Stationarity in Time Series

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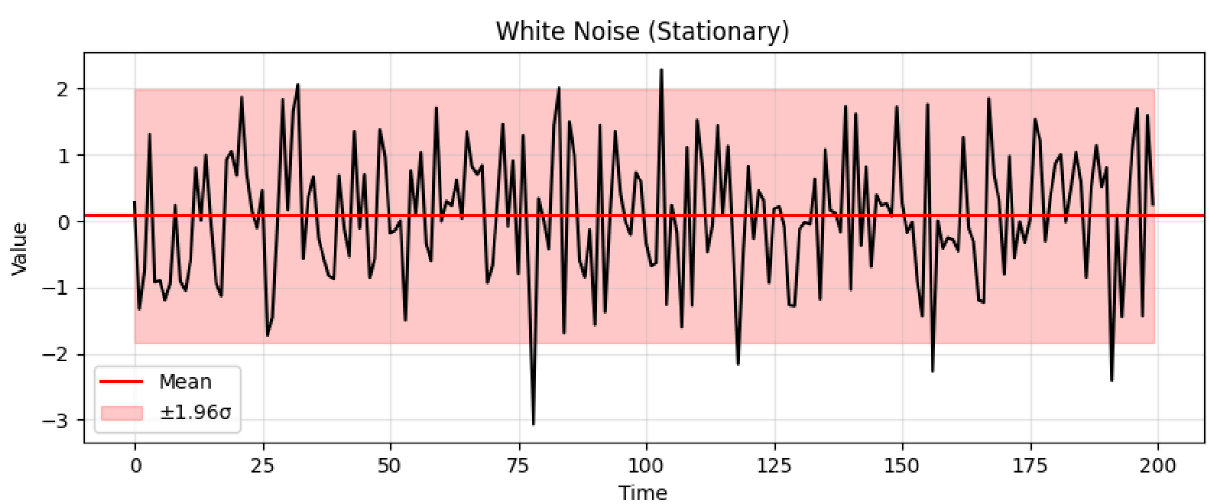
In [1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import adfuller, kpss
from statsmodels.graphics.tsaplots import plot_acf
from statsmodels.tsa.arima_process import ArmaProcess

# Utility for consistent plots
def run_sequence_plot(x, y, title, xlabel="Time", ylabel="Value"):
    fig, ax = plt.subplots(figsize=(10, 3.5))
    ax.plot(x, y, 'k-')
    ax.set_title(title)
    ax.set_xlabel(xlabel)
    ax.set_ylabel(ylabel)
    ax.grid(alpha=0.3)
    return ax
```

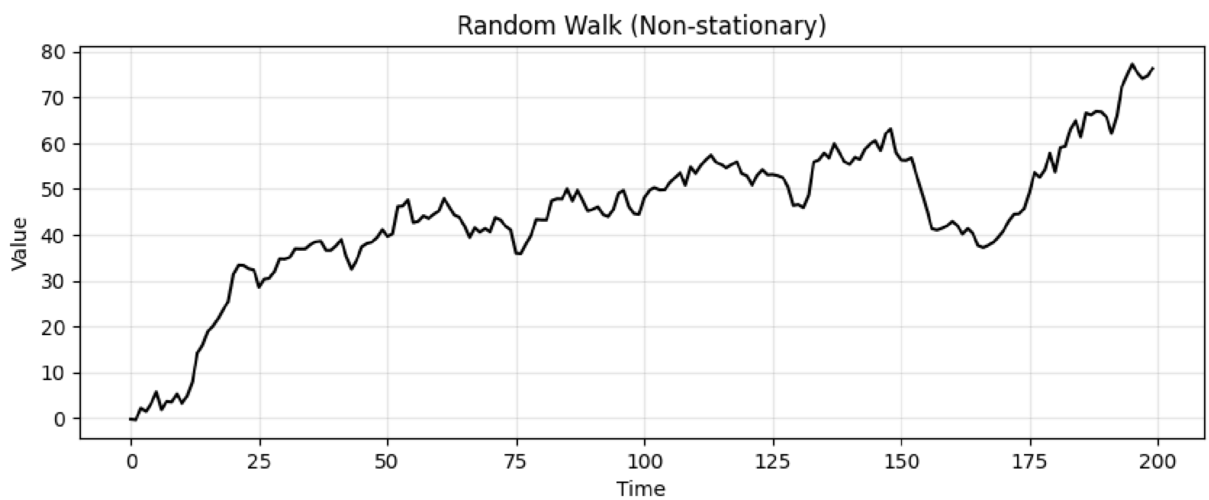
In [2]:
```
T = 200
time = np.arange(T)
white = np.random.normal(loc=0, scale=1.0, size=T)
ax = run_sequence_plot(time, white, "White Noise (Stationary)")
ax.axhline(white.mean(), color='red', label='Mean')
ax.fill_between(time,
                white.mean() - 1.96*white.std(),
                white.mean() + 1.96*white.std(),
                color='red', alpha=0.2, label='±1.96σ')
ax.legend()
plt.show()
```



Discussion: White Noise

- Fluctuations hover around a fixed mean (≈ 0) and stay within constant variance bands.
- No visible trend, no repeating patterns, and successive points show no dependence → ideal stationary behavior.
- All three weak-stationarity conditions (constant mean, constant variance, constant autocovariance) are met.

In [3]:
```
# build random walk
rand = np.empty(T)
seed = 0
for t in range(T):
    rand[t] = seed + np.random.normal(0, 2.5)
    seed = rand[t]
run_sequence_plot(time, rand, "Random Walk (Non-stationary)")
plt.show()
```

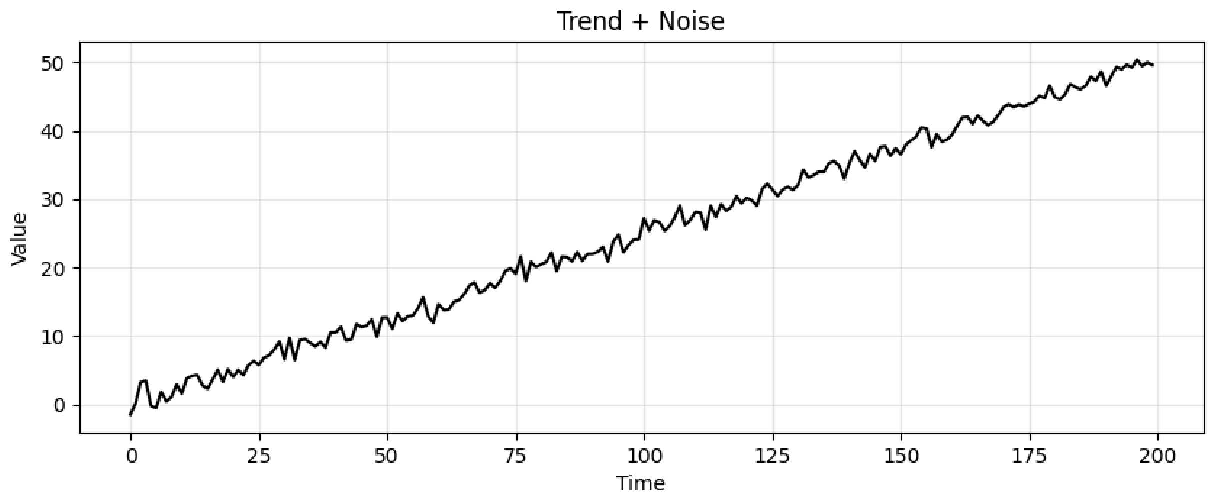


Discussion: Random Walk

- The series meanders away from its starting value with no pull-back → mean drifts over time.
- Variance grows larger as time progresses (wider swings later on) → violates constant-variance requirement.
- Strong persistence: every new point depends heavily on the last one, creating long trends.

Linear Trend

In [4]:
```
trend = np.linspace(0, 50, T) + np.random.normal(0,1,T)
run_sequence_plot(time, trend, "Trend + Noise")
plt.show()
```

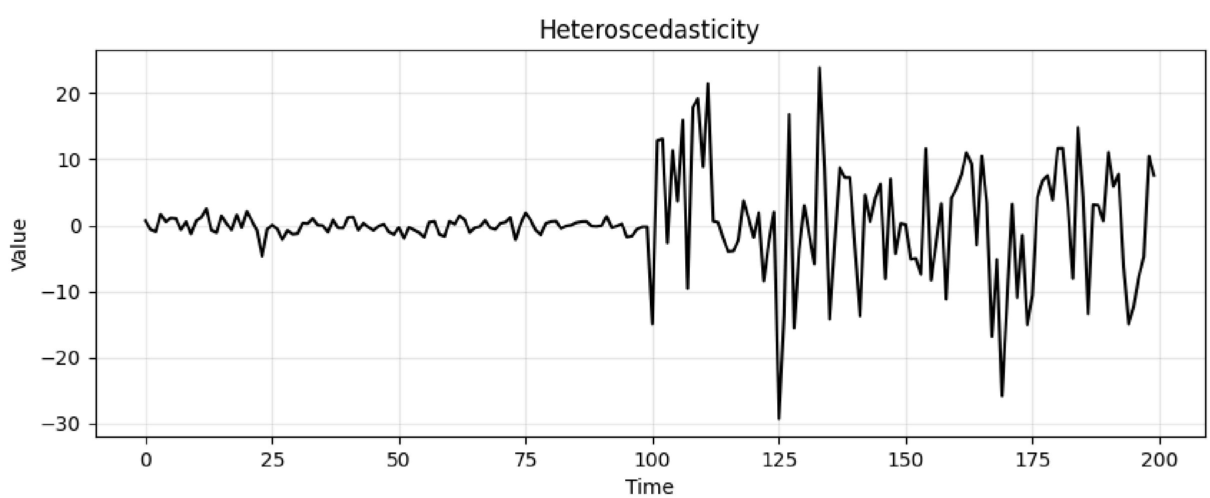


Discussion: Trend + Noise

- Despite random jitter, there's a clear upward slope → mean is not constant.
- Even if variance is roughly stable, the drifting level breaks stationarity.

Heteroscedastic Series*

In [5]:
```
level1 = np.random.normal(0,1,100)
level2 = np.random.normal(0,10,100)
het = np.concatenate([level1, level2])
run_sequence_plot(time, het, "Heteroscedasticity")
plt.show()
```



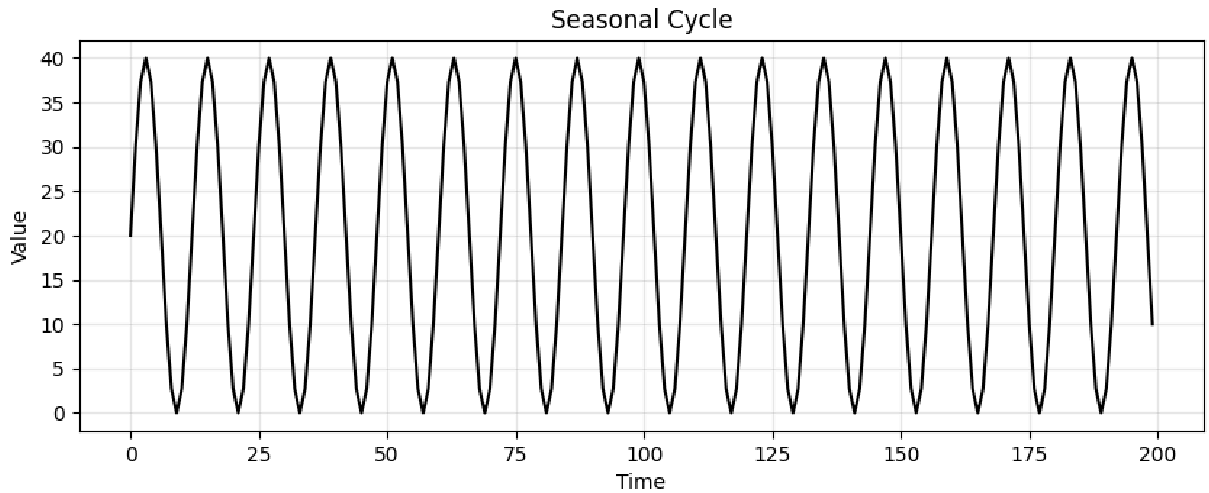
Discussion: Heteroscedastic Series

- First half is low-volatility, second half is high-volatility → variance changes over time.
- Mean may stay around zero, but changing spread alone breaks weak stationarity.

Pure Seasonality

In [6]:
```
season = 20 + 20*np.sin(2*np.pi*time/12)
run_sequence_plot(time, season, "Seasonal Cycle")
```

```
plt.show()
```

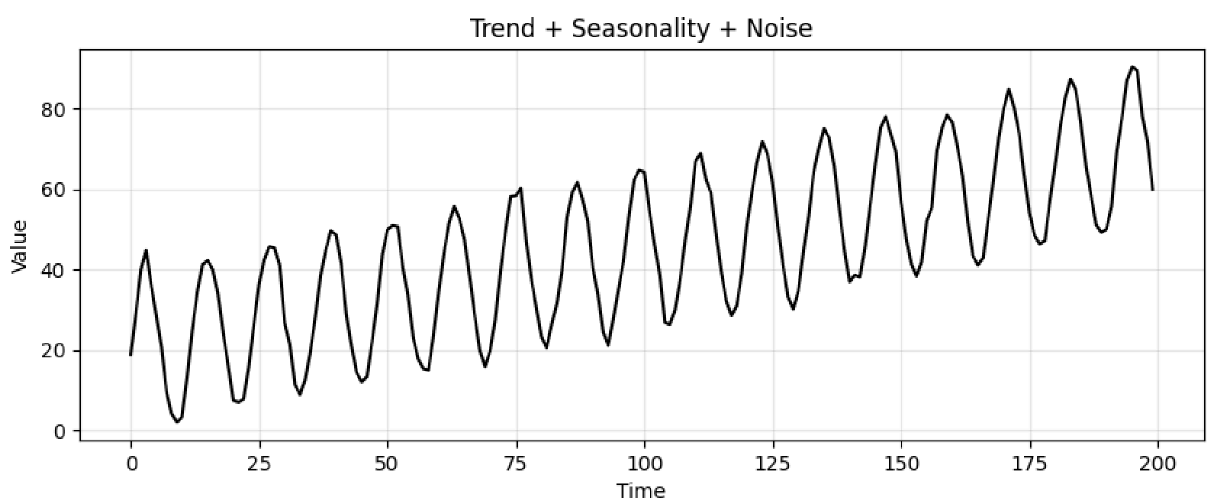


Discussion: Pure Seasonality

- The series oscillates with a fixed amplitude and period but the within-cycle mean shifts (peaks vs troughs).
- Over any sub-cycle, the average value moves away from the long-run center → violates constant-mean.

Trend + Seasonality

In [7]:
```
ts = trend + season + white
run_sequence_plot(time, ts, "Trend + Seasonality + Noise")
plt.show()
```



Discussion: Combined Trend & Seasonality

- Upward drift plus repeating peaks/troughs → both mean and periodic structure change over time.
- Two-step transformation needed: first remove seasonality, then difference or detrend to remove the remaining drift.

Detecting Stationarity

In [8]:
```
def stationarity_tests(x):
    adf_stat, adf_p, *_ = adfuller(x)
    kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
    print(f"ADF p-value: {adf_p:.4f}")
    print(f"KPSS p-value: {kpss_p:.4f}")

print("White noise tests:")
stationarity_tests(white)

print("\nRandom walk tests:")
stationarity_tests(rand)
```

```
White noise tests:
ADF p-value: 0.0000
KPSS p-value: 0.1000

Random walk tests:
ADF p-value: 0.2302
KPSS p-value: 0.0100
```

```
/var/folders/yj/3s0hc5nn3qlg4lqp7wmfgq_c0000gn/T/ipykernel_87838/368991436.p
y:3: InterpolationWarning: The test statistic is outside of the range of p-v
alues available in the
look-up table. The actual p-value is greater than the p-value returned.

  kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
/var/folders/yj/3s0hc5nn3qlg4lqp7wmfgq_c0000gn/T/ipykernel_87838/368991436.p
y:3: InterpolationWarning: The test statistic is outside of the range of p-v
alues available in the
look-up table. The actual p-value is smaller than the p-value returned.

  kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
```

Discussion: Stationarity Tests

- **White noise:** ADF $p < 0.05$ (reject non-stationary), KPSS $p > 0.05$ (fail to reject stationary) → confirms stationarity.
- **Random walk:** ADF $p > 0.05$, KPSS $p < 0.05$ → confirms non-stationarity.

Transformations to Achieve Stionarity

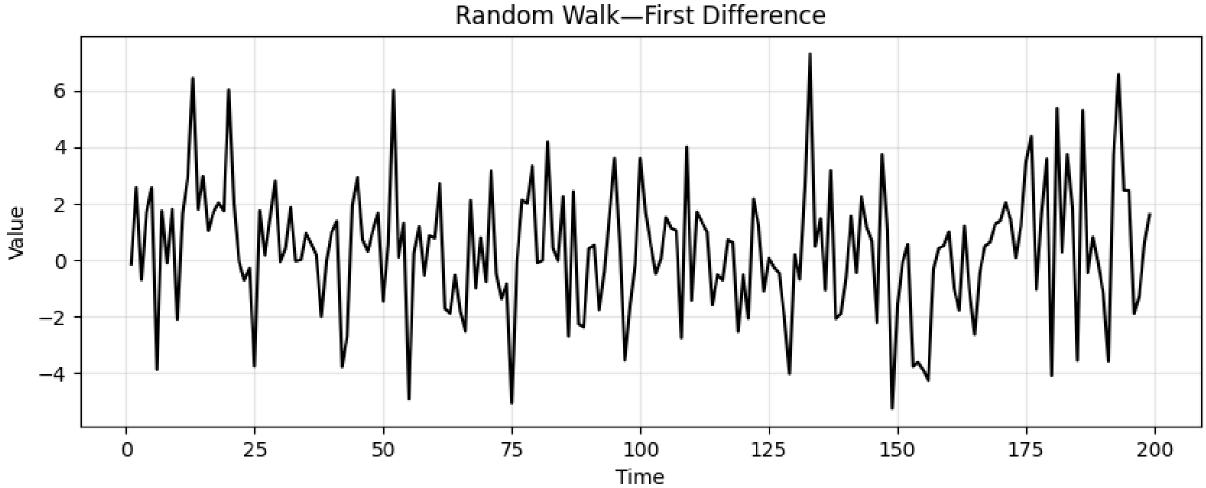
First Differencing

In [9]:
```
diff_rand = np.diff(rand)
run_sequence_plot(time[1:], diff_rand, "Random Walk—First Difference")
stationarity_tests(diff_rand)
plt.show()
```

```
ADF p-value: 0.0000
KPSS p-value: 0.1000
```

```
/var/folders/yj/3s0hc5nn3qlg4lqp7wmfgq_c0000gn/T/ipykernel_87838/368991436.p
y:3: InterpolationWarning: The test statistic is outside of the range of p-v
alues available in the
look-up table. The actual p-value is greater than the p-value returned.

  kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
```



Discussion: First Difference of Random Walk

- After differencing, the mean centers around zero and variance no longer explodes.
- Dependence on only the prior step remains, but no long-term drift.
- Series now approximates white-noise-like behavior → stationarity tests should pass.

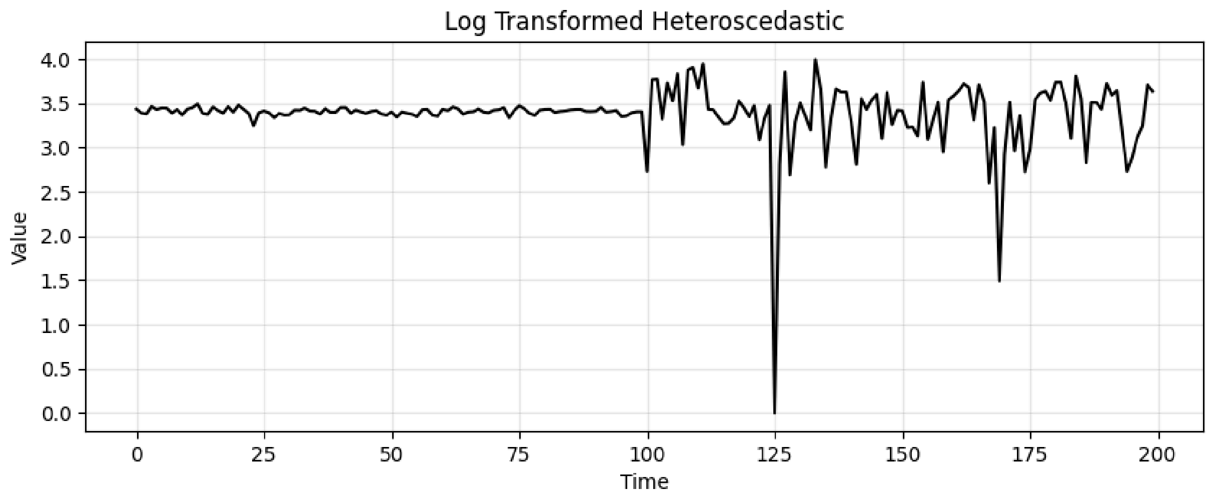
Log Transform

In [10]:
```
het_pos = het - het.min() + 1
log_het = np.log(het_pos)
run_sequence_plot(time, log_het, "Log Transformed Heteroscedastic")
stationarity_tests(log_het)
plt.show()
```

```
ADF p-value: 0.0000
KPSS p-value: 0.1000
```

```
/var/folders/yj/3s0hc5nn3qlg4lqp7wmfgq_c0000gn/T/ipykernel_87838/368991436.p
y:3: InterpolationWarning: The test statistic is outside of the range of p-v
alues available in the
look-up table. The actual p-value is greater than the p-value returned.

  kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
```



Discussion: Log-Transformed Heteroscedastic Series

- Applying log squashes the high-variance segment, bringing its spread closer to the low-variance segment.
- Variance is now more uniform across the entire series.
- Mean remains stable; this transform often suffices to restore stationarity when heteroscedasticity is the only issue.

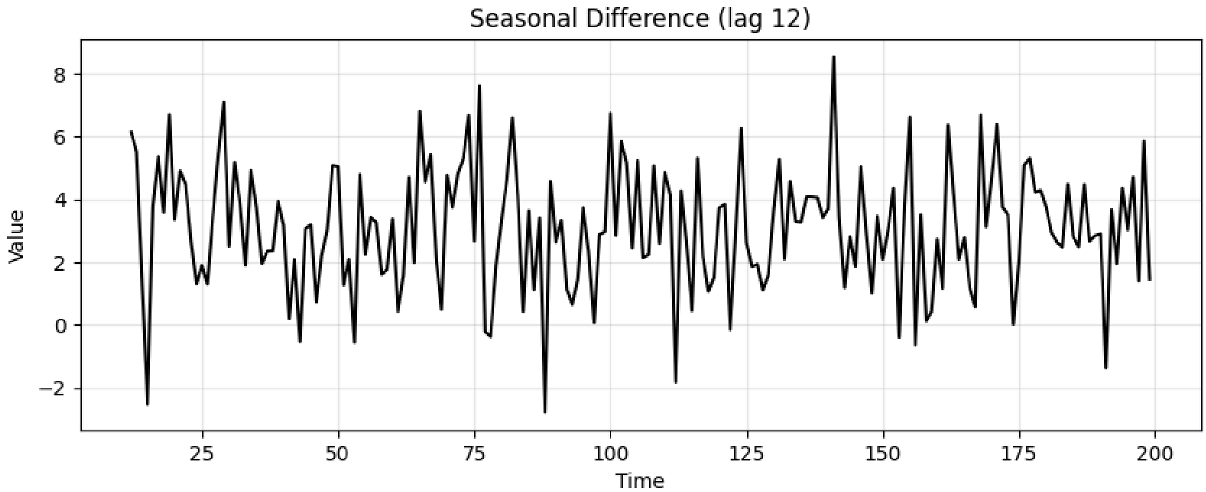
Seasonal Differencing

In [11]:
```
seas_diff = ts[12:] - ts[:-12]
run_sequence_plot(time[12:], seas_diff, "Seasonal Difference (lag 12)")
stationarity_tests(seas_diff)
plt.show()
```

```
ADF p-value: 0.0001
KPSS p-value: 0.1000
```

```
/var/folders/yj/3s0hc5nn3qlg4lqp7wmfgq_c0000gn/T/ipykernel_87838/368991436.p
y:3: InterpolationWarning: The test statistic is outside of the range of p-v
alues available in the
look-up table. The actual p-value is greater than the p-value returned.

  kpss_stat, kpss_p, *_ = kpss(x, regression='c', nlags='auto')
```



Discussion: Seasonal Differencing (lag 12)

- Subtracting each point from its value 12 steps ago effectively removes the repeating cycle.
- The resulting series fluctuates around a constant level with no obvious periodic swings.
- After this step (and perhaps a subsequent first difference), the series meets stationarity requirements.