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ALGEBRAIC SPACES

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PREFACE

These notes are based on lectures given at Yale University in the spring of 1969. Their object is to show how algebraic functions can be used systematically to develop certain notions of algebraic geometry, which are usually treated by rational functions by using projective methods. The global structure which is natural in this context is that of an algebraic space—a space obtained by gluing together sheets of affine schemes by means of algebraic functions.

I tried to assume no previous knowledge of algebraic geometry on the part of the reader but was unable to be consistent about this. The attempt only prevented me from developing any topic systematically. Thus, at best, the notes can serve as a naive introduction to the subject.

M. A.

Cambridge, Massachusetts
1971

1 ALGEBRAIC FUNCTIONS

It seems reasonable for us to restrict our attention here to algebraic geometry over an algebraically closed field k — the generalization to arbitrary ground fields is for the most part routine. I will try to bring out analogies with the theory of complex analytic spaces as we go along, and then we will specialize to the case that $k = \mathbf{C}$ is the field of complex numbers.

The basic building blocks in algebraic geometry are "affine schemes." An affine scheme V is determined by a system of polynomial equations

$$(1.1)\qquad \begin{array}{l} g_1(x_1, \ldots, x_n) = 0 \\ \vdots \qquad\qquad\quad \vdots \\ g_r(x_1, \ldots, x_n) = 0\,, \quad g_i \in k[x_1, \ldots, x_n]\,. \end{array}$$

(For the present, we are considering only schemes "of finite type over k.") It has an underlying point set $|V|$, the set of points $c = (c_1, \ldots, c_n) \in k^n$ which are solutions of (1.1):

$$g(c) = 0\,.$$

These are the "points" of V. But the point set does not determine the scheme. Instead, the notion of scheme is defined by the convention that two systems (1.1), (1.1′) shall determine the same scheme if and only if the ideal $I = (g_1, \ldots, g_r)$ generated in the polynomial ring $k[x]$ by $\{g_1, \ldots, g_r\}$ is the same as the corresponding ideal I'. When this is so, the underlying point sets are equal, as is easily seen. Conversely, if

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two ideals I,I′ determine the same point sets, then their radicals are equal: $\sqrt{I} = \sqrt{I'}$, where

$$\sqrt{I} = \{f \in k[x] \mid f^N \in I \text{ for some } N\}.$$

That is the Hilbert Nullstellensatz.

A scheme V whose ideal I is equal to its radical is said to be a "reduced" scheme. Thus the reduced schemes are the ones determined by their point sets. Sometimes they are called "varieties." But a general scheme V has certain infinitesimal properties which would be lost if it were replaced by the scheme V_{red} determined by $\sqrt{I}$. We do not need to worry about the nature of these properties for the present.

Let V be as above. Its "affine coordinate ring" is defined to be the ring

$$A = k[x_1, \ldots, x_n]/(g_1, \ldots, g_r).$$

It follows from the Nullstellensatz that the points $|V|$ are in one-to-one correspondence with the maximal ideals of A. If W is another affine scheme, given by the system of equations

$$(1.2) \qquad \begin{aligned} &h_1(y_1, \ldots, y_N) = 0 \\ &\quad\vdots \\ &h_s(y_1, \ldots, y_N) = 0 \qquad h \in k[y], \end{aligned}$$

then a "map" $V \to W$ is by definition nothing more than a k-algebra homomorphism

$$A \xleftarrow{\phi} B,$$

where $B = k[y]/(h)$ is the affine ring of W. Such a homomorphism is determined by polynomials

$$y_j(x) \quad j = 1, \ldots, N$$

subject to the condition that for $\nu = 1, \ldots, s$

$$h_\nu(y(x)) \equiv 0 \quad (\text{modulo } (g_1, \ldots, g_r)).$$

Of course, ϕ induces a map on the underlying point sets

$$|V| \to |W|.$$

Thus the category of affine schemes and maps between them is by definition the dual (arrow reversed) of the category of k-algebras which are quotients $A = k[x]/(g)$ of polynomial rings.

If we like, we can consider affine schemes without a fixed embedding into affine space. They correspond (again by definition) to k-algebras A of finite type, i.e., to ones isomorphic to some $k[x]/(g)$:

$$(\text{affine schemes}/k) \approx (k\text{-algebras})^\circ$$

(the symbol $^\circ$ denotes the dual category). The affine scheme V corresponding to an algebra A is usually denoted by

$$V = \operatorname{Spec} A .$$

The empty set of equations determines the scheme called affine n-space $E^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. Its underlying point set is of course the set of all n-tuples $c = (c_1, \ldots, c_n) \in k^n$. For any affine scheme $V = \operatorname{Spec} A$, a map from V to the "affine line" E^1,

$$V \to E^1$$

is given by a k-homomorphism $\phi : k[t] \to A$ (t "variable"), i.e., by an arbitrary choice of element $\phi(t) = a \in A$. Thus we may think of the elements of A as "functions" on V in the sense that they correspond to admissible maps to E^1. It is clear that such a function has a "value" at a point of V: Say that

$$A = k[x_1, \ldots, x_n]/(g) \quad \text{and that} \quad f(x_1, \ldots, x_n)$$

is a polynomial representing $\bar{f} \in A$. Let the point be $c = (c_1, \ldots, c_n)$. Then the value of $\bar{f}$ at c is just

$$\bar{f}(c) = f(c_1, \ldots, c_n) .$$

However, the values of $\bar{f}$ at the various points of V do not determine $\bar{f}$ unless V is reduced.

A "closed subscheme" of a scheme $V = \operatorname{Spec} A$ is a scheme of the form $W = \operatorname{Spec} A/\mathfrak{a}$, for some ideal $\mathfrak{a}$ of A. For instance, a presentation $A = k[x]/(g)$ represents V as a closed subscheme of affine space E^n.

Then $A/\mathfrak{a}$ becomes a quotient of $k[x]$ too, obtained by adding some more equations to (1.1). Thus the point set $|W|$ is a subset of $|V|$, i.e., the map $W \to V$ determined by the canonical projection $A \to A/\mathfrak{a}$ is one to one. The subsets of $|V|$ of this form are the closed sets of a topology on $|V|$, which is called the "Zariski topology."

The next thing to consider is the notion of functions defined "locally" at a point of a scheme, say at the origin $(0, \ldots, 0)$ in affine space E^n. We want an algebraic analogue of the ring

$$C < x_1, \ldots, x_n >$$

of convergent power series in $x_1, \ldots, x_n$. An obvious choice is the ring of rational functions

$$\frac{f(x)}{g(x)}, \quad x = x_1, \ldots, x_n,$$

where $f, g \in k[x]$ and $g(0) \neq 0$. This is the classical choice. However, it is not adequate for our purposes. Instead, we take the ring

$$k\{x_1, \ldots, x_n\}$$

of "algebraic functions," i.e., the subring of the ring $k[[x]]$ of formal power series consisting of those series $\phi(x)$ which are algebraically dependent on the coordinate functions x over k. In other words, those ϕ for which there exists a nonzero polynomial $f(x,y) \in k[x,y]$ with $f(x,\phi(x)) = 0$.

These algebraic functions may be defined implicitly by polynomial equations as follows: Consider a system of polynomial equations

$$(1.3) \qquad f(x,y) = 0 \quad f \in k[x,y],$$

where $x = x_1, \ldots, x_n$, $y = y_1, \ldots, y_N$, and $f = f_1, \ldots, f_N$. Suppose given a solution in k:

$$x = 0 \quad y = y^\circ$$

such that the jacobian with respect to y is not zero there:

$$\det\left(\frac{\partial f}{\partial y}\right)(0,y^\circ) \neq 0 .$$

Then the system (1.3) can be solved uniquely for power series y(x) with $y(0) = y^\circ$ in the usual way, by solving inductively for the coefficients of the series. It is not difficult to show that such implicitly defined series $y_\nu(x)$ are algebraic functions and that, conversely, any algebraic function can be obtained in this way.

Note that an algebraic function $\phi(x)$ is not in general single valued "near" the origin. There does not seem to be any reasonable way to choose algebraically a value for ϕ at the points other than the origin. Instead, the function must be taken as a multivalued function, together with a chosen "branch" at the origin. As a single-valued function, its natural domain of definition is not a subset of E^n at all but, rather, a certain affine scheme X lying over E^n in several sheets, together with a chosen point x above the origin. In fact, if, say, ϕ is one of the functions $y_\nu(x)$ above, then we can take for X the locus of zeros of $f_1, \ldots, f_N$ in E^{n+N}. On X, ϕ is one of the coordinate functions.

Returning to our affine scheme V defined by the equations (1.1), let $v \in V$ be a point, say the origin in E^n. We introduce the local ring of algebraic functions on V at v: it is just the residue ring

$$(1.4) \qquad \tilde{\mathcal{O}}_{V,v} = k\{x_1, \ldots, x_n\}/(g) .$$

We can also consider the complete local ring of V at v,

$$(1.5) \qquad \hat{\mathcal{O}}_{V,v} = k[[x_1, \ldots, x_n]]/(g) .$$

Let $V \to W$ be a map of affine schemes, and let $v \in V$ be a point mapping to $w \in W$. The map induces local homomorphisms

$$\tilde{\mathcal{O}}_{W,w} \to \tilde{\mathcal{O}}_{V,v} \quad \text{and} \quad \hat{\mathcal{O}}_{W,w} \to \hat{\mathcal{O}}_{V,v} .$$

A map is said to be "etale" at v if either of these maps is an isomorphism—it follows then that both are. This notion is analogous to that of local isomorphism for analytic spaces.

We will call "etale neighborhood" of v in V an (everywhere) etale map $V' \to V$, together with a choice of point $v' \in V'$ lying over v. Thus V' will lie over V in several sheets. The above discussion shows that given an element $\phi \in \tilde{\mathcal{O}}_{V,v}$, there is an etale neighborhood (V', v') of v in V such that the element

$$\phi \in \tilde{\mathcal{O}}_{V',v'} = \tilde{\mathcal{O}}_{V,v}$$

is induced by a (globally defined) function on V', i.e., by a map $V' \to E^1$.

Now I want to state a theorem which asserts that there are "enough" algebraic functions. Since it is already somewhat novel in the analytic situation, let me state it there first:

Theorem (1.6): Consider an arbitrary system of analytic equations

$$(*) \qquad f(x,y) = 0 ,$$

where $x = x_1, \ldots, x_n$; $y = y_1, \ldots, y_N$; and where $f = f_1, \ldots, f_m$ are convergent power series in x,y. Suppose given a formal power series solution

$$\bar{y}(x) = \bar{y}_1(x), \ldots, \bar{y}_N(x) \quad \text{of } (*): \quad f(x,\bar{y}(x)) = 0, \quad \text{where } \bar{y}_\nu(x)$$

are series without constant term. Let c be an integer. There exists a convergent series solution y(x) of (*) which agrees with $\bar{y}(x)$ up to terms of degree $\geqq c$:

$$y(x) \equiv \bar{y}(x) \quad (\text{modulo } (x)^c) .$$

Note that no requirement whatsoever is made on the system (*), and that the integers n,N,m are arbitrary. In order to put the result into perspective, we will give some examples of related problems, collected from various sources. They show that questions of this general type may be nontrivial:

Example 1: The algebraic differential equation

$$x^2 y' = y - x^2$$

has the unique formal series solution

$$y = \sum_{1}^{\infty} n!\,x^{n+1} ,$$

which is not convergent.

Example 2: Let $f(x) = a_1x + a_2x^2 + \cdots$ be a convergent series with $a_1 \neq 0$. View f as a map from a neighborhood of the origin in x-space to another neighborhood, and consider the problem of finding a new coordinate function y which linearizes f. This leads to the simple functional equation in y (the Schroeder equation):

$$f(y(x)) = y(a_1x) .$$

If a_1 is not a root of unity, then a formal series solution exists. If however $|a_1| = 1$, it may not be convergent.

Example 3: It is known (Osgood) that there are infinitely many analytically independent convergent series $y(x_1,x_2)$ in two variables. However, one can pose the following question (Abyankhar): Suppose that there is a formal power series relation among convergent series

$$y_1(x) , \ldots , y_N(x) \quad (x = x_1 , \ldots , x_n).$$

Does it follow that the series are analytically dependent? This has been proved recently by Abyankhar and van der Put for the case $N = 2$.

Example 4: Consider the situation of the above theorem. It may happen that some variables are missing in some of the formal series $\bar{y}_\nu(x)$. If so, can a convergent solution be found with the same property? This is not known.

Here is the algebraic result:

Theorem (1.7): Let $f(x,y) = 0$ be a system of polynomial equations in

$$x = x_1 , \ldots , x_n , \quad y = y_1 , \ldots , y_N ,$$

with coefficients in k. Let c be an integer. Given a formal power series solution $\bar{y}(x) \in k[[x]]$, there is an algebraic solution $y(x) \in k\{x\}$ such that

$$\bar{y}(x) \equiv y(x) \quad (\text{modulo } (x)^c) .$$

In this theorem, we do not need to assume that the power series are without constant term, since substitution into a polynomial is always defined.

The theorem implies that one can approximate by algebraic functions any structure given "formally," which can be described by a solution to a finite set of polynomial equations. Suppose we have some algebro-geometric structure S. Typically, S will be a type of structure consisting of some affine schemes and of some maps between them, having certain properties. Now such a structure can usually be described in terms of solutions of finitely many equations. For instance, an affine scheme V is defined by a system of polynomial equations (1.1). These polynomials have finitely many coefficients. To give a map $V \to W$, where W is defined by (1.2), we have to give polynomials $y_j(x)$ such that $h_\nu(y_j(x)) \equiv 0$ (modulo (g)), i.e., such that there exist polynomials $z_{\nu\mu}(x)$ with

$$h_\nu(y(x)) = \sum_\mu z_{\nu\mu}(x)\, g_\mu(x) \quad \text{for all } \nu .$$

Collecting terms of given degree, one sees that this equality is equivalent with the vanishing of certain polynomial equations in the coefficients of the polynomials g_μ, h_ν, y_j, $z_{\nu\mu}$. Moreover, given any other solution of this finite set of polynomial equations, we can construct a map of affine schemes $V' \to W'$ by viewing the solution as giving coefficients for new polynomials g'_μ, h'_ν, y'_j, $z'_{\nu\mu}$.

Suppose now that our original map of schemes $V \to W$ depends on formal parameters $t = t_1, \ldots, t_m$, i.e., that the coefficients are all power series in t. Then the above theorem guarantees the existence of a map $V' \to W'$ of schemes, depending on algebraic parameters, i.e., with coefficients in $k\{t\}$, which approximates the formal map $V \to W$ very closely. In this way, one can approximate a very general type of structure S algebraically. This is quite different from the analytic case: It is not at all clear, in general, how to describe an analytic space by means of finitely many quantities.

In practice, the properties which we want the approximation S′ to preserve are the most difficult to control by equations. They will usually be expressed by some dimensions (integers) which are not supposed to drop. As an example, consider the structure of a finite module $\bar{M}$ over the ring $k[[t]]$. It can be described as a quotient of a free module by finitely many linear relations, say

$$\begin{array}{l} \bar{a}_{11}v_1 + \cdots + \bar{a}_{1n}v_n = 0 \\ \vdots \qquad\qquad\quad \vdots \\ \bar{a}_{m1}v_1 + \cdots + \bar{a}_{mn}v_n = 0 \end{array}$$

where $\bar{a}_{ij} \in k[[t]]$. To approximate $\bar{M}$ algebraically, we can replace the $\bar{a}_{ij}$ by sufficiently near elements $a'_{ij} \in k\{t\}$, and take for M′ the resulting module.

But it is clear that this will not satisfy us if the set of relations is redundant, since the rank of M′ will drop unless we choose the a'_{ij} very carefully. If we want to control the rank of $\bar{M}$, we have to express it in terms of some polynomial equations involving the $\bar{a}_{ij}$ and possibly some auxiliary quantities.

Here is a straightforward way to control the rank: Say that $\bar{M}$ has rank r. Then there will exist a nonzero element $\bar{d} \in k[[t]]$ so that $\bar{M}$ becomes free of rank r when $\bar{d}$ is made invertible. More precisely, there will exist n − r of the elements

$$\ell_i = \bar{a}_{i1}v_1 + \cdots + \bar{a}_{in}v_n ,$$

say $i = 1, \ldots, n - r$, and r elements

$$w_i = \bar{c}_{i1}v_1 + \cdots + \bar{c}_{in}v_n , \quad i = 1, \ldots, r$$

which form a basis for the free module of rank n, and such that every ℓ_i is a linear combination of $\ell_1, \ldots, \ell_{n-r}$ (allowing $\bar{d}$ in the denominator). We may assume that the $\bar{c}_{ij}$ lie in $k[[t]]$. Writing this all out in terms of the coefficients and $\bar{d}$, we find that there exist elements $\bar{e}, \bar{b}_{ij} \in k[[t]]$ such that

$$(\det A)\,\bar{e} = \bar{d}^N$$

$$\bar{d}^N \ell_i = \sum_{j=1}^{n-r} \bar{b}_{ij} \ell_j ,$$

where A is the matrix

$$\begin{bmatrix} \bar{c}_{11} & \cdots & \bar{c}_{1n} \\ \vdots & & \vdots \\ \bar{c}_{r1} & \cdots & \bar{c}_{rn} \\ \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & & \vdots \\ \bar{a}_{n-r1} & \cdots & \bar{a}_{n-rn} \end{bmatrix} .$$

These equations are equivalent with a certain finite set of polynomial equations $P(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}) = 0$ in the elements $\bar{a}_{ij}$, $\bar{b}_{ij}$, $\bar{c}_{ij}$, $\bar{d}$, $\bar{e}$. We now take the equations

$$P(a, b, c, d, e) = 0$$

in the unknowns a_{ij}, b_{ij}, c_{ij}, d, e, and approximate the given formal solution algebraically.

2 ALGEBRAIC SPACES

Consider the problem of defining global objects by gluing affine schemes together. The idea is that such a global object should be obtained by identifying various "sheets" of some affine schemes, using algebraic functions. For intuition, we can relate this question to the analytic case: Let V be an affine scheme over **C**. Then V has an "underlying" structure of complex analytic space. To get it, we need only to view the defining equations (1.1) for V as a system of analytic equations.

Now consider the converse problem: Let X be an analytic space. What do we mean by an algebraic structure on X? What we want is an algebraic structure which is not necessarily affine, so the thing to do is to consider a covering of X by a finite set of affine schemes U_i. Now the disjoint union of finitely many affine schemes is again an affine scheme:

$$(\mathrm{Spec}\ A) \amalg (\mathrm{Spec}\ B) = \mathrm{Spec}\ (A \times B)\ .$$

Hence we may replace the collection $\{U_i\}$ by the single affine scheme $U = \amalg U_i$. Thus we want a surjective map

$$U \to X \tag{2.1}$$

with U an affine scheme. Strictly speaking, we mean, of course, a map from the analytic space underlying U to X. Then X may be obtained set-theoretically by identifying certain points of U.

In addition, we need to know that the identifications among the various U_i are algebraic in nature. This means the following: Consider the equivalence relation on U defined by the map to X:

$$U \times_X U = R \subset U \times U\ . \tag{2.2}$$

Here $U \times_X U = R$ is the set of pairs of points (u_1, u_2) which have the same image in X. It has a structure of analytic space. Now $U \times U$ is naturally an affine scheme, since U is one:

$$(\operatorname{Spec} A) \times (\operatorname{Spec} B) \approx \operatorname{Spec} (A \otimes_k B).$$

Hence what we require is simply that R be the analytic space associated to a closed subscheme of $U \times U$. This closed subscheme is then unique.

Finally, we have to put the appropriate requirement on the map $U \to X$. Here there are two choices: A structure of scheme on X is given by data as above, such that U is a disjoint union of affine schemes U_i which are mapped isomorphically to open subspaces of X. A structure of algebraic space on X is data as above, where the map of analytic spaces $U \to X$ is "etale," which means a local isomorphism, i.e., that every point $u \in U$ has an open neighborhood mapping isomorphically to its image.

Thus to give X a structure of algebraic space, we allow a single connected component U_i to lie over X in several sheets (always finitely many).

To define scheme or algebraic space abstractly, we just consider an equivalence relation with the appropriate conditions on it:

Definition (2.3): An "algebraic space" X consists of an affine scheme U and a closed subscheme $R \subset U \times U$ such that

(i) R is an equivalence relation.

(ii) The projection maps $p_i : R \to U$ $(i = 1,2)$ are etale.

The algebraic space X is a "scheme" if in addition

(iii) The restriction of R to each connected component of U is the trivial "diagonal" equivalence relation.

The notion of equivalence relation is the categorical one in the category of affine schemes, which we leave to the reader to make precise.

The underlying point set $|X|$ of an algebraic space is the set $|U|/|R|$.

Having defined algebraic spaces by means of "atlases," we have to do some work to define the notion of a "map" between algebraic spaces. Suppose we want to describe a map from an affine scheme V to X. Working set-theoretically for the moment, we have a diagram

$$\begin{array}{ccccc} & & & & V \\ & & & & \downarrow f \\ R & \rightrightarrows & U & \rightarrow & X . \end{array}$$

Form the fibered product $W = U \times_X V = \{(u,v) \mid \text{images in } X \text{ are equal}\}$. Then W maps surjectively to V, and it is clear that f is determined by the projection map $W \to U$. Now given a surjective map $W \to V$ and a map $W \to U$, the condition that there exist a map $f : V \to X$ with $W = U \times_X V$ is just that W identify with a subset of $U \times V$ which is stable with respect to the equivalence relation in the sense that if $(u_1,v) \in W$, then $(u_1,u_2) \in R$ if and only if $(u_2,v) \in W$. Thus we use the following definition:

Definition (2.4): Let X be the above algebraic space and let V be an affine scheme. A "map" $V \to X$ is a closed subscheme $W \subset U \times V$ such that

(i) The projection $W \to V$ is etale and surjective,

(ii) The two closed subschemes $R \times_U W$, $W \times_V W$ of $U \times U \times V$ are equal.

To extend this definition to arbitrary algebraic spaces, the crucial fact is

Descent, or Sheaf Property (2.5): (Grothendieck) Let $V' \to V$ be an etale surjective map of affine schemes. Then the sequence

$$\mathrm{Hom}\,(V,X) \to \mathrm{Hom}\,(V',X) \rightrightarrows \mathrm{Hom}\,(V' \times_V V', X)$$

is exact, i.e., a map $V \to X$ is given by a map $V' \to X$ such that the maps $V' \times_V V' \to X$ obtained by composition with the projections $V' \times_V V' \rightrightarrows V'$ are equal.

Here a diagram of sets

$$S \to S' \underset{\beta}{\overset{\alpha}{\rightrightarrows}} S''$$

is called "exact" if S maps injectively to S' and if its image is the set of elements $x \in S'$ with $\alpha(x) = \beta(x)$.

Let Y be another algebraic space, defined by the equivalence relation $S \subset V \times V$. Then we define Hom (Y,X) so as to make the diagram

(2.6) $\quad \mathrm{Hom}\,(Y,X) \to \mathrm{Hom}\,(V,X) \rightrightarrows \mathrm{Hom}\,(S,X)$

exact. Thus a map $Y \to X$ is given by a map $V \to X$ compatible with the equivalence relation S. This makes the algebraic spaces into a category.

The local ring of algebraic functions $\tilde{\mathcal{O}}_{X,x}$ of an algebraic space X at a point x is defined as $\tilde{\mathcal{O}}_{X,x} = \tilde{\mathcal{O}}_{U,u}$, where $u \in U$ is any point lying over x and where $\tilde{\mathcal{O}}_{U,u}$ is defined as in (1.4). The point x is said to be a "smooth' point, if

$$\tilde{\mathcal{O}}_{X,x} \approx k\{z_1, \ldots, z_d\}$$

for suitably chosen variables z_i. The integer d is then the "dimension" of X at x.

Various notions defined for affine schemes extend immediately to algebraic spaces. For instance, a map $f : Y \to X$ is "etale" at $y \in Y$ if the associated map on local rings

$$\tilde{\mathcal{O}}_{X,x} \to \tilde{\mathcal{O}}_{Y,y} \qquad (x = f(y))$$

is an isomorphism. A "closed subspace" of an algebraic space X can be defined as follows: say, $X = U/R$. Then a closed subspace X′ is just U'/R', where $U' \subset U$ is a closed subscheme stable under the equivalence relation. This means that the two induced subschemes $R \times_U U'$, $U' \times_U R$ of R, defined set-theoretically as

$$\{(u',v) \in R \mid u' \in U'\}$$
$$\{(v,u') \in R \mid u' \in U'\},$$

are equal.

Finally, we need the notion of "structure sheaf" $\mathcal{O}_X$ on an algebraic space X. This is by definition the functor which to every algebraic space V etale over X ($V \to X$ etale) associates the ring $\mathcal{O}(V)$ of functions on V (= maps $V \to E^1$). It has the sheaf property obtained by putting $X = E^1$ in (2.5).

The simplest example of a scheme which is not affine is the projective line $\mathbf{P}^1$. It is defined by the equivalence relation on $\mathbf{E}^1 \amalg \mathbf{E}^1 = U$ which identifies x with $1/x'$, if x,x' are coordinate functions on the two copies of $\mathbf{E}^1$. Thus the relation has four components $\mathbf{E}^1 \amalg \mathbf{E}^1 \amalg (\mathbf{E}^1 - 0) \amalg (\mathbf{E}^1 - 0)$. As a subset of $U \times U = 4(\mathbf{E}^1 \times \mathbf{E}^1)$, it is shown in figure 2.1.

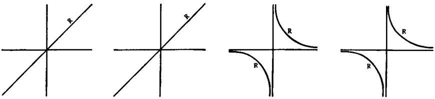

Figure 2.1.

Projective space $\mathbf{P}^n$ is defined by an atlas with $U = (n+1)\mathbf{E}^n$ in the standard way.

One can also find an "atlas" for $\mathbf{P}^1$ as algebraic space in which U is connected: consider the function

$$x = t/(t^3 + 2).$$

Its derivative vanishes at the points $t^3 = 1$, and for $t \neq 1, \zeta, \zeta^2$, ($\zeta = e^{2\pi i/3}$), every value of x including ∞ is taken on at least once. Thus we can view this function as defining a surjective map $U \to \mathbf{P}^1$, where

$$U = \operatorname{Spec} k\left[t, \frac{1}{t^3 - 1}\right],$$

which is the complement of the points $t = 1, \zeta, \zeta^2$ in $\mathbf{E}^1$. The resulting equivalence relation R is defined in $U \times U \subset \mathbf{E}^1 \times \mathbf{E}^1$ by the equation

$$(t_2 - t_1)(t_1 t_2^2 + t_1^2 t_2 - 2) = 0.$$

Its real locus looks like this:

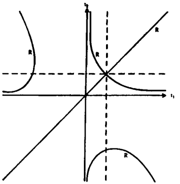

Figure 2.2 The curves for $t_i = 1$ (dashed) are omitted.

It is not easy to give examples of algebraic spaces which aren't schemes. (Since we have defined them by atlases, we should say more precisely, "aren't isomorphic to schemes.") In low dimensions, one hasn't any:

Theorem (2.7): (1) Every algebraic curve is a scheme. (2) Every nonsingular algebraic surface is a scheme. (3) Every algebraic space with a group structure is a scheme. However, (4) there exist singular algebra surfaces and nonsingular 3-dimensional algebraic spaces which aren't schemes.

We will see some explicit examples later.

Of course, even in dimension 1, there will in general be no globally defined functions on X (maps $X \to E^1$), other than the constant functions (elements of k). This is true already for $X = \mathbf{P}^1$, and has to do with the fact that the projection maps $R \rightrightarrows U$ do not have a constant number of points in their fibers. (Notice that in the above examples, there are vertical asymp totes on R, which give points $u \in U$ over which R has fewer than the maximum number of points.) If there were a constant number, there would be enough global functions to make X into an affine scheme. In fact, it would follow that the affine coordinate ring B of R is a finite A-module (U = Spec and one has the theorem of Grothendieck (SGAD, V):

Theorem (2.8): Let $R \subset U \times U$ be an equivalence relation such that the projection maps $R \to U$ are etale (or, more generally, flat). Say that $R = \operatorname{Spec} B$, $U = \operatorname{Spec} A$. If the projections make B into a finite A-module, then $X = U/R$ is an affine scheme.

The proof runs as follows: We have a pair of maps $f_1, f_2 : A \to B$ given by the projections. Let $A_0 \subset A$ be the subalgebra making the diagram

$$A_0 \to A \overset{f_1}{\underset{f_2}{\rightrightarrows}} B$$

exact, i.e.,

$$A_0 = \{a \in A \mid f_1(a) = f_2(a)\}.$$

It follows immediately from the definition of map of algebraic spaces that A_0 is the ring of global functions on X. The essential point is to show that A_0 contains enough elements. First of all, A is integral (hence finite) over A_0. For, let $a \in A$ and let $x \in X$. If $u_1, \ldots, u_n$ are the points of U lying over x, we can form the symmetric functions $s_\nu = s_\nu(a)$ of $a(u_1), \ldots, a(u_n)$. These depend on x alone, and a satisfies the equation

$$a^n + s_1 a^{n-1} + \cdots + s_n = 0. \tag{2.9}$$

To see that this works scheme-theoretically, one views B as a locally free A-module of rank n via the homomorphism f_1. Consider the A-linear map

$$\phi : B \to B$$

given by multiplication by $f_2(a)$. Using the fact that R is an equivalence relation, one can show that the polynomial

$$P(t) = t^n + s_1 t^{n-1} + \cdots + s_n$$

is the characteristic polynomial of ϕ. Hence $s_1, \ldots, s_n$ are well-defined elements of A, and (2.9) follows from the Cayley-Hamilton theorem.

It is easy to see, now, that A_0 separates points of X. For, let $x, x' \in X$ and let $u_1, \ldots, u_n$ and $u'_1, \ldots, u'_n$ be the points of U lying over x, x'

respectively. These points correspond to maximal ideals of A (the kernels of the maps $A \to k$ given by evaluation of a function at the point). By the Chinese remainder theorem, there is an element $a \in A$ such that

$$a(u_i) = 0$$
$$a(u_i') = 1 \quad \text{for } i = 1, \ldots, n.$$

Then $s_n = s_n(a)$ takes the values 0 at x and 1 at x′. The rest is just a question of sorting out the definitions.

Now for a general algebraic space X, the atlas may be chosen so that the projection maps $R \to U$ are n-to-one nearly everywhere for suitable n. Using this fact and the above theorem, one can show,

Corollary (2.10): Every algebraic space X has a dense open subset X′ which is an affine scheme.

In fact, the complement Y of X′ in X will have smaller "Krull dimension"—one less, in general. By making an algebraic space, we are adding something of lower dimension to an affine scheme "at infinity."

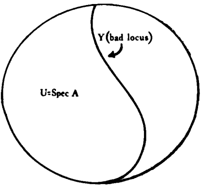

Figure 2.3.

3 CONSTRUCTION TECHNIQUES

We now want to consider a general construction problem. By this we mean a problem whose solution is the construction of an algebraic space or a scheme. Here are some examples to keep in mind:

1. Hilbert Space. Let X be a given algebraic space, assumed complete. The points of the space Hilb X shall correspond to closed subspaces of X.

2. Picard Scheme. Let X be complete. The points of Pic X are isomorphism classes of locally free, rank one sheaves of $\mathcal{O}_X$-modules (or "line bundles") on X.

3. Moduli. The points of the algebraic space are isomorphism classes of schemes of a certain type. This problem presents serious technical difficulties, and so we will not consider it here.

We recall that an algebraic space X is said to be "complete" if the following condition holds: Let C be a nonsingular algebraic curve, let c be a point of C, and let ϕ: $C - c \to X$ be a map. Then ϕ extends to a map $C \to X$.

In these examples, the problem is posed in such a way that the underlying point set is already given. The problem is to put an algebraic structure on this set. Now to determine such an algebraic structure, more data are required. We need to have a notion of "algebraic family" of points. The appropriate definitions turn out to be

1. Hilbert Space. Let V be an algebraic space. An algebraic family of closed subspaces Y_V of X, parametrized by V, is a closed subspace Y of $V \times X$ such that the projection map $Y \to V$ is "flat."

2. Picard Scheme. We suppose for simplicity that X has no global functions other than the "constant" ones—elements of k. (This is true if X is reduced and connected). Choose a base point $x \in X$. Let V be an algebraic space. An algebraic family of locally free rank-one sheaves on X, parametrized by V, is a locally free rank-one sheaf L on $V \times X$ whose restriction to $V \times x = V$ is the trivial sheaf $\mathcal{O}_V$.

The problem now becomes to construct a universal algebraic family. Thus in the Hilbert case, say, we want an algebraic space H and an algebraic family $C \subset H \times X$ of closed subspaces parametrized by H, having the universal property: if $Y \subset V \times X$ is any algebraic family, there is a unique map $V \to H$ such that the pull-back of C to $V \times X$ is Y.

Actually, both of the above examples are slightly too big to admit universal families. For instance, in the Hilbert case, H will have to contain a copy of X to parametrize points, a copy of a scheme parametrizing pairs of distinct points, etc. However, H will be a union of infinitely many open subsets which are algebraic spaces. We could restrict H enough by imposing some "degree" conditions on the subspaces, but this would require an analysis of these conditions and is not very natural. It is more appropriate to introduce a notion of algebraic space "locally of finite type," meaning a union of open subsets which are algebraic spaces (of finite type).

We can state the general construction problem we have in mind as follows: Consider an arbitrary contravariant functor

$$F : (\text{algebraic spaces})^\circ \to (\text{sets}) .$$

We want to "represent" F, i.e., to find an algebraic space Z locally of finite type and a functorial isomorphism

$$F(V) \approx \operatorname{Hom}(V,Z)$$

for every algebraic space V. The identity map $Z \xrightarrow{id} Z$ then plays the role of universal element. In case of the Hilbert functor, we put

$$\text{Hilb}(V) = \text{set of algebraic families of closed subspaces parametrized by } V.$$

If $f: V' \to V$ is a map, then Hilb (f) is the map Hilb (V) $\to$ Hilb (V′) defined by pull-back: a family $Y \subset V \times X$ is sent to $Y' = V' \times_V Y \subset V' \times X$.

Actually, it is enough to know such a functor F on affine schemes, provided it satisfies the descent condition analogous to (2.5). Thus we may as well work with a functor

$$F : (\text{affine schemes})^\circ \to (\text{sets})$$

or, equivalently,

$$F : (\text{k-algebras}) \to (\text{sets}) .$$

We will pass informally from one notation to the other so that, if

$$V = \text{Spec } A, \quad \text{we will write} \quad F(V) = F(A) .$$

One just has to keep straight the directions of the arrows: F is covariant in A and contravariant in V.

As in our examples, we are given the underlying point set of the desired algebraic space Z. It is

$$|Z| = F(\text{point}) = F(k) ,$$

and we want to find conditions on F which guarantee the existence of an algebraic structure on this point set.

One can begin by studying the infinitesimal structure of Z: its tangent space, etc. This is usually called "deformation theory." Here is the way it presents itself in algebraic geometry:

Consider a local k-algebra A which is finite dimensional as a vector space over k—for instance $A = k[t]/(t^n)$. Then, since our functor is defined on all affine schemes, we can substitute A into F to get a set F(A). The map $A \to k = A/\mathfrak{m}$ ($\mathfrak{m}$ the maximal ideal) gives us a map

$$F(A) \to |Z| = F(k) ,$$

and we call an element $\alpha \in F(A)$ with image z in $|Z|$ an "infinitesimal deformation" of z, parametrized by Spec A. These infinitesimal deformations give information about the structure of Z.

Suppose for the moment that Z exists, so that there is a map

$$\alpha : \operatorname{Spec} A \to Z .$$

What is such a map? Consider a system of polynomial equations with coefficients in A:

$$f_\nu(y_1, \ldots, y_N) = 0 , \quad \nu = 1, \ldots, N , \; f_\nu \in A[y] .$$

Let $y_1^\circ, \ldots, y_N^\circ$ be elements of A such that

$$f(y^\circ) \equiv 0 \pmod{\mathfrak{m}}$$

but

$$\det\left(\frac{\partial f}{\partial y}\right)(y^\circ) \not\equiv 0 \pmod{\mathfrak{m}} .$$

Then, since some power of the maximal ideal $\mathfrak{m}$ is the zero ideal, one sees immediately that Newton's method applied to y° converges to a solution in finitely many steps. Thus the system has a solution in A. This means that if we write V = Spec A, then the local ring $\tilde{\mathcal{O}}_{V,v}$ of algebraic functions on V at its unique point v is just the ring A itself.

Now any map $\alpha : V \to Z$ induces a map of local rings

$$\tilde{\mathcal{O}}_{V,v} = A \leftarrow \tilde{\mathcal{O}}_{Z,z} .$$

Conversely, given such a map of local rings, it is clear how to construct α: Take an affine scheme U etale over Z, with a point u lying over z, so that $\tilde{\mathcal{O}}_{Z,z} \approx \tilde{\mathcal{O}}_{U,u}$. Say that U = Spec B. Then we have

$$B \to \tilde{\mathcal{O}}_{U,u} = \tilde{\mathcal{O}}_{Z,z} \to A ,$$

which induces a map $U \leftarrow V$, and, by composition, we obtain

$$V \to U \to Z .$$

Thus an element $\alpha \in F(A)$ inducing z is just a homomorphism of local k-algebras

$$(3.1) \qquad \tilde{\mathcal{O}}_{Z,z} \to A .$$

Suppose for example that $A = k[t]/(t^2)$. Then the homomorphism will be of the form

$$f \to f(z) + t\, d(f) ,$$

where f(z) is the value of $f \in \tilde{\mathcal{O}}_{Z,z}$ at z (i.e., the residue of f modulo $\mathfrak{m}_z$) and where $d : \tilde{\mathcal{O}}_{Z,z} \to k$ is some map. Computation shows that

$$d(f + g) = d(f) + d(g), \quad d(fg) = f\, d(g) + g\, d(f), \quad d(c) = 0$$

if $c \in k$. Thus d is a k-derivation. By definition, such a d corresponds to a "tangent vector" to Z at z. Thus the geometric interpretation of Spec $k[t]/(t^2)$ is that it is a point together with a tangent vector. Note that the set of these k-derivations ($\approx$ set of deformations of z parametrized by $k[t]/(t^2)$) forms a finite dimensional vector space over k.

Returning to the general situation, suppose we restrict our attention to k-algebras A such that $\mathfrak{m}^{n+1} = 0$. An element $\alpha \in F(A)$ inducing z is called an "n-th order deformation." The ring

$$\tilde{\mathcal{O}}_{Z,z}/\mathfrak{m}^{n+1} = \mathcal{O}_n ,$$

which is a finite local k-algebra, is clearly universal for n-th order deformations, i.e., every homomorphism (3.1) factors through it. Its spectrum Spec $\mathcal{O}_n = Z_n$, which is a scheme having one underlying point, is called the n-th order neighborhood of z in Z. It is given together with the universal element, say

$$z_n \in F(Z_n) ,$$

corresponding to the "inclusion" map (Z_n is a closed subspace of Z)

$$Z_n \xrightarrow{z_n} Z .$$

Now going back to our general functor F, we can begin by constructing the infinitesimal neighborhoods Z_n of some point $z_0 = Z_0 \in F(k)$. There are very natural techniques which permit one to conclude their existence, notably the theorem below. Let us denote by $F_0(A)$ the set of elements inducing z_0 in $F(k)$ for a finite local k-algebra A. If Z exists, then the interpretation of $F_0(A)$ as $\mathrm{Hom}_k(\tilde{\mathcal{O}}_{Z,z}, A)$ shows that for any diagram

$$\begin{array}{ccc} A' & & B \\ & \searrow \;\; \swarrow & \\ & A & \end{array}$$

of such algebras we have

$$(*) \qquad F_0(A' \times_A B) \xrightarrow{\approx} F_0(A') \times_{F_0(A)} F_0(B).$$

Moreover, $F_0(k[t]/(t^2))$ is a finite dimensional vector space with its natural structure. It is not hard to show conversely that, if F_0 has these properties, then the infinitesimal neighborhoods Z_n exist. In practice, though, it is hard to check (*) in the general case. Schlessinger's theorem asserts that a special case, much easier to verify, is enough:

Theorem (3.2): (Schlessinger) The infinitesimal neighborhoods Z_n exist if F has the following properties:

(i) With the above notation, the map (*) is an isomorphism when $A' \to A$ is surjective and has a kernel of dimension one as vector space over k.

(ii) When (i) holds, the set of elements $\alpha \in F_0(k[t]/(t^2))$ forms a k-vector space. This space is finite dimensional.

Here is how the vector space structure is defined. Consider the ring

$$k[t,t'] = k[t] \times_k k[t'], \text{ where } t^2 = t'^2 = tt' = 0.$$

It maps to $k[t]$ by

$$(3.3) \qquad a + bt + b't' \mapsto a + (b + b')t.$$

Denote by $F_0(A)$ the set of elements of $F(A)$ which induce $z_0 \in F(k)$. By (i) of Theorem 3.2,

$$F_0(k[t,t']) = F_0(k[t]) \times F_0(k[t']).$$

Thus (3.3) defines a law of composition on $F_0(k[t])$, and it makes this set into an abelian group. Given $c \in k$, we get a map from $k[t]$ to itself sending

$$a + bt \mapsto a + cbt .$$

By functoriality this yields an operation of k on $F_0(k[t])$. It is easily checked that with these laws $F_0(k[t])$ becomes a k-vector space. Of course, if Z exists, then this vector-space structure is just the natural one on the set of derivations of

$$\tilde{\mathcal{O}}_{Z,z_0} \quad \text{into k. Thus} \quad F_0(k[t]) \quad (t^2 = 0)$$

is the tangent space to Z at z_0.

Assume now that one knows the existence of Z_n for each n. Then we obtain an inverse system of local rings

$$\cdots \to \mathcal{O}_{n+1} \to \mathcal{O}_n \to \cdots$$

(writing $Z_n = \operatorname{Spec} \mathcal{O}_n$). Let $\hat{\mathcal{O}}$ be the inverse limit of this system. This will be a complete local ring and should be the completion

$$\hat{\mathcal{O}}_{Z,z_0} \quad \text{of the local ring of} \quad Z \text{ at } z_0 .$$

Now we have a compatible system of universal elements $z_n \in F(\mathcal{O}_n)$. In other words, the map $\mathcal{O}_{n+1} \to \mathcal{O}_n$ sends $z_{n+1} \mapsto z_n$. Thus we are tempted to substitute $\hat{\mathcal{O}}$ into F, and to look for an element

(3.4) $\bar{z} \in F(\hat{\mathcal{O}})$

inducing z_n via the map $\hat{\mathcal{O}} \to \mathcal{O}_n$, for each n.

This brings up a very fundamental point: The ring $\hat{\mathcal{O}}$ is a k-algebra, but it is not of finite type. So, what does it mean to substitute $\hat{\mathcal{O}}$ into F? Suppose that F is defined for all k-algebras of finite type. Then we can extend it as a functor, completely formally, to all k-algebras A, by the definition

(3.5) $F(A) = \varinjlim_{i} F(A_i)$

where the limit is taken over the filtering family $\{A_i\}$ of subalgebras of A of finite type. It is very easy to check that this does define a functor.

Definition (3.6): (Grothendieck) Let F be a functor defined on a full subcategory of (k-algebras) containing all k-algebras of finite type. Then F is said to be "locally of finite presentation" if (3.5) holds whenever the left side is defined.

This condition is a very important one and usually does hold. Of course, if F were already given somehow on all k-algebras, it might not be easy to prove it locally of finite presentation.

At any rate, let us assume that substitution of $\hat{\mathcal{O}}$ into F is permitted. Still it will not be at all clear, in general, that an element $\bar{z}$ as above (3.4) exists. The existence of this element is a subtle point. Fortunately, a theorem of Grothendieck (EGA III.5) guarantees its existence in the cases of Hilbert and Picard functors.

Definition (3.7): We say that F has "effective formal moduli" at $z_0 \in F(k)$ if universal deformations $z_n \in F(\mathcal{O}_n)$ of order n exist for all n, and if there is an element

$$\bar{z} \in F(\hat{\mathcal{O}}) \quad (\hat{\mathcal{O}} = \varprojlim \mathcal{O}_n)$$

inducing z_n. If only the z_n exist, we just say that F has "formal moduli" at z_0.

Theorem (3.8): Suppose F has effective formal moduli $\bar{z} \in F(\hat{\mathcal{O}})$ at $z_0 \in F(k)$ and that F is locally of finite presentation. Then there is an affine scheme $V = \operatorname{Spec} B$, a point $v \in V$, and an element $z' \in F(V)$, which realizes the universal deformations z_n in the following sense: the map $B \to k$ determined by v sends z' to z_0, and moreover z' identifies $\tilde{\mathcal{O}}_{V,v}/\mathfrak{m}_v^{n+1}$ with $\mathcal{O}_n$ for every n.

Thus, in particular, $\hat{\mathcal{O}}_{V,v} \approx \hat{\mathcal{O}}$.

Assume again that Z exists. Then z' is a map $V \to Z$, and it induces an isomorphism $\hat{\mathcal{O}}_{Z,z} \to \hat{\mathcal{O}}_{V,v}$, i.e., it is etale at v. Thus it will be everywhere etale if we replace V by a suitable open set. Now by definition, an algebraic space is determined by an etale covering. Clearly, therefore, we will be able to conclude existence of Z from the above theorem, provided F has effective formal moduli at every point. Of course, there will be a few technical points to be checked. They are of less interest, however, so that we will suppress them here.

Corollary (3.9): The functors Hilb X and Pic X are representable as algebraic spaces locally of finite type. By (3) of (2.7), Pic X is a scheme.

Here is the proof of the theorem in the "unobstructed" case. By unobstructed case, we mean the case that the ring $\hat{\mathcal{O}}$ is a power series ring $k[[x_1, \ldots, x_n]]$, i.e., that the scheme Z, if it exists, is smooth at z_0. In general, $\hat{\mathcal{O}}$ will be a quotient of such a ring. The word unobstructed refers to the fact that there is no condition on a first-order deformation to extend it to arbitrarily high order.

Write $k[[x_1, \ldots, x_n]]$ as a union of its subalgebras A_i of finite type over k. Since F is assumed locally of finite presentation, the element $\bar{z}$ is represented by some element in an $F(A_i)$, say $\alpha \in F(A)$. Remember that this means the inclusion map

(3.10) $\quad \bar{\phi} : A \hookrightarrow \hat{\mathcal{O}}$

sends $\alpha \mapsto \bar{z}$. Now write A as a quotient of a polynomial ring:

$$A = k[y_1, \ldots, y_N]/(g_1, \ldots, g_m).$$

Then the inclusion map (3.10) corresponds to a solution $\bar{y}(x)$ of the system of equations $g(y) = 0$ in the ring $k[[x]]$. By theorem (1.7), there is a solution $y(x)$ by algebraic functions $y(x) \in k\{x\}$ with

$$y_\nu(x) \equiv \bar{y}_\nu(x) \quad (\text{modulo } (x)^2).$$

Let $\tilde{\mathcal{O}}$ be the local ring of algebraic functions of $E^n = \operatorname{Spec} k[x]$ at the origin. There is an affine scheme $V = \operatorname{Spec} B$ mapping to E^n together with a point $v \in V$ lying over the origin, at which $V \to E^n$ is etale, such that the $y_\nu(x)$ are global functions on V, i.e., are elements of B. This solution of the equations $g(y) = 0$ yields a map

$$\phi' : A \to B .$$

Now the local ring of V at v is $\tilde{\mathcal{O}}_{V,v} = \tilde{\mathcal{O}}$, and its completion is $\hat{\mathcal{O}}$:

$$B \to \tilde{\mathcal{O}}_{V,v} \subset \hat{\mathcal{O}}_{V,v} = \hat{\mathcal{O}}.$$

We claim that the element $z' \in F(V)$ induced by α via ϕ' realizes the universal deformations z_n.

Apply the universality of the elements $z_n \in F(\mathcal{O}_n)$. Let $z_n' \in F(\mathcal{O}_n)$ be the elements induced by z' (since $\mathcal{O}_n = \tilde{\mathcal{O}}_{V,v}/\mathfrak{m}_v^{n+1}$). By universality of z_n, there is a unique map

$$\epsilon_n : \mathcal{O}_n \to \mathcal{O}_n \quad \text{sending} \quad z_n \mapsto z_n' .$$

Because of the uniqueness, these maps are compatible; hence, passing to the inverse limit, they define a map

$$\hat{\epsilon} : \hat{\mathcal{O}} \to \hat{\mathcal{O}} = \hat{\mathcal{O}}_{V,v} \quad \text{sending} \quad z_n \to z_n' , \text{ all } n .$$

Thus we are through if we show that $\hat{\epsilon}$ is an isomorphism, since then the z_n' will obviously be universal too. Now by assumption, $y(x) \equiv \bar{y}(x)$ (modulo $(x)^2$). Therefore the two maps

$$A \to \tilde{\mathcal{O}}_{V,v}/\mathfrak{m}_v^2 = k\{x\}/(x)^2 = k[[x]]/(x)^2$$

induced by ϕ', $\bar{\phi}$ are <u>equal.</u> By universality, it follows that the map ϵ_1 is the identity. Therefore $\hat{\epsilon}$ is a map from $k[[x]]$ to itself which is congruent to the identity, modulo $(x)^2$. So it has the form

$$x_i \mapsto x_i + p_i(x)$$

where $p_i(x)$ are power series beginning with terms of degree two. One checks immediately that such a transformation is invertible, which completes the proof.

4 MODIFICATIONS

Two complete algebraic spaces X, X′ are called "birationally equivalent" if there are dense open subspaces $U \subset X$, $U' \subset X'$ which are isomorphic. The problem of classifying algebraic spaces can be split naturally into two parts:

(a) classify birational equivalence classes, and
(b) classify algebraic spaces which are birationally equivalent to a given one.

We are interested here in problems related to (b).

Definition (4.1): A "modification" is a proper map $f : X' \to X$ of algebraic spaces and a closed subset $Y \subset X$, such that the restriction of f to $U = X - Y$ is an isomorphism. The map f is also called a "dilatation" of Y in X, or a "contraction" of $Y' = f^{-1}(Y)$ in X'.

A map $f : X' \to X$ is called "proper" if the following condition holds: Let C be a curve and $c \in C$. Given a commutative diagram

$$\begin{array}{ccc} (C - c) & \xrightarrow{g} & X' \\ \cap & & \downarrow \\ C & \to & X, \end{array}$$

there is a (unique) extension of g to C. Thus an algebraic space is proper over the point Spec k if and only if it is complete.

Given any two embeddings of an algebraic space U as open subspaces of complete algebraic spaces X,X′, we can consider the closure (as algebraic space) X″ of U, as embedded diagonally into $X \times X'$. Then U is open in X″, and we obtain a diagram of modifications

$$(4.2) \qquad \begin{array}{ccccc} & & X'' & & \\ & \swarrow & & \searrow & \\ X - U = Y \subset X & & & & X' \supset Y' = X' - U. \end{array}$$

Thus all changes of the type we are interested in can be obtained in two steps, as a dilatation followed by a contraction, and so it is enough to consider modifications.

The simplest case of a modification would be one in which the map $f : X' \to X$ is finite-to-one above Y. If X is reduced, then these modifications can all be dominated by a single one—the "normalization" $\bar{X}$ of X: Let $X = U/R$, where $U = \operatorname{Spec} A$ and $R = \operatorname{Spec} B$. Let K be the product of the localizations of A at its minimal prime ideals (so that K is a product of fields). Call $\bar{A}$ the integral closure of A in K, and define $\bar{B}$ in the same way. Then the normalization $\bar{X}$ is the algebraic space defined by the equivalence relation $\operatorname{Spec} \bar{B} \subset (\operatorname{Spec} \bar{A}) \times (\operatorname{Spec} \bar{A})$. It is finite over X. Zariski's Main Theorem assets the following: Every finite-to-one birational proper map $X' \to X$ fits into a diagram

$$\begin{tikzcd} \bar{X} \arrow[d] \arrow[dr] & \\ X & X' \arrow[l] \end{tikzcd}.$$

This gives one good control over such modifications.

The opposite question, of finding finite-to-one contractions of a space X', is also not difficult to treat.

Now consider the case that f is not finite-to-one. A very important case is the "blowing-up" of a sheaf of ideals in X. I will describe it in the case that $X = \operatorname{Spec} A$ is affine: Let the ideal of A be generated by $u_0, \ldots, u_n \in A$. Then we view the u_i formally as homogeneous coordinates in projective space. This defines a closed subspace of $X \times \mathbf{P}^n$, whose $(n + 1)$ standard affine opens are

$$\operatorname{Spec} A\,[u_0/u_i, \ldots, u_n/u_i] \subset X \times \mathbf{E}^n .$$

Thus we are just formally adjoining the ratios of the various u_j. It can be shown that every modification of projective schemes is the blowing-up of a suitable ideal.

The Theorem of Resolution of Singularities (Hironaka) asserts that every reduced algebraic space $\mathbf{X}$ of characteristic zero has a blowing-up $X' \to X$ such that X' is smooth.

As we will see, the problem of contractions is, at least apparently, more subtle. Consider first the case that X', X are smooth complete algebraic surfaces, i.e., smooth complete 2-dimensional algebraic spaces. It is known that such an algebraic space X is always projective, hence is a scheme. In this simple case, one has a complete answer to the problem.

Recall that, given two curves C_1, C_2 on a complete smooth algebraic surface X', their intersection number $(C_1 \cdot C_2)$ is defined. If the curves have no component in common, $(C_1 \cdot C_2)$ is just the number of intersection points, counted with a suitable multiplicity to take into account the tangencies. The number is defined, however, for any pair of curves, though it may be negative.

<u>Castelnuovo Criterion (4.3)</u>: An irreducible curve C can be contracted to a smooth point $x \in X$ if and only if its genus is zero (i.e., $C \approx \mathbf{P}^1$) and the self-intersection number $(C^2) = (C \cdot C)$ is -1. Such a curve is called an "exceptional curve of the first kind." Conversely, the result of blowing-up the maximal ideal of a point $x \in X$ (X smooth) is a modification $f : X' \to X$, with X' smooth and such that $f^{-1}(x) = C$ is an exceptional curve of the first kind.

<u>Factorization theorem (4.4)</u>: (Zariski) Every modification $f : X' \to X$, where X', X are smooth surfaces, can be obtained as a sequence of contractions of exceptional curves of the first kind or, equivalently, as a sequence of blowings-up of points.

Thus such a general modification will contract a finite set of (possibly reducible) curves to points, each connected curve being a tree of rational curves obtained by successive blowing-up of points on it. The intersection numbers will satisfy certain conditions which can be completely elucidated. If you blow up a smooth point on a curve D, the self-intersection of its (proper) inverse image is one less than that of D. Thus in figure 4.1, the self-intersection numbers indicated are possible. There are also other possibilities.

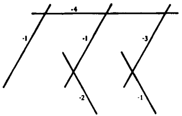

Figure 4.1.

Now suppose X is only assumed normal. Any surface X′ can be dominated by one which is smooth (resolution of singularities), so that it is natural to consider the case that X′ is still smooth. This case can be related to the general one as in (4.2). We therefore ask the following question: Which curves on a smooth surface X′ can be contracted to a normal point $x \in X$? Here is the answer:

Theorem (4.5):

(i) C must be connected.

(ii) Let the irreducible components of C be $C_1, \dots, C_r$. The intersection numbers must form a definite negative matrix $\| (C_i \cdot C_j) \|$.

(iii) Conversely, if (i), (ii) hold, then C may be contracted to a point **x** on an algebraic space X, and the modification $f : X' \to X$ so obtained is a blowing-up of a sheaf of ideals in X.

Since X′, being a smooth surface, is a scheme, it is natural to ask whether or not X is also a scheme. This turns out to be a rather subtle question, except in the following case, which is something of a curiosity.

Theorem (4.6): Suppose the ground field k is the algebraic closure of a finite field. Then every normal algebraic surface over k is a projective scheme. Hence the algebraic space X of (iii) of (4.5) is a scheme.

An example showing that this hypothesis on k is needed is the following:

Example: (Nagata-Mumford) Let C_0 be a smooth cubic curve in the plane $\mathbf{P}^2$, and let X′ be obtained from $\mathbf{P}^2$ by blowing up 10 points $p_1, \dots, p_{10}$

of C_0. Since $(C_0^2) = 9$, the inverse image C of C_0 has self-intersection number -1. Thus the hypotheses of (iii) of (4.5) are satisfied. Suppose C can be contracted to a point x of a scheme X. Then there will be an affine open set $U \subset X$ containing x. Let U' be its (isomorphic) inverse image in X'. The closed set $D = X' - U'$ will be a curve not meeting C. Let D_0 be its image in $\mathbf{P}^2$. This image meets C_0 only in the points $p_1, \ldots, p_{10}$. Say that the curve D_0 is of degree d. Then there exists a rational function on $\mathbf{P}^2$ whose zeros are the curve D and poles are dL, where L is the line at infinity. Restricting this function to C_0, we obtain a rational function on that curve whose divisor is

$$\sum_i r_i p_i - d(L \cdot C_0) \quad \text{for some } r_i \geqq 0 .$$

Now divisor classes of degree zero on C are parametrized by a group whose elements are the points of C (C is an elliptic curve, i.e., has genus 1). If k is not the algebraic closure of a finite field, this group will contain subgroups of arbitrary rank. Thus we can choose the points $p_1, \ldots, p_{10}$ in such a way that no equation of the form

$$\Sigma v_i p_i - d(L \cdot C_0) = 0$$

holds. Then it follows that the algebraic space X can not be a scheme. However, special choices of $p_1, \ldots, p_{10}$ result in a situation where X is indeed a scheme.

Corollary (4.7): There are deformations of 2-dimensional normal schemes, as algebraic spaces, which are not schemes.

In higher dimensions, Grauert has proved the following theorem for complex analytic spaces:

Theorem (4.8): (Grauert) Let X' be a nonsingular complex analytic space, and $Y' \subset X'$ a closed compact connected subspace. Suppose the normal bundle N of Y' in X' is negative. Then Y' can be contracted to a point analytically.

The same result (with Y′ complete) holds for algebraic spaces. Note that when X′ is a surface and Y′ = C a curve, the degree of the normal bundle (which is a line bundle on C) is just the self-intersection number (C^2). Thus negativity just means $(C^2) < 0$, and so this result generalizes the assertion (iii) of (4.5) for surfaces when C is irreducible and nonsingular. The negativity of the normal bundle is, however, no longer a necessary condition when dim X′ > 2.

I now want to state a general result from which these assertions for algebraic spaces follow. Let $f: X' \to X$, $Y \subset X$ be a modification. Consider the infinitesimal neighborhoods Y_n of $Y = Y_0$ in X. These are the closed subschemes of X defined by the powers of the ideal I of definition of Y, where we put, say, the reduced structure on Y. The collection of these neighborhoods defines what is called the "formal completion" $\mathfrak{X}$ of X along Y:

$$\mathfrak{X} = \{Y_n\}.$$

It is a "formal algebraic space." Now let $Y' = f^{-1}(Y)$, and let $\mathfrak{X}'$ be the formal completion of X′ along Y′. The map

$$\mathfrak{f}: \mathfrak{X}' \to \mathfrak{X}$$

induced by f is a formal modification, in a sense which is not worth while making precise here.

<u>Theorem (4.9)</u>:

(i) Let Y′ be a closed subspace of X′, and let $\mathfrak{f}: \mathfrak{X}' \to \mathfrak{X}$ be a formal modification, where $\mathfrak{X}'$ is the formal completion of X′ along Y′. There exists a modification $f: X' \to X$ inducing $\mathfrak{f}$. It is unique up to unique isomorphism.

(ii) Let Y be a closed subspace of X, and let $\mathfrak{f}: \mathfrak{X}' \to \mathfrak{X}$ be a formal modification, where $\mathfrak{X}$ is the formal completion of X along Y. There is a modification $f: X' \to X$ inducing $\mathfrak{f}$. It is unique up to unique isomorphism.

Thus a contraction or a dilatation which is given formally actually exists as an algebraic space.

The analog of Grauert's theorem follows easily from (i) of (4.9): Suppose for simplicity that Y' is of codimension one in X'. Let Y'_n denote the n-th order neighborhood, and $\mathcal{O}_n$ the structure sheaf of Y'_n. Then $\mathcal{O}_n$ maps to $\mathcal{O}_{n-1}$ and standard considerations show that the kernel of this map is the sheaf of sections of the n-th tensor power of the dual of the normal bundle:

$$0 \to N^{-\otimes n} \to \mathcal{O}_n \to \mathcal{O}_{n-1} \to 0 .$$

Since N is negative, $N^{-\otimes n}$ is very positive for large n, and this implies that the global sections of these sheaves form an exact sequence

$$(4.10) \qquad 0 \to H^{\circ}(Y, N^{-\otimes n}) \to A_n \to A_{n-1} \to 0 \quad (\text{n large}) ,$$

where A_n is the ring of global functions on Y'_n. This ring is a finite-dimensional local k-algebra since Y' is complete and connected.

Let

$$\bar{A} = \varprojlim A_n .$$

This is a complete local ring and defines a formal algebraic space $\mathfrak{X} = \{Y_n\} = \{\text{Spec } A_n\}$. Since A_n is the ring of global functions on Y'_n, we have an obvious map $Y'_n \to Y_n$, hence a map

$$\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X} ,$$

which maps Y' to a single point. What has to be shown is that $\mathfrak{f}$ is a formal modification. Now since we have not defined this notion, we can't give the proof. However, we can indicate the main point at which the negativity comes in. To show $\mathfrak{f}$ a formal modification, we must show that it does not collapse anything other than Y'. It turns out that this can be expressed as follows: Let $\mathfrak{Z}$ be the formal algebraic space defined by $k[[t]]$,

$$\mathfrak{Z} = \{\text{Spec } k[[t]]/(t^{n+1})\} .$$

A map $\mathfrak{Z} \to \mathfrak{X}$ is called a "formal branch" in $\mathfrak{X}$. The assertion is that a formal branch b in $\mathfrak{X}$ not entirely contained in the closed point lifts in only one way to $\mathfrak{X}'$. To see this, suppose we are given two liftings b_1, b_2 (figure 4.2). Suppose for simplicity that they pass through distinct points

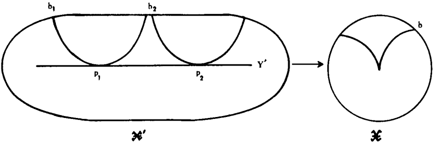

Fig. 4.2

p_1, p_2 of Y'. Since they agree in $\mathfrak{X}$, every formal function along Y', i.e., element of $\bar{A}$, has the same restriction to $\mathfrak{Z}$ via b_1 and b_2. Now for $n \gg 0$, the bundle $N^{-\otimes n}$ is very positive, and so there is a global section s of this bundle which is zero at p_1 but not a p_2. The element s represents an element in A_n by (4.10), and this element can be lifted to an element $\bar{a} \in \bar{A}$. Clearly $\bar{a}$ vanishes to higher order on b_1 at p_1 than on b_2 at p_2, which is a contradiction.

Theorem (4.9) is proved by appealing to (3.8). One can describe the functor which should be represented by the required modification a priori, and then it is a question of verifying that this functor is indeed representable. In case (i) for instance, the underlying point set is

$$|X| = |U| \cup |Y| ,$$

where $U = X' - Y'$. Let $V = \operatorname{Spec} A$ be an affine scheme. Then a map $g : V \to X$ will yield a closed subscheme $C = g^{-1}(Y)$. Let the formal completion of V along C be $\mathfrak{Z}$. Then we obtain a map $\mathfrak{Z} \to \mathfrak{X}$ and a map $V - C \to U$. The algebraic space X does not appear in these maps. Thus we can define a value of the functor on V as a collection

(a) a closed subset $C \subset V$,

(b) a map $V - C \to U$,

(c) a formal map $\mathfrak{Z} \to \mathfrak{X}$.

The data (b), (c) are required to satisfy a certain compatibility condition which we can't make precise here.

As an application of this theorem, we can characterize the compact analytic spaces which have an underlying algebraic structure: Consider the functor

$$\text{(complete algebraic spaces)} \xrightarrow{\text{an}} \text{(compact analytic spaces)}$$

taking an algebraic space to its underlying analytic space. Serre's GAGA theory asserts that this functor is fully faithful, i.e., that it defines an equivalence of the category of complete algebraic spaces with a full subcategory of the category of analytic spaces. It is therefore natural to ask for characterizations of the image. A classical result is:

Theorem (4.11): (Chow-Kodaira) A compact smooth analytic surface X is a projective scheme iff X has two algebraically independent meromorphic functions on it.

Of course, this is false if X is singular or of dimension > 2, since any algebraic space which is not a scheme gives a counterexample. The natural assertion is:

Theorem (4.12): Let X be a compact analytic space. Then X is the analytic space associated to an algebraic space if and only if every irreducible component C of X, with its reduced structure, has d algebraically independent meromorphic functions, $d = \dim C$.

We call these spaces "Moisezon spaces" because they have been extensively studied by Moisezon. Note that since very nonsingular algebraic surface is a projective scheme, this theorem generalizes the above result of Chow and Kodaira. Here is how the proof goes:

Lemma (4.13): (Moisezon) A closed analytic subspace of a Moisezon space is a Moisezon space.

Now a Moisezon space X is, more or less by definition, birationally equivalent to a projective scheme X′. Thus it can be related to X′ by a pair of modifications

$$\begin{tikzcd} & X'' \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

which will be isomorphisms outside of subspaces of lower dimension. By induction on the dimension and the above lemma, every infinitesimal neighborhood of these subspaces is algebraic. Hence the formal modifications induced by the above diagram are algebraic. Thus an application of each of the two parts of (4.9) yields the result.

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