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Multinomial convolution polynomials

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Abstract

In [9] Knuth shows how to derive the convolution formulas of Hagen, Rothe and Abel from Vandermonde's convolution and the binomial theorem for integer exponents. In the present paper, we shall first present a short and elementary proof of the multi-extension of the above convolution formulas, due to Raney and Mohanty. In the second part we shall present a multi-version of Knuth's approach to convolution polynomials and derive another short proof of the above formulas.

1. Introduction

Recall that a family of polynomials $(F_n(x))$ $(n \geqslant 0)$ is said to be of convolution type if $F_n(x)$ has degree $\leqslant n$ and satisfies the convolution condition

$$F_n(x+y) = \sum_{k=0}^{n} F_k(x) F_{n-k}(y).$$

The two typical families of convolution polynomials are the binomial theorem for integers

$$\frac{(x+y)^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$$

and Vandermonde's convolution

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.$$

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Some important extensions of the above identities have been given by Abel, Rothe and Hagen in the last century. More precisely, for $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$ let

$$A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n}.$$

In 1891, Hagen [7,5,6] proved the following identity:

$$\sum_{k=0}^{n}(p+qk)A_k(a,b)A_{n-k}(c,b) = \frac{p(a+c)+aqn}{a+c}A_n(a+c,b).$$

In the case $q=0$, this formula reduces to Rothe's convolution formula [17,5,16] dated back in 1793:

$$\sum_{k=0}^{n} A_k(a,b)A_{n-k}(c,b) = A_n(a+c,b).$$

Note that Hagen's formula implies also (see the examples at the end of this paper for the multinomial case) Abel's [1] identity:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k},$$

where $x, y, z \in \mathbb{C}$ and $n \in \mathbb{N}$.

In a *refreshing* paper about *convolution polynomials* Knuth [9] shows how to derive these seemingly non-trivial formulas from the basic binomial theorem for integers and Vandermonde's convolution.

Since the multi-extensions of the above identities are also known, it is natural to give a multi-extension of Knuth's results. To express them more concisely, we shall use the vector notation. Throughout this paper m will be a fixed natural number. For $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $\boldsymbol{b} = (b_1, \ldots, b_m) \in \mathbb{N}^m$ set $|\boldsymbol{a}| = \sum_{i=1}^m a_i$, $\boldsymbol{a}! = a_1! \cdots a_m!$, $\boldsymbol{a}+\boldsymbol{b} = (a_1+b_1, \ldots, a_m+b_m)$ and $\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^m a_ib_i$. We order the elements of $\mathbb{N}^m$ in *lexicographic order*, i.e., $\boldsymbol{a} < \boldsymbol{b}$ if $a_1 < b_1$ or there is a $i > 1$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ but $a_i < b_i$. Also for $1 \leqslant i \leqslant m$ let $\boldsymbol{e}_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$ where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Finally, for each complex number x and $\boldsymbol{n} \in \mathbb{Z}^m$ we define

$$\binom{x}{\boldsymbol{n}} = \begin{cases} \dfrac{x(x-1)\cdots(x-|\boldsymbol{n}|+1)}{\boldsymbol{n}!} & \text{if } \boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m, \\ 0 & \text{otherwise.} \end{cases}$$

Next define the multi-analogue of $A_n(a,b)$ by

$$A_{\boldsymbol{n}}(x,\boldsymbol{b}) = \frac{x}{x+\boldsymbol{b}\cdot\boldsymbol{n}}\binom{x+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}}.$$

The multi-extension of Rothe's convolution formula can then be stated as follows:

$$\sum_{\boldsymbol{k}} A_{\boldsymbol{k}}(a,\boldsymbol{b})A_{\boldsymbol{n}-\boldsymbol{k}}(c,\boldsymbol{b}) = A_{\boldsymbol{n}}(a+c,\boldsymbol{b}). \tag{1.1}$$

Remark. The author first learned this identity in July 1988 at Oberwolfach, where Louck [11] communicated it as a conjecture. It was shortly proved independently by Strehl, Paule and the author by using different methods and reported at the 20th session of le séminaire Lotharingien de Combinatoire held at Alghero, Sardegna, in September 1988. Shortly after, I noted that formula (1.1) was already established by Raney [15, Theorems 2.2 and 2.3] in 1960. We refer the reader to the recent paper by Strehl [18] for a comprehensive account of the various aspects of this formula. Curiously enough, Raney's derivation of (1.1) was entirely different from the three forementioned ones. In fact, Strehl's proof is of combinatorial nature [18], Paule's proof is based on the one variable Lagrange inversion formula [14] and the author's proof is inductive and will be reproduced below. Note that Chu [2] quoted Raney's identity explicitly in his paper to derive a combinatorial interpretation of the generalized Catalan numbers.

A multi-extension of Hagen's identity has also been given by Mohanty [12], by applying the *multivariable Lagrange inversion formula*, in the following form:

$$\sum_{\boldsymbol{k}} (p + \boldsymbol{q} \cdot \boldsymbol{k}) A_{\boldsymbol{k}}(a, \boldsymbol{b}) A_{\boldsymbol{n}-\boldsymbol{k}}(c, \boldsymbol{b}) = \frac{p(a+c) + a(\boldsymbol{q} \cdot \boldsymbol{n})}{a+c} A_{\boldsymbol{n}}(a+c, \boldsymbol{b}), \tag{1.2}$$

where $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{C}^m$ and $a, c, p \in \mathbb{C}$.

The aim of this paper is twofold. First we present a very short and elementary proof of (1.1) and (1.2) from scratch, which is similar in spirit to Good's short proof of the Dyson conjecture [4]. In the second part we shall present a natural multi-extension of Knuth's approach [9] to ordinary convolution polynomials. As a consequence of this generalization we will see that (1.2) and (1.1) are, respectively, equivalent to the following trivial convolution identities:

$$\frac{(x+y)^{|\boldsymbol{n}|}}{\boldsymbol{n}!} = \sum_{0 \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} \frac{x^{|\boldsymbol{k}|}}{\boldsymbol{k}!} \frac{y^{|\boldsymbol{n}-\boldsymbol{k}|}}{(\boldsymbol{n}-\boldsymbol{k})!} \tag{1.3}$$

and

$$\binom{x+y}{\boldsymbol{n}} = \sum_{0 \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} \binom{x}{\boldsymbol{k}} \binom{y}{\boldsymbol{n}-\boldsymbol{k}}. \tag{1.4}$$

As noticed by Knuth [9], in the special case when the nth convolutional $F_n(x)$ is of degree n, the sequence $(n!F_n(x))$ is said to be of *binomial type* [13]. We note that in the latter case, Joni [10] has given a multivariable generalization of the binomial polynomials.

2. A short proof of formula (1.2)

Notice that formula (1.2) is equivalent to (1.1) and to the following identity:

$$\sum_{\boldsymbol{k}} (\boldsymbol{q} \cdot \boldsymbol{k}) A_{\boldsymbol{k}}(a, \boldsymbol{b}) A_{\boldsymbol{n}-\boldsymbol{k}}(c, \boldsymbol{b}) = \frac{a(\boldsymbol{q} \cdot \boldsymbol{n})}{a+c} A_{\boldsymbol{n}}(a+c, \boldsymbol{b}). \tag{2.1}$$

A further look at (2.1) shows that formula (1.1) is actually a consequence of (2.1): exchanging a with c, and $\boldsymbol{k}$ with $\boldsymbol{n}-\boldsymbol{k}$ in (2.1) yields

$$\sum_{\boldsymbol{k}} (\boldsymbol{q}\cdot(\boldsymbol{n}-\boldsymbol{k}))A_{\boldsymbol{k}}(a,\boldsymbol{b})A_{\boldsymbol{n}-\boldsymbol{k}}(c,\boldsymbol{b}) = \frac{c(\boldsymbol{q}\cdot\boldsymbol{n})}{a+c}A_{\boldsymbol{n}}(a+c,\boldsymbol{b}). \tag{2.2}$$

By summing up the two above identities side by side and taking $\boldsymbol{q}=\boldsymbol{n}$ we get (1.1).

Therefore, it suffices to prove (2.1). However, it will be more interesting at this stage to first give an independent proof of (1.1) to illustrate our method. In the sequel, we assume that a belongs to $\mathbb{N}$. Let

$$S_{\boldsymbol{n}}(a,c,\boldsymbol{b}) = \sum_{\boldsymbol{k}} A_{\boldsymbol{k}}(a,\boldsymbol{b})A_{\boldsymbol{n}-\boldsymbol{k}}(c,\boldsymbol{b}).$$

First

$$S_{\boldsymbol{n}}(0,c,\boldsymbol{b}) = A_{\boldsymbol{n}}(c,\boldsymbol{b}), \qquad S_0(a,c,\boldsymbol{b}) = 1. \tag{2.3}$$

Next, by the definition of $A_n(a,\boldsymbol{b})$ we have

$$\begin{aligned} A_{\boldsymbol{n}}(a,\boldsymbol{b}) &= \frac{1}{a-1+\boldsymbol{b}\cdot\boldsymbol{n}}\left(a-1+\sum_{i=1}^{m}\frac{b_i n_i}{a+\boldsymbol{b}\cdot\boldsymbol{n}}\right)\binom{a+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}} \qquad (2.4)\\ &= \frac{1}{a-1+\boldsymbol{b}\cdot\boldsymbol{n}}\left[(a-1)\binom{a+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}}+\sum_{i=1}^{m} b_i\binom{a-1+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_i}\right]\\ &= A_{\boldsymbol{n}}(a-1,\boldsymbol{b})+\sum_{i=1}^{m} A_{\boldsymbol{n}-\boldsymbol{e}_i}(a+b_i-1,\boldsymbol{b}). \end{aligned}$$

The last line is due to

$$\binom{x}{\boldsymbol{n}} = \binom{x-1}{\boldsymbol{n}}+\sum_{i=1}^{m}\binom{x-1}{\boldsymbol{n}-\boldsymbol{e}_i},$$

so that

$$\begin{aligned} S_{\boldsymbol{n}}(a,c,\boldsymbol{b}) &= \sum_{\boldsymbol{k}}\left(A_{\boldsymbol{k}}(a-1,\boldsymbol{b})+\sum_{i=1}^{m} A_{\boldsymbol{k}-\boldsymbol{e}_i}(a+b_i-1,\boldsymbol{b})\right)A_{\boldsymbol{n}-\boldsymbol{k}}(c,\boldsymbol{b})\\ &= S_{\boldsymbol{n}}(a-1,c,\boldsymbol{b})+\sum_{i=1}^{m} S_{\boldsymbol{n}-\boldsymbol{e}_i}(a+b_i-1,c,\boldsymbol{b}). \qquad (2.5) \end{aligned}$$

Eqs. (2.3)–(2.5) show that $S_{\boldsymbol{n}}(a,c,\boldsymbol{b})$ and $A_{\boldsymbol{n}}(a,c,\boldsymbol{b})$ satisfy the same recurrence and the initial conditions. Therefore, they are equal.

Now we turn to (2.1). Note that (2.1) is equivalent to the following m identities:

$$\sum_{\boldsymbol{k}} k_j A_{\boldsymbol{k}}(a,\boldsymbol{b})A_{\boldsymbol{n}-\boldsymbol{k}}(c,\boldsymbol{b}) = \frac{a n_j}{a+c}A_{\boldsymbol{n}}(a+c,\boldsymbol{b}), \quad (1\leqslant j\leqslant m). \tag{2.6}$$

For $1 \leqslant j \leqslant m$, let $T_{\boldsymbol{n}}^{(j)}(a,c,\boldsymbol{b})$ and $S_{\boldsymbol{n}}^{(j)}(a,c,\boldsymbol{b})$ be, respectively, the left-hand and right-hand sides of (2.6). By definition, we first have

$$S_{\boldsymbol{n}}^{(j)}(a,c,\boldsymbol{b}) = (a-1)\binom{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j} + \binom{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j}. \tag{2.7}$$

Now write the first $\binom{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j}$ as

$$\binom{a-2+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j} + \sum_{i=1}^{m}\binom{a-2+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_i}$$

and the second one as

$$\left(\frac{\boldsymbol{b}\cdot(\boldsymbol{n}-\boldsymbol{e}_j)}{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}} + \frac{a-1+c+b_j}{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}}\right)\binom{a-1+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j}$$

$$= \sum_{i=1}^{m} b_i \binom{a-2+c+\boldsymbol{b}\cdot\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{e}_j-\boldsymbol{e}_i} + A_{\boldsymbol{n}-\boldsymbol{e}_j}(a-1+c+b_j,\boldsymbol{b}).$$

We see that

$$\begin{aligned} S_{\boldsymbol{n}}^{(j)}(a,c,\boldsymbol{b}) &= S_{\boldsymbol{n}}^{(j)}(a-1,c,\boldsymbol{b}) + \sum_{i=1}^{m} S_{\boldsymbol{n}-\boldsymbol{e}_i}^{(j)}(a-1+b_i,c,\boldsymbol{b}) \\ &\quad + A_{\boldsymbol{n}-\boldsymbol{e}_j}(a-1+c+b_j,\boldsymbol{b}). \end{aligned} \tag{2.8}$$

Also we have the boundary conditions

$$S_{\boldsymbol{n}}^{(j)}(0,c,\boldsymbol{b}) = 0 \quad \text{and} \quad S_{0}^{(j)}(a,c,\boldsymbol{b}) = 0. \tag{2.9}$$

On the other hand, by (2.4) we verify easily that $T_{\boldsymbol{n}}^{(j)}(a,c,\boldsymbol{b})$ satisfies the same recurrence relation (2.8) and boundary condition (2.9). This completes the proof of (2.6). □

3. Multi-convolution polynomials

A family of polynomials $\{F_n(x)\}_{n\geqslant 0}$ forms a *multinomial convolution family* if $F_n(x)$ has degree $\leqslant |\boldsymbol{n}|$ and if the convolution condition

$$F_{\boldsymbol{n}}(x+y) = \sum_{0\leqslant \boldsymbol{k}\leqslant \boldsymbol{n}} F_{\boldsymbol{k}}(x)F_{\boldsymbol{n}-\boldsymbol{k}}(y) \tag{3.1}$$

holds for all x and y and for all $\boldsymbol{n}\geqslant 0$. In the case that $m=1$ we recover the monomial *convolution family* studied by Knuth [9]. It is worth noticing that this definition is different from the definition of 'higher-dimensional polynomials of binomial type' studied by Joni [10]. Many of such families are known, and they appear frequently in the applications. For example, when $F_{\boldsymbol{n}}(x) = x^{|\boldsymbol{n}|}/\boldsymbol{n}!$, condition (3.1) is equivalent to the binomial theorem for integer exponents. We can also let $F_{\boldsymbol{n}}(x)$ be the multinomial coefficient $\binom{x}{\boldsymbol{n}}$; the corresponding identity (3.1) may be called multi-Vandermonde's convolution.

Knuth showed that convolution polynomials arise as coefficients in the xth power of a *one-variable* power series. Now we show that multi-convolution polynomials arise as coefficients in the xth power of a *several variables* power series. Let $\boldsymbol{z}^{\boldsymbol{n}} = z_1^{n_1} \dots z_m^{n_m}$. For a formal series $F(\boldsymbol{z}) = \sum_{\boldsymbol{n} \geqslant \boldsymbol{0}} F_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}$, let $[\boldsymbol{z}^{\boldsymbol{n}}]F(\boldsymbol{z})$ be the coefficient of $\boldsymbol{z}^{\boldsymbol{n}}$ in $F(\boldsymbol{z})$.

Theorem 1. *Let*

$$F(\boldsymbol{z}) = 1 + \sum_{\boldsymbol{n} \neq \boldsymbol{0}} F_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \tag{3.2}$$

be any power series with $F(0) = 1$. *Then the polynomials*

$$F_{\boldsymbol{n}}(x) = [\boldsymbol{z}^{\boldsymbol{n}}]F(\boldsymbol{z})^x \tag{3.3}$$

form a multi-convolution family. Conversely, for every convolution family $(F_{\boldsymbol{n}}(x))$ *there is a power series* $F(\boldsymbol{z})$ *such that* (3.3) *holds or it is identically zero.*

Proof. It is easy to verify that the coefficient of $\boldsymbol{z}^{\boldsymbol{n}}$ in $F(\boldsymbol{z})^x$ is indeed a polynomial in x of degree $\leqslant |\boldsymbol{n}|$, because

$$F(\boldsymbol{z})^x = 1 + \sum_{k \geqslant 1} \binom{x}{k} \left(\sum_{\boldsymbol{n} \neq \boldsymbol{0}} F_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \right)^k .$$

This construction produces a convolution family because of the rule $F(\boldsymbol{z})^{x+y} = F(\boldsymbol{z})^x F(\boldsymbol{z})^y$. Conversely, suppose that the polynomials

$$F_{\boldsymbol{n}}(x) = (f_{\boldsymbol{n}0} + f_{\boldsymbol{n}0} x + \cdots + f_{\boldsymbol{n}|\boldsymbol{n}|} x^{|\boldsymbol{n}|})/\boldsymbol{n}!$$

form a convolution family. The condition $F_{\boldsymbol{0}}(x) = F_{\boldsymbol{0}}(x)^2$ can hold only if $F_{\boldsymbol{0}}(x) = 0$ or $F_{\boldsymbol{0}}(x) = 1$. In the former case it is easy to prove by induction that $F_{\boldsymbol{n}}(x) = 0$ for all $\boldsymbol{n}$ because $F_{\boldsymbol{n}}(x) = \sum_{\boldsymbol{0} < \boldsymbol{k} < \boldsymbol{n}} F_{\boldsymbol{k}}(x) F_{\boldsymbol{n}-\boldsymbol{k}}(0)$. Otherwise, the condition

$$F_{\boldsymbol{n}}(2x) = 2F_{\boldsymbol{n}}(x) + \sum_{\boldsymbol{0} < \boldsymbol{k} < \boldsymbol{n}} F_{\boldsymbol{k}}(x) F_{\boldsymbol{n}-\boldsymbol{k}}(x) \tag{3.4}$$

for $\boldsymbol{n} > \boldsymbol{0}$ implies that $F_{\boldsymbol{n}}(0) = f_{\boldsymbol{n}0} = 0$ for $\boldsymbol{n} > \boldsymbol{0}$ by induction. For $k > 1$, by equating coefficients of x^k on both sides of (3.4) we find that the coefficient $f_{\boldsymbol{n}k}$ is forced to have certain values based on the coefficients of $F_{\boldsymbol{i}}(x)$ with $\boldsymbol{i} < \boldsymbol{n}$, because $2^k f_{\boldsymbol{n}k}$ occurs on the left and $2 f_{\boldsymbol{n}k}$ on the right. The coefficient $f_{\boldsymbol{n}1}$ can, however, be choosen freely. Any such choice must make

$$F_{\boldsymbol{n}}(x) = [\boldsymbol{z}^{\boldsymbol{n}}] \exp\left(x \sum_{\boldsymbol{n} \neq \boldsymbol{0}} f_{\boldsymbol{n}1} \boldsymbol{z}^{\boldsymbol{n}} \right)$$

by induction on $\boldsymbol{n}$. $\square$

Examples

The first example mentioned above, $F_{\boldsymbol{n}}(x) = x^{|\boldsymbol{n}|}/\boldsymbol{n}!$, comes from the power series $F(\boldsymbol{z}) = \mathrm{e}^{z_1+z_2+\cdots+z_m}$; the second example, $F_{\boldsymbol{n}}(x) = \binom{x}{\boldsymbol{n}}$, comes from $F(\boldsymbol{z}) = 1 + z_1 + z_2 + \cdots + z_m$.

As in the case of ordinary convolution polynomials [9], every multi-convolution family $(F_{\boldsymbol{n}}(x))$ satisfies another general convolution formula in addition to the one we began with. We have the following result.

Lemma 1 (A derived convolution). *If $\{F_{\boldsymbol{n}}(x)\}_{\boldsymbol{n}\geqslant 0}$ is a set of multi-convolution polynomials, then*

$$(x+y)\sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} k_i F_{\boldsymbol{k}}(x)F_{\boldsymbol{n}-\boldsymbol{k}}(y) = x n_i F_{\boldsymbol{n}}(x+y) \quad (1 \leqslant i \leqslant m).$$

or equivalently for all $\boldsymbol{q} \in \mathbb{C}^m$ we have

$$(x+y)\sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} (\boldsymbol{q}\cdot\boldsymbol{k}) F_{\boldsymbol{k}}(x)F_{\boldsymbol{n}-\boldsymbol{k}}(y) = x(\boldsymbol{q}\cdot\boldsymbol{n}) F_{\boldsymbol{n}}(x+y).$$

Proof. The alternative convolution formula is proved by differentiating the basic identity $F(\boldsymbol{z})^x = \sum_{\boldsymbol{n}\geqslant 0} F_{\boldsymbol{n}}(x) z_1^{n_1}\ldots z_m^{n_m}$ with respect to z_i $(1\leqslant i\leqslant m)$ and

$$x z_i \frac{\partial}{\partial z_i} F(\boldsymbol{z}) F(\boldsymbol{z})^{x-1} = \sum_{\boldsymbol{n}} n_i F_{\boldsymbol{n}}(x) z_1^{n_1}\ldots z_m^{n_m}.$$

Now observe that $\sum k_i F_{\boldsymbol{k}}(x)F_{\boldsymbol{n}-\boldsymbol{k}}(y)$ is the coefficient of $z_1^{n_1}\ldots z_m^{n_m}$ in the product $x z_i(\partial F(\boldsymbol{z})/\partial z_i)F(\boldsymbol{z})^{x+y-1}$ while $n_i F_{\boldsymbol{n}}(x+y)$ is the coefficient of $z_1^{n_1}\ldots z_m^{n_m}$ in $(x+y) z_i(\partial F(\boldsymbol{z})/\partial z_i)F(\boldsymbol{z})^{x+y-1}$. □

Lemma 2. *If $F_{\boldsymbol{n}}(x) = [\boldsymbol{z}^{\boldsymbol{n}}]F(\boldsymbol{z})^x$, then*

$$\frac{x}{x+n_i} F_{\boldsymbol{n}}(x+n_i) = [\boldsymbol{z}^{\boldsymbol{n}}]G_i(\boldsymbol{z})^x, \quad 1\leqslant i\leqslant m,$$

where $G_i(\boldsymbol{z}) = F(z_1,\ldots,z_iG_i(\boldsymbol{z}),\ldots,z_m)$.

Proof. Since $F_{\boldsymbol{n}}(x)$ is a polynomial of degree $\leqslant|\boldsymbol{n}|$, so it is sufficient to prove the above identity for the integer values k of x. Let $w(\boldsymbol{z}) = z_i/F(\boldsymbol{z})$, then $w(z_1,\ldots,z_iG_i(\boldsymbol{z}),\ldots,z_m) = z_i$. By Lagrange's inversion formula (see [3, p. 159]), we have

$$[z_i^{n_i}](z_i^k G_i^k(\boldsymbol{z})) = \frac{k}{n_i}[z_i^{n_i-k}]F(\boldsymbol{z})^{n_i}, \quad 1\leqslant i\leqslant m.$$

Consequently,

$$[z_i^{n_i}](G_i^k(\boldsymbol{z})) = \frac{k}{n_i+k}[z_i^{n_i}]F(\boldsymbol{z})^{n_i+k}, \quad 1\leqslant i\leqslant m.$$

It follows that for $1 \leqslant i \leqslant m$ we have

$$[z^{\boldsymbol{n}}]G_i(z)^x = \frac{k}{n_i + k}[z^{\boldsymbol{n}}]F(z)^{n_i+k} = \frac{k}{n_i + k}F_{\boldsymbol{n}}(n_i + k).$$

This completes the proof. □

Theorem 2. *Let $\{F_{\boldsymbol{n}}(x)\}_{\boldsymbol{n} \geqslant 0}$ be any family of polynomials in x such that $F_{\boldsymbol{n}}(x)$ has degree $\leqslant |\boldsymbol{n}|$. If*

$$F_{\boldsymbol{n}}(2x) = \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} F_{\boldsymbol{k}}(x)F_{\boldsymbol{n}-\boldsymbol{k}}(x) \tag{3.5}$$

holds for all $\boldsymbol{n}$ and x, then the following identities hold for all $\boldsymbol{n}$, x, y and t:

$$\frac{(x+y)F_{\boldsymbol{n}}(x+y+\boldsymbol{t}\cdot\boldsymbol{n})}{x+y+\boldsymbol{t}\cdot\boldsymbol{n}} = \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} \frac{xF_{\boldsymbol{k}}(x+\boldsymbol{t}\cdot\boldsymbol{k})}{x+\boldsymbol{t}\cdot\boldsymbol{k}} \cdot \frac{yF_{\boldsymbol{n}-\boldsymbol{k}}(y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k}))}{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})},$$

$$\frac{(\boldsymbol{q}\cdot\boldsymbol{n})F_{\boldsymbol{n}}(x+y+\boldsymbol{t}\cdot\boldsymbol{n})}{x+y+\boldsymbol{t}\cdot\boldsymbol{n}} = \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} \frac{(\boldsymbol{q}\cdot\boldsymbol{k})F_{\boldsymbol{k}}(x+\boldsymbol{t}\cdot\boldsymbol{k})}{x+\boldsymbol{t}\cdot\boldsymbol{k}} \cdot \frac{yF_{\boldsymbol{n}-\boldsymbol{k}}(y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k}))}{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})}.$$

Proof. The proof of Theorem 1 shows that the multinomial convolution condition could be weakened for (3.4), or (3.5). Hence (3.5) implies that polynomials $F_{\boldsymbol{n}}(x)$ form a set of convolution polynomials. Now, for $1 \leqslant i \leqslant m$, the process of going from $F_{\boldsymbol{n}}(x)$ to $xF_{\boldsymbol{n}}(x+n_i)/(x+n_i)$ of Lemma 2 can be iterated: another replacement gives $xF_{\boldsymbol{n}}(x+2n_i)/(x+2n_i)$, and after t_i iterations we discover that the polynomials $xF_{\boldsymbol{n}}(x+t_in_i)/(x+t_in_i)$ also form a convolution family. This holds for all positive integers t_i $(1 \leqslant i \leqslant m)$, and the convolution condition is expressible as a set of polynomial relations in t_i $(1 \leqslant i \leqslant m)$; therefore $xF_{\boldsymbol{n}}(x+t_in_i)/(x+t_in_i)$ is a convolution family for all complex numbers t_i $(1 \leqslant i \leqslant m)$. Thus by applying the above process we obtain succesfully a sequence of convolution polynomials families

$$\frac{x}{x+t_1n_1+\cdots+t_in_i}F_{\boldsymbol{n}}(x+t_1n_1+\cdots+t_in_i)$$

for $i = 1, 2, \ldots, m$ and the last family corresponds to the first identity. The second one follows by combining the first one with the derived convolution formula of Lemma 1. □

Examples

Let $F_{\boldsymbol{n}}(x) = \binom{x}{\boldsymbol{n}}$; then it follows from (1.4) and Theorem 2 that

$$\frac{x+y}{x+y+\boldsymbol{t}\cdot\boldsymbol{n}}\binom{x+y+\boldsymbol{t}\cdot\boldsymbol{n}}{\boldsymbol{n}}$$

$$= \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} \frac{x}{x+\boldsymbol{t}\cdot\boldsymbol{k}}\binom{x+\boldsymbol{t}\cdot\boldsymbol{k}}{\boldsymbol{k}} \frac{y}{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})}\binom{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})}{\boldsymbol{n}-\boldsymbol{k}},$$

$$\frac{(\boldsymbol{q}\cdot\boldsymbol{n})}{x+y+\boldsymbol{t}\cdot\boldsymbol{n}}\binom{x+y+\boldsymbol{t}\cdot\boldsymbol{n}}{\boldsymbol{n}}$$

$$=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}}\frac{\boldsymbol{q}\cdot\boldsymbol{k}}{x+\boldsymbol{t}\cdot\boldsymbol{k}}\binom{x+\boldsymbol{t}\cdot\boldsymbol{k}}{\boldsymbol{k}}\frac{y}{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})}\binom{y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k})}{\boldsymbol{n}-\boldsymbol{k}}.$$

Combining the above two identities yields (1.2).

Let $F_{\boldsymbol{n}}(x)=x^{\boldsymbol{n}}/\boldsymbol{n}!$, then it follows from (1.3) and Theorem 2 that

$$\frac{(x+y)(x+y+\boldsymbol{t}\cdot\boldsymbol{n})^{|\boldsymbol{n}|-1}}{\boldsymbol{n}!}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}}\frac{x(x+\boldsymbol{t}\cdot\boldsymbol{k})^{|\boldsymbol{k}|-1}}{\boldsymbol{k}!}\frac{y(y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k}))^{|\boldsymbol{n}-\boldsymbol{k}|-1}}{(\boldsymbol{n}-\boldsymbol{k})!},$$

$$\frac{(\boldsymbol{q}\cdot\boldsymbol{n})(x+y+\boldsymbol{t}\cdot\boldsymbol{n})^{|\boldsymbol{n}|-1}}{\boldsymbol{n}!}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}}\frac{(\boldsymbol{q}\cdot\boldsymbol{k})(x+\boldsymbol{t}\cdot\boldsymbol{k})^{|\boldsymbol{k}|-1}}{\boldsymbol{k}!}\frac{y(y+\boldsymbol{t}\cdot(\boldsymbol{n}-\boldsymbol{k}))^{|\boldsymbol{n}-\boldsymbol{k}|-1}}{(\boldsymbol{n}-\boldsymbol{k})!}.$$

Combining the above two identities yields.

$$\sum_{\boldsymbol{k}}(p+\boldsymbol{q}\cdot\boldsymbol{k})B_{\boldsymbol{k}}(x,\boldsymbol{t})B_{\boldsymbol{n}-\boldsymbol{k}}(y,\boldsymbol{t})=\frac{p(x+y)+x(\boldsymbol{q}\cdot\boldsymbol{n})}{x+y}B_{\boldsymbol{n}}(x+y,\boldsymbol{t}),\tag{3.6}$$

where $B_{\boldsymbol{n}}(x,\boldsymbol{t})=x(x+\boldsymbol{t}\cdot\boldsymbol{n})^{|\boldsymbol{n}|-1}/\boldsymbol{n}!$. If we take $\boldsymbol{q}=\boldsymbol{t}$ and $p=1$ in the above formula we obtain a multi-extension of Abel's identity:

$$(x+y+\boldsymbol{n}\cdot\boldsymbol{t})^{|\boldsymbol{n}|}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}(x+\boldsymbol{k}\cdot\boldsymbol{t})^{|\boldsymbol{k}|}y(y+(\boldsymbol{n}-\boldsymbol{k})\cdot\boldsymbol{t})^{|\boldsymbol{n}-\boldsymbol{k}|-1},$$

where $\binom{\boldsymbol{n}}{\boldsymbol{k}}=\prod_{i=1}^{m}\binom{n_i}{k_i}$ or equivalently

$$(x+y)^{|\boldsymbol{n}|}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}(x-\boldsymbol{k}\cdot\boldsymbol{t})^{|\boldsymbol{n}-\boldsymbol{k}|}y(y+\boldsymbol{k}\cdot\boldsymbol{t})^{|\boldsymbol{k}|-1}.$$

We can also derive (3.6) from the first example as follows. If we substitute a, c and $\boldsymbol{b}$ by xa, xc and $x\boldsymbol{b}=(xb_1,\ldots,xb_m)$ respectively, in (1.2), we get a polynomial identity in x. Equating the coefficients of $x^{|\boldsymbol{n}|}$ in both sides of this identity yields (3.6).

Note that formula (3.6) generalizes also Hurwitz's *multinomial* extension of Abel's identity [8]. To see this, setting $\boldsymbol{n}=\boldsymbol{1}=(1,\ldots,1)$, $\boldsymbol{q}=\boldsymbol{0}$ and $p=1$, then (3.6) reduces to

$$\sum_{\boldsymbol{k}\in\{0,1\}^m}x(x+\boldsymbol{k}\cdot\boldsymbol{t})^{|\boldsymbol{k}|-1}y(y+(\boldsymbol{1}-\boldsymbol{k})\cdot\boldsymbol{t})^{|\boldsymbol{1}-\boldsymbol{k}|-1}=(x+y)(x+y+|\boldsymbol{t}|)^{m-1}.$$

We have shown how to create new multinomial convolution families from given ones exactly as in the monomial case. The additional constructions mentioned by Knuth

[9] can also be extended. For example, if $F_n(x)$ and $G_n(x)$ are convolution families, then so is the family $H_n(x)$ defined by

$$H_n(x) = \sum_{k=0}^{n} F_k(x)G_{n-k}(x).$$

as did Knuth [9] in the case $m = 1$.

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