Integer sequences, algebraic series and differential operators[1]

PhD Defense

Sergey Yurkevich

University Paris-Saclay (Inria Saclay) and University of Vienna

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[1]Supervised by Alin Bostan and Herwig Hauser

Contents of the thesis I

Chapter 1: Introduction and summary of all chapters.

Chapter 2: "On a Class of Hypergeometric Diagonals", with A. Bostan, 2022. In: *Proceedings of the American Mathematical Society*, vol 150, pp. 1071–1897.

Chapter 3: Joint work with A. Bostan and J.-A. Weil, and: "The art of algorithmic guessing in gfun", 2022. In: *Maple Transactions*, vol 2, pp. 14421:1–14421:19.

Chapter 4: "A hypergeometric proof that Iso is bijective", with A. Bostan, 2022. In: *Proceedings of the American Mathematical Society*, vol 150, pp. 2131–2136.

Chapter 5: "Fast Computation of the N-th Term of a q-Holonomic Sequence and Applications", with A. Bostan, 2023. In *J. of Symbolic Comp.*, vol 115, pp. 96–123.

Contents of the thesis II

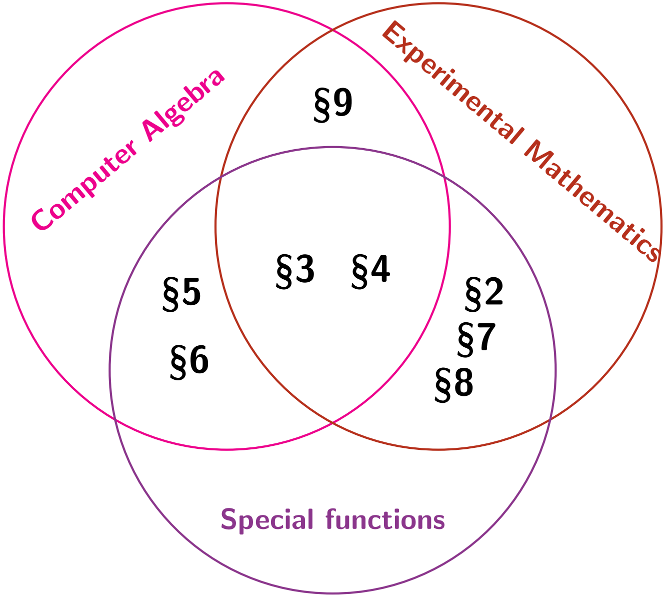
Chapter 6: "Beating binary powering for polynomial matrices", with A. Bostan and V. Neiger, 2023. To appear in the Proceedings of *ISSAC'23*.

Chapter 7: "On the q-analogue of Pólya's Theorem", with A. Bostan, 2023. In: *Electronic Journal of Combinatorics*, vol 30, pp. 2.9:1-9.

Chapter 8: "On the representability of sequences as constant terms", with A. Bostan and A. Straub, 2023. To appear in *Journal of Number Theory*.

Chapter 9: "An algorithmic approach to Rupert's problem", with J. Steininger, 2023, In: *Mathematics of Computation*, vol 92, pp. 1905–1929.

Chapter 10: A collection of 60 open problems and questions related to the thesis.


Computer Algebra
Experimental Mathematics
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Chapter 2: *Hypergeometric diagonals*

$$\mathrm{Diag}((1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N}) = {}_MF_{M-1}([u]\,;[v]\,;(-N)^N t).$$

Starting point

"Guessing – that's the important beginning of solving any problem."

- Starting point is the main identity from [Abdelaziz, Koutschan, Maillard, 2020]:

$$ {}_3F_2\left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9}\right]\ ; \left[1, \frac{2}{3}\right]\ ; 27\,t\right) = \mathrm{Diag}\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right) $$

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- Left-hand side is a generalized *hypergeometric function*:

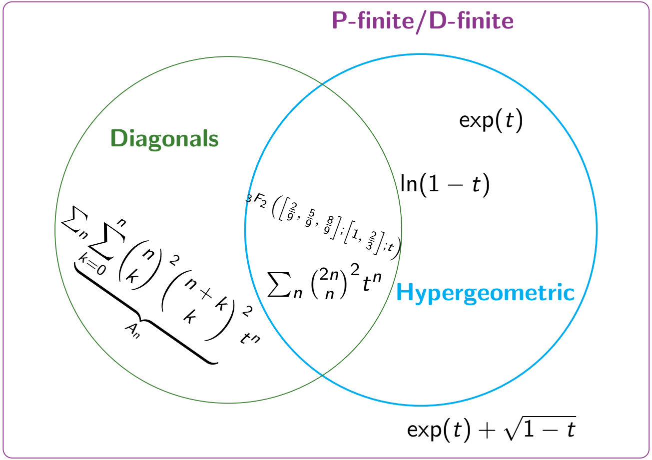
$$ {}_3F_2\left(\left[\frac{2}{9},\frac{5}{9},\frac{8}{9}\right];\left[1,\frac{2}{3}\right];27\,t\right) := \mathbf{1} + \frac{\mathbf{40}}{\mathbf{9}}t + \frac{\mathbf{5236}}{\mathbf{81}}t^2 + \cdots + \mathbf{a_n}t^n + \cdots . $$

$$ \frac{a_{n+1}}{a_n} = \frac{(9n+2)(9n+5)(9n+8)}{3(n+1)^2(9n+6)} $$

- Right-hand side is the diagonal of an *algebraic function*:

$$ \frac{(1-x-y)^{1/3}}{1-x-y-z} = \mathbf{1} + \frac{2}{3}x + \frac{2}{3}y + z + \frac{10}{9}xy + \frac{5}{3}xz + \cdots + \frac{\mathbf{40}}{\mathbf{9}}xyz + \cdots + \frac{\mathbf{5236}}{\mathbf{81}}x^2y^2z^2 + \cdots . $$

Setting



A sequence $(u_n)_{n\geq 0}$ is **P-finite** if it satisfies a linear recurrence with polynomial coefficients:

$$c_r(n)u_{n+r} + \cdots + c_0(n)u_n = 0.$$

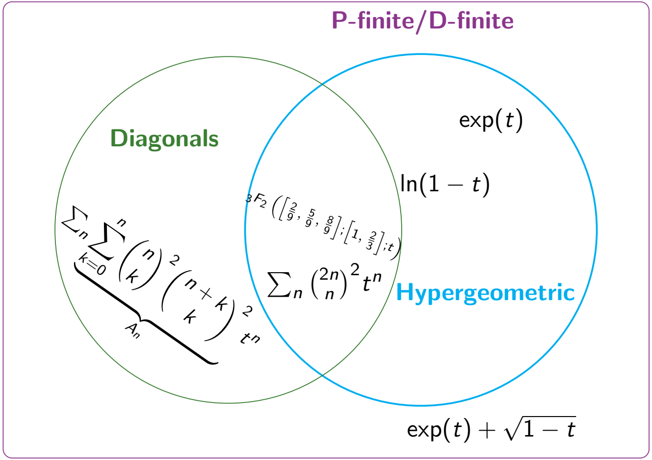
$(u_n)_{n\geq 0}$ is **hypergeometric** if $r = 1$.

Let $(\alpha)_n = \alpha\cdot(\alpha+1)\cdots(\alpha+n-1)$.

Then $u_n = \frac{(a)_n\cdot(b)_n}{(c)_n\cdot n!}$ satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0.$$

Setting



A series $f(t) \in \mathbb{Q}[\![t]\!]$ is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

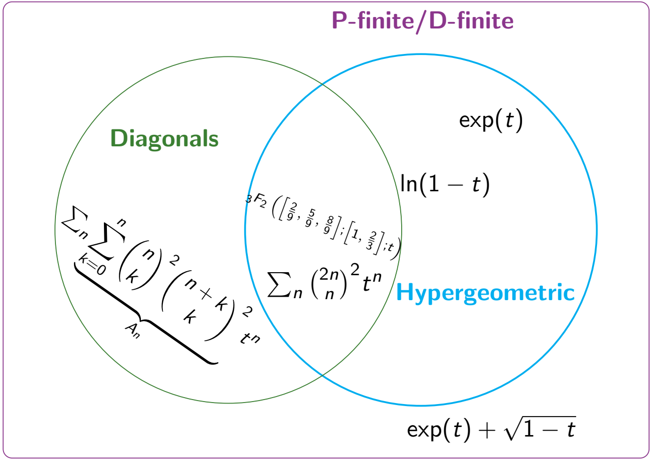
$$p_r(t)f^{(r)}(t) + \cdots + p_0(t)f(t) = 0.$$

Let $(\alpha)_n = \alpha \cdot (\alpha+1)\cdots(\alpha+n-1)$.

Then ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; t\right] := \sum_{n\geq 0} \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!} t^n$ satisfies

$$t(1-t)f''(t) + (c-(a+b+1)t)f'(t) - abf(t) = 0.$$

Setting



For a multivariate power series

$$f(x_1, \ldots, x_n) = \sum_{j_1,\ldots,j_n} f_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$$

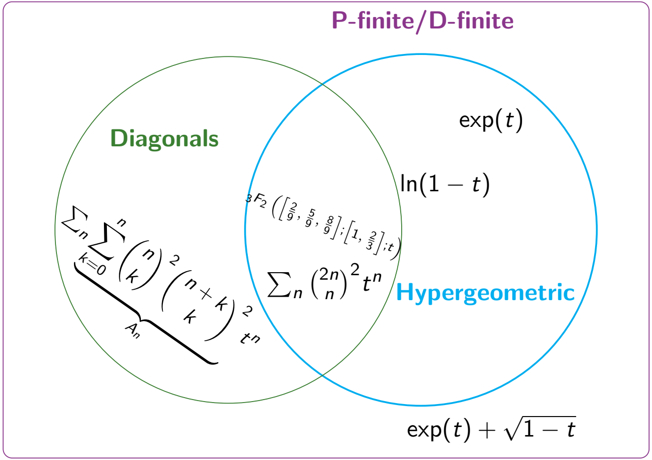
the diagonal is given by

$$\mathrm{Diag}(f) = \sum_j f_{j,j,\ldots,j} t^j \in \mathbb{Q}[\![t]\!].$$

Diagonals are series which can be written as diagonals of **multivariate algebraic** functions.

$$\mathrm{Diag}\left(\frac{1}{1-x-y}\right) = \mathrm{Diag} \sum_{i,j} \binom{i+j}{j} x^i y^j = \sum_n \binom{2n}{n} t^n = (1-4t)^{-1/2}$$

Setting



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Diagonals are series which can be written as diagonals of **multivariate algebraic** functions.

Christol's Conjecture [Christol, 1986]: Any convergent **D-finite** power series with *integer coefficients* is a **diagonal**. Specifically: ${}_3F_2\left(\left[\frac{1}{9},\frac{4}{9},\frac{5}{9}\right];\left[1,\frac{1}{3}\right],t\right)\in$ **Diagonals**.

Main result **A**: Hypergeometric diagonals

"First guess, then prove.
All great discoveries were made in this style."

Theorem (Bostan, Y., 2022)

The **diagonal** *of any finite product of algebraic functions of the form*

$$(1 - x_1 - \cdots - x_n)^R, \qquad R \in \mathbb{Q},$$

is a generalized **hypergeometric** *function with explicitly determined parameters.*

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- This vastly generalizes the main identity in [Abdelaziz, Koutschan, Maillard, 2020].
- We also settle down other memberships: E.g. ${}_3F_2\left([\frac{1}{4}, \frac{3}{8}, \frac{7}{8}]; [1, \frac{1}{3}], t\right) \in$ **Diagonals**.

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- **Main observation** for the proof:

$$[x_1^{k_1} \cdots x_N^{k_N}](1 + x_1)^{b_1}(1 + x_1 + x_2)^{b_2} \cdots (1 + x_1 + \cdots + x_N)^{b_N}$$
$$= \binom{b_N}{k_N}\binom{b_{N-1} + b_N - k_N}{k_{N-1}} \cdots \binom{b_1 + \cdots + b_N - k_N \cdots - k_2}{k_1}.$$

Chapter 3: *Dubrovin-Yang-Zagier numbers and algebraicity of D-finite functions*

$$(a_n)_{n\geq 0} = (1\,,\ -48300\,,\ 7981725900\,,\ -1469166887370000, \ldots)$$
$$(b_n)_{n\geq 0} = (1\,,\ -144900\,,\ 88464128725\,,\ -6227007345699000, \ldots)$$

Origin of a_n and b_n

"So this is a very mysterious example"

- In Arithmetic and Topology of Differential Equations, 2018 by Don Zagier:

$$u_{n-3} + 20\left(4500n^2 - 18900n + 19739\right)u_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)u_n +$$
$$+25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right)u_{n-1} = 0,$$

with initial terms $u_0 = 1$, $u_1 = -161/(2^{10} \cdot 3^5)$ and $u_2 = 26605753/(2^{23} \cdot 3^{12} \cdot 5^2)$.

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Problem (Zagier, 2018)

Find $(\alpha,\beta) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $u_n \cdot (\alpha)_n \cdot (\beta)_n \cdot \gamma^n \in \mathbb{Z}$ for some $\gamma \in \mathbb{Z}^*$.

$(x)_n := x\cdot(x+1)\cdots(x+n-1)$.

- [Yang and Zagier]: $a_n = u_n \cdot (3/5)_n \cdot (4/5)_n \cdot (2^{10}\cdot 3^5 \cdot 5^4)^n \in \mathbb{Z}$,
- [Dubrovin and Yang]: $b_n = u_n \cdot (2/5)_n \cdot (9/10)_n \cdot (2^{12}\cdot 3^5\cdot 5^4)^n \in \mathbb{Z}$.

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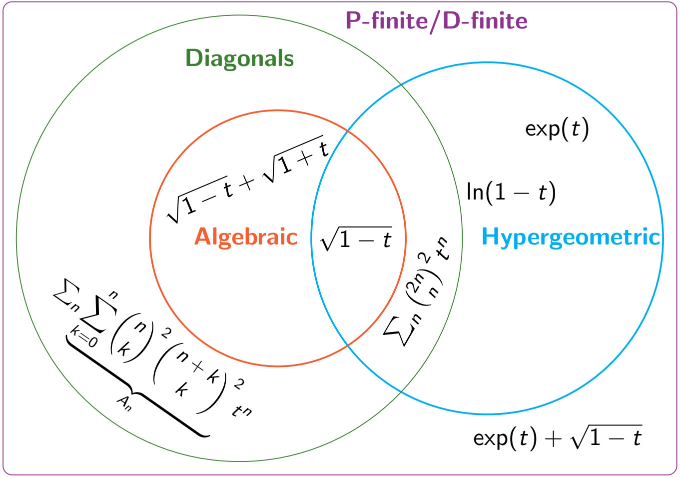
- [Dubrovin and Yang]: $b_n = u_n \cdot (2/5)_n \cdot (9/10)_n \cdot (2^{12}\cdot 3^5\cdot 5^4)^n \in \mathbb{Z}$.
- "Yang and I found a formula showing that the numbers a_n are integers [...]" "Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function [...] is actually **algebraic**!" [Zagier, 2018]

Definitions and interactions



[Abel, 1827]: **Algebraic** $\subseteq$ **D-finite**.

[Furstenberg, 1967]: **Algebraic** $\subseteq$ **Diagonals**.

[Singer 1979, 2014]: **D-finite** $f(t) \overset{?}{\in}$ **Algebraic**.

[Christol, 1984 and Lipshitz, 1988]: **Diagonals** $\subseteq$ **D-finite**.

[Petkovsek 1992]: **D-finite** $f(t) \overset{?}{\in}$ **Hypergeometric**.

[Beukers, Heckman, 1989]: **Algebraic** $\cap$ **Hypergeometric**.

[Bostan, Lairez, Salvy, 2017]: **Diagonals** $=$ **Multiple binomial sums**.

André-Christol Conjecture [André, 2004]:
D-finite $f(t) \in \mathbb{Z}[\![t]\!]$ convergent & minimal ODE ordinary in 0 $\Rightarrow$ $f(t)$ **Algebraic**

Main result **B**: Solving the mystery of a_n and b_n

"So this is a very mysterious example."

- "Yang and I found a formula showing that the numbers a_n are integers [...]" "Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function [...] is actually **algebraic**!"
- "My presumed arithmetic intuition [...] was entirely broken" – [Wadim Zudilin]

Problem

Investigate the nature of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and similar sequences.

Main result B: Solving the mystery of a_n and b_n

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Problem

Investigate the nature of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and similar sequences.

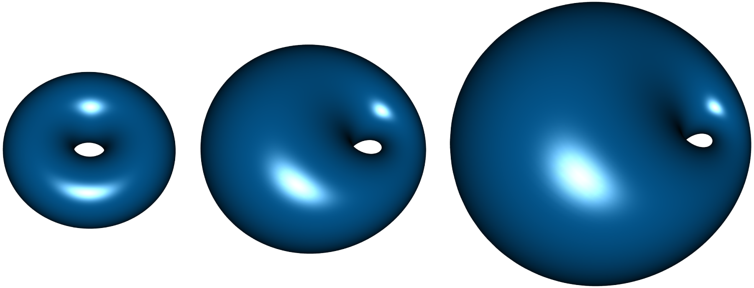
Theorem (Bostan, Weil, Y.)

The generating functions of both $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are **algebraic**.

Theorem (Bostan, Weil, Y.)

Seven more solutions to Zagier's problem: $(c_n)_{n\geq 0}, \ldots, (i_n)_{n\geq 0} \in \mathbb{Z}$.

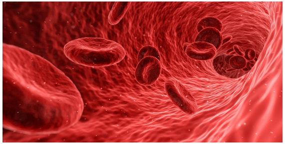
Chapter 4:
On the reduced volume of conformal transformations of tori



Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- *Canham model* predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the *Willmore energy*



$$W(S) := \int_S H^2 \mathrm{d}A, \quad (H \text{ is mean the curvature})$$

over orientable closed surfaces $S \subseteq \mathbb{R}^3$ with genus g, area A_0 and volume V_0.

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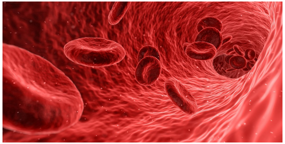

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over orientable closed surfaces $S \subseteq \mathbb{R}^3$ with genus g, area A_0 and volume V_0.

- [Willmore, 1965]: For a torus $T = T(R, r)$ the Willmore energy is:

$$W(T) = \frac{\pi^2 R^2}{r\sqrt{R^2 - r^2}} \rightsquigarrow \text{minimal for } R/r = \sqrt{2}.$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces in $\mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T_{\sqrt{2}}$.

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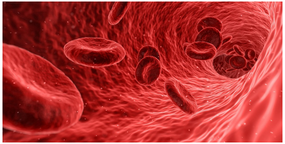

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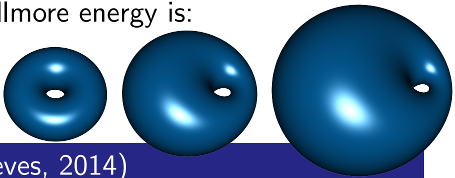

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces in $\mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T_{\sqrt{2}}$.

- $W(S)$ is invariant under Möbius transformations $\Rightarrow$ no uniqueness of the shape.

Main result **C**: Iso is bijective

"Nature is not generic."

- In Canham's model, instead of A_0 and V_0 rather prescribe the *isoperimetric ratio*:

$$\iota_0 := \pi^{1/6} \frac{\sqrt[3]{6V_0}}{\sqrt{A_0}} \in (0, 1].$$

Question

Is the minimizer of $W(S)$ with prescribed genus g and isoperimetric ratio ι_0 unique?

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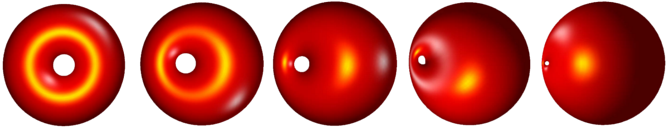
Theorem (Yu, Chen, 21; Melczer, Mezzarobba, 21; Bostan, Y., 22)

The shape of the projection of the Clifford torus to $\mathbb{R}^3$ is uniquely determined by ι_0. Thus, if $g = 1$ and $\iota_0^3 \in [3/(2^{5/4}\sqrt{\pi}), 1]$ then Canham's model has a unique solution.



Main result **C'**: Iso is bijective

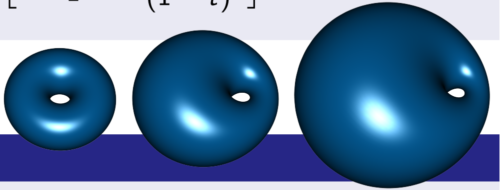
"I could never resist a definite integral."

Proposition (Bostan, Y., 2022)

The surface area $\sqrt{2}\pi^2 A(t^2)$ *and volume* $\sqrt{2}\pi^2 V(t^2)$ *of* $i_{(t,0,0)}(T_{\sqrt{2}})$ *are given by*

$$A(t) = \frac{4\left(1-t^2\right)}{\left(t^2-6t+1\right)^2} \cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};\frac{4t}{(1-t)^2}\right],$$

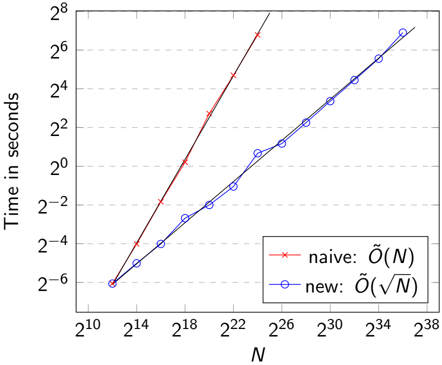
$$V(t) = \frac{2\left(1-t\right)^3}{\left(t^2-6t+1\right)^3} \cdot {}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2}\\ & 1\end{matrix};\frac{4t}{(1-t)^2}\right].$$



Theorem (Bostan, Y., 2022)

The function $\mathrm{Iso}(t)^2 = 36\pi\frac{V(t^2)^2}{A(t^2)^3}$ *is increasing on* $t \in (0, \sqrt{2}-1)$.

Chapter 5: *Computing terms in q-holonomic sequences*



Main result **D**: Sublinear algorithm for q-holonomic sequences

- A sequence $(u_n)_{n\geq 0} \in \mathbb{K}$ is **holonomic/P-finite** if it satisfies

$$c_r(n)u_{n+r} + \cdots + c_0(n)u_n = 0 \quad n \geq 0, \quad c_0(x), \ldots, c_r(x) \in \mathbb{K}[x].$$

Theorem (Strassen, 1977; Chudnovsky^2, 1988)

Given $N \in \mathbb{N}$, one can compute u_N in $\tilde{O}(\sqrt{N})$ arithmetic operations. *Naive: $O(N)$*

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Theorem (Strassen, 1977; Chudnovsky[2], 1988)

Given $N \in \mathbb{N}$, one can compute u_N in $\tilde{O}(\sqrt{N})$ arithmetic operations. *Naive: $O(N)$*

- A sequence $(u_n(q))_{n\geq 0} \in \mathbb{K}$ is called *q-holonomic* if for some $q \in \mathbb{K}$ it satisfies

$$c_r(q, q^n)u_{n+r} + \cdots + c_0(q, q^n)u_n = 0 \quad n \geq 0, \qquad c_0(x, y), \ldots, c_r(x, y) \in \mathbb{K}[x, y].$$

Theorem (Bostan, Y., 2023)

Given $N \in \mathbb{N}$, one can compute $u_N(q)$ in $\tilde{O}(\sqrt{N})$ arithmetic operations. *Naive: $O(N)$*

Idea: For $M(x) \in \mathbb{K}[x]^{r\times r}$ compute $M(q^{N-1}) \cdots M(q)M(1)$ using baby-steps/giant-steps.

Application: Evaluation of polynomials *"Do not waste a factor of two!"*

- **Task:** Given a polynomial $P(x) \in \mathbb{K}[x]$ and $q \in \mathbb{K}$, deduce $P(q) \in \mathbb{K}$ fast.

Application: Evaluation of polynomials

"Do not waste a factor of two!"

- **Task:** Given a polynomial $P(x) \in \mathbb{K}[x]$ and $q \in \mathbb{K}$, deduce $P(q) \in \mathbb{K}$ fast.
- Generically, Horner's rule needs $O(\deg P)$ operations.
- Our results imply that one can do better for large families of polynomials.

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- Our results imply that one can do better for large families of polynomials.
- For example, the truncated Jacobi theta function

$$\vartheta_N(x) := 1 + x + x^4 + x^9 + \cdots + x^{N^2}$$

evaluated at $q \in \mathbb{K}$ in $\tilde{O}(\sqrt{N})$ operations [Nogneng, Schost, 2018], [Bostan, Y., 2023].

Application: Evaluation of polynomials

"Do not waste a factor of two!"

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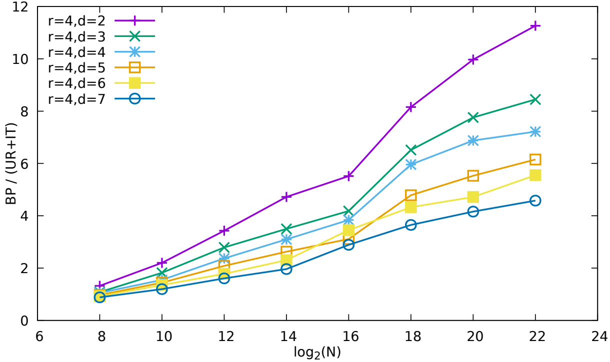
- **Method:** $\vartheta_N(q) = u_N$, where $u_n = \sum_{k=0}^{n} q^{k^2}$ is q-holonomic.
- [Bostan, Y., 2023]: Same complexity via unified algorithm for $\prod_{i=0}^{N}(x - a^i)$, or q-Hermite polynomials, or $\prod_{i=1}^{\infty}(1 - x^i)^3 \bmod x^N$, etc.

Chapter 6: *Computing terms in polynomial C-finite sequences*



Polynomial C-finite sequences: Example

- Fibonacci polynomials: $F_0(x) = 0, F_1(x) = 1$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$

  $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$ and $F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9$.

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- [Folkore]: Use binary powering to compute M_N, where $M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$:

$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) \cdot M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases} \qquad O(N \log(N))$$

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- **Idea:** Write $F_N(x) = f_0 + f_1 x + \cdots + f_N x^N$. Then $(f_k)_{k\geq 0}$ is **P-finite**:

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0,$$

$O(N)$

with $(f_0, f_1) = (1, 0)$ for odd N and $(f_0, f_1) = (0, N/2)$ for even N.

Main result E: Beating binary powering

"The development of fast algorithms is slow!"

A **polynomial C-finite sequence** $(u_n(x))_{n\geq 0} \in \mathbb{K}[x]^{\mathbb{N}}$ satisfies a recurrence

$$u_{n+r}(x) = c_{r-1}(x)u_{n+r-1}(x) + \cdots + c_0(x)u_n(x),$$

for some polynomials $c_0(x), \ldots, c_{r-1}(x) \in \mathbb{K}[x]$.

Theorem (Bostan, Neiger, Y., 2023)

Given a **polynomial C-finite sequence** $(u_n(x))_{n\geq 0}$*, one can compute* $u_N(x)$ *in* $O(N)$ *operations in* K.

Corollary

Given a polynomial matrix $M(x)$*, one can compute* $M(x)^N$ *in* $O(N)$ *field operations.*

Chapter 6:
On the q-analogue of Pólya's Theorem

Specifically, if n, k, a, b satisfy the conditions stated earlier, is the function

$$F(x, q) = \sum_{t=0}^{\infty} \begin{bmatrix} n + at \\ k + bt \end{bmatrix}_q x^t$$

algebraic? That is, does there exist a nonzero polynomial $P(x, y, z)$ whose coefficients are constants (say, complex numbers) such that $P(x, q, F(x, q)) = 0$, for all x and q?

[Aissen, 1979]

Main result **F**: A q-analogue of Pólya's theorem

"In mathematics often the simplest is the best."

- Consequence of Pólya's theorem [Pólya, 1922]:

Theorem (Pólya, 1922)

For admissible n, k, a, b, the function $F(x) := \sum_{j\geq 0} \binom{n+aj}{k+bj} x^j$ is **algebraic** *over $\mathbb{Q}(x)$.*

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- Aissen asked whether a q-analogue holds [Aissen, 1979]. We prove:

Theorem (Bostan, Y., 2022)

For admissible n, k, a, b, the function

$$F(x, q) := \sum_{j\geq 0} \begin{bmatrix} n+aj \\ k+bj \end{bmatrix}_q x^j \in \mathbb{C}[q][\![x]\!]$$

is **never algebraic** *over $\mathbb{Q}(q, x)$. If $q \in \mathbb{C}$, then $F(x, q)$ is algebraic iff q is root of unity.*

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- $u_n(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$, where $[n]_q! := (1+q)\cdots(1+q+\cdots+q^{n-1})$.
- **Idea:** It holds that $(u_n(q))_{n\geq 0}$ is ***q*-holonomic**.

Chapter 8:
Representation of sequences as constant terms

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \mathrm{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right].$$

Main result **G**: Describing **Constant terms** $\cap$ **C-finite sequences**

- A sequence $A(n)$ is a **constant term** if it can be represented as

$$A(n) = \mathrm{ct}[P(\boldsymbol{x})^n Q(\boldsymbol{x})],$$

where $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ are Laurent polynomials in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

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Specifically: Are the Fibonacci numbers a constant term sequence?

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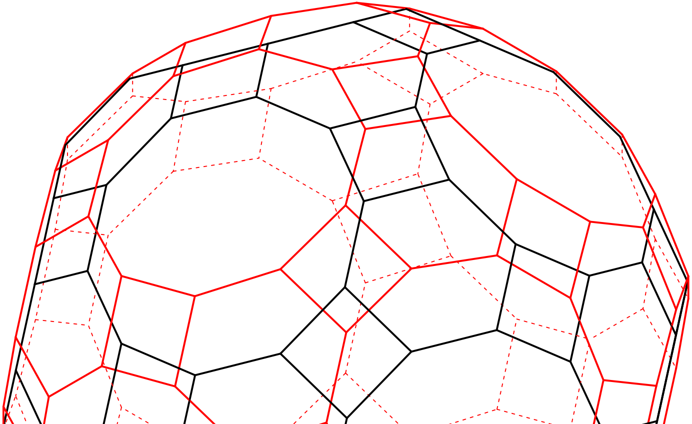
Specifically: Are the Fibonacci numbers a constant term sequence?

Theorem (Bostan, Straub, Y., 2023)

Let $A(n)$ be a **C-finite sequence**. *$A(n)$ is a* **constant term** *if and only if it has a single characteristic root λ and $\lambda \in \mathbb{Q}$.*

Chapter 9: *On Rupert's problem*

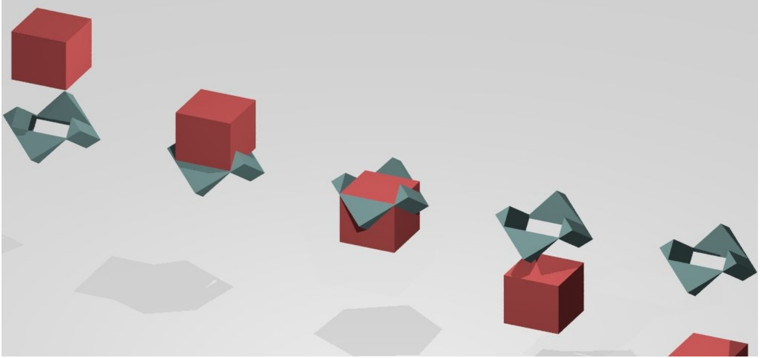


Summary and main result H: Deciding Rupertness

"It shows us 'what's out there'."

Definition

A convex polyhedron $\mathbf{P} \subseteq \mathbb{R}^3$ *is called* **Rupert** *if a hole with the shape of a straight tunnel can be cut into it such that a copy of* $\mathbf{P}$ *can be moved through this hole.*



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All **Platonic** *solids are* **Rupert**.

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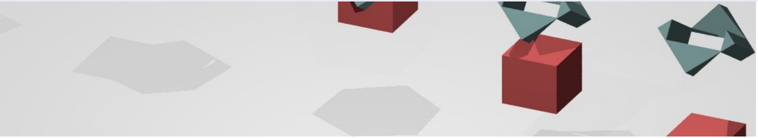
At least 9 **Archimedean** *solids are* **Rupert**.



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- [Steininger, Y., 22]: Practical algorithm and proof of algorithmic decidability.

Summary and conclusion

- **A** **Diagonals** of products of $(1 - x_1 - \cdots - x_n)^R$ are **hypergeometric** functions.
- **B** The generating functions of the Dubrovin-Yang-Zagier numbers are **algebraic**.
- **C** $\mathrm{Iso}_R(t)$ is a quotient of **hypergeometric** functions and increasing. Thus the shape of a projection of the Clifford torus is uniquely determined by its isoperimetric ratio.
- **D** We can compute the N-th term of a ***q*-holonomic sequence** faster than previously.
- **E** We can compute the N-th term of a **polynomial C-finite sequence** faster.
- **F** The q-analogue of Pólya's theorem holds if and only if q is a root of unity.
- **G** A **C-finite sequence** is a **constant term** iff it has 1 characteristic root λ and $\lambda \in \mathbb{Q}$.
- **H** Rupertness is decidable and the truncated icosidodecahedron is Rupert.

Perspectives and open questions

"Curiouser and curiouser!"

A? Describe **Diagonals** among **D-finite** functions.

B? Given a **D-finite** function, how to prove or disprove that it is **algebraic** in practice?

C? Given a **D-finite** function/**P-finite** sequence, how to prove that it is increasing?

D? Compute N-th terms in some **P-finite** sequences faster than in $\tilde{O}(\sqrt{N})$ operations.

E? Compute the N-th term of an integer **C-finite sequence** in $O(N)$ bit complexity.

F? Does there exist a suitable notion of "q-algebraicity"?

G? Describe **Constant terms** among **Diagonals** or **P-finite** sequences.

H? Prove or disprove that the Rhombicosidodecahedron is Rupert.

And many, many more...

Bonus: Definition of ${}_pF_q$ and algebraicity

The generalized **hypergeometric function** with parameters $a_1, \dots, a_p$ and $b_1, \dots, b_q$ is:

$$ {}_pF_q([a_1, \dots, a_p]\,;[b_1, \dots, b_q]\,;t) := \sum_{j \geq 0} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{t^j}{j!}, $$

where $(x)_n := x \cdot (x+1) \cdots (x+n-1)$ is the rising facorial.

- [Fürnsinn, Y., 2023] Can also handle the case: $a_j, b_k \notin \mathbb{Q}$ and $a_j - b_k \in \mathbb{Z}$.

Bonus: Definition of ${}_pF_q$ and algebraicity

Theorem (Christol, 1986 and Beukers, Heckman, 1989)

Assume that the rational parameters $\{a_1, \dots, a_p\}$ and $\{b_1, \dots, b_{p-1}, b_p = 1\}$ are disjoint modulo $\mathbb{Z}$. Let N be their common denominator. Then

$${}_pF_{p-1}([a_1, \dots, a_p], [b_1, \dots, b_{p-1}]; t) \qquad \textit{is}$$

- **algebraic** *if and only if for all $1 \leq r < N$ with $\gcd(r, N) = 1$ the numbers $\{\exp(2\pi i r a_j), 1 \leq j \leq p\}$ and $\{\exp(2\pi i r b_j), 1 \leq j \leq p\}$ interlace on the unit circle.*

- **globally bounded** *if and only if for all $1 \leq r < N$ with $\gcd(r, N) = 1$, one encounters more numbers in $\{\exp(2\pi i r a_j), 1 \leq j \leq p\}$ than in $\{\exp(2\pi i r b_j), 1 \leq j \leq p\}$ when running through the unit circle from 1 to $\exp(2\pi i)$.*

- [Fürnsinn, Y., 2023] Can also handle the case: $a_j, b_k \notin \mathbb{Q}$ and $a_j - b_k \in \mathbb{Z}$.

Bonus: DYZ-like numbers

Zagier's problem

Find $(\alpha, \beta) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $u_n \cdot (\alpha)_n \cdot (\beta)_n \cdot \gamma^n \in \mathbb{Z}$ for some $\gamma \in \mathbb{Z}^*$.

$$(x)_n := x \cdot (x+1) \cdots (x+n-1).$$

#	u	v	ODE order	degree	#	u	v	ODE order	degree
a_n	3/5	4/5	2	120	f_n	19/60	49/60	4	155520
b_n	2/5	9/10	4	120	g_n	19/60	59/60	4	46080
c_n	1/5	4/5	2	120	h_n	29/60	49/60	4	46080
d_n	7/30	9/10	4	155520	i_n	29/60	59/60	4	155520
e_n	9/10	17/30	4	155520					

Theorem (Bostan, Weil, Y., 2023)

The sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}, (c_n)_{n\geq 0}, \ldots, (i_n)_{n\geq 0}$ are solutions to Zagier's problem.

- Estimates for degrees based on numerical monodromy group computations.
- Proof of **algebraicity**: Done: a_n, b_n, c_n. In progress: $d_n, e_n, f_n, g_n, h_n, i_n$.