MASTER'S THEOREM :- (for Dividing func)

We take recurrence relation in general form:

$$T(n) = a\,T(n/b) + f(n)$$

$a \geq 1$, $b > 1$] let assume

$$f(n) = \Theta(n^{k} \log^{p} n)$$

find 2 Things :-

1. $\log_b a$
2. k $\longrightarrow$ NOTE :- * (k is actually a power of n) & (p is " " " of $\log n$)

3- Cases :-

1) If $\log_b a > k$ then $\Theta(n^{\log_b a})$

2) If $\log_b a = k$, Then

 (i) if $p > -1$ then $\Theta(n^k \log^{p+1} n)$

 (ii) if $p = -1$ then $\Theta(n^k \log \log n)$

 (iii) if $p < -1$ then $\Theta(n^k)$

3) if $\log_b a < k$, then

 (i) if $p \geq 0$ then $\Theta(n^k \log^p n)$

 (ii) if $p < 0$ then $O(n^k)$

Case 1 Examples

Example 1 :-

$$T(n) = 2T(n/2) + 1$$

$a = 2$

$b = 2$

$f(n) = \Theta(1)$

$= \Theta(n^0 \log^0 n)$

Now, $k = 0$, $p = 0$

then $\log_2^2 = 1 \geq k = 0$

So, this satisfies first condition.

Case 1 :- ~~$\Theta(\log_b^a)$~~

~~$= \Theta(1)$~~ ~~$\Theta(n$~~

$\Theta(n^{\log_b^a})$

$\Theta(n^1)$

$\Theta(n)$ Ans.

Example 2 :-

$$T(n) = 4T(n/2) + n$$

$a = 4$
$b = 2$

$\log_2^4 = 2$, $f(n) = O(n^1 \log^0 n)$

$K = 1$, $P = 0$

So, this is comes under

Case 1 :- $\overset{\log_2^4}{2} > \overset{K}{1}$

Now,

$$O(n^{\log_b^a})$$

$$= O(n^{\log_2^4})$$

$$= O(n^2) \text{ Ans.}$$

Example 3:—

$$T(n) = 8T(n/2) + n$$

$\log_2 8 = 3$, $f(n) = \Theta(n^1 \log^0 n)$

$k = 1$, $p = 0$

$3 > 1$

So, this is also Case 1 :—

$$\Theta(n^{\log_2 8})$$

$$\Theta(n^3)$$ Ans.

Case 2 Examples:

Example 4):-

$$T(n) = 2T(n/2) + n$$

$$\underbrace{\log_2^2 = 1 \quad , k = 1}_{=} \quad , \quad P = 0$$

Case 2 (i) $\Rightarrow$ $P > -1$

$$\Rightarrow \Theta(n^k \log^{P+1} n)$$

$$\Rightarrow \Theta(n^1 \log^{0+1} n)$$

$$\Rightarrow \Theta(n \log n) \text{ Ans}$$

Example 5):-

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

$$\underbrace{\log_2^4 = 2 \quad , \quad k = 2}_{=} \quad , \quad P = 0$$

Case 2(i):-

$$\Theta(n^2 \log n)$$

NOTE:- $\log_b^a = k$ & n^2 $\Longrightarrow$ Simply multiply by $\log n$

See example 5, 6, 7

Example 6 :- $T(n) = 4T(n/2) + n^2 \log n$ → Simply multiply by $\log n$

$\log_2^4 = 2$, $k = 2$, $p = 1$

$f(n) = O(n^2 \log^1 n)$

Case 2(i) ⇒ $\Theta(n^k \log^{p+1} n)$

⇒ $\Theta(n^2 \log^2 n)$ Ans.

Example 7 :- $T(n) = 4T(n/2) + n^2 \log^5 n$ → Simply multiply by $\log n$

$\log_2^4 = 2$, $k = 2$

Case 2(i) :-

⇒ $\Theta(n^2 \log^6 n)$

Example 8 :- $T(n) = 8T(n/2) + n^3$ — Simply multiply by $\log n$

$\log_2^8 = 3$, $k = 3$

Case 2(i)

$\Rightarrow \Theta(n^3 \log n)$

Example 9 :- $T(n) = 2T(n/2) + \frac{n}{\log n}$

$\log_2^2 = 1$, $k = 1$, $P = -1$

$\because f(n) = \Theta(n^1 \log^{-1} n)$

Case 2(ii) Apply :-

$\Rightarrow \Theta(n^k \log \log n)$

$\Rightarrow \Theta(n^1 \log \log n)$

$\Rightarrow \Theta(n \log \log n)$ Ans.

Example 10 :- $T(n) = 2T(n/2) + \frac{n}{\log^2 n}$

$= 2T(n/2) + n \log^{-2} n$

$\log_2^2 = 1$, $k = 1$, $P = -2$

Case 2(iii) Apply :- $\Theta(n)$ Ans.

Example of Case 3

Example 11:-

$T(n) = T(n/2) + n^2$

$\log_2^1 = 0 \quad \& \quad k = 2 \quad , \quad p = 0$

$\underbrace{0 \quad\quad k=2}_{<}$

Case 3(i) apply $\Rightarrow$

$\Rightarrow \Theta(n^k \log^p n)$

$= \Theta(n^2 \log^0 n)$

$\Rightarrow \Theta(n^2)$

Example 12:- $2T(n/2) + n^2$

||ly as Example 11 $\longrightarrow \Theta(n^2)$ Ans.

Example 13:- $T(n) = 2T(n/2) + n^2 \log n$

$\log_2^2 = 1 \quad , \quad k = 2 \quad , \quad p = 1$

Case 3(i) $\Rightarrow \Theta(n^2 \log n)$

Example 14:- $T(n) = 2T(n/2) + n^{2} \log^{2} n$

$\swarrow k \quad \swarrow p$

$\longrightarrow \Theta(n^2 \log^2 n)$ Ans.

Example 15!- $T(n) = 4T(n/2) + n^3$

$\log_2^4 = 2 \quad , \quad k = 3$

Case 3(i) Apply $\Rightarrow \Theta(n^3)$ Ans

Example 16:-

$$T(n) = 4T(n/2) + \frac{n^3}{\log n}$$

$\log_2^4 = 2$, $k=3$, $p=-1$

Case 3(iii) :-

$$O(n^k)$$

$$\Rightarrow O(n^3) \text{ Ans.}$$

(My Notebook Notes)

Example:-

(1) $T(n) = 9T(n/3) + n$

$\log_3^9 \Rightarrow 2 \quad , k = 1 \quad , p = 0$

$\because \frac{\log 9}{\log 3} \Rightarrow \frac{\log 3^2}{\log 3} \Rightarrow 2 \frac{\cancel{\log 3}}{\cancel{\log 3}}$

$2 > 1$

Case 1 Apply:-

$\Theta(n^{\log_b^a})$

$= \Theta(n^2)$

(2) $T(n) = T(2n/3) + 1$

$a = 1 \; , \; b = 3/2 \; , \; f(n) = 1$

$\log_{3/2}^1 \Rightarrow \frac{\log 1}{\log 3/2} \Rightarrow \frac{0}{\log 3/2} = 0 \quad , k = 0 , \; p = 0$

Case 2(i):- $\Theta(n^k \log^{p+1} n)$

$\Rightarrow \Theta(n^0 \log^1 n)$

$\Rightarrow \Theta(\log n)$ Ans

Example :-

$$T(n) = 2T(n/2) + n$$

$$\underbrace{\log_2^2 = 1 \quad , \quad k = 1}_{=} \quad , \quad p = 0$$

Case 2(i) Apply :-

$$\Theta(n^k \log^{p+1} n)$$

$$\rightarrow \Theta(n^1 \log^{0+1} n)$$

$$\rightarrow \Theta(n \log n) \text{ Ans.}$$

Masters' Theorem for Decreasing function :-

General form :- $T(n) = aT(n-b) + f(n)$

$a > 0$, $b > 0$ & $f(n) = O(n^k)$

where $k \geq 0$

Simply multiply by (n) with f(n)

$T(n) = T(n-1) + 1 \longrightarrow O(n)$

$T(n) = T(n-1) + n \longrightarrow O(n^2)$

$T(n) = T(n-1) + \log n \longrightarrow O(n \log n)$

$T(n) = 2T(n-1) + 1 \longrightarrow O(2^n)$

$T(n) = 3T(n-1) + 1 \longrightarrow O(3^n)$

$T(n) = 2T(n-1) + n \longrightarrow O(n * 2^n)$

Multiply by a^n with f(n)

Case 1 :-

if $a = 1 \rightarrow O(n^{k+1})$

or $O(n^k \cdot n)$

or $O(n * f(n))$

Case 2 :-

if $a > 1 \rightarrow O(n^k a^n)$

when used in recurrence (n-2) (n-3) (n-4) ← b

or $O(n^k a^{n/b})$

or $O(f(n) * a^{n/b})$

Case 3 :-

if $a < 1 \rightarrow O(n^k)$

or

$O(f(n))$