Linear independence of values of E-functions

1996 Sb. Math. 187 1197

(http://iopscience.iop.org/1064-5616/187/8/A03)

Sbornik: Mathematics **187**:8 1197–1211
Matematicheskiĭ Sbornik **187**:8 93–108 UDC 511.36

Linear independence of values of E-functions

Yu.V. Nesterenko and A.B. Shidlovskiĭ

Abstract. We prove a general theorem that establishes a relation between linear and algebraic independence of values at algebraic points of E-functions and properties of the ideal formed by all algebraic equations relating these functions over the field of rational functions. Using this theorem we prove sufficient conditions for linear independence of values of E-functions as well as for algebraic independence of values of subsets of them. The main result is an assertion stating that at all algebraic points, except finitely many, the values of E-functions are linearly independent over the field of all algebraic numbers if the corresponding functions are linearly independent over the field of rational functions. The theorem is applied to concrete E-functions.

Bibliography: 6 titles.

Introduction

In this paper we prove theorems concerning the linear and algebraic independence of the values at algebraic points of sets of E-functions forming solutions of a system of linear differential equations with coefficients in $\mathbb{C}(z)$.

E-functions were defined in 1929 by C.L. Siegel (see [1]). The basic examples of E-functions are given by $e^{\alpha z}$ with α an algebraic number and by entire generalized hypergeometric functions whose parameters are rational numbers.

In [1] Siegel considered the functions

$$K_\lambda(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(\lambda+1)\cdots(\lambda+n)} \cdot \left(\frac{z}{2}\right)^{2n}, \qquad \lambda \neq -1, -2, \dots, \tag{1}$$

which are solutions of the linear differential equation

$$y'' + \frac{2\lambda+1}{z} y' + y = 0, \tag{2}$$

and he proved the algebraic independence of the set of numbers $K_{\lambda_i}(\xi_j)$, $i = 1, \dots, m$, $j = 1, \dots, n$, for various values of $\lambda_i \in \mathbb{Q}$ and $\xi_j \in \mathbb{A}$ satisfying certain natural restrictions. (Here and below, $\mathbb{A}$ is the field of all algebraic numbers.)

This work was carried out with the partial support of the Russian Foundation for Fundamental Research (grant no. 94-01-00739) and the International Science Foundation (grant no. MNS 300).

AMS 1991 *Mathematics Subject Classification.* 11J91, 33C40.

Siegel pointed out the generality of his approach and subsequently published a corresponding general theorem. In 1954, Shidlovskiĭ weakened Siegel's condition in a substantial way and proved a theorem which, in essence, reduced the problem of studying algebraic independence of values of E-functions at algebraic points to the study of algebraic independence over $\mathbb{C}(z)$ of the functions themselves. In subsequent years many scientists have obtained various results regarding the arithmetical properties of values of E-functions. To become acquainted with the definition of the E-function, the history of the question and the main results on this topic one may consult the book [2].

Consider a family of E-functions

$$f_1(z),\ \ldots,\ f_m(z), \qquad m \geqslant 2, \tag{3}$$

that form a solution of the system of linear differential equations

$$y'_k = \sum_{i=1}^{m} Q_{ki} y_i, \qquad 1 \leqslant k \leqslant m, \qquad Q_{ki} \in \mathbb{C}(z), \tag{4}$$

and let $T(z) \in \mathbb{C}[z]$ be the least common denominator of all the functions Q_{ki}.

The problems of transcendental or algebraic independence of a set of numbers

$$f_1(\xi),\ \ldots,\ f_m(\xi), \tag{5}$$

$\xi \in \mathbb{A}$, $\xi T(\xi) \neq 0$, or of a subset of it, have largely been solved (see [2]). The problem of linear independence of the numbers (5), however, has been solved in certain specific situations only. The first result in this direction is the well-known Lindemann–Weierstrass theorem on linear independence over $\mathbb{A}$ of the numbers $e^{\alpha_1}, \ldots, e^{\alpha_m}$ for distinct $\alpha_1, \ldots, \alpha_m \in \mathbb{A}$. The book [2] contains some results on linear independence of numbers.

In [3] certain theorems regarding linear independence of the numbers (5) were proved and the following conjecture was put forward.

Conjecture A. *Under the above-mentioned conditions, the numbers* (5) *are linearly independent over* $\mathbb{A}$ *if and only if the functions* (3) *are linearly independent over* $\mathbb{C}(z)$.

This assertion has been proved in the following cases:

1) when the functions (3) are homogeneously algebraically independent over $\mathbb{C}(z)$;
2) when any subset of the functions (3) that is homogeneously algebraically independent over $\mathbb{C}$ is also homogeneously algebraically independent over $\mathbb{C}(z)$, that is, when the degree of homogeneous transcendence of the set of functions (3) over $\mathbb{C}(z)$ is equal to the degree of homogeneous transcendence of these functions over $\mathbb{C}$,

$$\operatorname{tr}\deg^0_{\mathbb{C}}\{f_1(z), \ldots, f_m(z)\} = \operatorname{tr}\deg^0_{\mathbb{C}(z)}\{f_1(z), \ldots, f_m(z)\}; \tag{6}$$

3) when $m = 2$.

In this paper we establish conditions of linear independence of the numbers (5) over $\mathbb{A}$ (Corollary 1 to Theorem 1) as well as the algebraic independence of the set of numbers (5) (Corollary 2 to Theorem 1). In particular, these conditions make it possible to prove that Conjecture A is true for all numbers $\xi \in \mathbb{A}$ except, possibly, finitely many (Theorem 2). Theorem 3 is an instance of the general assertion of Corollary 1, applied to specific functions. It shows that Conjecture A is true for the set of functions $K_{\lambda_i}(\xi_j z)$ (see (1)) with $\lambda_i \in \mathbb{Q}$, $\xi_j \in \mathbb{A}$ and $f_1(z) = 1$. We shall now state the assertions.

Let $\xi \in \mathbb{C}$ and let $\phi_\xi \colon \mathbb{C}[z, x_1, \dots, x_m] \to \mathbb{C}[x_1, \dots, x_m]$ be the homomorphism obtained by replacing the variable z by ξ in the polynomial ring $\mathbb{C}[z, x_1, \dots, x_m]$. For each ideal I of $\mathbb{C}[z, x_1, \dots, x_m]$, we denote by $I_\xi = \phi_\xi(I)$ the corresponding ideal in $\mathbb{C}[x_1, \dots, x_m]$.

Theorem 1. *Suppose that the E-functions* (3) *form a solution of the system of linear differential equations* (4), *and let $\mathfrak{P}$ be the prime ideal in $\mathbb{C}[z, x_1, \dots, x_m]$ generated by all polynomials that are homogeneous in the variables x_i and that vanish when the functions $f_1(z), \dots, f_m(z)$ are substituted for the variables $x_1, \dots, x_m$, respectively, and the algebraic number ξ is distinct from zero and the singular points of the system* (4) *and is such that the ideal $\mathfrak{P}_\xi$ is prime.*

Then for any homogeneous polynomial $P = P(x_1, \dots, x_m) \in \mathbb{A}[x_1, \dots, x_m]$ satisfying $P \notin \mathfrak{P}_\xi$ we have

$$P\big(f_1(\xi), \dots, f_m(\xi)\big) \neq 0.$$

If, under the conditions of Theorem 1, the functions $f_1(z), \dots, f_m(z)$ are homogeneously algebraically independent, then we can put $\mathfrak{P} = (0)$.

Note that if the polynomials $P_i(z, x_1, \dots, x_m)$, $1 \le i \le M$, form a basis of the ideal $\mathfrak{P}$, then

$$\mathfrak{P}_\xi = \big(P_1(\xi, x_1, \dots, x_m), \dots, P_M(\xi, x_1, \dots, x_m)\big),$$

and the fact that $\mathfrak{P}_\xi$ is a prime ideal can be effectively verified (see [4]).

Below (see §1, Lemma 2) we shall prove that the set of points ξ for which the ideal $\mathfrak{P}_\xi$ is not prime is finite.

Corollary 1. *If the conditions of Theorem* 1 *hold and the functions* (3) *are linearly independent over $\mathbb{C}(z)$, then the numbers* (5) *are linearly independent over $\mathbb{A}$.*

Proof. Assume, on the contrary, that the numbers (3) are linearly dependent over $\mathbb{A}$, that is, there are algebraic numbers $c_1, \dots, c_m$, not all equal to zero, such that

$$c_1 f_1(\xi) + \cdots + c_m f_m(\xi) = 0. \tag{7}$$

Put $L = c_1 x_1 + \cdots + c_m x_m \neq 0$. If $L \in \mathfrak{P}_\xi$, then, by the homogeneity of $\mathfrak{P}$ with respect to $x_1, \dots, x_m$, we find that

$$L = (z - \xi) \cdot L_1 + L_2, \qquad L_2 \in \mathfrak{P}, \tag{8}$$

where $L_1, L_2 \in \mathbb{C}[z, x_1, \dots, x_m]$ are linear forms in the variables x_i. Since, by the conditions of Corollary 1, the functions $f_1(z), \dots, f_m(z)$ are linearly independent over $\mathbb{C}(z)$, we can have $L_2 \in \mathfrak{P}$ only if $L_2 = 0$. But then (8) leads to a contradiction, since $L \in \mathbb{C}[x_1, \dots, x_m]$ by definition. So, $L \notin \mathfrak{P}_\xi$ and, by Theorem 1, (7) is impossible.

We can similarly prove that if, under the conditions of Theorem 1, the functions (3) are not related over $\mathbb{C}(z)$ by a homogeneous equation of degree r, then the numbers (5) are also not related over $\mathbb{A}$ by a homogeneous equation of degree r.

Note that the assertion of Corollary 1 remains true if the condition that $\mathfrak{P}_\xi$ is prime is weakened to the requirement that this ideal does not have linear divisors, that is, that any linear form $L = c_1x_1 + \cdots + c_mx_m$ is equal to $\mathfrak{P}_\xi$: $L = \mathfrak{P}_\xi$. This follows easily from the proof of Theorem 1 (see (23) and the subsequent argument).

We consider a concrete example relating to the functions (1).

Corollary 2. *Let $\lambda \in \mathbb{Q}\backslash\mathbb{Z}$, and let ξ be a non-zero algebraic number. The numbers*

$$1,\ K_\lambda(\xi),\ K'_\lambda(\xi),\ K_{-\lambda}(\xi),\ K'_{-\lambda}(\xi) \tag{9}$$

are linearly independent over $\mathbb{A}$ if and only if $\lambda \neq k + 1/2$, $k \in \mathbb{Z}$.

If $\lambda = n + 1/2$, $n \in \mathbb{Z}$, then the functions

$$K_\lambda(z),\ K'_\lambda(z),\ K_{-\lambda}(z),\ K'_{-\lambda}(z) \tag{10}$$

are linearly dependent over $\mathbb{C}(z)$. By the relations

$$2\lambda K_\lambda(z) + zK'_\lambda(z) = 2\lambda K_{\lambda-1}(z), \qquad zK_\lambda(z) = -2\lambda K'_{\lambda-1}(z) \tag{11}$$

(see [2], Chapter 6, §2, relations (42), (43)), which are true for all $\lambda \neq 0, -1, -2, \dots$, it suffices to prove this assertion for $\lambda = 1/2$ only. But, clearly, in this case it holds by (11). So, for certain polynomials $P_i(z) \in \mathbb{A}[z]$, $1 \leq i \leq 4$, we have the equality

$$P_1(z)K_\lambda(z) + P_2(z)K'_\lambda(z) + P_3(z)K_{-\lambda}(z) + P_4(z)K'_{-\lambda}(z) = 0.$$

Without loss of generality we may assume that the polynomials $P_i(z)$ are coprime. Since $\xi \in \mathbb{A}$, this implies a linear relation between the numbers (9) over $\mathbb{A}$.

If $\lambda \neq k + 1/2$, $k \in \mathbb{Z}$, then any three of the functions $K_\lambda(z), K'_\lambda(z), K_{-\lambda}(z)$, $K'_{-\lambda}(z)$ are algebraically independent over $\mathbb{C}(z)$ (see [2], Chapter 9, §7, Theorem 8) and the relation

$$K_\lambda(z)K'_{-\lambda}(z) - K'_\lambda(z)K_{-\lambda}(z) - 2\frac{\lambda}{z}K_\lambda(z)K_{-\lambda}(z) + 2\frac{\lambda}{z} = 0$$

holds (see [2], Chapter 9, §7, (73)).

Consequently, using the notation of Theorem 1, for the functions

$$1,\ K_\lambda(z),\ K'_\lambda(z),\ K_{-\lambda}(z),\ K'_{-\lambda}(z)$$

the ideal $\mathfrak{P}$ is the principal ideal generated by the irreducible polynomial

$$zx_1x_4 - zx_2x_3 - 2\lambda x_1x_3 + 2\lambda x_0^2. \tag{12}$$

For any $\xi \neq 0$ the polynomial $\xi x_1x_4 - \xi x_2x_3 - 2\lambda x_1x_3 + 2\lambda x_0^2$ is easily seen to be irreducible. Therefore the ideal

$$\mathfrak{P}_\xi = (\xi x_1x_4 - \xi x_2x_3 - 2\lambda x_1x_3 + 2\lambda x_0^2)$$

is prime and by Corollary 1 the numbers (9) are linearly independent over $\mathbb{A}$.

We shall prove a more general assertion in Theorem 3 below.

Corollary 3. *Under the conditions of Theorem* 1 *and if for some* r, $1 \le r < m$, *the functions*

$$f_1(z),\ \ldots,\ f_r(z)$$

are homogeneously algebraically independent over $\mathbb{C}(z)$ *and the numbers*

$$f_1(\xi),\ \ldots,\ f_r(\xi) \tag{13}$$

are homogeneously algebraically dependent over $\mathbb{A}$, *then there is an irreducible polynomial* $Q = Q(z, x_1, \ldots, x_m) \in \mathbb{A}[z, x_1, \ldots, x_m]$, *homogeneous in the variables* $x_1, \ldots, x_m$, *such that*

$$Q\big(z, f_1(z), \ldots, f_m(z)\big) = 0 \quad \textit{and} \quad Q(\xi, x_1, \ldots, x_m) \in \mathbb{A}[x_1, \ldots, x_r].$$

Proof. Let $P = P(x_1, \ldots, x_r) \in \mathbb{A}[x_1, \ldots, x_r]$ be a homogeneous irreducible polynomial satisfying

$$P\big(f_1(\xi), \ldots, f_r(\xi)\big) = 0.$$

By Theorem 1, $P \in \mathfrak{P}_\xi$, so that

$$P = (z - \xi) \cdot P_1 + Q_1, \qquad Q_1 \in \mathfrak{P},$$

where $P_1, Q_1 \in \mathbb{A}[z, x_1, \ldots, x_m]$ are polynomials that are homogeneous in the variables x_i and are such that $Q_1(\xi, x_1, \ldots, x_m) \ne 0$. The irreducibility of P implies that $Q_1 = a(z)Q$, where $a(z) \in \mathbb{A}[z]$, $Q \in \mathbb{A}[z, x_1, \ldots, x_m]$, with Q irreducible as well. The equality

$$Q(\xi, x_1, \ldots, x_m) = a(\xi)^{-1} \cdot P(x_1, \ldots, x_r)$$

completes the proof of Corollary 3.

We now assume that the functions (1) satisfy, in addition, the condition (6). Then the ideal $\mathfrak{P}$ is generated by homogeneous polynomials in the ring $\mathbb{C}[x_1, \ldots, x_m]$ and for any $\xi \in \mathbb{C}$ we have $\mathfrak{P}_\xi = \mathfrak{P} \cap \mathbb{C}[x_1, \ldots, x_m]$, a prime ideal. Thus, given the condition (6), the assertion of Theorem 1 holds for any algebraic number ξ distinct from zero and from the singular points of the system (4). In particular, we obtain

Corollary 4. *Suppose that the E-functions* (3) *form a solution of the system of linear differential equations* (4) *and satisfy condition* (6).

Then for any algebraic number ξ not equal to zero or to a singular point of the system (4):

1) *linear independence over* $\mathbb{C}(z)$ *of the functions* (3) *implies linear independence over* $\mathbb{A}$ *of their values* (5);
2) *the numbers* (13) *are homogeneously algebraically dependent over* $\mathbb{A}$ *if and only if there is an irreducible polynomial* $Q(z, x_1, \ldots, x_m) \in \mathbb{A}[z, x_1, \ldots, x_m]$, *homogeneous in the variables* $x_1, \ldots, x_m$, *such that*

$$Q\big(z, f_1(z), \ldots, f_m(z)\big) \equiv 0 \quad \textit{and} \quad Q(\xi, x_1, \ldots, x_m) \in \mathbb{A}[x_1, \ldots, x_r].$$

Note that the first assertion was also proved in [2] (see [2], Chapter 4, §8, Theorem 9).

The functions $e^{\alpha_i z}$, $1 \le i \le m$, for distinct algebraic numbers α_i, satisfy the conditions of Corollary 4. Taking in this case $\xi = 1$, assertion 1) implies the Lindemann–Weierstrass theorem.

As shown above, the set of points ξ for which the ideal $\mathfrak{P}_\xi$ is not prime, is finite. This makes it possible to prove the following theorem.

Theorem 2. *Suppose that the E-functions* (3) *form a solution of the system of linear differential equations* (4) *and are linearly independent over* $\mathbb{C}(z)$. *Then there is a finite and effectively definable set* $\Lambda \subset \mathbb{C}$ *such that for any algebraic number* $\xi \notin \Lambda$ *the numbers*

$$f_1(\xi),\ \ldots,\ f_m(\xi)$$

are linearly independent over the field of all algebraic numbers.

A similar assertion holds in the situation described by Corollary 3.

For concrete families of functions, when polynomials generating the ideal $\mathfrak{P}$ can be explicitly described, the set of points ξ for which the ideal $\mathfrak{P}_\xi$ is not prime can be explicitly indicated. As an example we give here the following assertion.

Theorem 3. *Let* $\lambda_1, \ldots, \lambda_m$ *be rational numbers not equal to negative integers and let* $\xi_1, \ldots, \xi_n$ *be non-zero algebraic numbers whose squares are distinct. Then the following three assertions are equivalent:*

1) *the numbers*

$$1,\ K_{\lambda_1}(\xi_j),\ K'_{\lambda_1}(\xi_j),\ \ldots,\ K_{\lambda_m}(\xi_j),\ K'_{\lambda_m}(\xi_j), \qquad 1 \leqslant j \leqslant n, \tag{14}$$

are linearly independent over $\mathbb{A}$;

2) *the functions*

$$1,\ K_{\lambda_1}(\xi_j z),\ K'_{\lambda_1}(\xi_j z),\ \ldots,\ K_{\lambda_m}(\xi_j z),\ K'_{\lambda_m}(\xi_j z), \qquad 1 \leqslant j \leqslant n,$$

are linearly independent over $\mathbb{C}(z)$;

3) *there do not exist two parameters* λ_i, λ_j *such that* $\lambda_i - \lambda_j \in \mathbb{Z}$.

This theorem generalizes the Lindemann–Weierstrass theorem. In fact, for $m=1$ and $\lambda_1 = -1/2$ we have

$$K_{-1/2}(z) = \cos z, \qquad K'_{-1/2}(z) = -\sin z,$$

so that

$$e^{iz} = K_{-1/2}(z) - iK'_{-1/2}(z), \qquad e^{-iz} = K_{-1/2}(z) + iK'_{-1/2}(z). \tag{15}$$

Let $\alpha_1, \ldots, \alpha_s$ be distinct algebraic numbers. By extending this set, if necessary, we can assume without loss of generality that $\alpha_1 = 0$, $s = 2n+1$, $\alpha_{n+j} = -\alpha_j$, $j = 2, \ldots, n+1$. Setting now $\xi_j = i\alpha_{j+1}$, $j = 1, \ldots, n$, we find by Theorem 3 that the numbers $e^{\alpha_1}, \ldots, e^{\alpha_s}$ are linearly independent over $\mathbb{A}$, which is the assertion of the Lindemann–Weierstrass theorem.

§1. Reduction of Theorem 2 to Theorem 1

Theorem 2 can be deduced from Theorem 1 using the following two lemmas.

Lemma 1. *The ideal $\mathfrak{P}$ is absolutely prime, that is, it remains prime under extension to an arbitrary ring $K[x_1,\dots,x_m]$, where K is a field containing $\mathbb{C}(z)$.*

Proof. We denote the field of fractions of the quotient ring $\mathbb{C}[z,x_1,\dots,x_m]/\mathfrak{P}$ by $F(\mathfrak{P})$. It follows from the definition of $\mathfrak{P}$ that the field $F(\mathfrak{P})$ is isomorphic to the field $\mathbb{C}(z,tf_1(z),\dots,tf_m(z))$, where t is a variable that is independent of z. This field has characteristic zero and is hence separable over $\mathbb{C}(z)$. To prove the lemma it now suffices to to prove that the field $\mathbb{C}(z)$ is algebraically closed in $F(\mathfrak{P})$ (see [5], Vol. 2, Chapter 7, §12, Theorem 39). Let $S \in \mathbb{C}(z,x_1,\dots,x_m)$ and let $\varphi = S(z,tf_1(z),\dots,tf_m(z))$ be an algebraic function of z. This function is single-valued, for the $f_i(z)$ are entire functions. Consequently, $\varphi \in \mathbb{C}(z)$, that is, $\mathbb{C}(z)$ is algebraically closed in $F(\mathfrak{P})$. This proves the lemma.

Corollary. *The functions $f_1(z),\dots,f_m(z)$ are linearly independent over the field $\mathbb{B}$ of all algebraic functions.*

In fact, if there were a linear relation

$$\alpha_1(z)f_1(z)+\cdots+\alpha_m(z)f_m(z)=0, \qquad \alpha_i(z)\in\mathbb{B},$$

then the ideal $\mathfrak{P}^e$, the extension of $\mathfrak{P}$ to the ring $\mathbb{B}[x_1,\dots,x_m]$, would contain a linear form. But it would then follow from the definition of these ideals that $\mathfrak{P}$ must also contain a linear form. This contradicts the assumption of linear independence of the $f_i(z)$ over $\mathbb{C}(z)$.

Lemma 2. *Let $\mathfrak{P}$ be an absolutely prime ideal in the ring $\mathbb{C}[z,x_1,\dots,x_m]$. Then there are only finitely many numbers $\xi \in \mathbb{C}$ for which the ideal $\mathfrak{P}_\xi$ is not prime.*

The proof of this Lemma (see [4], Theorem 16) also gives an effective procedure for finding those numbers ξ for which the ideal $\mathfrak{P}_\xi$ is not prime.

Corollary. *Under the conditions of Theorem 2, the set of numbers ξ for which the ideal $\mathfrak{P}_\xi$ is not prime, is finite.*

By Lemma 1, this assertion is a direct consequence of Lemma 2.

Let Λ denote the set of all singular points of the system of differential equations (4), including 0, and of all numbers $\xi \in \mathbb{A}$ for which the ideal $\mathfrak{P}_\xi$ is not prime. By the Corollary to Lemma 2, Λ is a finite set. Theorem 2 now follows immediately from Corollary 1 to Lemma 1.

§2. Proof of Theorem 1

We introduce some notations and definitions needed in the proof of the theorem. Let $\Bbbk = \mathbb{C}(z)$ be the field of rational functions and R the subfield of $\Bbbk$ consisting of the rational functions not having a singularity at the point ξ.

For each natural number N and homogeneous ideal $I \subset \Bbbk[x_1,\dots,x_m]$ we denote by $\mathfrak{L}_I(N)$ the linear space over $\Bbbk$ generated by the residues modulo I of the homogeneous polynomials of degree N in the ring $\Bbbk[x_1,\dots,x_m]$. Note that if $J \subset I$ are two homogeneous ideals, then we may assume that $\mathfrak{L}_I(N) \subset \mathfrak{L}_J(N)$.

Also, we set $\chi_I(N) = \dim \mathfrak{L}_I(N)$. By a theorem of Hilbert, for all sufficiently large N the quantity $\chi_I(N)$ is a polynomial in N (see [5], Vol. 2, Chapter 7, §12, Theorem 41). The degree of this polynomial is equal to the projective dimension of I (*ibid.*, Theorem 42). Thus,

$$\chi_I(N) = \alpha N^d + O(N^{d-1}), \qquad d = \dim I, \quad \alpha > 0. \tag{16}$$

Definition. By a *ξ-basis of the linear space* $\mathfrak{L}_I(N)$ we mean any family of elements

$$b_1, \ \dots, \ b_q \tag{17}$$

of this space that satisfies the three conditions:

1) the elements (17) are residues modulo I of certain homogeneous polynomials of degree N in the ring $\mathbb{C}[z][x_1, \dots, x_m]$;
2) the residue modulo I of each homogeneous polynomial of degree N in the ring $R[x_1, \dots, x_m]$ is a linear combination of the elements (17) with coefficients from R;
3) the elements (17) are linearly independent over $\mathbb{k}$.

Note that any ξ-basis is a basis in the usual sense of the linear space $\mathfrak{L}_I(N)$.

Lemma 3. *For any homogeneous ideal $I \subset \mathbb{k}[x_1, \dots, x_m]$ and natural number N the space $\mathfrak{L}_I(N)$ has a ξ-basis.*

Proof. There is a set of elements (17) with the properties 1) and 2). For example, the set of residues modulo I of all products of the powers

$$x_1^{k_1} \cdots x_m^{k_m}, \qquad k_i \geqslant 0, \quad k_1 + \dots + k_m = N.$$

Let (17) be one such set, having the smallest possible number of elements q. We claim that the set (17) thus defined also has the property 3). In fact, in the opposite case there would exist polynomials $p_1(z), \dots, p_q(z) \in \mathbb{C}[z]$ such that

$$p_1(z)b_1 + \dots + p_q(z)b_q = 0.$$

Moreover, without loss of generality we may assume that $p_q(\xi) \neq 0$. Then $p_i(z)/p_q(z)$ is in R, $i = 1, \dots, q-1$, and

$$b_q = -\frac{p_1(z)}{p_q(z)} \cdot b_1 - \dots - \frac{p_{q-1}(z)}{p_q(z)} \cdot b_{q-1},$$

contradicting the definition of q. This contradiction proves Lemma 3.

For an ideal $\mathfrak{A}$ in the ring $\mathbb{C}[z, x_1, \dots, x_m]$ we denote by $\mathfrak{A}^e$ its extension to the ring $\mathbb{k}[x_1, \dots, x_m]$.

We denote by s the degree of the polynomial P given in the conditions of Theorem 1 with respect to all variables $x_1, \dots, x_m$. We also assume that $A_1, \dots, A_u$ is a set of homogeneous polynomials in $\mathbb{C}[z][x_1, \dots, x_m]$ of degree $N - s$ whose residues modulo $\mathfrak{P}^e$ form a ξ-basis of the linear space $\mathfrak{L}_{\mathfrak{P}^e}(N-s)$. If $d = \dim \mathfrak{P}^e$ is the projective dimension of the ideal $\mathfrak{P}^e$, then by (16) we have

$$u = \chi_{\mathfrak{P}^e}(N-s) = \alpha(N-s)^d + O\big((N-s)^{d-1}\big) = \alpha N^d + O(N^{d-1}). \tag{18}$$

Suppose now that the assertion of Theorem 1 is not true, that is,

$$P\big(f_1(\xi), \dots, f_m(\xi)\big) = 0.$$

Let $J = (P, \mathfrak{P})$ be the homogeneous ideal generated in $\Bbbk[x_1, \dots, x_m]$ by the polynomial P and the polynomials in the ideal $\mathfrak{P}$. Since $P \notin \mathfrak{P}$, otherwise $P \in \mathfrak{P}^e$, we have $\dim J \leq d-1$ (see [5], Vol. 2, Chapter 7, §7, Corollary of Theorem 21). Let $B_1, \dots, B_v$ be a set of homogeneous polynomials of degree N in $\mathbb{C}[z][x_1, \dots, x_m]$ whose residues modulo J form a ξ-basis of the linear space $\mathfrak{L}_J(N)$. Equality (16) implies that

$$v = \chi_J(N) = O(N^{d-1}). \tag{19}$$

Lemma 4. *Under the conditions of Theorem 1, the residues of the polynomials* $P \cdot A_1, \dots, P \cdot A_u, B_1, \dots, B_u$ *modulo the ideal* $\mathfrak{P}^e$ *form a* ξ*-basis of the space* $\mathfrak{L}_{\mathfrak{P}^e}(N)$.

Proof. Let

$$a_1, \ \dots, \ a_u, \quad b_1, \dots, b_v, \tag{20}$$

be the corresponding residues of the polynomials $P \cdot A_1, \dots, P \cdot A_u, B_1, \dots, B_u$ modulo the ideal $\mathfrak{P}^e$.

If for certain polynomials $\gamma_1, \dots, \gamma_u, \delta_1, \dots, \delta_v \in \mathbb{C}[z]$ we have

$$\gamma_1 a_1 + \dots + \gamma_u a_u + \delta_1 b_1 + \dots + \delta_v b_v = 0,$$

then

$$\gamma_1 P A_1 + \dots + \gamma_u P A_u + \delta_1 B_1 + \dots + \delta_v B_v \in \mathfrak{P}^e, \tag{21}$$

and consequently

$$\delta_1 B_1 + \dots + \delta_v B_v \in J.$$

However, since the elements $b_1, \dots, b_v$ form a basis of $\mathfrak{L}_J(N)$, this membership relation implies that $\delta_1 = \dots = \delta_v = 0$. Then (21) implies

$$P \cdot (\gamma_1 A_1 + \dots + \gamma_u A_u) \in \mathfrak{P}^e$$

and, since $\mathfrak{P}$ is a prime ideal, $P \notin \mathfrak{P}$, $\mathfrak{P} \cap \mathbb{C}[z] = \{0\}$, we have

$$\gamma_1 A_1 + \dots + \gamma_u A_u \in \mathfrak{P}^e.$$

The residues of the polynomials A_j modulo $\mathfrak{P}^e$ form a basis of $\mathfrak{L}_{\mathfrak{P}^e}(N-s)$, hence $\gamma_1 = \dots = \gamma_u = 0$. Thus, the elements (20) are linearly independent over $\mathbb{C}(z)$.

We will now show that the second condition for being a ξ-basis holds for (20). Since the elements $b_1, \dots, b_v$ form a ξ-basis of $\mathfrak{L}_J(N)$, for any polynomial H in

$R[x_1, \dots, x_m]$ and certain rational functions $\beta_1, \dots, \beta_v \in R$ we have the membership relation

$$P_1 = H - \beta_1 B_1 - \cdots - \beta_v B_v \in J.$$

This implies that there are a polynomial $b(z) \in \mathbb{C}[z]$, $b(\xi) \neq 0$, and an integer $n \geq 0$ such that $b(z) \cdot P_1 \in \mathbb{C}[z, x_1, \dots, x_m]$ and for some $Q(z, x_1, \dots, x_m)$ in $\mathbb{C}[z, x_1, \dots, x_m]$ we have

$$(z - \xi)^n \cdot b(z) \cdot P_1 \equiv P \cdot Q \pmod{\mathfrak{P}}. \tag{22}$$

Let n be the smallest number for which such a representation exists.

We assume that $n \geq 1$. Then (22) implies that

$$P(x_1, \dots, x_m) \cdot Q(\xi, x_1, \dots, x_m) \in \mathfrak{P}_\xi. \tag{23}$$

By the conditions of Theorem 1, the ideal $\mathfrak{P}_\xi$ is prime and $P \notin \mathfrak{P}_\xi$. Therefore $Q(\xi, x_1, \dots, x_m) \in \mathfrak{P}_\xi$, and for some polynomial $G \in \mathbb{C}[z, x_1, \dots, x_m]$ the congruence

$$Q(z, x_1, \dots, x_m) \equiv (z - \xi) G(z, x_1, \dots, x_m) \pmod{\mathfrak{P}}$$

holds. Now (22) implies

$$(z - \xi)^n \cdot b(z) \cdot P_1 \equiv (z - \xi) \cdot G \cdot P \pmod{\mathfrak{P}}.$$

But $\mathfrak{P}$ is a prime ideal and $z - \xi \notin \mathfrak{P}$. Consequently,

$$(z - \xi)^{n-1} \cdot b(z) \cdot P_1 \equiv G \cdot P \pmod{\mathfrak{P}}.$$

This contradiction with the choice of n shows that, in fact, $n = 0$, that is,

$$b(z) \cdot P_1 \equiv P \cdot Q \pmod{\mathfrak{P}}.$$

The polynomial Q/b belongs to the ring $R[x_1, \dots, x_m]$. By the definition of a ξ-basis, for certain elements $\alpha_1, \dots, \alpha_u \in R$ the following congruence holds in the ring $R[x_1, \dots, x_m]$:

$$Q/b \equiv \alpha_1 A_1 + \cdots + \alpha_u A_u \pmod{\mathfrak{P}^e};$$

consequently, so does the congruence

$$H \equiv \beta_1 B_1 + \cdots + \beta_v B_v + \alpha_1 P A_1 + \cdots + \alpha_u P A_u \pmod{\mathfrak{P}^e}.$$

This completes the proof of Lemma 4.

We now turn immediately to the proof of Theorem 1.

Let $\mathbb{K}$ be a finite extension of the field $\mathbb{Q}$ containing the coefficients of all E-functions $f_j(z)$, the coefficients of the polynomial P and the number ξ. Also, let $\overline{f} = (f_1(z), \dots, f_m(z))$,

$$F_i(z) = P(\overline{f}) A_i(\overline{f}), \qquad i = 1, \dots, u,$$
$$F_{j+u}(z) = B_j(\overline{f}), \qquad j = 1, \dots, v,$$

and let $M = u + v$. Using (18), (19) we can assert that $M = \alpha N^d + O(N^{d-1})$. Let

$$D = \frac{\partial}{\partial z} + \sum_{k=1}^{m} \Bigl(\sum_{j=1}^{m} Q_{kj} x_j \Bigr) \frac{\partial}{\partial x_k}$$

be the differential operator corresponding to the system (4). Since ξ differs from the singular points of (4), the differential operator D acts in the ring $R[x_1, \ldots, x_m]$. In particular, this means that all the polynomials $D(PA_i)$, $D(B_j)$ belong to the ring $R[x_1, \ldots, x_m]$ and, by Lemma 4, each of them is congruent modulo $\mathfrak{P}$ some linear combination of the polynomials $P \cdot A_1, \ldots, P \cdot A_u, B_1, \ldots, B_v$ with coefficients in R. In turn, this means that the set of E-functions $F_1(z), \ldots, F_M(z)$ forms the solution of a system of linear differential equations with coefficients in $\mathbb{C}(z)$; moreover, ξ is not a singular point of this system. The functions $F_i(z)$ are linearly independent over $\mathbb{C}(z)$, since in the opposite case some linear combination of the polynomials $P \cdot A_1, \ldots, P \cdot A_u, B_1, \ldots, B_v$ with coefficients in $\mathbb{C}[z]$ would belong to $\mathfrak{P}$, contradicting Lemma 4. Under these conditions we can assert (see [2], Chapter 3, §11, Lemma 17) that ρ, the maximal number of $F_i(\xi)$ that are linearly independent over $\mathbb{K}$, satisfies the inequality

$$\rho = \operatorname{rg}\bigl(F_1(\xi), \ldots, F_M(\xi)\bigr) \geqslant \frac{M}{h} = \frac{\alpha}{h} \cdot N^d + O(N^{d-1}),$$

where $h = [\mathbb{K} : \mathbb{Q}]$.

However, since $P(f_1(\xi), \ldots, f_m(\xi)) = 0$, we have

$$\rho = \operatorname{rg}\bigl(F_{u+1}(\xi), \ldots, F_M(\xi)\bigr) \leqslant v = O(N^{d-1}).$$

These bounds for ρ contradict each other, which completes the proof of Theorem 1.

§3. Proof of Theorem 3

To prove Theorem 3 we shall show the following dependence between its assertions: 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 1).

1) $\Longrightarrow$ 2). This can be proved in the same way as Corollary 2 of Theorem 1.

2) $\Longrightarrow$ 3). If there existed parameters λ_i, λ_j with $\lambda_i - \lambda_j = k \in \mathbb{Z}$, then, by (11), the functions $K_{\lambda_i}, K'_{\lambda_i}, K_{\lambda_j}, K'_{\lambda_j}$ would be linearly dependent over $\mathbb{C}(z)$. This would imply that the functions participating in the second assertion of Theorem 3 would be linearly dependent over $\mathbb{C}(z)$.

3) $\Longrightarrow$ 1). To prove this we use Corollary 1 to Theorem 1. We assume that the conditions $\lambda_i - \lambda_j \notin \mathbb{Z}$, $1 \le i < j \le m$, hold. Adjoining to the set $\lambda_1, \ldots, \lambda_m$ the numbers $-1/2$ and 0 and $-\lambda_j$ for certain j if necessary, and extending in this way the set of functions under consideration, by (11) we can assume without loss of generality that $m = 2r - 2$, $r \ge 2$, $\lambda_1 = -1/2$, $\lambda_2 = 0$,

$$\begin{aligned} &\lambda_i + \lambda_j \notin \mathbb{Z}, && 1 \leqslant i < j \leqslant r, \\ &\lambda_{r+i} = -\lambda_{i+2}, && i = 1, \ldots, r-2. \end{aligned}$$

By (15), to complete the proof of Theorem 3 it suffices to prove linear independence over $\mathbb{A}$ of the set of values at $z = 1$ of the functions

$$\begin{gathered} 1,\ e^{\xi_j z},\ e^{-\xi_j z},\ K_0(\xi_j z),\ K_0'(\xi_j z), \\ K_{\lambda_i}(\xi_j z),\ K_{\lambda_i}'(\xi_j z),\ K_{-\lambda_i}(\xi_j z),\ K_{-\lambda_i}'(\xi_j z), \quad 3 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n. \end{gathered} \tag{24}$$

These functions form a solution of a system of $1 + 2mn$ linear homogeneous differential equations with coefficients in $\mathbb{C}(z)$ and with one singular point at $z = 0$.

We set $\xi_{n+j} = -\xi_j$, $j = 1, \dots, n$, $d = 2n$, $\overline{\xi} = (\xi_1, \dots, \xi_d)$, and denote by L the lattice in $\mathbb{Z}^d$ formed by the vectors $\overline{a} \in \mathbb{Z}^d$ satisfying $(\overline{a}, \overline{\xi}) = 0$.

For each vector

$$\overline{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d, \qquad \lambda_j \geqslant 0,$$

we set

$$\overline{x}^{\overline{\lambda}} = x_1^{\lambda_1} \cdots x_d^{\lambda_d} \in \mathbb{C}[z, x_1, \dots, x_d].$$

Lemma 5. *Let $\mathfrak{J}$ be the ideal generated in the ring $\mathbb{C}[z, x_1, \dots, x_d]$ by the polynomials $\overline{x}^{\overline{\lambda}} - \overline{x}^{\overline{\mu}}$ for which $\overline{\lambda} - \overline{\mu} \in L$. Then $\mathfrak{J}$ is a prime ideal.*

Proof. Let $\mathfrak{P}_1$ be the prime ideal generated by all polynomials $P \in \mathbb{C}[z, x_1, \dots, x_d]$ satisfying

$$P(z, e^{\xi_1 z}, \dots, e^{\xi_d z}) = 0.$$

The definition of the lattice L implies that $\mathfrak{J} \subset \mathfrak{P}_1$. Consider an arbitrary polynomial $P \in \mathfrak{P}_1$. It can be written as $P = Q + R$ with $Q \in \mathfrak{J}$ and

$$R = \sum_{\overline{\lambda} \in \mathcal{N}} p_{\overline{\lambda}}(z) \overline{x}^{\overline{\lambda}}, \qquad p_{\overline{\lambda}} \in \mathbb{C}[z],$$

where for any two vectors $\overline{\lambda}, \overline{\mu} \in \mathcal{N}$ we have $\overline{\lambda} - \overline{\mu} \notin L$ and $p_{\overline{\lambda}}(z) \neq 0$. Since $R = P - Q$ and $P \in \mathfrak{P}_1$, $Q \in \mathfrak{J} \subset \mathfrak{P}_1$, we have $R \in \mathfrak{P}_1$. Consequently,

$$\sum_{\overline{\lambda} \in \mathcal{N}} p_{\overline{\lambda}}(z) e^{(\overline{\xi}, \overline{\lambda}) z} = 0. \tag{25}$$

Since for any two vectors $\overline{\lambda}, \overline{\mu} \in \mathcal{N}$ we have $\overline{\lambda} - \overline{\mu} \notin L$, the numbers $(\overline{\xi}, \overline{\lambda})$, $\overline{\lambda} \in \mathcal{N}$, are all different. Now (25) implies that $p_{\overline{\lambda}}(z) = 0$, $\overline{\lambda} \in \mathcal{N}$. This contradiction shows that $R = 0$, that is, $P \in \mathfrak{J}$. Thus we have proved that $\mathfrak{J} = \mathfrak{P}_1$, that is, $\mathfrak{J}$ is a prime ideal. This proves Lemma 5.

In the sequel we shall need the notion of height of an ideal in a Noetherian ring. For prime ideals the corresponding definition can be found, for example, in [5], Vol. 1, Chapter 4, §14. For an arbitrary ideal $\mathfrak{U}$ the height is defined as the smallest of the heights of the prime ideals containing $\mathfrak{U}$. We denote the height of an ideal $\mathfrak{U}$ by $\operatorname{rg} \mathfrak{U}$.

Lemma 6. *Let $\mathfrak{R}$ be a Noetherian ring, A and B elements of $\mathfrak{R}$, $\mathfrak{B}$ a prime ideal in $\mathfrak{R}$, and $\mathfrak{U}$ the ideal in the ring of polynomials $\mathfrak{R}[y]$ generated by the elements of $\mathfrak{B}$ and the polynomial $By - A$, that is, $\mathfrak{U} = (\mathfrak{B}, By - A) \subset \mathfrak{R}[y]$. If*

$$\mathrm{rg}(\mathfrak{B}, A, B) \geqslant \mathrm{rg}\,\mathfrak{B} + 2, \tag{26}$$

then $\mathfrak{U}$ is a prime ideal in $\mathfrak{R}[y]$ and $\mathrm{rg}\,\mathfrak{U} \geq \mathrm{rg}\,\mathfrak{B} + 1$.

Proof. The inequality (26) implies that $B \notin \mathfrak{B}$ and that the extension $\mathfrak{B}^*$ of $\mathfrak{B}$ in the ring $\mathfrak{R}[B^{-1}]$ is a prime ideal. Let $\mathfrak{M}$ be the ideal in $\mathfrak{R}[y]$ generated by all polynomials $P \in \mathfrak{R}[y]$ satisfying the condition $P\,|_{y=A/B} \in \mathfrak{B}^*$. Clearly, $\mathfrak{M}$ is a prime ideal. We have $\mathfrak{U} \subset \mathfrak{M}$. For an arbitrary polynomial $P \in \mathfrak{M}$ and a certain positive integer r we have the membership relation $B^r \cdot P \in \mathfrak{U}$. Since $\mathfrak{M}$ has a finite basis, there is a positive integer r such that $B^r \cdot \mathfrak{M} \subset \mathfrak{U}$.

Let $\mathfrak{Q}$ be a prime ideal in $\mathfrak{R}[y]$ associated with $\mathfrak{U}$. Then $\mathrm{rg}\,\mathfrak{Q} \leq \mathrm{rg}\,\mathfrak{B} + 1$ (see [5], Chapter 4, §14, Theorem 30). If $B \in \mathfrak{Q}$, then $By - A \in \mathfrak{U} \subset \mathfrak{Q}$, $\mathfrak{B} \subset \mathfrak{U} \subset \mathfrak{Q}$ imply that $(\mathfrak{B}, A, B) \subset \mathfrak{Q}$ and, hence, $\mathrm{rg}(\mathfrak{B}, A, B) \leq \mathrm{rg}\,\mathfrak{Q} \leq \mathrm{rg}\,\mathfrak{B} + 1$, contradicting (26). This contradiction implies that $B \notin \mathfrak{Q}$, that is, the polynomial B is not contained in any prime ideal associated with $\mathfrak{U}$. The inclusion $B^r \cdot \mathfrak{M} \subset \mathfrak{U}$ now implies that $\mathfrak{M} \subset \mathfrak{U}$, that is $\mathfrak{M} = \mathfrak{U}$ and, hence, $\mathfrak{U}$ is a prime ideal.

The inequality for the heights clearly holds, since the polynomial $By - A$ belongs to the ideal $\mathfrak{U}$ and is not contained in the prime ideal $\mathfrak{B}^e$, the extension of $\mathfrak{B}$ in $\mathfrak{R}[y]$. This completely proves the lemma.

Lemma 7. *Using the notation of Lemma 6, let $\mathfrak{R} = \mathfrak{D}[x]$ be the ring of polynomials in a single variable over a Noetherian ring $\mathfrak{D}$, $\mathfrak{D} \supset \mathbb{C}$, and let $A = A(x) \in \mathfrak{D}[x]$, $B \in \mathfrak{D}$; let $\mathfrak{B}$ be the extension in $\mathfrak{R}$ of a prime ideal $\mathfrak{B}_0 \subset \mathfrak{D}$, $A(x) \notin \mathfrak{B}$, $B \notin \mathfrak{B}$. Assume also that the ideal in $\mathfrak{D}$ generated by the elements of $\mathfrak{B}_0$ and the coefficients of the polynomials $A(x)$ and B coincides with $\mathfrak{D}$. Then*

$$\mathrm{rg}\big(\mathfrak{B}, A(x), B\big) \geqslant \mathrm{rg}\,\mathfrak{B} + 2.$$

Proof. By passing to the quotient rings modulo the ideals $\mathfrak{B}_0$ and $\mathfrak{B}$ we may assume without loss of generality that $\mathfrak{B}_0 = (0)$, $\mathfrak{B} = (0)$. If the assertion of the Lemma were not true, then there would exist a prime ideal $\mathfrak{P}$ in $\mathfrak{D}[x]$ associated with the ideal $(A(x), B) \subset \mathfrak{D}[x]$ such that $\mathrm{rg}\,\mathfrak{P} \leq 1$. Since $B \in \mathfrak{P}$, there would exist a prime ideal $\mathfrak{f}$ in $\mathfrak{D}$ associated with the principal ideal (B) and such that the extension $\mathfrak{f}^e$ of $\mathfrak{f}$ in $\mathfrak{D}[x]$ is contained in $\mathfrak{P}$, that is, $\mathfrak{f}^e \subset \mathfrak{P}$. Since B is not a zero divisor in $\mathfrak{D}$, we have $\mathrm{rg}\,\mathfrak{f} \geq 1$. We would then have $1 \leq \mathrm{rg}\,\mathfrak{f} \leq \mathrm{rg}\,\mathfrak{f}^e \leq \mathfrak{P}$. This proves that $\mathfrak{P} = \mathfrak{f}^e$, that is, $\mathfrak{P}$ has a basis consisting of elements of $\mathfrak{D}$. The membership relation $A(x) \in \mathfrak{P}$ implies that all the coefficients of $A(x)$ belong to $\mathfrak{P}$. By requirement, the ideal generated by B and the coefficients of A contains 1. Consequently, $1 \in \mathfrak{P}$. This contradiction shows that $\mathrm{rg}\,\mathfrak{P} \geq 2$. This proves Lemma 7.

Remark. We shall now consider a particular case relating to the functions $K_\lambda(z)$. Let ξ, λ be non-zero complex numbers. Let $\mathfrak{D}$ be a Noetherian ring containing the ring of polynomials $\mathbb{C}[z, x_1, x_2]$, let $\mathfrak{R} = \mathfrak{D}[x_3]$ and let (see (12))

$$R = \xi z(x_1 x_4 - x_2 x_3) - 2\lambda(x_1 x_3 - 1) = Bx_4 - A,$$

where

$$B = \xi z x_1, \qquad A = (\xi z x_2 + 2\lambda x_1)x_3 - 2\lambda,$$

are polynomials in $\mathfrak{R}$. The coefficients $\xi z x_2 + 2\lambda x_1$ and 2λ of A generate in $\mathfrak{D}$ an ideal which coincides with the ring $\mathfrak{D}$ itself. Hence, for any prime ideal $\mathfrak{B}_0$ in $\mathfrak{D}$, by Lemma 7 the ring $\mathfrak{R}$ satisfies $\operatorname{rg}(\mathfrak{B}, A, B) \geq \operatorname{rg}\mathfrak{B} + 2$, and by Lemma 6 we find that the ideal $(\mathfrak{B}_0, R) \subset \mathfrak{D}[x_3, x_4]$ is prime.

Below we shall use this reasoning inductively, adjoining polynomials corresponding to various values of ξ and λ.

Consider in the ring $\mathbb{W}$ of polynomials in the variables $x_{u,v}$, $1 \leq u \leq 2m$, $1 \leq v \leq n$, and with coefficients in $\mathbb{C}[z]$ the prime ideal $\mathfrak{P}$ generated by the polynomials that vanish when $e^{\xi_j z}$ is substituted for $x_{1,j}$, $e^{-\xi_j z}$ is substituted for $x_{2,j}$, $K_0(\xi_j z)$ is substituted for $x_{3,j}$, $K_0'(\xi_j z)$ is substituted for $x_{4,j}$, and the functions $K_{\lambda_i}(\xi_j z)$, $K'_{\lambda_i}(\xi_j z)$, $K_{-\lambda_i}(\xi_j z)$, and $K'_{-\lambda_i}(\xi_j z)$ are substituted for $x_{4i-7,j}$, $x_{4i-6,j}$, $x_{4i-5,j}$, and $x_{4i-4,j}$, respectively $(3 \leq i \leq r,\ 1 \leq j \leq n)$.

Let $\mathfrak{I}$ be the ideal in the ring $\mathbb{V} = \mathbb{C}[z, x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}]$ consisting of the polynomials that vanish when $e^{\xi_j z}$ is substituted for $x_{1,j}$ and $e^{-\xi_j z}$ is substituted for $x_{2,j}$. By Lemma 5, this ideal has a basis consisting of polynomials not depending on z. The form of these polynomials is indicated in Lemma 5. It can be readily seen that $\mathfrak{P}$ contains all polynomials in $\mathfrak{I}$ as well as the following polynomials:

$$Q_{i,j} = \xi_j z(x_{4i-3,j}x_{4i,j} - x_{4i-2,j}x_{4i-1,j}) - 2\lambda_{i+1}(x_{4i-3,j}x_{4i-1,j} - 1),$$
$$2 \leqslant i \leqslant r-1, \qquad 1 \leqslant j \leqslant n.$$

Let $\mathfrak{U}$ be the ideal generated in $\mathbb{W}$ by the polynomials $Q_{i,j}$ and the polynomials in $\mathfrak{I}$. As has been shown above, $\mathfrak{U} \subset \mathfrak{P}$.

Proposition 1.

1) *The ideal $\mathfrak{U}$ is prime and $\mathfrak{U} = \mathfrak{P}$.*
2) *For any point $\xi \in \mathbb{C}$ not equal to zero, the ideal $\mathfrak{U}_\xi$ is prime.*

Proof. Renumber in some manner the quadruples of variables

$$(x_{4i-7,j}, x_{4i-6,j}, x_{4i-5,j}, x_{4i-4,j}), \quad 3 \leq i \leq r, 1 \leq j \leq n.$$

We extend $\mathbb{V}$ by successively adjoining these quadruples. Simultaneously we extend $\mathfrak{I}$ by adjoining the corresponding polynomials $Q_{i,j}$. In this manner we obtain, as shown in the above remark, a sequence of prime ideals. The last term in this sequence coincides with $\mathfrak{U}$. Thus, $\mathfrak{U}$ is prime. Moreover, Lemma 6 implies that by adjoining a polynomial $Q_{i,j}$ the height of the corresponding ideal increases by 1. Therefore $\operatorname{rg}\mathfrak{U} \geq \operatorname{rg}\mathfrak{I} + n(r-2)$.

Let s be the maximal number of elements $\xi_1, \ldots, \xi_n$ that are linearly independent over $\mathbb{Q}$. Then $\operatorname{rg}\mathfrak{I} = 2n - s$ and, hence,

$$\operatorname{rg}\mathfrak{U} \geqslant rn - s. \tag{27}$$

By [2], Chapter 7, §7, Theorem 9, the functions

$$e^{\xi_k z},\ K_0(\xi_j z),\ K_0'(\xi_j z),\ K_{\lambda_i}(\xi_j z),\ K_{\lambda_i}'(\xi_j z),\ K_{-\lambda_i}(\xi_j z),$$
$$1 \leqslant k \leqslant s, \qquad 3 \leqslant i \leqslant r, \qquad 1 \leqslant j \leqslant n,$$

are algebraically independent over $\mathbb{C}(z)$. Therefore,

$$\operatorname{rg} \mathfrak{P} \leqslant 2mn - s - 2n - 3(r-2)n = rn - s. \tag{28}$$

In view of the inclusion $\mathfrak{U} \subset \mathfrak{P}$, the inequalities (27) and (28) imply that $\mathfrak{U} = \mathfrak{P}$. This proves the first assertion.

The proof of the second assertion repeats verbatim the proof of the fact that $\mathfrak{U}$ is a prime ideal, after replacing the ring $\mathbb{V}$ by $\mathbb{V} = \mathbb{C}[x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}]$ and the polynomials $Q_{i,j}$ by $Q_{i,j}(\xi, x_{4i-3,j}, x_{4i-2,j}, x_{4i-1,j}, x_{4i,j})$. This proves Proposition 1.

We now turn to the proof of the implication 3) $\Rightarrow$ 1). The ideal $\mathfrak{U}^h$ in $\mathbb{C}[z, x_0, x_{1,1}, \dots, x_{2m,n}]$ generated by the polynomials that are homogeneous in $x_0, x_{i,j}$ and vanish when 1 is substituted for x_0, $e^{\xi_j z}$ is substituted for $x_{1,j}$, $e^{-\xi_j z}$ is substituted for $x_{2,j}$, $K_0(\xi_j z)$ is substituted for $x_{3,j}$, $K_0'(\xi_j z)$ is substituted for $x_{4,j}$, and the functions $K_{\lambda_i}(\xi_j z)$, $K_{\lambda_i}'(\xi_j z)$, $K_{-\lambda_i}(\xi_j z)$, and $K_{-\lambda_i}'(\xi_j z)$ are substituted for $x_{4i-7,j}$, $x_{4i-6,j}$, $x_{4i-5,j}$, and $x_{4i-4,j}$, respectively ($3 \leq i \leq r$, $1 \leq j \leq n$), is, by the first assertion of Proposition 1, the homogenization of $\mathfrak{U}$. For each $\xi \neq 0$ the ideal $(\mathfrak{U}^h)_\xi$ is the homogenization of $\mathfrak{U}_\xi$, which is prime by the first assertion of Proposition 1. Thus, the ideal $(\mathfrak{U}^h)_\xi$ is prime and the conditions of Theorem 1 hold for the collection of functions (24) for any $\xi \neq 0$, in particular for $\xi = 1$. The functions (24) are linearly independent over $\mathbb{C}(z)$. This follows from the fact that, as shown above, all homogeneous basis polynomials of $\mathfrak{U}^h$ have degree at least 2 with respect to $x_0, x_{i,j}$. Thus, for the collection of functions (24) and $\xi = 1$ all the conditions of Corollary 1 to Theorem 1 hold. By this Corollary, the values of the functions (24) at $z = 1$ are linearly independent over $\mathbb{A}$. As shown above, this suffices to complete the proof of Theorem 3.

Bibliography

[1] C. L. Siegel, "Über einige Anwendungen Diophantischer Approximationen", *Abh. Preuss. Akad. Wissenschaft. Phys.-Math. Kl.* **1** (1929–1930), 1–70.

[2] A. B. Shidlovskiĭ, *Transcendental numbers*, Nauka, Moscow 1987; English transl., de Gruyter, Berlin 1989.

[3] A. B. Shidlovskiĭ, "On the linear independence of the values of E-functions at algebraic points", *Mat. Zametki* **55**:2, (1994), 174–185; English transl. in *Math. Notes* **55** (1994).

[4] W. Krull, "Parameterspezialisierung in Polynomringen II", *Arch. Math.* **1**:2 (1948), 129–137.

[5] O. Zariski and P. Samuel, *Commutative algebra*, vols. 1–2, van Nostrand, New York 1958–1960.

[6] V. Kh. Salakhov, "A criterion for the algebraic independence of values of a class of hypergeometric E-functions", *Mat. Sb.* **181**:2 (1990), 189–211; English transl. in *Math. USSR-Sb.* **69** (1991).

Moscow State University

Received 12/JAN/96
Translated by R. A. M. HOKSBERGEN
Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX