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The Truncated Tetrahedron Is Rupert

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NOTES

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Abstract. A polyhedron $\mathcal{P}$ has the Rupert property if a straight tunnel can be made in it, large enough so that a copy of $\mathcal{P}$ can pass through this tunnel. Eight Archimedean polyhedra are known to have the Rupert property. In this note, we append the truncated tetrahedron to this list.

At the end of the 17th century, Wallis related that Prince Rupert of the Rhine, (Ruprecht von der Pfalz), proved he could cut a tunnel through a cube, permitting another congruent cube to pass through [6]. We call a solid having the Rupert property RP, that is, that in such a solid, a hole can be cut through which a copy of the solid can pass. It has been proved that all Platonic solids are RP [3–5]. In dimension $n \geq 3$, the n-cube is RP [2]. And in [1], Chai, Yuan, and Zamfirescu proved that eight among the thirteen Archimedean solids are RP. For the other five Archimedean solids (the truncated tetrahedron, the snub cube, the rhombicosidodecahedron, the truncated icosidodecahedron, and the snub dodecahedron), the problem remained open. We consider here the case of the truncated tetrahedron, and we claim that this solid also has the Rupert property.

More precisely, a convex solid $\mathcal{P}$ is RP if the following construction is possible: Let two vectors $\boldsymbol{n}_{\text{out}}$ and $\boldsymbol{n}_{\text{in}}$ be respectively orthogonal to two planes π_{out} and π_{in}. The orthogonal projections of $\mathcal{P}$ onto these two planes will give two convex sets, the outer projected set and the inner projected set. If there is an isometry μ such that the image of the inner projected set by μ is included in the interior of the outer projected set, then $\mathcal{P}$ is RP [1].

Theorem 1. *The truncated tetrahedron has the Rupert property.*

Proof. We take the truncated tetrahedron $\mathcal{T}$ whose vertices in $\mathbb{R}^3$ have coordinates given by the column named $\mathcal{T}$ in Table 1. In this polyhedron, the hexagonal face $GHIJKL$ is contained in the plane $x + y + z = -3$, and is parallel to the triangle ABC contained in the plane $x + y + z = 5$. Both planes are orthogonal to the vector $(1, 1, 1)$. See Figure 1.

For the inner projection, we take $\boldsymbol{n}_{\text{in}} = (1, 0, -1)$. This vector is orthogonal to the plane π_{in} having equation $x = z$, and an orthonormal basis of π_{in} is $(\boldsymbol{e}^1_{\text{in}}, \boldsymbol{e}^2_{\text{in}})$ with $\boldsymbol{e}^1_{\text{in}} = \frac{1}{\sqrt{6}}(-1, 2, -1)$ and $\boldsymbol{e}^2_{\text{in}} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Let φ_{in} be the function defined by

$$\forall\ \boldsymbol{V} \in \mathbb{R}^3, \varphi_{\text{in}}(\boldsymbol{V}) = (\langle \boldsymbol{V} \mid \boldsymbol{e}^1_{\text{in}}\rangle, \langle \boldsymbol{V} \mid \boldsymbol{e}^2_{\text{in}}\rangle) = \left(\frac{-x + 2y - z}{\sqrt{6}}, \frac{x + y + z}{\sqrt{3}}\right),$$

where $\langle\,|\,\rangle$ is the scalar product of $\mathbb{R}^3$. $\varphi_{\text{in}}(\boldsymbol{V})$ gives the coordinates in the basis $(\boldsymbol{e}^1_{\text{in}}, \boldsymbol{e}^2_{\text{in}})$ of the orthogonal projection of $\boldsymbol{V}$ onto the plane π_{in}. Before taking the projection of $\mathcal{T}$, we translate it so that the point J will be at the origin $(0, 0, 0)$ of $\mathbb{R}^3$. Let $\mathcal{T} - J$ denote this translated truncated tetrahedron. After having projected $\mathcal{T} - J$ onto π_{in} with φ_{in},

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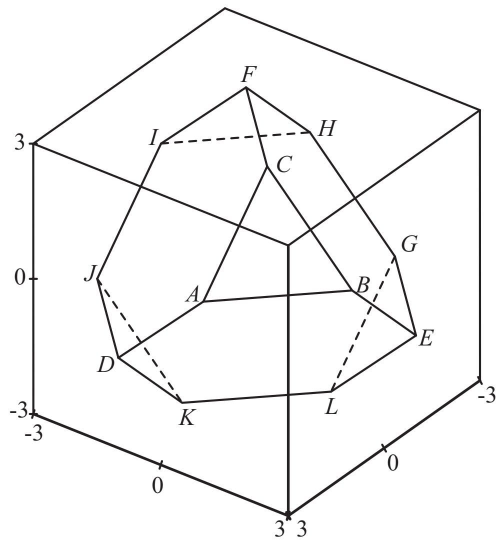

Figure 1. The truncated tetrahedron.

Table 1. Coordinates of vertices.

Vertex	$\mathcal{T}$	P_{in}	P_{out}
A	$(3, 1, 1)$	$(1.66299, 4.61880)$	$(1.48866, 4.64658)$
B	$(1, 3, 1)$	$(4.11248, 4.61880)$	$(4.30296, 4.63246)$
C	$(1, 1, 3)$	$(1.66299, 4.61880)$	$(2.65127, 4.51771)$
D	$(3, -1, -1)$	$(0.84650, 2.30940)$	$(0.16303, 2.38772)$
E	$(-1, 3, -1)$	$(5.74548, 2.30940)$	$(5.79162, 2.35950)$
F	$(-1, -1, 3)$	$(0.84650, 2.30940)$	$(2.48824, 2.12999)$
G	$(-3, 1, -1)$	$(4.92898, 0)$	$(5.62859, -0.02823)$
H	$(-3, -1, 1)$	$(2.47949, 0)$	$(3.97690, -0.14298)$
I	$(-1, -3, 1)$	$(0.03, 0)$	$(1.16261, -0.12887)$
J	$(1, -3, -1)$	$(0.03, 0)$	$(0, 0)$
K	$(1, -1, -3)$	$(2.47949, 0)$	$(1.65169, 0.11476)$
L	$(-1, 1, -3)$	$(4.92898, 0)$	$(4.46599, 0.10064)$

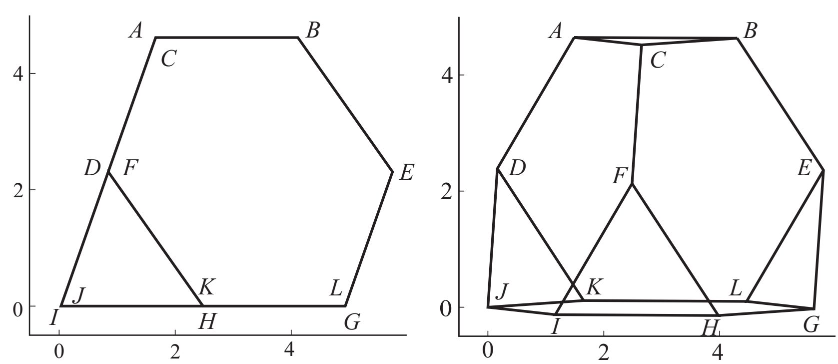

Figure 2. P_{in} (on the left) and P_{out} (on the right).

we translate the projected set by the vector $(0.03, 0)$ so that the projected point J is gently moved right to $(0, 0)$. So, the final inner projected set is $P_{\text{in}} = \varphi_{\text{in}}(\mathcal{T} - J) + (0.03, 0)$. (See the left part of Figure 2.) The coordinates of the projected vertices, up to five decimal places, are given by the column named P_{in} in Table 1.

For the outer projection, we consider the rotation R_1 around the axis $(-1, 1, 0)$ with the angle $\frac{1}{20}$ rad, followed by the rotation R_2 around the axis $(-1, -1, -1)$ with angle $\frac{1}{10}$ rad. Let $R = R_2 \circ R_1$. The matrices of the three rotations (when we apply them to a column vector on their right) are given to five decimal places:

$$R_1 = \begin{bmatrix} 0.99938 & -0.00062 & 0.03534 \\ -0.00062 & 0.99938 & 0.03534 \\ -0.03534 & -0.03534 & 0.99875 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.99667 & 0.05930 & -0.05597 \\ -0.05597 & 0.99667 & 0.05930 \\ 0.05930 & -0.05597 & 0.99667 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.99799 & 0.06062 & -0.01858 \\ -0.05866 & 0.99399 & 0.09247 \\ 0.02408 & -0.09120 & 0.99554 \end{bmatrix}.$$

Let $\boldsymbol{n} = (1, 1, -2)$. This vector is orthogonal to the plane $x + y - 2z = 0$, and an orthonormal basis of this plane is $\boldsymbol{e}^1 = \frac{1}{\sqrt{2}}(-1, 1, 0)$ and $\boldsymbol{e}^2 = \frac{1}{\sqrt{3}}(1, 1, 1)$. We set $\boldsymbol{n}_{\text{out}} = R(\boldsymbol{n})$. The vectors $\boldsymbol{e}^1_{\text{out}} = R(\boldsymbol{e}^1)$ and $\boldsymbol{e}^2_{\text{out}} = R(\boldsymbol{e}^2)$ form an orthonormal basis of the plane π_{out} orthogonal to $\boldsymbol{n}_{\text{out}}$. As for the inner projection, we translate the truncated tetrahedron $\mathcal{T}$ so that J will be at $(0, 0, 0)$. Let φ_{out} be the function defined by

$$\varphi_{\text{out}}(\boldsymbol{V}) = (\langle \boldsymbol{V} \mid \boldsymbol{e}^1_{\text{out}} \rangle, \langle \boldsymbol{V} \mid \boldsymbol{e}^2_{\text{out}} \rangle).$$

The final outer projected set is $P_{\text{out}} = \varphi_{\text{out}}(\mathcal{T} - J)$. (See the right part of Figure 2.) The coordinates of the projected vertices are given by the column named P_{out} in Table 1.

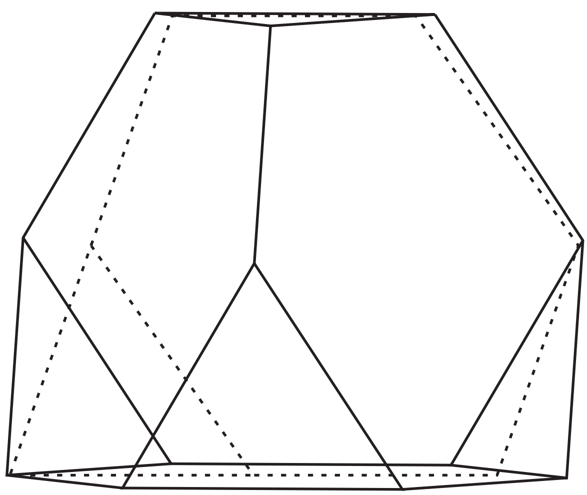

Figure 3. P_{out} and dashed P_{in}.

To compare P_{out} and P_{in}, we superimpose the two figures; see Figure 3. We can verify that P_{in} is in the interior of P_{out}. For example, the projection of the face ABC onto P_{in} is in the interior of P_{out}, since the ordinates of its vertices are less than 4.62

whereas the ordinates of the projections of A and B onto P_{out} are more than 4.63. The projection of the point J onto P_{in} is in the interior of P_{out} since it is on the right of the projection of the point J onto P_{out}. The projection E_{in} of the point E onto P_{in} is on the left of the projection $[G_{\text{out}}E_{\text{out}}]$ of the segment $[GE]$ onto P_{out} since $\det(E_{\text{out}} - G_{\text{out}}, E_{\text{in}} - G_{\text{out}}) \approx 0.10201$ and is positive.

■

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