4 Identity :→

Consider a non empty set A and a binary operation * on A. Then the operation * has an identity property if there exists an element e in A such that

$$a * e \text{ (right identity)} = e * a \text{ (left identity)}$$
$$= a \quad \forall a \in A.$$

Q:→ Consider the binary operation * on I, the set of +ve integers defined by $a * b = \frac{ab}{2}$. Determine the identity for the binary operation * if exists.

Ans:→ Let e be a +ve integer num,

$$e * a = a$$

$$\cancel{e}d = \cancel{a}2$$

$$\frac{ea}{2} = a \Rightarrow e = 2$$

Similarly,

$$a * e = a$$

$$\frac{ae}{2} = a \quad \text{or } e = 2$$

from (1) and (2) for e = 2, we have

$$e * a = a * e = a$$

∴ 2 is the identity element for *.

5 Idempotent :→

Consider a non-empty set A and a binary operation * on A. Then the operation * has the inverse property i.e.

$$a * a = a \quad \forall a \in A.$$

(7) Distributivity :-

Consider a non-empty set A and 2 binary operations $*$ and $+$ on A.

$$a * (b+c) = (a*b) + (a*c)$$

$$\text{and } (b+c) * a = (b*a) + (c*a)$$

(8) Cancellation :-

Consider a non-empty set A and a binary operation $*$ on A. Then, operation $*$ has the cancellation property, if we have

$$a * b = a * c \Rightarrow b = c$$

$$\text{and } b * a = c * a \Rightarrow b = c$$

Semi-Group :-

Let us consider an Algebraic System $(A, *)$, where $*$ is a binary operation on A. Then the system $(A, *)$ is said to be a semi-group, if it satisfies the following properties:

(1) The operation $*$ is a closed operation on set A.

(2) The operation $*$ is an associative operation.

Example:→ Consider an algebraic system $(A, *)$ ②
Where $A = \{1, 3, 5, 7, 9 \ldots\}$ the set of all +ve odd integers and $*$ is a binary operation means multiplication. Determine whether $(A, *)$ is a semi-group.

Ans:→ Closure:→ The operation $*$ is a closed operation because multiplication of two +ve odd integers is a +ve odd number.

Associative:→ The operation is an associative operation on set A. we have

$$(a * b) * c = a * (b * c)$$ → eg. $(1*3)*5 = 1*(3*5)$
$15 = 15$

Hence, algebraic system $(A, *)$ is a semigroup

Q-2. Let S be a semi-group with an identity element e and if b and b' are inverses of an element $a \in S$, then prove that $b = b'$ i.e. inverse are unique, if they exist.

Ans:→ Given b is an inverse of a, ∴, we have

$$a * b = e = b * a$$

Also, b' is an inverse of a, ∴, we have

$$a * b' = e = b' * a$$

Consider $b * (a * b') = b * e = b$ —①
$(b * a) * b' = e * b' = b'$ —②

Now, associativity holds in S.

$\therefore$ $\boxed{b = b'}$

Q:⇒ Let N be a set of +ve integers and let $*$ be the binary operation of (L.C.M) on N. find

(a) $4 * 6$, $3 * 5$, $9 * 18$, $1 * 6$

(b) Is $(N, *)$ a semi-group

(c) Is N commutative.

(d) find the identity element of N.

(e) Which elements of N have inverses?

Ans:⇒ Let $x, y \in N$ and $x * y =$ L.C.M of x and y.

$\therefore$ $4 * 6 =$ L.C.M of 4 and 6 $= 12$

$3 * 5 =$ L.C.M of 3 and 5 $= 15$

$9 * 18 =$ L.C.M of 9 and 18 $= 18$

$1 * 6 =$ L.C.M of 1 and 6 $= 6$

(b) We know that operation of L.C.M is associative i.e.

$a * (b * c) = (a * b) * c \quad \forall a, b, c \in N.$

$\therefore$ N is a semi-group under $*$.

(c) Also for $a, b \in N$

$a * b =$ LCM of a and b $=$ L.C.M of b and a $= b * a$

$\therefore$ N is commutative also.

(d) for $a \in N$, Consider $a * 1 =$ LCM of a and $1 = a$ (3)

Also, $1 * a =$ L.C.M of 1 and $a = a$

$\therefore \quad a * 1 = a = 1 * a$

i.e. 1 is the identity element of N.

(e) Consider $a * b = 1$ i.e. L.C.M of a and b is 1, which is possible if $a = 1$ and $b = 1$ i.e. the only element which has an inverse is 1 and it is its own inverse.

(d) Let e is the identity element of Q $\therefore$ for $a \in Q$, we have

$$a * e = a$$
$$a + e - ae = a$$
$$e - ea = 0$$
$$e(1 - a) = 0$$
$$e = 0 \quad \text{if } a \neq 1$$

$\therefore$ identity of Q is 0.

Q → Consider the set Q of rational no's and let * be the operation on Q defined by $a*b = a+b-ab$

(a) find $3*4$, $2*(-5)$, $7*\frac{1}{2}$

(b) Is $(Q,*)$ a semi group?

(c) Is Q commutative?

(d) find the identity element of Q.

Ans:→ (a) $a*b = a+b-ab$

$3*4 = 3+4-12 = -5$, $2*(-5) = 2+(-5)-(-10)$

$= 2-5+10 = 7$

$7*\frac{1}{2} = 7+\frac{1}{2}-\frac{7}{2} = 4$

(b) Q will be a semi group if it holds associativity under * for $a,b,c \in Q$

Consider $a*(b*c) = a*(b+c-bc)$

$= a+(b+c-bc) - a(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$ —(1)

Also, $(a*b)*c = (a+b-ab)*c$

$= a+b-ab+c-(a+b-ab)c$

$= a+b+c-ab-ac-bc+abc$

$= a+b+c-bc-ab-ac+abc$ —(2)

from (1) and (2)

$a*(b*c) = (a*b)*c$

(c) $a*b = a+b-ab = b+a-ba = b*a$

∴ Q is commutative

Monoid :->

Let us consider an algebraic system (A, o) where o is the binary operation on A. Then the system (A, o) is said to be a monoid if it satisfies the following properties:

(1) The operation o is a closed operation on set A.

(2) The operation o is an associative operxn.

(3) There exists an identity element w.r.t the operation o.

examples :- $(N, \times)$, $(Z, +)$, $(Q, +)$ are monoids.

Q:-> Consider an algebraic system $(I, +)$ where the set $I = \{0, 1, 2, 3, 4 \ldots\}$ the set of natural nos. including 0 and + is an addition operation. Determine whether $(I, +)$ is a monoid.

Ans:-> Closure property :-> The operation + is closed since sum of 2 natural no. is a natural no.

Associative :-> $(a+b)+c = a+(b+c)$

Identity :-> The element 0 is an identity element w.r.t operation +.

Hence, $(I, +)$ is a monoid.

Q-2. Let S be a finite set and $F(S)$ be the collection of all functions $f: S \to S$ under the operation of composition of functions. Show that $F(S)$ is a semi-group. Is $F(S)$ a monoid ?

Ans:→ Let $f, g, h \in F(S)$, then we know that composition of functions is associative

i.e.

$$f \circ (g \circ h) = (f \circ g) \circ h \quad \forall \; f, g, h \in F(S).$$

Hence $F(S)$ is a semi group. Also the identity fxn is an identity element of $F(S)$.

$\therefore$ $F(S)$ is a monoid.

Group

:→ Let us consider an algebraic system $(G, *)$ Where $*$ is the binary operation on G. Then the system $(G, *)$ is said to be a group. if it satisfies the following properties;

1. The operation $*$ is a closed operxn.
2. The operation $*$ is an associative operxn.
3. There exists an identity element w.r.t the operation

(2)

(4) for every $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1} * a$

$= a * a^{-1} = e$

ex:→ (1) The sets $(Q, +)$ $(R, +)$ and $(C, +)$ are groups under addition.

(2) The sets R^* (set of non-zero reals) Q^* (set of non-zero rationals) and C^* (set of non zero complex numbers) are groups under multiplication.

Q- Consider an algebraic system $(Q, *)$ where Q is the set of rational numbers and $*$ is a binary operation defined by

by

$$a * b = a + b - ab \quad \forall a, b \in Q$$

Determine whether $(Q, *)$ is a group.

Ans:→

Closure :- Since the elements $a * b \in Q$ for every $a, b \in Q$ hence the set Q is close under the operation $*$.

(2) **Associative** :→ Let us assume $a, b, c \in Q$, then we have

$$(a * b) * c = (a+b-ab) * c$$
$$= (a+b-ab)+c-(a+b-ab)c$$
$$= a+b-ab+c-ac-bc+abc$$
$$= a+b+c-ab-ac-bc+abc$$

Similarly, $a*(b*c) = a+b+c-ab-ac-bc+abc$

$\therefore$ $(a*b)*c = a*(b*c)$

$\therefore$ $*$ is associative.

(3) **Identity** :→ Let e is an identity element. Then we have $a*e = a \;\forall a \in Q$

$\therefore$ $a+e-ae = a$ or

$a-ae=0$

or $e(1-a)=0$ or $e=0$, if $1-a \neq 0$

Similarly, for $e*a = a \;\forall a \in Q$

we have $e=0$

$\therefore$ for $e=0$, we have $a*e = e*a = a$

Thus, 0 is the identity element.

Inverse :- Let us assume an element $a \in Q$.
Let a^{-1} is an inverse of a. Then we have

$$a * a^{-1} = 0$$

$$a + a^{-1} - a a^{-1} = 0$$

$$a^{-1}(1-a) = -a$$

or $$a^{-1} = \frac{a}{a-1}, \quad a \neq 1$$

Now $\frac{a}{a-1} \in Q$ if $a \neq 1$

$\therefore$ every element has inverse such that $a \neq 1$.

Since the algebraic system $(Q, *)$ satisfy all the properties of a group
Hence $(Q, *)$ is a group.