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A useful application of Brun's irrationality criterion

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Abstract

We show that Apéry's sequence of rational numbers that converge to $\zeta(3)$ contains a subsequence that satisfies an irrationality criterion of Brun.

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1. Introduction

In 1910, Viggo Brun established a sufficient condition for the limit of a convergent sequence of rational numbers to be irrational. In a later paper he described the result as "simple but unfortunately not very useful" since very few sequences satisfy the rather stringent criteria. His result is as follows.

Theorem 1 (*Brun's Irrationality Criterion, [1]*)**.** *Let (x_n) and (y_n) be strictly increasing sequences of natural numbers such that (x_n/y_n) is a strictly increasing sequence and tends to some limit L. If the sequence (δ_n) given by*

$$\delta_n = \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

is strictly decreasing then L is irrational.

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Whilst Brun did not consider the criterion very useful, it was pointed out in a recent note by Angelo Mingarelli [3] that it may be useful enough to give a new proof of Apéry's theorem that $\zeta(3)$ is irrational. In this paper we confirm Mingarelli's suspicions, proving that the sequences that Apéry defined in order to apply Dirichlet's criterion contain a subsequence that satisfies Brun's criterion. The result requires some of the same steps as Apéry's original proof, in particular the confirmation that his two rational sequences satisfy a certain recurrence relation and have explicitly bounded denominators. We omit reproving these details here, as they are proved mostly through exhausting algebraic manipulation rather than any crucial insights. The curious reader can find both proofs in Alfred van der Poorten's excellent paper on the theorem [5].

In the final part of the paper we will prove a second conjecture of Mingarelli that concerns the sequence δ_n defined above when the two sequences under consideration are Apéry's sequences. Mingarelli conjectured that in this case (δ_n) contains arbitrarily long runs of consecutive decreasing terms. This result is not sufficient to apply Brun's criterion, but is an interesting result nonetheless.

2. A proof of Apéry's theorem

In 1978 Roger Apéry defined a pair of sequences whose ratio converged to $\zeta(3)$ quickly enough to apply Dirichlet's criterion, and thus established the irrationality of $\zeta(3)$. The result came somewhat out of the blue, as did the sequences he defined. They involve an auxiliary function defined for integers $0 \le k \le n$ by

$$c_{n,k} = \sum_{\ell=1}^{n} \frac{1}{\ell^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}.$$

With this he then defined

$$a_n = \sum_{k=0}^{n} c_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The terms a_n are not in general integral, but if one defines

$$\nu_n = 2(\mathrm{lcm}[1, 2, \ldots, n])^3$$

then $\nu_n a_n$ and $\nu_n b_n$ are a pair of integer sequences with the property that

$$\left| \zeta(3) - \frac{\nu_n a_n}{\nu_n b_n} \right| \le \frac{c}{(\nu_n b_n)^{1+\delta}}$$

for all n and some explicit constants $c, \delta > 0$. This is Dirichlet's criterion for irrationality, and so $\zeta(3)$ is irrational.

The definitions above make it clear that $(\nu_n a_n)$ and $(\nu_n b_n)$ are increasing sequences of positive integers, and we will prove later that (a_n/b_n) is also an increasing sequence. So we seem tantalisingly close to applying Brun's criterion to these sequences. If we define the sequence (δ_n) by

$$\delta_n = \frac{\nu_{n+1}a_{n+1} - \nu_n a_n}{\nu_{n+1}b_{n+1} - \nu_n b_n}$$

then to fulfil the last of Brun's conditions we just need that (δ_n) is a strictly decreasing sequence. Alas, this is not the case. Indeed, the first counterexample comes early with $\delta_4 < \delta_5$. But δ_n does seem to have some general downward trends, and so Mingarelli made the following conjecture.

Conjecture 2 (*[3, Conjecture 5]*). *There is an increasing sequence n_k such that*

$$\delta_k := \frac{\nu_{n_{k+1}}a_{n_{k+1}} - \nu_{n_k} a_{n_k}}{\nu_{n_{k+1}}b_{n_{k+1}} - \nu_{n_k} b_{n_k}}$$

is a strictly decreasing sequence.

If this were true then we could apply Brun's criterion to the sequences $(\nu_{n_k}a_{n_k})_{k\in\mathbb{N}}$ and $(\nu_{n_k}b_{n_k})_{k\in\mathbb{N}}$ and establish the irrationality of $\zeta(3)$. In this section we prove the existence of just such a subsequence. Of crucial import will be the following recurrence relations, which we state here without proof, but whose proof can be found in, for example, [5].

Lemma 3. *Let*

$$P(n) = 34n^3 + 51n^2 + 27n + 5.$$

The sequences a_n and b_n satisfy the recurrence

$$(n+1)^3 u_{n+1} - P(n)u_n + n^3 u_{n-1} = 0.$$

This recurrence relation is arguably the hardest part of Apéry's original proof. Unfortunately we cannot avoid it by appealing to Brun's criterion rather than Dirichlet's. But we press on regardless.

For completeness we first establish that our sequences are the ones we should be looking at.

Lemma 4.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \zeta(3).$$

Proof. By the definitions of the three sequences we have

$$a_n = b_n \sum_{\ell=1}^{n} \frac{1}{\ell^3} + \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}.$$

For $1 \le m < n$ we have

$$\binom{n}{m}\binom{n+m}{m} \ge \binom{n}{1}\binom{n+1}{1} = n(n+1) > n^2.$$

While for $m = n$ we have

$$\binom{n}{m}\binom{n+m}{m} = \binom{2n}{n}.$$

This central binomial coefficient is easily checked to be at least n^2 since

$$\binom{2n}{n} \geq \left(\frac{2n}{n}\right)^n = 2^n \geq n^2.$$

And so

$$\sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \leq \sum_{m=1}^{k} \frac{1}{2n^2} \cdot \frac{(-1)^{m-1}}{m^3} \leq \frac{1}{2n^2} \sum_{m=1}^{k} \frac{1}{m^3}.$$

The last sum on the right is bounded by the infinite series $\sum m^{-3} = \zeta(3) < 2$. Putting this observation together with the above inequality back into our expression for a_n we arrive at

$$a_n < b_n \sum_{\ell=1}^{n} \frac{1}{\ell^3} + \frac{b_n}{n^2}.$$

To involve our proposed limit we write

$$\sum_{\ell=1}^{n} \frac{1}{\ell^3} = \zeta(3) - \sum_{\ell=n+1}^{\infty} \frac{1}{\ell^3}.$$

Hopefully the right-most sum is but an error term, which we can confirm by noting that

$$\sum_{\ell=n+1}^{\infty} \frac{1}{\ell^3} \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}.$$

And so

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \zeta(3)\right| &= \left|\frac{a_n}{b_n} - \sum_{\ell=1}^{n} \frac{1}{\ell^3} - \sum_{\ell=n+1}^{\infty} \frac{1}{\ell^3}\right| \\
&\leq \left|\frac{a_n}{b_n} - \sum_{\ell=1}^{n} \frac{1}{\ell^3}\right| + \left|\sum_{\ell=n+1}^{\infty} \frac{1}{\ell^3}\right| \\
&< \frac{3}{2n^2}.
\end{aligned}$$

In particular, $a_n/b_n \to \zeta(3)$. □

Next we establish another hypothesis of Brun's criterion, namely that (a_n/b_n) is an increasing sequence.

Lemma 5. *For $n \in \mathbb{N}$,*

$$a_{n+1}b_n - a_n b_{n+1} = \frac{6}{(n+1)^3}.$$

In particular, the sequence (a_n/b_n) is strictly increasing.

Proof. The proof of this uses the recurrence from Lemma 3, namely that if $P(n) = 34n^3 + 51n^2 + 27n + 5$ then

$$(n+1)^3 a_{n+1} - P(n)a_n + n^3 a_{n-1} = 0$$

and

$$(n+1)^3 b_{n+1} - P(n)b_n + n^3 b_{n-1} = 0$$

for all $n \geq 1$.

Multiplying the first of these recurrences by b_n, the second by a_n, and then subtracting gives

$$(n+1)^3(a_{n+1}b_n - b_{n+1}a_n) + n^3(a_{n-1}b_n - b_{n-1}a_n) = 0,$$

or that

$$(n+1)^3(a_{n+1}b_n - b_{n+1}a_n) = n^3(a_n b_{n-1} - b_n a_{n-1}).$$

Note that the right hand side of this identity is simply the left hand side with n replaced by $n-1$. In particular, using induction, we can repeatedly step down, replacing n by $n-1$ until we arrive at

$$(n+1)^3(a_{n+1}b_n - b_{n+1}a_n) = a_1 b_0 - a_0 b_1.$$

Substituting in the values of a_0, a_1, b_0, b_1 this gives us

$$a_{n+1}b_n - b_{n+1}a_n = \frac{6}{(n+1)^3}$$

for all $n \in \mathbb{N}$. $\square$

Now onto the rest of the proof. It will be useful to think of picking a subsequence (n_k) as first picking an element (n_1, n_2) from row n_1 of the following table:

$$\begin{array}{cccccc}
1,2 & 1,3 & 1,4 & 1,5 & 1,6 & \cdots \\
 & 2,3 & 2,4 & 2,5 & 2,6 & \cdots \\
 & & 3,4 & 3,5 & 3,6 & \cdots \\
 & & & 4,5 & 4,6 & \cdots \\
 & & & & & \ddots
\end{array}$$

We then go to row n_2 and pick an element (n_2, n_3) and so on. Since δ_k is a function of $\nu_{n_k}, \nu_{n_{k+1}}, a_{n_k}, a_{n_{k+1}}, b_{n_k}$, and $b_{n_{k+1}}$ it is better to think of it in terms of pairs (n_k, n_{k+1}) rather than the single index k, so we write

$$\delta(n_k, n_{k+1}) = \frac{\nu_{n_{k+1}} a_{n_{k+1}} - \nu_{n_k} a_{n_k}}{\nu_{n_{k+1}} b_{n_{k+1}} - \nu_{n_k} b_{n_k}}$$

and think of it as a function from elements of the above table to $\mathbb{Q}$. Our task is to find a subsequence (n_k) such that $\delta(n_k, n_{k+1}) > \delta(n_{k+1}, n_{k+2})$ for all k. We will achieve this in two parts. First we will show that if one fixes an $m \in \mathbb{N}$ then

$$\lim_{n\to\infty} \frac{\nu_{m+n}a_{m+n} - \nu_m a_m}{\nu_{m+n}b_{m+n} - \nu_m b_m} = \zeta(3).$$

So if we apply δ to the above table then the limit as we move right along any given row is $\zeta(3)$. Next we will show that given any m, $\delta(m, n) > \zeta(3)$ for all sufficiently large $n > m \geq 1$. And then we will be done, for if we apply δ to the above table then we can start by picking some (n_1, n_2) for which $\delta(n_1, n_2) > \zeta(3)$. (Spoiler: $(n_1, n_2) = (1, 2)$ will suffice.) We then need to pick an element (n_2, n_3) on row n_2 such that $\delta(n_1, n_2) > \delta(n_2, n_3)$. But by our lemmata $\delta(n_2, k) \to \zeta(3)$ as $k \to \infty$ and $\delta(n_2, k) > \zeta(3)$ for all sufficiently large k, hence we can always find an n_3 with $\delta(n_1, n_2) > \delta(n_2, n_3) > \zeta(3)$. We then go to row n_3 and so on. So let us do all that now.

Lemma 6. *Let x_n and y_n be positive sequences with y_n increasing and $y_n \to \infty$, and suppose $x_n/y_n \to L$. Fix some $m \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \frac{x_{m+n} - x_m}{y_{m+n} - y_m} = L.$$

Proof. Dividing both the numerator and denominator of

$$\frac{x_{m+n} - x_m}{y_{m+n} - y_m}$$

by y_{m+n} and then using the fact that $y_{m+n} \to \infty$ and $x_{m+n}/y_{m+n} \to L$ gives the result. □

Corollary 7. *For any fixed $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} \delta(m, m+n) = \zeta(3).$$

Proof. Setting $x_n = \nu_n a_n$ and $y_n = \nu_n b_n$ in the previous lemma and using the fact that $a_n/b_n \to \zeta(3)$ from Lemma 4, we have the required result. □

We now need to show that for any given $m \in \mathbb{N}$ we have that $\delta(m, n) > \zeta(3)$ for all sufficiently large n (although this inequality being true infinitely often would suffice). Rearranging this inequality it amounts to showing that for any given m, for all sufficiently large $n > m$ we have

$$\nu_n b_n \zeta(3) - \nu_n a_n < \nu_m b_m \zeta(3) - \nu_m a_m.$$

For fixed m the right hand side of this inequality is some positive constant. So if we can show that the left hand side tends to 0 then we will be done.

Lemma 8. *We have*

$$\nu_n b_n \zeta(3) - \nu_n a_n \ll \frac{\nu_n}{b_n}.$$

Proof. By Lemma 5,

$$a_n b_{n-1} - b_n a_{n-1} = \frac{6}{n^3}.$$

If we define $s_n = \zeta(3) - a_n/b_n$ then for any $n \geq 1$ we have

$$s_{n-1} - s_n = \frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{a_n b_{n-1} - b_n a_{n-1}}{b_{n-1} b_n} = \frac{6}{n^3 b_{n-1} b_n}.$$

By the absolute convergence of the series involved we can write

$$\begin{aligned}\zeta(3) - \frac{a_n}{b_n} &= s_n \\ &= s_n - s_{n+1} + s_{n+1} - s_{n+2} + s_{n+2} - s_{n+3} + s_{n+3} - \cdots \\ &= \sum_{m=n+1}^{\infty} (s_{m-1} - s_m) \\ &= 6 \sum_{m=n+1}^{\infty} \frac{1}{m^3 b_{m-1} b_m}.\end{aligned}$$

Since (b_n) is an increasing sequence we have

$$\begin{aligned}\zeta(3) - \frac{a_n}{b_n} &= 6 \sum_{m=n+1}^{\infty} \frac{1}{m^3 b_{m-1} b_m} \\ &< 6 \sum_{m=n+1}^{\infty} \frac{1}{m^3 b_n^2}.\end{aligned}$$

In particular, then,

$$\nu_n b_n \zeta(3) - \nu_n a_n < \frac{6\nu_n}{b_n} \sum_{m=n+1}^{\infty} \frac{1}{m^3} \leq \frac{6\zeta(3)\nu_n}{b_n} < \frac{8\nu_n}{b_n}. \quad \square$$

The above result is the same one that Apéry required in his proof. Ostensibly, though, we are now in a better position. In the original proof, in order to apply Dirichlet's irrationality criterion, one needs to show that for all sufficiently large n,

$$\left| \zeta(3) - \frac{\nu_n a_n}{\nu_n b_n} \right| \leq \frac{c}{(\nu_n b_n)^{1+\delta}}$$

for some constants $c, \delta > 0$. Rearranging this a bit and using the previous lemma this follows if it can be shown that there exists some constant $\varepsilon > 0$ such that $\nu_n = O(b_n^{1-\varepsilon})$. To apply Brun's criterion we just need that

$$\nu_n b_n \zeta(3) - \nu_n a_n \to 0,$$

which, again after rearrangement and using the last lemma, follows if we can show that $\nu_n = o(b_n)$. This is a strictly weaker statement than requiring that $\nu_n = O(b_n^{1-\varepsilon})$, so in theory Brun's criterion should now be easier to apply. In practice both ν_n and b_n approach geometric growth, and so to apply either criterion we need to show that if $\nu_n \ll t^n$ and $b_n \gg u^n$ then $t < u$, hence applying either criterion in this case turns out to an equivalent exercise.

Lemma 9. *With ν_n and b_n as above, we have*

$$\nu_n \ll 27^n$$

and

$$b_n \gg 28^n.$$

Proof. The prime number theorem fairly readily offers up the asymptotic upper bound $\mathrm{lcm}[1, \dots, n] \ll \exp((1+\varepsilon)n)$ for any $\varepsilon > 0$. But one need not apply such heavy machinery. In [2], Hanson gives an elementary proof in the vein of Chebyshev to attain the upper bound

$$\mathrm{lcm}[1, \dots, n] \le 3^n$$

for all sufficiently large n, whence

$$\nu_n = 2\mathrm{lcm}[1, \dots, n]^3 \ll 27^n.$$

The lower bound for b_n follows from the recurrence relation. Indeed, we will show that for all $n \ge 9$ we have

$$\frac{b_n}{b_{n-1}} > 28.$$

First we note that $b_9/b_8 = 28.6\dots$ so the claim holds for $n = 9$. Now suppose it holds for some $n \ge 9$. The recurrence relation for b_n tells us

$$\frac{b_{n+1}}{b_n} = \frac{P(n)}{(n+1)^3} - \frac{n^3}{(n+1)^3}\frac{b_{n-1}}{b_n}.$$

This with the inductive hypothesis leads to

$$\begin{aligned}\frac{b_{n+1}}{b_n} &\ge \frac{P(n)}{(n+1)^3} - \frac{n^3}{28(n+1)^3} \\ &= \frac{28P(n) - n^3}{28(n+1)^3}.\end{aligned}$$

After rearranging this, proving the claim amounts to proving that

$$P(n) \ge 28(n+1)^3 + n^3/28.$$

We actually prove the slightly stronger statement that

$$P(n) \ge 29n^3 + 84n^2 + 84n + 28 = 28(n+1)^3 + n^3.$$

To do this we simply step along the polynomial $P(n)$ replacing An^d by $(A-c)n^d + 9cn^{d-1}$ for some suitable c, using the fact that $n \ge 9$ to ensure that the inequality faces the right way. Hence we get:

$$\begin{aligned}34n^3 + 51n^2 + 27n + 5 &\ge 29n^3 + 96n^2 + 27n + 5 \\ &\ge 29n^3 + 84n^2 + 135n + 5 \\ &\ge 29n^3 + 84n^2 + 84n + 464\end{aligned}$$

which is big enough for our purposes. Hence $b_n \gg 28^n$. □

Corollary 10. *For any $m \in \mathbb{N}$ there is $N_m \in \mathbb{N}$ such for any $n \ge N_m$,*

$$\delta(m, n) > \zeta(3).$$

Proof. By simple rearrangement, $\delta(m, n) > \zeta(3)$ is true if and only if

$$\nu_n b_n \zeta(3) - \nu_n a_n < \nu_m b_m \zeta(3) - \nu_m a_m.$$

For fixed m, the right hand side of this inequality is just some positive constant ε_m. By the previous two lemmas the sequence on the left converges to zero and thus must eventually fall below and stay below ε_m. □

Theorem 11. *There is a sequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \geq 1$,*

$$\delta(n_k, n_{k+1}) > \delta(n_{k+1}, n_{k+2}).$$

Proof. We can take $n_1 = 1$ and then look for a value n_2 such that $\delta(1, n_2) > \zeta(3)$. In fact we can take $n_2 = 2$ since $\delta(1, 2)$ is, by a happy coincidence, the fifth convergent to $\zeta(3)$, and being an odd convergent is thus greater than $\zeta(3)$. We then need to find a value $n_3 > 2$ such that

$$\zeta(3) < \delta(2, n_3) < \delta(1, 2).$$

Between them, Corollaries 7 and 10 assure us of the existence of just such a value n_3. And so we proceed inductively to find the whole sequence. □

Corollary 12. *The number $\zeta(3)$ is irrational.*

Proof. Let (n_k) be the sequence given by Theorem 11. Let $\alpha_k = \nu_{n_k} a_{n_k}$ and $\beta_k = \nu_{n_k} b_{n_k}$. As before we know α_k and β_k are increasing sequences of integers and α_k/β_k is an increasing sequence with, by Lemma 4,

$$\lim_{k\to\infty} \frac{\alpha_k}{\beta_k} = \zeta(3).$$

Moreover,

$$\frac{\alpha_{k+1} - \alpha_k}{\beta_{k+1} - \beta_k} = \delta(n_k, n_{k+1})$$

is a decreasing sequence by Theorem 11. So, by Brun's irrationality criterion, $\zeta(3)$ is irrational. □

3. A geometric interpretation

The intuition behind Brun's criterion is disarmingly simple. Rational approximations a_n/b_n to a real number $\alpha > 0$ can be thought of as lattice points $(b_n, a_n) \in \mathbb{Z}^2$ such that the slope of the line joining the origin to (b_n, a_n) approaches the slope of the line $y = \alpha x$, i.e. α. If α is rational then there is a strip on either side of the line $y = \alpha x$ containing no lattice points. Brun's criterion says if there are infinitely many lattice points (b_n, a_n) whose abscissae increase with n such that all the points lie below the line $y = \alpha x$, but such that the distance from the points to the line decreases to zero with n, then α is irrational. Hence, if one draws the sequence out in $\mathbb{R}^2$ and joins each point (b_n, a_n) to (b_{n+1}, a_{n+1}) then the resulting polygon shows its convex side to the line $y = \alpha x$. The reason Brun claimed the criterion was not very useful is because most sequences of rational numbers that converge to a given irrational will show their concave side to the line, hence allowing the distance from the points (b_n, a_n) to the line to increase with n.

That is not to say that these Brun's criterion-satisfying sequences are hard to find. There is a canonical such sequence for any irrational number. If the convergents to an irrational number α are p_n/q_n then the points (q_{2n}, p_{2n}) all lie below the line $y = \alpha x$ (since the even convergents are less than α) and the distance from these points to the line $y = \alpha x$ decreases with n, since the convergents are those numbers that minimise this distance (see [4, Chapter 7]). Unfortunately if one can find a simple expression for all the even convergents to a number α then one will have presumably shown that they are infinite in number, and hence the number α is *a fortiori* irrational.

4. The downward trends of δ_n

Earlier we cryptically remarked that while the sequence δ_n defined by

$$\delta_n = \delta(n, n+1) = \frac{\nu_{n+1}a_{n+1} - \nu_n a_n}{\nu_{n+1}b_{n+1} - \nu_n b_n}$$

is not strictly decreasing since, for example, $\delta_4 < \delta_5$, it does nonetheless exhibit some general "downward trends". These trends were noticed in [3] wherein the following appeared.

Conjecture 13 (*[3, Conjecture 6]*). *For any $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$\delta_n > \delta_{n+1} > \cdots > \delta_{n+N-1}.$$

That is, there are arbitrarily long runs where this sequence is decreasing. In this section we establish this fact. The primary obstacle to the statement is the presence of the factors ν_n and ν_{n+1} in the definition of δ_n. So first we figure out how to get rid of them.

Lemma 14. *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$\mathrm{lcm}[1, \ldots, n] = \mathrm{lcm}[1, \ldots, n+1] = \cdots = \mathrm{lcm}[1, \ldots, n+N].$$

Proof. Since

$$\mathrm{lcm}[1, \ldots, n] \leq \mathrm{lcm}[1, \ldots, n+1] \leq \cdots \leq \mathrm{lcm}[1, \ldots, n+N]$$

it suffices to find an n such that $\mathrm{lcm}[1, \ldots, n] = \mathrm{lcm}[1, \ldots, n+N]$. Using the standard formula for lowest common multiples we require

$$\prod_{p \leq n} p^{\lfloor \log n/\log p \rfloor} = \prod_{p \leq n+N} p^{\lfloor \log(n+N)/\log p \rfloor}$$

where both products are taken over the primes. If we take $n = k! + k - N$ for some $k \geq N+2$ then we know that $n, n+1, \ldots, n+N$ are all composite since they are divisible by, respectively, $k-N, k-N+1, \ldots, k$. So we just need to find a k so that

$$\prod_{p<n} p^{\lfloor \log n/\log p \rfloor} = \prod_{p<n} p^{\lfloor \log(n+N)/\log p \rfloor}.$$

Clearly if

$$\left\lfloor \frac{\log(n+N)}{\log p} \right\rfloor = \left\lfloor \frac{\log n}{\log p} \right\rfloor$$

for all $p < n$ then we are done. This condition will fail precisely if there is an integer c such that

$$\frac{\log n}{\log p} < c \le \frac{\log(n+N)}{\log p},$$

so that

$$n < p^c \le n + N.$$

So if we can find a k such that there are no prime powers in the range $(k! + k - N, k! + k]$ then we are done.

We can write the integers in this range as $k! + (k - d)$ for $0 \le d \le N - 1$. And so we may factorise them as

$$k! + (k - d) = (k - d)[k(k-1)\cdots(k-d+1)(k-d-1)! + 1].$$

So if one of these numbers $k! + (k - d)$ is of the form p^c for some prime p then $k - d = p^a$ and

$$k(k-1)\cdots(k-d+1)(k-d-1)! + 1 = p^b$$

for integers $a + b = c$. But if $a > 1$ then by Wilson's theorem

$$(k-d-1)! \equiv 0 \pmod{p}$$

and so

$$k(k-1)\cdots(k-d+1)(k-d-1)! + 1 \equiv 1 \pmod{p}.$$

In particular if $k - d = p^a$ with $a > 1$ then $k! + (k - d)$ is not a prime power. The only problem would be if $k - d$ were prime for some d. If we take $k = (N + 1)! + N + 1$ then $k - d$ has to be composite for $0 \le d \le N - 1$, and so we are done. That is, if $n = ((N + 1)! + N + 1)! + (N + 1)! + N + 1$ then

$$\mathrm{lcm}[1, \ldots, n] = \mathrm{lcm}[1, \ldots, n+1] = \cdots = \mathrm{lcm}[1, \ldots, n+N]. \quad \square$$

Taking these values of n as our starting points, we have N terms in a row for which $\nu_{n+1} = \nu_n$. Thus we just need to show the sequence without the ν_n terms is decreasing. In fact much more is true. If Brun's criterion allowed rational sequences rather than just integral ones then *any* subsequence of Apéry's original sequences would suffice, thanks to the following.

Theorem 15. *Let (a_n) and (b_n) be the sequences as defined earlier, and let $n > m > \ell \ge 0$ be natural numbers. Then*

$$\frac{a_m - a_\ell}{b_m - b_\ell} > \frac{a_n - a_m}{b_n - b_m}. \tag{1}$$

In particular, then, by starting at those values n given by Lemma 14 we can cancel all the ν_n terms and then use this result to give Conjecture 13. The proof of Theorem 15 is in a few stages. First we generalise Lemma 5.

Lemma 16. *Let $Q_0(n) = 0$, $Q_1(n) = 1$, and for $k \geq 1$ let*

$$Q_{k+1}(n) = P(n+k)Q_k(n) - (n+k)^6 Q_{k-1}(n).$$

Then for all $n \geq 0$ and $k \geq 0$,

$$a_{n+k}b_n - a_n b_{n+k} = \frac{6Q_k(n)}{(n+1)^3(n+2)^3 \cdots (n+k)^3} = \frac{6n!^3 Q_k(n)}{(n+k)!^3}.$$

Proof. The identity is trivial for $k = 0$ and holds for $k = 1$ by Lemma 5. Now suppose it holds for some k and $k-1$ where $k \geq 1$. Then using Lemma 3

$$\begin{aligned} a_{n+k+1}b_n - a_n b_{n+k+1} &= \left(\frac{P(n+k)a_{n+k} - (n+k)^3 a_{n+k-1}}{(n+k+1)^3} \right) b_n \\ &\quad - a_n \left(\frac{P(n+k)b_{n+k} - (n+k)^3 b_{n+k-1}}{(n+k+1)^3} \right) \\ &= \frac{P(n+k)}{(n+k+1)^3}(a_{n+k}b_n - a_n b_{n+k}) \\ &\quad - \frac{(n+k)^3}{(n+k+1)^3}(a_{n+k-1}b_n - a_n b_{n+k-1}). \end{aligned}$$

By the inductive hypotheses this can be written

$$\begin{aligned} a_{n+k+1}b_n - a_n b_{n+k+1} &= \frac{6P(n+k)Q_k(n)}{(n+1)^3 \cdots (n+k+1)^3} \\ &\quad - \frac{6(n+k)^3 Q_{k-1}(n)}{(n+1)^3 \cdots (n+k-1)^3(n+k+1)^3} \end{aligned}$$

whence the result follows. □

We now prove Theorem 15 via the following three lemmas.

Lemma 17. *For all $n \geq 1$,*

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} > \frac{a_{n+1} - a_n}{b_{n+1} - b_n}.$$

Lemma 18. *For all $n > m \geq 0$,*

$$\frac{a_n - a_m}{b_n - b_m} > \frac{a_{n+1} - a_m}{b_{n+1} - b_m}.$$

Lemma 19. *For all $n > m + 1 > 0$,*

$$\frac{a_n - a_m}{b_n - b_m} > \frac{a_n - a_{m+1}}{b_n - b_{m+1}}.$$

Between them these three inequalities grant inequality (1) for any $\ell < m < n$. Indeed, repeated applications of Lemma 18 give us

$$\frac{a_m - a_\ell}{b_m - b_\ell} > \frac{a_{m+1} - a_\ell}{b_{m+1} - b_\ell} > \cdots > \frac{a_n - a_\ell}{b_n - b_\ell},$$

and then using Lemma 19 repeatedly gives us

$$\frac{a_n - a_\ell}{b_n - b_\ell} > \frac{a_n - a_{\ell+1}}{b_n - b_{\ell+1}} > \cdots > \frac{a_n - a_m}{b_n - b_m}.$$

Proof of Lemma 17. Multiplying by the denominators and simplifying, the inequality is equivalent to

$$(a_n b_{n+1} - a_{n+1} b_n) + (a_{n-1} b_n - a_n b_{n-1}) + (a_{n+1} b_{n-1} - a_{n-1} b_{n+1}) > 0.$$

Using Lemma 16 on the three bracketed terms, this in turn is equivalent to

$$\frac{6 Q_2(n-1)}{n^3 (n+1)^3} > \frac{6}{(n+1)^3} + \frac{6}{n^3}.$$

Using the fact that $Q_2(n-1) = P(n)$ this simplifies to the inequality

$$P(n) > n^3 + (n+1)^3.$$

But comparing coefficients of both sides this inequality very much holds. □

Proof of Lemma 18. Multiplying by the denominators and rearranging a bit, then using Lemma 16 shows that the claimed inequality is equivalent to

$$Q_{n+1-m}(m) > \frac{n!^3}{m!^3} + (n+1)^3 Q_{n-m}(m) \tag{2}$$

for any $n > m \geq 0$. We prove this by induction on n. The basis case is when $n = m + 1$, then the inequality simplifies to

$$P(n) > n^3 + (n+1)^3,$$

which as we noted above is true for all $n \geq 0$.

Now suppose that inequality (2) holds for some $n \geq m + 1$. Then using this and Lemma 16

$$\begin{aligned} Q_{n+2-m}(m) &= P(n+1) Q_{n+1-m}(m) - (n+1)^6 Q_{n-m}(m) \\ &> P(n+1) Q_{n+1-m}(m) + \frac{(n+1)!^3}{m!^3} - (n+1)^3 Q_{n+1-m}(m) \\ &= \frac{(n+1)!^3}{m!^3} + (P(n+1) - (n+1)^3) Q_{n+1-m}(m), \end{aligned}$$

and so we are done if $P(n+1) - (n+1)^3 > (n+2)^3$. But this is the same inequality we keep coming back to, so we have the result. □

Proof of Lemma 19. We want to prove that

$$\frac{a_n - a_m}{b_n - b_m} > \frac{a_n - a_{m+1}}{b_n - b_{m+1}}$$

whenever $n > m + 1 > 0$. Note that

$$\frac{a_n - a_m}{b_n - b_m} = \frac{(a_n - a_{m+1}) + (a_{m+1} - a_m)}{(b_n - b_{m+1}) + (b_{m+1} - b_m)}.$$

And so by the usual inequality for mediants we have the desired inequality if

$$\frac{a_{m+1} - a_m}{b_{m+1} - b_m} > \frac{a_n - a_{m+1}}{b_n - b_{m+1}}.$$

We prove this by induction on n. The basis case is $n = m + 2$, but this is precisely Lemma 17. Now suppose that it holds for some n. By this assumption and Lemma 18 we then have

$$\frac{a_{m+1} - a_m}{b_{m+1} - b_m} > \frac{a_n - a_{m+1}}{b_n - b_{m+1}} > \frac{a_{n+1} - a_{m+1}}{b_{n+1} - b_{m+1}}$$

which proves the result. □

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