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The Diagonal of a D-Finite Power Series Is D-Finite

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Let K be a field of characteristic zero, $x = x_1, ..., x_n$ several variables, and $K[[x]]$ the ring of formal power series in $x_1, ..., x_n$ over K. We call $f \in K[[x]]$ *D-finite* (or differentiably finite) if the set of all derivatives $(\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_n)^{i_n} f$ $(i_j \in \mathbf{N})$ lie in a finite-dimensional vector space over $K(x)$, the field of rational functions in $x_1, ..., x_n$. This is equivalent to saying that f satisfies a system of linear partial differential equations of the form

$$\left\{a_{in_i}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i} + a_{in_i-1}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i-1} + \cdots + a_{i0}(x)\right\} f = 0, \quad i = 1, ..., n, \quad (1)$$

where the $a_{ij}(x) \in K[x]$. We shall also write these equations as $A_i(x_1, ..., x_n; \partial/\partial x_i) f = 0$, $i = 1, ..., n$. The theory of D-finite power series in one variable is worked out in [9]. We call $f \in K[[x]]$ *rational* if $f \in K(x)$ and *algebraic* if it is algebraic over $K(x)$. If $f = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ we define the *primitive diagonal* $I_{12}(f) = \sum a_{i_1 i_1 i_3 \cdots i_n} x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n}$. The other primitive diagonals I_{ij} (for $i < j$) are defined similarly. By a *diagonal* we mean any composition of the I_{ij}, and by the complete diagonal (or just the diagonal) of f we mean $I_{12} I_{23} \cdots I_{n-1n}(f) = \sum a_{ii \cdots i} x_1^i$.

In this paper we will show (Theorem 1) than any diagonal of a D-finite power series is again D-finite. In [6] it is shown that the diagonal of a rational power series in two variables is algebraic and that in the case that K has characteristic $p \neq 0$ any diagonal of a rational power series in any number of variables is algebraic. (In characteristic 0 the diagonal of a rational power series in three variables need not be algebraic.) In [2, 3] it is shown, in the case that K has characteristic $p \neq 0$, that the diagonal of an algebraic power series in any number of variables is algebraic and that if $f \in \mathbf{Z}_p[[x]]$ is algebraic ($\mathbf{Z}_p$ the p-adic integers) then any diagonal of f is algebraic mod p^s (for all s). In [7, 10] it is claimed that the diagonal of a rational function in any number of variables is D-finite, but the proofs con-

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tain gaps which do not seem easy to fill. Doron Zeilberger has informed me that he is able to prove that the diagonal of a rational function is D-finite using Bernstein theory. We shall use a clever counting argument introduced in [7, 10] in Lemma 3 below. In [1] it is shown that the complete diagonals of a restricted class of rational power series are D-finite. The restriction can be avoided by the use of Dwork's paper [4]—see also [5]. Deligne has also pointed out (see the footnote on p. 5 of [1]) that the D-finiteness of the diagonals of rational power series can be deduced via resolution of singularities from the finiteness of cohomology for the complement of a hypersurface. While our proof below is elementary and more general, these methods give more information about the differential equations satisfied by the diagonals of rational power series.

Let $f \in K[[x_1, ..., x_n]]$ satisfy Eq. (1), and let s be a new variable. Define

$$F(s, x_1, x_3, ..., x_n) = \frac{1}{s} f\left(s, \frac{x_1}{s}, x_3, ..., x_n\right).$$

F is not a formal power series in $s, x_1, x_3, ..., x_n$, but is an element of the $K[s, x_1, x_3, ..., x_n]$-module M of all

$$G = \sum_{\substack{j \in \mathbf{Z} \\ i_2, ..., i_n \in N \\ j + i_2 \geqslant -k}} a_{j i_2 \cdots i_n} s^j x_1^{i_2} x_3^{i_3} \cdots x_n^{i_n}$$

for some $k \in \mathbf{N}$, depending on G. Let $\mathscr{D}$ be the ring of all linear partial differential operators in $\partial/\partial s$, $\partial/\partial x_1$, $\partial/\partial x_3, ..., \partial/\partial x_n$ with coefficients from $K[s, x_1, x_3, ..., x_n]$. Then M is a $\mathscr{D}$-module in the natural way. Notice that the coefficient of $1/s$ in F is just $I_{12}(f)$. Later we shall need

Lemma 1. *If* $0 \neq p \in K[s, x_1, x_3, ..., x_n]$ *and* $G \in M$ *satisfy* $pG = 0$ *then* $G = 0$.

Proof. For suitable k, $s^k G \in K[[s, x_1/s, x_3, ..., x_n]]$. Make the substitution $x_1 = su$, u a new variable, to get $0 = p(s, su, x_3, ..., x_n)$ $s^k G(s, u, x_3, ..., x_n) \in K[[s, u, x_3, ..., x_n]]$. The conclusion now follows from the observations that multiplication by s, and the substitution $x_1 = su$, are both one-to-one.

Lemma 2. *F is D-finite (in the variables $s, x_1, x_3, ..., x_n$).*

Proof. This is immediate from the fact that f is D-finite, by the chain rule.

Hence there are nonzero linear partial differential operators, with polynomial coefficients,

$$A\left(s, x_1, ..., x_n; \frac{\partial}{\partial s}\right) = L(s, x_1, x_3, ..., x_n)\left(\frac{\partial}{\partial s}\right)^m + \text{lower-order terms in } \frac{\partial}{\partial s}$$

and

$$B_i\left(s, x_1, ..., x_n; \frac{\partial}{\partial x_i}\right) = L_i(s, x_1, x_3, ..., x_n)\left(\frac{\partial}{\partial x_i}\right)^{m_i} + \text{lower-order terms in } \frac{\partial}{\partial x_i},$$

for $i = 1, 3, ..., n$ such that

$$\begin{aligned} AF &= 0 \\ B_iF &= 0 \qquad \text{for} \quad i = 1, 3, ..., n. \end{aligned} \tag{2}$$

LEMMA 3. *There are nonzero linear partial differential operators* $P_i(x_1, x_3, ..., x_n;\ \partial/\partial s,\ \partial/\partial x_i)$, *for* $i = 1, 3, ..., n$, *with coefficients from* $K[x_1, x_3, ..., x_n]$, P_i *containing only derivatives of the form* $(\partial/\partial s)^\beta\,(\partial/\partial x_i)^\gamma$ *such that*

$$P_i\left(x_1, x_3, ..., x_n; \frac{\partial}{\partial s}, \frac{\partial}{\partial x_i}\right) F = 0 \qquad \textit{for} \quad i = 1, 3, ..., n.$$

Proof. Without loss of generality we may assume that A and the B_i in (2) above all have the same leading coefficient, i.e., that $L_i = L$ for $i = 1, 3, ..., n$. Let all the coefficients in A and the B_i have total degrees $\leqslant d$. Let $D = (\partial/\partial s)^\beta\,(\partial/\partial x_1)^\gamma$. If $\beta \geqslant m$ we have

$$LDF = \sum P_\delta D_\delta F,$$

where the sum on the right-hand side is over $D_\delta = (\partial/\partial s)^{\delta_1}\,(\partial/\partial x_1)^{\delta_2}$ with $\delta_1 < \beta$ and $\delta_2 \leqslant \gamma$ and the P_δ are polynomials in s, x_1, $x_3, ..., x_n$ of total degree $\leqslant d$. We obtain this by applying $(\partial/\partial s)^{\beta - m}\,(\partial/\partial x_1)^\gamma$ to $AF = 0$. The similar statement holds if $\gamma \geqslant m_1$, but then we must use $B_1F = 0$ and the sum is over $\delta_1 \leqslant \beta$ and $\delta_2 < \gamma$.

Iterating the above we see that if $\beta+\gamma\leqslant N$ then

$$L^N\left(\frac{\partial}{\partial s}\right)^\beta\left(\frac{\partial}{\partial x_1}\right)^\gamma F=\sum P_\delta D_\delta F,$$

where now the sum is over all $D_\delta=(\partial/\partial s)^{\delta_1}(\partial/\partial x_1)^{\delta_2}$ with $\delta_1<m$ and $\delta_2<m_1$ and the polynomials P_δ have total degrees $\leqslant Nd$.

Now let

$$D=x_1^{\alpha_1}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\left(\frac{\partial}{\partial s}\right)^\beta\left(\frac{\partial}{\partial x_1}\right)^\gamma, \tag{3}$$

where $\sum\alpha_i+\beta+\gamma\leqslant N$. Then

$$L^N DF=\sum \bar{P}_\delta D_\delta F, \tag{4}$$

where the sum is over all $D_\delta=(\partial/\partial s)^{\delta_1}(\partial/\partial x_1)^{\delta_2}$ with $\delta_1<m$, $\delta_2<m_1$ and the total degrees of the $\bar{P}_\delta$ are all $\leqslant N(d+1)$. The number of monomials in $s, x_1, x_3, ..., x_n$ of degree $\leqslant N(d+1)$ is $\binom{N(d+1)+n+1}{n+1}$. Hence the vector space of all such $\bar{P}_\delta D_\delta$ has dimension $mm_1\binom{N(d+1)+n+1}{n+1}\leqslant c_1N^{n+1}$ for some fixed c_1 and all $N\geqslant 1$. On the other hand, the number of D's of the type (3) above is $\binom{N+n+2}{n+1}$, which is $>c_2N^{n+2}$ for some $c_2>0$. Hence for N large enough there are $a_{\alpha_1\alpha_3\cdots\alpha_n\beta\gamma}\in K$, not all zero, such that

$$L^N\sum_{\alpha_1+\cdots+\alpha_n+\beta+\gamma\leqslant N} a_{\alpha_1\cdots\alpha_n\beta\gamma}x_1^{\alpha_1}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\left(\frac{\partial}{\partial s}\right)^\beta\left(\frac{\partial}{\partial x_1}\right)^\gamma F=0.$$

Let

$$P_1=\sum a_{\alpha_1\cdots a_n\beta\gamma}x_1^{\alpha_1}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\left(\frac{\partial}{\partial s}\right)^\beta\left(\frac{\partial}{\partial x_1}\right)^\gamma.$$

Then we have $L^NP_1F=0$ and hence, by Lemma 1, that $P_1F=0$. The $P_3, ..., P_n$ are found in a similar way, using $\partial/\partial x_i$ and $B_i=0$ in place of $\partial/\partial x_1$ and $B_1=0$. This completes the proof of Lemma 3.

Now let the P_i be as in Lemma 3 and let $P_i=\sum_{j=\alpha_i}^{\beta_i}P_{ij}(x_1, x_3, ..., x_n; \partial/\partial x_i)(\partial/\partial s)^j$ with $P_{i\alpha_i}\not\equiv 0$. Notice that the coefficient of $1/s^{\alpha_i+1}$ in P_iF is $(-1)^{\alpha_i}\alpha_i!\,P_{i\alpha_i}I_{12}(f)$. Hence $I_{12}(f)$ satisfies the equations

$$P_{i\alpha_i}\left(x_1, x_3, ..., x_n; \frac{\partial}{\partial x_i}\right)I_{12}(f)=0 \qquad \text{for} \quad i=1, 3, ..., n,$$

i.e., $I_{12}(f)$ is D-finite.

Iterating we have

THEOREM 1. *If $f\in K[[x_1, ..., x_n]]$ is D-finite, and I is any diagonal, then $I(f)$ is D-finite.*

Remarks. (1) In the case that f is convergent for $|x_i| < a$ for all i, F is analytic for $0 < |s| < a$, $|x_1| < |s|\, a$, and $|x_i| < a$ for $i = 3, ..., n$, and we can avoid the use of module M and Lemma 1 is trivial.

(2) If $f = \sum a_\nu x^\nu$, $g = \sum b_\nu x^\nu$, ν a multi-index, then the Hadamard product $f * g = \sum a_\nu b_\nu x^\nu$. Since $f * g = I_{1n+1} I_{2n+2} \cdots I_{n2n} f(x_1, ..., x_n)$ $g(x_{n+1}, ..., x_{2n})$, it follows from Theorem 1 that if f and g are D-finite then so is $f * g$. (If f is just differentially algebraic and g is D-finite it doesn't follow that $f * g$ is differentially algebraic—see Proposition 6.3 of [8].)

(3) Instead of iterating the argument given for Theorem 1 one can do several steps at once. For example, if $f(x_1, x_2, x_3)$ is D-finite and one wants to show that the complete diagonal $I(f)$ is D-finite, one can consider $F(s, t, x_1) = (1/st)\, f(s, t/s, x_1/t)$ and use the argument in Lemma 3 to show that F satisfies an equation of the form

$$P\left(x_1; \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}\right) F = 0, \qquad P \not\equiv 0.$$

If $P = P_{\alpha\beta}(x_1;\ \partial/\partial x_1)(\partial/\partial s)^\alpha\, (\partial/\partial t)^\beta +$ higher-order terms in $\partial/\partial s$ or $\partial/\partial t$ with $P_{\alpha\beta} \not\equiv 0$ then by considering the coefficient of $(1/s^{\alpha+1})(1/t^{\beta+1})$ we get that $P_{\alpha\beta}(x_1; \partial/\partial x_1)\, I(f) = 0$. This will give us smaller bounds for the order and degree of the equation satisfied by $I(f)$ than those obtained by iterating.

(4) If $f(x) = \sum_{n_1, ..., n_k \geqslant 0} f(n_1, ..., n_k)\, x_1^{n_1} \cdots x_k^{n_k}$ is D-finite and $C \subseteq \mathbf{N}^k$ is defined by a finite set of inequalities of the form $\sum a_i n_i + b \geqslant 0$, where the a_i, $b \in \mathbf{Z}$, then

$$h(x) = \sum_{\substack{n_1, ..., n_k \geqslant 0 \\ C}} f(n_1, ..., n_k)\, x_1^{n_1} \cdots x_k^{n_k}$$

is also D-finite. To see this consider the case that C is defined by just one inequality $\sum_{i=1}^{l} \alpha_i n_i + \alpha_0 \geqslant \sum_{i=l+1}^{k} \beta_i n_i$, where the α_i, $\beta_i \in \mathbf{N}$. Let

$$g(x, s, t) = s^{\alpha_0} \prod_{i=1}^{l} \frac{1}{1 - x_i s^{\alpha_i}} \prod_{i=l+1}^{k} \frac{1}{1 - x_i t^{\beta_i}}$$

$$a(s, t) = \frac{1}{1-s}\frac{1}{1-st} = \sum_{i \geqslant j} s^i t^j$$

$$b(x, s, t) = a(s, t) \prod_{i=1}^{n} \frac{1}{1 - x_i}.$$

Then

$$\tilde{g}(x) = (g * b)(x, 1, 1) = \sum_{\substack{n_i \geqslant 0 \\ C}} x_1^{n_1} \cdots x_k^{n_k}$$

and $h(x) = f(x) * \tilde{g}(x)$. Iterating we get the result for general C.

(5) If $f(x)$, C are as above and the $m_i(n_1, \ldots, n_k) = \sum_{j=1}^{k} a_{ij} n_j + b_i$, where the a_{ij}, b_i are nonnegative elements of $\mathbf{Q}$, then

$$l(x) = \sum_{\substack{n_1, \ldots, n_k \geqslant 0 \\ C}} f(n_1, \ldots, n_k)\, x_1^{m_1(n_1, \ldots, n_k)} \cdots x_k^{m_k(n_1, \ldots, n_k)}$$

is also D-finite. Let $y_i(x) = x_1^{a_{i1}} \cdots x_k^{a_{ik}}$ and notice that $l(x) = x_1^{b_1} \cdots x_k^{b_k} h(y_1, \ldots, y_k)$, where h is as above.

(6) Remark (5) gives a positive answer to question 4(e) of [9]. For example, if the sequences $f_i(n)$, $i = 1, 2, 3$, are p-recursive (i.e., the $\sum f_i(n) x^n$ are D-finite) then $f(x, y, z) = \sum_{i,j,k \geqslant 0} f_1(i)\, f_2(j)\, f_3(k)\, x^i y^j z^k$ is D-finite. Let C be defined by $i + 2j = k$ (i.e., $i + 2j \geqslant k$ and $k \geqslant i + 2j$). Then

$$F(x) = \sum_{\substack{i, j, k \geqslant 0 \\ i + 2j = k}} f_1(i)\, f_2(j)\, f_3(k)\, x^{(i+k)/2}$$

is D-finite (taking $m_1(i, j, k) = (i + k)/2$ and $m_2 = m_3 = 0$). But $F(x) = \sum_{n \geqslant 0} \sum_{k=0}^{n} f_1(n-k)\, f_2(k)\, f_3(n+k)\, x^n$.

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Note added in proof. H. Sharif (University of Kent, Canterbury, England) has informed me (April 1987), that he has independently proved the theorem of this paper, by somewhat different methods.

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