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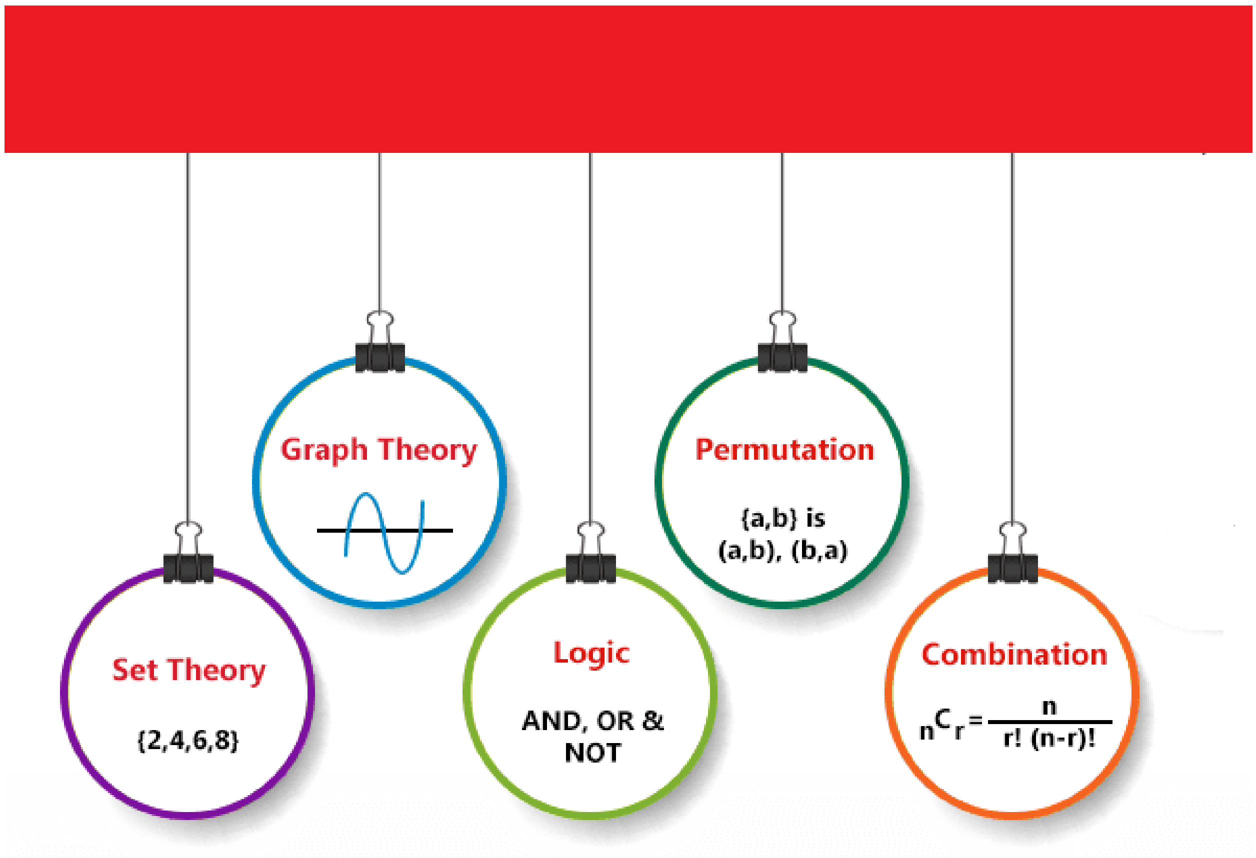
UNIVERSITY INSTITUTE OF ENGINEERING

BE: CSE (All IT branches)

Sub: Mathematics for computing **Sub. Code- 20SMT-125**



Basic Structure

Lecture -1

Introduction to Sets and Operations on Sets

Sets:

"A set is a Many that allows itself to be thought of as a One." (Georg Cantor)

Definition: - A set is a collection of well defined and different elements. A set can be written explicitly by listing its elements using curly bracket.

A set is typically determined by its distinct elements, or members, by which we mean that the order does not matter, and if an element is repeated several times, we only care about one instance of the element. We typically use the bracket notation { } to refer to a set.

Example:-
The sets {1, 2, 3} and {3, 1, 2} are the same, because the ordering does not matter. The set {1, 1, 1, 2, 3, 3, 3} is also the same set as {1, 2, 3}, because we are not interested in repetition: either an element is in the set, or it is not, but we do not count how many times it appears.

Representation of a Set

Sets can be represented in two ways:

(i) Roster or Tabular Form or Explicit Form
(ii) Set Builder Notation or Implicit Form

Roster or Tabular Form:-The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 2.1: Set of vowels in English alphabet, A = {a, e, i, o, u}

Example 2.2: Set of odd numbers less than 10, B = {1, 3, 5, 7, 9}

Set Builder Notation:-The set is defined by specifying a property that elements of the set have in common. The set is described as A = {x: p(x)}

Example 2.3: The set {a, e, i, o, u} is written as:

A = {x: x is a vowel in English alphabet}

Example 2.4: The set {1, 3, 5, 7, 9} is written as: B ={x: $1 \le x < 10$ and (x%2) $\neq$ 0}

Standard Notation of sets:-

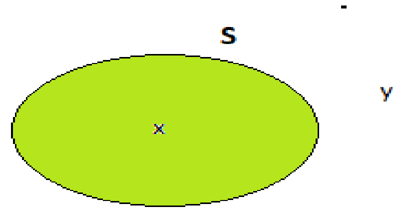
N = {1, 2, . . .}, the set of Natural numbers;
W = {0, 1, 2, . . .}, the set of whole numbers
Z = {0, 1, −1, 2, −2, . . .}, the set of Integers;
Q = { p q : p, q ∈ Z, q 6= 0}, the set of Rational numbers;
R = the set of Real numbers; and
C = the set of Complex numbers.

Cardinality of a Set: - Cardinality of a set S, denoted by |S|, is the number of distinct elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example: | {1, 4, 3, 5}| = 4, |{1, 2, 3,4,5,…}| = ∞

Membership

If an element x is a member of any set S, it is denoted by x∈ S and if an element y is not a member of set S, it is denoted by y ∉ S.



x ∈ S and y ∉ S

Example: If $S = \{1, 1.2, 1.7, 2\}$, $1 \in S$ but $1.5 \notin S$

Subset: - A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

Example 1: Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Example 2: Let, $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Proper Subset: - The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

Example: Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y.

Note:

- Every set is o subset of itself.
- The empty set is a subset of every set.
- The total number of subsets of a finite set containing n elements is 2^n.

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, universal, singleton set, etc.

a) **Finite Set:-**A set which contains a definite number of elements is called a finite set.

Example: $S = \{x \mid x \in N \text{ and } 70 > x > 50\}$

Infinite Set: - A set which contains infinite number of elements is called an infinite set.

Example: $S = \{x \mid x \in N \text{ and } x > 10\}$

Universal Set:-It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example: We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Empty Set or Null Set:-An empty set contains no elements. It is denoted by ∅. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example: $S = \{x \mid x \in N \text{ and } 7 < x < 8\} = \emptyset$

Singleton Set or Unit Set:-Singleton set or unit set contains only one element. A singleton set is denoted by {s}.

Example: $S = \{x \mid x \in N, 7 < x < 9\} = \{ 8 \}$

Equal Set:-If two sets contain the same elements they are said to be equal.

Example: If A = {1, 2, 6} and B = {6, 1, 2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set:-If the cardinalities of two sets are same, they are called equivalent sets.

Example: If A = {1, 2, 6} and B = {16, 17, 22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$

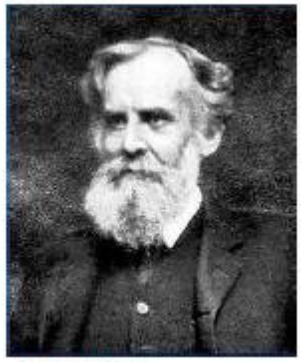
Disjoint Set:-Two sets A and B are called disjoint sets if they do not have even one element in common.

Overlapping Set:-Two sets that have at least one common element are called overlapping sets.

Example: Let, A = {1, 2, 6} and B = {6, 12, 42}.There is a common element '6'; hence these sets are overlapping sets.

Venn Diagrams:-

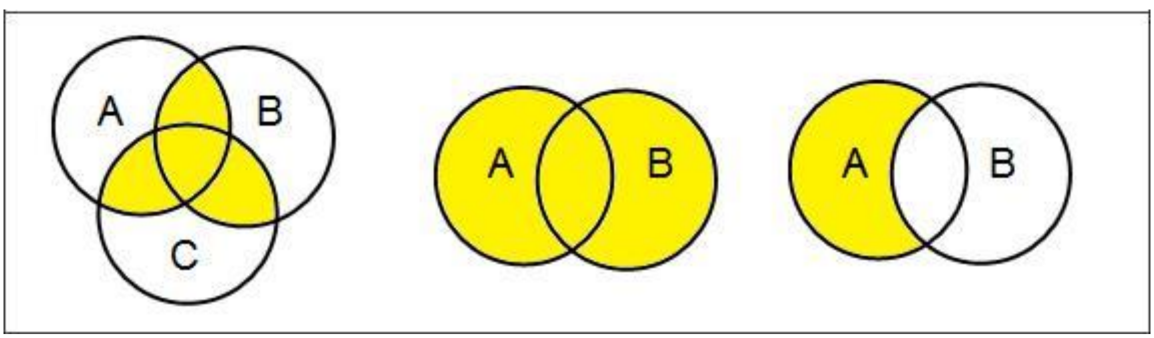
Venn diagram, invented in1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.



John Venn
(1834-1923)

In Venn-diagrams the universal set U is represented by point within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle.

Examples:-



Examples of Venn-diagrams

Set Operations

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian product.

Set Union:-The union of sets A and B (denoted by A ∪ B) is the set of elements which are in A, in B, or in both A and B. Hence,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Example: If A = {10, 11, 12, 13} and B = {13, 14, 15}, then A∪B = {10, 11, 12, 13,14, 15}. (The common element occurs only once)

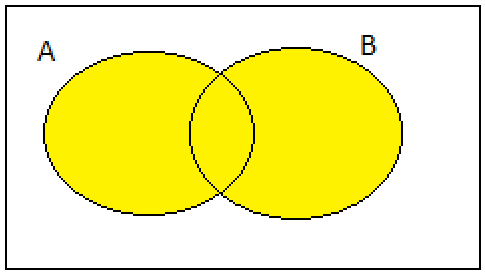


Figure: Venn diagram of A∪B

Set Intersection:-The intersection of sets A and B (denoted by A ∩ B) is the set of elements which are in both A and B. Hence,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example: If A = {11, 12, 13} and B = {13, 14, 15}, then A∩B = {13}.

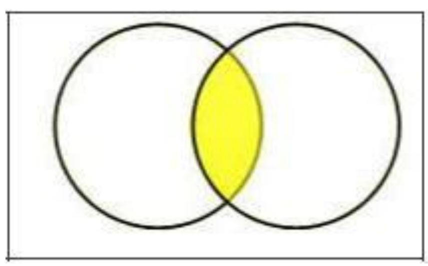
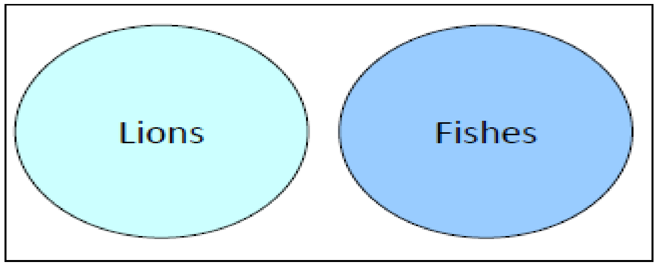


Figure: Venn diagram of A∩B

Note:-Two sets A and B are called **disjoint** sets if they do not have even one element in common i.e. n (A ∩ B) = Ø

Example:



Lions $\cap$ Fishes = $\varnothing$

Set Difference/ Relative Complement:-The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence,

$$A-B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Example: If A = {10, 11, 12, 13} and B = {13, 14, 15}, then (A−B) = {10, 11, 12} and (B−A) = {14,15}. Here, we can see (A−B) ≠ (B−A)

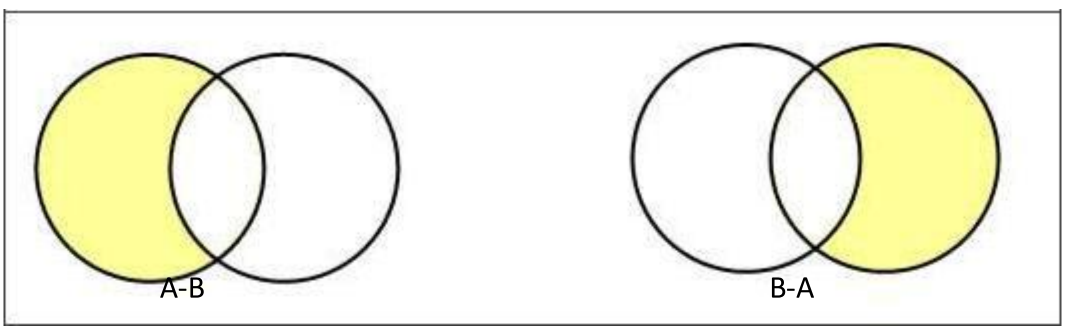


Figure: Venn diagram of A–B and B–A.

Complement of a Set:-The complement of a set A (denoted by A’) is the set of elements which are not in set A. Hence,

$$A' = \{x \mid x \notin A\}.$$

More specifically, A'= (U–A) where U is a universal set which contains all objects.

Example: If A ={x | x belongs to set of odd integers} then A' ={y | y does not belong to set of odd integers}

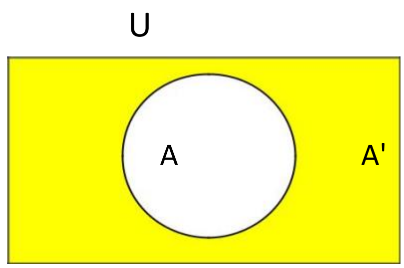


Figure: Venn diagram of A'

Some Properties of Complement of Sets:

- A ∪ A’ = U
- A ∩ A’ = Φ
- U’ = Φ
- Φ’ = U
- (A’)’ = A

Symmetric difference of two sets:-For any set A and B, their symmetric difference (A – B) ∪ (B – A) is defined as set of elements which do not belong to both A and B. It is denoted by A Δ B. Hence,

$$A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}.$$

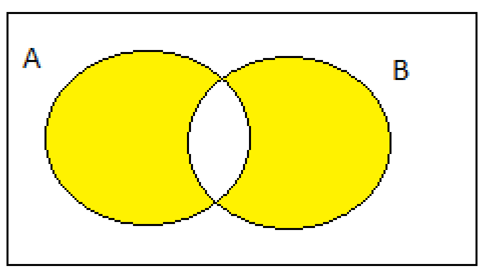


Figure: Venn diagram of A Δ B = (A – B) ∪ (B – A)

Example: If A = {10, 11, 12, 13} and B = {13, 14, 15}, then (A−B) = {10, 11, 12} and (B−A) = {14,15}.

A Δ B = (A – B) ∪ (B – A)={10,11,12,14,15}.

Lecture -2

Algebra of Sets, Combination of Sets and Duality

Algebra of Sets:

Algebra of sets explains the basic properties and laws of sets, i.e. the set -theoretic operations of union, intersection, and complementation. It also explains the relations of set equality and set inclusion. Systematic procedure for evaluating expressions, along with performing calculations which involve these operations and relations are included as well.

Laws of Algebra of Sets:

Some of the useful properties/operations on sets are as follows:

Letting A, B, C range over subsets of U.

1) **Idempotent Laws:** For any set A, we have

(a) $A \cup A = A$

(b) $A \cap A = A$

2) **Identity Laws:** For any set A, we have

(a) $A \cup \Phi = A$

(b) $A \cap U = A$

3) **Commutative Laws:** For any two sets A and B, we have

(a) $A \cup B = B \cup A$
(b) $A \cap B = B \cap A$

Proof: (a) $A \cup B = B \cup A$

Let x be any arbitrary element of $A \cup B$. Then

$x \in (A \cup B) = \{ x: x \in A \text{ or } x \in B \}$

⇨ $x \in B$ or $x \in A$

⇨ $x \in B \cup A$

Thus, $x \in (A \cup B) \Rightarrow x \in (B \cup A)$

⇨ $A \cup B \subsetneq B \cup A$ ………………….(1)
Similarly
$x \in (B \cup A) = \{x: x \in B \text{ or } x \in A\}$
⇨ $\{ x : x \in B \text{ or } x \in A \}$
⇨ $X \in (A \cup B)$
⇨ $x \in (B \cup A) \Rightarrow x \in (A \cup B)$
⇨ $B \cup A \subsetneq A \cup B$ ………………(2)

From equation (1) and (2) we get

$$A \cup B = B \cup A$$

Similarly we can prove part (b).

4) Associative Laws: For any three sets A, B and C, we have

(a) $A \cup (B \cup C) = (A \cup B) \cup C$
(b) $A \cap (B \cap C) = (A \cap B) \cap C$

Proof: (a) $(A \cup B) \cup C = A \cup (B \cup C)$

Let x be any arbitrary element of $(A \cup B) \cup C$. Then

⇨ $x \in (A \cup B) \cup C$
⇨ $x \in (A \cup B)$ or $x \in C$
⇨ $x \in A$ or $x \in B$ or $x \in C$
⇨ $x \in A$ or ($x \in B$ or $x \in C$)

- ⇨ $x \in A$ or ($x \in (B \cup C)$)
- ⇨ $x \in A \cup (B \cup C)$
- ⇨ $(A \cup B) \cup C \subsetneq A \cup (B \cup C)$…………(1)

Let $x \in A \cup (B \cup C)$

- ⇨ $x \in A$ or $x \in (B \cup C)$
- ⇨ $x \in A$ or $x \in B$ or $x \in C$
- ⇨ $x \in A$ or ($x \in B$) or $x \in C$

- ⇨ $x \in (A \cup B)$ or $x \in C$
- ⇨ $x \in (A \cup B) \cup C$
- ⇨ $A \cup (B \cup C) \subsetneq (A \cup B) \cup C$……………….(2)

From equation (1) and (2) we get

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Similarly, we can prove part (b).

5) Distributive Laws: If A, B and Care three sets, then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let x be any arbitrary element of $A \cup (B \cap C)$. Then

- ⇨ $x \in A \cup (B \cap C)$
- ⇨ $x \in A$ or $x \in B \cap C$
- ⇨ $x \in A$ or ($x \in B$ and $x \in C$)
- ⇨ $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$
- ⇨ $x \in (A \cup B)$ and $x \in (A \cup C)$
- ⇨ $A \cup (B \cap C) \subsetneq (A \cup B) \cap (A \cup C)$ ……………(1)

Now let $x \in (A \cup B) \cap (A \cup C)$

- ⇨ $x \in (A \cup B)$ and $x \in (A \cup C)$
- ⇨ ($x \in A$ or $x \in B$)and ($x \in A$ or $x \in C$)

- ⇨ $x \in A$ or ($x \in B$ and $x \in C$)
- ⇨ $x \in A$ or $x \in B \cap C$
- ⇨ $x \in A \cup (B \cap C)$
- ⇨ $(A \cup B) \cap (A \cup C) \subsetneq A \cup (B \cap C)$……………(2)

  From equation (1) and (2) we get

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Similarly, we can prove part (b).

6) De-Morgan's Laws: If A and B are two sets, then

(a) $(A \cup B)^C = A^C \cap B^C$

(b) $(A \cap B)^C = A^C \cup B^C$

Proof: (a) $(A \cup B)^C = (A^C \cap B^C)$

$x \in (A \cup B)^C = x \notin (A \cup B) = x \notin A$ and $x \notin B \Rightarrow x \in A^C$ and $x \in B^C \Rightarrow x \in A^C \cap B^C$

So $(A \cup B)^C$ is a subset of $(A^C \cap B^C)$

Similarly $(A^C \cap B^C)$ is a subset of $(A \cup B)^C$

Therefore, $(A \cup B)^C = (A^C \cap B^C)$.

(b) $(A \cap B)^C = A^C \cup B^C$

Let $x \in (A \cap B)^C \Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A^C$ or $x \in B^C = x \in A^C \cup B^C$

So $(A \cap B)^C$ is a subset of $A^C \cup B^C$

Similarly $A^C \cup B^C$ is a subset of $(A \cap B)^C$

Therefore $(A \cap B)^C = A^C \cup B^C$

Illustrative Examples:

Example: For sets A and B using properties of sets, prove that

(a) $A-(B \cup C) = (A-B) \cap (A-C)$

(b) $A-(B \cap C) = (A-B) \cup (A-C)$

(c) $(A \cup B)-C = (A-C) \cup (B-C)$

Proof:

(a) $A-(B \cup C) = A \cap (B \cup C)^C$

$= A \cap (B^C \cap C^C)$

$= (A \cap B^C) \cap (A \cap C^C)$

$= (A-B) \cap (A-C)$

(b) $A-(B \cap C) = A \cap (B \cap C)^C$

$= A \cap (B^C \cup C^C)$

$= (A \cap B^C) \cup (A \cap C^C)$

$= (A-B) \cup (A-C)$

(c) $(A \cup B)-C = (A \cup B) \cap C^C$

$= (A \cap C^C) \cup (B \cap C^C)$

$= (A-C) \cup (B-C)$

Example: For sets A and B prove that $A \cup B = (A-B) \cup (B-A) \cup (A \cap B)$.

Proof: R.H.S. $= (A-B) \cup (B-A) \cup (A \cap B)$

$= (A-B) \cup [(B-A) \cup (A \cap B)]$

$= (A \cap B^C) \cup [(B \cap A^C) \cup (B \cap A)]$

$= (A \cap B^C) \cup [(B \cap (A^C \cup A)]$

$= (A \cap B^C) \cup [(B \cap X]$ where X is universal set

$= (A \cap B^C) \cup B$

$= (A \cup B) \cap (B^C \cup B)$

$= (A \cup B) \cap X$

$= A \cup B$

= L.H.S.

Example: Let A and B be the following subsets of the real numbers defined as $A = \{x: 0<x<5\}$ and $B = \{x: 2<x<8\}$. Express $A \cup B$ as the union of three disjoint sets.

Solution: $A = \{x: 0<x<5\}$, $B = \{x: 2<x<8\}$.

As $A \cup B = (A-B) \cup (B-A) \cup (A \cap B)$.

Here $A-B = \{x: 0<x\le 2\}$, $B-A = \{x: 5\le x<8\}$ and $A \cap B = \{x: 0<x<2\}$

Therefore, $A \cup B = \{x: 0<x\le 2\} \cup \{x: 5\le x<8\} \cup \{x: 0<x<2\}$.

Duality

Suppose E is an equation of set algebra, the dual E* of E is the equation obtained by replacing each occurrence of ∪,∩, U and Φ in E by ∩,∪, Φ and U respectively.

Example:

1) The dual of $(\mathrm{A} \cup \mathrm{B}) \cap (\mathrm{A} \cup \mathrm{B}^{\mathrm{C}}) = \mathrm{A} \cup \Phi$ is $(\mathrm{A} \cap \mathrm{B}) \cup (\mathrm{A} \cap \mathrm{B}^{\mathrm{C}}) = \mathrm{A} \cap \mathrm{U}$).

2) The dual of $\mathrm{A} \cup \mathrm{A}^{\mathrm{c}} = \mathrm{X}$ is $\mathrm{A} \cap \mathrm{A}^{\mathrm{c}} = \emptyset$.

3) The dual of $\mathrm{A} \cup (\mathrm{B} \cap \mathrm{A}) = \mathrm{A}$ is $\mathrm{A} \cap (\mathrm{B} \cup \mathrm{A}) = \mathrm{A}$.

Lecture -3

Cardinality of Sets, Classes of Sets, Power Sets

Cardinality of Sets:

Cardinality of a Set: - Cardinality of a set S, denoted by |S|, is the number of distinct elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞. The cardinality of the set *A* is often notated as |*A*| or n(*A*)

Example: | {1, 4, 3, 5}| = 4, |{1, 2, 3, 4, 5,…}| = ∞

Illustrative Examples:

Example 1: Let A = {1, 2, 3, 4, 5, 6} and B = {2, 4, 6, 8}. What is the cardinality of B? A ⋃ B, A ⋂ B?

Solution:

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of A ⋃ B is 7, since A ⋃ B = {1, 2, 3, 4, 5, 6, 8}, which contains 7 elements.

The cardinality of A ⋂ B is 3, since A ⋂ B = {2, 4, 6}, which contains 3 elements.

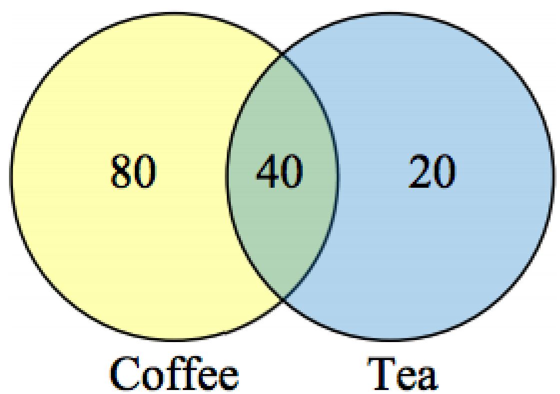
Example 2: What is the cardinality of *P* = the set of English names for the months of the year?
Solution: The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 3: A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

(a) Tea only
(b) Coffee only
(c) Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?
Solution: This question can most easily be answered by creating a Venn diagram.



We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

200 – 20 – 80 – 40 = 60 people who drink neither.

Example 4: A survey asks: “Which online services have you used in the last month?”

(a) Twitter

(b) Facebook
(c) Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Solution: Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice.
i.e. cardinality of $F \cup T$ = 70% + 40% – 20% = 90%

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T$ = 90%, the cardinality of $(F \cup T)^c$ must be the other 10%.

Classes of Sets

A class is a generalized set invented to get around Russell's antinomy while retaining the arbitrary criteria for membership which leads to difficulty for sets. The members of classes are sets, but it is possible to have the class C of "all sets which are not members of themselves" without producing a paradox (since C is a proper class (and not a set) it is not a candidate for membership in C). Since a big part of the point of set theory is to provide a simple, general toolkit for building *well-founded* mathematical structures, the ability to break it so easily was a serious problem. But no one wanted to give up set theory: it's too powerful, and too easy to use, to just give up on it because of this problem. So people set out to find a solution. The easiest one which avoids the problem was proposed by Alan Turing, and refined by Kurt Gödel. The idea is that instead of just having one kind of collection of things called a set, we'll create two different kinds of collections.

Any collection of things is called a *class*.

A collection of things which is also a *member* of some class is called a *set*.

A class which is *not* a set is called a *proper class*.

Outside set theory, the word "class" is sometimes used synonymously with "set". This usage dates from a historical period where classes and sets were not distinguished as they are in modern set-theoretic terminology. Many discussions of "classes" in the 19th century and earlier are really referring to sets, or perhaps rather take place without considering that certain classes can fail to be sets.

Power set of a set:

Definition. Let A be any set. The power set of A is the set P(A) = {B | B ⊂ A}. In words, the power set of A is the set of all the subsets of A.

Note: 1) For any set A we have ∅ ∈ P(A) and A ∈ P(A), so P(A) is nonempty for every set A. The power set of A is frequently denoted 2^A.

2) Power set of a finite set is finite.

Example:

1) $P(\{a,b\}) = \{\Phi, \{a\}, \{b\}, \{a,b\}\}$.
2) If A = {1,2,3} then $P(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
3) If $A = \Phi$ then $P(A) = \{\Phi\}$.

Number of Elements in Power Set:

For a given set S with n elements, number of elements in P(S) is 2^n.

As each element has two possibilities (present or absent}, possible subsets are 2×2×2.. n times = 2^n. Therefore, power set contains 2^n elements.

We can also justify this with the following theorem.

Theorem: Prove that a set containing n distinct elements has 2^n subsets *i.e.* $|P(A)| = 2^n$.

Proof: Let $A = \{a_1, a_2, a_3, \ldots \ldots, a_n\}$ where a_i's are distinct.

A selection of r objects from the elements of the set A can be made ${}^nC_r$ way where $0 \leq r \leq n$. Hence, there are ${}^nC_r$ subsets of a set A which contains r elements.

Therefore,

Number of subsets of A containing no element is ${}^nC_0$.

Number of subsets of A containing one element is ${}^nC_1$.

Number of subsets of A containing two elements is ${}^nC_2$.

……………………………………………………………………………………….

……………………………………………………………………………………….

Number of subsets of A containing n elements is ${}^nC_n$.

Hence, the total number of elements of A = ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n$

$$= 2^n$$

Theorem : For any sets A and B, A ⊂ B if and only if P(A) ⊂ P(B).

Proof: First assume that A ⊂ B and let X ∈ P(A).

By definition X ⊂ A, so as A ⊂ B implies that X ⊂ B.

Hence X ∈ P(B) and P(A) ⊂ P(B).

Conversely,

If we assume that P(A) ⊂ P(B), then A ∈ P(A) ⊂ P(B) and A ⊂ B by definition.

Theorem: For any sets A and B, P(A ∩ B) = P(A) ∩ P(B).

Proof: First note that A ∩ B ⊂ A , so previous theorem implies that

P(A ∩ B) ⊂ P(A).

The same reasoning shows that P(A ∩ B) ⊂ P(B), so we have

P(A ∩ B) ⊂ P(A) ∩ P(B).

Now suppose that $X \in P(A) \cap P(B)$, then $X \in P(A)$ and $X \in P(B)$.

By definition $X \subset A$ and $X \subset B$, so $X \subset A \cap B$.

It follows that $P(A \cap B) \supset P(A) \cap P(B)$, so $P(A \cap B) = P(A) \cap P(B)$ as desired.

Example: Is it true that power set of AUB is equal to union of power sets of A and B? Justify.

Solution: Let A = {a , b} , B = {c} then AUB = { a , b , c }

P(A) = { Ø , { a } , { b } , { a , b } }

P(B) = { Ø , { c } }

P(A) U P(B) = { Ø , { a } , { b } , { c } , { a , b } }

Whereas

P(AUB) = { Ø , { a } , { b } , { c } , { a , b } , { a , c } , { b , c } , { a , b , c } }

Therefore P(AUB) ≠ P(A) U P(B).

Countable set and its power set

A set is called countable when its element can be counted. A countable set can be finite or infinite

Example: set S1 = {a, e, i, o, u} representing vowels is a countably finite set. However, S2 = {1, 2, 3……} representing set of natural numbers is a countably infinite set.

Note

- Power set of countably finite set is finite and hence countable. For example, set S1 representing vowels has 5 elements and its power set contains 2^5 = 32 elements. Therefore, it is finite and hence countable.
- Power set of countably infinite set is uncountable. For example, set S2 representing set of natural numbers is countably infinite. However, its power set is uncountable.

Uncountable set and its power set

A set is called uncountable when its element can't be counted. An uncountable set can be always infinite.

Example: Set S containing all fractional numbers between 1 and 10 is uncountable.

Note

- Power set of uncountable set is always uncountable.

Example: Set S representing all fractional numbers between 1 and 10 is uncountable. Therefore, power set of uncountable set is also uncountable.

Lecture -4

Min Sets & Max Sets, Cartesian product

Partition of Sets:

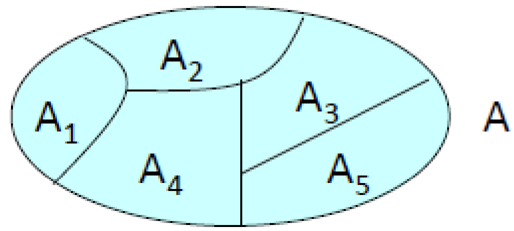
Definition Let S be a nonempty set. A **partition** of S is a subdivision of S into non overlapping, nonempty subsets. Precisely, a partition of S is a collection {Ai} of non empty subsets of S such that:

(i) Each a in S belongs to one of the A_i.

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_j \neq A_j \text{ then } A_j \cap A_k = \Phi$$

The subsets in a partition are called cells.



A is partitioned into five cells, A_1, A_2, A_3, A_4, A_5

Example: Consider the following collections of subsets of the set

S = {1, 2, …,10}:

(i) [{1, 3, 5 }, {2, 7}, {4, 8, 9, 10}]

(ii) [{2, 3, 7, 10}, {1, 4, 5, 8}, {6, 7, 9}]

(iii) [{1, 4, 5}, {2, 3, 7, 8, 10}, {6, 9}]

Then (i) is not a partition of *S* since 6 in *S* does not belong to any of the subsets. Furthermore, (ii) is not a partition of *S* since {2, 3, 7, 10} and {6, 7, 9} are not disjoint. On the other hand, (iii) is a partition of *S*.

Illustrative Examples:

Example: Let X={1,2,3,……,8,9}. Determine whether or not each of the following is a partition of X.

(a) { {1,3,6} , {2,8} , {5,7,9} }

(b) { {2,4,5,8} , {1,9} , {3,6,7} }

(c) { {1,5,7} , {2,4,8,9} , {3,5,6} }

Solution:

(a) { {1,3,6} , {2,8} , {5,7,9} } is not Partition. As {1,3,6} U {2,8} U {5,7,9} ≠ X

(b) { {2,4,5,8} , {1,9} , {3,6,7} } is Partition. As

{2,4,5,8} U {1,9} U {3,6,7} = X and

{2,4,5,8} ∩{1,9}=∅, {1,9}∩{3,6,7}=∅, {2,4,5,8} ∩{3,6,7} =∅

(d) { {1,5,7} , {2,4,8,9} , {3,5,6} } is not Partition. As {1,5,7} ∩{3,5,6}={5}≠∅

Example: Find all the partitions of the set A={ a , b , c , d }.

Solution: All the partitions of the set A are

$$P_1 = \{\{a\},\{b\},\{c\},\{d\}\}$$

$$P_2 = \{ \{a\},\{b,c,d\}\}$$

$$P_3 = \{ \{b\},\{a,c,d\}\}$$

$$P_4 = \{ \{c\},\{a,b,d\}\}$$

$$P_5 = \{ \{d\},\{a,b,c\}\}$$

$$P_6 = \{ \{a,b\},\{c,d\}\}$$

$$P_7 = \{ \{a,c\},\{b,d\}\}$$

$$P_8 = \{ \{a,d\},\{b,c\}\}$$

$$P_9 = \{ \{a\},\{b\},\{c,d\}\}$$

$$P_{10} = \{ \{a\},\{c\},\{b,d\}\}$$

$$P_{11} = \{ \{a\},\{d\},\{b,c\}\}$$

$$P_{12} = \{ \{b\},\{c\},\{a,d\}\}$$

$$P_{13} = \{ \{b\},\{d\},\{a,c\}\}$$

$$P_{14} = \{ \{c\},\{d\},\{a,b\}\}$$

$$P_{15} = \{ \{a,b,c,d\}\}$$

Minimum Set or Minset or Minterm:

Let A be any nonempty set and $B_1, B_2, \ldots\ldots, B_n$ be any subsets of A. Then the minimum set generated by the collection $\{B_1, B_2, \ldots\ldots, B_n\}$ is a set of the type $D_1 \cap D_2 \cap \ldots\ldots \cap D_n$ where each $D_1, D_2, \ldots\ldots, D_n$ is B_i or B_i^c for $i = 1,2,3,\ldots\ldots,n$.

Example: The minimum set generated by B_1 and B_2 are

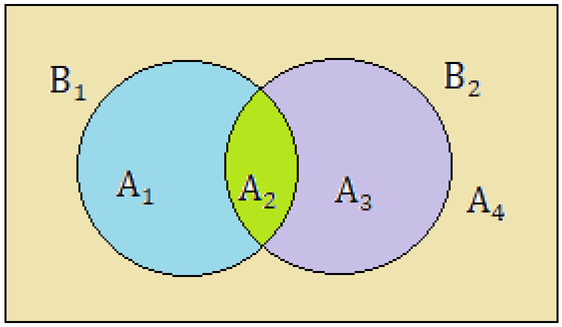
$$A_1 = B_1 \cap B_2^c$$

$$A_2 = B_1 \cap B_2$$

$$A_3 = B_1^c \cap B_2$$

$$A_4 = B_1^c \cap B_2^c$$

Venn diagram



Example: The minimum set generated by $B_{1,}, B_2$ and B_3 are

$$A_1 = B_1 \cap B_2 \cap B_3^c$$

$$A_2 = B_1 \cap B_2^c \cap B_3$$

$$A_3 = B_1 \cap B_2^c \cap B_3^c$$

$$A_4 = B_1^c \cap B_2 \cap B_3$$

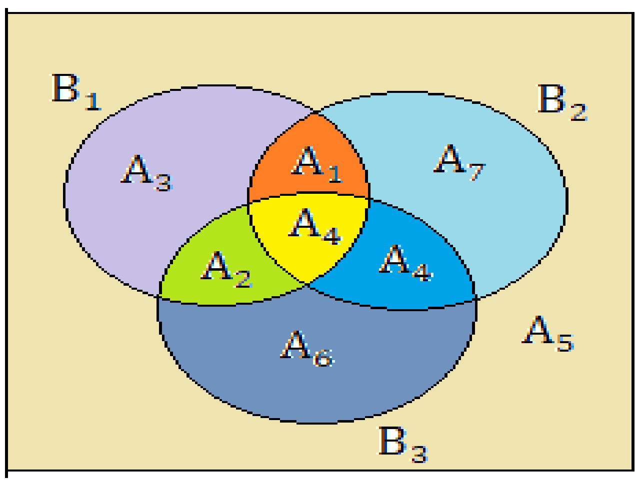
$$A_5 = B_1^c \cap B_2^c \cap B_3^c$$

$$A_6 = B_1^c \cap B_2^c \cap B_3$$

$$A_7 = B_1^c \cap B_2 \cap B_3^c$$

$$A_4 = B_1 \cap B_2 \cap B_3$$

Venn diagram



Remark: 1) The number of minsets generated by n sets is 2^n.

2) The collection of all the nonempty minsets generated y given subsets gives rise the partitions of the set.

Maximum Set or Maxset or Maxterm:

The dual of the Minset is the Maxset i.e. A maxset is the maximum or the largest set that is obtained by the union of any two or more subsets of a partition set.

Illustrative Examples:

Example: Partition the set A={ 1,2,3,4,5,6 } using the minsets generated by $B_1 = \{1,3,5\}$ and $B_2 = \{1,2,3\}$.

Solution: The number of minsets generated by B_1 and B_2 is 2^2=4. These are

$$A_1 = B_1 \cap B_2^c = \{5\}$$

$$A_2 = B_1 \cap B_2 = \{1,3\}$$

$$A_3 = B_1^c \cap B_2 = \{2\}$$

$$A_4 = B_1^c \cap B_2^c = \{4,6\}$$

Since $A_1 \cup A_2 \cup A_3 \cup A_4 = A$ and $A_i \cap A_j = \phi$ for $i \neq j$.

Therefore, $\{A_1, A_2, A_3, A_4\}$ is a partition of the set A.

Example: Let A={ 1,2,3 } using the minsets and maxsets generated by $B_1 = \{1,2\}$ and $B_2 = \{2,3\}$.

Solution: The number of minsets and maxsets generated by B_1 and B_2 is 2^2=4.

$B_1 = \{1,2\}$,$B_2 = \{2,3\}$, $B_1^c = \{3\}$ and $B_2^c = \{1\}$

Minsets are

$$A_1 = B_1 \cap B_2^c = \{1\}$$

$$A_2 = B_1 \cap B_2 = \{2\}$$

$$A_3 = B_1^c \cap B_2 = \{3\}$$

$$A_4 = B_1^c \cap B_2^c = \emptyset$$

Maxsets are

$$M_1 = B_1 \cup B_2^c = \{1,2\}$$

$$M_2 = B_1 \cup B_2 = \{1,2,3\}$$

$$M_3 = B_1^c \cup B_2 = \{2,3\}$$

$$M_4 = B_1^c \cup B_2^c = \{1,3\}$$

Cartesian product

René Descartes
(1596-1650)

René Descartes invented the Cartesian product. It derives the name from the same person. René formulated analytic geometry which helped in the origination of this concept which we further generalize in terms of direct product.

Ordered Pair:
An **ordered pair** (a, b) is a pair of objects. The order in which the objects appear in the pair is significant: the ordered pair (a, b) is different from the ordered pair (b, a) unless $a = b$.

Cartesian Product / Cross Product of two sets:-

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B. In terms of set-builder notation, that is

$$\mathrm{A} \times \mathrm{B} = \{(\mathrm{a}, \mathrm{b}): \mathrm{a} \in \mathrm{A} \text{ and } \mathrm{b} \in \mathrm{B}\}$$

Note:

1) $\mathrm{A} \times \mathrm{B}$ and $\mathrm{B} \times \mathrm{A}$ are different sets if sets $\mathrm{A} \neq \mathrm{B}$ i.e. The Cartesian product $A \times B$ is not commutative,

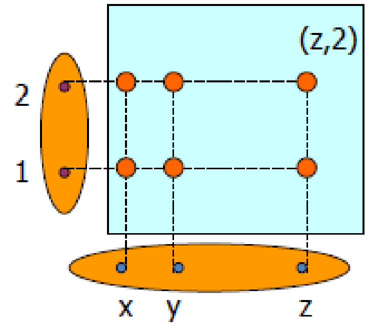
Example: If we take two sets A= {1, 2} and B= {x, y, z},

The Cartesian product of A and B is written as:

A×B= {(1, x), (1, y), (1, z), (2, x), (2, y), (2,z) }

The Cartesian product of B and A is written as:

B×A= {(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)}



2) $A \times B = \emptyset$ when one or both of A, B are empty.

Example :

$A = \{1,2\}; B = \{3,4\}$

$A \times B = \{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{3,4\} \times \{1,2\} = \{(3,1), (3,2), (4,1), (4,2)\}$

$A = B = \{1,2\}$

$A \times B = B \times A = \{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

$A = \{1,2\}; B = \emptyset$

$A \times B = \{1,2\} \times \emptyset = \emptyset$

$B \times A = \emptyset \times \{1,2\} = \emptyset$

3) The Cartesian product is not associative.

$$A \times (B \times C) \neq (A \times B) \times C$$

Example: If $A = \{1\}$, then $(A \times A) \times A = \{ ((1,1),1) \} \neq \{ (1,(1,1)) \} = A \times (A \times A)$.

4) $|A \times B| = |A| \cdot |B|$.

Example: Let A = {a, b} and B = {5, 6} then $A \times B = \{(a,5), (a,6), (b,5), (b,6)\}$.

Here $|A \times B| = 4 = 2.2 = |A| \cdot |B|$.

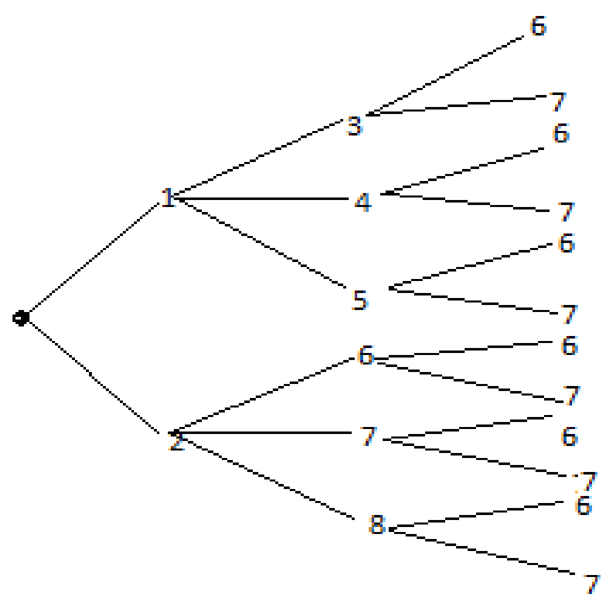
Cartesian product / Cross Product of n sets:-

The Cartesian product of n sets A_1, A_2.....A_n, denoted as $A_1 \times A_2 \times \ldots \times A_n$, can be defined as all possible ordered pairs $(x_1,x_2,\ldots x_n)$ where $x_1 \in A_1$, $x_2 \in A_2$, ... $x_n \in A_n$

.

Example: Let $A = \{1,2\}$, $B = \{3,4,5\}$ and $C = \{6,7\}$. Then $A \times B \times C$ consistsof all triple (a,b,c) , where a ∈, $b \in B$ and $c \in C$. Hence

$$A\times B\times C=\{(1,3,6),(1,3,7),(1,4,6),(1,4.7),(1,5,6),(1,5,7),(2,3,6),(2,3,7),(2,4,6),(2,4,7),(2,5,6),(2,5,7)\}$$



Lecture -5

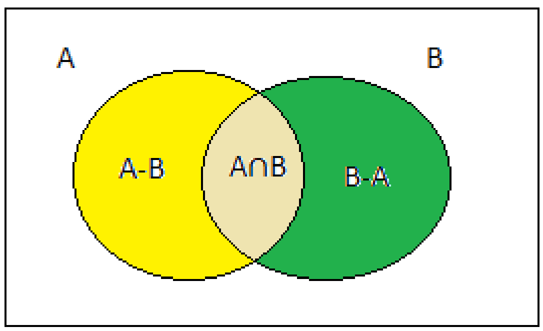
Principles of Inclusion & Exclusion

The Inclusion-Exclusion principle:-The Inclusion-exclusion principle computes the cardinal number of the union of multiple non-disjoint sets.

(1) For two sets A and B, the principle states:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: We know that $A \cup B$ is the union of three disjoint sets $A-B$, $A \cap B$ and $B-A$.



Therefore, $|A \cup B| = |A-B| + |B-A| + |A \cap B|$ (i)

Again A is union of $A-B$ and $A \cap B$, which are disjoint sets.

Therefore, $|A| = |A-B| + |A \cap B|$ (ii)

Similarly, $|B| = |B-A| + |A \cap B|$ (iii)

Adding (ii) and (iii) we get

$$|A| + |B| = |A-B| + |B-A| + 2|A \cap B|$$

$$= (|A-B| + |B-A| + |A \cap B|) + |A \cap B|$$

$$= |A \cup B| + |A \cap B|$$

$$\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|.$$

Hence Proved.

(2) For two disjoint sets A and B, the principle states:

$$|A \cup B| = |A| + |B|$$

Proof: Since A and B are disjoint sets, then $A \cap B = \emptyset \Rightarrow |A \cap B| = 0$

Therefore, $|A \cup B| = |A| + |B|$

Hence Proved.

(3) For three sets A, B and C, the principle states:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

(4) General Principle of Inclusion-Exclusion

Let $A_1, A_2, \dots \dots, A_n$ be n finite sets. Then

$$|A_1 \cup A_2 \cup \dots \dots \cup A_n|$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i,j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j \le k} |A_i \cap A_j \cap A_k|$$

$$- \cdots \dots \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \dots \cap A_n|$$

Illustrative Examples

Example 1: If A and B be two sets containing 3 and 6 elements respectively, what can be the maximum number of elements in $A \cup B$? Find also, the minimum number of elements in $A \cup B$.

Solution:

We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

This show that $n(A \cup B)$ is minimum or maximum according as

$n(A \cap B)$ is maximum or minimum respectively.

Case 1: When $n(A \cap B)$ is minimum, i.e. $n(A \cap B) = 0$. This is possible only when $A \cap B = \Phi$. In this case,

$$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$$

$$n(A \cup B)max = 9$$

Case 2: When $n(A \cap B)$ is maximum

This is possible only when $A \subset B$.

In this case $n(A \cap B) = 3$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6$$

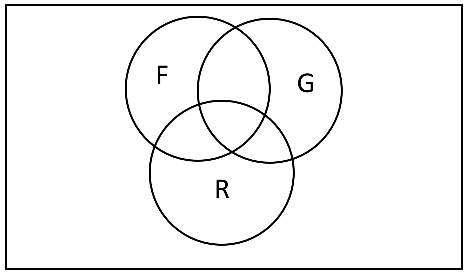
$$n(A \cup B)min = 6.$$

Example 2: Suppose that 100 of 120 students at a college take at least one of the languages French, German and Russian, also suppose 65 study French, 45 study German, 42 study Russian, 20 study French and German, 25 study French and Russian and 15 study German and Russian.

Find number of students who study all three languages and to fill in correct number of students in each of the eight regions.

Solution:

Let F, G and R denote the sets of students studying French, German and Russian respectively. Then the Venn diagram shown as below



$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$

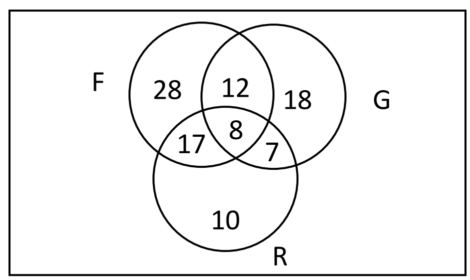
$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(F \cap G \cap R)$

$n(F \cap G \cap R) = 8$ study the three languages.

20 – 8 = 12 study French and German but not Russian.

25 – 8 = 17 study French and Russian but not German.

15 – 8 = 7 study German and Russian but not French.



65 – 12 – 8 – 17 = 28 study only French.

45 – 12 – 8 – 7 = 18 study only German.

42 – 17 – 8 – 7 = 10 study only Russian.

120 – 100 = 20 do not study any of the language.

18 + 10 + 28 = 56 students study only one language.

Example 3: Find the number of positive integer between 1 and 1000 which are divisible neither by 2 nor by 5.

Solution: Let A_1 and A_2 be the sets of positive integers from 1 to 1000 which are divisible by 2 and 5 respectively. Then

$|A_1| = \left\lfloor \frac{1000}{2} \right\rfloor = 500$ and $|A_2| = \left\lfloor \frac{1000}{5} \right\rfloor = 200.$

Integers in the set $A_1 \cap A_2$ are divisible by both 2 and 5 and we know that an integer is divisible by both 2 and 5 iff it is divisible by lcm[2,5]=10.

Therefore, $|A_1 \cap A_2| = \left\lfloor \frac{1000}{10} \right\rfloor = 100.$

By Inclusion-Exclusion Principle, the number of integers between 1 and 1000 which are divisible neither by 2 nor by 5 = $|\overline{A_1} \cap \overline{A_2}|$

= 1000-$(|A_1|+|A_2| - |A_1 \cap A_2|)$

= 1000-(500+200-100)

= 400