ON AN INTEGRAL IDENTITY

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ABSTRACT. We give three elementary proofs of a nice equality of definite integrals, which arises from the theory of bivariate hypergeometric functions, and has connections with irrationality proofs in number theory. We furthermore provide a generalization together with an equally elementary proof and discuss some consequences.

Introduction

The following infinite family of equalities between definite integrals was proven by S. B. Ekhad, D. Zeilberger and W. Zudilin in [5], using the Almkvist–Zeilberger creative telescoping algorithm [2] for symbolic integration:

(1)
$$\int_0^1 \frac{x^n (1-x)^n}{((x+a)(x+b))^{n+1}} dx = \int_0^1 \frac{x^n (1-x)^n}{((a-b)x + (a+1)b)^{n+1}} dx,$$

for any reals a > b > 0 and any nonnegative integer n.

As pointed out in [5], these integrals "are not taken from a pool of no-one-cares analytic creatures": they are related to rational approximations to some logarithmic values [1] and a trained eye could recognize in identity (1) a particular case of a known relation for Appell's bivariate hypergeometric function (see §4 below).

The proof provided in [5] is, without any doubt, elementary. It requires some clever (and at first sight, magic) auxiliary rational functions coming from the silicon guts of the first author of [5]. Of course, this is not objectable at all, and we are ourselves convinced that computer-assisted proofs are an increasingly important trend in mathematics. At the same time we think that, pour l'honneur de l'esprit humain, it is of some interest to offer a proof that a freshperson could not only follow but also create.

In Sections 1–3 of this note we provide three elementary proofs of the identity (1). We also generalize the identity. In Section 4 we discuss some direct consequences, emphasizing the relation of the identities with known identities for hypergeometric functions. We also give a couple of combinatorical identities, and finally conclude that the Legendre polynomials are eigenfunctions of a certain differential operator.

1. First proof: using a rational change of variables

Note that the natural range to assure the convergence of the integrals is a, b > 0 and $n \in \mathbb{R}_{>-1}$. Let us demonstrate firstly that an utterly simple change of variables proves the equality in this extended range and in fact an even more general equality.

Proof of (1) for a, b > 0 and $n \in \mathbb{R}_{>-1}$. With the rational change of variables

$$x = \frac{b(1-u)}{b+u}$$
, one gets $dx = -\frac{b(1+b)}{(b+u)^2} du$.

Note that it is a promising change because it takes 0, 1, b(a+1)/(b-a) and ∞ , which are the singularities of the second integrand, into 1, 0, -a and -b, respectively, which are singularities of the first integrand. The interval [0, 1] is preserved. Thus,

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a direct calculation yields

$$\int_0^1 \frac{x^n (1-x)^n}{\left((x+a)(x+b)\right)^{n+1}} \, \mathrm{d}x = \int_0^1 \frac{\left(\frac{b(1-u)}{b+u}\right)^n \left(1 - \frac{b(1-u)}{b+u}\right)^n}{\left[\left(\frac{b(1-u)}{b+u} + a\right) \left(\frac{b(1-u)}{b+u} + b\right)\right]^{n+1}} \frac{b(1+b)}{(b+u)^2} \, \mathrm{d}u$$

$$= \int_0^1 \frac{(1-u)^n u^n b^{n+1} (1+b)^{n+1} \, \mathrm{d}u}{\left[\left(b(1-u) + a(b+u)\right) \left(b(1-u) + b(b+u)\right)\right]^{n+1}}$$

$$= \int_0^1 \frac{u^n (1-u)^n}{\left((a-b)u + (a+1)b\right)^{n+1}} \, \mathrm{d}u,$$

which proves (1).

Actually, the same change of variables also proves a generalization of (1):

Theorem 1.1. If $a, b > 0, k, n \in \mathbb{R}$ and $s, \ell \in \mathbb{R}_{>-1}$, then

(2)
$$\int_0^1 \frac{x^{\ell} (1-x)^s}{(x+a)^{k+1} (x+b)^{n+1}} dx = \frac{(b+1)^{s-n}}{b^{n-\ell}} \int_0^1 \frac{x^s (1-x)^{\ell} (x+b)^{n+k-\ell-s}}{((a-b)x + (a+1)b)^{k+1}} dx.$$

We will come back to and draw some conclusions from this more general identity in Section 4.

2. Second proof: using indefinite integration

We still consider the extended convergence range a, b > 0 but now n is a nonnegative integer. The following proof is based on the generating functions [10] of the sequence of integrals when n varies.

Proof of (1) for a, b > 0 and $n \in \mathbb{Z}_{\geq 0}$. Let I_1 and I_2 be the generating functions of each side of (1) i.e., multiplying by t^n and summing from n = 0 to ∞ . They clearly converge uniformly for small values of t and we have $I_j(t) = \int_0^1 dx/P_j(x,t)$ where

(3)
$$P_1(x,t) = (x+a)(x+b) - tx(1-x)$$
 and $P_2(x,t) = (a-b)x + (a+1)b - tx(1-x)$.

If $P(x) = Ax^2 + Bx + C$ has no zero in [0, 1], and $\Delta = B^2 - 4AC > 0$, then, using standard integration techniques,

$$\int_0^1 \frac{\mathrm{d}x}{P(x)} = \frac{1}{r} \log \left(\frac{B + 2C + r}{B + 2C - r} \right) \quad \text{with} \quad r = \sqrt{\Delta}.$$

Note that for t small enough P_1 and P_2 fulfill these conditions. A calculation shows that both polynomials have the same discriminant Δ and the same values of B+2C. Hence $I_1(t)=I_2(t)$ in some interval containing the origin and then the integrals in (1), which are their Taylor coefficients, are equal.

3. Third proof: Creative Telescoping

We cannot resist the temptation to offer a third proof, in the spirit of the one in [5], but based on a different kind of "creative telescoping". The starting point is the same as in Section 2, namely that the family of identities (1) is equivalent to the fact that the two integrals $I_1(t)$ and $I_2(t)$ between x=0 and x=1 of the rational functions $F_1=1/P_1$ and $F_2=1/P_2$, with P_j as in (3), are equal.

Creative telescoping (this time in the classical "differential-differential" setting) shows that F_1 and F_2 satisfy the equalities

$$(t-2ab-a-b)F_j + (t^2-2t(2ab+a+b)+(a-b)^2)\frac{\partial F_j}{\partial t} + \frac{\partial}{\partial x}(F_jR_j) = 0$$

where $R_1(t,x)$ and $R_2(t,x)$ are the rational functions

$$R_1(t,x) = ((a+b+t+2)x + 2ab + a + b - t)x$$

and

$$R_2(t,x) = \frac{\left((2ab+a+b)t - (a-b)^2 \right)x^2 + b(a+1)(2ab+a+b-t)}{t+b-a}.$$

Hence, by integration between x = 0 and x = 1, one obtains that both I_1 and I_2 are solutions of the differential equation

$$(t - 2ab - a - b)I(t) + (t^2 - 2t(2ab + a + b) + (a - b)^2)I'(t) + 2 = 0$$

Therefore $I_1 = I_2$ by Cauchy's theorem, since $I_1 - I_2$ is the solution of a differential equation of order 1 with leading term non-vanishing at t = 0, and its evaluation at t = 0 is zero, as $I_1(0) = I_2(0) = \frac{1}{a-b} \ln \left(\frac{a(b+1)}{(a+1)b} \right)$.

4. Some consequences

4.1. **Appell's identity.** We will show that identity (2) from our Theorem 1.1 contains, as a particular case, a classical hypergeometric identity due to Appell. The relevant definitions to state this identity are the classical Gauss hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;t)$ and the Appell bivariate hypergeometric function $F_1(\alpha;\beta,\beta';\gamma;x,y)$, given respectively by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{t^n}{n!} \quad \text{and} \quad \sum_{m,n>0} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!},$$

where $(a)_n$ denotes the rising factorial $a(a+1)\cdots(a+n-1)$ for $n\in\mathbb{N}$ and it is assumed |t|,|x|,|y|<1 to assure the convergence.

It is very classical that these hypergeometric functions admit the following integral representations, which hold as soon as $\beta, \beta' > 0$ and $\gamma := \beta + \beta' > \alpha > 0$:

(4)
$${}_{2}F_{1}\left(\alpha,\beta;\gamma;z\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')} \int_{0}^{1} \frac{t^{\beta-1}(1-t)^{\beta'-1}}{\left(1-tz\right)^{\alpha}} dt$$

and

(5)
$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \frac{t^{\alpha - 1} (1 - t)^{\gamma - \alpha - 1}}{(1 - tx)^{\beta} (1 - ty)^{\beta'}} dt.$$

Equation (4) is due to Euler [3, Th. 2.2.1], and (5) to Picard [7], see also [4, Eq. (9)]. With these notations, we are able to state the following hypergeometric function identity, which appears on page 8 of Appell's classical memoir [4]:

Corollary 4.1 (Appell's identity). If α , β , $\beta' > 0$, $\beta + \beta' > \alpha$, |x| < 1 and |y| < 1 then

(6)
$${}_{2}F_{1}\left(\alpha,\beta;\beta+\beta';\frac{y-x}{y-1}\right) = (1-y)^{\alpha}F_{1}(\alpha;\beta,\beta';\beta+\beta';x,y).$$

Equivalently, in terms of integrals:

(7)
$$\int_{0}^{1} \frac{t^{\beta-1} (1-t)^{\beta'-1}}{((y-x)t+1-y)^{\alpha}} dt = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\beta+\beta'-\alpha)} \int_{0}^{1} \frac{t^{\alpha-1} (1-t)^{\beta+\beta'-\alpha-1}}{(1-tx)^{\beta} (1-ty)^{\beta'}} dt.$$

Remark 4.2. Identity (1) is a particular case of (7), with $\alpha = \beta = \beta' = n + 1$.

Proof of Corollary 4.1. If we set $n = \ell + s - k$ in (2), then the identity becomes (8)

$$\int_0^1 \frac{x^{\ell} (1-x)^s}{(x+a)^{k+1} (x+b)^{\ell+s-k+1}} \, \mathrm{d}x = \frac{(b+1)^{k-\ell}}{b^{s-k}} \int_0^1 \frac{x^s (1-x)^{\ell}}{((a-b)x + (a+1)b)^{k+1}} \, \mathrm{d}x.$$

Next we replace the integration variable x with 1-t in the first integral and by t in the second integral, then a with -1/x and b with -1/y, to deduce the following equivalent form for x, y < 0:

$$\int_0^1 \frac{t^\ell (1-t)^s}{\left(1 - \frac{y-x}{y-1}t\right)^{k+1}} \, \mathrm{d}t = (1-y)^{\ell+1} \int_0^1 \frac{t^\ell (1-t)^s}{(1-tx)^{k+1} (1-ty)^{\ell+s-k+1}} \, \mathrm{d}t,$$

that can be analytically continued to the values of x and y as in the statement. Inserting $\ell = \alpha - 1$, $k = \beta - 1$ and $s = \beta + \beta' - \alpha - 1$, using the integral representations (4) and (5) and the obvious symmetry ${}_2F_1(\alpha, \beta; \gamma; t) = {}_2F_1(\beta, \alpha; \gamma; t)$, we deduce (6).

The close relationship between (8) and the univariate and bivariate hypergeometric functions is shown in the previous proof of Corollary 4.1. Taking into account that (7) is a formulation of this result not involving any hypergeometric function, it appears as a natural problem to provide a more direct proof of (7). We present an independent proof involving basic real and complex analysis.

Alternative proof of (7). We write $\gamma = \beta + \beta'$ as before. Changing $t \mapsto 1 - t$ in the first integral of (7) and multiplying the identity by $\Gamma(\alpha)\Gamma(\gamma - \alpha)$, we want to prove that the functions

$$G_1(x,y) = \Gamma(\alpha)\Gamma(\gamma - \alpha) \int_0^1 \frac{t^{\gamma - \beta - 1}(1 - t)^{\beta - 1}}{(1 - ty - (1 - t)x)^{\alpha}} dt$$

and

$$G_2(x,y) = \Gamma(\beta)\Gamma(\gamma - \beta) \int_0^1 \frac{t^{\alpha - 1}(1 - t)^{\gamma - \alpha - 1}}{(1 - tx)^{\beta} (1 - ty)^{\gamma - \beta}} dt$$

coincide. Both are analytic functions of x and y on Re(x), Re(y) < 1/2 (for instance by Morera's theorem applied separately in x and y). Then it is enough to prove

$$\frac{\partial^{n+m} G_1}{\partial x^n \partial y^m}(0,0) = \frac{\partial^{n+m} G_2}{\partial x^n \partial y^m}(0,0) \quad \text{for every } m,n \in \mathbb{Z}_{\geq 0},$$

because in this case their Taylor coefficients coincide. Noting that the k-th derivative of $(1-x)^{-\delta}$ is $\frac{\Gamma(\delta+k)}{\Gamma(\delta)}(1-x)^{-\delta-k}$, this is the same as

$$\Gamma(\gamma - \alpha)\Gamma(\alpha + m + n)B(\gamma + m - \beta, \beta + n) = \Gamma(\beta + n)\Gamma(\gamma + m - \beta)B(\alpha + m + n, \gamma - \alpha)$$

with $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$. Using the well-known elementary evaluation $B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$, the proof is complete.

4.2. Combinatorial identities and Legendre polynomials. The equality (1) and its generalizations can be interpreted in combinatorial terms. If we in (2) set a = b > 0, $\ell = s = n$ and replace k by k - 1 then we get

(9)
$$\int_0^1 \frac{x^n (1-x)^n}{(x+b)^{n+k+1}} \, \mathrm{d}x = \frac{1}{b^k (b+1)^k} \int_0^1 \frac{x^n (1-x)^n}{(x+b)^{n-k+1}} \, \mathrm{d}x.$$

Corollary 4.3. The functions $f_n(x) = x^n(1+x)^n$ with n a nonnegative integer satisfy

$$f_k f_n^{(n+k)} = \frac{(n+k)!}{(n-k)!} f_n^{(n-k)}$$
 for $0 \le k \le n$.

Proof. By Taylor expansion at x = -b we have

$$x^{n}(1-x)^{n} = (-1)^{n} f_{n}(-x) = \sum_{m=0}^{2n} c_{m}(x+b)^{m}$$
 with $c_{m} = (-1)^{n+m} \frac{f_{n}^{(m)}(b)}{m!}$.

If we substitute this in (9) a term with $\log(b+1) - \log b$ appears in the LHS for m = n + k (the rest are rationals) and in the integral of the RHS for m = n - k. Hence $c_{n+k} = c_{n-k}/(b(b+1))^k$.

Essentially comparing coefficients one gets a triple binomial identity:

Corollary 4.4. For each k, ℓ, n nonnegative integers with $\ell, k \le n$

$$\sum_{m=0}^{k} \binom{k}{m} \binom{n}{m+\ell} \binom{2m+2\ell}{k+n} = \binom{n}{\ell} \binom{2\ell}{n-k}.$$

Remark 4.5. This hypergeometric identity can be rewritten as

$$_{3}F_{2}\left(-k,\ell-n,\ell+\frac{1}{2};\ell-\frac{k+n}{2}+1,\frac{1-n-k}{2}+\ell;1\right) = \frac{\binom{2\ell}{n-k}}{\binom{2\ell}{n+k}},$$

so it is a particular case of the Pfaff–Saalschütz identity [8, §2.3.1] for the evaluation at 1 of well-poised $_3F_2$'s. Alternatively, it can be automatically obtained using Zeilberger's creative telescoping algorithm [6].

Proof. In Corollary 4.3 change the variable $x \mapsto (x-1)/2$ to obtain, clearing denominators,

(10)
$$(n-k)!(x^2-1)^k \frac{\mathrm{d}^{n+k}}{\mathrm{d}x^{n+k}} (x^2-1)^n = (n+k)! \frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}} (x^2-1)^n.$$

We rewrite the left-hand side as

$$(n-k)! \sum_{\ell_1} (-1)^{k-\ell_1} \binom{k}{k-\ell_1} x^{2\ell_1} \cdot (n+k)! \sum_{\ell_2} (-1)^{n-\ell_2} \binom{n}{\ell_2} \binom{2\ell_2}{n+k} x^{2\ell_2-n-k},$$

and the right-hand side as

$$(n+k)!(n-k)! \sum_{\ell} (-1)^{n-\ell} \binom{n}{\ell} \binom{2\ell}{n-k} x^{2\ell-n+k}.$$

Comparing coefficients and renaming $\ell_1 = k - m$, $\ell_2 = m + \ell$ yields the result. \square

Remark 4.6. It turns out that (10) is a known identity expressing a symmetry of the associated Legendre polynomials. This can be obtained from Rodrigues' formula and from the general Legendre equation, see [9] for a simple elementary proof. The one obtained here (Corollary 4.3) is competitively simple.

The definition of the Legendre polynomials assures that they are eigenfunctions of the differential operator $\frac{d}{dx}(x^2-1)\frac{d}{dx}$. Although $\frac{d}{dx}$ and (x^2-1) do not commute and apparently there is not a simple formula for the commutator of their powers, Legendre polynomials are also eigenfunctions of a simple operator composed by powers of these operators.

Corollary 4.7. Let P_n be the n-th Legendre polynomial. Then for $0 \le k \le n$

$$L[P_n] = \frac{(n+k)!}{(n-k)!} P_n \qquad \text{where} \quad L = \frac{\mathrm{d}^k}{\mathrm{d}x^k} (x^2 - 1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}.$$

Proof. Recalling that P_n is proportional to $\frac{d^n}{dx^n}(x^2-1)^n$ (Rodrigues' formula) this follows taking the k-derivative of (10).

Actually, one could prove Corollary 4.3 from Corollary 4.7 by repeated integration, and noting that both sides in Corollary 4.3 are divisible by x^k .

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