# ON SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC SOLUTIONS

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**Introduction.** We consider second order linear differential operators

$$L = D^2 + B \cdot D + C,$$

$D = d/dx$, $B$, $C \in \mathbf{C}(x)$. The singular points of $L$ consist of the poles of $B$ and $C$ and possibly the point at infinity.

H. A. Schwarz [15] determined all such operators with three singular points whose kernel consists of algebraic functions. His method was to show that if $B$ and $C$ lie in $\mathbf{R}(x)$ then the monodromy group can be calculated from the group generated by the reflections relative to three circles which meet at angles determined by the exponent differences of $L$. He used this to show that the solutions of $L$ are all algebraic if and only if these angles coincide with the angles of a spherical triangle whose vertices are fixed points of three rotations any pair of which generates a finite rotation group.

The Schwarz list of 15 reduced curvilinear triangles contains a basic sublist of exponent differences

| | | | | group type | order $= M$ |
|---|---|---|---|---|---|
| I | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{n}$ | dihedral | $2n$ |
| II | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | tetrahedral | 12 |
| IV | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | octahedral | 24 |
| VI | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | icosohedral | 60 |

the first case corresponding to an infinite series of dihedral groups.

Klein asserted [10, pp. 132–135] that he had determined all differential equations of second order with only algebraic solutions. This widely accepted assertion is quite misleading. One may pose two questions

(0.1) Given $n + 1$ points $\{a_1, \ldots, a_n, \infty\}$, determine all second order linear differential equations with only algebraic solutions and with these points as singularities.

(0.2) Given a second order linear differential equation $L$, determine in a finite number of steps whether the solutions of $L$ are all algebraic.

The method of Klein is better adapted to problem (0.2) than to problem (0.1). His method (cf. Theorem 3.4 below) is based on the fact that (excluding the cyclic case) $L$ has only algebraic solutions if and only if it is a weak pullback by a rational map of an element in the basic Schwarz list. To decide whether $L$ as given by (3.3) is the pullback of $L_0$ is equivalent to the determination of the existence of a rational solution $x$ of the non-linear third order differential equation (3.7.3). With an a-priori bound for the degree of $x$ this can be reduced to a purely algebraic problem. Klein illustrated this program [12] by exhibiting cases XII, XIV, XV of Schwarz as pullbacks of VI (the other cases on the full Schwarz list had been previously checked off in that way by Brioschi [2]). Klein did not for example show by his method that the Schwarz list is exhaustive, a reasonable step in illustrating a credible program for problem (0.1).

It is our understanding that Problem 0.1 is still open for $n \geq 3$ (i.e. four or more singular points). In the present article we carry out the Klein program for Problem (0.2). The new ingredients of our work are

(0.3) A general formula (Section 1) for the degree of the pullback mapping.

(0.4) A decision procedure (Section 6) for the existence of an algebraic solution of a *linear* first order differential equation over a Riemann Surface. (Correction: This procedure is not new. See R. H. Hisch, Bull. Amer. Math. Soc. 76 (1970), pp. 605–608).

The degree calculation is needed to carry out Klein's method for checking whether $L$ is a pullback of II, IV or VI but the method breaks down if we must test whether the group of $L$ is one of the two infinite series. The cyclic case is treated in Section 4 but the main point in the dihedral case is the reduction (Section 5) to problem (0.4) mentioned above.

In a subsequent article we shall explain how Klein's program may be extended to second order differential equations defined over a Riemann Surface.

In Section 8 we indicate connections with Grothendiecke's conjecture and the work of Katz. We note that the conjecture is still open for four or more singular points (unless the differential equation "comes" from algebraic geometry). Even if verified it would not respond to Problem (0.2).

We note that Forsyth [3, p. 184] discussed Problem (0.2) along the lines of Klein. His treatment cannot be taken seriously.

We regard the present article as expository. Indeed a superior decision procedure (over the sphere) has been given by L. Fuchs (cf. items 19, 20, 21, 22, 23, 25, Vol. II of his collected works). This may be explained quite easily.

For each integer $m$ let $L_m$ be the linear differential equation of order $m + 1$ satisfied by all homogeneous forms of degree $m$ in solutions $y_1$, $y_2$ of the second order linear differential equation $L$ given by (4.1). If $L$ has only algebraic solutions then (cf. Theorem 3.8 below) $L_{12}$ must have at least one solution which is the radical of a rational function. If $L_{12}$ has such a solution but $L_2$ does not then (Fuchs p. 45 loc cit) all solutions of $L$ are algebraic. If $L_2$ has such a solution, $\phi$, but $L$ does not then $\phi^2$ is a rational function and $\phi = y_1y_2$, a product of independent solutions of $L$. Putting $\tau = y_2/y_1$ we obtain

$$\tau' = \frac{w}{\phi}\tau \tag{0.5}$$

where $w = y_1y_2' - y_2y_1'$ is a constant which may be calculated from

$$-w^2 = \phi^2\left[\left(\frac{\phi'}{\phi}\right)^2 + 2\left(\frac{\phi'}{\phi}\right)' - 4Q\right] \tag{0.6}$$

In this case Problem (0.2) is reduced to determining whether $\tau$ as given by (0.5) is algebraic, a problem evidently untreated by Fuchs. Our Section 6 serves then to fill in this gap in the treatment of Fuchs. If

$L$ has itself a solution which is the radical of a rational function then the solutions of $L$ are all algebraic only if $L$ satisfies the criteria for the cyclic case (cf. Section 4 below). Finally we note that the question of whether $L_{12}$ has a solution, $v$, which is the radical of a rational function is elementary since $v$ must be of the form

$$\prod_{i=1}^{n} (x - a_i)^{\alpha_i} g(x)$$

where $g$ is a polynomial whose degree as well as the values of the $\alpha_1$ are determined up to a finite set of possibilities by the exponents of $L$ at its singularities $\{a_1, \ldots, a_n, \infty\}$.

1. **Pullbacks on Riemann Surfaces.** Let $L$ be a second order differential equation with coefficients in an algebraic function field $K$ of characteristic zero and algebraically closed field of constants $k$. Hence choosing $x \in K$, $x \notin k$, we may write

$$L = D^2 + AD + B \tag{1.1}$$

where $A, B \in K$ and $D = d/dx$. Letting $\Gamma$ be the Riemann surface corresponding to $K$, we view $L$ as being *equivalent* to $L_1$ also defined over $K$ if a ratio of solutions (at some point of $\Gamma$) of $L$ is also a ratio of solutions of $L_1$. Thus in particular we identify $L$ with

$$L_1 = D^2 + B_1 \tag{1.2}$$

where

$$B_1 = B - \frac{1}{2} A' - \frac{1}{4} A^2 \qquad \left(A' = \frac{d}{dx} A\right),$$

since

$$L_1 = \frac{1}{\theta} \circ L \circ \theta \quad \text{where} \quad \theta'/\theta = -\frac{1}{2} A$$

(notice that $\theta$ does not lie in $K$ but is defined locally on $\Gamma$).

We shall assume that at each point, $P$, of $\Gamma$, $L$ has two independent solutions of the form

$$y_1 = t^{\alpha_i}(1 + b_{i1}t + b_{i2}t^2 + \cdots) \qquad i = 1, 2 \tag{1.3}$$

where $t$ is a local uniformizing parameter, $\alpha_i \in k$ and the $b_{i,j}$ also lie in $k$. Because of our definition, the pair $(\alpha_1, \alpha_2)$ is not well defined but their difference is well defined up to sign. For a fixed model we have $(\alpha_1, \alpha_2) = (0, 1)$ for almost all points and so if we choose any archimedean valuation of $k$ and setting $\gamma(P) = |\alpha_1 - \alpha_2|$ for each point $P$ of $\Gamma$, we see that $\gamma(P) - 1 = 0$ for almost all $P$. Let $S$ be any finite subset of $\Gamma$ and put

$$\Delta(S, L) = \sum_{P \in S} (\gamma(P) - 1). \tag{1.4}$$

For $S$ large enough we obtain a limiting value $\Delta(L)$ independent of $S$, but depending on the valuation of $k$ chosen for the construction.

Now let $K_0$ be an algebraic extension of $K$ and let $\pi$ denote the mapping $\Gamma_0 \rightarrow \Gamma$ of the corresponding Riemann surface. Let $L_0$ be the pullback of $L$ by $\pi$ (i.e. if $\tau$ is ratio of two solutions of $L$ then $\tau \circ \pi$ is ratio of two solutions of $L_0$). Using the same valuation of $k$ we may define $\Delta(L_0)$.

LEMMA 1.5 *Let $g$ (resp: $g_0$) be the genus of $\Gamma$ (resp: $\Gamma_0$), let $M$ be the mapping degree, then*

$$M[\Delta(L) - 2(g - 1)] = \Delta(L_0) - 2(g_0 - 1). \tag{1.5.1}$$

*Proof.* For $P \in \Gamma$, let $P_0$ be a point of $\Gamma_0$ above $P$. Let $e(P_0)$ be the relative ramification. If $\alpha_1, \alpha_2$ are exponents at $P$ of a model of $L$, then $\alpha_i \cdot e(P_0)$ are exponents at $P_0$ of $L_0$. Thus

$$\gamma(P_0) = \gamma(P) \cdot e(P_0)$$

which shows that

$$\sum_{P_0 | P} \gamma(P_0) = M \cdot \gamma(P). \tag{1.6}$$

If $S$ is any finite subset of $\Gamma$ and $S_0$ is the set of all points of $\Gamma$ above $S$, then

$$\Delta(S_0, L_0) + \text{card } S_0 = M(\Delta(S, L) + \text{card } S). \tag{1.7}$$

The Hurwitz genus formula states that

$$2(g_0 - 1) - 2M(g - 1) = M \text{ card } S - \text{card } S_0 \tag{1.8}$$

provided $S$ contains all points of $\Gamma$ which ramify in $\Gamma_0$. The lemma now follows by taking $S$, so large that in (1.7) we may replace $\Delta(S_0, L_0)$ (resp $\Delta(S, L)$) by $\Delta(L_0)$ (resp: $\Delta(L)$) and using (1.8) to eliminate the terms involving cardinality.

In the application below $g_0 = g_1 = 0$ and the exponent differences are all rational. Besides, coordinate functions $t$, $x$ will be fixed on $\Gamma_0$, $\Gamma$ (that is $K_0 = k(t)$, $K = k(x)$) and by the *weak pullback* of $L$ to $(\Gamma_0, t)$ we will denote the model of the pullback of $L$ to $\Gamma_0$, which is in the form (1.2) with respect to the coordinate function $t$.

The point in the lemma is that $L$, $L_0$, $\Gamma$ and $\Gamma_0$ may be given without the relation $\pi$ between $\Gamma$ and $\Gamma_0$ being known. In that situation for given set $S$ of $\Gamma$ there is no way to determine the lifting, $S_0$, to $\Gamma_0$. We can determine the set $T$ of singularities of $L$ and $T_0$ of singularities of $L_0$ but $T_0$ need not coincide with $\pi^{-1}(T)$, the problem being that ramification points of $\pi$ may produce singular points of $L_0$ and again if $\gamma(P) = 1/e$ and $P_0$ covers $P$ with ramification $e$ then $P_0$ is not a singularity of $L_0$ since the exponent difference at $P_0$ would be 1. The situation would be different if we viewed $L$ as determining the solution space rather than ratios of solutions.

2. **Finite homography groups.** We review (following Halphen [6]) the theory of such groups. Let $G$ be a finite group of homographies

$$t \to \frac{at + b}{ct + d}$$

where $a$, $b$, $c$, $d$ lie in $\mathbf{C}$ and $ad - cb \neq 0$. Thus $G$ may be viewed as a finite set of automorphisms of the function field $H = \mathbf{C}(t)$. By Lüroth's theorem, the fixed field $H^G$ is generated by a single element $y$ and so is of genus zero. Let $P_1, \ldots, P_s$ be the points of $H^G$ which ramify in $H$, let $M$ be the order of $G$ and let $e_i$ be the relative ramification of $H$ over $H^G$ at one point of $H$ above $P_i$. It then follows that there are $M/e_i$ places of $H$ above $P_i$ and each place has ramification $e_i$, which shows that if we use the Hurwitz genus formula we have

$$0 = 2(1 - M) + \sum_{i=1}^{s} (e_i - 1)M/e_i.$$

and so

$$\sum_{i=1}^{s} 1/e_i = s - 2 + \frac{2}{M}. \tag{2.1}$$

Since each $e_i \geq 2$, we obtain

$$s/2 \geq \Sigma\, 1/e_i > s - 2$$

which shows that $s = 2, 3$.

If $s = 2$, we use $e_i \leq M$ to conclude from (2.1) that

$$e_1 = e_2 = M$$

which shows that the Galois group $G$ coincides with the ramification group at $P_1$ and hence $G$ is cyclic.

If $s = 3$ then (2.1) becomes

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} = 1 + \frac{2}{M} \tag{2.2}$$

a diophantine equation with only 4 types of solutions as given by the table in the Introduction. (The point is that we cannot have $e_i \geq 3$ for all $i$ and so can assume $e_1 = 2$. Furthermore $e_2$, $e_3$ cannot both exceed 3. Thus $e_2 = 2$ or 3.)

Let $G$ be a finite group of homographies. Then for each $\theta \in G$ we may choose a matrix

$$A_\theta = \begin{pmatrix} a_\theta & b_\theta \\ c_\theta & d_\theta \end{pmatrix} \in SL(2, \mathbf{C}) \tag{2.3}$$

such that

$$\theta(t) = \frac{a_\theta t + b_\theta}{c_\theta t + d_\theta}. \tag{2.4}$$

Of course $A_\theta$ is only determined mod $\pm 1$ and hence if $\theta$, $\varphi \in G$, we have

$$A_\theta A_\varphi = h(\theta, \varphi) A_{\theta\varphi} \tag{2.5}$$

for suitable $h(\theta, \varphi) \in \{\pm 1\}$. It is clear that $h$ is a 2-cocycle for $G$ with coefficients in $\{\pm 1\}$ with trivial action of $G$ on $\{\pm 1\}$. (The example of the 4-group generated by $x \to \pm 1/x$ shows that $h$ need not be cohomologically trivial). For later use we define a mapping $k$ of $G \times G$ into $\pm 1$ and a mapping $k_0$ of $G$ into $\pm 1$ by the formulas

$$k(\phi, \theta) A_{\phi^{-1}\theta} = A_\phi^{-1} A_0 \tag{2.6.1}$$

$$k_0(\phi) = \prod_{\theta \in G} k(\phi, \theta) \tag{2.6.2}$$

For $u \in \mathbf{C}$, we define

$$F_u(t) = \prod_{\theta \in G} (t - \theta u) \tag{2.7}$$

if $\theta u$ is finite for all $\theta$. Whenever $\theta u$ is infinite we replace the factor $t - \theta u$ by $-(a_\theta u + b_\theta)$. Thus $F_u$ is a polynomial of degree $M - \sigma_u$ where $\sigma_u$ is the number of $\theta \in G$ such that $\theta u = \infty$. We note that

$$\pm F_{\varphi u}(t) = F_u(t)/(c_\varphi u + d_\varphi)^{\sigma_u}$$

for all $\phi$ in $G$, the sign being given by $\prod h(\theta\phi^{-1}, \phi)$, the product being over all $\theta$ such that $\theta u = \infty$.

If we put $t = t_1/t_2$, we obtain

$$\Delta_u t_2^M F_u(t) = \prod_{\theta \in G} [t_1(c_\theta u + d_\theta) - t_2(a_\theta u + b_\theta)] \tag{2.8}$$

where $M =$ order of $G$

$$\Delta_u = \prod (c_\theta u + d_\theta)$$

the product being over all $\theta \in G$ such that $\theta u \neq \infty$. If $\varphi \in G$ then putting

$$\varphi_1(t_1, t_2) = a_\varphi t_1 + b_\varphi t_2$$
$$\varphi_2(t_1, t_2) = c_\varphi t_1 + d_\varphi t_2$$

we obtain $\varphi(t) = \varphi_1(t_1, t_2)/\varphi_2(t_1, t_2)$ and so

$$\begin{aligned}
\Delta_u \varphi_2(t_1, t_2)^M F_u(\varphi(t)) &= \prod_{\theta \in G} [\varphi_1(t_1, t_2) \cdot (c_\theta u + d_\theta) \\
&\qquad - \varphi_2(t_1, t_2)(a_\theta u + b_\theta)] \\
&= \prod_{\theta \in G} [t_1(c_{\varphi^{-1}\theta} u + d_{\varphi^{-1}\theta}) \\
&\qquad - t_2(a_{\varphi^{-1}\theta} u + b_{\varphi^{-1}\theta})] k(\phi, \theta) \\
&= \Delta_u t_2{}^M F_u(t) k_0(\phi).
\end{aligned}$$

This shows that

$$F_u(\varphi(t)) = \frac{k_0(\phi)}{(c_\varphi t + d_\varphi)^M} F_u(t). \tag{2.9}$$

The key point is that the ratio between $F_u(\varphi(t))$ and $F_u(t)$ is independent of $u$. Thus if $v$ lies in another orbit, we see that $F_u/F_v$ is an absolute invariant under $G$, and so lies in $H^G$, the subfield of $H = \mathbf{C}(t)$ fixed by $G$.

Putting $x = F_u(t)/F_v(t)$ we see that $H \supset H^G \supset \mathbf{C}(x)$ and hence

$$\deg H/\mathbf{C}(x) \geq M$$

while the equation

$$0 = F_u(t) - xF_v(t)$$

is a polynomial in $t$ of degree $M$, which shows that

$$\deg \mathbf{C}(t)/\mathbf{C}(x) \leq M.$$

This shows that

$$H^G = \mathbf{C}(x).$$

LEMMA 2.10. *If $u$, $v$, $w$ are points of* $\mathbf{C}$ *then the polynomials $F_u$, $F_v$, $F_w$ are linearly dependent over* $\mathbf{C}$.

*Proof.* We may assume $u$, $v$, $w$ to lie in distinct $G$-orbits and deg $F_w = M$. Then $H^G$ is generated over $\mathbf{C}$ by $F_u/F_w$ and also by $F_u/F_v$. This shows that $F_u/F_w$ and $F_u/F_v$ are related by a homography. Hence there exist $a$, $b$, $c$, $d$ in $\mathbf{C}$ such that

$$\frac{F_u}{F_w} = \frac{aF_u + bF_v}{cF_u + dF_v}$$

Since $F_u$ and $F_w$ are relatively prime, we conclude that $F_w$ divides $cF_u + dF_v$ but deg $F_w = M \geq$ deg $(cF_u + dF_v)$ which shows that up to a non-zero constant factor,

$$F_w = cF_u + dF_v.$$

This completes the proof of the lemma.

THEOREM 2.11. *If $G$ is a finite non-cyclic group of homographies of order $M$ then there exist polynomials $P_1$, $P_2$, $P_3$ of degree $M/e_i$* ($i = 1, 2, 3$) (*or possibly $M/e_i - 1$ for one $i$*) *such that*

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} = 1 + \frac{2}{M} \tag{2.2}$$

$$P_1^{e_1} + P_2^{e_2} + P_3^{e_3} = 0, \qquad (P_i, P_j) = 1 \quad \text{for} \quad i \neq j \tag{2.11.1}$$

$$P_i(\theta t) = P_i(t)/(c_\theta t + d_\theta)^{M/e_i} \qquad \forall\theta \in G. \tag{2.11.2}$$

*Proof.* As before let $H = \mathbf{C}(t)$, $H^G$ the fixed field under $G$. The extension $H/H^G$ ramifies at 3 valuations of $H^G$. If $a_i$ is such a place of $H^G$, let $P_i$ be the polynomial whose roots are the finite extensions of $a_i$ to $H$. This means that the degree of $P_i$ is $M/e_i$ unless the infinite place of $H$ lies above $a_i$. It is clear that $P_i^{e_i}$ is an invariant constructed from an orbit and after adjusting constant factors, equation (2.11.1) follows from Lemma 2.10, equation (2.11.2) follows from (2.9) and (2.2) has been previously established.

To formulate the converse of the above theorem we consider solutions of the equation

$$Q_1^{e_1} + Q_2^{e_2} + Q_3^{e_3} = 0 \tag{2.12}$$

in $\mathbf{C}[t]$ such that $Q_1$, $Q_2$, $Q_3$ are relatively prime in pairs and such that if (say)

$$\deg Q_3^{e_3} < \deg Q_1^{e_1} = \deg Q_2^{e_2} = M \tag{2.13}$$

then

$$\deg Q_3^{e_3} \equiv M \bmod e_3 .$$

LEMMA 2.14.

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \geq 1 + \frac{2}{M} \tag{2.14.1}$$

*and equality holds if and only if the equation*

$$-x = \frac{Q_1^{e_1}}{Q_3^{e_3}}(t) \tag{2.15}$$

*defines an extension* $\mathbf{C}(t)$ *of* $\mathbf{C}(x)$ *ramified only at 0, 1,* $\infty$ *with ramification* $e_1$, $e_2$, $e_3$ *at each point of t-sphere above the indicated points of the x-sphere. If equality holds then there exists a constant c such that*

$$Q_2^{e_2-1} = c(e_1 Q_1' Q_3 - e_3 Q_3' Q_1) \tag{2.15.1}$$

*Proof.* We assume that either $\deg Q_i^{ei} = M$ for $i = 1, 2, 3$ or equation 2.13 holds. We put $(w_1, w_2, w_3) = (0, 0, 0)$ in the first case and $(0, 0, M\text{-}\deg Q_3^{e_3})/e_3$ in the second case.

We put $\sigma_\infty = 0$, in the first case and $\sigma_\infty = w_3 e_3 - 1$ in the second case. In either case it represents the contribution of the point $t = \infty$ to the Hurwitz genus formula for the genus of $\mathbf{C}(t)$ as an extension of $\mathbf{C}(x)$. This formula may be written

$$0 = 1 - M + \frac{1}{2}\sum_{i=1}^{3} N_i(e_i - 1) + \frac{1}{2}\sigma_\infty + E \tag{2.16}$$

where $N_i = \deg Q_i$ and $E \geq 0$ is a correction term introduced to allow for the possibility that

(a) Ramification may occur at points other than $x = 0, 1, \infty$
(b) the zeros of $Q_i$ may be non-simple.

Since

$$M = (N_i + w_i)e_i,$$

we deduce from (2.16)

$$M\left(\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1\right) = 2E + \begin{cases} 2 & \text{if} \quad w_3 = 0 \\ 1 + w_3 & \quad\quad w_3 \neq 0. \end{cases} \tag{2.17}$$

This proves the inequality which appears in the statement of the lemma. Equality holds if and only if both $E = 0$ and $w_3 = 0, 1$. This completes the proof of the lemma.

Now assume equality holds. Let

$$H = e_1 Q_1' Q_3 - e_3 Q_3' Q_1$$
$$K = e_2 Q_2' Q_3 - e_3 Q_3' Q_2$$

By dividing equation (2.12) by $Q_3^{e_3}$ and differentiating

$$Q_1^{e_1 - 1} H + Q_2^{e_2 - 1} K = 0.$$

Since $Q_1$ and $Q_2$ are relatively prime, $Q_2^{e_2-1}$ divides $H$. Since $Q_1$ and $Q_3$ are relatively prime, $H$ cannot be zero. Equation (2.15.1) now follow by comparison of degrees of two sides.

PROPOSITION 2.18. *Let $e_1$, $e_2$, $e_3$ be given positive integers. Let $\mathbf{C}(t)$ be algebraic over $\mathbf{C}(x)$ ramified at $x = a_1, a_2, a_3$ (and possibly* *elsewhere). Suppose that at each place of* $\mathbf{C}(t)$ *above* $a_i$ *the ramification index is divisible by* $e_i$.

Let $z$ be a multivalued function on the $x$-sphere having only $a_1$, $a_2$, $a_3$ as critical points, suppose each branch of $z$ is locally algebraic and that each branch at $a_i$ lies in $\mathbf{C}(((x - a_i)^{1/e_i}))$. Then $z \in \mathbf{C}(t)$.

*Proof.* The function $z$ is everywhere uniform on the $t$-sphere and hence is rational function of $t$.

*Definition.* A solution of (2.12) is called primitive if equality holds in (2.14.1).

LEMMA 2.19. *If* $(Q_1, Q_2, Q_3)$ *is a primitive solution of* (2.12) *and* $(P_1, P_2, P_3)$ *is an arbitrary one, then there exist polynomials* $f$, $g$ *such that*

$$P_i = Q_i\left(\frac{f}{g}\right) \cdot g^{M/e_i} \qquad i = 1, 2, 3.$$

*Proof.* Define $z$, $t$ algebraic over $\mathbf{Z}(x)$ by setting

$$\begin{cases} -x = \dfrac{Q_1{}^{e_1}}{Q_3{}^{e_3}}(z) \\[2ex] -x = \dfrac{P_1{}^{e_1}}{P_3{}^{e_3}}(t) \end{cases} \tag{2.19.1}$$

We apply proposition 2.18 and lemma 2.14 and conclude that

$$z \in \mathbf{C}(t), \qquad \text{i.e.} \quad z = f/g,$$

where $f$, $g \in \mathbf{C}[t]$, $(f, g) = 1$. Equation (2.19.1) gives

$$\frac{P_1{}^{e_1}}{P_3{}^{e_3}} = \frac{Q_1{}^{e_1}(f/g)}{Q_3{}^{e_3}(f/g)} = \frac{[g^{N_1}Q_1(f/g)]^{e_1}}{[g^{N_3}Q_3(f/g)]^{e_3}}\, g^{e_3w_3}$$

$$\frac{P_2{}^{e_2}}{P_3{}^{e_3}} = x - 1 = \frac{Q_2{}^{e_2}(f/g)}{Q_3{}^{e_3}(f/g)} = \frac{[g^{N_2}Q_2(f/g)]^{e_2}}{[g^{N_3}Q_3(f/g)]^{e_3}}\, g^{e_3w_3}$$

(with no loss in generality we take 3 as in the proof of Lemma 2.14).

By hypothesis $1 = (Q_1, Q_2)$. From this we deduce that $g^{N_1}Q_1(f/g)$ and $g^{N_2}Q_2(f/g)$ are relatively prime. Again $(P_1, P_2) = 1$ by hypothesis and the lemma follows from our relations.

COROLLARY 2.20. *If $(P_1, P_2, P_3)$ and $(Q_1, Q_2, Q_3)$ are both primitive solutions then the rational function $f/g$ represents a homography.*

*Proof.* In this situation, equation (2.19.1) shows that $z$ and $t$ are rational functions of each other, hence $z$ is homographic image of $t$. We can now formulate a converse of Theorem 2.11.

THEOREM 2.21. *If $(Q_1, Q_2, Q_3)$ is a primitive solution of* (2.12) *then the equation*

$$-x = Q_1^{e_1}(t)/Q_3^{e_3}(t) \tag{2.21.1}$$

*defines a galois extension* $\mathbf{C}(t)$ *of* $\mathbf{C}(x)$ *of degree $M$.*

*Proof.* Consider

$$h(Y) = xQ_3^{e_3}(Y) + Q_1^{e_1}(Y)$$

as polynomial in $Y$ with coefficients in $\mathbf{C}(x)$. Clearly $t$ is a root and by the corollary if $t_1$ is another root then $t_1$ is homographic image of $t$. Thus $h$ splits in $\mathbf{C}(t)$. It is easy to check that $h$ is irreducible in $\mathbf{C}(x)[Y]$ and so deg $\mathbf{C}(t)/\mathbf{C}(x) = m$ as asserted.

Note: The extension $\mathbf{C}(t)/\mathbf{C}(x)$ is ramified only at 0, 1, $\infty$. This completes the demonstration that the finite subgroups of the group of homographies are uniquely determined up to inner automorphisms by the invariants $e_1, e_2, e_3$.

The existence of the indicated groups is demonstrated by writing down a primitive solution in each case.

For future use we note that these groups correspond to the case in which $s = 3$ in equation (2.1) and hence to the *non-cyclic* finite groups of homographies.

1. Dihedral group of order $2n$

$$(X_1^n + X_2^n)^2 - (X_1^n - X_2^n)^2 = 4(X_1X_2)^n$$

2. Tetrahedral

$$
\begin{aligned}
12\sqrt{-3}f^2 &= \phi_1^3 - \phi_2^3 \\
f &= X_1X_2(X_1^4 - X_2^4) \\
\phi_1 &= X_1^4 + 2\sqrt{-3}\,X_1^2X_2^2 + X_2^4 \\
\phi_2 &= X_1^4 - 2\sqrt{-3}\,X_1^2X_2^2 + X_2^4
\end{aligned}
$$

3. Octahedral

$$
\begin{aligned}
W^3 - K^2 &= 108f^4 \\
f &= X_1X_2(X_1^4 - X_2^4) \\
W &= X_1^8 + 14X_1^4X_2^4 + X_2^8 \\
K &= X_1^{12} - 33X_1^8X_2^4 - 33X_1^4X_2^8 + X_2^{12}
\end{aligned}
$$

4. Icosahedral

$$
\begin{aligned}
1728f^5 &= T^2 + H^3 \\
f &= X_1X_2(X_1^{10} + 11X_1^5X_2^5 - X_2^{10}) \\
H &= -(X_1^{20} + X_2^{20}) + 228(X_1^{15}X_2^5 - X_1^5X_2^{15}) \\
&\quad - 494X_1^{10}X_2^{10} \\
T &= X_1^{30} + X_2^{30} + 522(X_1^{25}X_2^5 - X_1^5X_2^{25}) \\
&\quad - 10005(X_1^{20}X_2^{10} - X_1^{10}X_2^{20})
\end{aligned}
$$

3. **Pullbacks on the gauss sphere.** We consider hypergeometric differential equations in normalized form: For $\lambda$, $\mu$, $\nu$ elements of $\mathbf{C}$, let

$$
L_{\lambda,\mu,\nu} = D^2 + \frac{A}{x^2} + \frac{B}{(x-1)^2} + \frac{C}{x(x-1)} \tag{3.1}
$$

with

$$
D = \frac{d}{dx}
$$

$$4A = 1 - \lambda^2$$
$$4B = 1 - \mu^2$$
$$4C = \lambda^2 + \mu^2 - \nu^2 - 1.$$

This is the unique second order differential equation with rational coefficients, singular points only at 0, 1, $\infty$, with constant wronskian and with exponent differences $\lambda$, $\mu$, $\nu$ at 0, 1, $\infty$ respectively.

If $(Q_1, Q_2, Q_3)$ is a primitive solution of (2.12) then by setting

$$-x = Q_1^{e_1}(t)/Q_3^{e_2}(t) \tag{3.2}$$

we obtain an extension $K_0 = \mathbf{C}(t)$ of $\mathbf{C}(x) = K$ such that the pullback of $L_{\lambda,\mu,\nu} = L$ in the sense of Section 1 has $e_1\lambda$, $e_2\mu$, $e_3\nu$ as exponent differences. In particular if

$$(\lambda, \mu, \nu) = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}\right)$$

then the pullback has only unity as exponent difference, and no logarithmic singularities and is Fuchsian. Hence the pullback has no singularities and hence is $d^2/dt^2$.
The function $t$ is a ratio of solutions of this equation and so this algebraic function of $x$ is a ratio of solutions of $L_{1/e_1,1/e_2,1/e_3}$. We summarize:

THEOREM 3.2.1. *If* $\sum_{i=1}^{3}(1/e_i) > 1$ *then all solutions of* $L_{1/e_1,1/e_2,1/e_3}$ *are algebraic and t given by* (3.2) *is a ratio of solutions and the group of* $\mathbf{C}(t, x)/\mathbf{C}(x)$ *is dihedral, tetrahedral, octahedral, icosahedral depending on the values of* $(e_1, e_2, e_3)$.

Let

$$L = \frac{d}{dt^2} + Q(t) \tag{3.3}$$

be a *normalized* second order linear differential operator defined over $\mathbf{C}(t)$ whose solutions are *locally algebraic.* Let $\tau$ be a ratio of solutions and let $G$ be the group of homographies corresponding to the action of monodromy upon $\tau$. As is well known the finiteness of $G$ is equivalent to the condition that all solutions of $L$ be algebraic functions. We will refer to $G$ as the *projectivized* monodromy group of $L$.

THEOREM 3.4. (Klein) *If $G$ is finite but not cyclic then $L$ is the pullback by a rational map of one of the hypergeometric operators $L_{1/e_1,1/e_2,1/e_3}$ with $\sum_{i=1}^3 (1/e_i) > 1$. Conversely such a pullback has a finite projectivized monodromy group.*

*Proof.* The converse follows directly from Theorem 3.2.1.

If $G$ is finite non-cyclic then by Theorem 2.11 there exists a primitive solution of (2.12) for suitable $e_1$, $e_2$, $e_3$ such that $Q_1^{e_1}/Q_3^{e_3}$ is invariant under $G$. Thus

$$-\xi(t) = \frac{Q_1^{e_1}}{Q_3^{e_3}}(\tau(t)) \tag{3.5}$$

is invariant under analytic continuation on the $t$-sphere, and hence is a rational function of $t$. Comparing this with equation (3.2) we see that $\tau(t)$ is a ratio of solutions of $L_{1/e_1,1/e_2,1/e_3}$ if we replace $x$ by $\xi(t)$. This completes the proof of the theorem.

Suppose now that the projectivized group of $L$ is finite and not cyclic. Hence according to the theorem $L$ is a pullback of some $L_{1/e_1,1/e_2,1/e_3}$ by a rational map. This map need not be unique. To see this let $L_0 = d^2/dt^2$ so that $L_0$ is pullback of $L_{1/e_1,1/e_2,1/e_3}$ ($\sum (1/e_i) > 1$) by means of equation (3.2). If we put

$$t' = \sigma t$$

where $\sigma$ is an arbitrary homography and put

$$-X_1 = \frac{Q_1^{e_1}}{Q_3^{e_3}}(t') \tag{3.6}$$

then $L_0$ is again pullback of $L_{1/e_1,1/e_2,1/e_3}$ by $X_1$. If $X_1$ were to equal $X$ for all $\sigma$, we could conclude that $X$ is a constant.

The point in this example is that $L_0$ has the trivial projectivized group while $L_{1/e_1,1/e_2,1/e_3}$ has a non-trivial group. In general if $L$ is a pullback of $L_{1/e_1,1/e_2,1/e_2}$ then the group of $L$ is a subgroup of that of $L_{1/e_1,1/e_2,1/e_3}$.

THEOREM 3.7. (Klein [12]). *If $L$ and $L_{1/e_1,1/e_2,1/e_3}$ have the same group and if $e_1 \neq e_2$ then the pullback mapping is unique.*

*Proof.* By hypothesis there exists $x \in \mathbf{C}(t)$ such that $\tau$, a ratio of solutions of $L$, satisfies

$$-x = \frac{Q_1{}^{e_1}}{Q_3{}^{e_3}}(\tau). \tag{3.7.1}$$

Field theoretically we have the lattice shown.

$$\begin{tikzcd}
 & \mathbf{C}(\tau, t) \\
\mathbf{C}(\tau) \arrow[ur, dash] \arrow[d, dash] & \mathbf{C}(t) \arrow[u, dash] \\
\mathbf{C}(\tau) \cap \mathbf{C}(t) \arrow[ur, dash] \arrow[d, dash] & \\
\mathbf{C}(x) &
\end{tikzcd}$$

The projectivized group $G$ of $L$ coincides with the galois group of $\mathbf{C}(\tau, t)/\mathbf{C}(t)$ and hence by the usual identification with the group of $\mathbf{C}(\tau)/(\mathbf{C}(\tau) \cap \mathbf{C}(t))$. On the other hand the group of $L_{1/e_1,1/e_2,1/e_3}$ coincides with the group of $\mathbf{C}(\tau)/\mathbf{C}(x)$. The hypothesis of equality of groups is equivalent to the assertion that

$$C(\tau) \cap \mathbf{C}(t) = \mathbf{C}(x). \tag{3.7.2}$$

This condition fixes $x$ up to homography. By an elementary calculation, $L$ as given by (3.3) is the pullback of

$$L_0 = \frac{d^2}{dx^2} + q(x)$$

under the mapping $x = x(t)$ if

$$Q(t) = q(x) \cdot x'^2 + \frac{1}{2}\left(\frac{x''}{x'}\right)' - \frac{1}{4}\left(\frac{x''}{x'}\right)^2, \quad \left(x' = \frac{dx}{dt}\right) \cdot \tag{3.7.3}$$

a formula which may also be written in terms of the Schwarzian derivative. The key point is that

$$\frac{1}{2}\left(\frac{x''}{x'}\right)' - \frac{1}{4}\left(\frac{x''}{x'}\right)^2 = \frac{1}{2}\{x, t\} \tag{3.7.4}$$

remains invariant under homographies

$$x \mapsto \frac{ax+b}{cx+d}, \qquad ad - bc = 1$$

Since $x$ is fixed up to homography, equation 3.7.3 implies that if $y = y(t)$ is also a pullback mapping then

$$q(x)x'^2 = q(y)y'^2 \tag{3.7.5}$$

$$y = \frac{ax+b}{cx+d} \tag{3.7.6}$$

and $q$ is given by (3.1) with $(\lambda, \mu, \nu) = (1/e_1, 1/e_2, 1/e_3)$. Note that $A$, $B$, $C$ in (3.1) are each *distinct* from zero. This shows that

$$\frac{A}{x^2} + \frac{B}{(x-1)^2} + \frac{C}{x(x-1)}$$

$$= \left[\frac{A}{y^2} + \frac{B}{(y-1)^2} + \frac{C}{y(y-1)}\right]\frac{1}{(cx+d)^4} \tag{3.7.7}$$

We consider four cases. Since $a$ and $c$ cannot simultaneously be zero we exhaust all possibilities.

*Case* 1 $c$, $a$, $a$-$c$ all distinct from zero.

The right side of (3.7.7) lies in $x^{-4}\,\mathbf{C}[[x^{-1}]]$. The corresponding Laurent series for the right side is

$$(A + B + C)x^{-2} + (2B + C)x^{-3} \bmod x^{-4}$$

This shows that

$$A + B + C = 0$$
$$2B + C = 0$$

and hence $A = B$ contrary to hypothesis.

*Case* 2 $a = 0, c \neq 0.$

Here we may let $c = 1$, $b = -1$. The right side of 3.7.7 is

$$\frac{A}{(x+d)^2} + \frac{B}{(x+1+d)^2(x+d)^2} + \frac{C}{(x+1+d)(x+d)^2}$$

Comparing poles we see that as sets $\{0, 1\} = \{-d, -d-1\}$. Hence $d = -1$ and now multiplying both sides of (3.7.7) by $x^2$ and setting $x = 0$ we obtain $A = B$ contrary to hypothesis.

*Case* 3 $c = 0, a \neq 0.$

We may set $a = 1 = d$. Proceeding as in the previous case we obtain $\{0, 1\} = \{-b, 1-b\}$ as sets. Hence $b = 0$ and therefore $y = x$ as asserted.

*Case* 4 $a = c \neq 0.$

Here we may let $a = c = 1$, $d = b + 1$. Proceeding as in Case 2, $\{0, 1\} = \{-b, -b-1\}$ as sets. Hence $b = -1$ and now multiplying both sides of (3.7.7) by $(x-1)^2$ and setting $x = 1$, we obtain $A = B$ contrary to hypothesis.

THEOREM 3.8 (Klein [12]) *In the notation of equation 3.7.1, let*

$$\tau = y_1/y_2$$

*with $y_1$, $y_2$ solutions of L, and let*

$$w = y_2y_1' - y_1y_2',$$

*(prime denotes differentiation with respect to t). Let $Q_3(y_1, y_2)$ denote the form of degree $M/e_3$,*

$$Q_3(y_1, y_2) = y_2^{M/e_3} Q_3(\tau)$$

Then

$$Q_3(y_1, y_2) = C\left[\frac{w}{x'} x^{1-(1/e_1)}(1 - x)^{1-(1/e_2)}\right]^{M/2e_3} \tag{3.8.1}$$

*for some constant C.*

*Proof.* Taking the derivative of both sides of equation (3.7.1) and applying (2.15.1) gives

$$x' = C\frac{w}{y_2^2}\frac{Q_1^{e_1-1}Q_2^{e_2-1}}{Q_3^{e_3+1}}$$

We multiply by $y_2^2 Q_3^{2e_3/M}/x'$. This gives

$$Q_3(y_1, y_2)^{2e_3/M} = \frac{Cw}{x'}\left(\frac{Q_1^{e_1}}{Q_3^{e_3}}\right)^{1-(1/e_1)}\left(\frac{Q_2^{e_2}}{Q_3^{e_3}}\right)^{1-(1/e_2)}$$

Equation (3.8.1) now follows from 2.15, 2.12.

4. **Cyclic Case.** Let

$$L = D^2 - Q, \qquad D = \frac{d}{dt} \tag{4.1}$$

be a given 2nd order differential operator with $Q \in \mathbf{C}(t)$. We again assume that all solutions of $L$ are locally algebraic. Our object is to decide in a finite number of steps whether the projectivized group is cyclic. While there are of course an infinite set of cyclic groups, in the present situation there is an a-priori bound for the order of the group. The group $G$ is generated in any case by the transformations corresponding to local monodromy around singular points. In general this gives no upper bound for the order of $G$ but if $G$ is cyclic then its order is bounded by the least common multiple of the denominators of the exponent differences. This will not be used in the following.

Let $\tau$ be a ratio of solutions of $L$. If $G$ is cyclic then replacing $\tau$ by some homographic image if necessary, we may suppose that the elements of $G$ are of the form

$$\tau \mapsto w\tau \tag{4.2}$$

where $w$ runs through the $m$th roots of unity in $\mathbf{C}$. Hence $\tau^m$ is invariant under the monodromy group and thus an element of $\mathbf{C}(t)$, say

$$\tau^m = \xi \in \mathbf{C}(t). \tag{4.3}$$

Then by a classical calculation two independent solutions of $L$ are given by

$$\begin{aligned} u &= \sqrt[2]{1/(\xi^{1/m})}, \\ v &= u\xi^{1/m}. \end{aligned} \tag{4.4}$$

This shows that $u'/u$ and $v'/v$ lie in $\mathbf{C}(t)$.

THEOREM 4.4. *Let $L$ as given by* (4.1) *have solutions which are locally algebraic. Then the projectivized group of $L$ is cyclic if and only if the Riccati equation*

$$y' + y^2 = Q \tag{4.5}$$

*has two distinct rational solutions.*

*Proof.* The necessity of this condition has been explained. Conversely if $y$ is a rational solution of (4.5) then $y$ can have only simple poles since by hypothesis the singularities of $L$ are all regular which means that the poles of $Q$ are at most of order two. If we write

$$Q = \sum_{i=1}^{n} \frac{A_i}{(t-\alpha_i)^2} + \frac{B_i}{(t-\alpha_i)} \tag{4.6}$$

then we must have

$$\sum_{i=1}^{n} B_i = 0 \tag{4.7.1}$$

$$-4A_i = 1 - \lambda_i^2 \tag{4.7.2}$$

where $\lambda_i$ is the exponent difference at $\alpha_i$ ($i = 1, \ldots, n$)

$$1 - \lambda_\infty^2 = -4 \sum_{i=1}^{n} (A_i + \alpha_i B_i). \tag{4.7.3}$$

We may write a rational solution of (4.5) in the form

$$\eta = \sum_{i=1}^{n} \frac{c_i}{t - \alpha_i} + \sum_{j=1}^{N} \frac{1}{t - \theta_j} \tag{4.8}$$

where $\theta_1, \ldots, \theta_N$ are all distinct and are distinct from the singularities of $L$. Indeed equation (4.5) implies that

$$L = (D + \eta)(D - \eta) \tag{4.9}$$

and so each exponent of $D - \eta$ must be an exponent of $L$ which shows that

$$c_i = \frac{1 \pm \lambda_i}{2} \qquad i = 1, \ldots, n \tag{4.10}$$

and these numbers are rational by hypothesis. Now corresponding to the rational solution of (4.5) given by (4.8), there is an algebraic solution of $L$ given by

$$y = \prod_{j=1}^{N} (t - \theta_j) \cdot \prod_{i=1}^{n} (t - \alpha_i)^{c_i} \tag{4.11}$$

If equation (4.5) has two distinct solutions, then the ratio, $\tau$, of the corresponding solutions of $L$ would be the ratio of two functions of the form given by 4.11 and hence $\tau^m \in \mathbf{C}(t)$ for suitable integer $m$. This completes the proof of the Theorem.

COROLLARY 4.12. *Each rational solution of* (4.5) *is of the form of equation* (4.8) *where* $(c_1, \ldots, c_n)$ *satisfies* (4.10) *and*

$$N + \sum_{i=1}^{n} c_i + \frac{-1 \pm \lambda_\infty}{2} = 0.$$

*Proof.* We need only check the estimate for $N$. As in the proof of the theorem, the exponent of $D - \eta$ at infinity must also be an exponent of $L$ at infinity. The corollary follows directly.

What is the set of all rational solutions of the Riccati equation?

THEOREM 4.13. *If the Riccati equation* (4.5) *has more than two solutions in* $\mathbf{C}(t)$ *then the projectivized group of* $L$ *is trivial,* (*i.e. all the exponent differences are integers*).

*Proof.* Let $\eta_1, \eta_2, \eta_3$ be solutions of equation (4.5) and let $u_i$ be solution of $L$, such that

$$\eta_i = u_i'/u_i \qquad (i = 1, 2, 3)$$

Then

$$\eta_i - \eta_j = w_{i,j}/u_i u_j$$

where $w_{i,j}$ is the associated wronskian of $L$. Let $\tau = u_2/u_1$, so

$$\tau = \frac{1}{u_1 u_2} \Big/ \frac{1}{u_2 u_3} = \text{const.} \cdot \eta_1 - \eta_3/\eta_2 - \eta_3.$$

Thus $\tau$ lies in $\mathbf{C}(\eta_1, \eta_2, \eta_3)$. This completes the proof.

5. **Dihedral case. Part I.** Let $L$ be as in equation (4.1). We now wish to determine whether the projectivized group is dihedral.

THEOREM 5.1. *If the projectivized group of* $L$ *is dihedral then the Riccati equation* (4.5) *has a solution in a quadratic extension of* $\mathbf{C}(t)$.

*Proof.* The conventional hypergeometric differential operator

$$x(1 - x)D^2 + (\gamma - (\alpha + \beta + 1)x)D - \alpha\beta \tag{5.1.1}$$

has exponent differences $1 - \gamma$, $\gamma - \alpha - \beta$, $\alpha - \beta$ at $0, 1, \infty$ and no other singularities. Putting

$$\alpha = \frac{1 - \mu}{2}, \qquad \beta = -\frac{\mu}{2}, \qquad \gamma = \frac{1}{2}$$

we obtain a differential equation with exponent differences $\tfrac{1}{2}$, $\mu$, $\tfrac{1}{2}$

which by explicit calculations (going back to Gauss) has $(1 + \sqrt{x})^{\mu}$ and $(1 - \sqrt{x})^{\mu}$ as solutions. If we replace $x$ by $x(t)$ then the normalized pull-back is satisfied by $(1 \pm \sqrt{x})^{\mu}/\sqrt{w}$ where $w = (1 - x)^{\mu-1}(\sqrt{x})'$. From this the Riccati equation has solution

$$\eta = \frac{\mu}{2}\frac{x'}{x(1-x)}\sqrt{x} - \frac{1}{4}\frac{R'}{R} \tag{5.2}$$

where

$$\sqrt{R} = \frac{\mu}{2}\frac{x'}{x(1-x)}\sqrt{x}.$$

Now by Theorem 3.2, if $L$ has a dihedral group of order $2m$ then $L$ is the pullback of $L_{1/2,1/m,1/2}$ by a rational map $x = x(t)$. This completes the proof of the theorem.

We now study the existence of solutions of equations (4.5) in quadratic extensions.

LEMMA 5.3 (Fuchs). *Let $L_2$ be the third order linear differential equation satisfied by all binary quadratic forms in $y_1$, $y_2$, a pair of independent solutions of $L$. Suppose that the Ricatti equation* (4.5) *has no solution in* $\mathbf{C}(t)$. *Then equation* (4.5) *has a solution in a quadratic extension field of* $\mathbf{C}(t)$ *if and only if $L_2$ has a solution whose square lies in* $\mathbf{C}(t)$.

*Proof.* If equation (4.5) has a solution $\eta_1$ in a quadratic extension field, then we may write

$$\eta_1 = \gamma + \sqrt{R} \tag{5.3.1}$$

with $\gamma$ and $R$ in $\mathbf{C}(t)$. By hypothesis $\sqrt{R} \notin \mathbf{C}(t)$ and hence $\eta_2 = \gamma - \sqrt{R}$ is a distinct solution of (4.5). We may choose $y_i$ solution of $L$ $(i = 1, 2)$ such that

$$\eta_i = y_1'/y_i \tag{5.3.2}$$

and then

$$2\sqrt{R} = \eta_1 - \eta_2 = w/y_1y_2 \tag{5.3.3}$$

where $w$ is the wronskian of $L$. Since $w$ is a constant, we conclude that $1/\sqrt{R}$ is a constant multiple of $y_1y_2$ and hence satisfies $L_2$. This proves the assertion is one direction.

Conversely if $z \in \mathbf{C}(t)$ and $\sqrt{z}$ is a solution of $L_2$ then $\sqrt{z}$ is a quadratic form in solutions of $L$. Since such a quadratic form may be factored there are two possibilities:

$$\sqrt{z} = u_1{}^2 \tag{5.3.4}$$

$$\sqrt{z} = u_1u_2 \tag{5.3.5}$$

where $u_1$, $u_2$ are independent solutions of $L$. The first case is ruled out by the hypothesis that equation 4.5 has no solution $\frac{1}{4}\, z'/z \in \mathbf{C}(t)$. We now put $\eta_i = u_i'/u_i$ $(i = 1, 2)$ and calculate from (5.3.5),

$$\eta_1 + \eta_2 = \frac{1}{2}\frac{z'}{z}, \tag{5.3.6}$$

while precisely as 5.3.3

$$\eta_1 - \eta_2 = w/u_1u_2 = w/\sqrt{z} \tag{5.3.3$'$}$$

where again $w$ is a determination of the wronskian of $L$. Equations (5.3.3′), (5.3.6) show that (4.5) has a solution in a quadratic extension of $\mathbf{C}(t)$. This completes the proof of the lemma.

We now briefly discuss the existence of a solution $z$ of $L_2$ whose square lies in $\mathbf{C}(t)$. Such a solution must be of the form

$$t = \prod_{i=1}^{n} (t - \alpha_i)^{k_i} \circ g \tag{5.4}$$

where $\alpha_1, \ldots, \alpha_n$ are the finite singularities of $L$, $g$ is a polynomial different from zero at each $\alpha_i$. Each $k_i$ must be a half integer with

$$k_i = 1 \pm \lambda_i, 1 \tag{5.5}$$

while

$$\Sigma\, k_i + \deg g = 1, 1 \pm \lambda_\infty. \tag{5.6}$$

This gives a finite set of possibilities for the $k_i$ and the degree of $g$ and the determination of $g$ is reduced to a problem in linear algebra.

6. **Dihedral Case. Part II.** Let $L$ be a 2nd order differential operator as in equation (4.1). In Section 5 we showed that if $L$ has the dihedral group then $L$ decomposes into linear factors in a quadratic extension of $\mathbf{C}(t)$. Such decompositions were studied in that section. To determine whether the group is indeed dihedral we must now be prepared to answer the following question: Given $\eta$ algebraic over $\mathbf{C}(t)$ does the equation

$$y' = y\eta \tag{6.1}$$

have a solution algebraic over $\mathbf{C}(t)$? (For our application we may assume $\eta$ is in a quadratic extension of $\mathbf{C}(t)$). This is equivalent to the condition that there exist $m \in \mathbf{N}$ such that $m\eta$ is logarithmic derivative of element of $\mathbf{C}(t, \eta)$. The difficulty is that of finding an a-priori bound for $m$.

In the classical theory of hypergeometric functions this problem did not arise, as, for $n = 2$, the field $\mathbf{C}(t, \eta)$ is either $\mathbf{C}(\sqrt{t})$, $\mathbf{C}(\sqrt{t-1})$ or $\mathbf{C}(t, \sqrt{t(t-1)})$ and so in all cases of genus zero and hence equation (6.1) may be analyzed without difficulty.

We now consider a generalization of the above problem.

Let $\omega$ be a differential with at most simple poles and with rational residues at each pole on a curve $C$ defined over a field $K$ of characteristic zero. To determine in a finite number of steps whether $m\omega$ is a logarithmic differential for some (unknown) $m \in \mathbf{Z}$.

By solving this question we solve the problem of Section 5. As general references for this discussion see [13, 16]. At each point $P$ of $C$ we write

$$\omega = n_p \frac{dt_p}{t_p} + \omega_p. \tag{6.2}$$

where $t_p$ is a local parameter at $P$, $\omega_p$ is regular at $t_p$ and $n_p$ is a rational number which we may assume to be in $\mathbf{Z}$. Let

$$\mathfrak{L} = \Sigma\, n_p P \tag{6.3}$$

a divisor of $C$ of degree zero.

There are two steps

I. Find an integer $m$ such that either $m\mathfrak{L}$ is principal or $\mathfrak{L}$ is of infinite order in the Jacobian of $C$.

II. Decide whether $m\mathfrak{L}$ is principal and if it is find $\theta$ in the function field of $C$ such that

$$(\theta) = m\mathfrak{L}$$

Suppose I and II carried out, then put

$$\omega_1 = \frac{d\theta}{\theta} - m\omega. \tag{6.4}$$

We note that $\omega_1$ is a differential of the first kind. If $\omega_1 = 0$ then the problem has an affirmative solution. If $\omega_1 \neq 0$ then $N\omega_1$ is never logarithmic regardless of $N \in \mathbf{N}$ and hence the problem has a negative solution.

I. We choose a non-singular model for $C$ in $\mathbf{P}^3$. Let $K$ be a field of definition of both $C$ and of the divisor $\mathfrak{L}$.

*Case 1.* $K$ is an algebraic number field.

Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be primes of $K$ extending distinct rational primes $p_1$, $p_2$ such that the reductions $\overline{C_1}$, $\overline{C_2}$ of $C$ are non-singular. Let $G_j$ $(j = 1, 2)$ be the group of points on the Jacobian $J(\overline{C_j})$ of the reduced curve $\overline{C_j}$ which are rational over the residue class field $\overline{K_j}$ and which are of order prime to $p_j$. Let $e_j$ be the exponent of $G_j$. Thus $(e_j, p_j) = 1$ and

$$e_j G_j = 0. \tag{6.5}$$

LEMMA 6.6. *If the image of $\mathfrak{L}$ in $J(C)$ is of finite order then*

$$e_1 e_2 \mathfrak{L} \sim 0.$$

*Proof.* If $\mathfrak{L}$ is of finite order then there exists $p_1{}^i m$, $(p_1, m) = 1$ such that

$$p_1{}^i m\mathfrak{L} \sim 0. \tag{6.6.1}$$

Hence $p_1{}^i\mathfrak{L}$ is of order prime to $p_1$. But the mapping

$$J(C)(K) \to J(\overline{C}_1)(\overline{K}_1)$$

of the group of $K$-rational points of $J(C)$ into the group of $\overline{K}_1$-rational points of $J(\overline{C}_1)$, is injective on the subgroup of elements of order prime to $p_1$. The image of $p_1{}^i\mathfrak{L}$ is rational over $\overline{K}_1$ and of order prime to $p_1$, hence is annihilated by $e_1$. Thus

$$e_1 p_1{}^i \mathfrak{L} \sim 0 \tag{6.6.2}$$

likewise

$$e_2 p_2{}^j \mathfrak{L} \sim 0$$

for some unknown $j$. We choose $a$, $b \in \mathbf{Z}$ such that

$$a p_1{}^i + b p_2{}^j = 1$$

and conclude that

$$e_1 e_2 \mathfrak{L} = e_1 e_2 (a p_1{}^i \mathfrak{L} + b p_2{}^j \mathfrak{L}) \sim 0.$$

This completes the proof of the lemma.

*Case 2.* $K$ is finitely generated over $\mathbf{Q}$.

We choose a specialization $K \to K'$ such that $K'$ is an algebraic number field and $C \to C'$, a non-singular curve. Since

$$J(C)(K) \xrightarrow[\text{tors}]{} J(C')(K')_{\text{tors}}$$

is an injection, we may repeat case 1. This completes the treatment of problem I.

II. We choose a plane curve model for $C$ with only ordinary singular points $P_1, \ldots, P_n$ the multiplicity of $P_j$ being $s_j$. Let $D = \sum_{j=1}^{n} s_j P_j$. Put

$$L = L_0 - L_\infty$$

$L_0$ and $L_\infty$ being positive divisors.

Construct a curve $B$ having intersection with $C$ containing $D + L_0$. Write

$$C \cap B = D + L_0 + X,$$

where $X$ is a positive divisor. Let $l =$ degree $B$. Then

$$L_0 \sim L_\infty$$

if and only if there exists a curve $B'$ of degree $l$ such that

$$B' \cap C = L_\infty + D + X.$$

This construction involves only linear algebra and the calculation of $X$. If $L_0 \sim L_\infty$ then $L_0 - L_\infty = (B/B')$. The solution of part II involves the application of this procedure to $m\mathfrak{L}$.

7. **Decision Procedure.** Let $L$ be a second order differential operator defined over $\mathbf{C}(t)$. We may suppose $L$ is given by (3.3). Our object is to give a decision procedure for determining whether all the solutions are algebraic.

We may assume that all singularities of $L$ are regular, and that all exponents are rational numbers. Our object is to determine whether the projective monodromy group $G$ of $L$ is one of the five finite types. We apply the procedure of Section 4 to determine whether $G$ is cyclic. If it is we are done. Conversely if we exclude the possibility that $G$ is cyclic, we then use the procedure of Section 5, Section 6 to determine whether $G$ is dihedral. Having decided that $G$ is neither cyclic nor dihedral we consider successively the three remaining finite possibilities, i.e. $G$ may be the tetrahedral, octahedral or icosahedral group. The reason for proceeding in this sequence is we wish to use the uniqueness theorem of Klein (3.7).

We now consider whether $L$ is the pullback by $x = x(t)$ of $L_{1/e_1, 1/e_2, 1/e_3}$ (cf. equation 3.1) where

$$\left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}\right) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$$

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$$

and where we may assume that we have already excluded previous elements in this list of three possibilities.

We are to decide whether there exists a rational function $x$ which satisfies (3.7.3). Here $q(x)$ is given explicitly by (3.1), i.e.

$$4q(x) = \frac{1 - \dfrac{1}{e_1{}^2}}{x^2} + \frac{1 - \dfrac{1}{e_2{}^2}}{(x-1)^2} + \frac{\dfrac{1}{e_1{}^2} + \dfrac{1}{e_2{}^2} - \dfrac{1}{e_3{}^2} - 1}{x(x-1)}$$

Let $a_1, \ldots, a_n, \infty$ be the singularities of $L$. Let the exponent differences be $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_\infty$. We know that these numbers are rational and we take them to be *positive.* We define

$$\Delta(L) = (\lambda_1 + \cdots + \lambda_n + \lambda_\infty) - (n - 1)$$

Thus

$$\Delta(L_{1/e_1, 1/e_2, 1/e_3}) = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1.$$

It follows from Section 1 that if $L$ is a pullback of $L_{1/e_1, 1/e_2, 1/e_3}$ by $x = x(t)$ then the mapping degree (i.e. the maximum of the degrees of numerator and denominator) must be

$$M = \Delta(L)/\Delta(L_0).$$

We normalize by insisting that the denominator is monic and we interpret equation 3.7.3 as a set of algebraic relations among the at most $2M + 1$ unknown coefficients of the numerator and denominator of $x$ (we must consider $M + 1$ possibilities as the exact degree of the denominator is not fixed). We thus arrive at $M + 1$ algebraic sets which have in all, at most one rational point. By elimination theory we should not only be able to decide whether a rational solution of 3.7.3 exists but we should indeed be able to determine the solution itself.

8. **Conjecture of Grothendieck.** Let $L$ again be a second order differential operator but for simplicity we suppose that the coefficients lie in $\mathbf{Q}(t)$. For almost all $p$ we may consider $L_p$, the reduction of $L$ modulo $p$ on $\mathbf{F}_p(t)$. Clearly $L_p$ acts as linear operator on $\mathbf{F}_p(t)$ as vector space over $\mathbf{F}_p(t^p)$. Let $K_p$ be the dimension of the kernel of $L_p$ viewed as vector space over $\mathbf{F}_p(t^p)$.

As a special case of a general conjecture of Grothendieck, it is conjectured:

(8.1) If $K_p = 2$ for almost all $p$ then all the solutions of $L$ are algebraic functions.

The converse of this statement is known.

The general conjecture has been verified by Katz [8] for the case in which $L$ is a suitable direct factor of the Fuchs-Picard equation. In particular it is known to be valid in the case of second order operators with 3 singular points. The conjecture is completely open in the case of Heun's differential equation [17, p. 576] (second order, four singular points), the point being that in the case of 3 singular points, the integral formula of Euler reveals the cohomological nature of the gauss hypergeometric functions but no integral formula is known for the solutions of the Heun differential equation.

For the case of 3 singular points there is a simple algorithm for the calculation of $K_p$. For 4 or more singular points no such algorithm is known. Thus even if Grothendieck's conjecture were confirmed, it is not clear that it would give a response to problem (0.2).

It may be useful to explain briefly how one could use Katz's result to verify Schwarz's list. We know from Ihara that the differential equation for $F(a, b; c; x)$ has $K_p = 2$ if and only if the minimal representative mod $p$ of $c$ lies between the corresponding representatives of $a$ and of $b$. Using the criterion of Grothendieck it is easy to check that each equation in Schwarz's list has only algebraic solutions.

To show that Schwarz's list is exhaustive one must start with the fact that the list consists of *classes* of exponent differences. Two sets $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1', \lambda_2', \lambda_3')$ of exponent differences are said to be equivalent [14, p. 119] if

$$\lambda_1' = \epsilon_1 \lambda_i + \mu_i$$

where $\epsilon_i = \pm 1$, each $\mu_i \in \mathbf{Z}$ and $\mu_1 + \mu_2 + \mu_3 \equiv 0 \bmod 2$. The central point is that if the sets of exponent differences are equivalent then the monodromy groups are isomorphic. This can be shown either by Riemann's method of calculating the monodromy group or by Gauss's relations among contiguous hypergeometric functions [5, Section 34].

Consequently we may assume that $(\lambda_1, \lambda_2, \lambda_3)$ form a reduced set of exponent differences, i.e.

$$i \geq \lambda_1 > 0 \tag{8.2}$$

$$1 \geq \lambda_1 + \lambda_j \quad \text{for} \quad i \neq j. \tag{8.3}$$

We follow Schwarz's theory of curvilinear triangles to show that if all solutions are algebraic then

$$\sum_{i=1}^{3} \lambda_i > 1. \tag{8.4}$$

We now use Section 4 to show that the cyclic case can only occur with exponent differences $(1/n,\ 1,\ 1/n)$, and the method of Section 5 to show that the dihedral monodromy group can occur only if the exponent differences are $1/2$, $1/2$, $1/m$. This leaves only the tetrahedral, octahedral and icosahedral groups. The elements of these groups have orders 2, 3, 4, 5, and hence the $\lambda_i$ have these integers as denominators. This condition together with (8.2) shows that there are only a finite number of possibilities; use of (8.3) and (8.4) further reduces the list. Furthermore orders 4 and 5 do not occur simultaneously. By these considerations we arrive at the Schwarz list together with the additional candidates:

| Exponent differences | Spherical excess |
|---|---|
| $\frac{1}{2}, \frac{2}{5}, \frac{2}{5}$ | $\frac{3}{10}$ |
| $\frac{2}{3}, \frac{1}{3}, \frac{1}{4}$ | $\frac{1}{4}$ |
| $\frac{2}{5}, \frac{2}{5}, \frac{1}{3}$ | $\frac{2}{15}$ |
| $\frac{2}{3}, \frac{2}{5}, \frac{1}{5}$ | $\frac{4}{15}$ |
| $\frac{3}{5}, \frac{2}{5}, \frac{1}{5}$ | $\frac{1}{5}$ |
| $\frac{3}{5}, \frac{2}{5}, \frac{2}{5}$ | $\frac{2}{5}$ |
| $\frac{3}{5}, \frac{1}{3}, \frac{1}{3}$ | $\frac{4}{15}$ |

These are excluded by means of Grothendieck's criterion, i.e. by the converse of (8.1).

It is not clear that this mixture of methods of Schwarz and Katz is any improvement over the original method of Schwarz. We note that for the case of 4 singular points there is no theory of reduced set of exponent differences. In the case of 4 singular points, the monodromy group depends not only on the exponential differences and singular points but also upon an additional parameter. One does not know how to infer isomorphism of monodromy groups from relations concerning the parameters [1, p. 311-329].

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