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SOME ENUMERATIVE RESULTS IN THE THEORY OF FORMS

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In a recent note* I attempted to obtain the postulation formula for the Grassmannian of k-spaces in $[n]$ by the consideration of forms of a certain type in $k+1$ sets of $r+1$ homogeneous variables, which I called k-connexes. My attempt was not entirely successful; I obtained a formula for k-connexes which suggested what the required postulation formula should be, but was unable to prove it. D. E. Littlewood† has now written a paper to show that my problem is intimately connected with the theory of invariant matrices, and has thereby established the truth of the postulation formula which I had conjectured. Littlewood's proof requires a considerable knowledge of the theory of invariant matrices, and this paper results from an attempt to re-write his proof in a form which is intelligible to a student not having this specialized knowledge. Prof. H. W. Turnbull has pointed out to me the importance of the so-called k-connexes in the theory of forms, particularly in connexion with the Gordan-Capelli series, and for this reason I am taking the k-connexes as the principal topic of this paper, leaving the deduction of certain postulation formulae which are the more immediate concern of a geometer to the end.

1. Let

$$\begin{pmatrix} x_0^0 & \dots & x_r^0 \\ x_0^1 & \dots & x_r^1 \\ \vdots & & \vdots \\ x_0^k & \dots & x_r^k \end{pmatrix} \tag{1}$$

be a matrix of $k+1$ rows and $r+1$ columns, whose elements are indeterminates. A k-connex of type $(l_0, \dots, l_k)$ is a polynomial in the elements x_j^i which is homogeneous of degree l_i in $(x_0^i, \dots, x_r^i)$ $(i = 0, \dots, k)$, and which can be written in the form

$$f(x_\alpha^0, |\, x_\alpha^0 x_\beta^1 \,|, \dots, |\, x_{\alpha_0}^0 \dots x_{\alpha_k}^k \,|),$$

that is, a homogeneous polynomial of degree $\lambda_i = l_i - l_{i+1}$ $(\lambda_k = l_k)$ in the $\binom{r+1}{i+1}$ determinants $|\, x_{\alpha_0}^0 \dots x_{\alpha_i}^i \,|$ formed from the first $i+1$ rows of the matrix (1), for $i = 0, \dots, k$.

If $k = 0$, a k-connex is simply a form of degree l_0 in $(x_0^0, \dots, x_r^0)$, and the number of independent 0-connexes of type l_0 is just the number of linearly independent forms of degree l_0, that is $\binom{l_0+r}{r}$. A base for these connexes is therefore provided by the different power products such as $(x_0^0)^{\rho_0} \dots\dots (x_r^0)^{\rho_r}$, where $\Sigma \rho_i = l_0$. But if $k > 0$, the various power products such as $(x_0^0)^{\rho_0} \dots (x_r^0)^{\rho_r} \,|\, x_0^0 x_1^1 \,|^{\rho_{0,1}} \dots |\, x_{r-1}^0 x_r^1 \,|^{\rho_{r-1,r}} \dots |\, x_{r-k}^0 \dots\dots x_r^k \,|^{\rho_{r-k,\dots,r}}$, where

$$\Sigma \rho_{i_0 \dots i_s} = \lambda_s,$$

are not linearly independent; for instance, if $k = 1$, $\lambda_0 = \lambda_1 = 1$, $r = 2$, the three power products

$$x_0^0 \,|\, x_1^0 x_2^1 \,|, \quad x_1^0 \,|\, x_2^0 x_0^1 \,|, \quad x_2^0 \,|\, x_0^0 x_1^1 \,|$$

* W. V. D. Hodge, *Proc. Cambridge Phil. Soc.* 38 (1942), 129.

† D. E. Littlewood. *Proc. Cambridge Phil. Soc.* 38 (1942), 394.

are linearly related, their sum being zero. The main object of this paper is to find a set of power products, for k-connexes of type $(l_0, \ldots, l_k)$, which are linearly independent and which have the property that any k-connex of the given type can be expressed as a linear combination of them; that is, the power products form a 'fundamental base' for the k-connexes of type $(l_0, \ldots, l_k)$.

With a view to the geometrical applications to be made later, it is desirable to make a slight generalization of our problem. It will be observed that the variables x_i^1 $(i = 0, \ldots, r)$ occur only in determinants in which the variables x_i^0 appear, that the variables x_i^2 appear only in determinants in which both x_i^0 and x_i^1 appear, and so on. Therefore the problem is unaltered if we replace the second row of the matrix (1) by a linear combination of the first two rows, so chosen that $x_0^1 = 0$, the third row by a linear combination of the first three rows so chosen that $x_0^2 = x_1^2 = 0$, and so on. This leads us to consider a generalization of the k-connexes, in which the determinants, instead of being constructed from the matrix (1), are constructed from the matrix

$$\begin{pmatrix} x_0^0 & x_1^0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & x_r^0 \\ 0 & 0 & \ldots & 0 & x_{\alpha_{k-1}}^1 & \ldots & \ldots & \ldots & x_r^1 \\ \vdots & & & & & & & & \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & x_{\alpha_0}^k & \ldots & x_r^k \end{pmatrix}, \tag{2}$$

where $\alpha_0 > \alpha_1 > \ldots > \alpha_k = 0$ are any integers, chosen once and for all, which satisfy the inequalities
$$k - i \leqslant \alpha_i \leqslant r - i \quad (i = 0, \ldots, k-1).$$
The consideration of such connexes is suggested by geometrical considerations, some of which will be explained later. It is important to notice that if any determinant

$$\begin{vmatrix} x_{i_0}^0 & \ldots & x_{i_s}^0 \\ \vdots & & \\ x_{i_0}^s & \ldots & x_{i_s}^s \end{vmatrix} \quad (i_0 < i_1 < \ldots < i_s)$$

is not zero, then the determinant obtained by missing out the last row and column is also not zero. Indeed, we see that in the above determinant the zero elements in any column are at the bottom of the column. If ρ_j is the number of zeros in the column with suffixes i_j, we have
$$\rho_0 \geqslant \rho_1 \geqslant \ldots \geqslant \rho_s,$$
and a necessary and sufficient condition that the determinant does not vanish is
$$\rho_j \leqslant s - j \quad (j = 0, \ldots, s).$$
Since the corresponding numbers ρ_j' $(j = 0, \ldots, s-1)$ for the determinant

$$\begin{vmatrix} x_{i_0}^0 & \ldots & x_{i_{s-1}}^0 \\ \vdots & & \\ x_{i_0}^{s-1} & \ldots & x_{i_{s-1}}^{s-1} \end{vmatrix}$$

are given by $\qquad \rho_j' = \rho_j - 1, \quad \text{when} \quad \rho_j > 0, \qquad \rho_j' = 0, \quad \text{when} \quad \rho_j = 0,$

we have
$$\rho_j' \leqslant s - 1 - j \quad (j = 0, \ldots, s-1),$$
and hence this new determinant does not vanish.

2. *Let us consider any k-connex which is a power product of type $(l_0, \ldots, l_k)$.* We arrange the factors so that the determinants of order k come first, then the determinants of order $k-1$, and so on. Now represent each factor by a column of entries, the numbers in the column being the suffixes of the columns in the corresponding deter-

minant. Each column begins on the same upper level, and the order of the columns is the order of the factors. There results a *tableau* in which the elements are the numbers $0, \ldots, r$, and there are $k+1$ rows, the first having l_0 numbers, the second l_1 numbers, and so on. Thus corresponding to the power product

$$|x_0^0 x_1^1 x_2^2|\,|x_1^0 x_3^1|\,|x_1^0 x_3^1|\,x_4^0$$

we have the tableau

$$\begin{array}{cccc} 0 & 1 & 1 & 4 \\ 1 & 3 & 3 & \\ 2 & & & \end{array}$$

Corresponding to a given power product we get different tableaux by rearranging the order of determinants of the same order, or by re-arranging the suffices in any determinant. In general, we shall find it convenient to arrange the suffixes so that the numbers increase as we go down a column. Conversely, to any tableau of $k+1$ rows, having l_0 numbers in the first row, l_1 in the second, etc., we get a power product.

A tableau is said to be *standard* if the numbers in any column increase (strictly) as we go down the column, and the numbers in any row do not decrease as we go along the row. Thus if $k = 1$, $r = 2$, $l_0 = 2$, $l_1 = 1$, we get power products corresponding to the tableaux

$$\begin{pmatrix} 0 & 0 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 \\ 2 & \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & \end{pmatrix},$$

of which all but the last are standard. The power products corresponding to standard tableaux are called *standard power products*. We shall prove that the standard power products of type $(l_0, \ldots, l_k)$ form a fundamental base for k-connexes of this type*.

3. We first show that any k-connex can be expressed as a sum of standard products. Let

$$p_{i_0\ldots i_s} = \begin{vmatrix} x^0_{i_0} & \ldots & x^0_{i_s} \\ \vdots & & \\ x^s_{i_0} & \ldots & x^s_{i_s} \end{vmatrix}.$$

Then, if $s \leqslant t$,

$$\sum_{\rho=0}^{t+1} (-1)^\rho p_{i_0\ldots i_{s-1} j_\rho} p_{j_0\ldots j_{\rho-1} j_{\rho+1}\ldots j_{t+1}}$$

$$= \sum_{\rho=0}^{t+1} \sum_{\sigma=0}^{s} (-1)^{\rho+\sigma+s} \begin{vmatrix} x^0_{i_0} & \ldots & x^0_{i_{s-1}} \\ \vdots & & \vdots \\ x^{\sigma-1}_{i_0} & & x^{\sigma-1}_{i_{s-1}} \\ x^{\sigma+1}_{i_0} & & x^{\sigma+1}_{i_{s+1}} \\ \vdots & & \vdots \\ x^s_{i_0} & \ldots & x^s_{i_s} \end{vmatrix} x^\sigma_{j_\rho} \begin{vmatrix} x^0_{j_0} & \ldots & x^0_{j_{\rho-1}} & x^0_{j_{\rho+1}} & \ldots & x^0_{j_{t+1}} \\ \vdots & & & & & \vdots \\ x^t_{j_0} & \ldots & x^t_{j_{\rho-1}} & x^t_{j_{\rho+1}} & \ldots & x^t_{j_{t+1}} \end{vmatrix}$$

$$= \sum_{\sigma=0}^{s} (-1)^{\sigma+s} \begin{vmatrix} x^0_{i_0} & \ldots & x^0_{i_{s-1}} \\ \vdots & & \vdots \\ x^{\sigma-1}_{i_0} & \ldots & x^{\sigma-1}_{i_{s-1}} \\ x^{\sigma+1}_{i_0} & \ldots & x^{\sigma+1}_{i_{s-1}} \\ \vdots & & \vdots \\ x^s_{i_0} & \ldots & x^s_{i_{s-1}} \end{vmatrix} \begin{vmatrix} x^\sigma_{j_0} & \ldots & x^\sigma_{j_{t+1}} \\ x^0_{j_0} & \ldots & x^0_{j_{t+1}} \\ \vdots & & \vdots \\ x^t_{j_0} & \ldots & x^t_{j_{t+1}} \end{vmatrix} = 0. \tag{3}$$

* The idea of standard tableaux and of standard power products is due to A. Young, *Proc. London Math. Soc.* (2), 28 (1927), 255–292.

From this we can deduce a more general form of relation*. We use the generalized Kronecker $\delta^{j_0 \dots j_{t+1}}_{i_0 \dots i_{t+1}}$, and the summation convention. We show that, if $s \leqslant t$,

$$\delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_1 \dots i_\lambda l_0 \dots l_{s-\lambda}} p_{l_{s-\lambda+1} \dots l_{t+1} k_1 \dots k_{s-\lambda}} = 0. \tag{4}$$

When $\lambda = s$ equation (4) is the same as (3). Now, by (3),

$$\begin{aligned}
&\delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_1 \dots i_\lambda l_0 \dots l_{s-\lambda}} p_{l_{s-\lambda+1} \dots l_{t+1} k_1 \dots k_{s-\lambda}} \\
&\quad = \sum_{\mu=1}^{t-s+\lambda+1} \delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_0 \dots l_{s-\lambda-1} l_{s-\lambda+\mu}} p_{l_{s-\lambda+1} \dots l_{s-\lambda+\mu-1} l_{s-\lambda} l_{s-\lambda+\mu+1} \dots k_{s-\lambda}} \\
&\quad\quad + \sum_{\mu=1}^{s-\lambda} \delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_0 \dots l_{s-\lambda-1} k_\mu} p_{l_{s-\lambda+1} \dots k_{\mu-1} l_{s-\lambda} k_{\mu+1} \dots k_{s-\lambda}} \\
&\quad = -(t-s+\lambda+1)\, \delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_0 \dots i_\lambda l_0 \dots l_{s-\lambda}} p_{l_{s-\lambda+1} \dots l_{t+1} k_1 \dots k_{s-\lambda}} \\
&\quad\quad + \sum_{\mu=1}^{s-\lambda} (-1)^{t+\mu-1} \delta^{l_0 \dots l_{t+1}}_{j_0 \dots j_{t+1}} p_{i_0 \dots i_\lambda k_\mu l_0 \dots l_{s-\lambda-1}} p_{l_{s-\lambda} \dots l_{t+1} k_1 \dots k_{\mu-1} k_{\mu+1} \dots k_{s-\lambda}}.
\end{aligned}$$

Hence, if equation (4) is true when λ is replaced by $\lambda+1$, the equation holds as it stands. But we have seen that it is true when $\lambda = s$; therefore it is true for $\lambda = 0, \dots, s$.

4. Now consider any power product of type $(l_0, \dots, l_k)$, and construct the corresponding tableau. Let N be any number greater than 1. We define the *measure* of the tableau to be N^{l_0} times the sum of the numbers in the first column plus N^{l_0-1} times the sum of the numbers in the second column, plus etc. We show that, if the tableau is not standard, we can replace the power product by a sum of power products of smaller measure. Indeed, if the tableau is not standard there is a pair of adjacent columns

$$\begin{array}{cc}
i_0 & j_0 \\
\vdots & \vdots \\
\vdots & j_s \\
i_t &
\end{array}$$

$(s \leqslant t)$ such that, for some $\lambda \leqslant s$,

$$j_0 < j_1 < \dots < j_\lambda < i_\lambda < \dots < i_t.$$

We now consider the particular case of (4)

$$\delta^{l_0 \dots \dots l_{t+1}}_{j_0 \dots j_\lambda i_\lambda \dots i_t} p_{j_{\lambda+1} \dots j_s l_0 \dots l_\lambda} p_{l_{\lambda+1} \dots l_{t+1} i_0 \dots i_{\lambda-1}} = 0.$$

By means of this equation, we replace the factor $p_{i_0 \dots i_t} p_{j_0 \dots j_s}$ in the power product by a sum of terms, and the power product is replaced by a sum of power products each of which can be represented by a tableau which differs from the original tableau only in the two columns in question; these columns have been changed by interchanging certain of the numbers $j_0 \dots j_\lambda$ in the second column with a corresponding set of *larger* numbers in the first column, and then rearranging the numbers in each column so that they increase as we go down. The sum of the numbers in the first column has been reduced by σ, say, and the sum of the numbers in the second column has been increased by σ, and hence the measure of the tableau has been decreased by $\sigma(N^j - N^{j-1}) > 0$. Now the measure of any tableau is not negative; hence we cannot carry on this process indefinitely. Therefore we can reach a set of standard power products after a finite

* These identities should be compared with those given by H. W. Turnbull, *Determinants, Matrices and Invariants* (Blackie, 1928), pp. 45 et seq.

number of stages. This proves that any k-connex can be expressed as a sum of standard power products.

5. We must now prove that there is no linear relation connecting the standard power products of type $(l_0, \ldots, l_k)$. Our argument is by induction, first on k and then on r. The result is trivial in the case $k = 0$, for in this case the standard power products are $(x_0)^{\alpha_0} \ldots (x_r)^{\alpha_r}$ arranged according to the suffix $0, \ldots, r$. We therefore assume the truth of the theorem for $(k-1)$-connexes. We begin the induction on r by considering the case $r = \alpha_0$. Consider any relation

$$f(p_{i_0}, p_{i_0 i_1}, \ldots, p_{i_0 \ldots i_k}) = 0$$

which is satisfied by the standard power products of type $(l_0, \ldots, l_k)$. We consider the number of times r appears as a suffix in the various terms of f. We collect together all the terms in which r appears as a suffix of $l_k - \lambda_k$ of the factors $p_{i_0 \ldots i_k}$, of $l_{k-1} - \lambda_{k-1}$ of the factors $p_{i_0 \ldots i_{k-1}}$, etc., and denote the sum of such terms by $f_{\lambda_0 \ldots \lambda_k}$, so that

$$f = \sum_{\lambda_0 = l_1}^{l_0} \cdots \sum_{\lambda_k = 0}^{l_k} f_{\lambda_0 \ldots \lambda_k}.$$

We note that, when $r = \alpha_0$, $\qquad f_{\lambda_0 \ldots \lambda_k} = 0,$
except when $\lambda_k = 0$.

Let μ_k be the smallest value of λ_k for which some $f_{\lambda, \ldots, \lambda_k}$ is different from zero, μ_{k-1} the smallest value of λ_{k-1} for which some $f_{\lambda_0 \cdots \lambda_{k_1} \mu_k}$ is not zero, and so on. Then, when we substitute the determinantal expressions for $p_{i_0 \ldots i_s}$ in f, we get a polynomial in which the coefficient of

$$(x_r^k)^{l_k - \mu_k} (x_r^{k-1})^{l_{k-1} - \mu_{k-1}} \ldots (x_r^0)^{l_0 - \mu_0} \quad \text{is} \quad g(p_{i_0}, \ldots, p_{i_0 \ldots i_{k-1}}).$$

It is easily seen that g depends only on the terms in $f_{\mu_0 \ldots \mu_k}$. Each of these latter is a standard power product, represented by a tableau in which r appears $l_0 - \mu_0$ times at the end of the first row, $l_1 - \mu_1$ times at the end of the second row, and so on. The contribution of any one of these terms to g is obtained by striking out r wherever it appears in the tableau and constructing the power product (still standard) in $[r-1]$ which corresponds to the resulting tableau. Thus g is a sum of standard power products of type $(\mu_0, \ldots, \mu_k)$ in $[r-1]$ whose terms are in one-one correspondence with those of $f_{\mu_0 \ldots \mu_k}$, and indeed g is just obtained from $f_{\mu_0 \ldots \mu_k}$ by omitting the suffix r whenever it occurs. Thus g is identically zero if and only if $f_{\mu_0 \ldots \mu_k}$ is identically zero, and hence, in view of the definition of $\mu_0, \ldots, \mu_k$, g is zero if and only if f is zero. Hence, in order to prove that $f \equiv 0$, we have only to prove $g \equiv 0$.

Now f, by hypothesis, vanishes when we substitute the determinantal expressions for $p_{i_0 \ldots i_s}$. Hence g must vanish when we substitute the determinantal expression for $p_{i_0 \ldots i_s}$. Thus $g = 0$ is a relation connecting the standard power products in $[r-1]$. If $r = \alpha_0$, then $\mu_k = 0$, and these power products are $(k-1)$-connexes. But by the hypothesis of induction these are linearly independent. Hence $g \equiv 0$, and therefore $f \equiv 0$. We now assume the truth of the theorem for k-connexes in $[r-1]$. Once again, we see that a linear relation $f = 0$ connecting the standard power products of type $(l_0, \ldots, l_k)$ in $[r]$ implies a relation $g = 0$ connecting the standard power products of type $(\mu_0, \ldots, \mu_k)$ in $[r-1]$. Since, by the hypothesis of induction, these are linearly independent, it follows that $f \equiv 0$, and we have the theorem:

The standard power products of type $(l_0, \ldots, l_k)$ associated with the matrix (2) *form a fundamental base for the k-connexes of this type associated with the matrix.*

As an example, when we take the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ 0 & y_1 & y_2 \end{pmatrix},$$

and $l_0 = 2$, $l_1 = 1$, the base for the k-connexes consists of the power products corresponding to the tableaux

$$\begin{pmatrix} 0 & 0 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \end{pmatrix}.$$

If, on the other hand, we replace y_1 by zero, we get the same base, omitting the first three members, since $p_{01} = 0$.

6. We now use the theorem just proved to calculate the number of independent k-connexes of type $(l_0, \ldots, l_k)$ associated with the matrix (2). We denote this number by $\psi_r(l_0, \ldots, l_k; \alpha_0, \ldots, \alpha_{k-1})$. We shall prove that

$$\psi_r(l_0, \ldots, l_k; \alpha_0, \ldots, \alpha_{k-1}) = \begin{vmatrix} \binom{l_0+r}{r} & \binom{l_1-1+r}{r} & \cdots & \binom{l_k-k+r}{r} \\ \binom{l_0+r+1-\alpha_{k-1}}{r-\alpha_{k-1}} & \binom{l_1+r-\alpha_{k-1}}{r-\alpha_{k-1}} & \cdots & \binom{l_k-k+1+r-\alpha_{k-1}}{r-\alpha_{k-1}} \\ \vdots & \vdots & & \vdots \\ \binom{l_0+r+k-\alpha_0}{r-\alpha_0} & \binom{l_1+r+k-1-\alpha_0}{r-\alpha_0} & \cdots & \binom{l_k+r-\alpha_0}{r-\alpha_0} \end{vmatrix}, \tag{5}$$

where the elements are the usual binomial coefficients and $\binom{a}{b}$ is zero if $0 \leqslant a < b$.

When $k = 0$, $\psi_r(l)$ is the number of forms of degree l in $[r]$, and therefore

$$\psi_r(l) = \binom{l+r}{r}.$$

So the formula is correct when $k = 0$. We may therefore assume, as hypothesis of induction, that the formula is true for $(k-1)$-connexes. We next prove it true for k-connexes $[r]$, when $\alpha_0 = r$. We have to count the number of standard tableaux of type $(l_0, \ldots, l_k)$. Now, since $\alpha_0 = r$,
$$p_{i_0 \ldots i_k} = 0,$$
unless one suffix is r, and hence the last row of each tableau consists entirely of the number r. In the second last row there will be $l_{k-1} - \lambda_{k-1}$ numbers r at the right-hand side, where $l_k \leqslant \lambda_k \leqslant l_{k-1}$, and so on. When we omit the number r wherever it occurs, we get a standard tableau of type $(\lambda_0, \ldots, \lambda_{k-1})$. Conversely, from any standard tableau, of this type, we can construct one of type $(l_0, \ldots, l_k)$ by inserting the number r at the right of the rows. Hence

$$\psi_r(l_0, \ldots, l_k; r, \alpha_1, \ldots, \alpha_{k-1}) = \sum_{\lambda_0 = l_1}^{l_0} \cdots \sum_{\lambda_{k-1} = l_k}^{l_{k-1}} \psi_{r-1}(\lambda_0, \ldots, \lambda_{k-1}; \alpha_1, \ldots, \alpha_{k-1}).$$

Since
$$\sum_{\lambda=a}^{b} \binom{\lambda+c}{d} = \binom{b+c+1}{d+1} - \binom{a+c}{d+1},$$

we find that $\psi_r(l_0, \ldots, l_k;\ r, \alpha_1, \ldots, \alpha_{k-1})$ is given by a determinant of k rows and columns having

$$\binom{l_{j-1}+r+i-j-\alpha_{k-i+1}}{r-\alpha_{k-i+1}}-\binom{l_j+r+i-j-1-\alpha_{k-i+1}}{r-\alpha_{k-i+1}},$$

with $\alpha_k = 0$, as the element in the ith row and jth column. Now, when $\alpha_0 = r$, the last row of the determinant in (5) consists of units. Operate on this determinant by the operations col. 1–col. 2, col. 2–col. 3, ..., col. k–col. $(k+1)$, and then expand in terms of the last row. The resulting determinant is that obtained for

$$\psi_r(l_0, \ldots, l_k;\ r, \alpha_1, \ldots, \alpha_{k-1}).$$

Hence the formula (5) is true when $\alpha_0 = r$.

When $r > \alpha_0$, we proceed by induction on r. A standard tableau of type $(l_0, \ldots, l_k)$ which has r in $l_k - \lambda_k$ $(0 \leqslant \lambda_k \leqslant l_k)$ places in the last row, in $l_{k-1} - \lambda_{k-1}$ $(l_k \leqslant \lambda_{k-1} \leqslant l_{k-1})$ places in the second last row, and so on, reduces, when r is omitted, to a standard tableau associated with an $(r-1)$-space of type $(\lambda_0, \ldots, \lambda_k)$, and conversely such a tableau leads to one of type $(l_0, \ldots, l_k)$ by inserting r at the appropriate places. Therefore we have

$$\psi_r(l_0, \ldots, l_k;\ \alpha_0, \ldots, \alpha_k) = \sum_{\lambda_0=l_1}^{l_0} \cdots \sum_{\lambda_k=0}^{l_k} \psi_{r-1}(\lambda_0, \ldots, \lambda_k;\ \alpha_0, \ldots, \alpha_r).$$

The summation on the right can be carried out exactly as above, and we arrive at equation (5), which is therefore proved. The reader can easily verify that, when $\alpha_i = k-i$, formula (5) reduces to that previously obtained by Littlewood and me.

As an example, we may consider the cases cited at the end of § 5. If $y_1 \neq 0$, we have $r = 2$, $\alpha_0 = 1$, and our formula gives

$$\psi_2(2, 1;\ 1) = \begin{vmatrix} \binom{2+2}{2} & \binom{0+2}{2} \\ \binom{3+1}{1} & \binom{1+1}{1} \end{vmatrix} = 8;$$

and if $y_1 = 0$, we have $r = 2$, $\alpha_0 = 2$, and hence

$$\psi_2(2, 1;\ 2) = \begin{vmatrix} \binom{2+2}{2} & \binom{0+2}{2} \\ \binom{3+0}{0} & \binom{1+0}{0} \end{vmatrix} = 5;$$

results which are in agreement with those already found.

7. The most important geometrical applications arise when $l_0 = l_1 = \ldots = l_k = n$. It is now convenient to write

$$a_i = r - \alpha_i \quad (i = 0, \ldots, k),$$

and $\phi_{a_0 \ldots a_k}(n)$ for $\psi_r(n, n, \ldots, n;\ \alpha_0, \ldots, \alpha_{k-1})$. The equations

$$x_0 = 0, \quad \ldots, \quad x_{r-a_i-1} = 0,$$

determine in $[r]$ a linear space A_i of a_i dimensions. A_0 is contained in A_1, A_1 in A_2, ... A_{k-1} in A_k. We consider the Grassmannian coordinates $X_{i_0 \ldots i_k}$ of a k-space in $[r]$. Interpreting the $X_{i_0 \ldots i_k}$ as coordinates of a point in a space of $N = \binom{r+1}{k-1} - 1$ dimensions, we obtain, as the k-space varies in $[r]$, a locus in $[N]$ whose points are in one-one correspondence with the k-spaces in $[r]$. This is the Grassmannian $\Omega_{r,k}$ of k-spaces in $[r]$. $\Omega_{r,k}$ clearly satisfies the equation

$$\sum_{\rho=1}^{k+1}(-1)^\rho X_{i_0\ldots i_{k-1}j_\rho} X_{j_0\ldots j_{\rho-1}j_{\rho+1}\ldots j_{k+1}} = 0. \tag{6}$$

Now, if the k-spaces in $[r]$ are restricted to satisfy the conditions of meeting A_i in a space of i dimensions ($i = 0, \ldots, k$),

$$X_{i_0\ldots i_k} = 0 \tag{7}$$

whenever $(i_0, \ldots, i_k)$ is a set of suffixes for which the corresponding determinant $p_{i_0\ldots i_k}$, constructed from (2), vanishes. The equations (7) determine a sub-locus $\Omega_{a_0}, \ldots, {}_{a_k}$ of $\Omega_{r,k}$ whose points are in one-one correspondence with the k-spaces of $[r]$ which meet A_i in a space of i dimensions ($i = 0, \ldots, k$). $\Omega_{a_0,\ldots,a_k}$ is the Schubert variety corresponding to the Schubert condition $(a_0, \ldots, a_k)$ defined by $A_0, \ldots, A_k$.

Now consider in $[N]$ any primal $F = 0$, of degree n. What we have proved in §§4 and 5 is that F can be reduced, modulo the left-hand sides of (6) and (7), to a sum of power products corresponding to standard tableaux, and that if F vanishes on $\Omega_{a_0,\ldots,a_k}$ this reduced form of F must vanish identically. In other words, if F vanishes on $\Omega_{a_0,\ldots,a_k}$,

$$F = \Sigma A_{ij}\,\delta^{l_0\ldots l_{k+1}}_{j_0\ldots j_{k+1}} X_{i_0\ldots i_{k-1}l_0} X_{l_1\ldots l_{k+1}} + \Sigma B_i X_{i_0\ldots i_k},$$

where, in the second summation, the summation is over the suffixes which appear in (7). A_{ij}, B_i are forms in all the $X_{i_0\ldots i_k}$.

When $a_i = r-k+i$, $\Omega_{a_0,\ldots,a_k} = \Omega_{r,k}$. The result in this case is well known*.

8. Corresponding to each standard tableau of type $(n, n, \ldots, n)$ there corresponds a primal F of degree n in $[N]$ which does not contain $\Omega_{a_0,\ldots,a_k}$. Hence it follows that the number of conditions to be satisfied in order that a primal of degree n in $[N]$ contain $\Omega_{a_0,\ldots,a_k}$, that is, the *postulation* of $\Omega_{a_0,\ldots,a_k}$, for primals of degree n, is $\phi_{a_0\ldots a_k}(n)$. This number can now be written

$$\phi_{a_0\ldots a_k}(n) = \begin{vmatrix} \binom{n+a_k}{a_k} & \binom{n-1+a_k}{a_k} & \cdots & \binom{n-k+a_k}{a_k} \\ \binom{n+1+a_{k-1}}{a_{k-1}} & \binom{n+a_{k-1}}{a_{k-1}} & \cdots & \binom{n-k+1+a_{k-1}}{a_{k-1}} \\ \vdots & \vdots & & \vdots \\ \binom{n+k+a_0}{a_0} & \binom{n+k-1+a_0}{a_0} & \cdots & \binom{n+a_0}{a_0} \end{vmatrix}. \tag{8}$$

It should be noticed that this formula holds for all values of $n \geqslant 0$, and not merely for sufficiently large n, as usually happens with postulation formulae.

It is well known† that the postulation of a variety V of dimension t in $[r]$ for primals of sufficiently high degree n is given by the expression

$$\sum_{i=0}(-1)^i p_i \binom{n+t-i-1}{t-i},$$

* See, for example, R. Weitzenbock, *Proc. Akad. Wet. Amsterdam*, 39 (1936), 503.

† F. Severi, *Rend. Circ. Mat. Palermo*, 28 (1909), 33.

where $\pi_i = p_i - (-1)^i$ is the arithmetic genus of the variety of dimension i obtained by taking the section of V by $r-i$ general primes. Comparing this with the formula of postulation for $\Omega_{a_0,\ldots,a_k}$, we are able to obtain the arithmetic genera of the various linear sections of $\Omega_{a_0,\ldots,a_k}$. By a simple calculation we obtain

$$p_{t-i} = (-1)^{t-i}\sum_{j=0}^{i}(-1)^j \phi_{a_0\ldots a_k}(-j)$$

$$= (-1)^{t-i}\sum_{j=0}^{i}(-1)^j \begin{vmatrix} \binom{a_k-j}{a_k} & \cdots & \binom{a_k-k-j}{a_k} \\ \vdots & & \\ \binom{a_0+k-j}{a_0} & \cdots & \binom{a_0-j}{a_0} \end{vmatrix},$$

where $\binom{a}{b}$ is to be replaced by $(-1)^b\binom{b-1-a}{b}$ if a is negative. In this formula $\phi_{a_0\ldots a_k}(n)$ is defined by (8) for all values of n. In particular, p_0 is the order of $\Omega_{a_0,\ldots,a_k}$, and we obtain

$$p_0 = t!\begin{vmatrix} \frac{1}{a_0!} & \frac{1}{(a_0-1)!} & \cdots & \frac{1}{(a_0-k)!} \\ \vdots & & & \\ \frac{1}{a_k!} & \frac{1}{(a_k-1)!} & \cdots & \frac{1}{(a_k-k)!} \end{vmatrix},$$

where $(b!)^{-1}$ is to be replaced by zero if b is negative. The dimension t of $\Omega_{a_0,\ldots,a_k}$ is $t = \sum_0^k a_i - \frac{1}{2}k(k+1)$. This result is known in the Schubert calculus.

For $\Omega_{r,k}$ we obtain $\quad \pi_t = \ldots = \pi_{t-r} = 0, \quad \pi_{t-r-1} = 1.$

Now it is known* that the (virtual) canonical system on $\Omega_{r,k}$ is $|-sC|$, where $|C|$ is the system of prime sections. Moreover, all the (general) linear sections of $\Omega_{r,k}$ are regular. Hence the complete canonical system on the section of $\Omega_{r,k}$ by $r+1$ general primals is cut by the primals of degree $r+1-s$. Hence $s = r+1$.

* F. Severi, *Ann. Mat.* (3), 24 (1915).

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