APÉRY EXTENSIONS

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Abstract. We initiate a program to exhibit the Apéry classes of Fano varieties as mirror to limiting extension classes of higher cycles on Landau-Ginzburg models (and thus, in particular, as periods). Using a new technical result on the inhomogeneous Picard-Fuchs equations satisfied by higher normal functions, we illustrate this principle for several threefolds.

1. Introduction

Despite a smattering of examples in recent years [MW09, DK11, JW14, DK14], the role of algebraic cycles and their invariants in mirror symmetry remains something of a mystery. In this paper, we give evidence for a new link, itself not yet well-understood, in the context of Fano/LG-model duality.

One of the features of *local* mirror symmetry uncovered in [DK14, BKV17] was the entrance of *mixed* Hodge structures, whose extension classes are described on the B-model side by regulators on algebraic K-theory. These same regulator classes, called *higher normal functions* when they occur in families, are at the heart of the second author's interpretation [Ke17] of Apéry's irrationality proofs for $\zeta(2)$ and $\zeta(3)$. It was in an effort to "recombine" this with the first author's enumerative, A-model interpretation [Go09] of Apéry's recurrence (see also [Ga16]), that the animating slogan of this paper suggested itself:

Arithmetic Mirror Symmetry Conjecture: *For each Fano n-fold F° admitting a toric degeneration, its Apéry numbers arise as limits of (classical and higher) normal functions produced by cycles on a 1-parameter family of CY $(n-1)$-folds defined over $\bar{\mathbb{Q}}$, together with extension classes in the monodromy-invariant part of a limiting MHS of the family.*

(We beg the readers' indulgence in deferring the refinement and explication of this – deliberately and necessarily vague – statement to §5.2.) While computations by G. da Silva [dS19] appeared to support our conjecture for the *rational* Fano 3-folds in [Go09], there initially seemed to be little hope for the Apéry numbers $\frac{1}{10}\zeta(2), \frac{1}{7}\zeta(2)$ of the non-rational Fanos V_{10}, V_{14}, owing to the "deresonation off the motivic setting" seemingly required for their computation. Moreover, the model of [Ke17], in its limitation to K_n^{alg} of CY $(n-1)$-folds, could only produce rational multiples of $(2\pi\mathbf{i})^3$ or $\zeta(3)$ if $n = 3$. However, a new paradigm began to emerge around two years ago, allowing a much greater variety of cycles to enter. The main result of this article is thus the following

Theorem 1.1. *The Arithmetic Mirror Symmetry Conjecture (more precisely, Conjecture 5.3) holds for the five Mukai Fano threefolds.*[1]

[1]by definition, the rank-one Fano 3-folds arising as complete intersections in the Grassmanians of simple Lie groups other than projective spaces [Go09]; they are V_{10}, V_{12}, V_{14}, V_{16}, and V_{18}.

The Theorem is proved in §§5.3-5.5 (modulo a detail deferred to §6), using a new result on inhomogeneous Picard-Fuchs equations satisfied by higher normal functions (Theorem 5.1). In §6 we propose a theory of "Apéry extensions" on the B-model side, which encompasses these examples, and highlight some implications of the Conjecture. In §§2-4 we place our story in context, recalling the mixed Hodge theory of GKZ systems and local mirror symmetry, quantum $\mathcal{D}$-modules and Apéry constants of Fanos, and higher normal functions on Landau-Ginzburg models. In the rest of this Introduction, we would like to convey the idea of what an Apéry extension is and why it is important.

Let $\mathcal{X} \overset{\pi}{\to} \mathbb{P}^1$ be a family of compact CY $(n-1)$-folds with smooth total space and fibers[2] $X_t = \pi^{-1}(t)$, smooth off $\Sigma = \{0, t_1, \ldots, t_c, \infty\}$. Laurent polynomials $\phi(\underline{x}) \in \bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with reflexive Newton polytope Δ are a key source for such families, with $\mathcal{X}$ obtained by blowing up $\mathbb{P}_\Delta$ along $\overline{\{\phi = 0\}} \cap (\mathbb{P}_\Delta \backslash \mathbb{G}_m^n)$, and π extending $1/\phi$. In particular, the *mirror LG-model of a Fano* F° *degenerating to* $\mathbb{P}_{\Delta^\circ}$ (like those in [Go09, Ga16]) arises in this way.

The cohomologies of the fibration $\mathcal{X}_\mathcal{U} \overset{\pi_\mathcal{U}}{\to} \mathcal{U} := \mathbb{P}^1 \backslash \Sigma$ produce VHSs $\mathcal{H}^\ell = \mathcal{H}^\ell_f \oplus \mathcal{H}^\ell_v$ with "fixed" and "variable" parts. At each $\sigma \in \Sigma$, we have the LMHS functor ψ_σ and monodromies T_σ. All our families will have maximal unipotent monodromy at the "north pole" $\sigma = 0$;[3] for simplicity, here we also assume[4] $\mathrm{rk}(T_\sigma - I) = 1$ if $\sigma \neq 0, \infty$, and $\ker\{H^n(X_\sigma) \to \psi_\sigma \mathcal{H}^n\} = \{0\}$ $(\forall \sigma)$. Then we may write $\mathsf{A}^\dagger_\phi := H^n(\mathcal{X} \backslash X_0, X_\infty)$ as an extension

$$0 \to (\psi_\infty \mathcal{H}^{n-1}_v)^{T_\infty} \to \mathsf{A}^\dagger_\phi \to \mathrm{IH}^1(\mathbb{P}^1 \backslash \{0\}, \mathcal{H}^{n-1}_v) \to 0. \tag{1.1}$$

of MHS. Now the Apéry numbers of F° record limits of ratios of solutions to its quantum difference equation (Definition 3.5); and a first approximation to the Conjecture is that we should be able to find them in the extension class of (1.1).

Unfortunately, extension classes of MHS do not produce well-defined numbers. For instance, we have $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-a), \mathbb{Q}(0)) \cong \mathbb{C}/\mathbb{Q}(a)$, which (say) would make $\frac{1}{10}\zeta(2)$ trivial in $\mathbb{C}/\mathbb{Q}(2)$. This is where writing them *as limits of admissible normal functions* enters: if (1.1) arises as $\lim_{t \to 0} \nu(t)$ for some $\nu \in \mathrm{ANF}(\mathcal{H}^{n-1}_v(r))$, and $k := \mathrm{rk}((\psi_0 \mathcal{H}^{n-1}_v)^{T_0})$, then ν has a *unique* lift $\tilde{\nu}$ on the disk $|t| < |t_{k+1}|$ to a single-valued section of $\mathcal{H}^{n-1}_v$. Pairing this with a suitable section $\omega \in \Gamma(\mathbb{P}^1, \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{v,e})$ yields a *truncated higher normal function* (THNF) $V(t) = \langle \tilde{\nu}, \omega \rangle$ whose first k Taylor coefficients in t are well-defined complex numbers refining the information in $\mathsf{A}^\dagger_\phi$. So *higher normal functions get us from extension data to constants.* According to the Conjecture, these should recover the desired Apéry numbers for the right choice of HNF. Here are two candidates.

Consider the VMHS $\mathcal{A}^\sigma_\phi := H^n(\mathcal{X} \backslash X_\sigma, X_t)$ over U ($\sigma = 0$ or ∞). As an extension it reads

$$0 \to \mathcal{H}^{n-1}_v \to \mathcal{A}^\sigma_\phi \to \mathrm{IH}^1(\mathbb{P}^1 \backslash \{\sigma\}, \mathcal{H}^{n-1}_v) \to 0, \tag{1.2}$$

in which the IH term is a *constant* VMHS. Taking first $\sigma = 0$, $\mathsf{A}^\dagger_\phi = (\psi_\infty \mathcal{A}^0_\phi)^{T_\infty}$ is recovered as the "south pole" limit of $\mathcal{A}^0_\phi$. If $\mathcal{H}^{n-1}_v$ is extremal (cf. §6) with Hodge numbers all 1, then

[2] written $\tilde{X}_t$ in the body of the paper

[3] "North" refers to the infinity point of the Landau-Ginzburg potential; since we work primarily in a neighborhood of this point, however, it is $t = 0$ for us.

[4] in the Introduction, but not in the body of the paper

$\mathrm{IH}^1(\mathbb{P}^1\backslash\{0\}, \mathcal{H}_v^{n-1}) \cong \mathbb{Q}(-n)$, and (1.2) gives a normal function in $\mathrm{ANF}(\mathcal{H}_v^{n-1}(n))$. This can only come from a "K_n" cycle (in $\mathrm{CH}^n(\mathcal{X} \setminus X_0, n)$), recovering the paradigm of [Ke17].

The alternate ($\sigma = \infty$) perspective is to view $\mathsf{A}_\phi^\dagger = [(\psi_0 \mathcal{A}_\phi^\infty)^{T_0}]^\vee(-n)$ as a "north pole" limit. This can make a huge difference, since $\mathcal{A}_\phi^\infty$ and $\mathcal{A}_\phi^0$ *are not dual* in general (although the invariant parts of their limits are). Indeed, for any morphism $\mathbb{Q}(-a) \overset{\mu}{\hookrightarrow} \mathrm{IH}^1(\mathbb{P}^1\backslash\{\infty\}, \mathcal{H}_v^{n-1})$, the μ^*-pullback

$$0 \to \mathcal{H}_v^{n-1} \to \mu^* \mathcal{A}_\phi^\infty \to \mathbb{Q}(-a) \to 0 \tag{1.3}$$

of (1.2) belongs to $\mathrm{ANF}(\mathcal{H}_v^{n-1}(a))$, and (by Beilinson-Hodge again) should arise from a "K_{2a-n}" cycle (in $\mathrm{CH}^a(\mathcal{X}\backslash X_\infty, 2a-n)$). It is the extensions of VMHS (1.3) that we call *Apéry extensions* (Definition 6.4). In the families of $K3$s mirror to V_{10} and V_{14}, we have $a = 2$, and the corresponding normal functions do indeed arise from torically natural K_1-cycles whose THNFs have the north pole limits $\frac{1}{10}\zeta(2), \frac{1}{7}\zeta(2)$. This change in perspective came as a revelation since, for these and similar cases, the south-pole approach is not computationally viable (Remark 6.6).

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2. Generic Laurent polynomials

2.1. **GKZ system.** Fix a vector $\underline{a} \in \mathbb{C}^{N+1}$ and a convex polytope $\mathbb{\Delta} \subset \mathbb{R}^{n+1}$ containing the origin, with vertices in $\mathbb{Z}^{N+1}$. The corresponding toric variety $\mathbb{P}_{\mathbb{\Delta}}$ compactifies $\mathbb{G}_m^{N+1}$ (with coordinate $\underline{x}$). Let $\mathsf{M} \subseteq \mathbb{Z}^{N+1}$ denote the monoid generated by $\mathfrak{M} := \mathbb{\Delta} \cap (\mathbb{Z}^{N+1} \setminus \{\underline{0}\})$ and $\mathbb{L} \subset \mathbb{Z}^{|\mathfrak{M}|}$ the lattice of relations; we assume for simplicity that $\mathsf{M}^{\mathrm{gp}} = \mathbb{Z}^{N+1}$ and $\mathsf{M} = \mathbb{Z}^{N+1} \cap \mathrm{Cone}_{\underline{0}}(\mathbb{\Delta})$. The coefficients $\underline{\lambda}$ of the generic Laurent polynomial $f(\underline{x}) = \sum_{\underline{m}\in\mathfrak{M}} \lambda_{\underline{m}} \underline{x}^{\underline{m}}$ parametrize the affine parameter space on which we define the *GKZ system* of partial differential operators:

$$\begin{cases} Z_i = \sum_{\underline{m}\in\mathfrak{M}} m_i \delta_{\lambda_{\underline{m}}} + a_i & (i = 0, \ldots, N) \\ \Box_{\underline{\ell}} = \prod_{\ell_{\underline{m}}>0} \partial_{\lambda_{\underline{m}}}^{\ell_{\underline{m}}} - \prod_{\ell_{\underline{m}}<0} \partial_{\lambda_{\underline{m}}}^{-\ell_{\underline{m}}} & (\underline{\ell} \in \mathbb{L} \subset \mathbb{Z}^{|\mathfrak{M}|}) \end{cases} \tag{2.1}$$

Proposition 2.1. *For each relative cycle $\mathscr{C}$ on $(\mathbb{P}_{\mathbb{\Delta}} \setminus \{f = 0\}, \mathbb{D}_{\mathbb{\Delta}} \setminus \{f = 0\})$, the function*

$$\mathscr{P}_{\mathscr{C}}(\underline{\lambda}) = \int_{\mathscr{C}} \underline{x}^{\underline{a}} e^{f(\underline{x})} \mathrm{dlog}(\underline{x}) \tag{2.2}$$

is a (local) solution of (2.1).

Check: Applying Z_i to $\mathscr{P}_{\mathscr{C}}$ gives

$$\int_{\mathscr{C}} \underline{x}^{\underline{a}} (\delta_{x_i} f + a_i) e^f \mathrm{dlog}(\underline{x}) = \int_{\mathscr{C}} d[\underline{x}^{\underline{a}} e^f \mathrm{dlog}(\underline{x_{\hat{i}}})] = 0,$$

while applying $\Box_{\underline{\ell}}$ yields

$$\int_{\mathscr{C}} \underline{x}^{\underline{a}} (\underline{x}^{\sum_{\underline{m}:\, \ell_{\underline{m}}>0} \ell_{\underline{m}} \underline{m}} - \underline{x}^{\sum_{\underline{m}:\, \ell_{\underline{m}}<0} (-\ell_{\underline{m}}) \underline{m}}) e^f \mathrm{dlog}(\underline{x}) = 0.$$

The solutions are (analytically) local because the cycles $\mathscr{C}$, and hence their period integrals $\mathscr{P}$, have monodromy about divisors in $\mathbb{A}^{|\mathfrak{M}|}$. $\square$

Since the corresponding $\mathcal{D} = \mathbb{C}[\underline{\lambda}, \partial_{\underline{\lambda}}]$-module

$$\tau^{\underline{a},\mathbb{A}}_{\mathrm{GKZ}} := \mathcal{D}/\mathcal{D}\langle \{Z_i\}, \{\square_{\underline{\ell}}\}\rangle \tag{2.3}$$

is holonomic [Ad94, Thm 3.9], the (local) solutions module

$$\mathrm{Hom}_{\mathcal{D}}(\tau, \hat{\mathcal{O}}_{\underline{\lambda}^0}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{\underline{\lambda}^0} \otimes_{\mathbb{C}[\underline{\lambda}]} \tau, \mathbb{C}) \tag{2.4}$$

at a point $\underline{\lambda}^0 \in \mathbb{C}^{|\mathfrak{M}|}$ is finite-dimensional. We shall think of (2.3) and (2.4) as "cohomology" and "homology" respectively, motivated by the parametrization of solutions by relative cycles; this will be made more precise in §2.3.

2.2. **Periods and residues.** Given $\underline{m} \in \mathbb{Z}^{N+1}$, write $\deg(\underline{m}) =: \kappa$ for the minimal $\kappa \in \mathbb{Z}_{\geq 0}$ such that $\kappa\mathbb{A} \ni \underline{m}$. The ring $R = \mathbb{C}[\underline{\lambda}][\underline{x}^{\mathfrak{M}}]$, its Jacobian ideal $J_f = (\{\partial_{x_i} f\}_{i=0}^N)$, and the Jacobian ring R/J_f are thereby graded by degree. Moreover, sending $p(\underline{x}) \mapsto p(\underline{x})\underline{x}^{\underline{a}} e^{f(\underline{x})} \mathrm{dlog}(\underline{x})$ induces a grading on $\tau^{\mathbb{A}}_{\mathrm{GKZ}}$ and a graded isomorphism

$$\mathrm{gr}(R/J_f) \xrightarrow[\mathrm{gr}(\mathcal{D})]{\cong} \mathrm{gr}(\tau^{\mathbb{A}}_{\mathrm{GKZ}}). \tag{2.5}$$

Specializing $\underline{\lambda}$ to a very general point $\underline{\lambda}^0$ (and hence R to $R^0 = \mathbb{C}[\underline{x}^{\mathfrak{M}}]$), the graded pieces have dimensions

$$\dim_{\mathbb{C}}(R^0/J_f)_{(k)} = \sum_{j=0}^{N+1} (-1)^j \binom{N+1}{j} \dim(R^0_{(k-j)}), \tag{2.6}$$

(where $\dim(R^0_{(k-j)})$ counts the points of degree $k-j$ in M) with sum over k

$$\dim_{\mathbb{C}}(R^0/J_f) = (N+1)!\,\mathrm{vol}(\mathbb{A}). \tag{2.7}$$

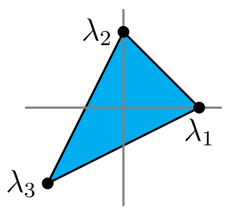
See [Ad94, (5.3) and Cor. 5.11].

Irregular case in mirror symmetry: A polytope $\Delta \subset \mathbb{R}^n$ with integer vertices is *reflexive* iff its polar polytope Δ° also has integer vertices; this implies that both have $\underline{0}$ as unique interior integer point. Fixing such a Δ, take $\mathbb{A} := \Delta$ and $\underline{a} = \underline{0}$. (Note that $N = n-1$.) Then we have a graded isomorphism of A- and B-model $\mathcal{D}$-modules

$$QH^*(\mathbb{P}_{\Delta^\circ}) \underset{\mathrm{gr}}{\cong} \tau^{\Delta}_{\mathrm{GKZ}} \tag{2.8}$$

with the grading by $\frac{\deg}{2}$ on the left-hand side (see [Ir19]).

Example 2.2. Let Δ be the triangle in the figure. Choose $\underline{\lambda}$ so that the cycle $\mathbb{T}^2 \cong S^1 \times S^1$ given by $|x_1| = |x_2| = 1$ avoids the zero-locus of $f = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1^{-1} x_2^{-1}$, so that $\mathscr{C} = \mathbb{T}_2$ is a relative cycle. By (2.6)-(2.7), the rank of τ is 3, with three graded pieces each of rank 1. Computing the period in (2.2) now gives



$$\tfrac{1}{(2\pi\mathbf{i})^2}\mathscr{P} = \tfrac{1}{(2\pi\mathbf{i})^2}\int_{\mathbb{T}^2} e^f \mathrm{dlog}(\underline{x}) = \tfrac{1}{(2\pi\mathbf{i})^2}\sum_{n\geq 0}\tfrac{1}{n!}\int_{\mathbb{T}^2} f^n \mathrm{dlog}(\underline{x}) = \sum_{m\geq 0}\tfrac{(\lambda_1\lambda_2\lambda_3)^m}{(m!)^3},$$

which is an irregular/exponential period. In particular, we see that $\tau^{\Delta}_{\mathrm{GKZ}}$ does not underlie a classical VHS or VMHS.

Regular case in mirror symmetry: With Δ as above, take $\mathbb{\Delta} \subset \mathbb{R}^{1+n}$ to be the convex hull of the origin and $\{1\} \times \Delta$ (so that now $N = n$), and put $\underline{a} := (1, \underline{0})$. We denote the resulting GKZ system by $\hat{\tau}^{\Delta}_{\mathrm{GKZ}}$. It has the same rank as $\tau^{\Delta}_{\mathrm{GKZ}}$ since $\mathrm{vol}(\mathbb{\Delta}) = \frac{1}{(n+1)!}\mathrm{vol}(\Delta)$. Rather than being isomorphic, the two are related (roughly) by Fourier-Laplace transform; and (as will be explained in §2.3) we have an isomorphism of $\mathcal{D}$-modules

$$QH_c^{*(+2)}(K_{\mathbb{P}_{\Delta^\circ}}) \underset{\mathrm{gr}}{\cong} \hat{\tau}^{\Delta}_{\mathrm{GKZ}}, \tag{2.9}$$

where $K_{\mathbb{P}_{\Delta^\circ}}$ is the total space of the canonical line bundle on $\mathbb{P}_{\Delta^\circ}$.

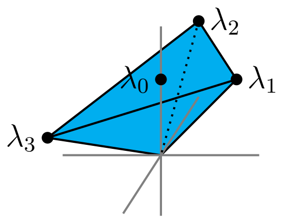
Now let $\phi_{\underline{\lambda}}(\underline{x}) = \sum_{m \in \Delta \cap \mathbb{Z}^n} \lambda_{\underline{m}} \underline{x}^{\underline{m}}$ be a general Laurent polynomial on Δ and Γ a relative n-cycle in $(\mathbb{P}_\Delta \setminus \{\phi = 0\}, \mathbb{D}_\Delta \setminus \{\phi = 0\})$. With $f = x_0 \phi(\underline{x})$ and $\mathscr{C} = \mathbb{R}_- \times \Gamma$, the periods in (2.2) take the form

(2.10)

$$\mathscr{P} = \int_{\mathscr{C}} x_0 e^f \tfrac{dx_0}{x_0} \wedge \mathrm{dlog}(\underline{x}) = \int_\Gamma \left(\int_{-\infty}^0 e^{x_0 \phi} dx_0 \right) \mathrm{dlog}(\underline{x}) = \int_\Gamma \tfrac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})} = 2\pi \mathbf{i} \int_\gamma \mathrm{Res}_{\phi=0} \left(\tfrac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})} \right)$$

if $\Gamma = \mathrm{Tube}(\gamma)$ for $\gamma \subset \{\phi = 0\}$. In particular, these are (regular) periods of a variation of mixed Hodge structure.

Example 2.3. With Δ as in Ex. 2.2, $\mathbb{\Delta}$ is the tetrahedron in the figure. Taking $\Gamma = \mathbb{T}^2$ in (2.10) and writing $t = \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0^3}$, Cauchy residue gives for $|t| < \frac{1}{27}$



$$\tfrac{1}{(2\pi \mathbf{i})^2} \mathscr{P} = \int_{\mathbb{T}^2} \frac{\mathrm{dlog}(\underline{x})/(2\pi \mathbf{i})^2}{\lambda_0 + (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1^{-1} x_2^{-1})} = \frac{1}{\lambda_0} \sum_{m \geq 0} \frac{(3m)!}{(m!)^3} t^m.$$

Since $\mathrm{rk}(\hat{\tau}^{\Delta}_{\mathrm{GKZ}}) = 3$, one expects 3 distinct periods related to the geometry of the family of elliptic curves $E_t = \overline{\{\phi_{\underline{\lambda}}(\underline{x}) = 0\}}$, which has a type I_9 singular fiber at $t = 0$.

Fix $\lambda_0 = 1$. If $\{\alpha, \beta\}$ is a symplectic basis for $H_1(E_t)$, with α vanishing at $t = 0$, we can take Γ to be $\mathrm{Tube}(\alpha) \simeq \mathbb{T}^2$ ($\mathscr{P}$ holomorphic in t), $\mathrm{Tube}(\beta)$ ($\mathscr{P} \sim \frac{9}{2\pi \mathbf{i}} \log(t)$), or $\sigma := \mathbb{R}_- \times \mathbb{R}_-$ ($\mathscr{P} \sim \frac{9}{2(2\pi \mathbf{i})^2} \log^2(t)$). Note that only the first two are "periods of E_t".

2.3. **Mixed Hodge theory of GKZ.** Continuing with the "regular case" above, and recalling that $\mathbb{D}_\Delta := \mathbb{P}_\Delta \setminus \mathbb{G}_m^n$, we set $X_{\underline{\lambda}} := \overline{\{\phi_{\underline{\lambda}}(\underline{x}) = 0\}}$ and $\partial X_{\underline{\lambda}} := X_{\underline{\lambda}} \cap \mathbb{D}_\Delta$. By Prop. 2.1, we know that at least some solutions of $\hat{\tau}^{\Delta}_{\mathrm{GKZ}}$ are parametrized by the choice of $\Gamma \in H_n(\mathbb{P}_\Delta \setminus X_{\underline{\lambda}}, \mathbb{D}_\Delta \setminus \partial X_{\underline{\lambda}})$, which (as a best-case scenario) suggests the following

Theorem 2.4 ([HLYZ16])**.** *We have a canonical isomorphism*

$$\hat{\tau}^{\Delta}_{GKZ} \cong H^n(\mathbb{P}_\Delta \setminus X_{\underline{\lambda}}, \mathbb{D}_\Delta \setminus \partial X_{\underline{\lambda}}), \tag{2.11}$$

in which the $\mathcal{D}$-module structure on the RHS is defined by the Gauss-Manin connection.

The connection to mirror symmetry is amplified by

Theorem 2.5 (Conjectured by [KKP17], proved by [Ha17] ($n = 3$) and [Sa18])**.**

$$\dim \mathrm{Gr}_F^{n-k} H^n(\mathbb{P}_\Delta \setminus X_{\underline{\lambda}}, \mathbb{D}_\Delta \setminus \partial X_{\underline{\lambda}}) = \dim H^{k,k}(\mathbb{P}_{\Delta^\circ}) = \dim H_c^{k+1,k+1}(K_{\mathbb{P}_{\Delta^\circ}}).$$

This refines (2.11) into a graded isomorphism strongly reminiscent of Griffiths's residue theory [Gr69], with gr_k (resp. multiplication by $x_0\underline{x}^{\underline{m}}$ as a map from $\mathrm{gr}_k \to \mathrm{gr}_{k+1}$) on the left matching Gr_F^{n-k} (resp. $\nabla_{\partial_{\lambda_{\underline{m}}}}\colon \mathrm{Gr}_F^{n-k} \to \mathrm{Gr}_F^{n-k-1}$) on the right.

However, the RHS of (2.11) is a (variation of) *mixed* Hodge structure, with a nontrivial *weight* filtration. While intersection theory on the A-model $K_{\mathbb{P}_{\Delta^\circ}}$ allows us to compute a basis of solutions to GKZ via mirror symmetry (Thm. 2.6 below), it is unclear how to see the weight filtration directly in these terms. To elaborate, we pose two questions:

(1) How might one isolate the highest weight part Gr^W_{n+1} of (2.11) (i.e., $H^{n-1}(X_{\underline{\lambda}})$) within the setting of GKZ solutions?

Under mirror symmetry we have the correspondences:
- $\underline{m} \in \mathfrak{M} \longleftrightarrow$ divisors $[D_{\underline{m}}] \in H^2(\mathbb{P}_{\Delta^\circ})$;
- relations $\ell \in \mathbb{L} \longleftrightarrow$ curves $[C_{\underline{\ell}}] \in H_2(\mathbb{P}_{\Delta^\circ})$; and
- *Mori cone* $\mathbb{L}_{\geq 0} \subset \mathbb{L} \longleftrightarrow$ effective curve classes.

We assume $\mathbb{L}_{\geq 0}$ is simplicial with basis $\{\underline{\ell}^{(i)}\}$, and put $t_i := \underline{\lambda}^{\underline{\ell}^{(i)}}$ (e.g. $t = \frac{\lambda_1\lambda_2\lambda_3}{\lambda_0^3}$ above), $\tau_i := \frac{\log(t_i)}{2\pi \mathbf{i}}$. The isomorphism class of $X_{\underline{\lambda}} (=: X_{\underline{t}})$ depends only on $\underline{t}$.

Theorem 2.6 ([HLY96]). *The ($\mathbb{C}$-linear combinations of) periods $\mathscr{P}$ of $\hat{\tau}^{\Delta}_{GKZ}$ are the ($\mathbb{C}$-linear combinations of) coefficients of cohomology classes in*

$$\mathscr{B}_\Delta := \sum_{\ell \in \mathbb{L}_{\geq 0}} \frac{\prod_{\underline{m}\colon \ell_{\underline{m}}<0} D_{\underline{m}}(D_{\underline{m}}-1)\cdots(D_{\underline{m}}+\ell_{\underline{m}}+1)}{\prod_{\underline{m}\colon \ell_{\underline{m}}>0}(D_{\underline{m}}+1)\cdots(D_{\underline{m}}+\ell_{\underline{m}})}(D_{\underline{0}}-1)\cdots(D_{\underline{0}}+\ell_{\underline{0}})\underline{\lambda}^{\underline{\ell}+\underline{D}} \in H^*(\mathbb{P}_{\Delta^\circ}) \otimes \mathbb{C}[[\underline{t}]][\underline{\tau}].$$

Conjecture 2.7 (Hyperplane Conjecture [HLY96, LZ16]). *The periods of (∇-flat sections of) $H^{n-1}(X_{\underline{t}})$ are the ($\mathbb{C}$-linear combinations of) coefficients of cohomology classes in $\mathscr{B}_\Delta \cup [X^\circ]$, where $X^\circ \subset \mathbb{P}_{\Delta^\circ}$ is an anticanonical hypersurface.*

Example 2.8. With Δ as in Examples 2.2-2.3, we have $\mathbb{P}_{\Delta^\circ} = \mathbb{P}^2$, $[X^\circ] = 3[H]$ (for H a hyperplane in $\mathbb{P}^2$), and

$$\mathscr{B}_\Delta = [1](\text{holo. period}) + [H](\log\ \text{period}) + [H]^2(\log^2\ \text{period}).$$

In this case

$$\mathscr{B}_\Delta \cup [X^\circ] = [H](\text{holo. period}) + [H]^2(\text{log period}),$$

and so the hyperplane conjecture correctly asserts that the holomorphic and log periods are the actual periods of $H^1(E_t)$.

(2) Can we compute the remaining GKZ periods, especially those which yield extension classes of Gr^W_{n+1} by other weight-graded pieces?

Here "compute" means *using the A-model.* We know at present of no (even conjectural) *intrinsic* A-model description of the full weight filtration. An *extrinsic* one, which we shall now sketch, was obtained in [BKV17] by presenting $K_{\mathbb{P}_{\Delta^\circ}}$ as the large-fiber-volume limit of compact elliptically-fibered Calabi-Yau $(n+1)$-folds. (Though [op. cit.] treats the case $n = 2$, this works in general.) To obtain these families of higher-dimensional CYs, let $\diamond \subset \mathbb{R}^2$ be the convex hull of $\{(-1, 1), (-1, -1), (2, -1)\}$, and $\hat{\Delta} \subset \mathbb{R}^{n+2}$ be the convex hull of

$\Delta \times (-1,-1)$ and $\underline{0} \times \diamond$. There are torically-induced morphisms $\mathbb{P}_{\hat{\Delta}} \to \mathbb{P}_{\Delta}$ and $\mathbb{P}_{\hat{\Delta}^\circ} \to \mathbb{P}_{\Delta^\circ}$ which restrict to elliptic fibrations on anticanonical (CY-)hypersurfaces $\hat{X}, \hat{X}^\circ$.

In particular, write $\hat{X}_{\underline{t},s}$ for the closure of the zero-locus of $\Phi(\underline{x},u,v) := \mathsf{a} + \mathsf{b}u^2v^{-1} + \mathsf{c}u^{-1}v^{-1} + \phi_{\underline{\lambda}}(\underline{x})u^{-1}v^{-1}$, where $s := \frac{\lambda_0 \mathsf{b}^2\mathsf{c}^3}{\mathsf{a}^6}$. Instead of the large complex-structure limit ($\underline{t} \to \underline{0}$ and $s \to 0$), we take only $s \to 0$. This has the effect of degenerating the generic fiber of $\hat{X}_{\underline{t},s} \to \mathbb{P}_{\Delta}$ and decompactifying that of $\hat{X}^\circ \to \mathbb{P}_{\Delta^\circ}$, resulting in the diagram

(2.12)
$$\begin{tikzcd}
 & K_{\mathbb{P}_{\Delta^\circ}} \arrow[dl, two heads, "\mathbb{A}^1"'] & & \hat{X}_{\underline{t},0} \arrow[dr, two heads, "\mathrm{I}_5"] & \\
\mathbb{P}_{\Delta^\circ} & & \overset{\text{Batyrev}}{\underset{\text{mirror}}{\leftrightsquigarrow}} & & \mathbb{P}_{\Delta} \\
 & \hat{X}^\circ \subset \mathbb{P}_{\hat{\Delta}^\circ} \arrow[ul, two heads, "\text{ell.}"] \arrow[uu, dashed, "s\to 0"'] & & \mathbb{P}_{\hat{\Delta}} \supset \hat{X}_{\underline{t},s} \arrow[uu, dashed, "s\to 0"] \arrow[ur, two heads, "\text{ell.}"'] &
\end{tikzcd}$$

with solid arrows labeled by generic fiber type. The singular CY $\hat{X}_{\underline{t},0}$ has the *Hori-Vafa model*

$$Y_{\underline{t}} := \hat{X}_{\underline{t},0} \cap (\mathbb{G}_m^n \times \mathbb{A}^2) = \{\phi_{\underline{\lambda}}(\underline{x}) + uv = 0\},$$

a smooth noncompact CY $(n+1)$-fold, as a Zariski open subset, and one has the

Theorem 2.9 ([DK11, BKV17])**.** *There are isomorphisms of* $\mathbb{Q}$*-VMHS*

$$H^{n+1}(\hat{X}_{\underline{t},0}) \cong H_{n+1}(Y_{\underline{t}})(-n-1) \cong H^n(\mathbb{P}_\Delta \setminus X_{\underline{t}}, \mathbb{D}_\Delta \setminus \partial X_{\underline{t}}). \tag{2.13}$$

Now by Theorem 2.4, the RHS of (2.13) identifies with $\hat{\tau}^{\Delta}_{\mathrm{GKZ}}$. On the other hand, Iritani's results [Ir11] on $\hat{\Gamma}$-integral structure allow us to explictly compute the LHS of the isomorphism

$$QH^{\mathrm{even}}(\hat{X}^\circ) \cong H^{n+1}(\hat{X}_{\underline{t},s}) \tag{2.14}$$

of A- and B-model $\mathbb{Z}$-VHS. Taking LMHS on both sides as $s \to 0$

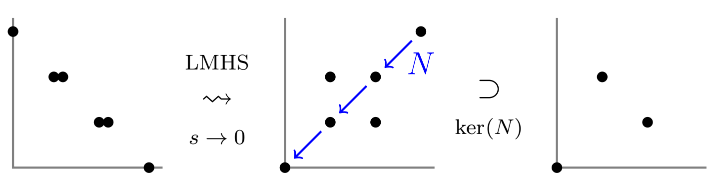
$$\psi_s QH^{\mathrm{even}}(\hat{X}^\circ) \cong \psi_s H^{n+1}(\hat{X}_{\underline{t},s}), \tag{2.15}$$

the (unipotent) monodromy invariants

$$QH_c^{*(+2)}(K_{\mathbb{P}_{\Delta^\circ}}) \cong H^{n+1}(\hat{X}_{\underline{t},0}) \tag{2.16}$$

must agree as $\mathbb{Q}$-VMHS.

Example 2.10. For $\mathbb{P}_{\Delta^\circ} = \mathbb{P}^2$, the Hodge-Deligne diagrams[5] for (2.14)-(2.16) are



Here $n = 2$ and $h^{2,1}(\hat{X}^\circ_{t,s}) = 2$, while each of the Gr^k_F $(k = 0,1,2)$ in Theorem 2.5 (visible in the right-most diagram) has rank 1.

[5]The number of dots in the (p,q) spot represents $h^{p,q}$ of the given MHS.

The upshot is that we recover the isomorphism $QH_c^{*(+2)}(K_{\mathbb{P}_{\Delta^\circ}}) \cong \hat{\tau}^{\Delta}_{\mathrm{GKZ}}$ claimed in (2.9), while promoting it to an isomorphism of $\mathbb{Q}$-VMHS. Moreover, we obtain the promised A-model description of the weight filtration on $\hat{\tau}^{\Delta}_{\mathrm{GKZ}}$ as the monodromy weight filtration $M_\bullet = W(N)[-n-1]_\bullet$ on LHS(2.16) $\subset$ LHS(2.15). We may therefore use [Ir11] to compute $W_\bullet \hat{\tau}^{\Delta}_{\mathrm{GKZ}}$, and the associated "mixed-$\mathbb{Q}$-periods", in terms of the intersection theory of $\mathbb{P}_{\hat{\Delta}^\circ}$ and Gromov-Witten theory of $\hat{X}^\circ$, restricted to classes of curves whose volume remains finite in the $s \to 0$ limit. This boils down to intersection theory and *local* GW theory of $\mathbb{P}_{\Delta^\circ}$. (The reader who wants to see this worked out in detail in some $n = 2$ cases may consult [BKV17].)

So far, we have said nothing about the extensions of MHS in (2.11) which these mixed periods are supposed to help us compute. (For instance, the right-hand term of Example 2.10 can be viewed as the dual of the extension associated to the regulator of a family of K_2^{alg} classes on the family E_t of elliptic curves.) The analysis of these VMHS undertaken in §§3-4 works in a "more general" setting which allows us to *drop* the genericity assumption on ϕ.

3. Special Laurent polynomials

3.1. **Landau-Ginzburg models.** Instead of starting with a reflexive polytope and letting ϕ vary over the corresponding parameter space minus discriminant locus, we begin by fixing a Laurent polynomial $\phi \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We assume that its Newton polytope Δ (the convex hull of those $\{\underline{m}\}$ for which $\underline{x}^{\underline{m}}$ has nonzero coefficient) is reflexive; and fixing a maximal projective triangulation $\mathrm{tr}(\Delta^\circ)$, we also assume that the associated toric n-fold $\mathbb{P}_\Delta := \mathbb{P}_{\Sigma(\mathrm{tr}(\Delta^\circ))}$ is smooth.[6] Write $\mathbb{D}_\Delta := \mathbb{P}_\Delta \setminus \mathbb{G}_m^n$ as before, $X_t \subset \mathbb{P}_\Delta$ for the Zariski closure of $\{1 = t\phi(\underline{x})\}$, and $Z := \mathbb{D}_\Delta \cap X_0$ for the base locus of the resulting pencil.

As in [DvdK98, Thm. 4], we may fix a sequence of blow-ups of $\mathbb{P}_\Delta$ (typically along successive proper transforms of components of Z) with composition $\beta \colon \mathcal{X} \to \mathbb{P}_\Delta$, such that:

- $\mathcal{X}$ is smooth;
- $\frac{1}{\phi(\underline{x})}$ extends to a holomorphic map $\pi \colon \mathcal{X} \to \mathbb{P}^1$; and
- $\mathrm{dlog}(\underline{x}) := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ extends to holomorphic form on $\mathcal{X} \setminus \pi^{-1}(0)$.

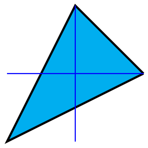
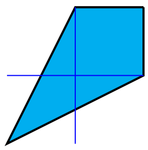
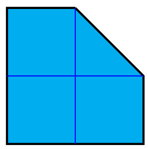
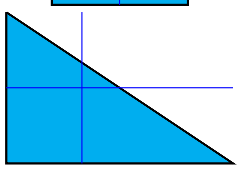
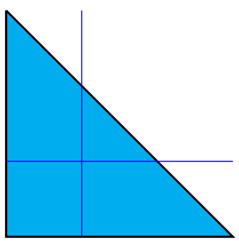
We shall assume that β may be chosen in such a way that this extended form is nowhere vanishing, so that the $\tilde{X}_t := \pi^{-1}(t)$ are Calabi-Yau for t not in the discriminant locus Σ. (This is weaker than assuming ϕ "generic", and implies that $\beta_t := \beta|_{\tilde{X}_t} \colon \tilde{X}_t \twoheadrightarrow X_t$ is a crepant resolution for $t \notin \Sigma$.) Despite the notation, $\tilde{X}_t$ is *not* smooth for $t \in \Sigma$.

Definition 3.1. (a) The *compact LG-model* associated to ϕ is the family $\pi \colon \mathcal{X} \to \mathbb{P}^1_t$ of CY $(n-1)$-folds $\tilde{X}_t$ just constructed. We may view its total space $\mathcal{X}$ as a smooth compactification of the pencil $\{1 = t\phi(\underline{x})\} \subset \mathbb{G}_m^n \times (\mathbb{P}^1_t \setminus \{0\})$, and $\tilde{X}_0$ as a blow-up of $\mathbb{D}_\Delta$.

(b) The (noncompact) *LG-model* associated to ϕ is the restriction $\mathcal{X} \setminus \tilde{X}_0 \to \mathbb{P}^1_t \setminus \{0\}$ of π.

[6] We will write $\mathbb{P}'_\Delta := \mathbb{P}_{\Sigma(\Delta^\circ)}$ for the singular toric n-fold (of which $\mathbb{P}_\Delta$ is a blow-up).

Example 3.2. Here are some Laurent polynomials in 2 variables (for $n = 3$, see §§5.3-5.5), together with the Kodaira types of the singular fibers of π (first and last at $t = 0$ resp. ∞):

i	$\Delta^{(i)}$	$\phi^{(i)}$	singular fibers
1		$x + y + \frac{1}{xy}$	$\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1(, \mathrm{I}_0)$
2		$16x + y - 3xy$ $-6 + \frac{1}{xy}$	$\mathrm{I}_8, \mathrm{I}_1, \mathrm{I}_1, \mathrm{II}$
3		$\frac{(1-x)(1-y)(1-x-y)}{xy}$	$\mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_5$
4		$x + y + \frac{1}{x^2y^3}$ $-4^{\frac{1}{3}}3^{\frac{1}{2}}$	$\mathrm{I}_6, \underbrace{\mathrm{I}_1, \ldots, \mathrm{I}_1}_{5 \text{ times}}, \mathrm{I}_1$
5		$\frac{1}{xy}F_3$ (F_3 a general cubic)	$\mathrm{I}_3, \underbrace{\mathrm{I}_1, \ldots, \mathrm{I}_1}_{9 \text{ times}}(, \mathrm{I}_0)$
6		$\frac{(1+x+y)^3}{xy}$	$\mathrm{I}_3, \mathrm{I}_1, \mathrm{IV}^*$

For instance, the last two share the same $\mathbb{P}_\Delta \cong \mathbb{P}^2$, but have different $\mathcal{X}$'s: obtained by blowing up at 9 distinct points (for the general cubic), vs. blowing up three times at each of three points.

3.2. **Variation of Hodge structure.** On a neighborhood of $t = 0$, consider the family of vanishing $(n-1)$-cycles γ_t on $\tilde{X}_t$ whose image under Tube: $H_{n-1}(\tilde{X}_t, \mathbb{Z}) \to H_n(\mathcal{X} \setminus \tilde{X}_t, \mathbb{Z})$ is $[\beta^{-1}(\mathbb{T}^n)]$, where $\mathbb{T}^n := \cap_{i=1}^n \{|x_i| = 1\}$. The family of holomorphic forms

$$\omega_t := \frac{1}{(2\pi\mathbf{i})^{n-1}} \mathrm{Res}_{\tilde{X}_t} \left(\frac{\mathrm{dlog}(\underline{x})}{1 - t\phi(\underline{x})} \right) \in \Omega^{n-1}(\tilde{X}_t) \tag{3.1}$$

then has the holomorphic period

$$A(t) := \int_{\gamma_t} \omega_t = \frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathbb{T}^n} \frac{\mathrm{dlog}(\underline{x})}{1 - t\phi(\underline{x})} = \sum_{k \geq 0} a_k t^k, \tag{3.2}$$

where $a_k = [\phi^k]_{\underline{0}}$ are the constant terms in powers of ϕ.

Writing $\pi_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ for the restriction of π over $\mathcal{U} := \mathbb{P}^1 \setminus \Sigma$, the local system $\mathbb{H}^{n-1} := R^{n-1}(\pi_{\mathcal{U}})_*\mathbb{Q}$ has maximal unipotent monodromy[7] at $t = 0$. It underlies a (polarized) VHS with sheaf of holomorphic sections $\mathcal{H}^{n-1} \cong \mathbb{H}^{n-1} \otimes \mathcal{O}_{\mathcal{U}}$ and Gauss-Manin connection ∇. In fact, we will work with the sub-local-system $\mathbb{H}_v^{n-1}$ orthogonal to the fixed part $\mathbb{H}_f^{n-1} = H^0(\mathcal{U}, \mathbb{H}^{n-1})$. The corresponding sub-VHS $\mathcal{H}_v^{n-1} \subseteq \mathcal{H}^{n-1}$ contains the Hodge line $\mathcal{H}^{n-1,0} = (\pi_{\mathcal{U}})_*\Omega^{n-1}_{\mathcal{X}_{\mathcal{U}}}$, and for $n \leq 3$ is always irreducible. On the level of $d_{\pi_{\mathcal{U}}}$-closed-form representatives, the polarization $\langle\ ,\ \rangle\colon \mathcal{H}_v^{n-1} \times \mathcal{H}_v^{n-1} \to \mathcal{O}$ is simply given by $\langle \omega, \eta\rangle = \int_{\tilde{X}_t} \omega_t \wedge \eta_t$.

For simplicity, we shall henceforth *assume* that $\mathcal{H}_v^{n-1}$ is irreducible, not just as a $\mathbb{Q}$-VHS but as a $\mathcal{D}$-module (or $\mathbb{C}$-VHS). Let $L \in \mathbb{C}[t, \delta_t]$ be the differential operator with $(\mathcal{H}_v^{n-1}, \nabla) \cong \mathcal{D}/\mathcal{D}L$, of degree d and order $r = \mathrm{rk}(\mathbb{H}_v^{n-1})$, normalized so that the coefficient of δ_t^r is 1 at $t = 0$.

A putative mirror to the LG-model is given by the following folklore

Conjecture 3.3. *There exists a Fano n-fold* $(\mathrm{X}^\circ, \omega)$*, determined by the triple* $(\mathbb{P}_\Delta, \mathbb{D}_\Delta, Z)$ *and admitting a toric degeneration to* $\mathbb{P}'_{\Delta^\circ}$*, from which one may recover* $\mathcal{H}_v^{n-1}$*. (In particular, for generic* ϕ*, we have* $\mathrm{X}^\circ = \mathbb{P}_{\Delta^\circ}$*.)*

Conversely, it is hoped that by studying "special" Laurent polynomials and classifying the associated local systems, one obtains a classification of Fano varieties admitting a toric degeneration. While Conjecture 3.3 is vague as stated, it will be refined below: a mechanism for recovering $\mathcal{H}_v^{n-1}$ (in some cases) is given in Conjecture 3.4; while the Hodge-theoretic sense in which X° is mirror to $\mathcal{X} \setminus \tilde{X}_0 \to \mathbb{A}^1$ is the subject of Conjecture 4.2.

3.3. **Quantum $\mathcal{D}$-module.** For simplicity, we assume in this subsection that the Picard rank $\rho(\mathrm{X}^\circ) = 1$. In the standard way [Go07], one uses genus-zero Gromov-Witten theory to construct a quantum product "$\star$" on $H^*(\mathrm{X}^\circ) \otimes \mathbb{C}[s^{\pm 1}]$. This is endowed with a $\mathcal{D}$-module structure by letting δ_s act via $1 \otimes \delta_s - (K_{\mathrm{X}^\circ}\star) \otimes 1$.

Now let $\mathcal{H}_v^{n-1}$ be as in Conjecture 3.3, and L be the corresponding Picard-Fuchs equation, with Fourier-Laplace transform $\hat{L}$. (Here we recall that the FL-transform and its inverse are given on functions/solutions[8] by

$$\hat{f}(s) := \frac{1}{2\pi\mathbf{i}} \oint f(t) e^{s/t} \frac{dt}{t} \quad \text{and} \quad \check{F}(t) := \frac{1}{t}\int_0^\infty F(s) e^{-s/t} ds, \tag{3.3}$$

and on operators by replacing $\partial_t \leftrightarrow -s$ and $t \leftrightarrow \partial_s$.) Then we have the following amplification of that Conjecture when $\omega = -K_{\mathrm{X}^\circ}$:

Conjecture 3.4. *As* $\mathcal{D}$*-modules,* $H^*(\mathrm{X}^\circ) \otimes \mathbb{C}[t^{\pm 1}] \cong \mathcal{D}/\mathcal{D}\hat{L}$*.*

That is, by applying inverse FL-transform to $\hat{L}$ we should obtain $\mathcal{H}_v^{n-1}$. This has been checked for an enormous number of Fano varieties of dimension 2 and 3 [CCGGK13].

Assuming this holds for X°, we write $(\hat{L} =) \sum_{i,j} \beta_{ij} t^i \delta_t^j$ for the (irregular) differential operator killing the generator $1 \otimes 1$ of the quantum $\mathcal{D}$-module, and convert this into an

[7]In this paper, a unipotent monodromy operator $T = e^N$ is *maximally unipotent* if $N^{n-1} \neq 0$.

[8]If $f(t) = \sum_k c_k t^k$ is a power-series, this gives $\hat{f}(s) = \sum_k \frac{c_k}{k!} s^k$.

(irregular) *quantum recursion*

$$\hat{R}\colon \quad \sum_{i,j} \beta_{ij}(k-i)^j \hat{u}_{k-i} = 0 \ \ (\forall k) \tag{3.4}$$

by applying $\hat{L}$ to a power series $\sum_k \hat{u}_k s^k$. We consider a basis of solutions $\{\hat{u}^{(i)}_{\cdot}\}_{i=0}^{d-1}$, defined over the same field as L (typically $\mathbb{Q}$), with $\hat{u}^{(i)}_k = 0$ for $k < i$ and $\hat{u}^{(i)}_i = \frac{1}{i!}$. Regularizing via the inverse FL transform, $\hat{L}, \hat{R}$ become L, R, with solutions $u_k = k!\hat{u}_k$; in particular, we have $u^{(i)}_k = 0$ for $k < i$ and $u^{(i)}_i = 1$.

We shall take the basis to be chosen so that $u^{(0)}_k = a_k$ is as in (3.2), and impose one more assumption: that the $\{a_k\}$ are nonzero. There are various ways to further normalize $u^{(1)}_{\cdot}, \ldots, u^{(d-2)}_{\cdot}$. For instance, there are $r_0 := \mathrm{rk}((\psi_0 \mathcal{H}^{n-1}_v)^{T_0}) \leq d$ independent holomorphic solutions to $L(\cdot) = 0$ at the origin, which we take to be given by the generating series of $u^{(0)}_{\cdot}, \ldots, u^{(r_0-1)}_{\cdot}$.[9] The remaining $d - r_0$ generating series will then be solutions to inhomogeneous equations $L(\cdot) = g_i(t) \in \mathbb{C}[t]$.[10] In particular, it will be important in §5 that when $d = 2$ and $r = n$ ($\implies P_0(T) = T^n$), $\sum_{k \geq 1} u^{(1)}_k t^k$ solves $L(\cdot) = t$.

Slightly generalizing the definition in [Go09], we propose

Definition 3.5. The *Apéry constants* of X° are the limits

$$\alpha^{(i)}_{\mathrm{X}^\circ} := \lim_{k\to\infty} \frac{\hat{u}^{(i)}_k}{\hat{u}^{(0)}_k} = \lim_{k\to\infty} \frac{u^{(i)}_k}{u^{(0)}_k}, \tag{3.5}$$

for $1 \leq i \leq d-1$. (When $d = 2$, we simply write $\alpha_{\mathrm{X}^\circ}$.)

Remark 3.6. The closely related definition in [Ga16] (of an Apéry class $A(\mathrm{X}^\circ) \in H^*_{\mathrm{prim}}(\mathrm{X}^\circ)$, with the constants appearing as its coefficients) only considers the first $r_0 - 1$ Apéry constants, corresponding to solutions of the homogeneous equation. Hoewver, the focus in [op. cit.] is on large-dimensional examples for which $r_0 = d$; taking hyperplane sections preserves d as well as the $\alpha^{(i)}$ (in our sense), even as r_0 decreases. Since four of the five 3-dimensional examples we consider in §5 are indeed obtained as multisections of homogeneous Fano varieties with $(\dim(H^*_{\mathrm{prim}}(\mathrm{X}^\circ) =) \, r_0 = 2 = d$, the "inhomogeneous" Apéry constants for the 3-folds in [Go09] are connected to the constants in [Ga16] in this way (albeit with a slightly different normalization).

Remark 3.7. Adding a constant c to ϕ conjugates $\hat{L}$ by e^{cs}, which does not affect the Apéry constants. We may thus choose the constant term to make $\phi = 0$ (i.e. $\tilde{X}_\infty$) singular.

By "specializing" Laurent polynomials, we hope not just to classify Fanos but to arrive at a B-model, Hodge-theoretic interpretation of their Apéry numbers. But there is a new twist. Consider the simplest case, where $d = 2$ and $r_0 = 1$, and write $u^{(0)}_k = a_k$, $u^{(1)}_k =: b_k = 0, 1, \ldots,$

[9]If $(\psi_0 \mathcal{H}^{n-1}_v)^{T_0}$ is Hodge-Tate, with r_0 distinct graded pieces $\{\mathbb{Q}(-p_i)\}_{i=0}^{r_0-1}$ (with $p_0 = 0$), one can take $\sum_k u^{(i)}_k t^k$ ($i = 0, \ldots, r_0 - 1$) to be the $\mathbb{C}$-periods of ω, against local sections $\varphi^{(i)}$ of $\mathbb{H}^{n-1}_{v,\mathbb{C}}$ passing through $\mathbb{C}(-p_i)$ at $t = 0$.

[10]Writing $L = \sum_{\ell=0}^d t^\ell P_\ell(\delta_t)$, it is reasonable to expect that $P_0(T) = \prod_{i=0}^{r_0-1}(T-i)^{n-2p_i}$, and then we may assume that $\sum_k u^{(i)}_k t^k$ ($i = r_0, \ldots, d-1$) solves $L(\cdot) = P_0(i) t^i$.

and $\alpha_{X^\circ} = \lim_{k\to\infty} \frac{b_k}{a_k}$. While $A(t) = \sum_{k\geq 0} a_k t^k$ is just the holomorphic period, the $\{b_k\}$ and hence α_{X° are *not* visible from $\mathcal{H}_v^{n-1}$ alone. It is for this reason that we turn to variations of MHS in the next section.

4. Higher normal functions

4.1. **Variation of mixed Hodge structure.** Fix a Laurent polynomial subject to the assumptions in §3.1, and write $X_t^* = X_t \cap \mathbb{G}_m^n$ for the level sets of $\frac{1}{\phi}$.

Proposition 4.1. *As MHS, $H^n(\mathcal{X} \setminus \tilde{X}_0, \tilde{X}_t) \cong H^n(\mathbb{G}_m^n, X_t^*)$ for $t \neq 0$.*

Proof. In order that the extended dlog$(\underline{x})$ on $\mathcal{X} \setminus \tilde{X}_0$ satisfy the nonvanishing assumption, the sequence of blow-ups in β must be centered along subschemes of successive proper transforms of $\mathbb{D}_\Delta$. Hence the restriction of π to the exceptional divisor $\mathcal{E} \subset \mathcal{X}$ of β is locally constant over $\mathbb{P}^1 \setminus \{0\}$. In particular, writing $\mathcal{E}_t := \mathcal{E} \cap \tilde{X}_t$, $\mathcal{E}_0$ is a deformation retract of $\mathcal{E} \setminus \mathcal{E}_t$ for any $t \neq 0$.

Since $\mathbb{G}_m^n = \mathcal{X} \setminus (\tilde{X}_0 \cup \mathcal{E})$ and $X_t^* = \tilde{X}_t \setminus \mathcal{E}_t$, we have

$$H^n(\mathbb{G}_m^n, X_t^*) \cong H^n(\mathcal{X} \setminus (\tilde{X}_0 \cup \mathcal{E}), \tilde{X}_t \setminus \mathcal{E}_t) \cong H_n(\mathcal{X} \setminus \tilde{X}_t, \tilde{X}_0 \cup (\mathcal{E} \setminus \mathcal{E}_t))(-n)$$

and $H^n(\mathcal{X} \setminus \tilde{X}_0, \tilde{X}_t) \cong H_n(\mathcal{X} \setminus \tilde{X}_t, \tilde{X}_0)(-n)$, which fit together in the long-exact sequence

$$\to H_n(\mathcal{E} \setminus \mathcal{E}_t, \mathcal{E}_0) \to H_n(\mathcal{X} \setminus \tilde{X}_t, \tilde{X}_0) \to H_n(\mathcal{X} \setminus \tilde{X}_t, \tilde{X}_0 \cup (\mathcal{E} \setminus \mathcal{E}_t)) \to H_n(\mathcal{E} \setminus \mathcal{E}_t, \mathcal{E}_0) \to$$

whose end terms are zero by the deformation retract property. $\square$

Now consider the VMHS

$$\mathcal{V}_{\phi,t} := H^n(\mathcal{X} \setminus \tilde{X}_0, \tilde{X}_t) \cong H^n(\mathbb{G}_m^n, X_t^*) \tag{4.1}$$

over $\mathcal{U} := \mathbb{P}^1 \setminus \Sigma$, with dual

$$\mathcal{V}_{\phi,t}^\vee(-n) \cong H^n(\mathcal{X} \setminus \tilde{X}_t, \tilde{X}_0) \cong H^n(\mathbb{P}_\Delta \setminus X_t, \mathbb{D}_\Delta \setminus Z). \tag{4.2}$$

If ϕ is generic, the right-hand term of (4.2) is nothing but a restriction of $\hat{\tau}_{\mathrm{GKZ}}^\Delta$. This suggests the following generalization of Theorem 2.5:

Conjecture 4.2 ([KKP17])**.** *For $t \notin \Sigma$, we have for each k*

$$\mathrm{rk}(\mathrm{Gr}_F^{n-k} \mathcal{V}_\phi^\vee(-n)) = \dim(H^{k,k}(\mathrm{X}^\circ)). \tag{4.3}$$

Remark 4.3. By Serre duality, (4.3) would imply that $\mathrm{rk}(\mathrm{Gr}_F^{n-k}\mathcal{V}_\phi) = \mathrm{rk}(\mathrm{Gr}_F^k \mathcal{V}_\phi)$ for all of our LG-models, which has been proved by Harder [Ha17].

In order to relate limits of extension classes in (4.1)-(4.2) to Apéry constants of X°, we shall need to kill off intermediate extensions which would otherwise "obstruct" these classes. This will be accomplished by placing a "K-theoretic" constraint on the Laurent polynomial:

Definition 4.4. We say ϕ is *tempered* if the coordinate symbol $\{x_1, \ldots, x_n\} \in H_{\mathcal{M}}^n(\mathbb{G}_m^n, \mathbb{Q}(n))$ lifts to a class in $H_{\mathcal{M}}^n(\mathcal{X} \setminus \tilde{X}_0, \mathbb{Q}(n))$.

Henceforth we shall be mainly concerned with the case where ϕ is tempered. When $n = 2$, this is just the condition that the edge polynomials of ϕ be cyclotomic [RV99]; some methods for checking temperedness for $n = 3, 4$ are given in [DK11, §]. Up to scale, tempered reflexive Laurent polynomials are defined over $\bar{\mathbb{Q}}$ [op. cit., Prop. 4.16] and are thereby rigid.

4.2. **Admissible and geometric normal functions.** A reference for the material that follows is [KP11, §2.11-12].

Definition 4.5. A *higher normal function* on $\mathcal{U}$ is (equivalently)

(i) a VMHS of the form $0 \to \mathcal{H} \xrightarrow{\imath} \mathcal{V} \to \mathbb{Q}(0) \to 0$, or
(ii) a holomorphic, horizontal section ν of $J(\mathcal{H}) := \mathcal{H}/(F^0\mathcal{H} + \mathbb{H})$,

where $\mathcal{H}$ is a polarizable VHS of pure weight $-r < -1$.

Here *horizontal* means that, for each local holomorphic lift $\tilde{\nu}$ to $\mathcal{H}$, we have $\nabla\tilde{\nu} \in F^{-1}\mathcal{H}$. For instance, given (i) we may locally lift $1 \in \mathbb{Q}(0)$ to $\nu_F \in F^0\mathcal{V}$ and $\nu_{\mathbb{Q}} \in \mathbb{V}$ (the local system underlying $\mathcal{V}$), then locally define $\tilde{\nu}$ (hence ν as in (ii)) by $\imath(\tilde{\nu}) = \nu_{\mathbb{Q}} - \nu_F$.

Let $\mathcal{V}_e$ denote Deligne's canonical extension of $\mathcal{V}$ to $\mathbb{P}^1$. Fixing disks $D_\sigma \subset \mathbb{P}^1$ at each $\sigma \in \Sigma\,(= \mathbb{P}^1 \setminus \mathcal{U})$, with coordinate t_σ, we write $T_\sigma = e^{N_\sigma}T_\sigma^{\mathrm{ss}}$ for the monodromy of $\mathbb{V}$ on $D_\sigma^* = D_\sigma \setminus \{\sigma\}$, and $M_\bullet^\sigma$ for the monodromy-weight filtration of the LMHS $\psi_\sigma\mathcal{H}$. Suppose now that there exist "lifts of 1":

- $\nu_F^\sigma \in \Gamma(D_\sigma, \mathcal{V}_e)$ – holomorphic, single-valued, with $\nu_F^\sigma|_{D_\sigma^*}$ in $F^0\mathcal{V}$
- $\nu_{\mathbb{Q}}^\sigma \in \Gamma(\widetilde{D_\sigma^*}^{\mathrm{un}}, \mathbb{V})^{T_\sigma^{\mathrm{ss}}}$ – flat, multivalued, with $N_\sigma\nu_{\mathbb{Q}}^\sigma \in M_{-2}^\sigma\psi_\sigma\mathcal{H}$.

Then we may confer on $\mathcal{V}_e|_\sigma$ the status of a MHS $\psi_\sigma\mathcal{V}$ as follows:

- the weight filtration $M_\bullet^\sigma$ extends that on $\psi_\sigma\mathcal{H}$, adding $\nu_{\mathbb{Q}}^\sigma$ to M_0^σ;
- the Hodge filtration $F_\sigma^\bullet$ extends that on $\psi_\sigma\mathcal{H}$, adding $\nu_F^\sigma(\sigma)$ to F_σ^0;
- the $\mathbb{Q}$-structure $(\psi_\sigma\mathcal{V})_{\mathbb{Q}}$ is easiest to describe after a base-change (to kill off T_σ^{ss}), as the specialization of $\exp(-\frac{\log(t_\sigma)}{2\pi\mathbf{i}}N_\sigma)\mathbb{V} \subset \mathcal{V}_e$ at σ.

Definition 4.6. The HNF ν is *admissible*, written $\nu \in \mathrm{ANF}(\mathcal{H})$, if this LMHS $\psi_\sigma\mathcal{V}$ (equivalently, ν_F^σ and $\nu_{\mathbb{Q}}^\sigma$) exists at each $\sigma \in \Sigma$. If, in addition, we may choose $\nu_{\mathbb{Q}}^\sigma$ so that $N_\sigma\nu_{\mathbb{Q}}^\sigma = 0$, then the *limit* $\lim_\sigma \nu \in J((\psi_\sigma\mathcal{H})^{T_\sigma})$ is defined;[11] otherwise, ν is *singular* at σ.

Remark 4.7. Writing $\mathcal{H} = \mathcal{H}_f \oplus \mathcal{H}_v$ for the decomposition into fixed and variable parts (with $\mathcal{H}_f = H_f \otimes \mathcal{O}_{\mathcal{U}}$), we claim that

$$0 \to J(H_f) \to \mathrm{ANF}(\mathcal{H}) \to Hg(H^1(\mathcal{U}, \mathcal{H})) \to 0 \tag{4.4}$$

is exact. Indeed, since $\mathrm{ANF}(\mathcal{H}) \cong \mathrm{Ext}^1_{\mathrm{AVMHS}(\mathcal{U})}(\mathbb{Q}(0), \mathcal{H})$, this follows at once from the spectral sequence

$$R\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), -) \circ R\Gamma_{\mathcal{U}} \implies R\mathrm{Hom}_{\mathrm{AVMHS}(\mathcal{U})}(\mathbb{Q}(0), -)$$

and triviality of $\mathrm{Ext}^{i>1}_{\mathrm{MHS}}$, by using the identifications $H^0(\mathcal{H}) = H_f$, $J(H_f) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H_f)$, and $Hg(H^1(\mathcal{H})) \cong \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\mathcal{H}))$.

We say that $\nu \in \mathrm{ANF}(\mathcal{H}^{2p-r}(p))$ is of *geometric origin* when it arises from a motivic cohomology class in

$$H^{2p-r+1}_{\mathcal{M}}(\mathcal{X}_{\mathcal{U}}, \mathbb{Q}(p)) \cong \mathrm{Gr}^p_\gamma K^{\mathrm{alg}}_{r-1}(\mathcal{X}_{\mathcal{U}})_{\mathbb{Q}} \cong \mathrm{CH}^p(\mathcal{X}_{\mathcal{U}}, r-1). \tag{4.5}$$

The most convenient representatives are found in the right-hand term, the *higher Chow groups* of Bloch [Bl86, Bl94], which (in their cubical formulation) are defined as the $(r-1)^{\mathrm{st}}$

[11] $\lim_\sigma \nu$ is given by $\imath(\widetilde{\lim_\sigma \nu}) = \nu_{\mathbb{Q}}^\sigma - \nu_F^\sigma(0)$, as in the passage from (i) to (ii) above.

homology of a complex $(Z^p(X,\bullet),\partial)$ of codim.-p cycles on $X\times\square^\bullet$, where $\square:=\mathbb{P}^1\setminus\{1\}$.[12] Given a cycle $\mathcal{Z}$ in (4.5), its restrictions $\mathcal{Z}_t\in \mathrm{CH}^p(\tilde{X}_t,r-1)$ have (for each $t\in\mathcal{U}$) Abel-Jacobi/regulator invariants[13]

$$\mathrm{AJ}(\mathcal{Z}_t)=:\nu_{\mathcal{Z}}(t)\in J(H^{2p-r}(\tilde{X}_t)(p)). \tag{4.6}$$

By [BZ90, Thm. 7.3], these glue together into an admissible normal function, so that $\mathcal{Z}\mapsto\nu_{\mathcal{Z}}$ defines a map

$$\mathrm{AJ}_\phi\colon \mathrm{CH}^p(\mathcal{X}_{\mathcal{U}},r-1)\to \mathrm{ANF}(\mathcal{H}^{2p-r}(p)). \tag{4.7}$$

Composing with projection to $\mathrm{ANF}(\mathcal{H}_v^{2p-r}(p))\cong Hg(H^1(\mathcal{U},\mathcal{H}_v^{2p-r}(p)))$ (cf. Remark 4.7) defines AJ_ϕ^v and $\nu_{\mathcal{Z}}^v$, for which we have the following special case of the Beilinson-Hodge Conjecture:

Conjecture 4.8 (BHC). *For $\mathcal{X}_{\mathcal{U}}$ defined over $\bar{\mathbb{Q}}$, AJ_ϕ^v is surjective. That is, admissible and geometric HNFs with values in $\mathcal{H}_v^{2p-r}(p)$ are the same thing.*

The equivalence in Definition 4.5, as well as the notion of admissibility, persist with $\mathcal{H}$ merely a VMHS; we shall loosely refer to sections of $J(\mathcal{H})$ in this more general setting as a *mixed HNF.*

4.3. $\mathcal{V}_\phi$ as a (mixed) higher normal function.

First we set (dually)

$$\begin{cases}\overline{H}^\ell(X_t^*):=\mathrm{coker}\{H^\ell(\mathbb{G}_n^m)\to H^\ell(X_t^*)\},\\ \underline{H}_\ell(X_t^*):=\ker\{H_\ell(X_t^*)\to H_\ell(\mathbb{G}_m^n)\}.\end{cases} \tag{4.8}$$

Since X_t^* is affine and $H^n(\mathbb{G}_m^n)\cong\mathbb{Q}(-n)$, the isomorphism in (4.1) yields at once an exact sequence of MHS

$$0\to\overline{H}^{n-1}(X_t^*)\to\mathcal{V}_{\phi,t}\to\mathbb{Q}(-n)\to 0, \tag{4.9}$$

[12] Elements of $Z^p(X,n)$ must meet all faces $X\times\square^m$ (defined by setting $\square$-coordinates to 0 or ∞) properly, and ∂ is given by an alternating sum of intersections with codim.-1 faces. See [DK11, §1] for a brief introduction to higher cycles and their Hodge-theoretic invariants.

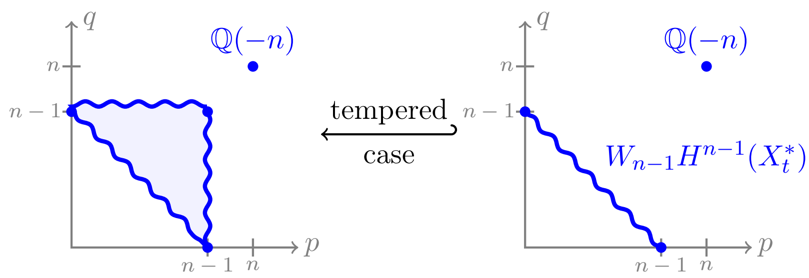
[13] These may be computed as the class of a closed $(2p-r-1)$-current $(2\pi\mathbf{i})^p\delta_\Gamma+(2\pi\mathbf{i})p-r+1(\mathcal{Z}_t)_*R_{r-1}$, where R_{r-1} is a standard $(r-2)$-current on $\square^{r-1}$, and Γ is a chain bounding on $(\mathcal{Z}_t)_*\mathbb{R}^{r-1}_{<0}$ [loc. cit.]. One defines these *regulator currents* inductively by

$$R_\ell(x_1,\ldots,x_\ell):=\log(x_1)\tfrac{dx_1}{x_1}\wedge\cdots\wedge\tfrac{dx_\ell}{x_\ell}-2\pi\mathbf{i}\delta_{T_{x_1}}\cdot R_{\ell-1}(x_2,\ldots,x_\ell),$$

where $T_x:=x^{-1}(\mathbb{R}_{<0})$; they satisfy

$$d[R_\ell]=\mathrm{dlog}(\underline{x})-(2\pi\mathbf{i})^\ell\delta_{\cap_{i=1}^\ell T_{x_i}}+\textstyle\sum_{i=1}^\ell(-1)^iR_{\ell-1}(x_1,\ldots,\widehat{x_i},\ldots,x_\ell)\delta_{(x_i)}.$$

exhibiting $\mathcal{V}_\phi$ as a mixed HNF, with Hodge-Deligne diagram of the form on the left:



Proposition 4.9. *If ϕ is tempered, then $\mathcal{V}_\phi$ has a (pure) sub-HNF as shown on the right.*

Proof. Each $\gamma \in \underline{H}_{n-1}(X_t^*, \mathbb{Q})$ may be written as $\partial\mu$ for a n-chain μ on $\mathbb{G}_m^n$. The extension class $\nu_\phi(t)$ of (4.9) in

$$J(\overline{H}^{n-1}(X_t^*)(n)) \cong \mathrm{Hom}(\underline{H}_{n-1}(X_t^*, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(n))$$

is then computed on γ (using Stokes's theorem) by

$$\langle \tilde{\nu}_\phi(t), \gamma\rangle = \int_\mu \mathrm{dlog}(\underline{x}) \underset{\mathbb{Q}(n)}{\equiv} \int_\mu d[R_n] = \int_\gamma R_n|_{X_t^*} = \langle \mathrm{AJ}(\{\underline{x}\}|_{X_t^*}), \gamma\rangle.$$

Therefore ν_t is the *geometric* (mixed) HNF associated to the coordinate symbol $\{\underline{x}\} = \{x_1, \ldots, x_n\} \in \mathrm{CH}^n(\mathbb{G}_m^n, n)$ (i.e., the graph of this n-tuple, viewed as a cycle in $\mathbb{G}_m^n \times \square^n$).

If ϕ is tempered, then $\{\underline{x}\}$ lifts to $\xi \in \mathrm{CH}^n(\mathcal{X} \setminus \tilde{X}_0, n)$, whose restrictions $\xi_t \in \mathrm{CH}^n(X_t^*, n)$ compute $\nu_\phi(t)$ via the composition

$$\mathrm{CH}^n(\tilde{X}_t, n) \stackrel{\mathrm{AJ}}{\to} J(H^{n-1}(\tilde{X}_t)(n)) \to J(W_{n-1}H^{n-1}(X_t^*)) \hookrightarrow J(\overline{H}^{n-1}(X_t^*)(n)).$$

So the extension of $\mathbb{Q}(-n)$ by $H^{n-1}(X_t^*)/W_{n-1}$ in (4.8) splits ($\forall t$). $\square$

Remark 4.10. It follows from Proposition 4.15 below that (for $t \in \mathcal{U}$) $W_{n-1}H^{n-1}(X_t^*) \cong \mathcal{H}_{v,t}^{n-1}$. (One may also show this directly.)

Having exhibited $\mathcal{V}_\phi$ as the *regulator extension*, we turn to its dual

$$0 \to \mathbb{Q}(0) \to \mathcal{V}_{\phi,t}^\vee(-n) \to \underline{H}_{n-1}(X_t^*)(-n) \to 0, \tag{4.10}$$

which identifies with the localization sequence

$$0 \to H^n(\mathbb{P}_\Delta, \mathbb{D}_\Delta) \to H^n(\mathbb{P}_\Delta \setminus X_t, \mathbb{D}_\Delta \setminus Z) \stackrel{\mathrm{Res}}{\to} \ker\{H^{n-1}(X_t, Z) \stackrel{\imath_*}{\to} H^{n+1}(\mathbb{P}_\Delta, \mathbb{D}_\Delta)\}(-1) \to 0.$$

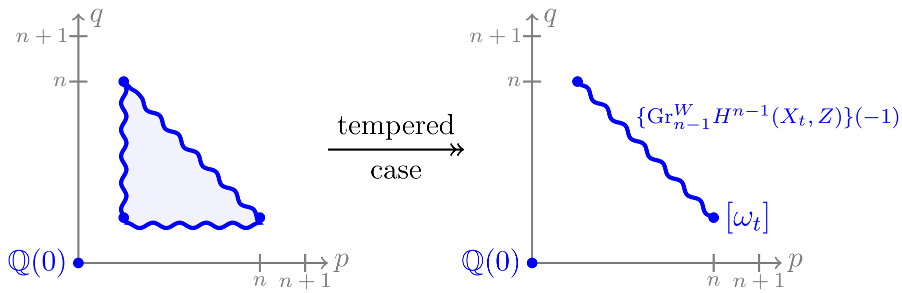
Writing $\Omega_t := \frac{\mathrm{dlog}(\underline{x})}{1-t\phi(\underline{x})} \in \Omega^n(\mathbb{P}_\Delta \setminus X_t)$ (so that $(2\pi\mathbf{i})^{n-1}\omega_t = \mathrm{Res}(\Omega_t)$), we obtain periods of the extension by lifting $(2\pi\mathbf{i})^{n-1}[\omega_t] \in F^n\{\ker(\imath_*)(-1)\}$ to $[\Omega_t] \in F^nH^n(\mathbb{P}_\Delta \setminus X_t, \mathbb{D}_\Delta \setminus Z; \mathbb{C})$ and pairing with the lift of $1^\vee \in \mathbb{Q}(0)^\vee$ to $T_{\underline{x}} := \cap_{i=1}^n T_{x_i} \in H_n(\mathbb{P}_\Delta \setminus X_t, \mathbb{D}_\Delta \setminus Z; \mathbb{Q})$. This yields

$$\int_{(-1)^{n-1}T_{\underline{x}}} \Omega_t = \int_{\mathbb{P}_\Delta} \tfrac{d[R_n]}{(-2\pi\mathbf{i})^n} \wedge \Omega_t = \int_{\mathbb{P}_\Delta} R_n \wedge \tfrac{d[\Omega_t]}{(2\pi\mathbf{i})^n} \equiv \langle \tilde{\nu}_\phi(t), [\omega_t]\rangle, \tag{4.11}$$

where the last equality only holds if $T_{\underline{x}} \cap X_t^* = \emptyset$, and only modulo *relative* periods $(2\pi\mathbf{i})^n \int_\eta \omega_t$ (with $\eta \in H_{n-1}(X_t, Z; \mathbb{Q})$).

Remark 4.11. The left-hand term of (4.11) is a special case of the GKZ integral (2.10), but with non-general ϕ. This type of integral also appears in Feynman integral computations [BKV15, BKV17].

When ϕ is tempered (so that $\tilde{\nu}_\phi(t) \in H^{n-1}(\tilde{X}_t, \mathbb{C})$), and certain technical assumptions hold (cf. [BKV15, §4.2]), the last equality of (4.11) holds modulo usual periods of ω_t. The VMHS picture is of course dual to that above:[14]



It will be convenient to enshrine the right-hand term of (4.11) in a

Definition 4.12. The *truncated higher normal function* (THNF) associated to a tempered ϕ is (any branch of) the multivalued function $V_\phi(t) := \langle \tilde{\nu}^v_\phi(t), [\omega_t] \rangle$.

Later we shall choose a branch of V_ϕ; but independent of this choice, it follows from [dAMS08] that the THNF satisfies an inhomogeneous Picard-Fuchs equation

$$LV_\phi(t) = g_\phi(t) \tag{4.12}$$

where $g_\phi \in \bar{\mathbb{Q}}(t)$ and L (from §3.2) depend only on ϕ.

Remark 4.13. Suppose $\mathrm{rk}(\mathbb{H}^{n-1}_v) = n$, write $L = \sum_{i=0}^n q_{n-i}(t)\delta_t^i$ (with $q_0(0) = 1$), and let $\mathcal{Y}(t) := \langle (2\pi \mathbf{i})^{n-1}\omega_t, \nabla^{n-1}_{\delta_t}\omega_t \rangle$ denote the Yukawa coupling. Taking $\gamma_t^\vee \in (\mathbb{H}^{n-1}_v)_{T_0}$ a local generator with $\langle \gamma_t, \gamma_t^\vee \rangle = 1$, define $D_\phi \in \mathbb{Q}^*$ by $N_0^{n-1}\gamma_t^\vee =: D_\phi \gamma_t$. By [DK11, Cor. 4.5], we have $g_\phi(t) = q_0(t)\mathcal{Y}(t)$. Moreover, if the $\{\tilde{X}_\sigma\}_{\sigma \in \Sigma \setminus \{0,\infty\}}$ have only nodal singularities, then by [Ke20, Prop. 7.1] $\mathcal{Y}(t) = \frac{D_\phi}{q_0(t)}$.

4.4. **$\mathcal{V}_\phi$ at infinity.** While ν_ϕ is singular at 0, we can compute its limit at $t = \infty$. First we shall isolate a part of the extension that splits off whether or not ϕ is tempered (which we don't assume here).

Definition 4.14. For $\sigma \in \Sigma$, the (pure weight ℓ) *phantom cohomology*

$$\mathrm{Ph}^\ell_\sigma := \ker\{H^\ell(X_\sigma) \to \psi_\sigma \mathcal{H}^\ell\} = \mathrm{im}\{H_{2n-\ell}(X_\sigma)(-n) \to H^\ell(X_\sigma)\} \tag{4.13}$$

measures the cycles that vanish on the nearby fiber. For any subset $\Sigma' \subseteq \Sigma$ (e.g. $\Sigma^* := \Sigma \setminus \{0, \infty\}$), put $\mathrm{Ph}^\ell_{\Sigma'} := \oplus_{\sigma \in \Sigma'} \mathrm{Ph}^\ell_\sigma$.

(We shall also write $\mathcal{X}_S$ resp. $\tilde{X}_S$ for $\pi^{-1}(S)$ when S is open resp. finite.)

[14]In view of Remark 4.10 and the polarization, we have $\mathrm{Gr}^W_{n-1}H^{n-1}(X_t, Z) \cong \mathcal{H}^{n-1}_{v,t}$.

Proposition 4.15. *In* AVMHS($\mathcal{U}$) *we have* $\mathcal{V}_\phi = \mathcal{A}_\phi^\dagger \oplus \mathrm{Ph}^n_{\Sigma\setminus\{0\}}$, *where* $\mathrm{Ph}^n_{\Sigma\setminus\{0\}}$ *is constant and* $\mathcal{A}^\dagger_{\phi,t}$ *is an extension of* $\mathrm{IH}^1(\mathbb{P}^1\setminus\{0\},\mathcal{H}_v^{n-1})$ (*also constant*) *by* $\mathcal{H}_v^{n-1}$. *Viewing* $\mathcal{A}_\phi^\dagger$ *instead as an extension of* $\mathbb{Q}(-n)$ *recovers* ν_ϕ.

Proof. By the Decomposition Theorem (cf. [KL19, (5.9)]), for any proper algebraic subset $\mathcal{S}\subset\mathbb{P}^1$ we have

$$H^\ell(\mathcal{X}_\mathcal{S}) \cong H^\ell_f \oplus \mathrm{IH}^1(\mathcal{S},\mathcal{H}^{\ell-1})\oplus \mathrm{Ph}^\ell_{\Sigma\cap\mathcal{S}} \tag{4.14}$$

as MHS. The long exact sequence associated to $(\mathcal{X}_\mathcal{S},X_t)$ (for $t\in\mathcal{U}$) therefore exhibits $H^n(\mathcal{X}_\mathcal{S},\tilde{X}_t)$ as an extension of $\mathrm{IH}^1(\mathcal{S},\mathcal{H}^{n-1})\oplus\mathrm{Ph}^n_{\Sigma\cap\mathcal{S}}$ by $\mathcal{H}_v^{n-1}$. But as a subMHS of $H^n(\mathcal{X}_\mathcal{S})$, $\mathrm{Ph}^n_{\Sigma\cap\mathcal{S}}$ is the image of $H_n(\tilde{X}_{\Sigma\cap\mathcal{S}})(-n)\cong H^n_{\tilde{X}_{\Sigma\cap\mathcal{S}}}(\mathcal{X}_\mathcal{S})\cong H^n_{\tilde{X}_{\Sigma\cap\mathcal{S}}}(\mathcal{X}_\mathcal{S},\tilde{X}_t)$ under the Gysin map, which obviously factors through $H^n(\mathcal{X}_\mathcal{S},X_t)$, splitting that part of the extension. Finally, if $\mathcal{S}=\mathbb{P}^1\setminus\{0\}$ then $\mathrm{IH}^1(\mathcal{S},\mathcal{H}_f^{n-1})\cong H^1(\mathcal{S})\otimes H_f^{n-1}=\{0\}$, and so $\mathrm{IH}^1(\mathcal{S},\mathcal{H}^{n-1})=\mathrm{IH}^1(\mathcal{S},\mathcal{H}_v^{n-1})$. □

Remark 4.16. The remarkable thing about $\mathcal{A}_\phi^\dagger = \mathcal{V}_\phi/\mathrm{Ph}^n_{\Sigma\setminus\{0\}}$ is that it elucidates what temperedness achieves. Comparing with (4.9), we see that it is built out of the three parts

$$\tfrac{W_{2n}}{W_{2n-2}}\mathcal{A}_\phi^\dagger \cong \mathrm{Gr}^W_{2n}\mathrm{IH}^1(\mathbb{P}^1\setminus\{0\},\mathcal{H}_v^{n-1})\cong\mathbb{Q}(-n) \tag{4.15}$$

$$\tfrac{W_{2n-2}}{W_{n-1}}\mathcal{A}_\phi^\dagger \cong W_{2n-2}\mathrm{IH}^1(\mathbb{P}^1\setminus\{0\},\mathcal{H}_v^{n-1})\cong \overline{H}^{n-1}(X_t^*)/W_{n-1} \tag{4.16}$$

$$\mathrm{Gr}^W_{n-1}\mathcal{A}_\phi^\dagger\cong\mathcal{H}_v^{n-1}. \tag{4.17}$$

Temperedness splits the extension of (4.15) by (4.16), which is a *constant* extension since it appears inside $\mathrm{IH}^1(\mathbb{P}^1\setminus\{0\},\mathcal{H}_v^{n-1})$.

Note that if $\sigma\in\Sigma\cap\mathcal{S}$, the same computation (together with[15] Clemens-Schmid) exhibits $H^n(\mathcal{X}_\mathcal{S},\tilde{X}_\sigma)$ as the direct sum of $\mathrm{Ph}^n_{\Sigma\cap\mathcal{S}\setminus\{\sigma\}}$ with an extension of $\mathrm{IH}^1(\mathcal{S},\mathcal{H}^{n-1})$ by $(\psi_\sigma\mathcal{H}_v^{n-1})^{T_\sigma}$. When $\mathcal{S}=\mathbb{P}^1\setminus\{0\}$ this yields the

Corollary 4.17. *The* MHS $\mathsf{A}_\phi^\dagger := H^n(\mathcal{X}\setminus\tilde{X}_0,\tilde{X}_\infty)/\mathrm{Ph}_{\Sigma^*}$ *is isomorphic to* $(\psi_\infty\mathcal{A}_\phi^\dagger)^{T_\infty}$, *hence computes* $\lim_\infty\nu_\phi$.

Dually, we may define

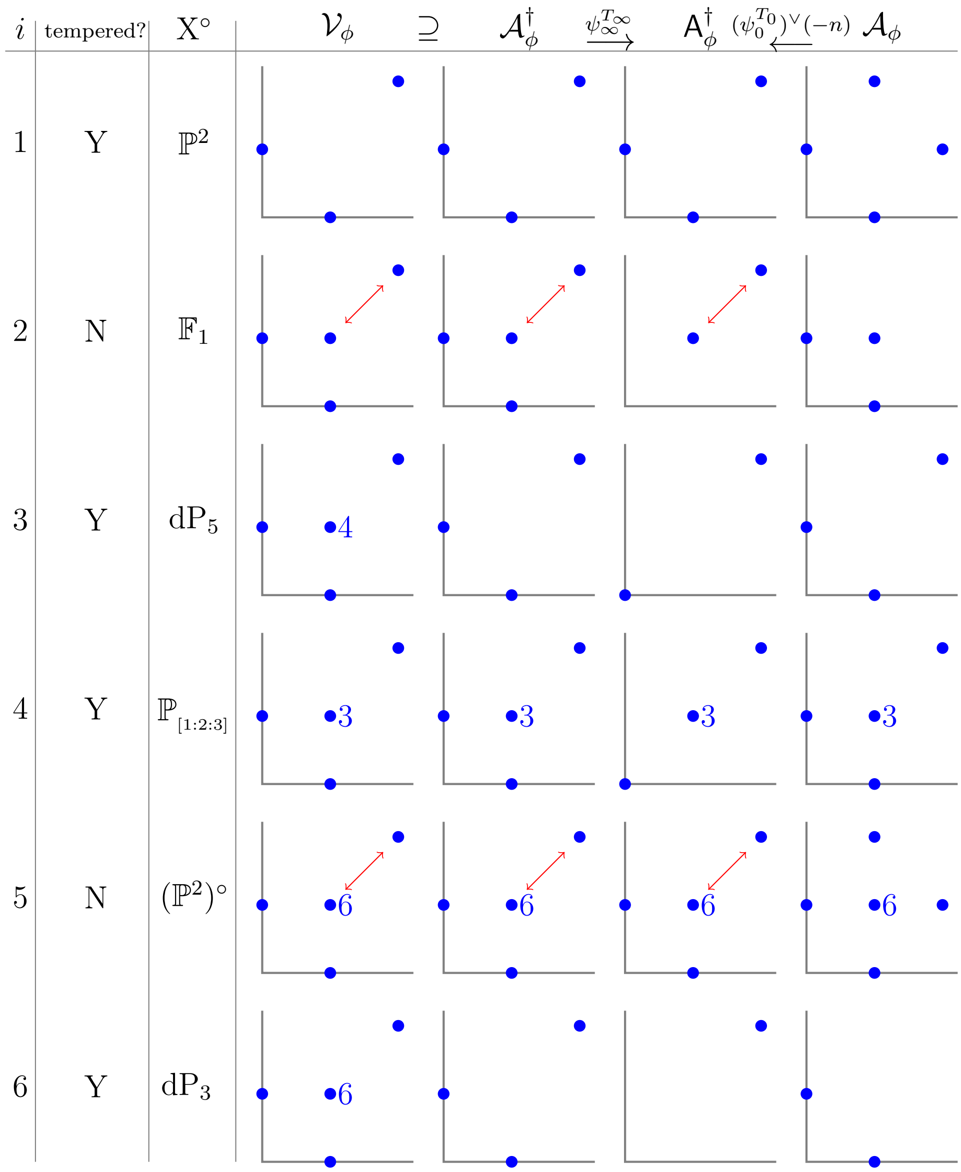
$$\mathsf{A}_\phi := H^n(\mathcal{X}\setminus\tilde{X}_\infty,\tilde{X}_0)/\mathrm{Ph}_{\Sigma^*}\cong(\mathsf{A}_\phi^\dagger)^\vee(-n), \tag{4.18}$$

which is itself obtained as the limit at 0 (more precisely, $(\psi_0\mathcal{A}_\phi)^{T_0}$) of

$$\mathcal{A}_{\phi,t} := H^n(\mathcal{X}\setminus\tilde{X}_\infty,\tilde{X}_t)/\mathrm{Ph}_{\Sigma\setminus\{\infty\}}. \tag{4.19}$$

[15] Here we need not assume unipotent monodromies; see [KL19].

Example 4.18. For the six $n = 2$ Laurent polynomials in Example 3.2, the Fano varieties[16] and VMHS Hodge-Deligne diagrams are:



The red arrows in cases (2) and (5) denote nontorsion extensions of (4.15) by (4.16), reflecting the nontemperedness. (These extensions record, in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2), \mathbb{Q}(-1)) \cong \mathbb{C}/\mathbb{Q}(-1)$, the logarithms of the toric boundary coordinates of the base locus $X_t \cap \mathbb{D}_\Delta$.) In the other cases, the limit $\mathsf{A}_\phi^\dagger$ contains only torsion extensions.[17] In *all* cases, we have $\mathrm{rk}(\mathrm{Gr}_F^k \mathcal{V}_\phi) = \dim(H^{k,k}(\mathrm{X}^\circ))$ in accordance with Conjecture 4.2.

[16] This is meant in the weak sense of Conjectures 3.3 and 4.2 only, although Conjecture 3.4 does in fact hold (modulo constant terms) for $\phi^{(i)}$ if $i = 1, 3, 4, 6$. (The nontempered examples were included for variety.)

[17] In case (1) (X_∞ smooth) this is by a computation in $K_2(X_\infty)$; in (3) and (4) (X_∞ singular), it is because $K_3^{\mathrm{ind}}(\mathbb{Q})$ is torsion. Later we will see how torsion extensions actually may lift to well-defined invariants in $\mathbb{C}$.

More striking is the disparity in form between $\mathcal{A}_\phi^\dagger$ and $\mathcal{A}_\phi$. While both share $\mathrm{Gr}_1^W \cong \mathcal{H}^1$ and $\mathrm{Gr}_2^W \cong \mathrm{IH}^1(\mathbb{P}^1, \mathcal{H}^1)$, we have $\frac{W_4}{W_2}\mathcal{A}_\phi^\dagger \cong \mathbb{Q}(-2)$ vs. $\frac{W_4}{W_2}\mathcal{A}_\phi \cong (\psi_\infty \mathcal{H}^1)_{T_\infty}(-1)$. Only in cases (3) and (4) does $\mathcal{A}_\phi$ yield a "K_2-type" normal function in $\mathrm{ANF}(\mathcal{H}^1(2))$, which for (3) is due to an involution of $\mathcal{X}$ over $t \mapsto -\frac{1}{t}$ [Ke17, §5.2]. However, we may regard (in (2), (4), and (5)) extensions of $\mathbb{Q}(-1) \subseteq \mathrm{IH}^1(\mathbb{P}^1, \mathcal{H}^1)$ by $\mathcal{H}^1$ as "K_0-type" normal functions, whose image in $\mathrm{ANF}(\mathcal{H}^1(1))$ generate the Mordell-Weil group ($\otimes\mathbb{Q}$) of π. That their limits at 0 capture part (or all, as in (2)) of $\lim_\infty \nu_\phi$ is essentially the fact that the limits in X_0 of Abel-Jacobi of differences of sections in $X_t \cap \mathbb{D}_\Delta$ are given by ratios of toric coordinates on $\mathbb{D}_\Delta$.

As mentioned in the Introduction, $\mathcal{A}_\phi^\dagger$ and $\mathcal{A}_\phi$ do not share the dual relationship (4.18) with their limits. Indeed, as we have just seen, $\mathcal{A}_\phi$ is not even an HNF in a canonical way (unlike $\mathcal{A}_\phi^\dagger$). However, for $n \geq 3$, it is precisely this lack of canonicity which makes $\mathcal{A}_\phi$ better adapted to exhibiting $\lim_\infty \nu_\phi$ in terms of limits of truncated HNFs.

5. The Conjecture and some Fano threefold examples

In this section we state a precise but restricted version of the Arithmetic Mirror Symmetry Conjecture (see §5.2), and then prove it when X° is one of the Mukai Fano threefolds V_{2N} ($5 \leq N \leq 9$) [Go09]. For each of these, [FANO] provides many LG-models of the form in Definition 3.1 – corresponding to the many possible toric degenerations $\mathbb{P}_{\Delta^\circ}$ of X° – satisfying Conjectures 3.3 and 3.4. They are found by taking ϕ to be (up to an additive constant) the Minkowski polynomial [CCGGK13] for the corresponding (reflexive) Δ, which is tempered in view of [dS19, Prop. 2.4].

Our job is then to exhibit $\mathcal{A}_{\phi,t}$ as a geometric HNF in the sense of (4.7), and the Apéry constant $\alpha_{\mathrm{X}^\circ}$ as the limit at $t = 0$ of the corresponding THNF, canonically normalized as described in §5.1. In contrast to $\mathcal{A}_{\phi,t}^\dagger$, this cannot arise from the lift of the coordinate symbol $\{x_1, x_2, x_3\}$ to $\mathrm{CH}^3(\tilde{X}_t, 3)$, since that HNF is singular at 0. Rather, we are looking for an extension

$$0 \to \mathcal{H}_v^2(p) \to \mathcal{A}_{\phi,t} \to \mathbb{Q}(0) \to 0 \tag{5.1}$$

arising from

$$\mathcal{Z} \mapsto \nu_{\mathcal{Z}} \colon \mathrm{CH}^p(\mathcal{X} \setminus \tilde{X}_\infty, r-1) \to \mathrm{ANF}(\mathcal{H}_v^{2p-r}(p))$$

with $2p - r = 2$, which forces $(p, r) = (3, 3)$ ($\mathcal{Z}_t$ belongs to the K_3^{alg} of the $K3$ fibers $\tilde{X}_t$) or $(2, 1)$ ($\mathcal{Z}_t$ lies in K_1 of the fibers).[18] It is these cycles $\mathcal{Z}$ which (in §§5.3-5.5) we will show how to construct in each case.

5.1. **The inhomogeneous equation of a normal function.** Given $\nu \in \mathrm{ANF}(\mathcal{H}_v^{n-1}(p))$, let $\tilde{\nu} := \nu_{\mathbb{Q}} - \nu_F$ be a multivalued holomorphic lift to $\mathcal{H}_v^{n-1}$. (Here v can be a higher or classical normal function, i.e. $p \geq \frac{n}{2}$.) We may generalize Definition 4.12 and (4.12) by setting

$$V(t) := \langle \tilde{\nu}(t), [\omega_t] \rangle$$

[18]Taking $p > 3$ yields $F^{-1}\mathcal{H}_v^2(p) = \{0\}$, making the extension class of (5.1) horizontal (by transversality) with rational monodromy (images under $T_\sigma - I$), hence trivial (since monodromy acts irreducibly on $\mathcal{H}_v^2$).

and $g(t) := LV(t) \in \mathbb{C}(t)$, which is zero iff ν is torsion [dAMS08].[19] (Note that since $\langle F^1, \omega\rangle = 0$ and $L\langle \mathbb{H}_v^{n-1}, \omega\rangle = 0$, g is independent of the choices of $\nu_{\mathbb{Q}}$ and ν_F.) In a special case, in which ν is singular at 0, we have a formula for $g(t)$ (Remark 4.13).

The next result summarizes what we can say more generally about this inhomogeneous term. It is motivated as follows. Suppose ν is nonsingular at 0 (Definition 4.6), so that the truncated NF has a power-series expanion $V(t) = \sum_{k\geq 0} v_k t^k$ there. If one knows L and can bound the degree of g (by some m), then we only need $\{v_k\}_{k=0}^m$ to compute g.

For its statement, we shall assume only that:

- $\{\tilde{X}_t\}$ is a family of CY $(n-1)$-folds over $\mathbb{P}^1$ (smooth off Σ);
- $\{\omega_t\}$ is a section of $\mathcal{H}_{v,e}^{n-1,0} \cong \mathcal{O}_{\mathbb{P}^1}(h)$, with divisor $h[\infty]$;
- $L = \sum_{j=0}^d t^j P_j(\delta_t) \in \mathbb{C}[t, \delta_t]$ is its PF operator, of degree d; and
- $\mathbb{H}_v^{n-1}$ has maximal unipotent monodromy at 0.

This is somewhat more general than the setting of the rest of this paper, which takes $\{\tilde{X}_t\}$ to arise from the level sets of a Laurent polynomial; in this case we have $h = 1$ (see [Ke20, Ex. 4.5]), and frequently only nodal singularities on the $\{\tilde{X}_\sigma\}_{\sigma\in\Sigma^*}$.

Theorem 5.1. *Assume ν is nonsingular away from 0 and ∞. Then g is a polynomial of degree $\leq d-h$. If ν is also nonsingular at 0, then $t \mid g$. If ν is also nonsingular at ∞ and T_∞ is unipotent, then* $\deg(g) \leq d-h-1$.

Proof. Let u be a local coordinate on a disk D_σ about $\sigma \in \Sigma$, and $\mathcal{H}_e$ resp. $\mathcal{H}^e$ the canonical resp. dual-canonical extensions of $\mathcal{H}_v^{n-1}|_{D_\sigma^*}$ to D_σ. (That is, the eigenvalues of ∇_{δ_u} are in $(-1, 0]$ resp. $[0, 1)$.) Assuming ν is nonsingular at σ, we may choose $\nu_{\mathbb{Q}}$ so that $N_\sigma \nu_{\mathbb{Q}} = 0$; thus $\tilde{\nu}$ is T_σ-invariant, and extends to a section of $\mathcal{H}^e$. Since ω is a section of $\mathcal{H}_e$, and $\langle\, ,\rangle$ extends to $\mathcal{H}^e \times \mathcal{H}_e \to \mathcal{O}$, $V = \langle \tilde{\nu}, \omega\rangle$ extends to a holomorphic function on D_σ. For $\sigma \in \Sigma^*$, we have $L \in \mathbb{C}[u, \partial_u]$ hence $g|_{D_\sigma}$ holomorphic. At $\sigma = 0$, maximal unipotency forces the indicial polynomial $P_0(T)$ to be divisible by T, so that L sends $\mathcal{O}(D_0) \to t\mathcal{O}(D_0)$ and $g(0) = 0$. If $\sigma = \infty$ and $u = t^{-1}$, our assumption that $(\omega) = h[\infty]$ gives $V|_{D_\infty} \in u^h \mathcal{O}(D_\infty)$; applying $L = \sum_{j=0}^d u^{-j} P_j(-\delta_u)$ yields $g|_{D_\infty} \in u^{h-d}\mathcal{O}(D_\infty)$.

We can refine the result at ∞, and deal with singularities at 0 and ∞, by writing $\tilde{\nu}$ and ω locally in terms of bases of the canonical extension. With σ, u as above, $\mathbb{H}_{v,\mathbb{C}}^{n-1} = \mathbb{H}_{\mathbb{C}}^{\mathrm{un}} \oplus \mathbb{H}_{\mathbb{C}}^{\mathrm{non}}$ decomposes into unipotent (T_σ^{ss}-invariant) and nonunipotent parts, with (multivalued) bases $\{\mathsf{e}_i\}$ and $\{\mathsf{e}_j^*\}$, the latter chosen so that $T_\sigma^{\mathrm{ss}} \mathsf{e}_j^* = \zeta_k^{a_j} \mathsf{e}_j^*$ ($\zeta_k := e^{\frac{2\pi \mathbf{i}}{k}}$). Writing $\ell(u) := \frac{\log(u)}{2\pi\mathbf{i}}$, a basis of $\mathcal{H}_e = \mathcal{H}_e^{\mathrm{un}} \oplus \mathcal{H}_e^{\mathrm{non}}$ is given by $\tilde{\mathsf{e}}_i := e^{-\ell(u)N_\sigma}\mathsf{e}_i$ and $\tilde{\mathsf{e}}_j^* := e^{-\ell(u)N_\sigma} u^{-\frac{a_j}{k}} \mathsf{e}_j^*$, which have the property that $\nabla_{\delta_u}\tilde{\mathsf{e}}_i, \nabla_{\delta_u}\tilde{\mathsf{e}}_j^* \in \mathcal{H}_e$. Admissibility says that the Hodge lift takes the form

$$\nu_F(u) = u\sum_i f_i(u)\tilde{\mathsf{e}}_i + u\sum_j f_j^*(u)\tilde{\mathsf{e}}_j^* + \tilde{\mathsf{e}}_{\mathbb{C}} \in \Gamma(D_\sigma, \mathcal{V}^e),$$

where $\mathsf{e}_{\mathbb{C}}$ is a $\mathbb{C}$-lift of 1 to $\mathbb{V}_{\mathbb{C}}^{\mathrm{un}}$ and $\tilde{\mathsf{e}}_{\mathbb{C}} := e^{-\ell(u)N_\sigma}\mathsf{e}_{\mathbb{C}}$. If ν is nonsingular at σ, then $\tilde{\mathsf{e}}_{\mathbb{C}} = \mathsf{e}_{\mathbb{C}}$ and $\nabla_{\delta_u}\tilde{\mathsf{e}}_{\mathbb{C}} = 0$; if it is singular at σ, then we may assume $\tilde{\mathsf{e}}_{\mathbb{C}} = \mathsf{e}_{\mathbb{C}} + \ell(u)\tilde{\mathsf{e}}_1$, so that $\nabla_{\delta_u}\tilde{\mathsf{e}}_{\mathbb{C}} \in \mathcal{H}_e^{\mathrm{un}}$. Write $\mathrm{ord}_\sigma(\omega) =: o$ (this is h if $\sigma = \infty$ and 0 if $\sigma = 0$).

[19] CHECK REF

Replacing $\tilde{\nu}$ by $\hat{\nu} := \mathsf{e}_{\mathbb{C}} - \nu_F$ changes it by a $\mathbb{C}$-period hence does not affect g. Writing $L = \sum_{k\geq 0} q_k^\sigma(u)\delta_u^k$, we have

$$g = L\langle \hat{\nu}, \omega\rangle = \sum_{k\geq 1}\sum_{j=1}^{k} q_k^\sigma(u)\langle \nabla_{\delta_u}^j \hat{\nu}, \nabla_{\delta_u}^{k-j}\omega\rangle$$

since $\nabla_L\omega = 0$. Clearly $\nabla_{\delta_u}^{k-j}\omega \in u^o\mathcal{H}_e$, while $\nabla_{\delta_u}^j\hat{\nu} \in u\mathcal{H}_e = u\mathcal{H}_{\mathrm{un}}^e \oplus \mathcal{H}_{\mathrm{non}}^e$ resp. $\mathcal{H}_e^{\mathrm{un}} \oplus \mathcal{H}_e^{\mathrm{non}} = \mathcal{H}^e$ for ν nonsingular resp. singular at σ. Hence $\langle \nabla_{\delta_u}^j\hat{\nu}, \nabla_{\delta_u}^{k-j}\omega\rangle$ belongs to $u^{o+1}\mathcal{O}(D_\sigma)$ if ν is nonsingular *and* T_σ is unipotent, and otherwise to $u^o\mathcal{O}(D_\sigma)$. For $\sigma = \infty$, multiplying by $q_k^\sigma(u)$ introduces u^{-d}. The result follows. □

Remark 5.2. Different choices of $\nu_{\mathbb{Q}}$ yield branches of V that differ by $\mathbb{Q}(p)$-periods $(2\pi\mathbf{i})^p\int_{\varphi_t}\omega_t$, $\varphi_t \in H_{n-1}(\tilde{X}_t, \mathbb{Q})$. If the $\{T_\sigma - I\}_{\sigma\in\Sigma^*}$ have rank one, and there are d of them (i.e. $\mathcal{H}_v^{n-1}$ has no "removable singularities"), and one $\sigma_0 \in \Sigma^*$ has greater modulus than the others, then we say that $\mathcal{X}$ is of *normal conifold type.* In this case V **can be chosen uniquely** by maximizing its radius of convergence; that is, there is a unique branch which is single-valued on the complement of the interval $[\sigma_0, \infty]$.

5.2. **The arithmetic mirror symmetry conjecture.** Rather than reiterating the general but vague version from the Introduction, we give a more precise variant in a restricted setting. Assume that our Fano variety and LG-model satisfy the following:

- $H^*_{\mathrm{prim}}(\mathrm{X}^\circ)$ and $(\psi_0\mathcal{H}_v^{n-1})^{T_0}$ are Hodge-Tate of rank r_0, with isomorphic associated gradeds (as predicted by Conjecture 4.2);
- $H^n_{\mathrm{prim}}(\mathrm{X}^\circ) = \{0\}$ (if n is even), and $\rho(\mathrm{X}^\circ) = 1$;
- $\mathcal{X}$ is of normal conifold type (Remark 5.2), and satisfies Conjecture 3.4;
- ϕ (and thus $\mathcal{X}$, and L) is defined over $\bar{\mathbb{Q}}$;
- $d = r_0 + 1$; and
- $P_0(d-1) \neq 0$.

Referring to §3.3, we write $b_j := u_j^{(d-1)}$ and $B(t) := \sum_{j\geq d-1} b_j t^j$, so that $\alpha_{\mathrm{X}^\circ}^{(d-1)} = \lim_{j\to\infty}\frac{b_j}{a_j}$ and $LB = P_0(d-1)t^{d-1}$.

Conjecture 5.3. (a) *The first $d-2$ Apéry constants $\{\alpha_{\mathrm{X}^\circ}^{(i)}\}_{i=1}^{d-2}$ are (up to $\bar{\mathbb{Q}}^*$-multiples) extension classes in $(\psi_0\mathcal{H}_v^{n-1})^{T_0}$ which are torsion (i.e. powers of $2\pi\mathbf{i}$) if ϕ is tempered.*

(b) *There is (up to scale) a unique HNF $\nu \in \mathrm{ANF}(\mathcal{H}_v^{n-1}(p))\setminus\{0\}$ singular only at $t = \infty$, for some (unique) $p \in [\frac{n+1}{2}, n] \cap \mathbb{Z}$. This HNF is motivic, i.e. arises from some $\mathcal{Z} \in \mathrm{CH}^p(\mathcal{X}\setminus\tilde{X}_\infty, 2p-n)_{\mathbb{Q}}$; and $LV = -\mathfrak{k}t^{d-1}$ for some*[20] $\mathfrak{k} \in \bar{\mathbb{Q}}^*$.

(c) *Normalize V uniquely as in Remark 5.2, and set $\hat{V}(t) := \frac{P_0(d-1)}{\mathfrak{k}}V(t)$. Then $\alpha_{\mathrm{X}^\circ}^{(d-1)} = \hat{V}(0) + \sum_{i=1}^{d-2}\beta_i\alpha_{\mathrm{X}^\circ}^{(i)}$, where $\beta_i \in \bar{\mathbb{Q}}\langle \hat{V}(0), \hat{V}'(0), \ldots, \hat{V}^{(i)}(0)\rangle$.*

Remark 5.4. Stated in this way, the thrust of the Arithmetic Mirror Symmetry Conjecture is somewhat obscured. What it really says us that given a Fano X° with H^*_{prim} as above, of rank one less than the degree of its quantum differential equation, there *exists* an LG-model

[20]More precisely, $\mathfrak{k}$ should belong to the common field of definition of $\mathcal{X}$ and $\mathcal{Z}$.

$\mathcal{X}$ (and cycle $\mathcal{Z}$) satisfying the remaining hypotheses together with the content of Conjecture 5.3.

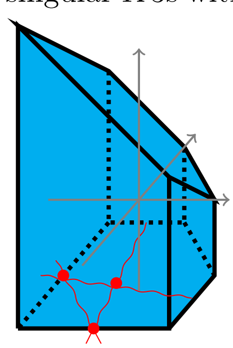
In the three subsections that follow, we check this conjecture in several cases with $n = 3 = r$ and $d = 2$. The situation simplifies, since $P_0(1) = 1$ and there is only one Apéry constant $\alpha_{\mathrm{X}^\circ}$; moreover, Theorem 5.1 guarantees that $LV = -\mathfrak{k}t$ for some $\mathfrak{k} \in \mathbb{C}^*$ once we have a HNF of the type described. So it remains to produce $\mathcal{Z}$ (hence ν), and show $\mathfrak{k} \in \bar{\mathbb{Q}}^*$ and $\alpha_{\mathrm{X}^\circ} = \hat{V}(0)$ in each case; we defer the uniqueness to §6.

Remark 5.5. A general result (for $n = r$ and $d = 2$) encompassing these cases appeared in [Ke20, Thm. 10.9], making essential use of Theorem 5.1 above. It reduces the Arithmetic Mirror Symmetry Conjecture to the existence of a "good" LG-model and the Beilinson-Hodge Conjecture; once the cycle is found, the equality $\alpha_{\mathrm{X}^\circ} = \hat{V}(0)$ is automatic. But we shall explicitly compute $\hat{V}(0)$ below in each case anyway, both to demonstrate concretely what the conjecture says, and to check that $\mathfrak{k} \neq 0$ and the cycle indeed produces a nontrivial HNF. Moreover, with [loc. cit.] in hand, one may regard these computations of $\hat{V}(0)$ as illustrations of a regulator calculus which is available for calculating the Apéry constant when modular and other methods (such as in [Go09]) are unavailable.

5.3. K_1 **of a** $K3$**: the** V_{10} **HNF.** The irrational Fano threefold $V_{10} = G(2,5) \cap \mathsf{Q} \cap \mathsf{P}_1 \cap \mathsf{P}_2$ (quadric and linear sections of the Plücker embedding) has a mirror LG model with discriminant locus $\Sigma = \{0, \sigma_+, \sigma_-, \infty\}$ (where $\sigma_\pm = \frac{-11 \pm 5\sqrt{5}}{4}$), given by the Laurent polynomial

$$\phi(\underline{x}) = \frac{(1-x_3)(1-x_1-x_3)(1-x_2-x_3)(1-x_1-x_2-x_3)}{-x_1x_2x_3}. \tag{5.2}$$

Namely, compactifying $\{1 = t\phi(\underline{x})\}$ in $\mathbb{P}_\Delta$ yields a family $\{X_t\}_{t \in \mathbb{P}^1}$, whose fibers over $\mathbb{P}^1 \backslash \Sigma$ are singular $K3$s with one A_3 and six A_1 singularities; these are resolved (to Picard-rank 19 $K3$s)



by $\beta\colon \tilde{X}_t \to X_t$. The Newton polytope Δ, together with a portion

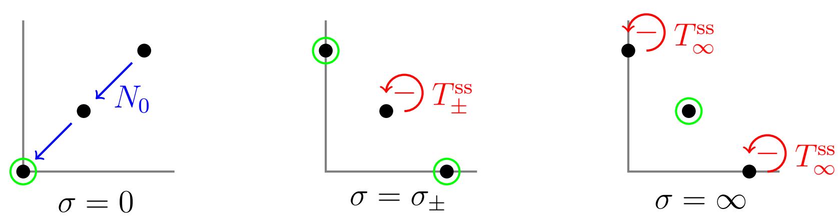
$$X_t \cap \{x_3 = 0\} = \{x_1 = 1\} \cup \{x_2 = 1\} \cup \{x_1 + x_2 = 1\} =: \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$$

of the base locus (red), are displayed in the figure. The PF operator and period sequence are given by

$$L = \delta_t^3 - 2t(2\delta_t + 1)(11\delta_t^2 + 11\delta_t + 3) - 4t^2(\delta_t + 1)(2\delta_t + 3)(2\delta_t + 1),$$

$$a_k := [\phi^k]_{\underline{0}} = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{k!2k!}{i!^2 j!^2 (k-i)!(k-j)!(k-i-j)!} = 1,\ 6,\ 114,\ \ldots;$$

while the monodromy operators $T_0, T_\pm, T_\infty$ have Jordan forms $J(3), (-\mathbf{1}) \oplus \mathbf{1}^2, (-\mathbf{1})^2 \oplus \mathbf{1}$ and LMHS types



where the T_σ-invariant classes are circled. The Apéry constant is $\alpha = \frac{1}{10}\zeta(2)$ [Go09].

To construct the cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{X} \setminus \tilde{X}_\infty, 1)$ we shall make use of the rational curves $\{\mathcal{C}_i\}$. On X_t, a higher Chow cycle is given by $(\mathcal{C}_1, g_1 := \frac{x_2}{x_2-1}) + (\mathcal{C}_2, g_2 := \frac{x_1-1}{x_1}) + (\mathcal{C}_3, g_3 := \frac{x_1}{x_1-1})$, since the sums of divisors cancel on X_t. To lift this to a cycle $\mathcal{Z}_t$ on $\tilde{X}_t$ (say, for $t \notin \Sigma$), one adds two more terms $(D_1, f_1) + (D_2, f_2)$ supported on the exceptional divisors over the nodes of X_t at $\mathcal{C}_1 \cap \mathcal{C}_3$ and $\mathcal{C}_2 \cap \mathcal{C}_3$. These $\{\mathcal{Z}_t\}$ are the restrictions of an obvious precycle $\mathcal{Z}$ on $\mathcal{X}$, whose boundary fails to vanish only on $\tilde{X}_\infty$.[21]

The next step is to find a family of closed 2-currents R_t on $\tilde{X}_t$ representing the class $\nu_{\mathcal{Z}}(t) \in J(H^2(\tilde{X}_t)(1))$, or more precisely a lift to $H^2(\tilde{X}_t, \mathbb{C})$ which is single-valued on $D_{|\sigma_-|}$. Writing $\mu := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 1,\ 1 - x_2 \leq x_1 \leq 1\}$, for $|t| \ll 1$ let Γ_t denote the branch of $\{(x_1, x_2, x_3) \in \tilde{X}_t \mid (x_1, x_2) \in \mu\}$ with x_3 small. Then we have[22] $R_t = (2\pi\mathbf{i})^2 \delta_{\Gamma_t} + 2\pi\mathbf{i} \sum_{i=1}^3 \log(g_i)\delta_{\mathcal{C}_i} + 2\pi\mathbf{i} \sum_{i=1}^2 \log(f_i)\delta_{D_i}$, which yields the THNF

$$V(t) = \langle [R_t], [\omega_t] \rangle = (2\pi\mathbf{i})^2 \int_{\Gamma_t} \omega_t = \tfrac{1}{2\pi\mathbf{i}} \int_\mu \int_{|x_3|=\epsilon} \tfrac{\mathrm{dlog}(\underline{x})}{1-t\phi} = \sum_{k\geq 0} t^k \int_\mu [\phi^k]_{x_3^0} \tfrac{dx_1}{x_1} \wedge \tfrac{dx_2}{x_2} =: \sum_{k \geq 0} v_k t^k.$$

(Here $[-]_{x_3^0}$ takes terms of the Laurent polynomial constant in x_3.) By Theorem 5.1, it suffices to compute

$$v_0 = \int_0^1 \int_{1-x_2}^1 \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} = -\int_0^1 \log(1-x_2) \frac{dx_2}{x_2} = \mathrm{Li}_2(1) = \zeta(2) \quad \text{and}$$

$$v_1 = \int_0^1 \int_{1-x_2}^1 \left\{ \begin{matrix} x_1^{-1}(4x_2^{-2} - 6 + 2x_2) + \\ (-6x_2^{-1} + 6 - x_2) + x_1(2x_2^{-1} - 1) \end{matrix} \right\} \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} = -10 + 6\zeta(2)$$

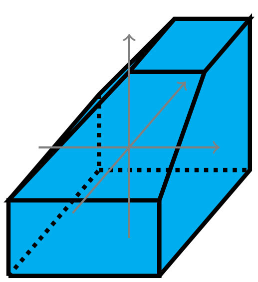
to conclude that $LV = -10t$. Normalization therefore yields

$$\hat{V}(t) = \tfrac{1}{10}\zeta(2) + (-1 + \tfrac{3}{5}\zeta(2))t + \cdots, \tag{5.3}$$

as desired.

5.4. K_3 **of a** $K3$**: the** V_{12} **HNF.** The LG mirror of the rational Fano $V_{12} = OG(5, 10) \cap \mathsf{P}_1 \cdots \cap \mathsf{P}_7$ is given by

$$\phi(\underline{x}) = \frac{(1-x_1)(1-x_2)(1-x_3)(1-x_1-x_2+x_1x_2-x_1x_2x_3)}{-x_1x_2x_3}. \tag{5.4}$$



This time the Picard-rank 19 $K3$s $\tilde{X}_t$, smooth for $t \notin \Sigma = \{0, \sigma_+, \sigma_-, \infty\}$ ($\sigma_\pm = (-1 \pm \sqrt{2})^4$), resolve 7 A_1 singularities on X_t. The family $\mathcal{X}$ is birational to that of [BP84] and underlies the proof of irrationality of $\zeta(3)$ [Ke17]; indeed, $\alpha = \frac{1}{6}\zeta(3)$ [Go09]. Its PF operator

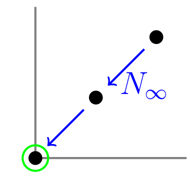
$$L = \delta_t^3 - t(2\delta_t + 1)(17\delta_t^2 + 17\delta_t + 5) + t^2(\delta_t + 1)^3$$

[21] The successive blowups along the components of the base locus occurring in the construction of $\mathcal{X}$ generate additional exceptional curves on $\tilde{X}_\infty$ which disconnect the 5-gon $D_1 \cup D_2 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, and it is on these that $\partial\mathcal{Z}$ is supported.

[22] In the context of regulator currents, $\log(-)$ means the single-valued branch with discontinuity along $\mathbb{R}_-$.

and (Apéry) period sequence

$$a_k := \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \binom{k+\ell}{\ell}^2 = 1,\ 5,\ 73,\ \ldots$$



$\sigma = \infty$

reflect a VHS with monodromies of the same types as in §5.3 except at $t=\infty$ (where we get maximal unipotent monodromy).

Since ϕ is tempered, the symbol $\{\underline{x}\}$ lifts to $\xi \in \mathrm{CH}^3(\mathcal{X}\setminus \tilde{X}_0, 3)$. The birational map $\mathcal{I}\colon (\underline{x},t)\mapsto \left(\frac{x_3}{1-x_3}, \frac{-(1-x_1)(1-x_2)}{1-x_1-x_2+x_1x_2-x_1x_2x_3}, \frac{x_1}{1-x_1}, \frac{1}{t}\right)$ from $\mathcal{X}$ to itself, viewed as a correspondence, allows us to define $\mathcal{Z} := \mathcal{I}^*\xi \in \mathrm{CH}^3(\mathcal{X}\setminus\tilde{X}_\infty, 3)$. The resulting THNF

$$V(t) = \langle \tilde{\nu}_{\mathcal{Z}}(t), \omega_t\rangle \overset{\mathcal{I}}{=} \langle \tilde{\nu}_\phi(t^{-1}), t^{-1}\omega_{t^{-1}}\rangle = \int_{X_{t^{-1}}} R_3(\underline{x})\wedge d\left[\tfrac{1}{(2\pi\mathbf{i})^3}\tfrac{\mathrm{dlog}(\underline{x})}{t-\phi(\underline{x})}\right] = \int d\left[\tfrac{R_3(\underline{x})}{(2\pi\mathbf{i})^3}\right]\wedge \tfrac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})-t}$$

$$= \int_{\mathbb{R}^3_-} \tfrac{\mathrm{dlog}(\underline{x})}{t-\phi(\underline{x})} = -\sum_{k\geq 0} t^k \int_{\mathbb{R}^3_-} \tfrac{\mathrm{dlog}(\underline{x})}{(\phi(\underline{x}))^{k+1}} = \sum_{k\geq 0} t^k \left(\int_{[0,1]^3} \tfrac{\prod_{i=1}^3 X_i^k(1-X_i)^k dX_i}{(1-X_3(1-X_1X_2))^{k+1}}\right) =: \sum_{k\geq 0} v_k t^k$$

has $v_0 = 2\zeta(3)$ and $v_1 = -12+10\zeta(3)$, which (again by Theorem 5.1) is enough to conclude that $LV = -12t$. But then normalization gives

$$\hat{V}(t) = \tfrac{1}{6}\zeta(3) + (-1+\tfrac{5}{6}\zeta(3))t + \cdots\,, \tag{5.5}$$

and in particular $\hat{V}(0) = \alpha$.

5.5. **HNFs for V_{14}, V_{16}, V_{18}.** Again, the LG models are families of Picard-rank 19 $K3$s. The irrational case $V_{14} = G(2,6)\cap \mathsf{P}_1\cap\cdots\cap\mathsf{P}_5$ is similar to V_{10}, with Laurent polynomial

$$\phi(\underline{x}) = \frac{(1-x_1-x_2-x_3)\{(1-x_2-x_3)(1-x_3)^2 - x_2(1-x_1-x_2-x_3)\}}{-x_1x_2x_3},$$

discriminant locus $\Sigma = \{0, \frac{1}{27}, -1, \infty\}$, and PF operator

$$L = \delta_t^3 - t(1+2\delta_t)(13\delta_t^2+13\delta_t+4) - 3t^2(\delta_t+1)(3\delta_t+4)(3\delta_t+2).$$

The monodromy types are the same as for V_{10}, except for T_∞, which acts on $(\psi_\infty\mathcal{H}^2_v)^{2,0}$ resp. $(\psi_\infty\mathcal{H}^2_v)^{0,2}$ by $e^{-\frac{2\pi\mathbf{i}}{3}}$ resp. $e^{\frac{2\pi\mathbf{i}}{3}}$.

The toric boundary divisor $x_1=0$ intersects X_t in $\mathcal{C}_1 = \{x_2 = 1-x_3\}$ and $\mathcal{C}_2 = \{x_2 = (1-x_3)^2\}$, and a cycle $\mathcal{Z}\in\mathrm{CH}^2(\mathcal{X}\setminus\tilde{X}_\infty, 1)$ is given by $(\mathcal{C}_1, \frac{x_3}{1-x_3}) + (\mathcal{C}_2, \frac{1-x_3}{x_3})$. Arguing as before, this yields

$$V(t) = \sum_{k\geq 0} v_k t^k = \sum_{k\geq 0} t^k \int_\mu [\phi^k]_{x_1^0} \tfrac{dx_2}{x_2}\wedge\tfrac{dx_3}{x_3}\,,$$

where $\mu := \{(x_1,x_2)\in\mathbb{R}^2 \mid 0\leq x_3\leq 1,\ (1-x_3)^2\leq x_2\leq 1-x_3\}$. We compute $v_0 = \zeta(2)$ and

$$v_1 = \int_0^1\int_{(1-x_3)^2}^{1-x_3} \left\{\begin{matrix} 2x_2x_3^{-1} + (-x_3+4-3x_3^{-1}) \\ +x_2^{-1}x_3^{-1}(x_3-1)^3 \end{matrix}\right\} \frac{dx_2}{x_2}\wedge\frac{dx_3}{x_3} = -7+4\zeta(2)\,,$$

hence that $LV = -7t$. Renormalizing this gives

$$\hat{V}(t) = \tfrac{1}{7}\zeta(2) + (-1+\tfrac{4}{7}\zeta(2))t + \cdots\,,$$

and $\hat{V}(0) = \frac{1}{7}\zeta(2)$ indeed matches the α from [Go09].

Turning to $V_{16} = LG(3,6) \cap \mathsf{P}_1 \cap \mathsf{P}_2 \cap \mathsf{P}_3$ and $V_{18} = (G_2/P_2) \cap \mathsf{P}_1 \cap \mathsf{P}_2$, we use

$$\phi(\underline{x}) = \tfrac{(1-x_1-x_2-x_3)(1-x_1)(1-x_2)(1-x_3)}{-x_1x_2x_3} \quad \text{resp.} \quad \tfrac{(x_1+x_2+x_3)(x_1+x_2+x_3-x_1x_2-x_2x_3-x_1x_3+x_1x_2x_3)}{-x_1x_2x_3}$$

from [dS19] for our LG models, with $\Sigma = \{0, 12 \pm 8\sqrt{2}, \infty\}$ resp. $\{0, 9 \pm 6\sqrt{3}, \infty\}$ and

$$L = \delta_t^3 - 4t(1+2\delta_t)(3\delta_t^2 + 3\delta_t + 1) + 16t^2(\delta_t+1)^3$$
$$\text{resp. } \delta_t^3 - 3t(1+2\delta_t)(3\delta_t^2 + 3\delta_t + 1) - 27t^2(\delta_t+1)^3.$$

The monodromy/LMHS types are the same as for V_{12}; we write $N = 6$, 8, or 9 for V_{2N}, and put $I(t) := \frac{1}{M_N t}$ with $M_N = 1, 16, -27$ respectively. In each case there is an isomorphism $\mathcal{H}_v^2 \cong I^*\mathcal{H}_v^2$ of $\mathbb{Q}$-VHS.[23] For $N = 8, 9$ this is not an integral isomorphism so is induced by correspondences $\mathcal{I}, \mathcal{I}^{-1} \in Z^2(\mathcal{X} \times I^*\mathcal{X})_{\mathbb{Q}}$ (with $\mathcal{I}^* \circ (\mathcal{I}^{-1})^* = \mathrm{id}_{\mathcal{H}_v^2}$) rather than a birational map; nevertheless, we may still define $\mathcal{Z} := \mathcal{I}^*\xi \in \mathrm{CH}^3(\mathcal{X}\backslash\tilde{X}_\infty, 3)$. Here we normalize $\mathcal{I}$ to pull back an integral generator ζ_s of $(\mathbb{H}_v^2)^{T_\infty}$ back to $\gamma_t \in (\mathbb{H}_v^2)^{T_0}$, where $s = I(t)$.

Since the integrals $\int_{\mathbb{R}^3_-} \frac{\mathrm{dlog}(\underline{x})}{(\phi(\underline{x}))^{k+1}}$ are quite difficult for $N = 8, 9$, we use a different strategy than in §5.4. As a section of $\mathcal{H}_{v,e}^{2,0} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, ω_t has divisor $[\infty]$, and so $(\mathcal{I}^{-1})^*\omega_{I(t)} = C_N t\omega_t$ for some $C_N \in \mathbb{C}^*$. Write $\zeta_s^\vee \in (\mathbb{H}_v^2)_{T_\infty}$ for the element dual to ζ_s, so that $\lim_{s\to\infty} s\omega_s = \frac{-1}{(2\pi\mathbf{i})^2}\mathrm{Res}_{\tilde{X}_\infty}(\frac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})}) = \mathfrak{r}_N \zeta_\infty^\vee$ in $H_2(\tilde{X}_\infty)$ where $\mathfrak{r}_N := \frac{-1}{(2\pi\mathbf{i})^3}\mathrm{Res}_p^3(\frac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})}) = 1, \frac{1}{2}$, resp. $\frac{1}{\sqrt{-3}}$ (for some triple-normal-crossing point $p \in \tilde{X}_\infty$). This yields

$$C_N = \lim_{t\to 0} \tfrac{1}{t}\langle \gamma_t, (\mathcal{I}^{-1})^*\omega_{I(t)}\rangle = \lim_{s\to\infty} M_N s\langle (\mathcal{I}^{-1})^*\gamma_{I(s)}, \omega_s\rangle = \lim_{s\to\infty} M_N\langle \zeta_s, s\omega_s\rangle = M_N\mathfrak{r}_N.$$

Write Λ for L with t replaced by s, we have $L = \frac{-1}{M_N s}\Lambda\frac{1}{s}$. Applying this to

$$V(t) = \langle \tilde{\nu}_{\mathcal{Z}}(t), \omega_t\rangle = \tfrac{1}{C_N t}\langle \mathcal{I}^*\tilde{\nu}_\phi(s), (\mathcal{I}^{-1})^*\omega_s\rangle = \tfrac{M_N s}{C_N}\langle \tilde{\nu}_\phi(s), \omega_s\rangle \tag{5.6}$$

yields $LV = \frac{-1}{C_N s}\Lambda\langle\tilde{\nu}_\phi(s), \omega_s\rangle = \frac{-D_N}{C_N s} = -\frac{D_N}{\mathfrak{r}_N}t$, where $D_N = 12$, 16, resp. 9 is the constant from Remark 4.13. Moreover, thinking of $\zeta_\infty^\vee$ as a "membrane stretched once around X_∞", taking $\lim_{s\to\infty}$ of (5.6) gives

$$V(0) = -\tfrac{1}{\mathfrak{r}_N}\int_{X_\infty} R_3(\underline{x}) \wedge \tfrac{1}{(2\pi\mathbf{i})^2}\mathrm{Res}_{X_\infty}\left(\tfrac{\mathrm{dlog}(\underline{x})}{\phi(\underline{x})}\right) = \int_{\zeta_\infty^\vee} R_3(\underline{x})|_{X_\infty} \tag{5.7}$$

for $\zeta_\infty^\vee$ in suitably general position;[24] and $\hat{V}(0) = \frac{\mathfrak{r}_N}{D_N}V(0)$.

We now use (5.7) to verify that $\hat{V}(0)$ recovers the Apéry constants in [Go09]. For $N = 6$, the computation in [Ke17, §5.3] (with $\zeta_\infty^\vee = -\psi$) gives $\int_{\zeta_\infty^\vee} R_3(\underline{x}) = 2\zeta(3)$, recovering $\hat{V}(0) = \frac{\zeta(3)}{6}$. For $N = 8$, putting $\zeta_\infty^\vee$ in general position is tricky so we use the first expression in (5.7). As $R_3(\underline{x}) = \log(x_1)\frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} + 2\pi\mathbf{i}\log(x_2)\frac{dx_3}{x_3}\delta_{T_{x_1}} + (2\pi\mathbf{i})^2\log(x_3)\delta_{T_{x_1}\cap T_{x_2}}$ is nontrivial

[23] This is easiest to see from the differential equation, but also follows from the fact that (for all five cases) the LG model of V_{2N} realizes the canonical weight-2 rank-3 VHS over $X_0(N)^{+N}$, which for N composite has an additional Fricke involution.

[24] Compare [Ke17, Thm. 4.2(b) + Cor. 4.3], which this generalizes.

only on the component $\{x_1 = 1 - x_2 - x_3\} \subset X_\infty$, with only its third term surviving against the $(2,0)$ residue form, this yields

$$V(0) = -2\int_{T_{1-x_2-x_3}\cap T_{x_2}} \log(x_3)\tfrac{dx_2\wedge dx_3}{(1-x_2)(1-x_3)(x_2+x_3)} = -2\int_1^\infty \tfrac{\log(x_3)}{1-x_3}\left(\int_{1-x_3}^0 \tfrac{dx_2}{(1-x_2)(x_2+x_3)}\right)dx_3$$

$$\underset{u=x_3^{-1}}{=} 4\int_0^1 \frac{\log^2(u)}{1-u^2}du = 4(\mathrm{Li}_3(1) - \mathrm{Li}_3(-1)) = 7\zeta(3),$$

hence $\hat{V}(0) = \frac{7}{32}\zeta(3)$.

Finally, for $N = 9$ we first replace $\{\underline{x}\}$ (hence ξ, and $\mathcal{Z}$) by the equivalent symbol $\{\underline{z}\}$, where $z_1 = \frac{-x_3}{x_1+x_2}$, $z_2 = -\frac{x_1}{x_2}$, and $z_3 = \frac{x_1x_2}{x_1+x_2}$. In these coordinates,

$$\phi(\underline{x}(\underline{z})) = z_1^{-1}z_3^{-1}(1-z_1)\{(1-z_3) - z_1(1-(1-z_2)z_3)(1-(1-z_2^{-1})z_3)\}$$

and so $X_\infty = X'_\infty \cup X''_\infty = \{z_1 = 1\} \cup \{z_1 = \varphi(z_2,z_3) := \frac{1-z_3}{(1-(1-z_2)z_3)(1-(1-z_2^{-1})z_3)}\}$, with $\mathcal{C}_\infty := X'_\infty \cap X''_\infty$ described by $z_3 = \mathfrak{z}(z_2) := \frac{1-z_2-z_2^{-1}}{(1-z_2)(1-z_2^{-1})}$. Clearly $R_3(\underline{x})|_{X'_\infty} = 0$. For $\zeta_\infty^\vee \cap X''_\infty$, which must bound on $\mathcal{C}_\infty$, we may take the 2-chain parametrized by $(z_2, z_3) = \{(-e^{-\mathbf{i}\theta}, \rho\mathfrak{z}(-e^{-\mathbf{i}\theta})) \mid \theta \in [-\frac{\pi}{3}, \frac{\pi}{3}],\ \rho \in [0,1]\}$. This yields

$$V(0) = \int_{\zeta_\infty^\vee\cap X''_\infty} \log(\varphi(z_2,z_3))\frac{dz_2}{z_2}\wedge\frac{dz_3}{z_3} = \int_{\partial(\zeta_\infty^\vee\cap X''_\infty)} \begin{pmatrix}\mathrm{Li}_2(\mathfrak{z}(z_2)) - \mathrm{Li}_2((1-z_2)\mathfrak{z}(z_2))\\ -\mathrm{Li}_2((1-z_2^{-1})\mathfrak{z}(z_2))\end{pmatrix}\frac{dz_2}{z_2}$$

$$= \int_{e^{-\frac{\pi\mathbf{i}}{3}}}^{e^{\frac{\pi\mathbf{i}}{3}}} (4\log(1-u) - \log(u))\log(u)\frac{du}{u} = \left[4\mathrm{Li}_3(u) - 4\mathrm{Li}_2(u)\log(u) + \tfrac{1}{3}\log^3(u)\right]_{e^{-\frac{\pi\mathbf{i}}{3}}}^{e^{\frac{\pi\mathbf{i}}{3}}} = \tfrac{4\pi^3\mathbf{i}}{27},$$

whereupon $\hat{V}(0) = \frac{V(0)}{9\sqrt{-3}} = \frac{4\pi^3}{3^5\sqrt{3}} = \frac{1}{3}L(\chi_3, 3)$.

6. Apéry and normal functions

In this brief final section, we introduce a framework for studying the normal functions arising in connection with the Arithmetic Mirror Symmetry Conjecture (including the examples in §5), and propose some terminology.

Definition 6.1. The *Apéry motive* is $\mathsf{A}_\phi := H^n(\mathcal{X}\setminus\tilde{X}_\infty, \tilde{X}_0)/\mathrm{Ph}_{\Sigma^*}$ from (4.18), or (if one prefers) its underlying mixed motive.

We dig into its structure a bit: there are exact sequences of MHS

(6.1)

$$\begin{array}{ccccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ & & & & (\psi_\infty\mathcal{H}_v^{n-1})_{T_\infty}(-1) & & \\ & & & & \uparrow & & \\ 0 \longrightarrow & (\psi_0\mathcal{H}_v^{n-1})^{T_0} & \longrightarrow & \mathsf{A}_\phi & \longrightarrow & \mathrm{IH}^1(\mathbb{A}^1, \mathcal{H}_v^{n-1}) & \longrightarrow 0 \\ & & & & \uparrow & & \\ & & & & \mathrm{IH}^1(\mathbb{P}^1, \mathcal{H}_v^{n-1}) & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where $(\cdot)_{T_\infty} = \mathrm{coker}(T_\infty - I)$, $(\cdot)^{T_0} = \ker(T_0 - I)$, and $\mathbb{A}^1$ means $\mathbb{P}^1 \setminus \{t = \infty\}$. The parabolic cohomology $\mathrm{IH}^1(\mathbb{P}^1, \mathcal{H}_v^{n-1})$ is pure of weight n and rank

$$ih^1(\mathbb{P}^1, \mathcal{H}_v^{n-1}) = \sum_{\sigma \in \Sigma} \mathrm{rk}(T_\sigma - I) - 2r. \tag{6.2}$$

Definition 6.2. $\mathcal{H}_v^{n-1}$ (or ϕ) is *extremal* if (6.2) is zero.

Recall that if ϕ is tempered, the coordinate symbol $\{\underline{x}\}$ lifts to $\xi \in \mathrm{CH}^n(\mathcal{X} \setminus \tilde{X}_0, n)$. The cycle class of $\mathrm{Res}(\xi) \in \mathrm{CH}^{n-1}(\tilde{X}_0, n-1)$ yields an embedding $\mathbb{Q}(-n) \hookrightarrow H^{n+1}_{\tilde{X}_0}(\mathcal{X}) \cong H_{n-1}(\tilde{X}_0)(-n)$, or dually[25] a splitting

$$\varepsilon \colon (\psi_0 \mathcal{H}_v^{n-1})^{T_0} \twoheadrightarrow \mathbb{Q}(0). \tag{6.3}$$

Suppose then that ϕ is tempered and extremal, and that (for some $p \in \mathbb{N}$) there exists an embedding

$$\mu \colon \mathbb{Q}(-p) \hookrightarrow (\psi_\infty \mathcal{H}_v^{n-1})_{T_\infty}(-1). \tag{6.4}$$

Then from (6.1) we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}(0) \arrow[r] \arrow[d, equal] & \mu^*\varepsilon_*\mathsf{A}_\phi \arrow[r] \arrow[d, hook] & \mathbb{Q}(-p) \arrow[r] \arrow[d, hook, "\mu"] & 0 \\
0 \arrow[r] & \mathbb{Q}(0) \arrow[r] & \varepsilon_*\mathsf{A}_\phi \arrow[r] & (\psi_\infty \mathcal{H}_v^{n-1})_{T_\infty}(-1) \arrow[r] & 0 \\
0 \arrow[r] & (\psi_0 \mathcal{H}_v^{n-1})^{T_0} \arrow[r] \arrow[u, two heads, "\varepsilon"] & \mathsf{A}_\phi \arrow[r] \arrow[u, two heads] & \mathrm{IH}^1(\mathbb{A}^1, \mathcal{H}_v^{n-1}) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{6.5}$$

with exact rows. Under $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-p), \mathbb{Q}(0)) \cong \mathbb{C}/\mathbb{Q}(p)$, define

$$\alpha_\phi(\mu) \in \mathbb{C}/\mathbb{Q}(p) \tag{6.6}$$

to be the image of the extension class of the top row of (6.5).

Example 6.3. The Laurent polynomials considered in §§5.3-5.5 are tempered and extremal, with $(\psi_\infty \mathcal{H}_v^2)_{T_\infty}(-1) \cong \mathbb{Q}(-2)$ resp. $\mathbb{Q}(-3)$ for V_{10}, V_{14} resp V_{12}, V_{16}, V_{18}. (Indeed, μ and ε are both isomorphisms.) In view of (6.10) below, in each case $\alpha_\phi(\mu)$ is just $V(0)$ viewed modulo $\mathbb{Q}(p)$. But for V_{10}, V_{14}, and V_{18}, $V(0)$ is *in* $\mathbb{Q}(p)$ and so $\alpha_\phi(\mu)$ is trivial!

From the example we see the importance of presenting (6.6) *as a limit of a HNF*, since by canonically normalizing the latter (Remark 5.2) we may then refine (6.6) to a well-defined complex number. To do this, note that the same proof as for Proposition 4.15 expresses the VMHS $\mathcal{A}_{\phi,t} := H^n(\mathcal{X} \setminus \tilde{X}_\infty, \tilde{X}_t)/\mathrm{Ph}_{\Sigma \setminus \{\infty\}}$ as an extension

$$0 \to \mathcal{H}_v^{n-1} \to \mathcal{A}_\phi \to \mathrm{IH}^1(\mathbb{A}^1, \mathcal{H}_v^{n-1}) \to 0\,. \tag{6.7}$$

So we arrive at this article's eponymous

[25] The first map in the portion $H_{n+1}(\tilde{X}_0)(-n) \to H^{n-1}(\tilde{X}_0) \to (\psi_0 \mathcal{H}_v^{n-1})^{T_0} \to 0$ of the Clemens-Schmid sequence has pure weight $n-1$, and so the second map has a splitting $(\psi_0 \mathcal{H}_v^{n-1})^{T_0} \hookrightarrow H^{n-1}(\tilde{X}_0)$ which is an isomorphism in weights $< n-1$. Dualizing the embedding yields $H^{n-1}(\tilde{X}_0) \twoheadrightarrow \mathbb{Q}(0)$, and (6.3) is the composition.

Definition 6.4. The pullback

$$0 \to \mathcal{H}_v^{n-1} \to \mu^*\mathcal{A}_\phi \to \mathbb{Q}(-p) \to 0 \tag{6.8}$$

of (6.7) under (6.4) is called an *Apéry extension.*

We may view (6.8) as a higher normal function

$$\nu_\mu \in \mathrm{ANF}(\mathcal{H}_v^{n-1}(p)) \tag{6.9}$$

which is singular at $t=\infty$ and only there,[26] and we define and normalize $V_\mu(t) := \langle \nu_{\mu,t}, \omega_t\rangle$ as in §5.1. Since $\mu^*\mathsf{A}_\phi \cong (\psi_0\mu^*\mathcal{A}_\phi)^{T_0}$ and ε is induced by pairing with $\lim_{t\to 0}\omega_t$, we conclude that

$$V_\mu(0) \mapsto \alpha_\phi(\mu) \quad \text{ under } \quad \mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Q}(p). \tag{6.10}$$

At least when ϕ is tempered and extremal, and $\mathrm{IH}^1(\mathbb{A}^1, \mathcal{H}_v^{n-1})$ is split Hodge-Tate, it is these $V_\mu(0)$ which are expected to produce (up to $\bar{\mathbb{Q}}^*$) the interesting Apéry constants. (In the absence of these conditions, of course, the situation will be more complicated.)

Conversely, any $\nu \in \mathrm{ANF}(\mathcal{H}_v^{n-1}(p))$ nonsingular off ∞ arises as in (6.8). In our fine examples, $\mathrm{IH}^1(\mathbb{A}^1, \mathcal{H}_v^2) \cong (\psi_\infty \mathcal{H}_v^2)_{T_\infty}(-1)$ has rank one. So this finishes off the uniqueness part of Conjecture 5.3 in each case, completing the proof of Theorem 1.1.

Now according to the BHC (since $\mathcal{X}\backslash\tilde{X}_\infty$ is defined over $\bar{\mathbb{Q}}$), there exists a higher cycle $\mathcal{Z}_\mu \in \mathrm{CH}^p(\mathcal{X}\backslash\tilde{X}_\infty, 2p-n)_\mathbb{Q}$ with $\nu_\mu = \nu_{\mathcal{Z}_\mu}$. This provides a mechanism for explaining the arithmetic content of the Apéry constants. Let $K \subset \bar{\mathbb{Q}}$ be the field of definition of $\mathcal{Z}_\mu$. By [7K, Cor. 5.3ff], $V_\mu(0)$ may be interpreted as the image of $\mathcal{Z}_0 := \imath^*_{\tilde{X}_0}\mathcal{Z}_\mu$ under

$$H^n_{\mathcal{M}}(\tilde{X}_{0,K}, \mathbb{Q}(p)) \overset{\mathrm{AJ}}{\to} J(H^{n-1}(\tilde{X}_0)(p)) \overset{\langle\cdot,\omega_0\rangle}{\twoheadrightarrow} \mathbb{C}/\mathbb{Q}(p)\,, \tag{6.11}$$

where the second map comes from temperedness of ϕ. Since $\tilde{X}_0 = \cup_i Y_i$ is a NCD, we have a spectral sequence $E_1^{a,b} = Z^p(\tilde{X}_0^{[a]}, 2p-n-b) \underset{a+b=*}{\Longrightarrow} H^{n+*}_{\mathcal{M}}(\tilde{X}_0, \mathbb{Q}(p))$ where $\tilde{X}_0^{[a]} := \amalg_{|I|=a+1}(\cap_{i\in I} Y_i)$. The induced filtration $\mathscr{W}_\bullet$ [op. cit., §3] has bottom piece

$$\begin{aligned}\mathscr{W}_{-n+1}H^n_{\mathcal{M}}(\tilde{X}_{0,K}, \mathbb{Q}(p)) &\cong \mathrm{coker}\{\mathrm{CH}^p(\tilde{X}_{0,K}^{[-n+2]}, 2p-1) \to \mathrm{CH}^p(\tilde{X}_{0,K}^{[-n+1]}, 2p-1)\} \\ &\cong \mathrm{CH}^p(\mathrm{Spec}(K), 2p-1)\,,\end{aligned}$$

and (6.11) restricts to the Borel regulator on this piece.

Example 6.5. In §§5.3-5.5, $\mathcal{Z}_0$ belongs to $\mathscr{W}_{-2}H^3_{\mathcal{M}}(\tilde{X}_{0,K}, \mathbb{Q}(p))$, with $K = \mathbb{Q}(\sqrt{-3})$ for V_{18} and $K = \mathbb{Q}$ for the other V_{2N}'s. Since each $\alpha_{\mathrm{X}^\circ}$ is also real by construction, and $\mathfrak{k}$ belongs to K, Borel's theorem (together with our conjecture) *forces* $\alpha_{\mathrm{X}^\circ}$ to be in $\mathbb{Q}(2)$ (V_{10}, V_{14}), $\zeta(3)\mathbb{Q}$ (V_{12}, V_{16}), and $\sqrt{-3}\mathbb{Q}(3)$ (V_{18}) respectively, before any computation is done.

Remark 6.6. We finally owe the reader an explanation regarding the flip in perspective from $\psi_\infty\mathcal{A}_\phi^\dagger$ (and the limit of the coordinate-symbol normal function at ∞) to $\psi_0\mathcal{A}_\phi$ (and the limit of Apéry normal functions at 0), specifically the "computational nonviability" claimed for the former. First of all, if we choose the lift $\tilde{\nu}_\phi$ to be single-valued around ∞, it is a section of the dual-canonical extension $(\mathcal{H}_v^{n-1})^e$; since ω is a section of $\mathcal{H}_{v,e}^{n-1}$ with a simple

[26]see the proof of [Ke20, Thm. 10.6].

zero at ∞, they do indeed pair to a holomorphic function V_ϕ on a disk D_∞, but one with $V_\phi(\infty) = 0$. Replacing ω by $\hat{\omega} := t\omega$ gives $\hat{V}_\phi := tV_\phi$, from which we can in principle read off the limiting extension class if we know the limits of the invariant periods of $\hat{\omega}$ at ∞ (which is already nontrivial); and this was essentially the method used for V_{16} and V_{18}.

But this approach becomes problematic when T_∞ is non-unipotent, intuitively because $\hat{\omega}$ then has periods which blow up at ∞, and we lack a suitable representative for $\hat{\omega}(\infty)$ on $\tilde{X}_\infty$. More precisely, if we let $\rho\colon D_\infty \to D_\infty$ be the base-change (ramified at ∞) which kills T_∞^{ss}, the pullback $\rho^*\omega$ is *not* a section of $(\rho^*\mathcal{H}_v^{n-1})_e$ over D_∞, and we cannot use [7K, Cor. 5.3] to compute $\hat{V}_\phi(\infty)$. So while, for (say) V_{10} and V_{14}, one can show (abstractly, from its inhomogeneous equation) that $\hat{V}_\phi(\infty)$ is a nonzero complex number, it does not seem nearly as accessible as the $V_\mu(0)$ values computed in §§5.3 and §§5.5.

Remark 6.7. In addition to its implications for the arithmetic of $\alpha_{\mathrm{X}^\circ}$, the Conjecture appears to produce interesting algebro-geometric predictions about Fanos. To just give the idea in the simplest possible case, suppose F^m is a Fano m-fold, with $H^*_{\mathrm{prim}}(\mathsf{F}^m)$ of rank two, concentrated in weights 0 and $2w$, with degree 2 QDE. Let P_i denote general hyperplanes in some $\mathbb{P}^M$ in which F^m is minimally embedded, and accept the idea that – as long as $\mathsf{F}^{m-\ell} := \mathsf{F}^m \cap \mathsf{P}_1 \cap \cdots \cap \mathsf{P}_\ell$ remains Fano – the Conjecture continues to hold and the Apéry constant α remains unchanged (cf. Remark 3.6). At first, $\alpha = \alpha_{\mathsf{F}^m}$ is computed by the LMHS of the LG model; but after hyperplane sections kill off the second Lefschetz string in H^*, $\alpha = \alpha_{\mathsf{F}^{m-\ell}}$ is computed by the limit of a nontorsion extension of $\mathcal{H}^{m-\ell-1}$ by $\mathbb{Q}(-w)$ (even if α is "torsion"). As soon as $F^{w-1}\mathcal{H}^{m-\ell-1} = \{0\}$, however, Griffiths transversality forces such normal functions to be flat and thus torsion. So for the Conjecture is to be consistent, $\mathsf{F}^{m-\ell}$ cannot be Fano for $m - \ell < w$; that is, the *index* $\mathtt{i}(\mathsf{F}^m)$ is $\leq m - w + 1$. (Recall that $\mathtt{i}$ is defined by $-K_F = \mathtt{i}\mathsf{h}$.) A quick perusal of examples in this paper suggests that this is sharp: $G(2,5)$, $OG(5,10)$, and $LG(3,6)$ (but not $G(2,6)$ or G_2/P_2) each have rank two H^*_{prim} and $d = 2$; while their respective $(m, w, \mathtt{i})$ are $(6,2,5)$, $(10,3,8)$, resp. $(6,3,4)$.

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