)$ is divisible by m

$\Rightarrow \quad b-c = mk_2 \quad \text{—— (ii)}$

Now, (i) + (ii)

$$a-b+b-c = mk_1 + mk_2$$

$$a-c = mk_1 + mk_2$$

$$a-c = m(k_1+k_2)$$

$$a-c = mk_3$$

$$\Rightarrow \quad a \equiv c \pmod{m}$$

Equivalence classes of R :→

Q→ What is equivalence class of 2 w.r.t congruence modulo 5?

Ans :→ 2 mod 5 = 2

dividend divisor R
$2 \div 5 = 0.4$
$0 \times 5 = 0$
$2 - 0 = 2$
Modulo method

$[2] = \{ \ldots -8, -3, 2, 7, 12 \}$

$-5 \quad +5$

Q→ What is equivalence class of 4 w.r.t congruence modulo 5?

Ans :→ 4 mod 5 = 4

$[4] = \{ \ldots -6, -1, 4, 9, 14, \ldots \}$

$-5 \quad +5$

- **Properties of Equivalence Class :→**

① Every element $a \in A$ is a member of equivalence class $[a]$

$\forall a \in A, \ a \in [a]$

② Two elements $a, b \in A$ are equivalent if and if they belong to the same equivalence class.

$\forall a, b \in A$ if $[a] = [b]$

(3) Every 2 equivalence Classes [a] and [b] are either equal or disjoint

$\forall a, b \in A$, $[a] = [b]$ or

$[a] \cap [b] = \phi$

Partition Set

→ Set is a collection of elements

$A = \{a, b, c, d\}$

$a_1 = \{a, b\}$ $a_2 = \{c, d\}$

Two conditions :→

(1) $\boxed{a_1 \cup a_2 = A}$

$a_1 \cup a_2 = \{a, b, c, d\} = A$

(2) $\boxed{a_1 \cap a_2 = \phi}$ (no element is common)

$a_1 \cap a_2 = \phi$

that means a_1 and a_2 are Partition set of A.

Ques:→ Let $X = \{1, 2, 3 \ldots 9\}$ Determine whether or not each of the following is a partition set of X :-

(i) $\{ (1,3,6) \ (2,8) \ (5,7,9) \}$

Solution:→(i) $\{(1,3,6) \cup \{2,8\} \cup \{5,7,9\}$

$= \{1, 2, 3, 5, 6, 7, 8, 9\}$

$\neq X$

because 4 is missing ⇒ We will not check 2nd condition

2) $(2,4,5,8)$ $(1,9)$ $(3,6,7)$

1 $= \{2,4,5,8\} \cup \{1,9\} \{3,6,7\}$

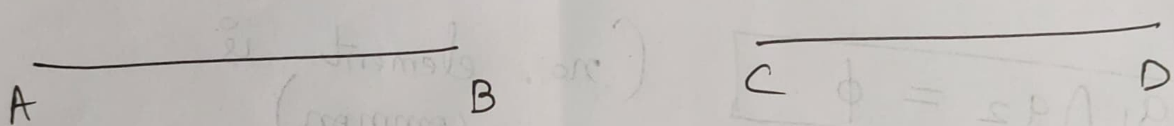
$= \{1,2,3,4,5,6,7,8,9\}$

$= X$ ✓

2 $= \phi$ ✓

Congruence :→

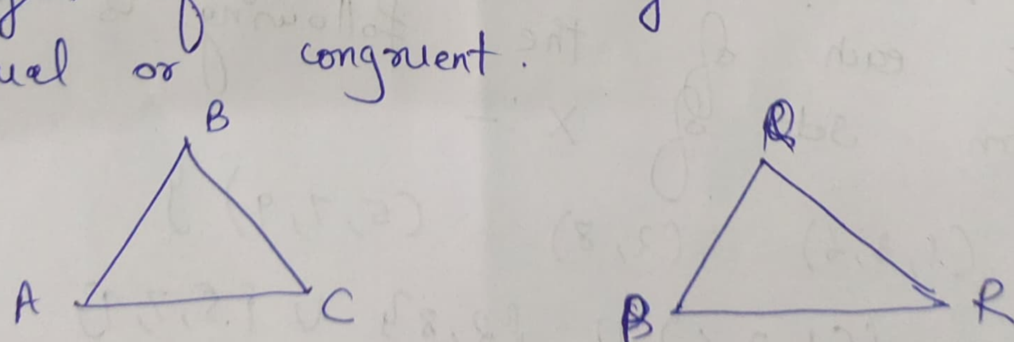
1 **Congruence of Line Segments :→**



2 lines are congruent, if they have same length.

2 **Congruence of 2 Triangles :→**

→ if corresponding sides and corresponding angles of the 2 triangles are equal or congruent.



$\Delta ABC \cong \Delta PQR$

Consider 3 right angle Δ T_1 with (4) sides 3, 4, 5 T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10 which triangles among T_1, T_2 and T_3 are related?

Ans :→ Two triangles are similar if ratio of their sides are equal

for Δ T_1 and T_3

Ratio of sides $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$

Hence T_1 and T_3 are related.

→ $\boxed{\text{Compatible relation}}$

Types of functions

(1) Injective (One - One fxns) :-

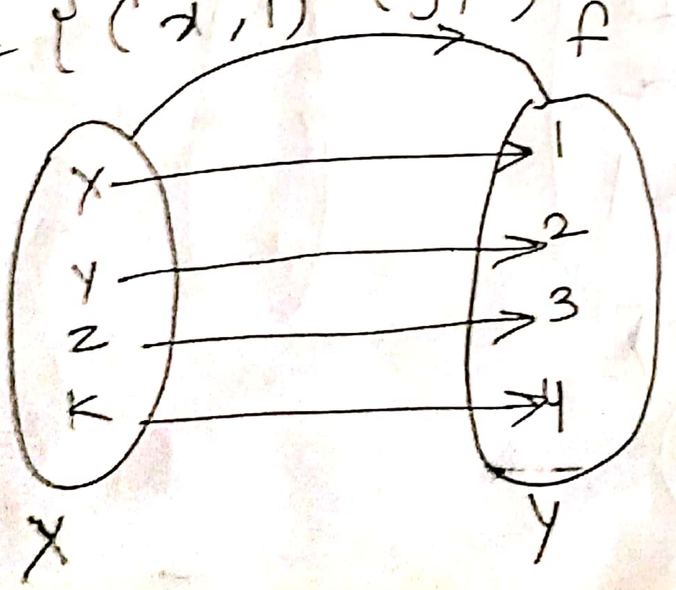
The fxn f is called one-to one or injective if different elements in X have different images in Y i.e. if

$$\boxed{\begin{gathered} f(x_1) = f(x_2) \\ \Rightarrow x_1 = x_2 \end{gathered}}$$

→ or we can say that every element of domain X has a unique image in the co-domain Y and there is no element of Y which is image of more than one element of domain X.

eg. Consider $X = \{x, y, z, k\}$ and $Y = \{1, 2, 3, 4\}$ and f is fxn from X to Y such that

$f = \{(x,1)\ (y,2)\ (z,3)\ (k,4)\}$



(one - one fxn)

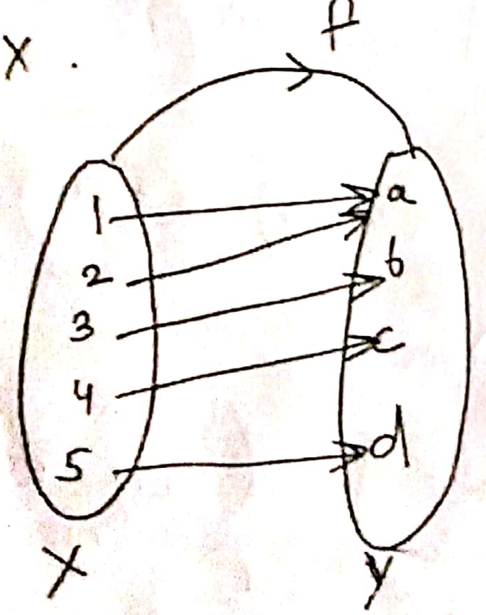
② Surjective (Onto fxn)

Let $f: X \to Y$. The function f is called surjective fxn if each element in Y is the image of atleast one element in X. In other words, the range of f is equal to codomain Y.

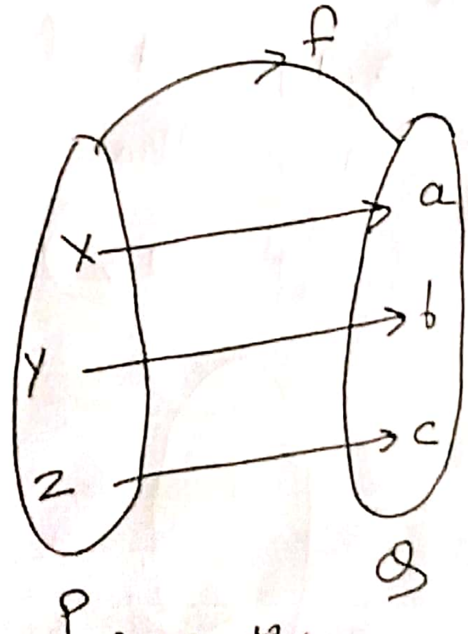
i.e. $\forall y \in Y$, $y = f(x)$ for some $x \in X$

Q- Consider $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{a, b, c, d\}$ and $f = \{(1,a)\ (2,a)\ (3,b)\ (4,c)\ (5,d)\}$

→ It is surjective fxn as every element of Y is the image of some element of X.



(Onto fxn)



(bijective fxn)

(c) Bijective (One-to One Onto) fxns :- (2)

A fxn which is both injective (one-to one) and surjective is called (bijective) fxn.

eg. Consider $P = \{x, y, z\}$ $Q = \{a, b, c\}$

$f: P \rightarrow Q$ Such that

$f = \{(x, a)\ (y, b)\ (z, c)\}$

The f is one-to one and also it is onto. so it is a bijective fxn.

(d) Into fxns :→

Let $f: X \rightarrow Y$. The fxn f is called an into fxn if the range of f is not equal to the co-domain of Y.

Note :- (if fxn is not onto, then it is called as into)

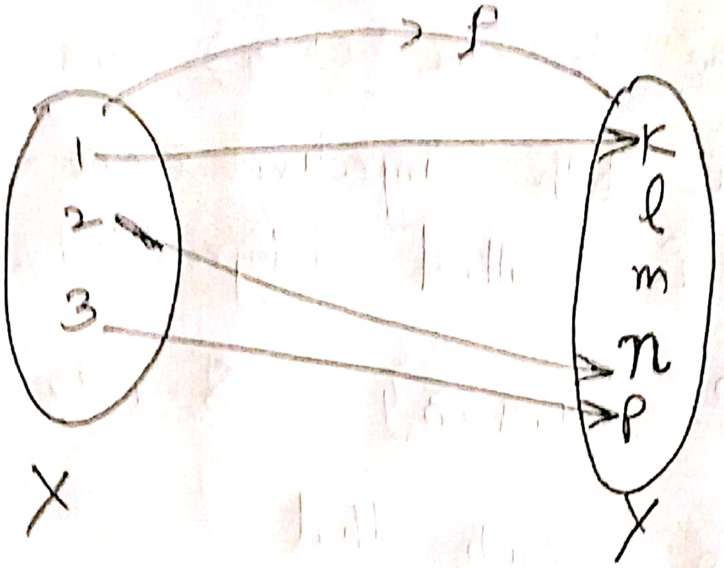
eg. Consider $X = \{1, 2, 3\}$, $Y = \{k, l, m, n, p\}$ and $f: X \rightarrow Y$ such that

~~$f = \{(k, l)\ ($~~

$f = \{(1, k)\ (2, n)\ (3, p)\}$

In the function f, the range ie. $\{k, n, p\} \neq$ Codomain of Y ie. $\{k, l, m, n, p\}$

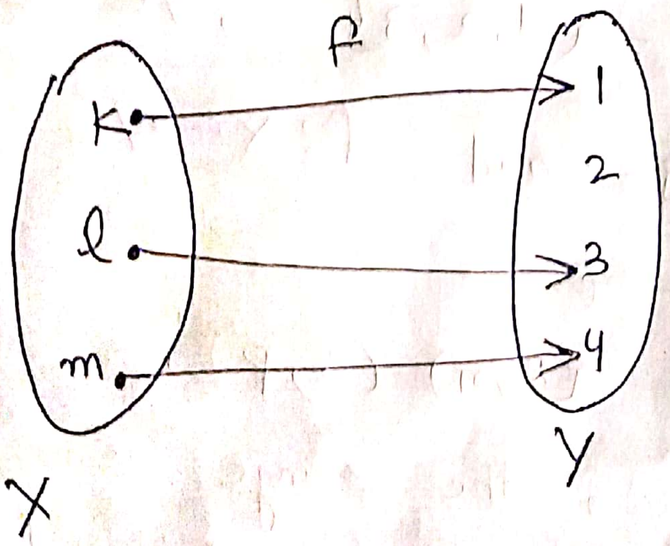
∴ it is an into fxn



(e) <u>One - One Into fxns</u> :→

Let $f: X \to Y$. The fxn f is called one-one into function if it is one-one ~~and~~ but not onto.

eg. Consider $X = \{k, l, m\}$, $Y = \{1, 2, 3, 4\}$ and $f: X \to Y$ such that
$f = \{(k, 1)\ (l, 3)\ (m, 4)\}$
The fxn f is one-one into fxn.



(f) Many One fxns :→ Let $f : X \to Y$. The fxn f is said to be many one fxn if there exist 2 or more than 2 diff elements in X having same image in Y.

eg. Consider $X = \{1, 2, 3, 4, 5\}$, $Y = \{x, y, z\}$ and $f : X \to Y$ such that $f = \{(1, x) (2, x) (3, x) (4, y) (5, z)\}$

fig 3.12

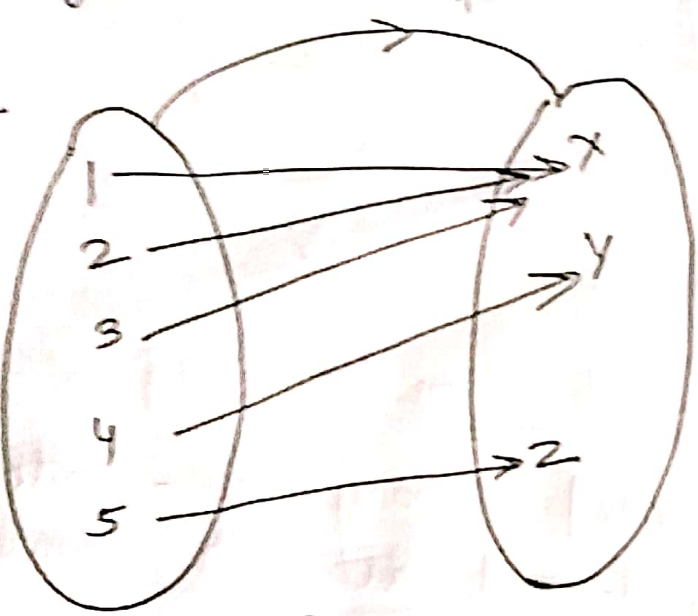


fig 3.12

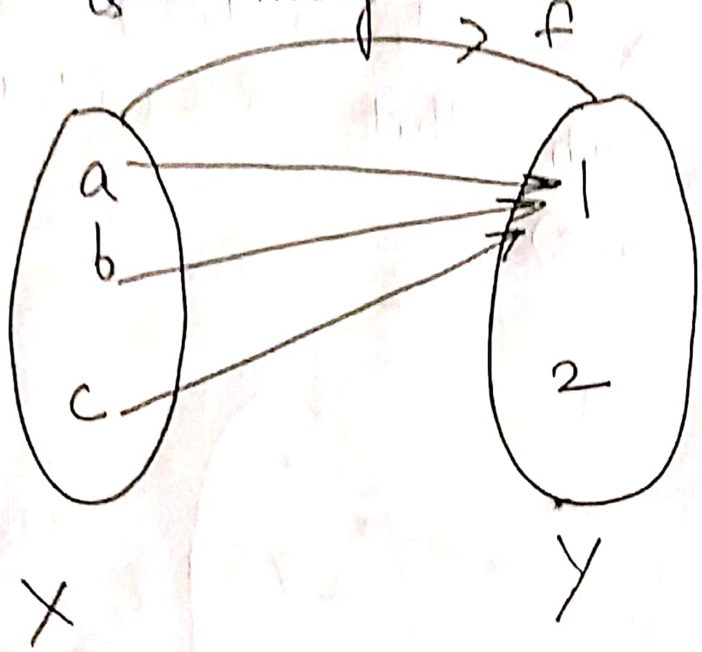
Note :→ if fxn is not one-one, then it is called as many-one

(g) Many One Into fxns :→
Let $f : X \to Y$. The function f is called many-one into fxn if and only if it is both many one and into function.

eg Consider $X = \{a, b, c\}$, $Y = \{1, 2\}$ and $f : X \to Y$ such that

$f = \{(a,1)\ (b,1)\ (c,1)\}$

As the function f is many - one and into, so it is many - one into fxn.

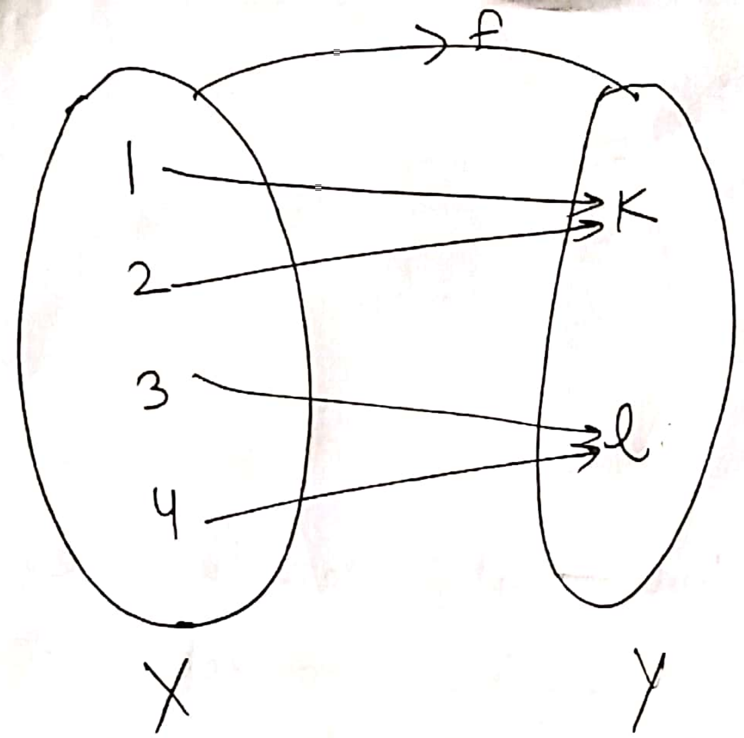


(d) Many One Onto fxn :- Let $f : X \to Y$. The function is called many - one - onto fxn if and if only if it is both many one and onto.

eg. Consider $X = \{1, 2, 3, 4\}$ $Y = \{k, l\}$ and $f : X \to Y$ such that

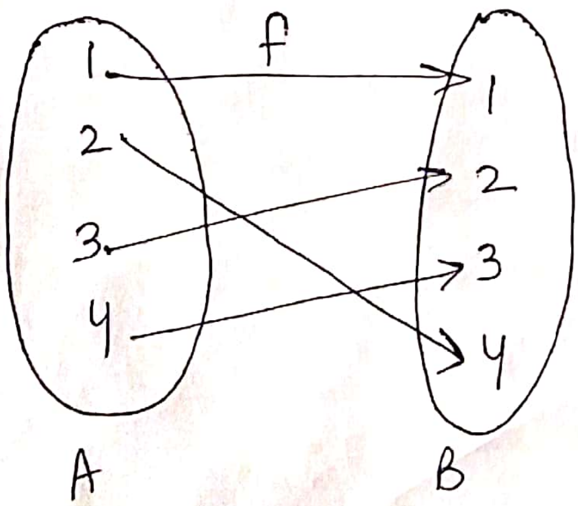
$f = \{(1,k)\ (2,k)\ (3,l)\ (4,l)\}$

→ The fxn f is many - one (as 2 elements have same image in Y) and it is onto (as every element of Y is the image of some element X).



Q:⇒ Let $A = B = \{1,2,3,4\}$. Define fxns $f: A \to B$ (if possible) such that

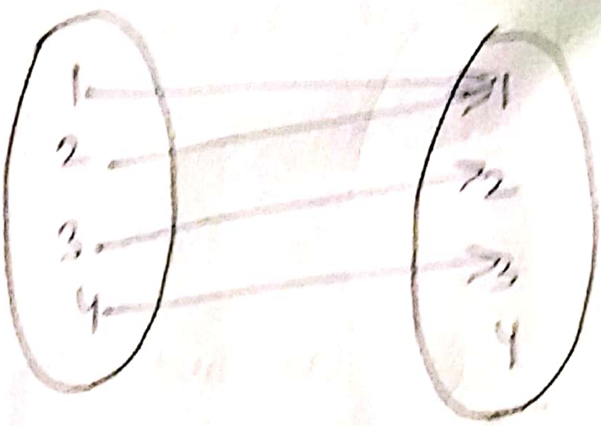
(i) f is one-one and onto



(i) $f = \{(1,1)$
$(2,4)$
$(3,2)$
$(4,3)$
is one to one and onto

② f is neither one-to one ~~and~~ nor onto.

$f = \{(1,1)\ (2,1)\ (3,2)\ (4,3)$



(iii) f is onto but not one-to one

→ It is not possible on set A = B = {1, 2, 3, 4}

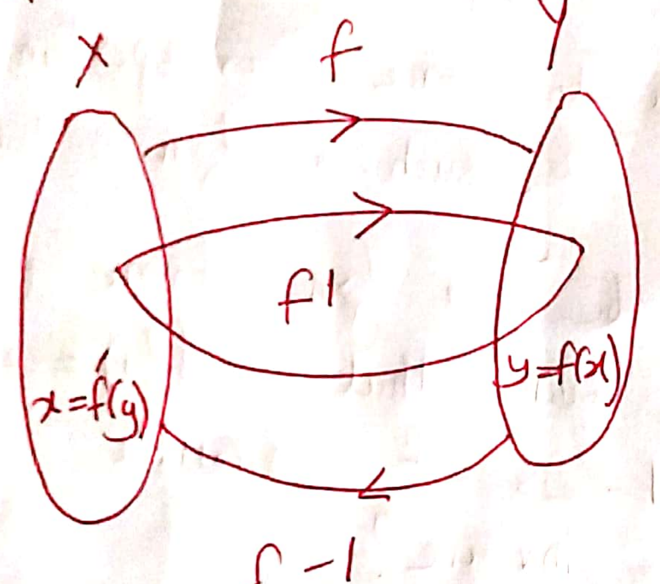
(iv) f is one-to one but not onto

→ The fxn is not possible on the set A = B = {1, 2, 3, 4}

1

Inverse function

If $X \to Y$ is a bijection then there always exist pre image $f^{-1}(y)$ of each element y of Y and this will be a unique element of Y.



Ex: Let $f: R \to R$ & $f(x) = 2x-3$

We know that it is one-one and onto function.

So, f^{-1} exist.

So, $y = f(x) = x = f^{-1}(y)$

$y = 2x - 3$

$2x = y+3$

$x = \frac{y+3}{2}$

$f^{-1}(y) = \frac{y+3}{2}$

- A one-to one correspondance is called invertible because we can define an inverse of this fxn.
- A fxn is not invertible if it is not a one-to one correspondance, because the inverse of such a fxn does not exist.

Q:→ Let f be a fxn from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Is f invertible and if it is, What is its inverse?

Ans:→ The function f is invertible because it is a one-one correspondance. The inverse function f^{-1} reverses the correspondance given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Q:→ Let $f: Z \rightarrow Z$ be such that $f(x) = x+1$. Is f invertible, and if it is, what is its inverse?

Ans:→ The function f has an inverse because it is a one-to one correspondance. To reverse the correspondance, suppose that y is the image of x, so that $y = x+1$. Then $x = y-1$. This means that $y-1$ is the unique element of Z that is sent to y by f. Consequently $f^{-1}(y) = y-1$.

Q:-> Let f be the fxn from R to R with $f(x) = x^2$. Is f invertible?

Ans:-> Because $f(-2) = f(2) = 4$, f is not one-to one. If an inverse fxn were defined, it would have to assign 2 elements to 4. Hence f is not invertible.

Composition of functions ①

Er. Meenakshi (E14673)

Def:→ Let $f: A \to B$ and $g: B \to C$ be 2 fxns. Then the function $gof : A \to C$, defined by
$$(gof)(x) = g(f(x)) \text{ for all } x \in A$$
is called the composition of f and g.

→ gof is defined only if range R_f is a subset of domain of D_g. Also gof is a function from A to C.

gof is the notation for the composition of f and g.

Eg:→ Let $f: \{1,2,3\} \to \{a,b\}$ is a fxn defined by $f(1) = a$, $f(2) = a$, $f(3) = b$ and $g: \{a,b\} \to \{5,6,7\}$ is defined by $g(a) = 5$, $g(b) = 7$

Find gof and fog.

→Sol^n $R_f = \{a, b\}$
$D_g = \{a, b\}$

Since $R_f \subseteq D_g$ $\therefore$ gof is defined

and $gof : \{1,2,3\} \to \{5,6,7\}$

Now $(gof)(1) = gf(1) = g(a) = 5$
$(gof)(2) = gf(2) = g(a) = 5$
$(gof)(3) = gf(3) = g(b) = 7$

To find fog, $R_g = \{5, 6, 7\}$
$D_f = \{1, 2, 3\}$

Since $R_g \not\subseteq$ domain D_f

$\therefore$ fog is not defined.

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Q:→ Let $f: \{1,2,3\} \to \{a,b\}$ be a fxn defined by $f(1) = a$ $f(2) = b$ $f(3) = b$

Let $g: \{a,b\} \to \{5,6,7\}$ be a fxn defined by $g(a) = 5$, $g(b) = 7$. find gof, fog.

Ans→ $f: \{1,2,3\} \to \{a,b\}$ and $g: \{a,b\} \to \{5,6,7\}$

gof :→ $R_f = \{a,b\} = D_g$

$\therefore$ gof is defined

i.e. $gof: \{1,2,3\} \to \{5,6,7\}$

$gof(1) = g(f(1)) = g(a) = 5$

$gof(2) = g(f(2)) = g(b) = 7$

$gof(3) = g(f(3)) = g(b) = 7$

$\therefore gof = \{5,7\}$

fog :→ $R_g = \{5,6,7\}$

$D_f = \{1,2,3\}$

Since $R_g \not\subseteq$ Domain D_f

$\therefore$ fog is not defined.

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Q:→ Let $f : R \to R$ be defined by $f(x) = x^3$ and $g : R \to R$ be defined by $g(x) = 3x+1$. Find $(gof)(x)$, $(fog)(x)$. Is $gof = fog$?

Sol:→ $f : R \to R$, $g : R \to R$

$R_f = R$, $D_g = R$

$\therefore$ gof is defined.

Also, $R_g = R$, $D_f = R$

$\therefore$ fog is defined.

Now $(gof)(x) = g(f(x)) = g(x^3)$

$= 3x^3+1$

$(fog)(x) = f(g(x)) = f(3x+1)$

$= (3x+1)^3$

Since $(gof)(x) = 3x^3+1$

and $(fog)(x) = (3x+1)^3$

$\therefore$ $gof \neq fog$.

Q:→ⓐ If $f(x) = x^2$, $g(x) = x^3$ find $(fogofog)(x)$.

Ans:→ $f(g(f(g(x)))) = f(g(f(x^3)))$

$= f(g(x^6))$

$f(x) = x^2,\ f(x^3)$
$= (x^3)^2 = x^6$

$= f(x^3)^6$

$= f(x^{18})$

$= x^{36}$

$g(x) = x^3;$
$g(x^6) = (x^3)^6$
$= x^{18}.$

(b) If $f(x) = x+1$, $g(x) = x+3$,
find $(f \circ f \circ f \circ f)(x)$

$\Rightarrow f(f(f(f(x))))$

$= f(f(f(x+1)))$

$= f(f(x+2))$

$= f(x+3)$

$= x+4$

(c) If $f(x) = 4x-1$, $g(x) = x^2+2$.
find $(f \circ (f \circ g))(1)$.

$(f \circ f \circ g)(1) = f(f(g(1)$

$= f(f(3))$

$= f(11)$

$= 44 - 1 = 43$

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(d) If $f(x) = \frac{x}{x+1}$

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$g(x) = \frac{1}{x-1}$ find $(fog)(x)$

$\rightarrow (fog)(x) = f(g(x))$

$= f\left(\frac{1}{x-1}\right)$

$= \frac{\frac{1}{x-1}}{\frac{1}{x-1} + 1}$

$= \frac{\frac{1}{x-1}}{\frac{1+x-1}{x-1}} = \frac{1}{x}$

(e) If $f(x) = x+5$, $g(x) = x^2$, find $(gof)(x)$ where $f: R \to R$, $g: R \to R$ is given.

$(gof)(x) = g(f(x)) = g(x+5)$

$= (x+5)^2 = x^2+25+10x$

(f) If $f: R \to R$ and $g: R \to R$ are 2 functions defined by $f(x) = \sin x$ and $g(x) = x^2$. find fog and gof.

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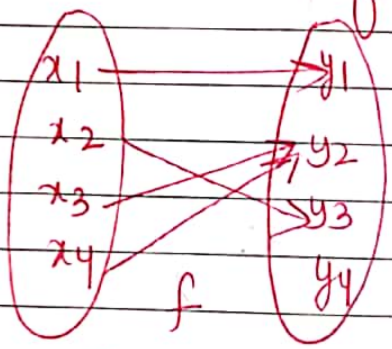
Is $f \circ g = g \circ f$?

Ans → $(f \circ g)(x) = f(g(x)) = f(x^2)$
$= \sin x^2$

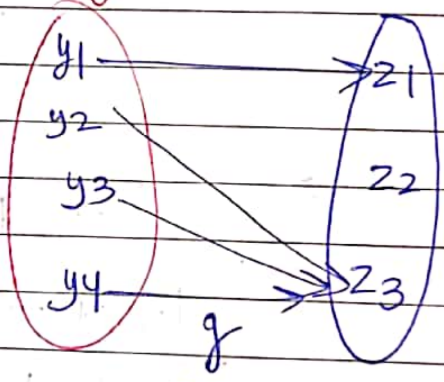
$(g \circ f)(x) = g(f(x))$
$= g(\sin x) = (\sin x)^2$
$= \sin^2 x$

Thus $g \circ f \neq f \circ g$.

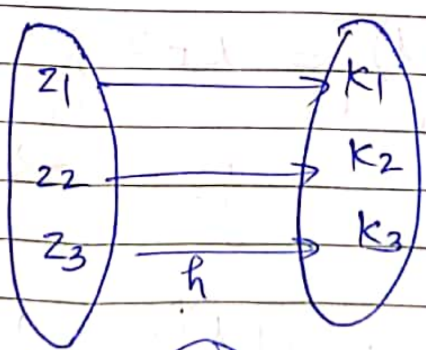
Q- Consider the functions f, g and h



(a)

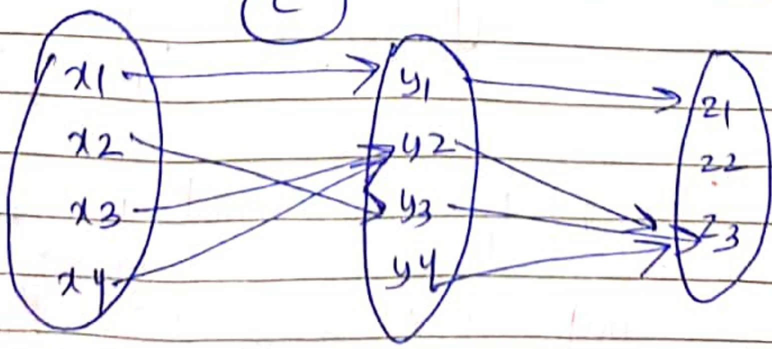


(b)



(c)

(i) gof



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$(g \circ f)(x_1) = g[f(x_1)]$
$= g(y_1) = z_1$

$(g \circ f)(x_2) = g[f(x_2)]$
$= g(y_3) = z_3$

$(g \circ f)(x_3) = g[f(x_3)]$
$= g(y_2) = z_3$

$(g \circ f)(x_4) = g[f(x_4)]$
$= g(y_2) = z_3$

So, $g \circ f = \{(x_1, z_1)\ (x_2, z_3)$
$(x_3, z_3)\ (x_4, z_3)\}$

(2) $h \circ (g \circ f)$

first find the composition of f with g and then with h.

$h \circ (g \circ f)(x_1) = h \circ [g(f(x_1))]$
$= h \circ [g(y_1)]$
$= h(z_1) = k_1$

$h \circ (g \circ f)(x_2) = h \circ [g(f(x_2)]$
$= h \circ [g(y_3)] = h(z_3)$
$= k_3$