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On factorials which are products of factorials

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Abstract

In this paper, we look at the Diophantine equation

$$n! = \prod_{i=1}^{t} a_i! \qquad n > a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t \geqslant 2. \tag{0·1}$$

Under the ABC conjecture, we show that it has only finitely many nontrivial solutions. Unconditionally, we show that the set of n for which the above equation admits an integer solution $a_1, \ldots, a_t$ is of asymptotic density zero.

1. *Introduction*

For a positive integer n we write $P(n)$ for the largest prime factor of n. Erdős and Graham [**5**, p. 70] make the observation that if it were known that

$$\frac{P(n(n+1))}{\log n} \longrightarrow \infty \qquad \text{as } n \to \infty, \tag{1·1}$$

then the Diophantine equation (0·1) would have only finitely many nontrivial solutions (see also [**4**]). A trivial solution has $a_1 = n - 1$. Thus, $n = a_2! \cdots a_t!$. They also mention that Hickerson conjectures that $16! = 14!5!2!$ is the largest nontrivial solution of equation (0·1).

Not only that it is not known whether estimate (1·1) holds, but it does not even seem to follow from known conjectures. In fact, the classical ABC conjecture barely implies that $P(n(n+1)) \geqslant (1 + o(1)) \log n$ as $n \to \infty$, which is much weaker than estimate (1·1). However, here we show that while the ABC conjecture is not strong enough to imply estimate (1·1), it is sufficient to imply that equation (0·1) admits only finitely many nontrivial solutions.

We recall that the ABC conjecture, or variants of it, has been previously used to infer that polynomial-factorial Diophantine equations have only finitely many integer solutions. For example, M. Overholt [**10**] used a weak form of the ABC conjecture due to Szpiro to show that the equation

$$n^2 - 1 = m! \tag{1·2}$$

has only finitely many integer solutions (n, m), while Luca [**8**] showed, under the full ABC conjecture, that if $Q(X) \in \mathbf{Q}[X]$ is a polynomial with rational coefficients and degree $\geqslant 2$, then the Diophantine equation

$$Q(n) = m! \tag{1·3}$$

has only finitely many integer solutions (n, m) with m positive. Unconditionally, Berend and Osgood [**2**] showed that the set of positive integers m such that equation (1·3) has an integer solution n is of asymptotic density zero. We refer the reader to [**1**] for several related results and an extensive bibliography on such problems. Here we show, unconditionally, that a similar result holds for solutions of equation (0·1), namely that the set of positive integers n whose factorial admits a representation of the form (0·1) for some integers $a_1, \ldots, a_t$ is of asymptotic density zero, and we give an upper bound on the counting function of this set.

Throughout this paper, we use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their regular meanings. We recall that $A \ll B$ is equivalent to $B \gg A$ and to $A = O(B)$, and means that there exists a constant α such that $|A| \leqslant \alpha B$. The constants implied by these symbols are absolute. For a positive real number x we write $\log x$ for the maximum between the natural logarithm of x and 1. For a positive integer $k \geqslant 2$, we write $\log_k x$ for the kth iterate of the function log. Thus, $\log_k x \geqslant 1$ holds for all positive real numbers x and $\log_k x$ coincides with the kth iterate of the natural logarithm function once x is sufficiently large. We use $\alpha, \beta, \ldots$ for positive constants. For a subset $\mathcal{A}$ of the positive integers we write $\mathcal{A}(x)$ for the finite set $\mathcal{A} \bigcap [1, x]$.

2. *Results*

We let $(n, a_1, \ldots, a_t)$ be a solution of equation (0·1). We write $k := n - a_1$, put $m := a_1 + 1$ and $a := a_2$. Then equation (0·1) becomes

$$a! \cdots a_t! = m(m+1)\cdots(m+k-1) \quad \text{and} \quad m > a = a_2 \geqslant \cdots \geqslant a_t \geqslant 2. \tag{2·1}$$

Assuming that the solution of equation (0·1) is nontrivial, we may also assume that $k \geqslant 2$.

We will first look for an upper bound on k in terms of a. Since $m > a_2$, it also follows that $m > k$. Indeed, if $m \leqslant k$, then $m + k - 1 \geqslant 2m - 1$, and by Bertrand's postulate, there exists a prime among $m, m+1, \ldots, m+k-1$, which cannot divide the left-hand side of equation (2·1) because $m > a$.

We may assume that $a \to \infty$, since if a remains bounded, then $P(m(m+1))$ remains bounded, and it is known that this can happen in only finitely many instances. Thus, this would imply that m is bounded, and since $m > k$, we get that n is also bounded.

As we have said in the Introduction, Erdős showed in [**4**], and Erdős and Graham mentioned it in [**5**], that limit (1·1) would imply the finiteness of the number of nontrivial solutions to equation (2·1). We record this as follows.

LEMMA 1. *Let $f : \mathbf{N} \to \mathbf{R}_+$ be any function that tends to infinity when n tends to infinity. Then, there are only finitely many nontrivial solutions of equation* (2·1) *with $a > f(n)\log n$.*

From now on, we assume that $a \leqslant f(n)\log n$, where f will be chosen later. We now note trivially that $k \leqslant 2a$. Indeed, if $k > 2a$, then, again by Bertrand's postulate, there exists a prime number in $(a, 2a)$. This prime, let's call it p, divides $k!$; hence, it also divides the right-hand side of equation (2·1), but obviously not the left.

A better upper bound on k in terms of a can be easily obtained from a result from [**7**]. Indeed, Laishram and Shorey showed in [**7**] that the product

$$m(m+1)\cdots(m+k-1) \quad \text{for} \quad m > k$$

is divisible by at least $\pi(k) + \lfloor 3\pi(k)/2 \rfloor - 1$ primes with finitely many exceptions for the pair (m, k). This implies for our nontrivial solutions to equation (2·1) that

$$\pi(a)(1 + o(1)) \geqslant \frac{5\pi(k)}{2} \tag{2·2}$$

as $a \to \infty$. In what follows, we prove a better upper bound on k in terms of a than inequality (2·2).

LEMMA 2. *The estimate*

$$k \ll \frac{\pi(a) \log_3 n}{\log_2 n}.$$

holds.

Proof. We may assume that $k > 1$ and that $a \to \infty$. Inequality (2·2) shows that $k \ll a \ll f(n) \log n$. For each prime number $p \leqslant k$, let $i_p \in \{0, \ldots, k-1\}$ be such that the exponent of p in the factorisation of $m + i_p$ is maximal. Let $\mathcal{S} = \{0, 1, \ldots, k-1\} \backslash \{i_p : p \leqslant k\}$. An elementary argument of Erdős (see, for example, [**3**, lemma 2]) shows that if we write

$$m + i = b_i c_i,$$

where $P(b_i) \leqslant k$ and c_i is free of primes $p \leqslant k$, then

$$\prod_{i \in \mathcal{S}} b_i | k!$$

Let $\mathcal{T} = \{i \in \mathcal{S} : b_i > k^2\}$. The above inequality together with the trivial inequality $k! < k^k$ shows that

$$\#\mathcal{T} < k/2.$$

Thus,

$$\ell := \#(\mathcal{S} \backslash \mathcal{T}) \geqslant \frac{k}{2} - \pi(k) \geqslant \frac{k}{3},$$

provided that $k \geqslant k_0$ is sufficiently large. From now on, we assume that $k \geqslant k_0$.

For a positive real number y and a positive integer n we write $\omega_y(n)$ for the number of distinct prime factors $p > y$ of n. We put

$$\omega(i) := \omega_{>k}(m + i), \quad \text{for} \quad i \in \mathcal{S} \backslash \mathcal{T},$$

and we assume that

$$\omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_\ell,$$

are all the values of $\omega(i)$ for $i \in \mathcal{S} \backslash \mathcal{T}$ ordered in nondecreasing order. Let j_1 and j_2 be such that $\omega(j_1) = \omega_1$ and $\omega(j_2) = \omega_2$. We look at the equation

$$(m + j_1) - (m + j_2) = (j_1 - j_2).$$

This can be rewritten as

$$\left|1 - b_{j_2} c_{j_2} b_{j_1}^{-1} c_{j_1}^{-1}\right| = \frac{|j_1 - j_2|}{m + j_1} \ll \frac{k}{n} = \exp(-(1 + o(1)) \log n).$$

We now use the facts that $\max\{b_{j_1}, b_{j_2}\} \leqslant k^2$, that $P(c_{j_1} c_{j_2}) \leqslant a \leqslant f(n) \log n$ and a linear form in logarithms due to Matveev [**9**], to infer that the inequality

$$\left|1 - b_{j_2} c_{j_2} b_{j_1}^{-1} c_{j_1}^{-1}\right| \geqslant \exp\left(-(\alpha \log(f(n) \log n))^{2\omega_2} \log(k^2)\right),$$

holds, where α is an absolute constant. Setting $f(n) = \log n$, we immediately get that

$$(1 + o(1)) \log n \leqslant 4(2\alpha \log_2 n)^{2\omega_2} \log_2 n,$$

which leads to

$$\omega_2 \gg \frac{\log_2 n}{\log_3 n}.$$

On the one hand, the expression

$$m(m+1)\cdots(m+k-1)$$

has at most $\pi(a)$ prime factors. On the other hand, it has at least

$$(\ell - 1)\omega_2 \gg \frac{k \log_2 n}{\log_3 n}$$

prime factors $> k$. This leads to the inequality

$$k \ll \frac{\pi(a) \log_3 n}{\log_2 n},$$

which completes the proof of Lemma 2.

In particular, taking $f(n) = \log_3 n$, and using Lemma 1, we get

$$k \ll \frac{\log n (\log_3 n)^2}{(\log_2 n)^2}. \tag{2·3}$$

We now recall the statement of the ABC conjecture. For any nonzero integer n we write $N(n) = \prod_{p|n} p$ for the *algebraic radical* of n.

CONJECTURE 1. *For every $\varepsilon > 0$, there exists a constant $\beta := \beta(\varepsilon)$ depending only on ε such that whenever A, B and C are three coprime nonzero integers with $A + B = C$, then the inequality*

$$\max\{|A|, |B|, |C|\} < \beta N(ABC)^{1+\varepsilon} \tag{2·4}$$

holds.

THEOREM 1. *Assume that Conjecture* 1 *holds. Then equation* (0·1) *has only finitely many nontrivial solutions.*

Proof. It is clear that if a prime p divides two of the numbers $m+i$ for $i \in \{0, \ldots, k-1\}$, then $p \leqslant k$. A simple counting argument shows that

$$\prod_{i=0}^{k-1} N(m+i) \leqslant \left(\prod_{p \leqslant a} p\right) \prod_{p \leqslant k} p^{\lfloor k/p \rfloor}.$$

Hence,

$$\prod_{i=0}^{k-1} N(m+i) \leqslant \exp\left(\sum_{p \leqslant a} \log p + k \sum_{p \leqslant k} \frac{\log p}{p}\right)$$

$$\leqslant \exp(O(a + k \log k)).$$

Let j_1 and j_2 be the two indices such that $N(m + j_1) \leqslant N(m + j_2)$ are the smallest two among $\{N(m+i) : i = 0, \ldots, k-1\}$. Then

$$N(m + j_2) \leqslant \left(\prod_{i=0}^{k-1} N(m+i)\right)^{1/k-1} = \exp\left(O\left(\frac{a}{k} + \log k\right)\right).$$

We now write $d = \gcd(j_1 - j_2, m + j_1)$, and apply the ABC conjecture with $\varepsilon = 1$ to the equation

$$\frac{m + j_1}{d} - \frac{m + j_2}{d} = \frac{j_1 - j_2}{d},$$

to get that

$$\frac{m}{d} \ll \left(N(m + j_1)N(m + j_2)\frac{|j_1 - j_2|}{d}\right)^2 \ll \frac{1}{d}\exp\left(O\left(\frac{a}{k} + \log k\right)\right).$$

Since $m \geqslant n/2$, we get that

$$\log n \ll \frac{a}{k} + \log k.$$

Since obviously $\log k < \log_2 n$ holds for large n (see inequality (2·3)), the above estimate leads to

$$a \gg k \log n. \tag{2·5}$$

On the other hand,

$$a \log a = (1 + o(1)) \log(a!) \leqslant k \log n,$$

leading to $a \log a \ll k \log n$, which together with inequality (2·5) gives $a \ll 1$, and completes the proof of Theorem 1.

Remark 1. A close analysis of our argument shows that Theorem 1 remains valid even with a weaker ABC conjecture, namely a conjecture asserting that inequality (2·4) holds with some constants $\varepsilon > 0$ and $\beta > 0$.

We now write $\mathcal{A}$ for the set of all positive integers n whose factorials admit a representation of the form (0·1) for some integers $a_1 \geqslant \cdots \geqslant a_t \geqslant 2$. Clearly,

$$\mathcal{A} = \mathcal{N} \cup \mathcal{T},$$

where $\mathcal{N}$ and $\mathcal{T}$ are the subsets of $\mathcal{A}$ formed by those n which contribute to a *nontrivial* or a *trivial* solution of equation (0·1), respectively. Note that since $16! = 14!5!2!$ and also $16! = 15!2!2!2!2!$, it follows that $\mathcal{N}$ and $\mathcal{T}$ are not disjoint. Theorem 1 shows that $\#\mathcal{N} = O(1)$ under the ABC conjecture. In what follows, we establish unconditional bounds on $\#\mathcal{N}(x)$ and $\#\mathcal{T}(x)$ as $x \to \infty$. We start with the following result.

THEOREM 2. *If $f : \mathbf{R}_+ \to \mathbf{R}_+$ is any function tending to infinity with x, then the estimate*

$$\#\mathcal{N}(x) \leqslant \exp\left(\frac{f(x)\log x}{\log_2 x}\right) \tag{2·6}$$

holds as $x \to \infty$.

Proof. Assume that $n \leqslant x$. If $k \geqslant 2$, then $P(n(n-1)) \leqslant a \leqslant f(x)\log x$ holds with $O(1)$ exceptions. With $u = n$ and $v = n - 1$, we are led to a solution of the equation $u - v = 1$ in positive integers $u,\ v$ whose prime factors belong to the finite set $\mathcal{P} = \{p : p \leqslant f(x)\log x\}$. By known results about $\mathcal{S}$-unit equations (see [**6**], for example), we get that the number of solutions of such an equation is

$$\leqslant \exp\left(O\left(\#\mathcal{P}\right)\right) = \exp\left(O\left(\pi\left(f(x)\log x\right)\right)\right) = \exp\left(O\left(\frac{f(x)\log x}{\log_2 x}\right)\right). \tag{2·7}$$

The fact that we can remove the symbol O from the final inequality (2·7) follows from the fact that f is arbitrary.

We shall later improve Theorem 2 without appealing to results from the theory of $\mathcal{S}$-unit equations. For the moment, we look at the size of $\mathcal{T}(x)$. Note that Theorem 2 with $f(x) = \log_3 x$ together with Theorem 3 below show that $\mathcal{A}$ is of asymptotic density zero, as mentioned in the Abstract and the Introduction.

THEOREM 3. *The inequalities*

$$\frac{\sqrt{\log x}}{\log_2 x} \ll \log\left(\#\mathcal{T}(x)\right) \ll \frac{\sqrt{\log x}\log_3 x}{\log_2 x} \tag{2·8}$$

hold as $x \to \infty$.

Proof. We start with the upper bound. Using the fact that $\log(a!) \gg a\log a$, it follows that in order to give an upper bound on $\#\mathcal{T}(x)$, it suffices to give an upper bound on the number of products of the form

$$n = \prod_{i=1}^{s}(b_i!)^{j_i}, \quad \text{where} \quad 2 < b_1 < \cdots < b_s,$$

$j_1, \ldots, j_s$ are positive integers, and

$$\sum_{i=1}^{s} j_i b_i \log b_i \ll \log x. \tag{2·9}$$

We let α be the constant implied in the above inequality (2·9). Inequality (2·9) implies that

$$b_s < 2\alpha\frac{\log_2 x}{\log_3 x}$$

holds when x is sufficiently large. We put $y = \sqrt{\log x}$ and split the range of the variables b_i in four, as follows:

$$\begin{aligned}
\mathcal{B}_1 &= \left\{b : y\log_2 x < b < 2\alpha\frac{\log x}{\log_2 x}\right\},\\
\mathcal{B}_2 &= \{b : y < b \leqslant y\log_2 x\},\\
\mathcal{B}_3 &= \left\{b : \frac{y}{\log_2 x} < b \leqslant y\right\},\\
\mathcal{B}_4 &= \left\{b : 2 \leqslant b \leqslant \frac{y}{\log_2 x}\right\}
\end{aligned}$$

and we put

$$B_\ell := \prod_{i:b_i\in\mathcal{B}_\ell}(b_i!)^{j_i} \quad \text{for} \quad \ell = 1,\ 2,\ 3,\ 4.$$

Here, and in what follows, we use the notation $\{i : b_i \in \mathcal{B}_\ell\}$ to mean that the range is restricted to those i for which the corresponding b_i belongs to $\mathcal{B}_\ell$. We now estimate the number of values that each B_i can assume. If $i = 1$, then

$$\sum_{i:b_i\in\mathcal{B}_1} j_i b_i \log b_i \gg \left(\sum_{i:b_i\in\mathcal{B}_1} j_i\right) y(\log_2 x)^2,$$

therefore inequality (2·9) implies that the inequality

$$\sum_{i:b_i\in\mathcal{B}_1} j_i < \beta \frac{y}{(\log_2 x)^2}$$

holds for large x with some constant β. Thus,

$$\#\mathcal{B}_1 \leqslant \sum_{s\leqslant \beta \frac{y}{(\log_2 x)^2}} \binom{\left\lfloor 2\alpha \frac{\log x}{\log_2 x}\right\rfloor + s - 1}{s} = \exp\left(O\left(\frac{y}{\log_2 x}\right)\right). \tag{2·10}$$

If $i = 2$, then

$$\sum_{i:b_i\in\mathcal{B}_2} j_i b_i \log b_i \gg \left(\sum_{i:b_i\in\mathcal{B}_2} j_i\right) y \log_2 x,$$

therefore inequality (2·9) implies that the inequality

$$\sum_{i:b_i\in\mathcal{B}_2} j_i < \gamma \frac{y}{\log_2 x}$$

holds for large x with some constant γ. Thus,

$$\#\mathcal{B}_2 \leqslant \sum_{s\leqslant \gamma \frac{y}{\log_2 x}} \binom{\lfloor y \log_2 x\rfloor + s - 1}{s} = \exp\left(O\left(\frac{y \log_3 x}{\log_2 x}\right)\right). \tag{2·11}$$

If $i = 3$, then

$$\sum_{i:b_i\in\mathcal{B}_3} j_i b_i \log b_i \gg \left(\sum_{i:b_i\in\mathcal{B}_3} j_i\right) y,$$

therefore inequality (2·9) implies that the inequality

$$\sum_{i:b_i\in\mathcal{B}_3} j_i < \delta y$$

holds for large x with some constant δ. We now write

$$B_3 = B_3' B_3'',$$

where $P(B_3') \leqslant y/\log_2 x$ and B_3'' is free of primes $\leqslant y/\log_3 x$. Note that

$$B_3'' = \prod_{\frac{y}{\log_2 x} < p \leqslant y} p^{c_p},$$

where

$$c_p = \sum_{i:b_i\in\mathcal{B}_3} j_i \left\lfloor \frac{b_i}{p} \right\rfloor.$$

It now follows immediately that

$$c_p \leqslant \left(\sum_{i: b_i \in \mathcal{B}_3} j_i \right) \log_2 x \leqslant \delta y \log_2 x.$$

Hence, if we write $\mathcal{B}_3''$ for the set of all possible values of the numbers B_3'', we then have

$$\#\mathcal{B}_3'' \leqslant \sum_{s \leqslant \delta y \log_2 x} \binom{\pi(y) + s - 1}{s} = \exp\left(O\left(\frac{y \log_3 x}{\log_2 x} \right) \right). \tag{2·12}$$

Finally, $B_4 B_3'$ is just a $y/\log_2 x$-smooth number of size $\leqslant x$, i.e., it is of the form

$$\prod_{p \leqslant \frac{y}{\log_2 x}} p^{d_p},$$

where $d_p \leqslant \log_2 x$. Hence, writing $z = y/\log_2 x$, we get that the number of such numbers is at most

$$\Psi(x, z) \leqslant \left(\frac{\log x}{\log 2} + 1 \right)^{\pi(y)} = \exp\left(O\left(\frac{y}{\log_2 x} \right) \right). \tag{2·13}$$

From estimates (2·10), (2·11), (2·12) and (2·13), we get that

$$\#\mathcal{T}(x) \ll (\#\mathcal{B}_1(x)) \cdot (\#\mathcal{B}_2(x)) \cdot \left(\#\mathcal{B}_3''(x)\right) \cdot \Psi(x, z) \leqslant \exp\left(O\left(\frac{y \log_3 x}{\log_2 x} \right) \right),$$

which completes the proof of the upper bound in (2·8).

For the lower bound, we let $\mathcal{P}$ be any set of primes in the interval $(y/2, y)$. The number of such sets is

$$\begin{aligned} \#\{\mathcal{P} : \mathcal{P} \text{ set of primes in } (y/2, y)\} &= 2^{\pi(y) - \pi(y/2)} \\ &= \exp\left(\log 2 (1 + o(1)) \frac{y}{\log_2 x} \right). \end{aligned} \tag{2·14}$$

To each such set $\mathcal{P}$, we associate the positive integer

$$n_{\mathcal{P}} = \prod_{p \in \mathcal{P}} p!.$$

Using unique factorisation, it is clear that the numbers $n_{\mathcal{P}}$ are distinct for distinct values of $\mathcal{P}$, and certainly

$$\log n_{\mathcal{P}} \leqslant \sum_{y/2 \leqslant p \leqslant y} p \log p \leqslant \frac{1}{2} (\pi(y) - \pi(y/2)) y \log_2 x = \frac{1}{2}(1 + o(1)) y^2 < \log x,$$

when x is large. This shows that when x is large, $\mathcal{T}(x)$ contains all the numbers $n_{\mathcal{P}}$ for such subsets $\mathcal{P}$, and the desired lower bound from (2·8) now follows from inequality (2·14).

Finally, we improve the result of our Theorem 2 to the following.

THEOREM 4. *The estimate*

$$\#\mathcal{A}(x) = \exp\left(O\left(\frac{\log x (\log_3 x)^2}{(\log_2 x)^2} \right) \right) \tag{2·15}$$

holds as $x \to \infty$.

Proof. Let x be a large real number. By estimate (2·8), it suffices to prove the inequality (2·15) with $\mathcal{A}(x)$ replaced by $\mathcal{N}(x)$. Let $n = m + k - 1 \in \mathcal{N}(x)$ be part of a solution of (2·1) with some $k \geqslant 2$. By inequality (2·3), we have that

$$k \ll \frac{\log x(\log_3 x)^2}{(\log_2 x)^2}. \tag{2·16}$$

Let α be the constant implied by the above $\ll$. We then have,

$$\log(n(n-1)\cdots(n-k+1)) \leqslant \alpha \left(\frac{\log x \log_3 x}{\log_2 x}\right)^2. \tag{2·17}$$

Let

$$y := \exp\left(\alpha \left(\frac{\log x \log_3 x}{\log_2 x}\right)^2\right).$$

Then $n(n-1)\cdots(n-k+1)$ is an element of $\mathcal{T}(y)$, and by estimate (2·8) the number of possible values is

$$\#\mathcal{T}(y) = \exp\left(O\left(\frac{\sqrt{\log y}\log_3 y}{\log_2 y}\right)\right) = \exp\left(O\left(\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right)\right).$$

Given $N \in \mathcal{T}(y)$ and k satisfying inequality (2·16), the equation

$$n(n-1)\cdots(n-k+1) = N$$

has at most k solutions because it is a polynomial equation of degree k in n. Thus,

$$\#\mathcal{T}(x) \leqslant \left(\sum_{k \leqslant \alpha \frac{\log x(\log_3 x)^2}{\log_2 x)^2}} k\right) \#\mathcal{T}(y) = \exp\left(O\left(\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right)\right),$$

which completes the proof of Theorem 4.

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