CONDITIONS FOR THE SUM OF A POWER SERIES TO BE ALGEBRAIC AND RATIONAL

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Let the power series

$$f(z) = \sum_{k \geq 0} \alpha_k z^k \tag{1}$$

converge in a neighborhood of zero. We consider the question of the structure of the coefficients of (1) under the condition that its sum f(z) is an algebraic function.

Various conditions for the sum of the series (1) to be rational are known. For example, the rationality of the function f(z) is equivalent with the fact that $\alpha_k = e(k)$ for all indices k, where e(x) is a quasipolynomial. Another criterion for rationality is the vanishing of the sequence of Hanekl determinants: $\det(d_{ij})_{i,j=1}^n = 0$ for $n > n_0$, where $d_{ij} = \alpha_{i+j-2}$ [1].

In [2] some rationality conditions are considered for a double power series

$$F(z_1, z_2) = \sum_{m,\, n \geq 0} a_{mn} z_1^m z_2^n, \tag{2}$$

which is convergent in a neighborhood of zero. They are gotten by applying a one-dimensional criterion with respect to each variable of the series (2).

It is shown (Theorem 1) that the existence of a rational function (2), for which (1) is "the diagonal," i.e.,

$$\sum_{k \geq 0} \alpha_k z^k = \sum_{k \geq 0} a_{kk} z^k \tag{3}$$

is a necessary and sufficient condition for the function (1) to be algebraic (from which the analogous assertion for Puiseux series $f(z) = \sum_{k \geq 0} \alpha_k z^{k/m}$ follows). Theorem 2 gives a necessary geometric condition for the rationality of (2), connected with the structure of the set of zeros of its homogeneous polynomials.

In [3] the asymptotics of the diagonal coefficients of the power series of a rational function of two variables is considered. By Theorem 1 this lets one investigate the asymptotics of the Taylor coefficients of an algebraic function.

The proof of Theorem 1 is based on the resolution of singularities. In connection with this, in Sec. 1 we give an integral formula for a branch of an implicit function (Proposition 1) which is needed. In [4] (cf. also [5, Sec. 21]) a formula is constructed for a branch of an implicit function with the help of an integral representation of sums of branches corresponding to different edges of the Newton diagram, from which one then gets the Lagrange decomposition of this branch. Proposition 1 can also be used for the expansion of an implicit function in a Lagrange series.

1. Integral Representation of a Holomorphic Branch of an Implicit Function

We denote by $\mathcal{O}$ the ring of germs of holomorphic functions of two variables at the point $(0, 0) \in \mathbb{C}^2$, $E = \{g \in \mathcal{O}\colon g(w, z) = (w - h(z))^n u(w, z),\ h, u \in \mathcal{O},\ u(0, 0) \neq 0,\ h(0) = 0\}$, [i.e., E is the set of germs $g \in \mathcal{O}$, for which the equation $g(w, z) = 0$ has a unique and holomorphic solution $w = h(z)$]. We consider an implicit function $w = \varphi(z)$ such that

$$f(w, z) = 0, \tag{4}$$

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where $f \in \mathcal{O}$, $f(0, 0) = 0$. Among the branches of an implicit function passing through the point $(0, 0)$, let there be a branch $w = \varphi_1(z)$, which is holomorphic at zero. To get an integral representation of this branch it is necessary to distinguish it from the other branches of an implicit function passing through $(0, 0)$.

Resolution of the singularities of the holomorphic curve $\{f(w, z) = 0\}$ is given by a multiple σ-process [6, Sec. 8B; 7]. Here we need to distinguish the holomorphic branch $w = \varphi_1(z)$ with only the help of "unilateral" transformations $(w, z) \to (wz, z)$ and $(w, z) \to (w + a, z)$ (compare [7]). We denote by $\mu(1) = [(w, z) \to (wz, z)]$ the monoidal change of variables and by $\ell(a) = [(w, z) \to (w + a, z)]$ the linear change of variables which carry the function $s(w, z)$ into the function $s_1(w, z) \equiv s(wz, z)$ and $s_2(w, z) \equiv s(w + a, z)$, respectively. We denote the composition of k transformations $\mu(1)$ by $\mu(k)$ $(k \geq 1)$, so that $\mu(k) = [(w, z) \to (wz^k, z)]$ obviously. The transformation $\ell(a_m) \circ \mu(k_m) \circ \ldots \circ \ell(a_1) \circ \mu(k_1)$ has the form $[(w, z) \to (r(w, z), z)]$, where

$$r(w, z) = z^{k_1}(\ldots(z^{k_m}(w + a_m) + a_{m-1})\ldots) + a_1). \tag{5}$$

We shall call a polynomial of the form (5) a resolving polynomial for the branch $w = \varphi_1(z)$ of the implicit function with respect to the equation (4), which is holomorphic at zero, if $f(r(w, z), z) = z^k g(w, z)$, where $g \in E$, $k \in \mathbf{N}$, and $\varphi_1(z) = r(h(z), z)$; here $w = h(z)$ is the unique holomorphic solution of the equation $g(w, z) = 0$, $h(0) = 0$.

Proposition 1. Let $r(w, z)$ be a resolving polynomial for a branch, holomorphic at zero, of the implicit function with respect to (4). Then for all $|z| < \delta(\rho)$ one has $\varphi_1(z) = r(h(z), z)$, where

$$h(z) = \frac{1}{2\pi i n} \int_{|w|=\rho} \frac{w\,[f(r(w, z), z)]'_w\, dw}{f(r(w, z), z)}, \tag{6}$$

ρ is sufficiently small, n is the multiplicity of the zero $w = \varphi_1(z)$ of the function $f(w, z)$.

A resolving polynomial can be constructed in the following way. According to the Weierstrass preparation theorm [8], we have $f(w, z) = z^\ell \psi(w, z) V(w, z)$, where $V(w, z) = w^k + a_{k-1}\cdot(z)w^{k-1} + \ldots + a_0(z)$ is a Wierstrass pseudopolynomial, $V \in \mathcal{O}$, $\psi \in \mathcal{O}$, $\psi(0, 0) \neq 0$, $a_j(0) = 0$ $(j = 0, 1, \ldots, k - 1)$. Since $r(0, 0) = 0$, $\psi(0, 0) \neq 0$, it suffices to construct a resolving polynomial $r(w, z)$ for the branch $w = \varphi_1(z)$ with respect to the equation $V(w, z) = 0$.

In a neighborhood of zero we have a single-valued decomposition

$$V(w, z) = \prod_{j=1}^{d} (w - \varphi_j(z))^{s_j}, \quad s_1 + \ldots + s_d = k.$$

Let $\varphi_1(z) = \sum_{k \geq 1} \beta_k z^{n_k}$ be the holomorphic branch considered of the implicit function, $\varphi_j(z) = \sum_{k \geq 1} b_{jk} z^{\lambda_{jk}}$ be the expansions of the other roots of the pseudopolynomial V in fractional powers $(2 \leq j \leq d)$. One can get any number of terms of these series with the help of the Newton diagram [7]. We shall assume that the coefficients β_k and b_{jk} of these series are enumerated without omissions, i.e., between any nonzero coefficients of the series there are no coefficients equal to zero. We take the number $m = \max_{j=\overline{2,d}} \max\{\nu\colon \beta_1 = b_{j1}, n_1 = \lambda_{j1}, \ldots, \beta_\nu = b_{j\nu}, n = \lambda_{j\nu}\} + 1$. Then the polynomial

$$r(w, z) = z^{n_1}(z^{n_2-n_1}(\ldots(z^{n_m-n_{m-1}}(w + \beta_m) + \beta_{m-1})\ldots) + \beta_1)$$

will be resolving for the branch $w = \varphi_1(z)$. In fact, the transformation $R = [(w, z) \to (r(w, z), z)]$ carries the factor $w - \varphi_1(z)$ of the pseudopolynomial V into the function $z^{n_m}\cdot(w - \sum_{k \geq m+1} \beta_k z^{n_k - n_m})$. Further, for the function $\varphi_j(z)$ $(2 \leq j \leq d)$, let p be the smallest number for which $\beta_p \neq b_{jp}$ or $n_p \neq \lambda_{jp}$ $(1 \leq p \leq m)$; then the transformation R carries the factor $w - \varphi_j(z)$ into the function

$$\begin{aligned} r(w, z) - \varphi_j(z) &= \beta_1 z^{n_1} + \ldots + \beta_m z^{n_m} + wz^{n_m} - \sum_{k \geq 1} b_{jk} z^{\lambda_{jk}} = \\ &= \beta_p z^{n_p} + \ldots + \beta_m z^{n_m} + wz^{n_m} - \sum_{k \geq p} b_{jk} z^{\lambda_{jk}} = \\ &= z^{n_p}\left[\beta_p - b_{jp} z^{\lambda_{jp} - n_p} + wz^{n_m - n_p} + \sum_{k \geq p+1} \beta_k z^{n_k - n_p} - \sum_{k \geq p+1} b_{jk} z^{\lambda_{jk} - n_p}\right] \equiv z^{n_p} \Phi_j(w, z). \end{aligned}$$

Since $\beta_p \neq b_{jp}$ or $n_p \neq \lambda_{jp}$, and moreover $\beta_p \neq 0$, in a sufficiently small bidisc $\{|w| \leq \rho, |z| \leq \rho\}$ there are no solutions of the equation $\Phi_j(w, z) = 0$. From this it is clear that the pseudopolynomial V, and hence also the function f, reduces the change R to the form $z^{k_1} g(w, z)$, $g \in E$. Moreover, the unique solution of the equation $g = 0$ is holomorphic and has the form $w = h(z) \equiv \sum_{k \geq m+1} \beta_k z^{n_k - n_m}$ so

$$\varphi_1(z) = \sum_{k \geq 1} \beta_k z^{n_k} = \beta_1 z^{n_1} + \ldots + \beta_m z^{n_m} + z^{n_m} h(z) = r(h(z), z).$$

Since $g \in E$, $g = (w - h(z))^n u(w, z)$, by the logarithmic residue theorem

$$h(z) = \frac{1}{2\pi i n} \int_{|w|=\rho} \frac{w g'_w(w, z)\, dw}{g(w, z)},$$

and since $f(r(w, z)) = z^{k_1} g(w, z)$, we have $w g_w'(w, z)/g(w, z) = w[f(r(w, z), z)]_w'/f(r(w, z), z)$, so we get (6). Proposition 1 is proved.

2. Connection of Algebraic Functions of One Variable with Rational Functions of Two Variables

THEOREM 1. In order that the function (1) be a branch which is holomorphic at zero of an algebraic function, it is necessary and sufficient that it be "the diagonal" of some rational function, holomorphic at zero, of the form (2), i.e., that there exist a rational function of two variables (2), for which (3) holds.

Proof. Sufficiency. Let $F(z_1, z_2)$ be a rational function, holomorphic at zero, having a series expansion (2); then for z sufficiently close to zero, one has [9, 10]

$$\sum_{k \geq 0} a_{kk} z^k = \frac{1}{2\pi i} \int_{|w|=\rho} F\left(w, \frac{z}{w}\right) \frac{dw}{w}.$$

The numerator of the rational function $F(w, z/w)$ decomposes into factors of the form $w - \varepsilon_j(z)$, so calculating the integral as the sum of residues at the poles of $w = \varepsilon_j(z)$, $|\varepsilon_j(z)| < \rho$, we get an algebraic function of z.

Necessity. Let f(z) be a branch, holomorphic at zero, of an algebraic function, which decomposes into a series (1), and is defined by a polynomial $P(w, z)$: $P(f(z), z) \equiv 0$. Without loss of generality one can assume that $f(0) = 0$, $P(0, 0) = 0$. According to Sec. 1, for the function $w = f(z)$ there exists a resolving polynomial $r(w, z)$ with respect to the equation $P(w, z) = 0$, such that $P(r(w, z), z) = z^k g(w, z)$, $g \in E$, $g(w, z) = (w - h(z))^n u(w, z)$, $h(0) = 0$, $u(0, 0) \neq 0$, $f(z) = r(h(z), z)$, where

$$nh(z) = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{w\, [P(r(w, z), z)]_w'\, dw}{P(r(w, z), z)}. \tag{7}$$

The rational integrand in (7) will be denoted by $F_3(w, z)$ and we let $F_2(w, z) \equiv wF_3(w, z)$. It is easy to see that $F_2(w, z) = nw^2/(w - h(z)) + w^2 u_w'/u$, from which, considering that $h(0) = 0$, $u(0, 0) \neq 0$, it follows that the rational function $F_1(z_1, z_2) \equiv F_2(z_1, z_1 z_2)$ is bounded, hence holomorphic in a neighborhood of the point (0, 0). By virtue of this, the rational function $F(z_1, z_2) \equiv r(F_1(z_1, z_2)/n, z_1 z_2)$ is also holomorphic at the point (0, 0), hence it decomposes into a series (2).

Calculating the integral

$$\frac{1}{2\pi i} \int_{|w|=\rho} F\left(w, \frac{z}{w}\right) \frac{dw}{w},$$

we get, on the other hand, that the function $\sum_{k \geq 0} a_{kk} z^k$ is the diagonal of the series (2) and, on the other hand,

$$\frac{1}{2\pi i} \int_{|w|=\rho} F\left(w, \frac{z}{w}\right) \frac{dw}{w} = \frac{1}{2\pi i} \int_{|w|=\rho} r\left(\frac{F_2(w, z)}{n}, z\right) \frac{dw}{w}.$$

Since the resolving polynomial $r(\alpha, \beta)$ is linear in α, it is easy to see that the last integral is equal to $r(h(z), z) = f(z)$, where the function $h(z)$ is given by (7). Theorem 1 is proved.

In [9] power series over a field of characteristic $p > 0$ are considered. It is shown that in this case the diagonal of the double power series of an algebraic function is also an algebraic function. We give an example, showing that for the field C this is not so.

Example. Let

$$F(z_1, z_2) = ((1 - z_1 - z_2)^2 - 4z_1z_2)^{-1/2} = \sum_{m,\, n\geqslant 0} \binom{m+n}{n}^2 z_1^m z_2^n$$

be an algebraic function; then its diagonal is equal to $f(z) = \sum_{k\geqslant 0} \binom{2k}{k}^2 z^k$. Since $\binom{2k}{k}^2 =$ $[(2k-1)!!\,2^k]^2/(k!)^2$, one has $f(z) = \sum_{k\geqslant 0} [(2k-1)!!/(k!)2^k]^2 (16z)^k = G(1/2, 1/2, 1; 16z)$, where $G(\alpha, \beta, \gamma; \zeta)$ is the Gauss hypergeometric function. For $\alpha = \beta = 1/2$, $\delta = 1$ this function is defined by the completely elliptic integral

$$G(1/2, 1/2, 1; \zeta) = \frac{2}{\pi}\int_0^1 ((1-x^2)(1-\zeta x^2))^{-1/2}\, dx$$

and has a logarithmic singularity.

3. A Necessary Condition for the Rationality of a Double Power Series

We associate with the series (2) the set

$$\mathfrak{M}_F = \bigcup_{g_k \not\equiv 0} \{z \in \mathrm{C}:\ g_k(z, 1) = 0\},$$

where $g_k(z_1, z_2) = \sum_{m=0}^{k} a_{m\,k-m} z_1^m z_2^{k-m}$ is a homogeneous polynomial of degree k, and let $\mathfrak{M}_F'$ be the collection of limit points of the set $\mathfrak{M}_F$.

For any closed set $M \subset \mathrm{C}$ there exists an entire function $F(z_1, z_2)$ for which $\mathfrak{M}_F' = M$.

In fact, since the set M is closed, there exists a countable set S such that $S' = M$. We enumerate the set S as a sequence $\{\zeta_k\}_{k=1}^{\infty}$ in such a way that $|\zeta_k| \leq k$, $k \in \mathrm{N}$; then the series in homogeneous polynomials

$$F(z_1, z_2) = \sum_{k\geqslant 1} \left[(k+1)^{-k}(k!)^{-1} \prod_{j=1}^{k} (z_1 - \zeta_j z_2)\right]$$

converges uniformly for all z_1, z_2. The function $F(z_1, z_2)$ is entire and for it $\mathfrak{M}_F = \{\zeta_k\}_{k=1}^{\infty}$ and $\mathfrak{M}_F' = M$.

THEOREM 2. If in (2) $F(z_1, z_2)$ is a rational function which is holomorphic in a neighborhood of zero, then $\mathfrak{M}_F' \subset \gamma \cup \Gamma$, where Γ is a finite set, $\gamma = \bigcup_{k=1}^{n} \{z \in \mathrm{C}:\ |\varphi_k| = |\psi_k|\}$; $\varphi_1, \psi_1, \ldots, \varphi_n, \psi_n$ are functions which are holomorphic outside the set Γ.

Proof. In (2) let $F(z_1, z_2) = P(z_1, z_2)/Q(z_1, z_2)$ bet a rational function. By Cauchy's formula we have $g_k(z, 1) = \mathrm{res}_{t=0}(P(zt, t)/Q(zt, t)t^{k+1})$. Applying the theorem on the complete sum of residues and calculating the residues at the poles $t = t_j(z)$ $(1 \leq j \leq n)$ of the function P/Qt^{k+1}, different from $t = 0$, and also considering that the residue at the point $t = \infty$ is equal to zero for $k > \deg P - \deg Q$, we get that

$$g_k(z, 1) = a_1(z, k)(t_1(z))^{-k} + \ldots + a_n(z, k)(t_n(z))^{-k} \tag{8}$$

for $k > k_0$, where $t_j(z)$ is an algebraic function, $a_j(z, k)$ is a polynomial in the integral variable k with coefficients which are algebraic functions of z; $1 \leq j \leq n$. Let Γ be the set consisting of the zeros of the coefficients of the highest powers of the variable k of the polynomials $a_j(z, k)$, and also of the singular points of the functions $1/t_j(z)$, $\alpha_j(z, k)$ $(1 \le j \le n)$. Obviously the set Γ is finite. If

$$z_0 \in \bigcup_{j=1}^n \{1/|t_j(z)| > \max(1/|t_i(z)|;\ i = 1, \ldots [j] \ldots, n)\} \setminus \Gamma,$$

then $z_0 \equiv \mathfrak{M}'_F$. In fact, to be definite, let $1/|t_1(z)| > \max(1/|t_j(z)|;\ j = 2, \ldots, n)$; then for all $k \ge N$ and all z from some neighborhood U of the point z_0 one has

$$|a_1(z, k)|\,|t_1(z)|^{-k} > \sum_{j=2}^n |a_j(z, k)|\,|t_j(z)|^{-k}.$$

It follows from this and (8) that in the neighborhood U only a finite number of the polynomials $g_k(z, 1)$ vanish and $z_0 \equiv \mathfrak{M}'_F$. Consequently, $\mathfrak{M}'_F \subset \bigcup_{i<j} \{|t_i| = |t_j|\} \bigcup = \gamma \bigcup \Gamma$. The construction of the set $\gamma \bigcup \Gamma$ $(\mathfrak{M}'_F \subset \gamma \bigcup \Gamma)$ can be made somewhat more detailed.

We note that a system of weighted homogeneous polynomials of a double power series of a rational function has the analogous property.

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