Public presentation of "Integer sequences, algebraic series and differential operators"

Sergey Yurkevich

Universität Wien, Austria and Inria, France

Friday 2nd July, 2021

Administrative Information

- *Cotutelle* between Inria Saclay (France) and University of Vienna (Austria).
  Defense in Austria.
- *Advisors*: Alin Bostan and Herwig Hauser.
- *Duration*: March 2020 – August 2023. In total 12 months in Paris.
- *Funding*: FWF-Project P-31338 and DOC fellowship.

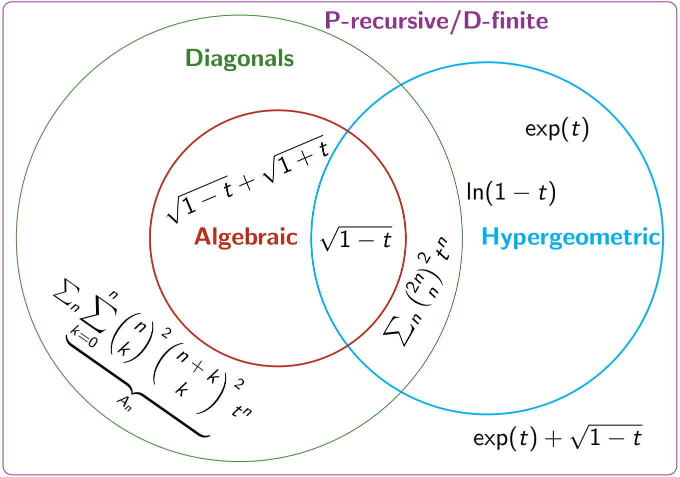
Motivating Example: Apéry's miracle

- Define the sequences $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ by the recursion

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3 u_{n-1} = 0, \quad n \geq 1,$$

  with initial conditions $(A_0, A_1) = (1, 5)$ and $(B_0, B_1) = (0, 6)$.
- $(A_n)_{n\geq 0} = (1, 5, 73, 1445, 33001, \dots)$ & $(B_n)_{n\geq 0} = (0, 6, \frac{351}{4}, \frac{62531}{36}, \frac{11424695}{288}, \dots)$.
- One finds that:
  - $A_n \in \mathbb{Z}$ for all $n \geq 0$ and $d_n^3 B_n \in \mathbb{Z}$ for all $n \geq 0$, where $d_n = \mathrm{lcm}\{1, 2, \dots, n\}$.
  - $B_n / A_n \to \zeta(3) := \sum_{k\geq 1} k^{-3}$ as $n \to \infty$.
- The facts above imply that $\zeta(3) \notin \mathbb{Q}$ [Apéry, 1979].
- Natural questions:
  - Why is $A_n \in \mathbb{Z}$ and what can we say about $f_{2,2}(t) := \sum_{n\geq 0} A_n t^n$?
  - Can we generalize this proof/method?
- In fact, deep theory responsible for this proof [Beukers, 1983].
- *Aim of thesis*: Study and understand this and similar phenomena!

Definitions and interactions



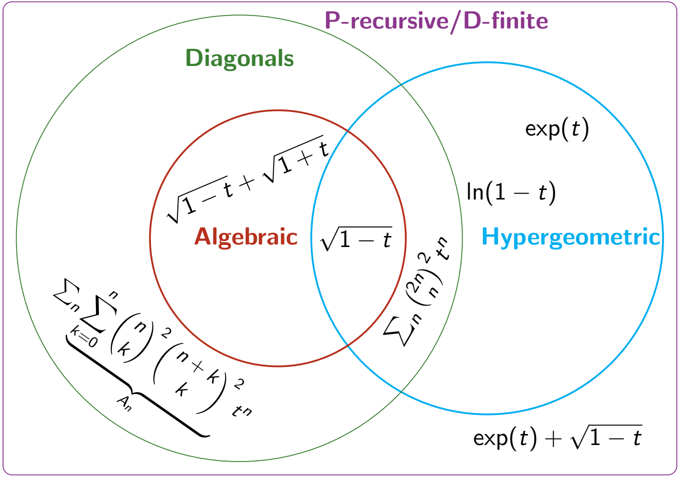
A sequence $(u_n)_{n\geq 0}$ is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_d(n)u_{n+d} + \cdots + c_0(n)u_n = 0.$$

$(u_n)_{n\geq 0}$ is **hypergeometric** if $d = 1$.

$u_n = 1/n!$ satisfies $nu_n = u_{n-1}$.

Definitions and interactions

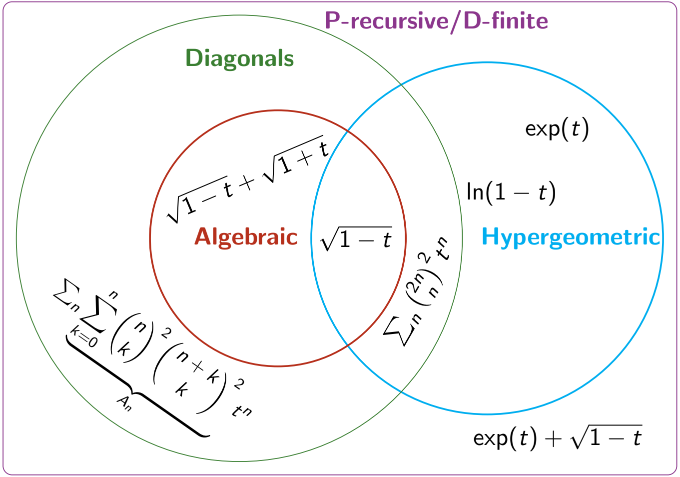


A power series $f(t) \in \mathbb{Q}[\![t]\!]$ is called **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(t)f^{(n)}(t) + \cdots + p_0(t)f(t) = 0.$$

$\exp(t)$ satisfies $\exp'(t) = \exp(x)$.

Definitions and interactions



For a multivariate power series

$$f(x_1, \ldots, x_n) = \sum_{j_1, \ldots, j_n} f_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}$$

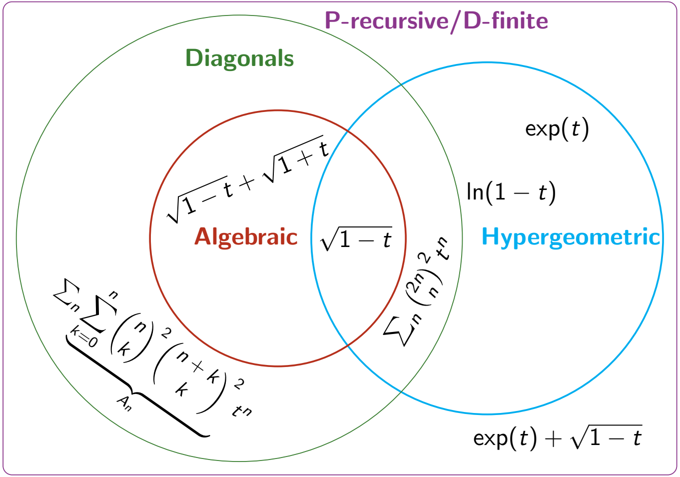
the diagonal is given by

$$\mathrm{Diag}(f) = \sum_j f_{j,j,\ldots,j} t^j \in \mathbb{Q}[[t]].$$

Diagonals are series which can be written as diagonals of multivariate *rational* functions.

$$\mathrm{Diag}\left(\frac{1}{1-x-y}\right) = \mathrm{Diag} \sum_{i,j} \binom{i+j}{j} x^i y^j = \sum_n \binom{2n}{n} t^n = (1-4t)^{-1/2}$$

Definitions and interactions



[Abel, 1827]:
Algebraic $\subseteq$ **D-finite**.

[Furstenberg, 1967]:
Algebraic $\subseteq$ **Diagonals**.

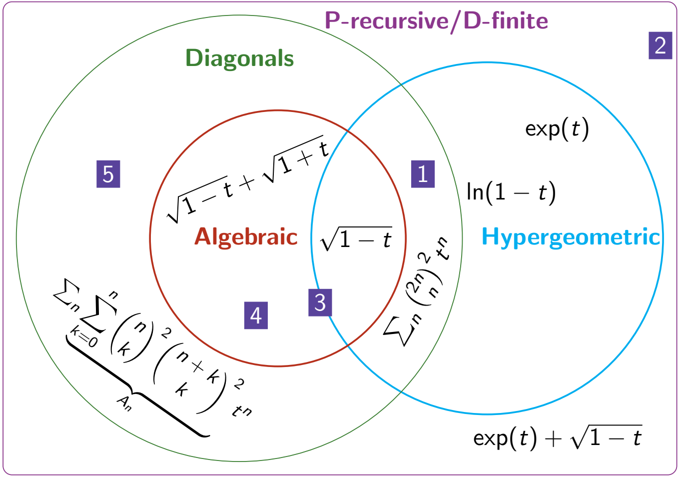
[Lipshitz, 1988]:
Diagonals $\subseteq$ **D-finite**.

[Beukers,Heckman, 1989]:
Algebraic $\cap$ **Hypergeometric**.

[Bostan,Lairez,Salvy, 2015]:
Diagonals $=$ **Multiple binomial sums**.

Christol's Conjecture [Christol, 1987]: A convergent **D-finite** power series with **integer coefficients** is a **Diagonal**.

Definitions and interactions



1. Hypergeometric diagonals
2. Computing N-th term of a q-P-recursive sequence
3. Bézivin's conjecture
4. Zagier's problem
5. Diagonal representations

Hypergeometric Diagonals: Towards Christol's conjecture

- Joint work with Alin Bostan.
- Generalization, extension and simplification of the main result of [Abdelaziz, Koutschan, Maillard, 2020] on Christol's conjecture.
- Main theorem:

Theorem (Bostan, Y., 2020)

$$\mathrm{Diag}\left(\prod_{i=1}^{n}(1+x_1+\cdots+x_i)^{b_i}\right) = {}_MF_{M-1}(u\,;v\,;t)$$

is a hypergeometric function with explicitly given parameters.

- Corollary: Christol's conjecture holds for a large class of hypergeometric functions, e.g. ${}_3F_2([1/9,4/9,7/9]\,;[2/3,1]\,;t)$. (${}_3F_2([1/9,4/9,5/9]\,;[1/3,1]\,;t)$ still open!)
- Accepted for publication in *Proceedings of the American Mathematical Society*.

Computing the N-th term of a q-P-recursive sequence

- Joint work with Alin Bostan.
- Adaptation of known results about complexity of computation of N-th terms in P-recursive sequences to their q-analogues.

Theorem (Bostan, Y., 2020)

Let $q \in \mathbb{K}$ and $(u_n)_{n \geq 0}$ be q-P-recursive sequence of order r. Let $N \in \mathbb{N}$. Then u_N can be computed in $\tilde{O}(r^\theta \sqrt{N})$ operations in $\mathbb{K}$.

- Naive and previously best known complexity: $O(N)$.
- Uses ideas of [Strassen, 1977]&[Chudnovsky^2, 1988] exploited in [Bostan, 2020].
- Applications: e.g. evaluation of polynomials or fast computation of p-curvatures.
- Accepted for publication in *Journal of Symbolic Computation*.

On Bézivin's conjecture

- Joint work with Herwig Hauser.

Conjecture (Bézivin, 1991)

If a differential operator L has a basis of series solutions with integer coefficients, then all solutions to $Ly = 0$ are algebraic.

- Deep commutative algebra involved.
- Grothendieck-Katz conjecture $\Rightarrow$ Bézivin's conjecture. Equivalence open.
- New elementary proof of Bézivin's conjecture for equations of order one.
- Current work todo:
  - Effective version of the proof.
  - Extend the proof to special cases of equations of order two.
  - Comparison with work of Katz, Honda and Chudnovsky[2].

Zagier's problem

- Joint work with Alin Bostan and Jacques-Arthur Weil.
- Origin: integral over a moduli space ("topological ODE") [Bertola, et. al, 2015].
- In [Zagier, 2018]:

$$c_{n-3} + 20\left(4500n^2 - 18900n + 19739\right) c_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)c_n$$
$$25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right) c_{n-1} = 0,$$

with initial terms $c_0 = 1, c_1 = -161/(2^{10}\cdot 3^5)$ and $c_2 = 26605753/(2^{23}\cdot 3^{12}\cdot 5^2)$.

- Task: Find $(u,v)\in\mathbb{Q}^*\times\mathbb{Q}^*$ such that $c_n\cdot(u)_n\cdot(v)_n\cdot w^n\in\mathbb{Z}$ for some $w\in\mathbb{Z}^*$.
- [Yang and Zagier]: $a_n = c_n\cdot(3/5)_n\cdot(4/5)_n\cdot(2^{10}\cdot 3^5\cdot 5^4)^n\in\mathbb{Z}$,
- [Dubrovin and Yang]: $b_n = c_n\cdot(2/5)_n\cdot(9/10)_n\cdot(2^{12}\cdot 3^5\cdot 5^4)^n\in\mathbb{Z}$.

Theorem (Bostan, Weil, Y.; *work in progress*)

There are 7 more pairs (u,v) for which $c_n\cdot(u)_n\cdot(v)_n\cdot w^n\in\mathbb{Z}$ for some $w\in\mathbb{Z}^*$. All yield algebraic generating functions.

Diagonal representation of generalized Apéry numbers

- Joint work with Duco van Straten.
- Open problem: Construct a power series which is a diagonal, but cannot be written as the diagonal of a rational function in three or less variables.

Theorem (van Straten, Y.; *work in progress*)

The generating function of the generalized Apéry numbers

$$f_{\alpha,\beta}(t) = \sum_{n\geq 0}\sum_{k\geq 0}\binom{n}{k}^{\alpha}\binom{n+k}{k}^{\beta}t^n,$$

for $\alpha, \beta \in \mathbb{N}$, is a diagonal of a rational power series in $\alpha+\beta$ variables, and not less.

- $f_{2,2}(t)$ is the generating function of the Apéry numbers.

Methodology

- Interplay: theoretical and applied mathematics
- Algebraic geometry
  - Connection to periods and modular forms.
  - Picard-Fuchs equations.
  - Connection to Gromov-Witten theory.
- Computer science
  - Efficient algorithms & symbolic computation
  - Experimental mathematics
  - Applications

Bonus: Diagonal representations of the Apéry numbers

- It holds that

$$\operatorname{Diag}\frac{1}{1-w(1+x)(1+y)(1+z)(xyz+yz+y+z+1)} = \sum_n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 t^n.$$

- It holds [Straub, 2014] that

$$\operatorname{Diag}\frac{1}{(1-x-y)(1-z-w)-xyzw} = \sum_n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 t^n.$$

Bonus: More definitions and facts

- A power series $f(t) \in \mathbb{Q}[\![t]\!]$ is called D-algebraic if
$$P(t, f(t), f'(t), \dots, f^{(n)}(t)) = 0,$$
for some polynomial $P(t, z_1, \dots, z_n) \neq 0$. $\tan(x)$ satisfies $\tan'(x) = \tan(x)^2 + 1$.
- Multiple binomial sums: Class of multivariate sequences, containing the binomial coefficient sequence, closed under pointwise addition, pointwise multiplication, linear change of variables and partial summation [Bostan,et.al,2015].
- $f \in \mathbb{Q}[\![t]\!]$ is called *globally bounded* if it has finite non-zero radius of convergence and $\beta \cdot f(\alpha \cdot t) \in \mathbb{Z}[\![t]\!]$ for some non-zero $\alpha, \beta \in \mathbb{Z}$.
- Let $\mathbb{K}$ be a field, and $q \in \mathbb{K}$. A sequence $(u_n(q))_{n \geq 0}$ in $\mathbb{K}^{\mathbb{N}}$ is called *q-holonomic* if there exist $r \in \mathbb{N}$ and polynomials $c_0(x, y), \dots, c_r(x, y)$ in $\mathbb{K}[x, y]$, with $c_r(x, y) \neq 0$, such that $c_r(q, q^n) u_{n+r}(q) + \cdots + c_0(q, q^n) u_n(q) = 0$, $n \geq 0$.
- A power series $f(t) = \sum_n a_n t^n \in \mathbb{Q}[\![t]\!]$ is a G-function if it has a positive radius of convergence, is D-finite, and both $|a_n|$ and the common denominator of $(a_1, \dots, a_n)$ are bounded by C^n for some C depending only on f.

Bonus: Finer classification