Chapter 1.2

Predicates and Quantifiers

Propositional logic is not enough to express the meaning of all statements in Mathematics and natural language.

Ex: Is "$x > 1$" True or false?

Is "x is a great tennis player" True or false?

⇒ **Predicate**

A predicate $P(x)$ is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variable,

where $P(x)$ is a propositional function and x is a predicate variable.

⇒ **Domain**

The domain of a predicate variable is the set of all possible values that may be substituted in place of variables.

Eg: "x is great tennis player"

[x is set of all human names]

⇒ Quantifier

Quantifiers are words that refer to quantities such as "some" or "all" and indicate how frequently a certain statement is true.
There are two types:-
(i) Universal Quantifier
(ii) Existential Quantifier

⇒ Universal Quantifier

The phrase "for all" denoted by $\forall$ is called universal quantifier.
Eg:-
Let "All students are smart"
Let $P(x)$ denote "x is smart"
Then the above sentence can be written as $\boxed{\forall x\, P(x)}$

⇒ Existential Quantifier

The phrase "there exists" denoted by $\exists$ is called existential Quantifier.
Eg:-
Let "There exists x such that "$x^2 = 9$"
Let $P(x) =$ "$x^2 = 9$"
Then above sentence can be written as $\boxed{\exists x\, P(x)}$.

Ques Let $D = \{1, 2, 3, 4, \ldots, 9\}$ Determine the truth value of each of the following statements.

(a) $(\forall x \in D), x + 4 < 15$
$\Rightarrow$ True, for every number in D satisfies $x + 4 < 15$

(b) $(\exists x \in D), x + 4 = 10$
$\Rightarrow$ True, for if $x = 6 \Rightarrow 6 + 4 = 10$

(c) $(\forall x \in D), x + 4 \leq 10$
$\Rightarrow$ False if $x = 7 \Rightarrow 7 + 4 = 11$
$x = 8 \Rightarrow 8 + 4 = 12$
$x = 9 \Rightarrow 9 + 4 = 13$

(d) $(\exists x \in D), x + 4 \cancel{>} 15$
$\Rightarrow$ False $x = 9 \Rightarrow 9 + 4 = 13 < 15$
False

$\Rightarrow$ Negation of Quantified Statement
Consider the statement
"All students in the class have taken a course in Discrete Mathematics"
This statement can be written as
$\boxed{\forall x \; P(x)}$
where $P(x) \rightarrow$ "x has taken a course in discrete mathematics"

Its negation reads
"It is not the case that all students in the class have taken a course in discrete mathematics"

This is equivalent to
"There is a student in the class who has not taken a course in discrete mathematics."

This is simply the existential quantification of the negation of the original propositional function, namely

$$\boxed{\exists x \sim P(x)}$$

This example illustrates the following equivalence

$$\boxed{\sim \forall x\, P(x) \equiv \exists x \sim P(x)} \quad \text{— } ①$$

Example 2:

$$\exists x\, P(x)$$

$$\boxed{\sim \exists x\, P(x) \equiv \forall x \sim P(x)} \quad \text{— } ②$$

① and ② Negation of Quantified Statements

Statement		Negation	
All true	$\forall x\, F(x)$	$\exists x[\sim F(x)]$	Atleast 1 False
Atleast 1 False	$\exists x[\sim F(x)]$	$\forall x\, F(x)$	All true
All false	$\forall x[\sim F(x)]$	$\exists x\, F(x)$	Atleast 1 True
Atleast 1 True	$\exists x\, F(x)$	$\forall x[\sim F(x)]$	All false

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⇒ All human beings are mortal

$P(x)$: x is mortal

⇒ $\boxed{\forall x\, P(x)}$

⇒ Some human beings are mortal

$P(x)$: x is mortal

$\boxed{\exists x\, P(x)}$

⇒ All students are successful

$P(x)$: x is successful

$\boxed{\forall x\, P(x)}$

⇒ Some students are successful

$P(x)$: x is successful

$\boxed{\exists x\, P(x)}$

⇒ Every person is precious

Rephrase: For every x, if x is a person then x is precious.

$M(x)$: x is a person

$A(x)$: x is precious

$\forall x\, [M(x) \rightarrow A(x)]$

⇒ Every student is clever

$M(x)$: x is student

$A(x)$: x is clever

$\forall x\, [M(x) \rightarrow A(x)]$

$\forall x \rightarrow$ implies
$\exists x \rightarrow \wedge$ and

$\Rightarrow$ All men are mortal
M(x): x is man
A(x): x is mortal
$\forall x [M(x) \rightarrow A(x)]$

$\Rightarrow$ There exists a student
P(x): x is a student
$\boxed{\exists x P(x)}$

$\Rightarrow$ Some students are clever
M(x): x is a student
P(x): x is clever
$\exists x [M(x) \wedge P(x)]$

$\Rightarrow$ Some students are not successful
M(x):- x is a student
P(x): x is successful
$\exists x [M(x) \wedge \sim P(x)]$

$\Rightarrow$ Not all birds can fly
M(x): x is a bird
P(x): x can fly

$\cancel{\exists x [M(x) \wedge \sim P(x)]}$
$\sim \forall x [M(x) \rightarrow P(x)]$

$\Rightarrow$ There is a student who likes Maths but not Geography
M(x): x is a student
A(x): x likes Maths
B(x): x ~~doesnot~~ likes Geography.
$\exists x [M(x) \wedge A(x) \wedge \sim B(x)]$

⇒ Let $P(x)$: x can speak Tamil
$Q(x)$: x know the language C

(a) There is a student who can speak Tamil and who knows C.

$$\exists x(P(x) \wedge Q(x))$$

(b) There is a student who can speak Tamil but does not know C.

$$\exists x(P(x) \wedge \sim Q(x))$$

(c) ~~Either~~ Every student can either speak Tamil or know C.

$$\forall x(P(x) \vee Q(x))$$

(d) No student can speak Tamil or knows C.

$$\sim[\forall x(P(x) \vee Q(x))]$$

⇒ All birds can fly.
$B(x)$: x is a bird
$F(x)$: x can fly.

$$\forall x[B(x) \rightarrow F(x)]$$

⇒ Some men are genius
$M(x)$: x is a man
$G(x)$: x is genius

$$\exists x(M(x) \wedge G(x))$$

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⇒ Some numbers are not rational.

$N(x)$: x is a number

$R(x)$: x is rational

$\exists x[N(x) \wedge \sim R(x)]$

⇒ All mammals are animals.

$M(x)$: x is a mammal

$A(x)$: x is an animal

$\forall x[M(x) \rightarrow A(x)]$

⇒ No natural number is negative.

$N(x)$: x is a natural number

$A(x)$: x is negative

$\sim[\forall x[N(x) \rightarrow A(x)]]$

Negation Questions

⇒ All integers are greater than 8

$P(x)$: x is an integer

$Q(x)$: x is greater than 8

$\forall x(P(x) \rightarrow Q(x))$

$= \forall x(\sim P(x) \vee Q(x))$

Negation: $\sim[\forall x(\sim P(x) \vee Q(x))]$

$= \exists x(P(x) \wedge \sim Q(x))$

Ans

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⇒ Not all men are genius

$P(x)$: x is a man

$Q(x)$: x is genius

$\sim\forall x(P(x) \rightarrow Q(x)) \Rightarrow \sim\forall x(\sim P(x) \vee Q(x))$

$\Rightarrow \forall x \sim(\sim P(x) \vee Q(x))$

Negation: $\sim[\forall x \sim(\sim P(x) \vee Q(x))]$

$\Rightarrow \exists x(P(x) \rightarrow Q(x))$ or $\exists x(\sim P(x) \vee Q(x))$

Ans

⇒ $(\forall x\, P(x)) \vee (\exists y\, P(y))$

$\sim[(\forall x\, P(x)) \vee (\exists y\, P(y))]$

$\sim[\forall x\, P(x)] \wedge \sim[\exists y P(y)]$

$\exists x \sim P(x) \wedge \forall y \sim P(y)$

⇒ $\forall x \in R \quad x>5 \rightarrow x^3>125$

$\forall x \in R \quad \sim(x>5) \vee x^3>125$

Negation:-

$\sim[\forall x \in R \quad \sim(x>5) \vee x^3>125]$

$\exists x \in R \quad x>5 \wedge x^3 \leq 125$

Ans

⇒ Some students are intelligent

$P(x)$: x is a student

$Q(x)$: x is intelligent

$\exists x(P(x) \wedge Q(x))$

Negation :- $\forall x(\sim P(x) \vee \sim Q(x))$

⇒ Any integer is either positive or negative
$P(x)$: x is integer
$A(x)$: x is positive
$B(x)$: x is negative

$\forall x (P(x) \rightarrow (A(x) \vee B(x)))$
$\forall x [\sim P(x) \vee (A(x) \vee B(x))]$
~~$\forall x [\sim P(x) \vee (A(x) \vee B(x))]$~~

Negation:-
$\exists x \sim [\sim P(x) \vee (A(x) \vee B(x))]$
$\exists x [P(x) \wedge \sim (A(x) \vee B(x))]$
$\exists x [P(x) \wedge \sim A(x) \wedge \sim B(x)]$

⇒ All flowers are red.
$P(x)$: x is a flower
$Q(x)$: x is red
$\forall x (P(x) \rightarrow Q(x)) \Rightarrow \forall x [\sim P(x) \vee Q(x)]$
Negation: $\exists x [P(x) \wedge \sim Q(x)]$

⇒ Some boys are intelligent
$P(x)$: x is a boy
$Q(x)$: x is intelligent
$\exists x [P(x) \wedge Q(x)]$
Negation: $\forall x [\sim P(x) \vee \sim Q(x)]$

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⇒ Every girl is not clever

$P(x)$: x is a girl

$Q(x)$: x is clever

$\forall x(P(x) \rightarrow \sim Q(x)) \Rightarrow \forall x[\sim P(x) \vee \sim Q(x)]$

Negation: $\exists x[P(x) \wedge Q(x)]$

⇒ Some numbers are not irrational.

$P(x)$: x is a number

$Q(x)$: x is irrational

$\exists x(P(x) \wedge \sim Q(x))$

Negation: $\forall x[\sim P(x) \vee Q(x)]$

Rules of Inference

1. Addition (a) $\dfrac{p}{p \vee q}$ (b) $\dfrac{q}{p \vee q}$

2. Simplification (a) $\dfrac{p \wedge q}{p}$ (b) $\dfrac{p \wedge q}{q}$

3. Conjunction

$$\begin{array}{c} p \\ q \\ \hline p \wedge q \end{array}$$

4. Modus Ponens

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline q \end{array}$$

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5. Modus tollens
$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \sim p \end{array}$$

6. Hypothetical syllogism
$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ p \rightarrow r \end{array}$$

7. Disjunctive syllogism
$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ q \end{array}$$

8. Constructive Dilemma
$$\begin{array}{l} (p \rightarrow q) \wedge (r \rightarrow s) \\ \underline{p \vee r} \\ q \vee s \end{array}$$

9. Distructive Dilemma
$$\begin{array}{l} (p \rightarrow q) \wedge (r \rightarrow s) \\ \underline{\sim q \vee \sim s} \\ \sim p \vee \sim r \end{array}$$

Questions on Rules of Inference

1. If Ram works hard, he will get a job. Ram works hard, therefore, he will get a job. Check validity.

Sol. p: Ram works hard
q: He will get a job

Premises: $p \rightarrow q$, p Conclusion: q

1. $p \rightarrow q$ Premise
2. p Premise
3. q Modus Ponens using 1 and 2

The argument is valid.

Q2. If a man is a bachelor, he is unhappy. If a man is unhappy, he dies young. Therefore, bachelors die young. Check validity.

Sol. p: A man is a bachelor
q: A man is unhappy
r: He dies young

Premises: $p \rightarrow q$, $q \rightarrow r$ Conclusion: $p \rightarrow r$

1. $p \rightarrow q$ Premise
2. $q \rightarrow r$ Premise
3. $p \rightarrow r$ Hypothetical Syllogism using 1 and 2

The argument is valid.

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Q3 If this number is divisible by 6, then it is divisible by 3. This number is not divisible 3. Therefore, this number is not divisible by 6.

Sol. p: The number is divisible by 6
q: The number is divisible by 3

Premises: $p \rightarrow q, \sim q$ Conclusion: $\sim p$

1. $p \rightarrow q$ Premises
2. $\sim q$ Premises
3. $\sim p$ Modus tollens using 1 and 2

The argument is valid.

Q4 Either Ram is not guilty or Shyam is telling the truth. Shyam is not telling the truth. Therefore, Ram is not guilty.

Sol. p: Ram is not guilty
q: Shyam is telling the truth

premises: $p \vee q, \sim q$ Conclusion: p

1. $p \vee q$ Premise
2. $\sim q$ Premise
3. p Disjunctive Syllogism using 1 and 2

The argument is valid

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Pigeon Hole

If n pigeon hole are occupied by $kn+1$ or more pigeons then atleast one pigeonhole is occupied by $k+1$ or more pigeons.

Q1 Find the minimum numbers of teachers in a college to be sure that 4 of them are born in same month.

$n=12 \quad k+1=4 \quad k=3$

$kn+1 = 12\times3+1 = 37$ Ans.

Q2. A box contains 10 blue balls, 20 red balls, 8 green balls, 15 yellow balls and 25 white balls. How many balls must we choose to ensure that we have 12 balls of same colour.

$n=5 \quad k+1=12 \quad k=11$

$kn+1 = 11\times5+1 = 56$ Ans.

Q3. Prove that among 1,00,000 people there are two who are born on same time.

Let $P=\{P_1, P_2, \dots, P_{100000}\}$ be the set of people

Let $H=\{H_1, H_2, \dots, H_{24}\}$ be the set of all hours in a day.

By extended pigeon hole principle, there are atleast $\left\lceil \frac{|P|}{|H|} \right\rceil$ person during the same hour.

$= \frac{100000}{24} = 4167$

Now $Q=\{Q_1, Q_2, \dots Q_{4167}\}$ and $M=\{M_1, M_2, \dots M_{60}\}$

Then number of person born in same minute are atleast

$= \left\lceil \frac{4167}{60} \right\rceil = 70$

Again $R = \{R_1, R_2 \ldots R_{70}\}$ and $S = \{S_1, S_2 \ldots S_{60}\}$

Then number of person born is same time (same second) are atleast

$= \left\lceil \frac{70}{60} \right\rceil = 2$