Q:→ Show that $(R, +)$ is a group.

Closure law :→ $a+b \in R \quad \forall\ a, b \in R$

$\because$ Sum of 2 real nos is a real number. So, closure law holds.

Associative law :→ $a+(b+c) = (a+b)+c$

$\therefore$ A.L. holds

Identity :→ $a+0 = a = 0+a$ for all $a \in R$

$\therefore$ 0 is the identity element.

Inverse :→ $a * a' = a' * a = e$ → Or $\underline{a+(-a)=0}$

$a+(-a) = (-a)+a$
$= 0$

$\therefore$ $(-a)$ is the inverse of element.

$\therefore$ $(R, +)$ is a group.

Q:→ Show that $(Z_6, +_6)$ is a group.

$Z_6 = \{0, 1, 2, 3, 4, 5\}$

$5 +_6 4 = 3$

$5 +_6 5 = 4$

$\frac{10}{6}$ → Remainder

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Closure :- Since all entries in the table are elements of Z_6, so closure law holds.

Associative :- $a +_6 (b +_6 c) = (a +_6 b) +_6 c$
$\forall \; a, b, c \in Z_6$

Identity Element :- 0 is the identity element

Inverse element :- The inverse is that when combined with operation gives us identity

Inverse of 0 is 0.
" " 1 is 5
" " 2 is 4
" " 3 is 3
" " 4 is 2
" " 5 is 1

So $(Z_6, +_6)$ is a group.

Scanned with CamScanner

Q:⇒ Show that $G = \{1, 2, 3, 4, 5\}$ is not a group under addition modulo 6.

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

$\because$ $0 \notin G$

$\therefore$ closure law does not hold

$\Rightarrow$ $(G, +_6)$ is not a group

Q:⇒ Show that $G = \{1, 2, 3, 4, 5\}$ is not a group under multiplication modulo 6.

$\times_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

$\because$ $0 \notin G$

$\therefore$ closure law does not hold

$\Rightarrow$ $(G, \times_6)$ is not a group.

Subgroup:→ Let us consider a group $(G, *)$. Also, let $S \subseteq G$; then $(S, *)$ is called a subgroup if it satisfies the following conditions: ①

1) The operation $*$ is closed operation on S.
2) The operation $*$ is an associative operation
3) As e is an identity element belonged to G. It must belong to the set S i.e. The identity element of $(G, *)$ must belongs to $(S, *)$.
4) For every element $a \in S$; a^{-1} also belongs to S.

Q- Let $(I, +)$ be a group, where I is the set of all integers and $(+)$ is an addition operation. Determine whether the following subsets of G are subgroups of G.

a) The set $(G_1, +)$ of all odd integers
b) The set $(G_2, +)$ of all +ve integers.

Ans:→ (a) The set G_1 of all odd integers is not a subgroup of G. It does not satisfy the closure property, since addition of 2 odd integers is always even.

(b) Closure property:→ The set G_2 is closed under the operation $+$, since addition of 2 even integers is always even.

(c) Associative property:→ The operation $+$ is associative since $(a+b)+c = a+(b+c)$ for every $a, b, c \in G_2$.

Scanned with CamScanner

(4) Identity :→ The element 0 is the identity element Hence $0 \in G_2$.

(5) Inverse :– The inverse of every element $a \in G_2$ is $-a \notin G_2$. Hence, the inverse of every element does not exists.

Since the system $(G_2, +)$ does not satisfy all the conditions of subgroups. Hence $(G_2, +)$ is not a subgroup of $(I, +)$

Abelian Group :→ Let us consider an algebraic system $(G, *)$ where $*$ is the binary operxn on G. Then the system $(G, *)$ is said to be an abelian group if it satisfies all the properties of the group plus an additional property:

(i) The operation $*$ is commutative

i.e. $a * b = b * a \quad \forall a, b \in G$.

Q:⇒ Consider an algebraic system $(G, *)$ where G is the set of all non-0 real numbers (2) and $*$ is a binary operation defined by $a * b = \frac{ab}{4}$. Show that $(G, *)$ is an abelian group.

Ans:⇒ **Closure property** :⇒ The set G is closed under the operation $*$. Since $a * b = \frac{ab}{4}$ is a real number. Hence belongs to G.

Associative :- $(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{(ab)c}{16}$
$= \frac{abc}{16}$

Similarly $a * (b * c) = a * \left(\frac{bc}{4}\right)$
$= \frac{a(bc)}{16} = \frac{abc}{16}$

Identity :- To find the identity element, let us assume that e is a +ve real no. Then for $a \in G$.

$e * a = a \Rightarrow \frac{ea}{4} = a$

or $e = 4$

Similarly, $a * e = a$

$\frac{ae}{4} = a$ or $e = 4$

Thus, the identity of an element in G is 4.

Inverse :- Let us assume that $a \in G$. if $a^{-1} \in G$ is an inverse of a then $a * a^{-1} = 4$

$\Rightarrow \frac{aa^{-1}}{4} = 4$ or $a^{-1} = \frac{16}{a}$

Similarly, $a^{-1} * a = 4$ gives

$\frac{a^{-1}a}{4} = 4$ or $a^{-1} = \frac{16}{a}$

Thus, the inverse of an element a in G is $\frac{16}{a}$.

Commutative :→ The operation $*$ on G is commutative

Since $a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$

Thus, the algebraic system $(G, *)$ is closed, associative, has identity element has inverse and commutative. Hence the system $(G, *)$ is an abelian group.

~~Q→ Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and~~

Q:→ Let (G, o) be a group. Show that if (G, o) is an Abelian group then $(a o b)^2 = a^2 o b^2$ for all a and b in G.

Let us assume that G is an Abelian Group

Solⁿ:→ $(a o b)^2 =$

$(a o b) o (a o b) = a o (b o a) o b$ [∵ o is associative]

$= a o (a o b) o b =$

$(a o a) o (b o b) = a^2 o b^2$

Hence $(a \circ b)^2 = a^2 \circ b^2$

COSETS

Let G be a group, H be a subgroup of G and $a \in G$. Then the set $Ha = \{ha : h \in H\}$ is called a <u>right coset</u> of H in G generated by a.

Similarly, the set $aH = \{ah : h \in H\}$ is called a <u>left coset</u> of H in G generated by a.

* aH and Ha are subsets of G.
* If e is the identity element, then

$He = H = eH$.

So, H itself is a right as well as left coset.

* If G is the abelian group, then $aH = \overset{G}{Ha}$ *

* ~~then~~ The right coset of H in G generated by a is $H + a = \{h + a : h \in H\}$, If the group operation is addition.

Similarly, the left coset is

$a + H = \{a + h ; h \in H\}$

Q:→ Let

$G = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$ under $+$,

$H = \{ \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \}$

find left and right cosets.

$H+0 = \{ \ldots -9, -6, -3, 0, 3, 6, 9, \ldots \} = H$

$H+1 = \{ \ldots -8, -5, -2, 1, 4, 7, 10 \}$

$H+2 = \{ \ldots -7, -4, -1, 2, 5, 8, 11 \}$

$H+3 = \{ \ldots -6, 0, 3, 6, 9, 12 \ldots \} = H$

$H+4 = \{ \ldots -5, -2, 1, 4, 7, 10, \ldots \} = H+1$

$\Rightarrow$ H, $H+1$, $H+2$ are three distinct right cosets.

$\because$ G is an abelian group (if we add 2 no's result will be same)

$\downarrow$

$a * b = b * a$

$\therefore$ H, $1+H$, $2+H$ are left cosets.