Beating binary powering for computing the Nth power[1]

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[1] Joint work with Alin Bostan and Vincent Neiger.

Motivating example: three sequences, three problems

- Fibonacci polynomials:

$$F_0(x) = 0, F_1(x) = 1 \text{ and } F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \text{ for } n \geq 0$$

- Euclidean division for bivariate polynomials:

$$R_n(x, y) = y^n \bmod y^2 - xy - 1$$

- Powers of a polynomial matrix:

$$M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$$

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$$F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8 \text{ and } F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9.$$

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$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) \cdot M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases}$$

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- Write $F_N(x) = f_0 + f_1 x + \cdots + f_N x^N$. Then $(f_k)_{k\geq 0}$ satisfy:

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0,$$

with $(f_0, f_1) = (1, 0)$ for odd N and $(f_0, f_1) = (0, N/2)$ for even N.

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Example: $F_9(x)$

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- For $N = 9$ we have: $f_0 = 1$, $f_1 = 0$ and:

$$f_{k+2} = \frac{(10+k)(8-k)}{4(k+1)(k+2)} f_k.$$

- $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$.

Polynomial C-finite sequences

- A **polynomial C-finite sequence** $(u_n(x))_{n\geq 0} \in \mathbb{K}[x]^{\mathbb{N}}$ satisfies a recurrence

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Theorem (Bostan, Neiger, Y., 2023)

Let $d, r \in \mathbb{N}$. There exists an algorithm solving in $O(N)$ operations $(\pm, \times, \div)$ in $\mathbb{K}$:

- SeqTerm: *Given a* **polynomial C-finite** *sequence $(u_n(x))_{n\geq 0}$ of order and degree at most r and d, compute the Nth term $u_N(x)$.*
- BivModPow: *Given polynomials $Q(x,y)$ and $P(x,y)$ in $\mathbb{K}[x,y]$ of degrees in y and x at most r and d, with $P(x,y)$ monic in y, compute $Q(x,y)^N \bmod P(x,y)$.*
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$$\begin{tikzcd}
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\boxed{\textsc{BivModPow}} \arrow[rr, Rightarrow, "\text{Cayley-Hamilton Thm.}"'] & & \boxed{\textsc{PolMatPow}} \arrow[ul, Rightarrow, "\text{Companion matrix}"']
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PROBLEM 4

What is the coefficient of x^{3000} in the expansion of the polynomial

$$(x+1)^{2000}(x^2+x+1)^{1000}(x^4+x^3+x^2+x+1)^{500}$$

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- The full coefficient of x^{3000} could be computed by [Flajolet, Salvy, 1997] in 15sec!

SeqTerm in $O(N)$

Lemma

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Sketch of proof.

The vector space spanned over $\mathbb{K}(x)$ by $(f^{(i)}(x))_{i\geq 0}$ is finite-dimensional over $\mathbb{K}(x, a(x))$ which is itself finite-dimensional over $\mathbb{K}(x)$. $\square$

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Set $g(x) = x^n$ which satisfies $xg'(x) = ng(x)$. $\square$

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- Recall: If $(u_n(x))_{n\geq 0}$ is **polynomial C-finite** then:

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- Write $u_N(x) = c_0 + c_1x + c_2x^2 + \cdots$. Then: $(c_k)_{k\geq 0}$ satisfies "**small**" **recursion**.

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- Write $u_N(x) = c_0 + c_1 x + c_2 x^2 + \cdots$. Then: $(c_k)_{k\geq 0}$ satisfies "**small**" **recursion**.
- Compute initial terms and unroll $\Rightarrow$ all c_i in $O(N)$ arithmetic operations
$\Rightarrow u_N(x)$ in $O(N)$ arithmetic complexity.

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- Creative Telescoping finds:

$$\Big(\underbrace{p_k(n,x)\partial_x^k + \cdots + p_0(n,x)}_{\text{“Telescoper”}} \Big) \frac{U(x,y)}{y^{n+1}} = \partial_y \Big(\underbrace{C(n,x,y)}_{\text{“Certificate”}} \Big).$$

- By Cauchy's integral theorem: $\big((p_k(n,x)\partial_x^k + \cdots + p_0(n,x)\big) u_n = 0.$

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- By Cauchy's integral theorem: $\big((p_k(n,x)\partial_x^k + \cdots + p_0(n,x)\big)\, u_n = 0.$
- Can prove for reduction based Creative Telescoping:

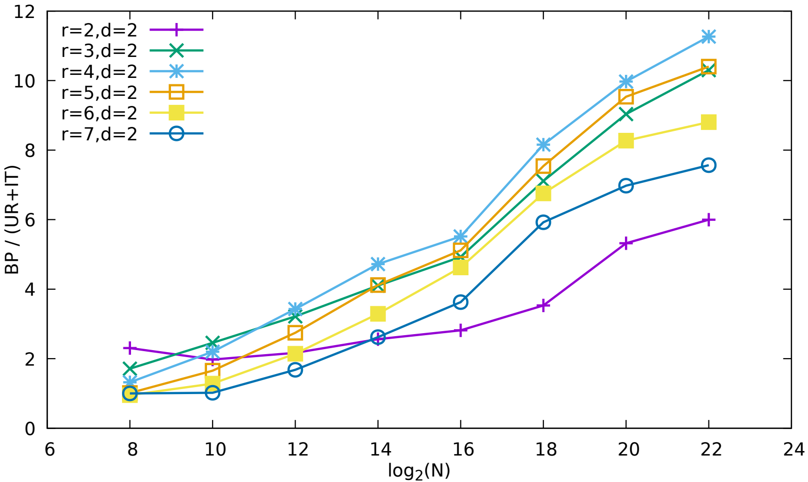
Order and degree of the Telescoper are independent of n.

Algorithm by example: Fibonacci polynomials

- $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ with $F_0(x) = 0, F_1(x) = 1$.
- Generating function: $$\sum_{k\geq 0} F_k y^k = \frac{1}{1-xy-y^2}.$$
- Hence: $$F_n = \frac{1}{2\pi i}\oint_{|y|=\epsilon} \frac{1}{(1-xy-y^2)y^{n+1}}\mathrm{d}y.$$

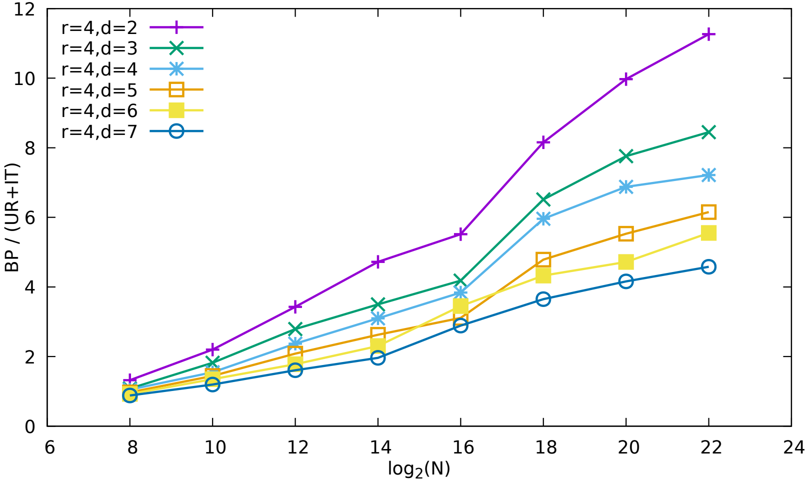
Precomputation

- `DEtools[Zeilberger](1/(1-x*y-y^2)/y^n, x, y, Dx);` $O(1)$
$$(x^2+4)F_n''(x)^2 + 3xF_n'(x) + (1-n^2)F_n(x) = 0.$$
- `gfun[diffeqtorec](deq, F(x), u(k));` $O(1)$
$$4(k+1)(k+2)f_{k+2} - (n+k+1)(n-k-1)f_k = 0.$$
- Compute f_0, f_1 by binary powering mod x^2. $O(\log(N))$
- Unroll. $O(N)$



- $M(x) \in \mathbb{K}[x]^{r\times r}$.
- Want: $M(x)^N$.
- $\deg M(x) = 2$.
- $r = 2, \dots, 7$.
- $N = 2^8, 2^{10}, \dots, 2^{22}$.

BP: Time for binary powering.
UR+IT: Time for unrolling + computing initial terms.



- $M(x) \in \mathbb{K}[x]^{4\times 4}$.
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Summary and future work

- SeqTerm, BivModPow and PolMatPow can be solved in complexity $O(N)$.

- $M(x)^N$ can be computed faster than with binary powering, in practice and theory.

- Many future works:
  - More detailed complexity (w.r.t. r, d).
  - The Kth coefficient of the Nth term.
  - More general sequences.
  - Connection to the Jordan–Chevalley decomposition.

Bonus: What if unrolling is impossible (singularities)?

- Consider $u_n = 2^n + x^n + x^{2n}$.
- **C-finite** recursion:

$$u_{n+3}(x) - (x^2 + x + 2)u_{n+2}(x) + x(x^2 + 2x + 2)u_{n+1}(x) - 2x^3 u_n(x) = 0.$$

- We find the **small ODE**: $x^2 u_n'''(x) - 3x(n-1)u_n''(x) + (2n-1)(n-1)u_n'(x) = 0$,
- For $u_n(x) = \sum_{k \geq 0} c_{n,k} x^k$ obtain the recursion: $(2n-k)(n-k)kc_{n,k} = 0$.
- **Problem:** Cannot unroll (for $k = 0$ and $k = N$ and $k = 2N$)!
- **Solution:** Define $v_n(x) = u_n(x+1)$. Then for $v_n(x) = \sum_{k \geq 0} d_{n,k} x^k$:

$$(k+1)(k+2)d_{n,k+2} - (k+1)(3N - 2k - 1)d_{n,k+1} + (2n-k)(n-k)d_{n,k} = 0.$$

  Compute $v_n(x)$, then compute u_N and u_{2N} via $c_{N,i} = \sum_{k \geq 0} d_{N,k} \binom{k}{i} (-1)^{k-i}$.
- This strategy works in general because the **ODE** has finitely many singularities.

Bonus: Some precomputation timings

		Maple				Sage	Mathematica					
r	d	redct	HT	ZB	c_t	ct	FCT	CT	HCT	ℓ	d_n	d_x
2	2	0.0	0.1	0.0	0.1	0.5	0.2	0.2	0.2	2	2	16
	4	0.0	0.0	0.0	0.1	0.6	0.4	0.4	0.3	2	2	34
	6	0.0	0.0	0.0	0.1	0.6	0.7	0.5	0.5	2	2	52
	8	0.0	0.0	0.0	0.1	0.8	1.0	0.7	0.7	2	2	70
3	1	0.0	0.2	0.0	0.5	2.0	2.0	1.3	1.3	3	5	24
	2	0.0	0.1	0.8	3.4	3.1	4.0	2.6	2.5	3	5	54
	3	0.1	0.2	0.8	9.3	5.6	10	5.7	5.4	3	5	84
	4	0.1	0.5	18	19	8.2	17	9.4	8.9	3	5	114
	5	0.2	1.1	5.1	32	12	25	14	14	3	5	144
	6	0.5	1.7	9.8	49	17	35	19	20	3	5	174
4	1	0.4	2.9	23	117	20	31	25	25	4	9	58
	2	1.7	17	410	749	45	101	96	95	4	9	128
	3	4.4	43			89	295	376	373	4	9	198
	4	12	82			172	388	752	693	4	9	268
	5	18	128			280	635			4	9	338
5	1	11	34	538		163	847	780		5	14	115
	2	64	183			515				5	14	250
	3	159	526							5	14	385
	4	345								5	14	520

- Want $M(x)^N$, with $M(x) \in \mathbb{K}[x]^{r\times r}$, degree d.
- Seconds for Telescoper of
$$\frac{P(x,y)}{y^{n+1}Q(x,y)},$$
$Q(x,y)$ is the char. poly.
- redct: [Bostan, Chyzak, Lairez, Salvy,'18]. HermiteTelescoping (HT): [Bostan, Lairez, Salvy,'13]. Zeilberger (ZB): [DETools]. c_t: [Chyzak, '00]. ct: [Kauers,Mezzarobba,'19]. CT: [Koutschan, '10].