Biomembranes and creative telescoping[1]

Seminar Algebra and Discrete Mathematics (Linz, Austria)

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[1] Joint work with Alin Bostan and Thomas Yu.

Motivating examples

- Recurrence for **Apéry numbers**:

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{satisfies} \quad (n+1)^3 A_{n+1} = (17n^2+17n+5)(2n+1)A_n - n^3 A_{n-1}.$$

- Generating function of **moments**:

$$m_n = \int_0^1 x^n \cdot \sqrt[3]{x(1-x)}\,\mathrm{d}x \quad \text{satisfies} \quad \sum_{k\geq 0} m_k t^k = c \cdot {}_2F_1\left[\begin{matrix} 1 & \frac{4}{3} \\ & \frac{8}{3} \end{matrix}; t\right].$$

- **Surface area** a projection to $\mathbb{R}^3$ of the **Clifford torus**:

$$\int_0^{2\pi}\int_0^{2\pi} \frac{(\sqrt{2}+\sin v)\,\mathrm{d}u\,\mathrm{d}v}{(1+2t(\sqrt{2}+\sin v)\cos u + t^2(3+2\sqrt{2}\sin v))^2}$$

$$= \frac{4\sqrt{2}\pi^2\,(1-t^2)}{(t^2-6t+1)^2}\;{}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; \frac{4t}{(1-t)^2}\right]$$

Algorithmic proofs

$$A_n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=:a_{n,k}} \quad \text{satisfies} \quad (n+1)^3 A_{n+1} = (17n^2+17n+5)(2n+1)A_n - n^3 A_{n-1}.$$

[van der Poorten, 1978]:

> Neither Cohen nor I had been able to prove (5) or (5′) in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b_n'\}$ satisfies the recurrence (4). This more or less broke the dam and (5′) and (5) were quickly conquered.

Algorithmic proofs

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`> Zeilberger(a, n, k, N);` finds in < 0.02 seconds:

$$L = (n+2)^3 N^2 - (17n^2 + 51n + 39)(2n+3)N + (n+1)^3 \text{ and}$$

$$C = (k^2 - 3/2k - 2n^2 - 6n - 4)k^4(16n+24)/(k-n-1)/(k-n-2),$$

with the property that ($N \cdot a_{n,k} := a_{n+1,k}$ and $K \cdot a_{n,k} := a_{n,k+1}$):

$$L \cdot \binom{n}{k}^2 \binom{n+k}{k}^2 = (K-1) \cdot C \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Sum over k from 0 to n and conclude.

Algorithmic proofs

$$m_n = \int_0^1 \underbrace{x^n \cdot \sqrt[3]{x(1-x)}}_{=: f_n(x)}\,\mathrm{d}x \quad \text{satisfies} \quad \sum_{k\geq 0} m_k t^k = {}_2F_1\left[\begin{matrix} 1 \ \frac{4}{3} \\ \frac{8}{3} \end{matrix}; t\right] \cdot \frac{2\pi^2}{15\Gamma(2/3)^3}.$$

`> creative_telescoping(f,n::shift,x::diff);` finds in < 0.1 seconds:

$$L = (3n+8)N - (3n+4) \quad \text{and} \quad C(x) = 3x(x-1),$$

with the property that ($N \cdot f_n(x) = f_{n+1}(x)$):

$$L \cdot x^n \sqrt[3]{x(1-x)} = \partial_x\left(C(x) \cdot x^n \sqrt[3]{x(1-x)}\right)$$

It follows that $L \cdot \int_0^1 x^n \sqrt[3]{x(1-x)}\mathrm{d}x = 0$ and hence $(3n+8)m_{n+1} = (3n+4)m_n$.

Algorithmic proofs

$$\int_0^{2\pi}\int_0^{2\pi} \frac{(\sqrt{2}+\sin v)\,\mathrm{d}u\,\mathrm{d}v}{(1+2t(\sqrt{2}+\sin v)\cos u+t^2(3+2\sqrt{2}\sin v))^2}$$

$$= \frac{4\sqrt{2}\pi^2\left(1-t^2\right)}{\left(t^2-6t+1\right)^2}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};\frac{4t}{(1-t)^2}\right].$$

Algorithmic proofs

$$\oint_\gamma \frac{2(2\sqrt{2}\,y - y^2 + 1)x\,\mathrm{d}x\mathrm{d}y}{\left(2\sqrt{2}\,t^2 x\,y^2 + 2\sqrt{2}\,t\,x^2 y - t\,x^2 y^2 - 2\sqrt{2}\,t^2 x - 2t^2 xy + 2\sqrt{2}\,ty + t\,x^2 - t\,y^2 - 2yx + t\right)^2}$$

$$= \frac{4\sqrt{2}\pi^2\left(1-t^2\right)}{\left(t^2-6t+1\right)^2}\ {}_2F_1\!\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};\frac{4t}{(1-t)^2}\right].$$

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> `FindCreativeTelescoping[F, {Der[x], Der[y]}, Der[t]];` finds in 10 seconds:

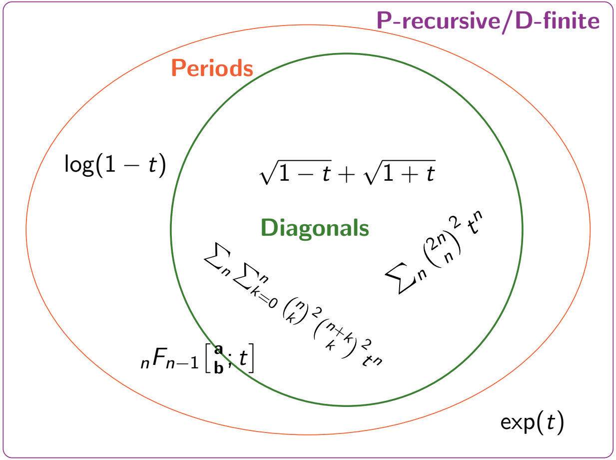
$$L = t\left(3t^2-1\right)\left(9t^4-2t^2+1\right)\left(3t^2+1\right)^2\partial_t^2 + \left(3t^2+1\right)\left(729t^8+162t^6-192t^4+38t^2-1\right)\partial_t$$
$$+12t\left(324t^8+333t^6+51t^4-53t^2+1\right),$$ and

$C_1, C_2 \in \mathbb{Q}(x, y, t)$ with the property that:

$$L \cdot F = \partial_x C_1 + \partial_y C_2.$$

Therefore it follows that $L \cdot \oint_\gamma F = 0$. Solving $Ly = 0$ we find the right-hand side.

Definitions and interactions



A power series $f(t) \in \mathbb{Q}[\![t]\!]$ is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(t) f^{(n)}(t) + \cdots + p_0(t) f(t) = 0.$$

This equation can be rewritten: $L \cdot f = 0$,

$$L = p_n(t)\partial_t^n + \cdots + p_0(t) \in \mathbb{Q}[t][\partial_t].$$

Let $(\alpha)_n = \alpha \cdot (\alpha+1) \cdots (\alpha+n-1)$.
Then ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; t\right] := \sum_{n\geq 0} \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!} t^n$ satisfies

$$t(1-t)f''(t) + (c-(a+b+1)t)f'(t) - abf(t) = 0.$$

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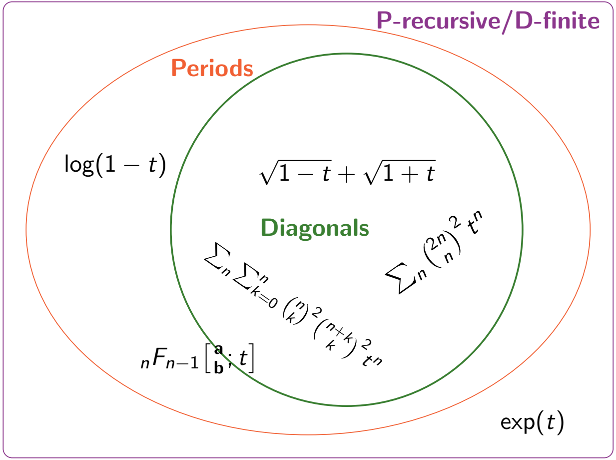

A sequence $(u_n)_{n\geq 0}$ is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_r(n)u_{n+r} + \cdots + c_0(n)u_n = 0.$$

Let $(\alpha)_n = \alpha \cdot (\alpha+1) \cdots (\alpha+n-1)$.

Then $u_n = \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!}$ satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0.$$

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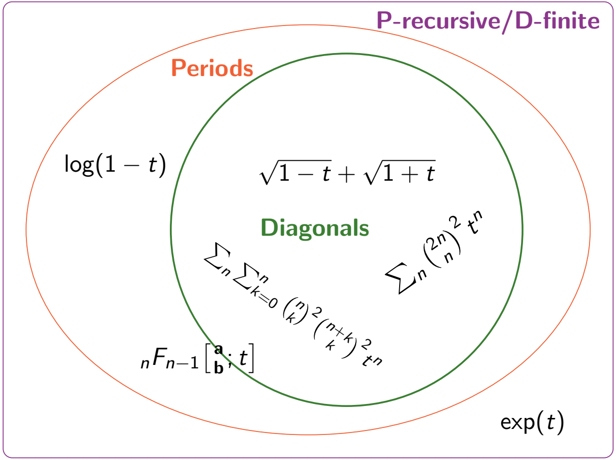


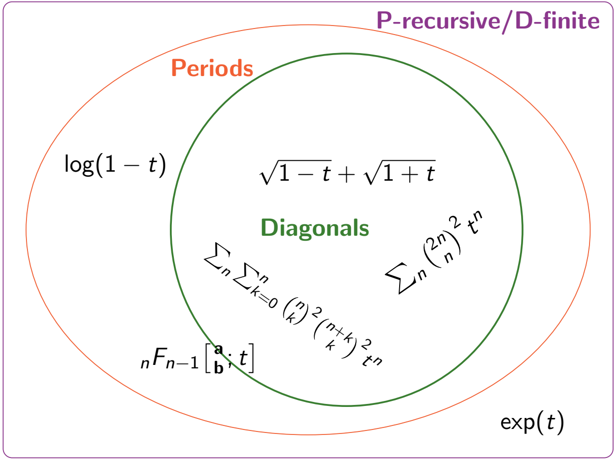
A power series $f(t) \in \mathbb{Q}[\![t]\!]$ is called a **Period function** if it is an integral of a rational function in t and $x_1, \ldots, x_n$ over a semi-algebraic set.

$$p(t) = 4\int_0^1 \sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x$$
$$= 4 \oiint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-t^2x^2}{(1-x^2)y^2}} \quad \text{and}$$
$$((t-t^3)\partial^2 + (1-t^2)\partial + t)\cdot p = 0,$$
$$p(t) = 2\pi - \frac{\pi}{2}t^2 - \frac{3\pi}{32}t^4 - \cdots.$$

André-Bombieri-Katz's theorem: A **Period function** is a G-function [André, 1989].

Bombieri-Dwork conjecture: Any G-function is a **Period function**.

Definitions and interactions



A power series $f(t) \in \mathbb{Q}[\![t]\!] = \sum_k u_k t^k$ is called a **Diagonal** if there exists a rational function

$$R = \sum_{i_1,\dots,i_n \geq 0} c_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Q}(x_1, \dots, x_n)$$

such that

$$f(t) = \mathrm{Diag}(R) := \sum_{k \geq 0} c_{k,\dots,k} t^k.$$

Equivalently [Bostan, Lairez, Salvy 2017], $(u_k)_{k\geq 0}$ is a **multiple binomial sum**.

$$\mathrm{Diag}\frac{1}{1-x-y} = \mathrm{Diag}\sum_{i,j\geq 0}\binom{i+j}{j}x^i y^j = \sum_{k\geq 0}\binom{2n}{n}t^k = \frac{1}{\sqrt{1-4t}}$$

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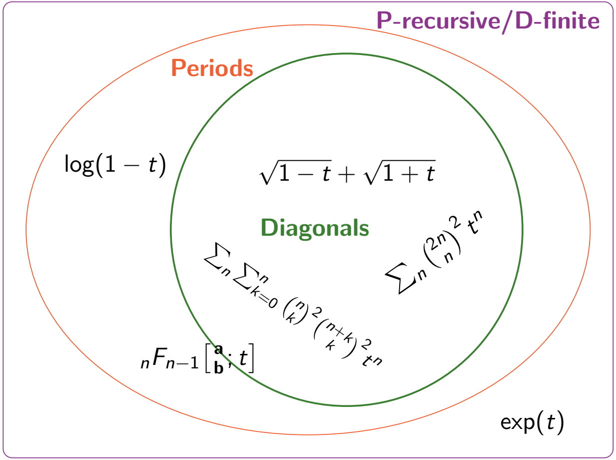

A power series $f(t) \in \mathbb{Q}[[t]] = \sum_k u_k t^k$ is called a **Diagonal** if there exists a rational function

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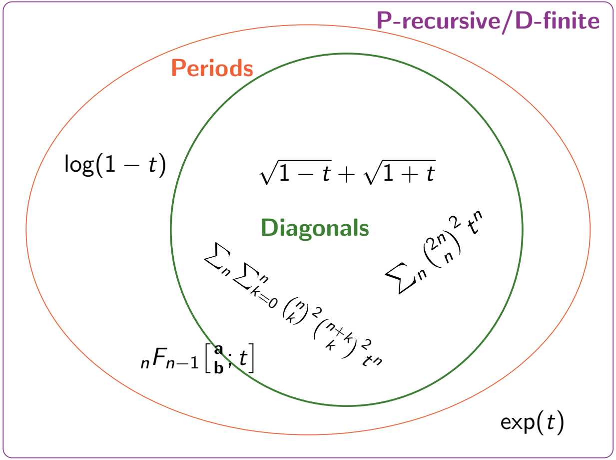
$$\mathrm{Diag}\frac{1}{1-x-y} = [x^{-1}]\frac{1}{x}\frac{1}{1-x-t/x} = \frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{\mathrm{d}x}{x-x^2-t} = (1-4t)^{-\frac{1}{2}}$$

Definitions and interactions



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Christol's conjecture: A convergent **D-finite** power series in $\mathbb{Z}[\![t]\!]$ is a **Diagonal.**

Principle of Creative Telescoping

- Goal: Given a **Period function** or **Diagonal**, find an annihilating ODE.

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- More precisely: Given $R \in \mathbb{Q}(x_1, \ldots, x_n\,; t)$ and a closed cycle $\gamma \subseteq \mathbb{C}^n$, find

$$L = p_n(t)\partial_t^n + \cdots + p_0(t) \in \mathbb{Q}[t][\partial_t], \quad \text{such that} \quad L \cdot \oint_\gamma R \mathrm{d}x = 0.$$

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- Note: $\int_\gamma \partial_{x_i} C \mathrm{d}x = \int_{\partial\gamma} C \mathrm{d}x = \int_\emptyset C \mathrm{d}x = 0$ for any rational function $C \in \mathbb{Q}(\mathbf{x}\,, t)$.
- So we need to find

$$L \in \mathbb{Q}[t][\partial_t], \text{ and } C_1, \ldots, C_n \in \mathbb{Q}(x_1, \ldots, x_n\,, t), \text{ such that}$$
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Principle of Creative Telescoping

$$\sum_{k=0}^{n} p_k(t)\frac{\mathrm{d}^k R}{\mathrm{d}t} = \partial_{x_1} C_1 + \cdots + \partial_{x_n} C_1 \Rightarrow \left(\sum_{k=0}^{n} p_k(t)\partial_t^k\right) \cdot \oint_\gamma R\mathrm{d}x = 0.$$

The telescoper and certificates always exist and can be found **algorithmically**

The Almkvist-Zeilberger algorithm [1990] *"I could never resist a definite integral."*

Input: A hyperexponential function $H(t,x)$, i.e. $\partial_t H/H$ and $\partial_x H/H \in \mathbb{Q}(t,x)$.
Output: A linear differential operator $P(t,\partial_t) \in \mathbb{Q}[t][\partial_t]$ and $G(t,x) \in \mathbb{Q}(t,x)$, s.t.

$$P \cdot H = \partial_x \left(G \cdot H \right).$$

Algorithm: Let $\mathbb{L} = \mathbb{Q}(t)$. For $r = 0, 1, 2, \ldots$ do:

1. Compute $a(t,x) = \partial_x H/H$ and $b_k(t,x) = \partial_t^k H/H$ for $k = 0, \ldots, r$.
2. Decide whether the (ordinary, linear, inhomogeneous, parametrized) diff. equation

$$\partial_x G + a(t,x) G = \sum_{k=0}^{r} c_k(t) b_k(t,x)$$

   has a rational solution $G \in \mathbb{L}(x)$ for some $c_0(t), \ldots, c_r(t) \in \mathbb{L}$ not all zero.
3. If found solution in (2), return $P = \sum_{k=0}^{r} c_k \partial_t^k$ and G; else increase r and repeat.

Some history of Creative Telescoping

- Indefinite integration/summation and working examples

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

  - Sums: [Bernoulli, Fasenmyer, Gosper,...]
  - Integrals: [Legendre, Ostrogradsky, Hermite, Picard, Manin, Griffiths, Feynman, ...]

$$\int \frac{P(x)}{Q(x)}\mathrm{d}x = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\mathrm{d}x \qquad \int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-xt)}} = \pi\, {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; t\right]$$

- Algorithmic Creative Telescoping (algorithmic definite summation&integration):
  - 1G: brutal elimination: [Fasenmyer, 1947], [Zeilberger, 1990], [Takayama, 1990]
  - 2G: rational solutions of linear ODEs: [Zeilberger, 1990], [Almkvist,Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]
  - 3G: 2G + linear algebra + bounds: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]
  - 4G: based on (Hermite- and generalized Griffiths-Dwork) reduction [Bostan, Chen, Chyzak, Kauers, Koutschan, Li, Lairez, Salvy, Singer,...]

Creative Telescoping and de Rham cohomology

"the certificate is not needed, its existence and regularity are sufficient."

- Let $\mathbb{L} = \mathbb{Q}(t)$, $f \in L[x_0, \ldots, x_n] = \mathbb{L}[x]$ and $\gamma \subseteq \mathbb{C}^n$ a closed n-cycle.
- Denote by $\mathbb{L}[x, 1/f]_p = \{F \in \mathbb{L}[x, 1/f] : F(\lambda x) = \lambda^p F(x), \forall \lambda \in \mathbb{Q}(t)\}$.
- We wish to compute the differential equation satisfied by

$$\oint_\gamma F(t; x_0, \ldots, x_n)\mathrm{d}x, \text{ where } F = a/f^\ell \in \mathbb{L}[x, 1/f]_{-n-1}.$$

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- Therefore we wish to find a non-trivial element in

$$H_f^{\mathrm{pr}} := \mathbb{L}[x, 1/f]_{-n-1}/D_f, \text{ where } \quad D_f := \mathrm{span}_{\mathbb{Q}}(\{\partial_{x_i} C : C \in \mathbb{L}[x, 1/f]_{-n}\})$$

- Generalized Griffiths-Dwork Reduction: $F \mapsto [F]$, s.t. $\oint_\gamma F\mathrm{d}x = 0 \iff [F] = 0$.

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Theorem [Griffiths 1969, Bostan, Lairez, Salvy 2013, Lairez 2016]

Assume that $\mathbb{L}[x]/\langle \partial_{x_0} f, \dots, \partial_{x_n} f\rangle$ is finite-dimensional over $\mathbb{L}$. Then H_f^{pr} is finitely generated over $\mathbb{L}$. Moreover the *Generalized Griffiths-Dwork Reduction* can be used to compute the (minimal regular) telescoper.

Issues with singularities: non-regular certificates

"the certificate is not needed, its existence and regularity are sufficient."

- The following example originates in [Picard, 1899]: Let $P_t(u) = u^3 + t$, then

$$F = \frac{x-y}{z^2 - P_t(x)P_t(y)}$$

$$= \partial_x \frac{2P_t(x)}{(x-y)(z^2 - P_t(x)P_t(y))} + \partial_y \frac{2P_t(y)}{(x-y)(z^2 - P_t(x)P_t(y))} + \partial_z \frac{3(x^2+y^2)z}{(x-y)(z^2 - P_t(x)P_t(y))},$$

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- So one has $1 \cdot F = \partial_x C_1 + \partial_y C_2 + \partial_z C_3$, **however**:

$$\oint_\gamma F \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \neq 0 \quad \text{for some } \gamma \subseteq \mathbb{C}^3.$$

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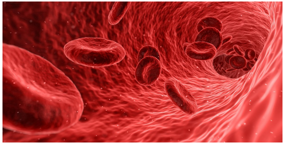
- Conclusion: Certificates are important.
  A certificate is called **regular** if it has no other poles than F.

Motivation and Introduction

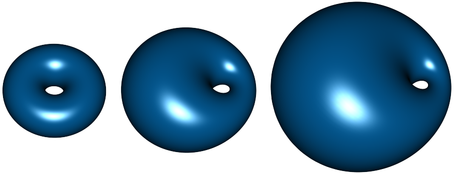
"Why do all humans have the same biconcave shaped red blood cells?"

- *Canham model* predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the *Willmore energy*



$$W(S) := \int_S H^2 \mathrm{d}A, \quad (H \text{ is the mean curvature})$$

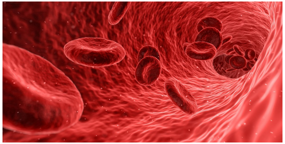
over orientable closed surfaces $S \subseteq \mathbb{R}^3$ with genus g, area A_0 and volume V_0.



Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- *Canham model* predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the *Willmore energy*



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- [Willmore, 1965]: For a torus $T = T(R,r)$ the Willmore energy is:

$$W(T) = \frac{\pi^2 R^2}{r\sqrt{R^2 - r^2}} \rightsquigarrow \text{minimal for } R/r = \sqrt{2}.$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces in $\mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T_{\sqrt{2}}$.

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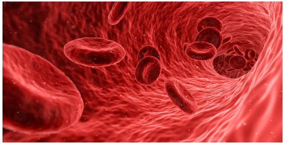

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Across all closed surfaces in $\mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T_{\sqrt{2}}$.

- $W(S)$ is invariant under Möbius transformations $\Rightarrow$ no uniqueness of the shape.

[Yu, Chen, 2021]: All projections of the (Clifford) torus

- The Clifford torus CT is defined as the following set in $\mathbb{S}^3$:

$$\mathrm{CT} := \{[\cos u, \sin u, \cos v, \sin v]^T/\sqrt{2} : u, v \in [0, 2\pi]\} \subseteq \mathbb{R}^4$$

- The torus with minor radius 1 and major radius $R > 1$:

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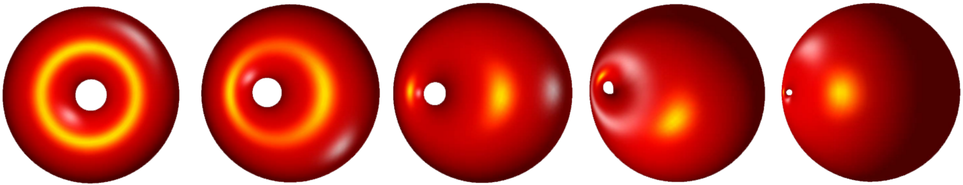
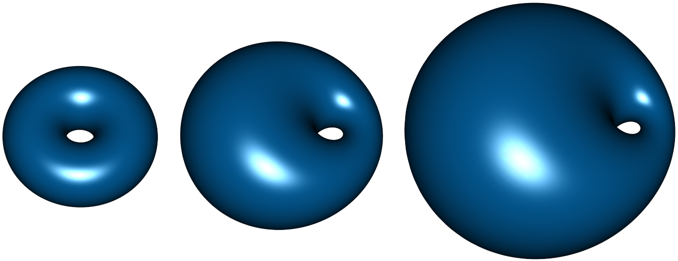
- $\mathrm{inv}_{(x,y,z)}$ is the inversion map about the unit sphere centered at $(x, y, z) \in \mathbb{R}^3$.
- The set of all shapes of stereographic projections of CT to $\mathbb{R}^3$ is parameterized by

$$\{\mathrm{inv}_{(t,0,0)}(T_{\sqrt{2}}) : t \in [0, \sqrt{2} - 1)\}.$$





$$W(\mathrm{inv}_{(x,y,z)}(T)) = W(T) = \int_T H^2 \mathrm{d}A = 2\pi^2.$$

[Willmore, 1965] and [Marques, Neves, 2014]

Then we have

(17) $$\tau(f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} H^2 b(a + b\cos u)\,du\,dv.$$

After some computation we find, on writing $b/a = c$, that

(18) $$\tau(f) = \frac{\pi}{2c\sqrt{1-c^2}}.$$

It is easy to see that $\tau(f) \to \infty$ both as $c \to 0$ and as $c \to 1$.

The minimum value of (f) occurs when $c = 1/\sqrt{2}$, when the value of $\tau(f)$ is π.

It seems reasonable to interpret $\tau(f)$ as a measure of the „niceness" of the shape of the surface $f(S)$, and to argue heuristically that a small value of $\tau(f)$ corresponds to a simple shape for $f(S)$. This suggests that (13) with $b/a = 1/\sqrt{2}$ gives the nicest shape for an embedded torus. However, whether or not $\tau(T) = \pi$ remains an open question. The problem for surfaces of genus $p \geq 2$ remains unsolved.

THE BLOG SCIENCE

Math Finds the Best Doughnut

After a 47-year search, mathematicians Fernando C. Marques and André Neves have found the best doughnut, or at least the best geometric shape for a doughnut.

By Frank Morgan, Contributor

Atwell Professor of Mathematics, Emeritus, Williams College; Editor-in-Chief, Notices of the American Mathematical Society

[Marques, Neves, 2014]: *Let $\Sigma \subseteq \mathbb{S}^3$ be an embedded closed surface of genus $g \geq 1$. Then $W(\Sigma) \geq 2\pi^2$ and the equality holds if and only if Σ is the Clifford torus up to conformal transformations of $\mathbb{S}^3$.*

Uniquness with prescribed isoperimetric ratio

- In Canham's model, instead of A_0 and V_0 rather prescribe the *isoperimetric ratio*:

$$\iota_0 := \pi^{1/6} \frac{\sqrt[3]{6V_0}}{\sqrt{A_0}} \in (0, 1].$$

Question

Is the minimizer of $W(S)$ with prescribed genus g and isoperimetric ratio ι_0 unique?

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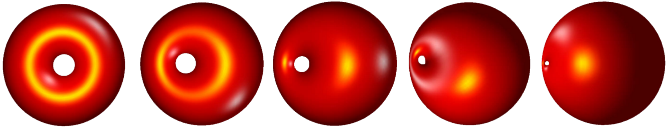
Theorem (Yu, Chen, 22; Melczer, Mezzarobba, 22; Bostan, Y., 22)

The shape of the projection of the Clifford torus to $\mathbb{R}^3$ is uniquely determined by ι_0. Thus, if $g = 1$ and $\iota_0^3 \in [3/(2^{5/4}\sqrt{\pi}), 1)$ then Canham's model has a unique solution.



Summary of [Yu, Chen, 22] and [Melczer, Mezzarobba, 22]

- Let $\iota(S) := \pi^{1/6}\sqrt[3]{6V(S)}/\sqrt{A(S)} \in (0,1]$, and $\tau := 3/(2^{5/4}\sqrt{\pi}) \approx 0.712$. Define

$$\begin{aligned}\mathrm{Iso}\colon [0,\sqrt{2}-1) &\to [\tau,1),\\ t &\mapsto \iota(\mathrm{inv}_{(t,0,0)}(T_{\sqrt{2}}))^3\end{aligned}$$

- $\sqrt{2}\pi^2 A(t^2)$ is the surface area and $\sqrt{2}\pi^2 V(t^2)$ is the volume of $\mathrm{inv}_{(t,0,0)}(T_{\sqrt{2}})$.

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- [Yu, Chen, 22]: Enough to show: $\mathrm{Iso}(t)$ is strictly increasing. Moreover,

$$\frac{V(t^2)A(t^2)}{2\pi^4}\frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{Iso}(t)^2) = 72t + 1932t^3 + 31248t^5 + \cdots =: \sum_{n\geq 0} a_n t^n$$

is a **D-finite** function. Enough to show: $a_n > 0$ for all $n \geq 0$.

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  is a **D-finite** function. Enough to show: $a_n > 0$ for all $n \geq 0$.
- [Melczer, Mezzarobba, 22]: Rigorous asymptotics & error bounds: $a_n > 0$.

  Therefore, $\mathrm{Iso}(t)$ is increasing.

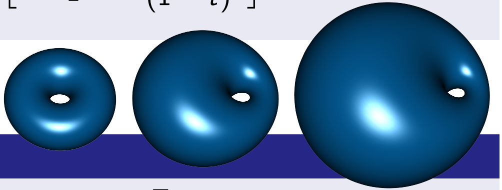
Closed form solution

Proposition (Bostan, Y., 2022)

The surface area $\sqrt{2}\pi^2 A(t^2)$ *and volume* $\sqrt{2}\pi^2 V(t^2)$ *of* $\mathrm{inv}_{(t,0,0)}(T_{\sqrt{2}})$ *are given by*

$$A(t) = \frac{4\,(1-t^2)}{(t^2-6t+1)^2} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; \frac{4t}{(1-t)^2}\right],$$

$$V(t) = \frac{2\,(1-t)^3}{(t^2-6t+1)^3} \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ & 1 \end{matrix} ; \frac{4t}{(1-t)^2}\right].$$



Corollary

The function $\mathrm{Iso}(t)^2 = 36\pi\frac{V(t^2)^2}{A(t^2)^3}$ *is increasing on* $t \in (0, \sqrt{2}-1)$.

Proof of closed-form for $V(t)$

- Let $Q(u, v, r;\ t) = 1 + 2t(\sqrt{2} + r\sin v)\cos u + t^2(2 + r^2 + 2\sqrt{2}r\sin v)t^2$. Then

$$\sqrt{2}\pi^2 V(t^2) = \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r\sqrt{2} + r^2\sin(v)}{Q(u, v, r\,;t)^3} \mathrm{d}u\mathrm{d}v\mathrm{d}r$$
$$= \int_0^1 \oint_{|x|=|y|=1} F(x, y, r;\ t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r = 2 + 48t^2 + \frac{1269}{2}t^4 + \cdots .$$

for some $F(x, y, r;\ t) \in \mathbb{Q}(x, y, r, t, \sqrt{2})$. Thus $V(t)$ is a **period function**.

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- **First try:** Use creative telescoping on the triple integral:

```
> FindCreativeTelescoping[F, {Der[x], Der[y], Der[r]}, Der[t]];
```

  finds $C_1, C_2, C_3 \in \mathbb{Q}(x,y,r,t)$ such that $F = \partial_x C_1 + \partial_y C_2 + \partial_r C_3$.

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- **Second try:** Find a closed form for $\oint_\gamma F\, \mathrm{d}x\mathrm{d}y$ and integrate $\mathrm{d}r$ "by hand".

```
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  finds $L \in \mathbb{Q}[r, t][\partial_t]$ and $C_1, C_2 \in \mathbb{Q}(x, y, r, t)$ s.t. $L \cdot F = \partial_x C_1 + \partial_y C_2$.
- The common denominator of C_1 and C_2 is

$$\mathrm{denom}(F) \cdot x \cdot y \cdot (1 + 2\sqrt{2}y - y^2) \cdot H(t, r).$$

Proof of closed-form for $V(t)$

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- The common denominator of C_1 and C_2 has

$$\mathrm{denom}(F) \cdot x \cdot y \cdot (1 + 2\sqrt{2}y - y^2) \cdot H(t,r) \cap \gamma = \emptyset.$$

$$\sqrt{2}\pi^2 V(t^2) = \int_0^1 \underbrace{\oint_{|x|=|y|=1} F(x,y,r;\ t)\,\mathrm{d}x\mathrm{d}y}_{=:G(r,t)}\,\mathrm{d}r.$$

$G(r,t)$ satisfies $\left(P_2(r,t)\partial_t^2 + P_1(r,t)\partial_t + P_0(r,t)\right)G(r,t) = 0$. Then:

$$G(r,t) = Q_1 \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ & 1 \end{matrix}; \phi_1\right] + Q_2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{3}{2} \\ & 1 \end{matrix}; \phi_2\right],$$

for some (explicit) $Q_1, Q_2, \phi_1, \phi_2 \in \mathbb{Q}(r,t)$. Then we also find:

$$\int_0^s G(r,t)\mathrm{d}r = \frac{2(s-t^2)^3}{(2-s)t^4 - 6t^2 + 1} \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ & 1 \end{matrix}; \frac{4t^2 s}{(1-t^2(2-s))^2}\right].$$

Finally: $\sqrt{2}\pi^2 V(t^2) = \int_0^1 G(r,t)\mathrm{d}r$, so set $s = 1$ above.

Iso is bijective

Proposition

Let

$$A(t) = \frac{4\left(1-t^2\right)}{\left(t^2-6t+1\right)^2} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; \frac{4t}{(1-t)^2}\right],$$

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We need to show that

$$z \mapsto \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2} \\ & 1\end{matrix}; \frac{4z}{(1-z)^2}\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2} \\ & 1\end{matrix}; \frac{4z}{(1-z)^2}\right]^3} \cdot \left(\frac{1-z}{1+z}\right)^3$$

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is increasing on $z \in [0, 3-2\sqrt{2})$. Let $x = 4z/(1-z)^2$, then it remains to show that

$$h\colon x \mapsto \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2} \\ & 1\end{matrix}; x\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2} \\ & 1\end{matrix}; x\right]^3} \cdot (x+1)^{-3/2}$$

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is increasing on $[0, 1)$. Observe: h can be written as $h(x) = \mathbf{g(x)}^2/\mathbf{f(x)}^3$, where

$$g(x) = {}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2} \\ & 1\end{matrix}; x\right] \cdot (x+1)^{-3/2} \quad \text{and} \quad f(x) = {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2} \\ & 1\end{matrix}; x\right] \cdot (x+1)^{-1/2}$$

To show: $\mathbf{g(x)}$ is increasing and $\mathbf{f(x)}$ is decreasing on $(0, 1)$.

Iso is bijective

Proposition

Let $a \geq 0$ and let $w_a : [0,1] \to \mathbb{R}$ be defined by

$$w_a(x) = {}_2F_1\left[\begin{matrix} -a \quad -a \\ 1 \end{matrix}; x\right] \cdot (x+1)^{-a}.$$

Then w_a is: decreasing if $0 < a < 1$; increasing if $a > 1$; constant if $a \in \{0,1\}$.

Clearly, $\mathbf{g(x)} = w_{3/2}(x)$ and $\mathbf{f(x)} = w_{-1/2}(x)$.

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Proof.

$$\frac{w_a'(x) \cdot (x+1)^{a+1}}{a \cdot (a-1) \cdot (1-x)^{2a}} = {}_2F_1\left[\begin{matrix} a+1 & a \\ & 2 \end{matrix}; x\right]. \qquad \square$$

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The general case $R > 1$

Recall:

$$T_R := \left\{ [(R + \cos v)\cos u, (R + \cos v)\sin u, \sin v]^T : u, v \in [0, 2\pi] \right\} \subseteq \mathbb{R}^3, \quad \text{and}$$

$\mathrm{inv}_{(x,y,z)}$ is the inversion about the unit sphere centered at (x, y, z).

Question

Are there closed formulas for the volume and surface area of $\mathrm{inv}_{(x,y,z)}(T_R)$ for any R?
Is $\mathrm{Iso}_R(t)$ increasing in t for any $R > 1$?

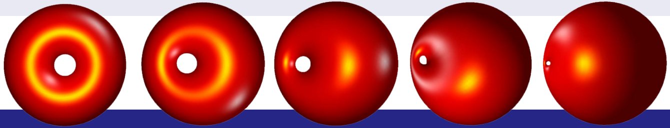
Computing the isoperimetric ratio

Theorem (Bostan, Yu, Y., 2023)

The surface area $A_R(t^2)R\pi^2$ and volume $V_R(t^2)R\pi^2$ of $\mathrm{inv}_{(t,0,0)}(\frac{T_R}{R^2-1})$ are given by

$$A_R(t) = \frac{4\,(1-(R^2-1)t^2)}{(1-2(R^2+1)t+(R^2-1)^2t^2)^2}\cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2} \\ & 1\end{matrix}; \frac{4t}{(1-(R^2-1)t)^2}\right],$$

$$V_R(t) = \frac{2\,(1-(R^2-1)t)^3}{(1-2(R^2+1)t+(R^2-1)^2t^2)^2}\cdot {}_3F_2\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2} & \frac{3}{2R^2-4}+1 \\ & 1 & \frac{3}{2R^2-4}\end{matrix}; \frac{4t}{(1-(R^2-1)t)^2}\right].$$



Corollary

For $R>1$ the function $\mathrm{Iso}_R^2(t^2) = 36\pi\frac{V_R(t^2)^2}{A_R(t^2)^3}$ is increasing on $t \in (0, (R+1)^{-1})$.

Theorem

For $R > 1$ the function $\mathrm{Iso}_R^2(t^2) = 36\pi \frac{V_R(t^2)^2}{A_R(t^2)^3}$ is increasing on $t \in (0, (R+1)^{-1})$, with

$$A_R(t) = \frac{4\,(1-(R^2-1)t^2)}{(1-2(R^2+1)t+(R^2-1)^2t^2)^2} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; \frac{4t}{(1-(R^2-1)t)^2}\right],$$

$$V_R(t) = \frac{2\,(1-(R^2-1)t)^3}{(1-2(R^2+1)t+(R^2-1)^2t^2)^2} \cdot {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2R^2-4}+1 \\ & 1 & \frac{3}{2R^2-4} \end{matrix}; \frac{4t}{(1-(R^2-1)t)^2}\right].$$

First perform the substitution $x = 4t^2/((1-(R^2-1)t^2)^2$. It remains to show that:

$$h(x) := {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]^2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]^{-3} \cdot (1+(R^2-1)\cdot x)^{-3/2}$$

is increasing on $x \in (0, 1)$ for all $R > 1$.

$$h(x) := {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix} ; x\right]^2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; x\right]^{-3} \cdot (1+(R^2-1)\cdot x)^{-3/2}$$

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$$h(x) := {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]^2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]^{-3} \cdot (1+(R^2-1)\cdot x)^{-3/2}$$

is increasing on $x \in (0,1)$ for all $R > 1$. Note that $h(x) = \mathbf{g(x)}^2/\mathbf{f(x)}^3$, where

$$g(x) := \frac{{}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]}{(1+x)^{3/4}\cdot(1+(R^2-1)\cdot x)^{3/4}} \quad \text{and} \quad f(x) := {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]\cdot(x+1)^{-1/2}.$$

$$h(x) := {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]^2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]^{-3} \cdot (1+(R^2-1)\cdot x)^{-3/2}$$

is increasing on $x \in (0,1)$ for all $R > 1$. Note that $h(x) = \mathbf{g(x)}^2/\mathbf{f(x)}^3$, where

$$g(x) := \frac{{}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]}{(1+x)^{3/4}\cdot(1+(R^2-1)\cdot x)^{3/4}} \quad \text{and} \quad f(x) := {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]\cdot(x+1)^{-1/2}.$$

We already saw: $\mathbf{f(x)}$ is decreasing.

$$h(x) := {}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]^2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right]^{-3} \cdot (1+(R^2-1)\cdot x)^{-3/2}$$

is increasing on $x \in (0,1)$ for all $R > 1$. Note that $h(x) = \mathbf{g(x)}^2/\mathbf{f(x)}^3$, where

$$g(x) := \frac{{}_3F_2\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)}+1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix}; x\right]}{(1+x)^{3/4}\cdot(1+(R^2-1)\cdot x)^{3/4}} \quad \text{and} \quad f(x) := {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix}; x\right] \cdot (x+1)^{-1/2}.$$

We already saw: $\mathbf{f(x)}$ is decreasing. For $\mathbf{g(x)}$ it holds that:

$$\frac{4\cdot \mathbf{g'(x)} \cdot (1+x)^{7/4}\cdot(1+(R^2-1)\cdot x)^{7/4}}{3\cdot(1-x)^2\cdot(R^2-1)} =: \sum_{n\geq 0} u_n(R)x^n, \quad \text{and}$$

$$u_{n+1}(R)/u_n(R) = (2n-1)(2n+1)\,p_{n+1}(R)/(4(n+2)(n+1)\,p_n(R)),\ u_0(R) = 1, \text{ where}$$

$$p_n(R) := 4(R^4+4R^2-4)n^3 + 6(R^4+R^2-2)n^2 + (2R^4-13R^2+10)n - 3R^2+3 > 0.$$

Summary and conclusion

- Creative Telescoping is a powerful tool for dealing with **Period functions**.
- Implemented versions of Creative Telescoping exist (both 2G and 4G). They are useful in practice and can solve non-trivial problems.
- The surface area and volume of any stereographic projection to $\mathbb{R}^3$ of the Clifford torus can be expressed in terms of **hypergeometric functions**.
- The Canham model in genus 1 has a unique solution when $\iota_0^3 \in \left(\frac{3}{2^{5/4}}\pi^{-\frac{1}{2}}, 1\right)$.