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Sums of Powers of Binomials, Their Apéry Limits, and Franel’s Suspicions

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We explicitly determine the Apéry limits for the sums of powers of binomial coefficients. As an application, we prove a weak version of Franel’s conjecture on the order of the recurrences for these sequences. Namely, we prove the conjectured minimal order under the assumption that such a recurrence can be obtained via creative telescoping.

1 Introduction

More than a century ago, Franel [10, 11] investigated the sums of integral powers of binomial coefficients

$$A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^{s}. \tag{1}$$

The special cases $A^{(1)}(n) = 2^n$ and $A^{(2)}(n) = \binom{2n}{n}$ are simple. On the other hand, the numbers $A^{(3)}(n)$, known as *Franel numbers* [23, A000172], cannot be expressed as a finite linear combination of hypergeometric terms [19, p. 160]. We will refer to the numbers $A^{(s)}(n)$ as the generalized Franel numbers. Long before the computer-algebra era, Franel [10] computed recurrences for $A^{(3)}(n)$ as well as, in the second note [11], for

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$A^{(4)}(n)$. Based on these findings, he predicted—quite optimistically—a general shape of the recursion for general s. Since then, explicit recurrences for $A^{(s)}(n)$ have been computed using creative telescoping by Perlstadt [18] for $s = 5, 6$ and, likewise, by McIntosh [14] for $s \leq 10$. Creative telescoping, which we briefly review in Section 2, is a powerful computer-algebra technique introduced by Zeilberger [28] that can, for fixed integer s, algorithmically determine a recurrence satisfied by $A^{(s)}(n)$. More specifically, given a hypergeometric term like $a(n,k) = \binom{n}{k}^s$, it produces an operator $P(n,N)$ (here, N is the shift operator in n: $Na(n,k) := a(n+1,k)$), as well as another hypergeometric term $b(n,k)$, such that

$$P(n,N)a(n,k) = b(n,k+1) - b(n,k). \tag{2}$$

Summing the relation (2) (with some care and under some mild assumptions; see the beginning of Section 2) over all integers k, the contribution of $b(n,k)$ telescopes away, allowing us to conclude that $A^{(s)}(n)$ is annihilated by the operator $P(n,N)$; in this case, we say that $A^{(s)}(n)$ satisfies the *telescoping recurrence equation* $P(n,N)A^{(s)}(n) = 0$. Notice that this telescoping equation is based on the representation (1), as it uses the operator $P(n,N)$ for the hypergeometric term $a(n,k) = \binom{n}{k}^s$. Using a different hypergeometric representation—and such exist (for example, $A^{(3)}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}$)—may potentially lead to a different operator.

Franel's suspicions about the form of linear recurrence with polynomial coefficients for $A^{(s)}(n)$ are not supported by computations in [14, 18] with the exception of one particular aspect, its order. Specifically, Franel conjectured it to be *equal* to $\lfloor (s+1)/2 \rfloor$. While the fact that the order of the recurrence is bounded from above by this quantity is shown to be true by Stoll [24], who indicates that the earlier proof of Cusick [7] has a gap, it remains open to demonstrate that, in general, no recurrence of lower order exists. The possibility for $A^{(s)}(n)$ for $s \geq 3$ to satisfy a recurrence of order 1, equivalently, to be a hypergeometric term in the single variable n, can be ruled out using the algorithm `Hyper` [19], when s is fixed. On the other hand, using congruential properties, Yuan, Lu, and Schmidt [26] prove that, for *any* $s \geq 3$, the sequence $A^{(s)}(n)$ cannot satisfy a recurrence of order 1. This implies that Franel's recurrences of order 2 for $s = 3$ and $s = 4$ are of minimal order. In general, to prove that the order $\lfloor (s+1)/2 \rfloor$ recurrence constructed in [24] for the sequence $A^{(s)}(n)$ is of minimal order, it is sufficient to show that the corresponding recurrence operator is irreducible (although this is not a necessary condition). For fixed (and sufficiently small) s, the latter task is again accessible for modern computer-algebra algorithms [4, 29] (for an explicit example of bounding the possible degree of a lower-order recurrence, we also refer to the proof of Proposition 8.4 in [3, pp. 692–694]). One goal of this paper is to address the problem for *generic s* by showing the following general result.

Theorem 1.1. Any telescoping recurrence satisfied by $A^{(s)}(n)$ based on the representation (1) has order at least $\lfloor (s+1)/2 \rfloor$.

In particular, in light of [24], this implies that Franel's conjecture on the exact order is true if the minimal-order recurrence satisfied by $A^{(s)}(n)$ is a telescoping recurrence equation. We refer to Remark 4.1 for evidence that this is the case.

Remark 1.2. One way of establishing lower bounds on the order of recurrences satisfied by a D-finite sequence $A(n)$ comes from the observation by McIntosh [14, Section 4.1, p. 27] that, if the sequence has the property that $A(n+1)/A(n) \to \mu$ where μ is an algebraic number of degree d, then $A(n)$ cannot satisfy a recurrence defined over $\mathbb{Q}$ of order less than d. For the generalized Franel numbers $A^{(s)}(n)$, however, it follows from (21) that $A^{(s)}(n+1)/A^{(s)}(n) \to 2^s$, so that this criterion is of no help.

Apéry's groundbreaking proof [2, 21] of the irrationality of $\zeta(3)$ is centred around the fact that

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}, \tag{3}$$

where the sequences

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \tag{4}$$

and $B(n)$ both are solutions to the three-term recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1} \tag{5}$$

with initial conditions $A(0) = 1$, $A(1) = 5$ as well as $B(0) = 0$ and $B(1) = 1$. Limits, like (3), of quotients of solutions to a common linear recurrence are referred to as *Apéry limits*. For an introduction to such limits we refer to [5] as well as to the papers [1, 25]. The main goal of this paper is to explicitly determine the Apéry limits associated to the generalized Franel numbers $A^{(s)}(n)$. In fact, we will then prove Theorem 1.1, discussed above, in Section 4 as an application of these Apéry limits.

It was conjectured in [5] that, for $s \geq 2m+1$, the minimal-order recurrence satisfied by $A^{(s)}(n)$ has Apéry limits that are rational multiples of $\zeta(2), \zeta(4), \ldots, \zeta(2m)$. More precisely, this means that the recurrence has rational solutions $A_j^{(s)}(n)$, where $j \in \{0, 1, \ldots, m\}$, (with $A_0^{(s)}(n) = A^{(s)}(n)$) such that

$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} \in \pi^{2j}\mathbb{Q}.$$

In Theorem 1.3, we prove this conjecture, with the minimal-order recurrence replaced by the minimal-order telescoping recurrence, and explicitly describe all of these Apéry limits. In particular, in terms of

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^{s} \left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s} = A^{(s)}(n,-t), \tag{6}$$

we identify specific solutions $A_j^{(s)}(n) \in \mathbb{Q}$ as the coefficients in the t-expansion

$$A^{(s)}(n,t) = \sum_{j\geq 0} A_j^{(s)}(n) t^{2j}. \tag{7}$$

Theorem 1.3. Any telescoping recurrence satisfied by $A^{(s)}(n)$ based on the representation (1) is solved, for large enough n, by the sequences $A_j^{(s)}(n) \in \mathbb{Q}$ defined in (7), where $j \in \{0, 1, \ldots, \lfloor (s-1)/2 \rfloor\}$. Furthermore, we have

$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = \varphi_j^{(s)} \pi^{2j}, \tag{8}$$

where $\varphi_j^{(s)} \in \mathbb{Q}_{>0}$ is the coefficient of t^{2j} in the power series of $(t/\sin(t))^s$.

Proof. First, we show in Theorem 2.1 that, for large enough n, $A^{(s)}(n,t)$ satisfies any such telescoping recurrence up to terms that are $O(t^s)$. Consequently, the sequence $A_j^{(s)}(n)$, which by (7) is the coefficient of t^{2j} of $A^{(s)}(n,t)$ as a power series in t, solves this recurrence for large n provided that $0 \leq 2j < s$.

Second, we prove in Theorem 3.1 that

$$\lim_{n\to\infty} \frac{A^{(s)}(n,t)}{A^{(s)}(n)} = \left(\frac{\pi t}{\sin(\pi t)}\right)^s, \tag{9}$$

and that the convergence is locally uniform in t (restricted to the unit ball $|t| < 1$). Recall that, if analytic functions f_n converge locally uniformly to a function f, then f is analytic and the derivatives of f_n converge to the corresponding derivatives of f. Since the terms on the left-hand side of (9) are analytic in t, locally uniform convergence allows us to compare the derivatives on both sides, so that (8) follows. We note that

$$\frac{\pi t}{\sin(\pi t)} = \sum_{j=1}^{\infty}\left(2 - \frac{1}{2^{2j-2}}\right)\zeta(2j)t^{2j} = \sum_{j=1}^{\infty}\left(\frac{1}{2^{2j-1}} - 1\right)\frac{B_{2j}}{(2j)!}(2\pi i t)^{2j},$$

where the expansion in terms of zeta values makes it transparent that $\varphi_j^{(s)}$ is positive, while the rationality of the $\varphi_j^{(s)}$ is obvious from the series rewritten in terms of Bernoulli numbers. ■

Example 1.4. Note that $A^{(s)}(n,t)$, as defined in (6), has the initial values

$$A^{(s)}(0,t) = 1, \quad A^{(s)}(1,t) = \frac{1}{(1-t)^s} + \frac{1}{(1+t)^s} = 2\sum_{j\ge 0}\binom{2j+s-1}{2j}t^{2j}.$$

Consequently, for $j \ge 1$, the sequences $A_j^{(s)}(n)$ have the initial values $A_j^{(s)}(0) = 0$ and $A_j^{(s)}(1) = 2\binom{2j+s-1}{2j}$.

Example 1.5. Let us consider the special case $j = 1$ of Theorem 1.3. As noted in Example 1.4, we have $A_1^{(s)}(1) = s(s+1)$. In terms of $B^{(s)}(n) = A_1^{(s)}(n)/A_1^{(s)}(1)$, the initial values are normalized to $B^{(s)}(0) = 0$ and $B^{(s)}(1) = 1$, and the Apéry limit (8) takes the form

$$\lim_{n\to\infty} \frac{B^{(s)}(n)}{A^{(s)}(n)} = \frac{1}{s(s+1)}\frac{s}{6}\pi^2 = \frac{\zeta(2)}{s+1}, \tag{10}$$

which matches [5, Conjecture 9] (we note that this conjecture further claims that the sequence $B^{(s)}(n)$ is the unique solution of the minimal-order recurrence satisfied by $A^{(s)}(n)$ with the above properties). The cases $s = 3$ and $s = 4$ of (10) had been numerically observed by Cusick [21, p. 202], while the case $s = 5$ appears as a conjecture in [1, Section 4.1]. The case $s = 3$ was previously proved by Zagier [27] using modular forms.

Remark 1.6. Dougherty-Bliss and Zeilberger [8] explore Apéry limits related to those of Example 1.5 in a different direction. They construct a particular sequence $\tilde{B}^{(s)}(n) \in \mathbb{Q}$ such that (10) holds with $B^{(s)}(n)$ replaced by $\tilde{B}^{(s)}(n)$. For fixed s, the sequence $\tilde{B}^{(s)}(n)$ is D-finite, which implies that $A^{(s)}(n)$ and $\tilde{B}^{(s)}(n)$ satisfy a common linear recurrence (namely, the recurrence obtained from the least common left multiple of the two individual recurrence operators), but that recurrence is not minimal unless $\tilde{B}^{(s)}(n)$ happens to solve the minimal recurrence satisfied by $A^{(s)}(n)$ (that this is not the case is readily checked for small s). We note that one also obtains the limits (10) for the alternative choice $\tilde{B}^{(s)}(n) = A^{(s)}(n)b(n)/(s+1)$ where $b(n)$ is any holonomic sequence such that $b(n) \to \zeta(2)$ as $n \to \infty$. For instance, one could choose $b(n) = \sum_{k=1}^{n} \frac{1}{k^2}$ or $b(n) = 3\sum_{k=1}^{n} \frac{1}{k^2}\binom{2k}{k}^{-1}$, where the latter is due to Apéry [2, 21] and converges at an exponential rate. On the other hand, Dougherty-Bliss and Zeilberger [8] hope that their construction has the potential for better irrationality measures.

Example 1.7. Likewise, for the case $j = 2$ of Theorem 1.3, we have $A_2^{(s)}(1) = s(s+1)(s+2)(s+3)/12$. If we let $C^{(s)}(n) = A_2^{(s)}(n)/A_2^{(s)}(1)$, then the initial values are normalized to $C^{(s)}(0) = 0$ and $C^{(s)}(1) = 1$, and the Apéry limit (8) takes the form

$$\lim_{n\to\infty} \frac{C^{(s)}(n)}{A^{(s)}(n)} = \frac{12}{s(s+1)(s+2)(s+3)} \frac{s(5s+2)}{360}\pi^4 = \frac{3(5s+2)}{(s+1)(s+2)(s+3)}\zeta(4),$$

which matches [5, Conjecture 11].

2 Solutions of the Telescoping Recurrence

We refer to [6, 13, 19, 28] for general introductions to creative telescoping. For our purposes, suppose that we are interested in a sequence

$$A(n) = \sum_{k=\alpha}^{\beta-1} a(n,k).$$

If $a(n,k)$ is an appropriate hypergeometric term, then creative telescoping algorithmically determines operators $P(n,N)$ as well as another hypergeometric term $b(n,k)$, such that

$$P(n,N)a(n,k) = b(n,k+1) - b(n,k). \tag{11}$$

Moreover, the term $b(n,k)$ as produced by creative telescoping is of the form $b(n,k) = R(n,k)a(n,k)$ for some rational function $R(n,k)$. When the hypergeometric term $a(n,k)$ is defined over the ring $\mathbb{Z}$ (and this is specifically the case of our interest here, although the argument below extends to other rings), that is, when both $a(n+1,k)/a(n,k)$ and $a(n,k+1)/a(n,k)$ are quotients of polynomials from $\mathbb{Z}[n,k]$, we can take $P(n,N) \in \mathbb{Z}[n,N]$ and we have $b(n,k)$ defined over $\mathbb{Z}$. We note that, given $P(n,N)$ and $R(n,k)$, an identity like (11) can be verified by dividing both sides by $a(n,k)$, upon which one obtains an identity between rational functions. For that reason, $R(n,k)$ is referred to as the *certificate* of the telescoping relation (11).

It follows from the telescoping nature of (11) that, after summing over k,

$$P(n,N)\sum_{k=\alpha}^{\beta-1} a(n,k) = b(n,\beta) - b(n,\alpha), \tag{12}$$

assuming that $b(n,k)$ is finite for the involved values of n and k.

For our present purposes, $a(n,k) = \binom{n}{k}^s$. We say that $P(n,N)$ is a *telescoping recurrence operator* for the generalized Franel numbers $A^{(s)}(n)$ if

$$P(n,N)\binom{n}{k}^s = b(n,k+1) - b(n,k), \tag{13}$$

where $b(n,k)/a(n,k) = R(n,k)$ is a rational function. We next show that it follows from (13) not only that $P(n,N)A^{(s)}(n) = 0$ but that the same recurrence is also solved by $A^{(s)}(n,t)$, as defined in (6), up to terms of order t^s or higher. Equivalently, the sequences $A_j^{(s)}(n) \in \mathbb{Q}$, as in (7), are solutions for $j \in \{0,1,\ldots,\lfloor (s-1)/2 \rfloor\}$.

Theorem 2.1. For fixed s, suppose that $P(n,N)$ is a telescoping recurrence operator for the generalized Franel numbers $A^{(s)}(n)$. Then, for large enough n, as $t \to 0$,

$$P(n,N)A^{(s)}(n,t) = O(t^s). \tag{14}$$

Proof. Using the reflection formula

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)},$$

we find that, for $n \in \mathbb{Z}_{\geq 0}$,

$$\binom{n}{-t} = \frac{\Gamma(n+1)}{\Gamma(n+t+1)\Gamma(1-t)} = \frac{\sin(\pi t)}{\pi} \frac{\Gamma(n+1)\Gamma(t)}{\Gamma(n+t+1)} = \frac{\sin(\pi t)}{\pi} \frac{n!}{t(t+1)\cdots(t+n)}. \tag{15}$$

Consequently, for $k \in \mathbb{Z}$ such that $0 \leq k \leq n$,

$$\begin{aligned} \binom{n}{k-t} &= \frac{\sin(\pi t)}{\pi t} \frac{(-1)^k n!}{(t-k)\cdots(t-1)(t+1)\cdots(t+n-k)} \\ &= \frac{\sin(\pi t)}{\pi t} \binom{n}{k} \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-1}, \end{aligned} \tag{16}$$

where we used $\sin(\pi(t-k)) = (-1)^k \sin(\pi t)$. If α and β are integers such that $\alpha \leq 0$ and $\beta > n$, we therefore have

$$A^{(s)}(n,t) = \left(\frac{\pi t}{\sin(\pi t)}\right)^s \sum_{k=0}^{n} \binom{n}{k-t}^s = \left(\frac{\pi t}{\sin(\pi t)}\right)^s \sum_{k=\alpha}^{\beta-1} \binom{n}{k-t}^s + O(t^s),$$

where the first equality is a consequence of (16), while the second follows from the added binomial coefficients being $O(t)$ as $t \to 0$. The claim (14) therefore follows if we can show that

$$P(n,N) \sum_{k=\alpha}^{\beta-1} \binom{n}{k-t}^s = O(t^s)$$

for large enough n.

Since (13) after dividing by $\binom{n}{k}^s$ is a rational-function identity in n and k, the relation (13) continues to hold if we replace k by $k-t$, for an indeterminate t, resulting in

$$P(n,N)\binom{n}{k-t}^s = b(n,k+1-t) - b(n,k-t). \tag{17}$$

Because $b(n,t)/a(n,t) = R(n,t)$ is a rational function, while $a(n,t)$ is an entire function in t for each $n \geq 0$, we conclude that, for each large enough n (so that the denominator of $R(n,t)$ cannot vanish for all t), $b(n,t)$ can have at most finitely many poles as a function of t. However, for fixed such n, the function $b(n,t+1) - b(n,t)$ is entire in t (since the left-hand side in (17) is a linear combination of entire functions), hence $b(n,t)$ cannot have poles at all. In particular, for large enough n, $b(n, t)$ is itself an entire function in t, and we can then sum (17) over k to obtain

$$P(n, N) \sum_{k=\alpha}^{\beta-1} \binom{n}{k-t}^s = b(n, \beta - t) - b(n, \alpha - t). \tag{18}$$

It therefore remains to show that $b(n, \alpha - t)$ and $b(n, \beta - t)$ are each $O(t^s)$ as $t \to 0$ for some integral $\alpha \leq 0$ and $\beta > n$ of our choosing. To that end, fix n and note that, if $\alpha \leq 0$ is an integer, then

$$b(n, \alpha - t) = R(n, \alpha - t) \binom{n}{\alpha - t}^s$$

is $O(t^s)$ as $t \to 0$, because the binomial coefficient

$$\binom{n}{\alpha - t} = \frac{(-1)^{\alpha+1}}{\binom{n-\alpha}{n}} t + O(t^2)$$

is $O(t)$, provided that the rational function $r(t) := R(n, t)$ (which is well defined for large enough n) does not have a pole at $t = \alpha$. This is necessarily the case for $\alpha \leq 0$ of large enough absolute value because $r(t)$ can have at most finitely many poles. The same argument applies to show that $b(n, \beta - t)$ is $O(t^s)$ for large enough integral $\beta > n$. ∎

Remark 2.2. The proof above shows that Theorem 2.1 is true for all $n \in \mathbb{Z}_{\geq 0}$ if the denominator of the rational certificate $R(n, k)$ has no factor of the form $n - n_0$ for some $n_0 \in \mathbb{Z}_{\geq 0}$ (so that $b(n, t)$ is an entire function in t for all $n \in \mathbb{Z}_{\geq 0}$). The computations mentioned in Remark 4.1 below show that, for $s \leq 20$, the minimal recurrence is telescoping and that, up to a constant multiple, the denominator of $R(n, k)$ is $(n-k+1)_m^s$. It is natural to expect that these observations continue to be true for all s.

Remark 2.3. With (16) in mind, we note that the rational function

$$\frac{n!}{t(t+1)\cdots(t+n)} = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{t+k} = \frac{\pi}{\sin(\pi t)} \binom{n}{-t},$$

and its powers play a role of building bricks in constructions of $\mathbb{Q}$-linear forms in zeta values [16, 30].

3 Proof of the Apéry Limits

This section is devoted to a proof of the following result that, together with Theorem 2.1, establishes the Apéry limits associated to the generalized Franel numbers $A^{(s)}(n)$ as claimed in Theorem 1.3.

Theorem 3.1. For any $s \in \mathbb{Z}_{>0}$, we have

$$\lim_{n\to\infty} \frac{A^{(s)}(n,t)}{A^{(s)}(n)} = \left(\frac{\pi t}{\sin(\pi t)}\right)^s, \tag{19}$$

where the convergence is locally uniform in t (restricted to the unit ball $|t| < 1$).

That is, we wish to show that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s}}{\sum_{k=0}^{n}\binom{n}{k}^s} = \left(\frac{\pi t}{\sin(\pi t)}\right)^s, \tag{20}$$

and that the convergence is locally uniform in t.

The asymptotics for sums of powers of binomials are known to be, for fixed s,

$$\sum_{k=0}^{n}\binom{n}{k}^s = \frac{2^{ns}}{\sqrt{s(\pi n/2)^{s-1}}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{21}$$

For instance, a more precise estimate with additional terms (and which applies to more general binomial sums) is derived by McIntosh [15]. Slightly weaker estimates are derived in [12, pp. 486–489], with full details provided in the case $s = 1$, as well as in [9]. In each case, the analysis rests on the fact that the binomial sum is dominated by those terms with $k \approx n/2$. However, the precise choice of cut-off for the dominant part of the sum differs between the various approaches. In [15] the dominant terms are those corresponding to k satisfying $\left|k - \frac{n}{2}\right| \le \varepsilon n$ for suitable $\varepsilon > 0$, while in [12] this condition is replaced with $\left|k - \frac{n}{2}\right| \le \varepsilon n^{1/2}$. On the other hand, in [12], one restricts to those k in the set

$$K_{n,\varepsilon} = \left\{k \in \mathbb{Z} : \left|k - \frac{n}{2}\right| \le n^{1/2+\varepsilon}\right\}.$$

It is this latter choice that is most suitable for our present purposes.

Naturally, our strategy to establish the limit (20) is to exploit the fact that the sums on the left-hand side are concentrated around $k \approx n/2$. For those k and large n, we have

$$\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right) \approx \prod_{j=1}^{\infty}\left(1-\frac{t}{j}\right)\left(1+\frac{t}{j}\right) = \frac{\sin(\pi t)}{\pi t}.$$

On the other hand, this is not true if k is not sufficiently close to $n/2$; however, we will show that the contribution from these k is overall negligible. To make this precise, we begin by observing the following desired behaviour for $k \in K_{n,\varepsilon}$.

Lemma 3.2. Fix $\varepsilon \in [0, 1/2)$ and $\tau > 0$. Then, for all integers $n \geq 0$, all $k \in K_{n,\varepsilon}$ and all $|t| \leq \tau$, we have

$$\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right) = \frac{\sin(\pi t)}{\pi t}\left(1+O\left(\frac{1}{n^{1/2-\varepsilon}}\right)\right),$$

where the implied constant depends on ε and τ (but not on t or k).

Proof. To begin with, note that

$$\prod_{j=1}^{n}\left(1+\frac{t}{j}\right) = \frac{(n+t)!}{n!\,t!}.$$

In light of the classical

$$\frac{1}{\Gamma(1+t)\Gamma(1-t)} = \frac{\sin(\pi t)}{\pi t},$$

we therefore need to show that

$$\frac{(k-t)!}{k!}\frac{(n-k+t)!}{(n-k)!} = 1 + O\left(\frac{1}{n^{1/2-\varepsilon}}\right). \tag{22}$$

To this end, recall Stirling's formula in its logarithmic form, namely,

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{n}\right). \tag{23}$$

With the assumption that $t = O(1)$, we deduce from (23) that

$$\frac{(n+t)!}{n!} = n^t\left(1+O\left(\frac{1}{n}\right)\right) \tag{24}$$

and, therefore,

$$\frac{(k-t)!}{k!}\frac{(n-k+t)!}{(n-k)!}=\left(\frac{n}{k}-1\right)^t\left(1+O\left(\frac{1}{k}\right)+O\left(\frac{1}{n-k}\right)\right).$$

The assumption $k\in K_{n,\varepsilon}$ implies that $k=\frac{n}{2}+O(n^{1/2+\varepsilon})$ and, in particular,

$$\frac{n}{k}-1=\frac{n}{\frac{n}{2}+O(n^{1/2+\varepsilon})}-1=1+O\left(\frac{1}{n^{1/2-\varepsilon}}\right),$$

leading us to the claimed relation (22). ■

On the other hand, for $k\notin K_{n,\varepsilon}$, the products can be bounded using the following simple observation.

Lemma 3.3. Fix $\tau\in(0,1)$. For all integers $n>0$ and all $|t|\le\tau$, we have

$$\left|\prod_{j=1}^{n}\left(1+\frac{t}{j}\right)\right|^{-1}=O(n^{\tau}),$$

where the implied constant depends on τ (but not on t).

Proof. Because $|t|\le\tau<1$, we have

$$\left|\prod_{j=1}^{n}\left(1+\frac{t}{j}\right)\right|^{-1}\le\prod_{j=1}^{n}\left(1-\frac{\tau}{j}\right)^{-1}$$

and it can be deduced from (24) that

$$\prod_{j=1}^{n}\left(1-\frac{\tau}{j}\right)^{-1}=\Gamma(1-\tau)n^{\tau}\left(1+O\left(\frac{1}{n}\right)\right).$$ ■

In particular, for $0<k\le n$ and $|t|\le\tau<1$, we conclude from Lemma 3.3 the crude bound

$$\left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s}=O(n^{2s}),\tag{25}$$

which could be easily strengthened but which suffices for our purposes.

We now write

$$a_{s,t}(k,n) = \binom{n}{k}^s \left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s}$$

and follow the approach in [12, pp. 486–489] to prove the following.

Lemma 3.4. Fix $s > 0$, $\tau \in (0,1)$ and $\varepsilon \in (0,1/6)$. Then, for all integers $n \geq 0$ and all $k \in K_{n,\varepsilon}$ and all t such that $|t| \leq \tau$, we have

$$\sum_{k=0}^{n} a_{s,t}(k,n) = \frac{2^{ns}}{\sqrt{s(\pi n/2)^{s-1}}}\left[\frac{\pi t}{\sin(\pi t)}\right]^s \left(1 + O\left(\frac{1}{n^{1/2-3\varepsilon}}\right)\right), \tag{26}$$

where the implied constant depends on s, ε, and τ (but not on t).

Proof. Proceeding as in [12], we obtain that, for $k \in K_{n,\varepsilon}$,

$$\binom{n}{k}^s = \left[\frac{2^{n+1}}{\sqrt{2\pi n}} e^{-2(k-n/2)^2/n}\right]^s \left(1 + O\left(\frac{1}{n^{1/2-3\varepsilon}}\right)\right). \tag{27}$$

Combined with Lemma 3.2, we find

$$a_{s,t}(k,n) = \left[\frac{2^{n+1}}{\sqrt{2\pi n}} e^{-2(k-n/2)^2/n} \frac{\pi t}{\sin(\pi t)}\right]^s \left(1 + O\left(\frac{1}{n^{1/2-3\varepsilon}}\right)\right).$$

In other words, we have, for $k \in K_{n,\varepsilon}$,

$$a_{s,t}(k,n) = b_{s,t}(k,n) + O(c_{s,t}(k,n)),$$

where

$$b_{s,t}(k,n) = \left[\frac{2^{n+1}}{\sqrt{2\pi n}} e^{-2(k-n/2)^2/n} \frac{\pi t}{\sin(\pi t)}\right]^s$$

and (because $\pi t/\sin(\pi t) = O(1)$)

$$c_{s,t}(k,n) = 2^{ns} e^{-2s(k-n/2)^2/n} \frac{1}{n^{(s+1)/2-3\varepsilon}}.$$

We now apply the tail-exchange method and estimate

$$\sum_k a_{s,t}(k,n) = \sum_k b_{s,t}(k,n) + O\left(\sum_{k\notin K_{n,\varepsilon}} |a_{s,t}(k,n)|\right) + O\left(\sum_{k\notin K_{n,\varepsilon}} |b_{s,t}(k,n)|\right) + O\left(\sum_{k\in K_{n,\varepsilon}} |c_{s,t}(k,n)|\right). \tag{28}$$

The idea, of course, being that the first term on the right-hand side of (28), namely

$$\begin{aligned}\sum_k b_{s,t}(k,n) &= \left[\frac{2^{n+1}}{\sqrt{2\pi n}}\,\frac{\pi t}{\sin(\pi t)}\right]^s \sum_k e^{-2s(k-n/2)^2/n}\\ &= \left[\frac{2^{n+1}}{\sqrt{2\pi n}}\,\frac{\pi t}{\sin(\pi t)}\right]^s \sqrt{\frac{\pi n}{2s}}\left(1+O\left(e^{-\frac{n\pi^2}{2s}}\right)\right)\\ &= \frac{2^{ns}}{\sqrt{s(\pi n/2)^{s-1}}}\left[\frac{\pi t}{\sin(\pi t)}\right]^s\left(1+O\left(e^{-\frac{n\pi^2}{2s}}\right)\right),\end{aligned} \tag{29}$$

provides the asymptotics for the left-hand side of (28) while the other terms are negligible in comparison. Indeed,

$$\sum_{k\in K_{n,\varepsilon}} |c_{s,t}(k,n)| \le \sum_k c_{s,t}(k,n) = \frac{2^{ns}}{n^{(s+1)/2-3\varepsilon}} \sum_k e^{-2s(k-n/2)^2/n}$$

is asymptotically smaller than (29) provided that $3\varepsilon < \frac{1}{2}$ (note that adding this contribution to (29) requires adjusting the error term in (29) to the one claimed in (26)). Likewise,

$$\sum_{k\notin K_{n,\varepsilon}} |b_{s,t}(k,n)| = \left|\frac{2^{n+1}}{\sqrt{2\pi n}}\,\frac{\pi t}{\sin(\pi t)}\right|^s \sum_{k\notin K_{n,\varepsilon}} e^{-2s(k-n/2)^2/n}$$

is asymptotically smaller than (29) because the right-hand side sum is $O(n^{-M})$ for all M (here, we use that $\varepsilon > 0$). Thirdly, by (25),

$$\sum_{k\notin K_{n,\varepsilon}} |a_{s,t}(k,n)| = O(n^{2s}) \sum_{k\notin K_{n,\varepsilon}} \binom{n}{k}^s$$

and the right-hand side sum is bounded by n times its largest term, which is bounded by the one corresponding to $k^* = \lfloor \frac{n}{2} + n^{1/2+\varepsilon} \rfloor \in K_{n,\varepsilon}$. In particular, applying (27) to that term reveals that

$$\sum_{k \notin K_{n,\varepsilon}} |a_{s,t}(k,n)| = \binom{n}{k^*}^s O(n^{2s+1}) = \left[\frac{2^{n+1}}{\sqrt{2\pi n}} e^{-2(k^*-n/2)^2/n} \right]^s O(n^{2s+1})$$

is asymptotically smaller than (29) as well. ■

Combining (21) and (26), we conclude the desired limit (20), including the required uniform convergence.

4 Lower Bounds for Telescoping Recurrences

We are now in a position to apply the results on Apéry limits to prove Theorem 1.1. That is, we wish to conclude that any telescoping recurrence satisfied by $A^{(s)}(n)$ has order at least $\lfloor (s+1)/2 \rfloor$.

Proof of Theorem 1.1. By Theorem 1.3, any telescoping recurrence satisfied by $A^{(s)}(n)$ is also solved, for large enough n, by the $\lfloor (s+1)/2 \rfloor$ sequences $A_j^{(s)}(n) \in \mathbb{Q}$ defined in (7), where $j \in \{0, 1, \ldots, \lfloor (s-1)/2 \rfloor\}$. We recall from [19, Theorem 8.2.1] that a recurrence with polynomial coefficients has order r if and only if the space of its solutions, upon identifying sequences that eventually agree, has dimension r.

Therefore, to conclude that any telescoping recurrence satisfied by $A^{(s)}(n)$ has order at least $r = \lfloor (s+1)/2 \rfloor$, it suffices to show that the r solutions $A_j^{(s)}(n)$, $j \in \{0, 1, \ldots, r-1\}$, upon this identification, are linearly independent. As these solutions are rational-valued, assuming their linear dependence, there must necessarily exist a dependence relation over $\mathbb{Q}$. This means that

$$0 = \sum_{j=0}^{r-1} \lambda_j A_j^{(s)}(n), \quad \lambda_j \in \mathbb{Q}, \tag{30}$$

for large enough n. Now, upon dividing (30) by $A^{(s)}(n)$ and taking the limit as $n \to \infty$, we find out that

$$0 = \lim_{n\to\infty} \sum_{j=0}^{r-1} \lambda_j \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = \sum_{j=0}^{r-1} \lambda_j \varphi_j \pi^{2j},$$

where the latter equality uses the Apéry limits established in Theorem 1.3. Since $\lambda_j \varphi_j \in \mathbb{Q}$, the transcendence of π implies that $\lambda_j \varphi_j = 0$ for all $j \in \{0, 1, \ldots, r-1\}$. We know that $\varphi_j \neq 0$, so we must have $\lambda_j = 0$ for all $j \in \{0, 1, \ldots, r-1\}$, proving the desired linear independence of the r solutions $A_j^{(s)}(n)$. ■

Remark 4.1. The computations of Perlstadt [18] and McIntosh [14] show that a telescoping recurrence equation of (the conjectured to be minimal) order $m = \lfloor (s+1)/2 \rfloor$ exists for $s \leq 10$. We have extended these computations to all $s \leq 20$ using Koutschan's implementation [13] `HolonomicFunctions` in Mathematica and, in each case, obtained a recurrence of order m (the minimality of these recurrence operators was then confirmed using the function `MinimalRecurrence` from the `LREtools` Maple package).

These computations suggest that a minimal-order recurrence of $A^{(s)}(n)$ can always be obtained via creative telescoping. Moreover, Alin Bostan observes that this minimal recurrence of order m has polynomial coefficients of degree

$$d = \begin{cases} \frac{1}{3}m(m^2-1)+1, & \text{for even } s, \\ \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{2}{3}m + \frac{(-1)^m - 1}{4}, & \text{for odd } s. \end{cases}$$

In particular, the degree appears to grow like $s^3/24$ (rather than being bounded by $s-1$ as Franel incorrectly predicted in [11]). As for the rational certificate, when written in lowest terms and with integer coefficients, we further find out that its denominator is given by

$$(n-k+1)_m^s = \prod_{j=1}^{m} (n-k+j)^s$$

and, thus, has degree ms in each of n and k. The corresponding numerator has degree $ms + \delta_2(s)$ in the variable k (as used in Remark 2.2), where the delta notation is for $\delta_r(s) = 0$ unless r divides s in which case $\delta_r(s) = 1$. At the same time, its degree in n is $d + s(s-1-\delta_2(s))/2 - \delta_6(s)$. Moreover, the numerator is

$$k^s \prod_{j \geq 1} (n+j)^{\max(0, s+2-4j-(-1)^s)}$$

times a (large) irreducible factor. These observations hold true for $s \leq 20$, and it is natural to expect that the patterns persist for larger s as well.

If desired, the above computations can readily be extended to larger s. Readers interested in computing telescoping recurrence equations for large s might find value in considering a guess-and-prove approach (with the above observations taken into account) such as described, for instance, in [20] for a different hypergeometric sum.

5 Conclusions

We have explicitly determined the Apéry limits associated to the generalized Franel numbers, resolving the explicit conjectures in [5]. As a novel application of Apéry limits, we proved in Theorem 1.1 that Franel's conjecture is true if the minimal-order recurrence satisfied by $A^{(s)}(n)$ is a telescoping recurrence equation. It would be useful to establish general conditions under which it can be guaranteed that creative telescoping is able to determine a recurrence of minimal order. As a rare result of this type, we mention that Schneider [22, Corollary 7.4] proves that creative telescoping finds a minimal (inhomogeneous) recurrence for certain sums over hypergeometric terms $a(n,k)$ where the summation bounds are independent of n but finite. On the other hand, we refer to Paule [17, Section 11.2] for an example in which creative telescoping is not able to find a recurrence of minimal order. We echo Chyzak's [6, p. 52] comment that "a theoretical explanation is still missing and would be welcome in order to design algorithms for minimal-order annihilators." From a different point of view, it is not necessarily that creative telescoping suffers from missing a minimal-order recurrence for a given D-finite sequence $A(n)$ but that the sequence itself always possesses multiple hypergeometric representations, also as multiple binomial sums, and that we *a priori* have no knowledge on which of those the algorithm will produce the optimal outcome.

As mentioned in the introduction, Stoll [24], as well as Cusick [7], construct recurrences for the generalized Franel numbers (of the conjectured order). It would be of interest to see if these constructions can be augmented to show that they actually result in telescoping recurrences.

As noted in the introduction (see also Remark 1.2 there), there is a shortage of general results that make it possible to prove lower bounds on the order of recurrences satisfied by D-finite sequences. We expect that the present approach can be applied to other families of binomial sums to compute the corresponding Apéry limits and to prove lower bounds for their minimal telescoping recurrences.

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