VARIATIONS ON A THEME OF POLYA

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THE THEME

For $n > 0$, we define the q-factorial f_n by $f_n(q) = \prod_{i=1}^{n} (q^i - 1)$, and we set $f_0(q) = 1$. Thus f_n is a polynomial of degree $\frac{1}{2}n(n+1)$. For integers k, n with $0 \le k \le n$, we define the q-binomial $b_{n,k}$ by $b_{n,k} = f_n/(f_k f_{n-k})$. If $k < 0$ or $k > n$, we set $b_{n,k} = 0$. A standard notation for $b_{n,k}(q)$ is $\begin{bmatrix} n \\ k \end{bmatrix}$. The notational analogy with the ordinary binomial coefficient is strengthened by the fact that when q is a prime power, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of subspaces of dimension k in an n-dimensional vector space over the field with q elements. We extend the domain of $b_{n,k}$ to $q = 1$, by continuity. Some well-known properties of the q-factorials and q-binomials are listed here for reference.

$$b_{n,k}(1) = \binom{n}{k} \tag{1}$$

$$f_n(q) = q^{(1/2)n(n+1)} f_n(1/q) \tag{2}$$

$$b_{n,k}(q) = q^{k(n-k)} b_{n,k}(1/q) \tag{3}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{6}$$

$$f_{n+1}(q) = (q^{n+1} - 1) f_n(q) \tag{7}$$

From (**5**) or (**6**) it is easy to see that $b_{n,k}$ is a polynomial of degree $k(n-k)$ with positive-integer coefficients.

Let $\begin{bmatrix} k+n \\ k \end{bmatrix} = \sum_{j=0}^{kn} A_{k,n,j} q^j$. From (**1**) it follows that,

$$\sum_{j=0}^{kn} A_{k,n,j} = \binom{k+n}{k} \tag{8}$$

The right-hand side of (**8**) is the number of paths (joining $(0, 0)$ to (k, n)—see second section for definition). One such path is illustrated in FIGURE 1, with $k = 8$, $n = 6$. With each path to (k, n), we associate an area (under the path). In FIGURE 1, the area is shaded.

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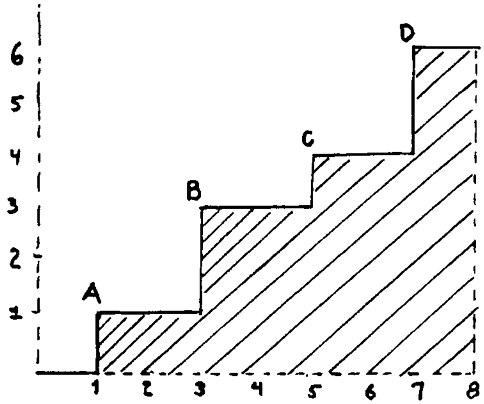

FIGURE 1.

Polya [1] gives combinatorial significance to the coefficients $A_{k,n,j}$ by showing that $A_{k,n,j}$ is the number of paths to (k, n) of area j. In FIGURE 1, the area of the illustrated path is 22 (square units). Polya's result can be restated in the language of partitions as: $A_{k,n,j}$ is the number of partitions of j into at most k parts, with each part at most n [2].

PATHS

For $k \geq 0$, $n \geq 0$, $k + n > 0$, by a path to (k, n) we mean a connected union of $k + n$ unit segments satisfying:

(i) Each segment has endpoints with integer coordinates.
(ii) One segment has $(0, 0)$ as an endpoint.
(iii) One segment has (k, n) as an endpoint.

The point (x, y) is a northwest corner of a path to (k, n) if and only if (x, y), $(x, y - 1)$, and $(x + 1, y)$ are all points of the path. In the figure the northwest corners are labeled A, B, C, and D. A path to (k, n) is determined by its set of northwest corners. If the set of northwest corners is $\{(x_i, y_i) | i = 1, 2, \ldots, p\}$ the labeling can be uniquely chosen so that the sequences (in this paper all sequences are finite) $\langle x_i \rangle$ and $\langle y_i \rangle$ are strictly increasing and satisfy,

$$0 \leq x_i < k, \qquad 0 < y_i \leq n \tag{9}$$

for all i. Conversely each pair of finite sequences of integers, which satisfy (9), correspond to the set of northwest corners of a path to (k, n). In what follows, we denote the path in question by $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, n)$—commas are omitted when only single-digit numerals are used. Thus the path in the figure is $1357 \text{---} 1346 - (8, 6)$. By the weight of a point (x, y) we mean $x + y$; by the weight of a path we mean the sum of the weights of its northwest corners. By a *subdiagonal path* we mean a path each of whose points (x, y) satisfies $x \geq y$. A necessary and sufficient condition that a path $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, n)$ be subdiagonal is that $x_i \geq y_i$, for all i and $k \geq n$.

THE FIRST VARIATION

THEOREM 1. The number of paths to (k, n) with weight j, is $A_{k,n,j}$.

Proof: Let $B_{k,n,j}$ = the *number* of paths to (k, n) of weight j. Let $X_{k,n,j}$ be the *set* of paths to (k, n) of weight j. There is an injective map $\alpha: X_{k,n-1,j} \to X_{k,n,j}$ given by $x_1, \ldots, x_p --- y_1, \ldots, y_p - (k, n-1) \mapsto x_1, \ldots, x_p --- y_1, \ldots, y_p - (k, n)$.

There is also an injective map $\beta: X_{k-1,n,j-n} \to X_{k,n,j}$ given by $x_1, \ldots, x_p --- y_1, \ldots, y_p - (k-1, n) \mapsto$

$$\begin{cases} x_1 + 1, x_2 + 1, \ldots, x_p + 1 --- y_2 - 1, \ldots, y_p - 1, n - (k, n), \text{ if } y_1 = 1 \\ 0, x_1 + 1, \ldots, x_p + 1 --- y_1 - 1, \ldots, y_p - 1, n - (k, n), \text{ if } y_1 \neq 1 \end{cases}$$

Further, $\alpha(X_{k,n-1,j}) \cup \beta(X_{k-1,n,j-n}) = X_{k,n,j}$ and $\alpha(X_{k,n-1,j}) \cap \beta(X_{k-1,n,j-n}) = \phi$; hence,

$$B_{k,n,j} = B_{k,n-1,j} + B_{k-1,n,j-n} \tag{10}$$

Next it is easy to verify that,

$$B_{0,n,0} = B_{k,0,0} = 1 \tag{11}$$

and for $j > 0$ or $j < 0$,

$$B_{0,n,j} = B_{k,0,j} = 0 \tag{12}$$

The counterpart of **(10)**,

$$A_{k,n,j} = A_{k,n-1,j} + A_{k-1,n,j-n} \tag{13}$$

follows from **(6)**, while,

$$A_{0,n,0} = A_{k,0,0} = 1 \tag{14}$$

and for $j \neq 0$,

$$A_{0,n,j} = A_{k,0,j} = 0 \tag{15}$$

follows immediately from the definition of $A_{k,n,j}$. Hence, $A_{k,n,j} = B_{k,n,j}$. □

q-CATALAN NUMBERS

The *Catalan numbers* C_k are a family of integers whose ubiquity and interest rival those of the binomial coefficients. There are a number of ways of expressing C_k in terms of binomial coefficients; we list three here:

$$C_k = \binom{2k}{k} \Big/ \binom{k+1}{1} \tag{16}$$

$$C_k = \binom{2k+1}{k} \Big/ \binom{2k+1}{1} \tag{17}$$

$$C_k = \binom{2k}{k} - \binom{2k}{k+1} \tag{18}$$

We use **(16)** to motivate the definition of q-Catalan numbers c_k given by,

$$c_k = \begin{bmatrix} 2k \\ k \end{bmatrix} \Big/ \begin{bmatrix} k+1 \\ 1 \end{bmatrix} \tag{19}$$

It can be shown that $c_k = \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \Big/ \begin{bmatrix} 2k+1 \\ 1 \end{bmatrix}$, and that, if $d_k = \begin{bmatrix} 2k \\ k \end{bmatrix} - \begin{bmatrix} 2k \\ k+1 \end{bmatrix}$, then,

$$d_k = q^k c_k \tag{20}$$

Clearly d_k is a polynomial of degree k^2. From the partition version of Polya's theorem, we obtain $A_{k,k,j} = A_{k+1,k-1,j}$, for $j < k$. Hence, by **(20)**, c_k is a polynomial of degree $k^2 - k$; we write,

$$c_k = \sum_{j=0}^{k^2-k} C_{k,j} q^j \tag{21}$$

It follows that,

$$\sum_{j=0}^{k^2-k} C_{k,j} = C_k \tag{22}$$

The Second Variation

Let $p < k$. A sequence $a_1, \ldots, a_{2p}$ satisfying:

(i) $1 \le a_i \le k$, for $i \le 2p$
(ii) $a_i < a_{i+2}$, for $i \le 2p - 2$
(iii) $a_{2i-1} < a_{2i}$, for at least one i

is called the *core* of a path $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_p, k - (k, k)$ [resp. $0, u_1, \ldots, u_{p-1}, k \text{---} v_1, \ldots, v_{p+1} - (k+1, k-1)$], if $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ (resp. $u_1, \ldots, u_{p-1}$ and $v_1, \ldots, v_{p+1}$) are complementary subsequences of $a_1, \ldots, a_{2p}$.

As an illustration, let $p = 4, k = 8$; then 1, 2, 2, 3, 4, 4, 5, 7 is the core of each of the following paths.

01234---24578 – (8, 8)	01248---23457 – (9, 7)
01245---23478 – (8, 8)	02348---12457 – (9, 7)
01247---23458 – (8, 8)	02458---12347 – (9, 7)
02345---12478 – (8, 8)	02478---12345 – (9, 7)
02457---12348 – (8, 8)*	
02347---12458 – (8, 8)*	

Lemma 1. Let $e_1, \ldots, e_{p+1}$ and $g_1, \ldots, g_{p+1}$ be strictly increasing sequences (in any linearly ordered set). Let w and $a_1, \ldots, a_p$ be complementary subsequences of $e_1, \ldots, e_{p+1}$ and let w and $b_1, \ldots, b_p$ be complementary subsequences of $g_1, \ldots, g_{p+1}$. Then $g_i \ge e_i$, for all i, implies $b_i \ge a_i$, for all i.

Proof:

Case 1. $w = e_s = g_s$; then,

$$b_m - a_m = \begin{cases} g_m - e_m, & \text{if } m < s \\ g_{m+1} - e_{m+1}, & \text{if } m \ge s \end{cases}$$

* The nonadmissible paths (see definition of "admissible" just before Lemma 2).

Case 2. $e_s = w = g_t$, $s > t$; then,

$$b_m - a_m = \begin{cases} g_m - e_m, & \text{if } m < t \\ g_{m+1} - e_m, & \text{if } t \le m < s \\ g_{m+1} - e_{m+1}, & \text{if } s < m \end{cases}$$

Case 3. $e_s = w = g_t$, $s < t$. This does not arise since,

$$g_t > g_s \geq e_s. \quad \square$$

We call a path $0, x_1, \ldots, x_2 \text{---} y_1, \ldots, y_p, k - (k, k)$ *admissible*, if the path $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k)$ is *not* subdiagonal.

LEMMA 2. Given Γ: $a_1, \ldots, a_{2p}$, $p < k$, $j = \sum_{i=1}^{2p} a_i$. The number of paths to $(k+1, k-1)$ of weight $j + k$ and core Γ equals the number of admissible paths to (k, k) with weight $j + k$ and core Γ.

Proof:

Case 1. The a_i are all distinct. Without loss of generality, we may assume that $a_i = i$. The number of paths to (k, k) (admissible or not) is $\binom{2p}{p}$. Of these the number of nonadmissible paths is C_p. Therefore, the number of admissible paths is $\binom{2p}{p} - C_p$. The number of paths to $(k+1, k-1)$ is $\binom{2p}{p+1}$. The result follows from (**18**).

The remark about nonadmissible paths follows readily from the correspondence $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k) \mapsto G$, where G is the linear array with $2p + 1$ entries, p of them P and $p + 1$ of them z, such that the occurrences of P are in positions $y_1, \ldots, y_p$. An example is $346 \text{---} 124 - (8, 8) \mapsto PPzPzzz$. Each such G is the Polish notation for a formal product with $p + 1$ factors in a free nonassociative binary system with a single generator z. Further, all such formal products arise. It is well-known that the number of such formal products is C_p.

Case 2. There is repetition among the a_i. Then the number of a_i whose value occurs only once must be even, say $2r$, with $1 \le r < p$. In this case, if $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k)$ has core Γ [resp. $0, u_1, \ldots, u_{p-1}, k \text{---} v_1, \ldots, v_{p+1} - (k+1, k-1)$ has core Γ], those integers which occur twice among the a_i (each value of an a_i occurs either once or twice) must occur among both the x_i and y_i (resp. u_i and v_i). The argument of *Case* 1 applies with p replaced by r. The question of admissibility is the only delicate part of the argument and is handled by LEMMA 1. $\square$

THEOREM 2. The number of subdiagonal paths to (k, k) of weight j is $C_{k,j}$.

Proof: Let $E_{k,j}$ be the number of subdiagonal paths to (k, k) of weight j. From (**20**) it follows that,

$$C_{k,j} = A_{k,k,j+k} - A_{k+1,k-1,j+k} \tag{23}$$

It suffices to prove that,

$$E_{k,j} = A_{k,k,j+k} - A_{k+1,k-1,j+k} \tag{24}$$

Let $Y_{k,j}$ be the set of subdiagonal paths to (k, k) of weight j: $X_{k,k,j+k}$ is the disjoint union of four sets of paths:

T_1: paths of the form $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_p, k - (k, k)$, which are not admissible.
T_2: paths of the form $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_p, k - (k, k)$, which are admissible.
T_3: paths of the form $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k)$, with $x_1 > 0$.
T_4: paths of the form $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_{p+1} - (k, k)$, with $y_{p+1} < k$.

and $X_{k+1,k-1,j+k}$ is the disjoint union of three sets of paths.

S_2: paths of the form $0, x_1, \ldots, x_{p-1}, k \text{---} y_1, \ldots, y_{p+1} - (k + 1, k - 1)$.
S_3: paths of the form $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k + 1, k - 1)$, with $x_1 > 0$.
S_4: paths of the form $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_{p+1} - (k + 1, k - 1)$, with $x_p < k$.

The remainder of the proof consists in showing that there are bijective maps:

$$\alpha\colon T_1 \to Y_{k,j}$$
$$\beta\colon T_2 \to S_2$$
$$\gamma\colon T_3 \to S_3$$
$$\delta\colon T_4 \to S_4$$

The map α is defined by $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_p, k - (k, k) \mapsto x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k)$.
The map γ is defined by $x_1, \ldots, x_p \text{---} y_1, \ldots, y_p - (k, k) \mapsto y_1, \ldots, y_p \text{---} x_1, \ldots, x_p - (k + 1, k - 1)$.
The map δ is defined by $0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_{p+1} - (k, k) \mapsto 0, x_1, \ldots, x_p \text{---} y_1, \ldots, y_{p+1} - (k + 1, k - 1)$.

We do not exhibit the map β, but its existence is guaranteed by LEMMA 2. The existence of the maps α, β, γ, and δ implies (**24**); thus the theorem is established. □

We note that an analog of THEOREM 2 with weight replaced by area is not valid. It suffices to examine the case for $j = 1$.

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