Algebraic Structures

- Defxn
- elementary properties of algebraic Structures
- Semi group
- Monoid
- Group
- homomorphism
- isomorphism and automorphism
- Subgroups
- normal subgroups, cyclic groups.

• **Algebraic Structure:** →

→ if there exists a system such that it consists of a non-empty set and one or more operations on that set, then that system is called an algebraic system. It is generally denoted by $(A, op_1, op_2 \dots op_n)$ where A is a non-empty set and $op_1, op_2 \dots op_n$ are operations on A. An algebraic system is also called an algebraic structure because the operations on Set A define structure on the elements of A.

(2) Binary operation

Consider a non-empty set A and a function such that $f: A \times A \rightarrow A$, then f is called binary operation on A whose domain is the set of ordered pairs of elements of A.

If $*$ is a binary operation on A, then it may be written as $a * b$.

- A binary operation can be denoted by any of the symbols $+, -, *, \oplus, \triangle, \boxdot, \vee, \wedge$ etc.

- TABLES OF OPERATION

Consider a non-empty finite set $A = \{a_1, a_2, a_3 \ldots a_n\}$

A binary operation $*$ on A can be described by means of table as shown below:

$*$	a_1	a_2	a_3		a_n
a_1	$a_1 * a_1$	$a_1 * a_2$			
a_2	$a_2 * a_1$	$a_2 * a_2$			
a_3			$a_3 * a_3$		
a_n					$a_n * a_n$

Q:→ Consider the set $A = \{1, 2, 3\}$ and a binary operation $*$ on the set A defined by $a * b = 2a + 2b$. Represent operation $*$ as a table on A. (2)

Ans:→ The table of operations is shown as below:

$*$	1	2	3
1	4	6	8
2	6	8	10
3	8	10	12

• **Properties of Binary Operations :→**

• **Closure Property :→** Consider a non-empty set A and a binary operation $*$ on A. Then A is closed under the operation $*$, if $a * b \in A$, where a and b are elements of A.

eg. the operation of addition on the set of integers is a closed operation i.e. if $a, b \in Z$, then $a + b \in Z \ \forall a, b \in Z$.

Ans:→ Consider the set $A = \{-1, 0, 1\}$. Determine whether A is closed under (i) addition
2) multiplication.

① The sum of the elements is $(-1)+(-1) = -2$ and $1+1=2$ does not belong to A. Hence A is not closed under ~~multi~~ addition.

② The multiplication of every 2 elements of the set are

$-1 * 0 = 0$; $-1 * 1 = -1$; $-1 * -1 = 1$

$0 * -1 = 0$; $0 * 1 = 0$; $0 * 0 = 0$

$1 * -1 = -1$; $1 * 0 = 0$; $1 * 1 = 1$

Since each multiplication belongs to A, hence A is closed under multiplication.

<u>Q2.</u> Consider the set $A = \{1, 3, 5, 7, 9 \ldots\}$, the set of odd +ve integers. Determine whether A is closed under (1) addition (2) multiplication

<u>Ans:</u> → (i) The set A is not closed under addition because the addition of 2 odd numbers produces an even number which does not belong to A.

(ii) The set A is closed under the operation multiplication because the multiplication of 2 odd nos produces an odd no. So, for every $a, b \in A$ we have $a * b \in A$.

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(2) __Associative property__ :→ Consider a non-empty set A and a binary operation $*$ on A. Then, the operation $*$ on A is associative, if for everyone $a, b, c \in A$, we have $(a * b) * c = a * (b * c)$ (3)

Q:→ Consider the binary operation $*$ on Q, the set of rational numbers defined by

$$a * b = a + b - ab \quad \forall\ a, b \in Q$$

Determine whether $*$ is associative.

(b) Consider the binary operation $*$ on the set N of +ve integers defined by

$$a * b = a^b$$

Determine whether $*$ is associative?

__Ans:__→ (a) $(a * b) * c = (a + b - ab) * c$

$a * b$

Put the value

$$= \underbrace{(a + b - ab)}_{a} + \underbrace{c}_{b} - \underbrace{(a + b - ab)\,c}_{ab}$$

$$= a + b - ab + c - ca - bc + abc$$

$$= a + b + c - ab - ac - bc + abc$$

Similarly, we have

$$a * (b * c) = (a + b + c - ab - ac - bc + abc)$$

$$\therefore (a * b) * c = a * (b * c)$$

Hence $*$ is associative.

(6) $*$ will be associative if

$$a * (b * c) = (a * b) * c \quad \forall a, b, c \in N.$$

Take $a = 2$, $b = 2$, $c = 3$ and consider

$$a * (b * c) = 2 * (2 * 3)$$
$$= 2 * 2^3$$
$$= 2 * 8 = 2^8 = 256$$

$$(a * b) * c = (2 * 2) * 3$$
$$= 2^2 * 3$$
$$= 4 * 3$$
$$= 4^3 = 64$$

Hence $a * (b * c) \neq (a * b) * c$

~~Hence~~ $\therefore$ $*$ is non-associative.

- <u>Commutative Property</u> :→

Consider a non-empty set A and a binary operation $*$ on A. $*$ is commutative if for every $a, b \in A$, we have $a * b = b * a$

Q:→ (a) Consider the binary operation $*$ on Q, the set of rational numbers, defined by

$$a * b = a^2 + b^2 \quad \forall a, b \in Q$$

Determine whether $*$ is commutative? (4)

(b) Consider $S = \{a, b, c, d\}$ and $*$ be a binary operation on S defined as shown in the following table.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	a	b
c	c	b	a	a
d	d	a	a	a

Determine (i) whether $*$ is associative?
(ii) whether $*$ is commutative?

Sol'n (a) Let us assume some elements $a, b \in Q$,
then by definition
$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$
Hence $*$ is commutative.

(b) (i) Let $a, b, c \in S$ and consider
$$b * (c * c) = b * a = b$$
and
$$(b * c) * c = a * c = c$$
$$b * (c * c) \neq (b * c) * c$$
Thus, $*$ is non-associative.

(ii) $b * c = a$ and $c * b = b$
$$\Rightarrow b * c \neq c * b$$
$\therefore$ $*$ is non-commutative.

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Ans - (2) Let $a, b, c \in \mathbb{Q}$, then by defxn, we have

$$(a * b) * c = \left(\frac{ab}{2}\right) * c$$

$$= \frac{\frac{ab}{2} \cdot c}{2} = \frac{abc}{4}.$$

Similarly, $a * (b * c) = a * \left(\frac{bc}{2}\right)$

$$= \frac{\frac{abc}{2}}{2} = \frac{abc}{4}$$

$\therefore$ $a * (b * c) = a * (b * c)$

Hence $*$ is associative.

Q ⇒ Consider the binary operation $*$ and $\mathbb{Q}$, set of rational no's defined by

$$a * b = \frac{ab}{2} \quad \forall\, a, b \in \mathbb{Q}$$

Determine whether $*$ is (i) associative (ii) Commutative.

(i) Let $a, b \in \mathbb{Q}$, then we have

$$a * b = \frac{ab}{2} = b * a = \frac{ba}{2}$$

$$= b * a$$

Hence $*$ is commutative

(4) **Identity** :→

Consider a non empty set A and a binary operation $*$ on A. Then the operation $*$ has an identity property if there exists an element e in A such that

$$a * e \text{ (right identity)} = e * a \text{ (left identity)}$$
$$= a \quad \forall\, a \in A.$$

Q:→ Consider the binary operation $*$ on I, the set of +ve integers defined by $a * b = \frac{ab}{2}$. Determine the identity for the binary operation $*$ if exists.

Ans:→ Let e be a +ve integer num,

$$e * a = a$$

$$\frac{ea}{2} = a \Rightarrow e = 2$$

$e\cancel{a} = \cancel{a}2$

Similarly,

$$a * e = a$$

$$\frac{ae}{2} = a \quad \text{or } e = 2$$

from (1) and (2) for $e = 2$, we have

$$e * a = a * e = a$$

$\therefore$ 2 is the identity element for $*$.

(5) **Idempotent** :→

Consider a non-empty set A and a binary operation $*$ on A. Then the operation $*$ has the inverse property i.e.

$$a * a = a \quad \forall\, a \in A.$$

(7) Distributivity :→

Consider a non-empty set A and 2 binary operations $*$ and $+$ on A.

$$a * (b+c) = (a*b) + (a*c)$$

and $$(b+c) * a = (b*a) + (c*a)$$

(8) Cancellation :→

Consider a non-empty set A and a binary operation $*$ on A. Then, operation $*$ has the cancellation property, if we have

$$a * b = a * c \Rightarrow b = c$$

and $$b * a = c * a \Rightarrow b = c$$

Semi-Group :→

Let us consider an Algebraic System $(A, *)$, where $*$ is a binary operation on A. Then the system $(A, *)$ is said to be a Semi-group, if it satisfies the following properties:

(1) The operation $*$ is a closed operation on set A.

(2) The operation $*$ is an associative operation.

Example:→ Consider an algebraic system $(A, *)$ ②
Where $A = \{1, 3, 5, 7, 9 \ldots\}$ the set of all +ve odd integers and $*$ is a binary operation means multiplication. Determine whether $(A, *)$ is a semi-group.

Ans:→ Closure:→ The operation $*$ is a closed operation because multiplication of two +ve odd integers is a +ve odd number.

Associative:→ The operation is an associative operation on set A. we have

$$(a * b) * c = a * (b * c) \to \text{eg.}$$

$(1 * 3) * 5 = 1 * (3 * 5)$
$15 = 15$

Hence, algebraic system $(A, *)$ is a semigroup

Q-2. Let S be a semi-group with an identity element e and if b and b' are inverses of an element $a \in S$, then prove that $b = b'$ i.e. inverse are unique, if they exist.

Ans:→ Given b is an inverse of a, ∴, we have

$$a * b = e = b * a$$

Also, b' is an inverse of a, ∴, we have

$$a * b' = e = b' * a$$

Consider $b * (a * b') = b * e = b$ —— ①
$(b * a) * b' = e * b' = b'$ —— ②

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Now, associativity holds in S.

$\therefore$ $\boxed{b = b'}$

Q:⇒ Let N be a set of +ve integers and let * be the binary operation of (L.C.M) on N. find

(a) $4 * 6$, $3 * 5$, $9 * 18$, $1 * 6$

(b) Is $(N, *)$ a semi-group.

(c) Is N commutative.

(d) find the identity element of N.

(e) Which elements of N have inverses?

Ans:⇒ Let $x, y \in N$ and $x * y =$ L.C.M of x and y.

$\therefore$ $4 * 6 =$ L.C.M of 4 and 6 $= 12$

$3 * 5 =$ L.C.M of 3 and $5 = 15$

$9 * 18 =$ L.C.M of 9 and $18 = 18$

$1 * 6 =$ L.C.M of 1 and $6 = 6$

(b) We know that operation of L.C.M is associative i.e.

$a * (b * c) = (a * b) * c \quad \forall a, b, c \in N.$

$\therefore$ N is a semi-group under $*$.

(c) Also for $a, b \in N$

$a * b =$ LCM of a and $b =$ L.C.M of b and $a = b * a$

$\therefore$ N is commutative also.

For $a \in N$, Consider $a * 1 =$ LCM of a and $1 = a$ (3)

Also, $1 * a =$ L.C.M of 1 and $a = a$

$\therefore \quad a * 1 = a = 1 * a$

i.e. 1 is the identity element of N.

(e) Consider $a * b = 1$ i.e. L.C.M of a and b is 1, which is possible if $a = 1$ and $b = 1$ i.e. the only element which has an inverse is 1 and it is its own inverse.

(d) Let e is the identity element of Q $\therefore$ for $a \in Q$, we have

$$a * e = a$$
$$a + e - ae = a$$
$$e - ea = 0$$
$$e(1 - a) = 0$$
$$e = 0 \quad \text{if } a \neq 1$$

$\therefore$ identity of Q is 0.

Q→ Consider the set Q of rational no's and let $*$ be the operation on Q defined by $a * b = a + b - ab$

(a) find $3*4$, $2*(-5)$, $7*\frac{1}{2}$

(b) Is $(Q, *)$ a semi group?

(c) Is Q commutative?

(d) Find the identity element of Q.

Ans→ (a) $a*b = a+b-ab$

$3*4 = 3+4-12 = -5$, $2*(-5) = 2+(-5)-(-10)$
$= 2-5+10 = 7$

$7*\frac{1}{2} = 7+\frac{1}{2}-\frac{7}{2} = 4$

(b) Q will be a semi group if it holds associativity under $*$ for $a, b, c \in Q$

Consider $a*(b*c) = a*(b+c-bc)$
$= a+(b+c-bc) - a(b+c-bc)$
$= a+b+c-bc-ab-ac+abc$ — (1)

Also, $(a*b)*c = (a+b-ab)*c$
$= a+b-ab+c-(a+b-ab)c$
$= a+b+c-ab-ac-bc+abc$
$= a+b+c-bc-ab-ac+abc$ — (2)

from (1) and (2)

$a*(b*c) = (a*b)*c$

(c) $a*b = a+b-ab = b+a-ba = b*a$

∴ Q is commutative

Monoid :→

Let us consider an algebraic system (A, o) where o is the binary operation on A. Then the system (A, o) is said to be a monoid if it satisfies the following properties:

1. The operation o is a closed operation on set A.
2. The operation o is an associative operxn.
3. There exists an identity element w.r.t the operation o.

examples :- $(N, \times)$, $(Z, +)$, $(Q, +)$ are monoids.

Q:→ Consider an algebraic system $(I, +)$ where the set $I = \{0, 1, 2, 3, 4 \ldots\}$ the set of natural nos. including 0 and $+$ is an addition operation. Determine whether $(I, +)$ is a monoid.

Ans:→ Closure property :→ The operation $+$ is closed since sum of 2 natural no. is a natural no.

Associative :→ $(a+b)+c = a+(b+c)$

Identity :→ The element 0 is an identity element w.r.t operation $+$.

Hence, $(I, +)$ is a monoid.

Q-2. Let S be a finite set and F(S) be the collection of all functions $f : S \to S$ under the operation of composition of functions. Show that F(S) is a semi-group. Is F(S) a monoid?

Ans:- Let $f, g, h \in F(S)$, then we know that composition of functions is associative i.e.

$$f \circ (g \circ h) = (f \circ g) \circ h \quad \forall f, g, h \in F(S).$$

Hence F(S) is a semigroup. Also the identity fxn is an identity element of F(S).

$\therefore$ F(S) is a monoid.

Group

→ Let us consider an algebraic system $(G, *)$ where $*$ is the binary operation on G. Then the system $(G, *)$ is said to be a group if it satisfies the following properties;

1. The operation $*$ is a closed operxn.
2. The operation $*$ is an associative operxn.
3. There exists an identity element w.r.t the operation

(ii) for every $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = e$

ex:→ (1) The sets $(Q, +)$ $(R, +)$ and $(C, +)$ are groups under addition.

(2) The sets R^* (set of non-zero reals) Q^* (set of non-zero rationals) and C^* (set of non zero complex numbers) are groups under multiplication.

Q- Consider an algebraic system $(Q, *)$ where Q is the set of rational numbers and $*$ is a binary operation defined by

$$a * b = a + b - ab \quad \forall\, a, b \in Q$$

Determine whether $(Q, *)$ is a group.

Ans:→
<u>Closure</u> :- Since the elements $a * b \in Q$ for every $a, b \in Q$ hence the set Q is close under the operation $*$.

(2) **Associative** :→ Let us assume $a, b, c \in Q$, then we have

$$(a*b)*c = (a+b-ab)*c$$
$$= (a+b-ab)+c-(a+b-ab)c$$
$$= a+b-ab+c-ac-bc+abc$$
$$= a+b+c-ab-ac-bc+abc$$

Similarly, $a*(b*c) = a+b+c-ab-ac-bc+abc$,

$\therefore \quad (a*b)*c = a*(b*c)$

$\therefore$ $*$ is associative.

(3) **Identity** :→ Let e is an identity element. Then we have $a*e = a \ \forall a \in Q$

$\therefore \quad a+e-ae = a$ or

$e - ae = 0$

or $\quad e(1-a) = 0$ or $e = 0$, if $1-a \neq 0$

Similarly, for $e*a = a \ \forall a \in Q$

we have $e = 0$

$\therefore$ for $e = 0$, we have $a*e = e*a = a$

Thus, 0 is the identity element.

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Inverse :- Let us assume an element $a \in Q$.
Let a^{-1} is an inverse of a. Then we have

$$a * a^{-1} = 0$$

$$a + a^{-1} - a a^{-1} = 0$$

$$a^{-1}(1-a) = -a$$

$$\text{or} \quad a^{-1} = \frac{a}{a-1}, \quad a \neq 1$$

Now $\frac{a}{a-1} \in Q$ if $a \neq 1$

$\therefore$ every element has inverse such that $a \neq 1$.

Since the algebraic system $(Q, *)$ satisfy all the properties of a group.
Hence $(Q, *)$ is a group.