Упражнение 4.1

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x^{(i)}$$

$W_k = V_0 + L(V_1 \ldots V_k) - k$-мерное линейное многообразие

Будем искать V_0 как решение задачи минимизации L_0:

$$V_0 = \underset{a_0 \in R^n}{\operatorname{argmin}} \left(\sum_{i=1}^{N} \operatorname{dist}^2(x_i, L_0) \right) =$$

$$= \underset{a_0 \in R^n}{\operatorname{argmin}} \left(\sum_{i=1}^{N} \| x^{(i)} - a_0 \|^2 \right) = \frac{1}{N} \sum_{i=1}^{N} x^{(i)} =$$

$$= \bar{x}$$

Задача 1

а) $a \in R^n$, $x \in R^n$ $\quad \frac{\partial (a^T x)}{\partial x} = a$

$a^T = (a_1, a_2 \ldots a_n)$

$x = (x_1, x_2 \ldots x_n)^T$

$a^T x = (a_1 x_1, a_2 x_2 \ldots a_n x_n)^T$

$$\frac{\partial (a^T x)}{\partial x} = (a_1, a_2 \ldots a_n)^T = (a^T)^T = a$$

б) $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ $\qquad \frac{\partial(Ax)}{\partial x} = A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix} \qquad x = \begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix} \rightarrow$$

$$\rightarrow Ax = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ & \dots & & \\ a_{n1}x_1 & \dots & & a_{nn}x_n \end{pmatrix}$$

$$\frac{\partial(Ax)}{\partial x} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{1n} & \dots & a_{nn}\end{pmatrix} = A$$

в) $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ $\qquad \frac{\partial(x^T A x)}{\partial x} = (A + A^T)x$

из (б)

$$x^T A = \begin{pmatrix} a_{11}x_1 & \dots & a_{1n}x_n \\ & \dots & \\ a_{n1}x_n & \dots & a_{nn}x_n \end{pmatrix} = T$$

$$Tx = \begin{pmatrix} a_{11}x_1^2 & \dots & a_{1n}x_1x_n \\ & \dots & \\ a_{n1}x_nx_1 & \dots & a_{nn}x_n^2 \end{pmatrix} \rightarrow \frac{\partial(Tx)}{\partial x} =$$

$$= \begin{pmatrix} 2a_1x_1 & \dots & (a_{1n}+a_{n1})x_n \\ (a_{n1}+a_{1n})x_1 & \dots & 2a_nx_n \end{pmatrix} = (A^T + A)x$$

A

Если $A = A^T$, то $A + A^T = 2A$

г) $x \in R^n$ $\quad \dfrac{\partial \|x\|^2}{\partial x} = 2x$

$\|x\|^2 = (x_1^2 + \dots + x_n^2)^T$

$\dfrac{\partial \|x\|^2}{\partial x} = (\cancel{2x_1 + 2x_2 +}\ 2x_1\ 2x_2\ \dots 2x_n)^T = 2x$

д) $g(x)$ — скаляр, $x \in R^n$ $\quad \dfrac{\partial g(x)}{\partial x} = \operatorname{diag}(g'(x))$

$g(x) = (g(x_1) \dots g(x_n))^T$

$$\frac{\partial g(x)}{\partial x} = \begin{pmatrix} g'_{x_1}(x_1) & \underset{=0}{g'_{x_1}(x_2)} & \dots & \underset{=0}{g'_{x_1}(x_n)} \\ & \dots & & \vdots \\ \underset{=0}{g'_{x_n}(x_1)} & \dots & & g'_{x_n}(x_n) \end{pmatrix} =$$

$$= \begin{pmatrix} g'_{x_1}(x_1) & 0 & \dots & 0 \\ 0 & g'_{x_2}(x_2) & \dots & 0 \\ 0 & \dots & & g'_{x_n}(x_n) \end{pmatrix} = \operatorname{diag} g'(x)$

2) $h: R^n \to R^m$, $g: R^m \to R^n$,

$x \in R^n$

$h(x) = (h(x_1) .. h(x_n))^T$ $g(h(x)) =$

$= g(h(x_1) .. g(h(x_n))^T$

$$\frac{\partial g(h(x))}{\partial x} = \frac{\partial (g(h(x_i))^T}{\partial x}_{i=\overline{1,n}}$$

производная сложной ф-ии $(f(g(x)))'_x =$

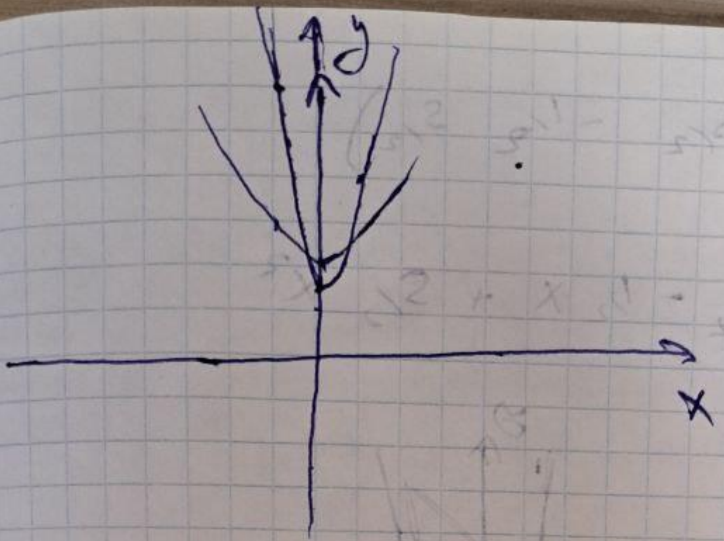
$= \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial x} \to$ тогда . из $g) \frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial x}$

Задача 3

x	1	1	0	0	-1
y	4	4	0	2	6

$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2$

$$X^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} 4 \\ 4 \\ 0 \\ 2 \\ 6 \end{pmatrix}$$



$$X^T X = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{pmatrix} \qquad X^T y = \begin{pmatrix} 16 \\ 2 \\ 14 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & -4 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right) \to \beta = (1 \ -1 \ 4)^T$$

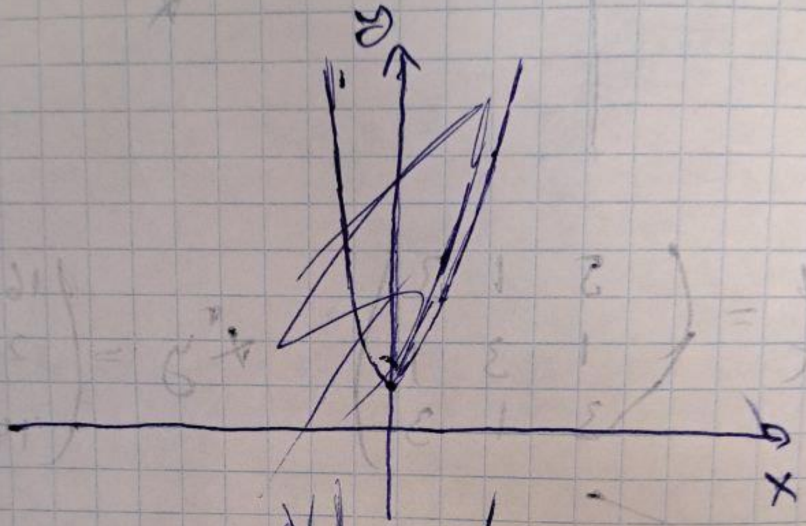
Т.о. $f(x) = 1 - x + 4x^2$

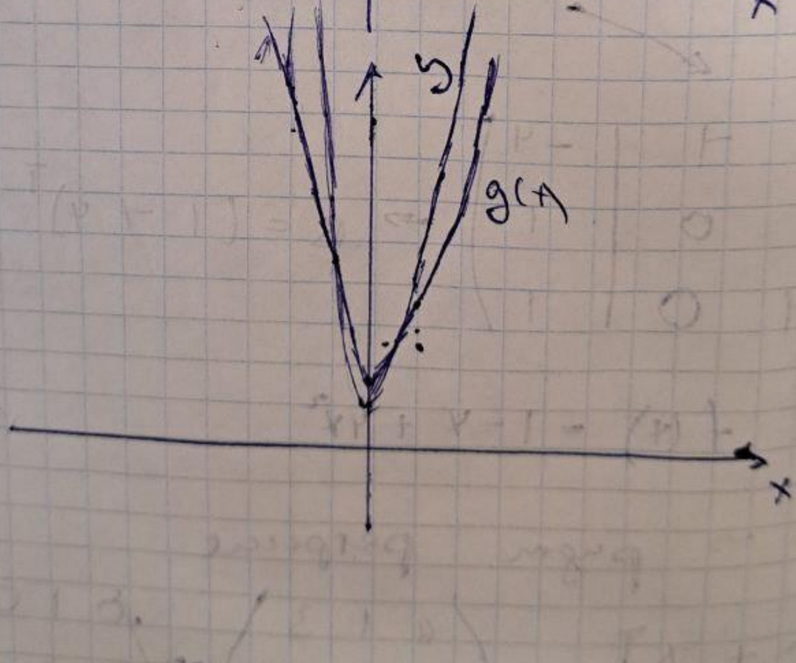
$\lambda = 1 \to$ ридж регрессия

$$X^T X + \lambda I = \begin{pmatrix} 6 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix} \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & -1/2 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 5/2 \end{array}\right) \to$$

$\rightarrow \quad \beta = \left(3/2 \quad -1/2 \quad 5/2 \right)$

$g(x) = 3/2 - 1/2\, x + 5/2\, x^2$





Задатак 9

x_1	0	1	0	2	2	2	4	3
x_2	-1	0	0	0	1	0	1	2
y	0	0	0	0	0	1	1	1

$\hat{P}_r(y=0) = 6/8$ $\quad\quad$ $\hat{P}_r(y=1) = 3/8$

$\hat{\mu}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad\quad$ $\hat{\mu}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\hat{\Sigma}_0 = \frac{1}{N_0 - 1} \sum_{y^{(i)}=0} (x^{(i)} - \hat{\mu}_0)(x^{(i)} - \hat{\mu}_0)^T =$$

$$= \frac{1}{4} \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\hat{\Sigma}_1 = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Оцењивање матрице коваријације

$$\Sigma = \frac{1}{N-K} \sum_{K} \sum_{y_i = k} (x^{(i)} - \hat{\mu}_k)(x^{(i)} - \hat{\mu}_k)^T =$$

$$= \frac{1}{8}\begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3/4 & 3/8 \\ 3/8 & 1/2 \end{pmatrix}$$

$$\hat\Sigma_0^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad \hat\Sigma_1^{-1} = \begin{pmatrix} 1/6 & -1/3 \\ -1/3 & 2/3 \end{pmatrix},$$

$$\hat\Sigma^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad \hat\Sigma_1 = \begin{pmatrix} \cancel{16}\,1/6 & -1/3 \\ -1/3 & 2/3 \end{pmatrix},$$

$$\hat\Sigma^{-1} = \begin{pmatrix} 32/15 & -16/15 \\ -8/5 & 16/15 \end{pmatrix}$$

Линейн дискр. ф-ии:

$$\delta_0(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 32/15 & -16/15 \\ -8/5 & 16/15 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} -$$

$$-\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 32/15 & -16/15 \\ -8/5 & 16/15 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \ln 5 - \ln 8 =$$

$$= \begin{pmatrix} 32/15\, x_1 - 8/5\, x_2 \\ -16/15\, x_1 + 16/15\, x_2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{32}{15} & -\frac{16}{15} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus$$

$$\oplus \ln 5 - \ln 8 = \frac{32}{15}x_1 - \frac{8}{5}x_2 - \frac{16}{15} + \ln 5 - \ln 8$$

$$\sigma_1(x) = \frac{32}{5}x_1 - \frac{56}{15}x_2 - 4 + \ln 3 - \ln 8$$

Разд. пов-ть $\sigma_0(x) = \sigma_1(x)$

$$\frac{32}{15}x_1 - \frac{8}{5}x_2 - \frac{16}{15} + \ln 5 = \frac{16}{3}x_1 - \frac{56}{15}x_2 - 4 + \ln 3$$

$$\frac{48}{15}x_1 - \frac{32}{15}x_2 - \frac{44}{15} + \ln \frac{3}{5} = 0$$

$$48x_1 - 32x_2 - 44 + 15\ln 3/5 = 0$$

Квадрат. дискр. ф-ии:

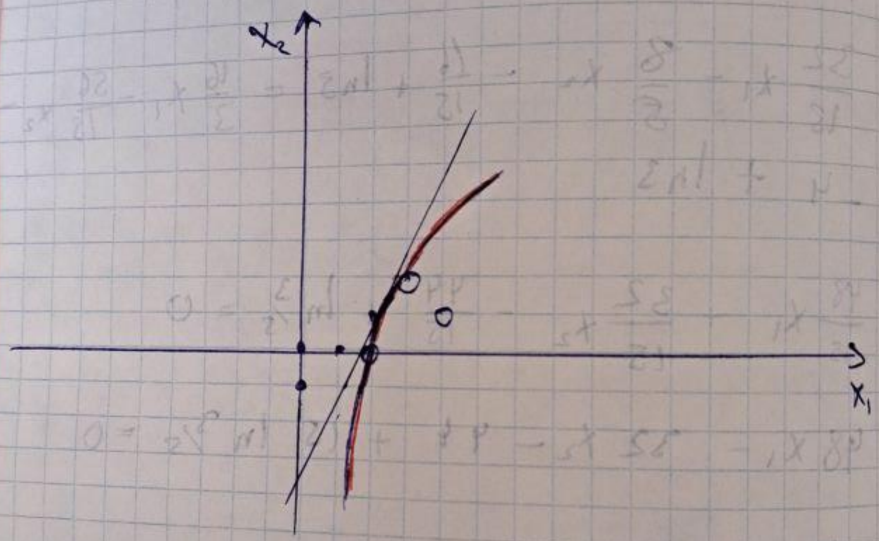
$$\sigma_0(x) = -1/2 - \frac{56}{15}x_2 + \frac{4}{3}x_1 + \ln 5/8 - 4x_1^2 + 16/15\, x_2^2$$

$$\sigma_1(x) = -x_1^2 + 4/3\, x_1 + \frac{16}{15}x_2^2 + \frac{32}{15}x_2 - \frac{1}{2}\ln 3/2$$

Разделяющая пов-ть:

$$-1/2 - \frac{56}{15}x_2 + 4/3\, x_1 + \ln 5/8 - 4x_1^2 + \frac{16}{15}x_2^2 = -x_1^2 + 4/3\, x_1 + 16/15\, x_2^2 + \frac{32}{15}x_2 - 1/2 \ln 3/2$$

$$3x_1^2 + \frac{58}{15}x_2 - \frac{1}{2}(\ln 3/2 - 1) - \ln 9/8 = 0$$



Задание 15

$\hat{P}_r(y=0) = 1/2$ $\quad$ $\hat{P}_r(y=1) = 1/2$

$\hat{P}_r(x_1=0 \mid y=0) = 3/5$ $\quad$ $\hat{P}_r(x_1=1 \mid y=0) = 2/5$

$\hat{P}_r(x_2=0 \mid y=0) = 2/5$ $\quad$ $\hat{P}_r(x_2=1 \mid y=0) = 3/5$

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От

$\frac{9}{8} = 0$

x_1

$= 0) = \frac{2}{5}$

$= 0) = \frac{3}{5}$

$\hat{P}_r(x_1 = 0 \mid y = 1) = \frac{3}{5}$ $\quad$ $\hat{P}_r(x_1 = 1 \mid y = 1) = \frac{2}{5}$

$\hat{P}_r(x_2 = 0 \mid y = 1) = 0$ $\quad$ $\hat{P}_r(x_2 = 1 \mid y = 1) = 1$

$$\hat{P}_r(y = 0 \mid x_1 = 1, x_2 = 1) = \frac{\frac{2}{5} \cdot \frac{3}{5} \cdot \frac{1}{2}}{\frac{3}{25} + \frac{1}{5}} =$$

$$= \frac{\frac{6}{50}}{\frac{8}{25}} = \frac{3 \cdot 25}{25 \cdot 8} = \frac{3}{8}$$

$$P_r(y = 1 \mid x_1 = 1, x_2 = 1) = \frac{\frac{2}{5} \cdot 1 \cdot \frac{1}{2}}{\frac{3}{25} + \frac{1}{5}} = \frac{\frac{1}{5}}{\frac{8}{25}} =$$

$$= \frac{1 \cdot \cancel{25}^{5}}{\cancel{5}_{1} \cdot 8} = \frac{5}{8}$$

Ответ: $\frac{3}{8}$; $\frac{5}{8}$