Problem 1.

$$\frac{\partial^2 y}{\partial x^2} + y = 8 \, , \quad 5 < x < 7$$

$$\left[y(x)\right]_{x=5} = 1 \qquad \left[\frac{\partial y(x)}{\partial x}\right]_{x=7} = -2$$

1. Multiply by weight function
2. Integrate across domain
3. Integrate by parts to transfer derivative to weight function.

$$w \frac{\partial^2 y}{\partial x^2} + wy - 8w = 0$$

$$\int_5^7 \left[w \frac{\partial^2 y}{\partial x^2} + wy - 8w\right] dx = 0$$

$$\begin{array}{ccc} w & + & \dfrac{\partial^2 y}{\partial x^2} \\ w' & - & \dfrac{\partial y}{\partial x} \end{array}$$

weak form:

$$\int_5^7 \left[-\frac{\partial w}{\partial x}\frac{\partial y}{\partial x} + wy - 8w\right] dx + \left[w \frac{\partial y}{\partial x}\right]_5^7 = 0$$

$$\int_5^7 \left[-\frac{\partial w}{\partial x}\frac{\partial y}{\partial x} + wy - 8w\right]dx + w(-2) - \underbrace{w\left[\frac{\partial y}{\partial x}\right]_{x=5}}_{0} = 0$$

$$\underbrace{\int_5^7 \left[-\frac{\partial w}{\partial x}\frac{\partial y}{\partial x} + wy\right]dx}_{B(w,u)} = \underbrace{\int_5^7 8w\,dx + 2w}_{l(w)}$$

↑

Bilinear and symmetric $\Rightarrow$

quadratic functional:

$$I(u) = \frac{1}{2}B(u,u) - l(u)$$

$$= \frac{1}{2}\int_5^7 \left[-\left(\frac{\partial^2 y}{\partial x^2}\right) + y^2\right]dx - \int_5^7 8y\,dx - 2y$$

Problem 2.

$$\frac{\partial^2}{\partial x^2}\left\{(1+x)\frac{\partial^2 \omega}{\partial x^2}\right\} - q = 0 \quad , \quad 1 < x < 2$$

$$\omega(x)\Big|_{x=1} = 1$$

$$\frac{\partial \omega(x)}{\partial x}\Big|_{x=1} = 1$$

$$(1+x)\frac{\partial^2 \omega}{\partial x^2}\Big|_{x=2} = M_0$$

$$\frac{\partial}{\partial x}\left\{(1+x)\frac{\partial^2 \omega}{\partial x^2}\right\}\Big|_{x=2} = V_0$$

$$0 = \int_1^2 v\left[\frac{\partial^2}{\partial x^2}\left((1+x)\frac{\partial^2 \omega}{\partial x^2}\right) - q\right] dx$$

$$\begin{array}{ccc} v & + & \frac{\partial^2}{\partial x^2}(\;) \\ v' & - & \frac{\partial}{\partial x}(\;) \\ v'' & + & (\;) \end{array}$$

$$0 = \int_1^2 \left[\frac{\partial^2 v}{\partial x^2}(1+x)\frac{\partial^2 \omega}{\partial x^2} - vq\right] dx$$

$$- \left[\frac{\partial v}{\partial x}(1+x)\frac{\partial^2 w}{\partial x^2}\right]_1^2 + \left[v\frac{\partial}{\partial x}\left((1+x)\frac{\partial^2 w}{\partial x^2}\right)\right]_1^2$$

Primary variables $w(x)$ and $\frac{\partial w(x)}{\partial x}$ specified at $x=1$

$$\Rightarrow \quad v(1) = \left.\frac{\partial v}{\partial x}\right|_{x=1} = 0$$

$$0 = \int_1^2 \left[\frac{\partial^2 v}{\partial x^2}(1+x)\frac{\partial^2 w}{\partial x^2} - vq \right] dx - \left.\frac{\partial v}{\partial x}\right|_{x=2} M_0$$

$$+ v(2) V_0$$

$$B(v,w) = \int_1^2 \left[\frac{\partial^2 v}{\partial x^2}(1+x)\frac{\partial^2 w}{\partial x^2} \right] dx$$

$$l(v) = \int_1^2 vq\, dx + \left.\frac{\partial v}{\partial x}\right|_{x=2} M_0 - v(2) V_0$$

$I(w,w)$ for symmetric, bilinear:

$$I(w,w) = \frac{1}{2} B(w,w) - l(w)$$

$$= \frac{1}{2}\int_1^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 (1+x)\, dx - \int_1^2 wq\, dx$$

$$- \left.\frac{\partial w}{\partial x}\right|_{x=2} M_0 + w(2) V_0$$

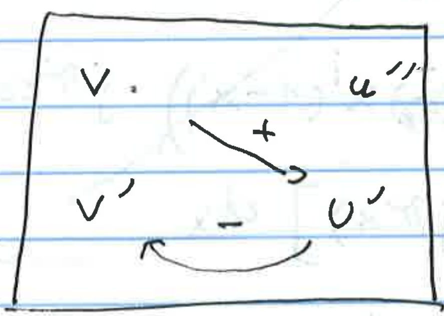
Problem 3.

$$\frac{\partial^2 u}{\partial x^2} - u = 0, \quad 0 < x < 1$$

$u(0) = 0$

$u(1) = 1$

$$0 = \int_0^1 \left[v \frac{\partial^2 u}{\partial x} - uv \right] dx = 0$$



$$0 = \int_0^1 \left[- \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} - uv \right] dx + \underbrace{\left[v \frac{\partial u}{\partial x} \right]_0^1}_{0}$$

$$B(w,u) = \int_0^1 \left[- \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} - uv \right] dx$$

~~$l(w) = 0$ symmetric, bilinear $\Rightarrow$~~

$$\cancel{I(u) = \frac{1}{2} \int_0^1 \left[- \frac{\partial^2 u}{\partial x^2} - u^2 \right] dx}$$

$$B_{ij} = B(\phi_i, \phi_j) = \int_0^1 \left[- \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} - \phi_i \phi_j \right] dx$$

$$F_i = - B(\phi_i, \phi_0) = - \int_0^1 \left[\frac{-\partial \phi_i}{\partial x} \frac{\partial \phi_0}{\partial x} - \phi_i \phi_0 \right] dx$$

$\phi_0(0) = 0$, $\phi_0(1) = 1$, $\phi_i(0) = \phi_i(1) = 0$

$(i = 1, 2, \ldots, n)$

$\phi_0 = x \qquad \phi_i = x^i(1-x)$

$$B_{ij} = \int_0^1 \left[-\frac{\partial (x^i(1-x))}{\partial x} \frac{\partial}{\partial x} \sin(j\pi x) - x^i(1-x)\sin(j\pi x) \right] dx$$

$$= \int_0^1 \left[-\left(i x^{i-1}(1-x) - x^i(1-x) \right) \cdot j\pi \cos(j\pi x) - x^i(1-x)\sin(j\pi x) \right] dx$$

$\phi_j = \sin(j\pi x) \qquad \phi_0 = x$

$$B_{ij} = \int_0^1 \left[-\frac{\partial}{\partial x} \sin(i\pi x) \frac{\partial}{\partial x} \sin(j\pi x) - \sin(i\pi x)\sin(j\pi x) \right] dx$$

$\sin'(i\pi x) \qquad \sin'(j\pi x)$

$\sin''(i\pi x)$

$$\int_0^1 \Big[-i\pi\cos(i\pi x)\big(j\pi\cos(j\pi x)\big) - \sin(i\pi x)\sin(j\pi x) \Big] dx$$

If $i \neq j$, $\int_0^1 \cos(i\pi x)\cos(j\pi x)\,dx = 0$

Same for sin

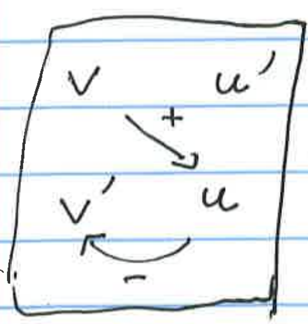
$$\Rightarrow \int_0^1 -i^2\pi^2\cos^2(i\pi x) - \sin^2(i\pi x)\,dx$$

$$= 0.5\left[-i^2\pi^2 - 1\right]$$

$$F_i = \int_0^1 \left[+ \frac{\partial}{\partial x}\sin(i\pi x)\,\overbrace{\frac{\partial}{\partial x} x} + \sin(i\pi x)\,x \right] dx$$

$$= \int_0^1 \Big[\underbrace{+\cos(i\pi x)\,i\pi}_{\Rightarrow 0} + \sin(i\pi x)\,x \Big] dx$$

$$= \int_0^1 \Big[\underbrace{+\sin(i\pi x)}_{u'}\,\underbrace{x}_{v} \Big] dx \quad \Big\| \quad u = \frac{-\cos(i\pi x)}{i\pi}$$



$$= \underbrace{\int_0^1 + \frac{\cos(i\pi x)}{i\pi}\,dx}_{0} - \left[\frac{\cos(i\pi x)}{i\pi} x \right]_0^1$$

$$= \frac{-\cos(i\pi)}{i\pi}$$

For $N = 2$:

$$B_{ii} = \begin{bmatrix} \frac{-\pi^2 - 1}{2} & 0 \\ 0 & \frac{-4\pi^2 - 1}{2} \end{bmatrix}$$

$$F_i = \begin{bmatrix} \frac{+1}{\pi} \\ \frac{-1}{2\pi} \end{bmatrix}$$

$$0.5 \begin{bmatrix} -\pi^2 - 1 & 0 \\ 0 & -4\pi^2 - 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} +1/\pi \\ -1/2\pi \end{bmatrix}$$

$$c_1 = (+1/\pi) / (0.5 \cdot (-\pi^2 - 1))$$

$$= \frac{-2}{\pi^3 + \pi}$$

$$c_2 = (-1/(2\pi)) / (0.5 \cdot (-4\pi^2 - 1))$$

$$= \frac{2}{8\pi^3 + 2\pi}$$