Algorithms and moduli spaces for differential equations

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Publication date:
2004

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Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
vrijdag 22 oktober 2004
om 16.15 uur

door

Maint Berkenbosch

geboren op 17 april 1977
te Heerenveen

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Introduction

This thesis treats several questions concerning linear differential equations. We mainly consider differential equations over $k(z)$, with k some field of characteristic zero. Such equations are of the form $L(y) := a_n y^{(n)} + \cdots + a_0 y = 0$, $a_i \in k(z)$, here $y^{(n)}$ denotes the n-th derivative of y with respect to some differentiation on $k(z)$, for example $\frac{d}{dz}$ or $z\frac{d}{dz}$. In this introduction we will give an overview of the problems treated in this thesis.

- Can one give explicit solutions of a second order equation over $k(z)$?

To answer this question, we first have to define what we mean by an explicit solution. It is quite natural to allow logarithms, exponentials and algebraic equations in the description of a solution. This leads to the notion of *Liouvillian* solutions. For example, the equation $4zy'' + 2y' = y$ has the Liouvillian solutions $\{e^{\sqrt{z}}, e^{-\sqrt{z}}\}$, but the Airy-equation $y'' = zy$ has no Liouvillian solutions. The famous Kovacic algorithm [Ko86] determines if a second order equation has Liouvillian solutions. If they exist, the algorithm gives them explicitly. One problem in actually implementing the algorithm is that the solutions may involve some algebraic field extensions of the field of constants k. An algorithm which determines for a given second order linear differential equation the possible extensions of the constant field, is given in Section 1.1.

Another problem is how to obtain a compact presentation of Liouvillian solutions. One way of doing this in the case of second order equations, is by writing such solutions in terms of the solutions of certain standard equations. *Klein's theorem* precisely states that this is possible. We present a new proof of this. The method of representing solutions via Klein's theorem works well because the standard equations are quite simple, and so are their solutions. In section 1.2 we give explicit formulas for the transformations involved (the so-called pullback formulas).

One can ask whether this method can be extended to third order equations.

- Is there a variant of Klein's theorem for third order equations?

In Section 1.3 we show that indeed there is. The difference with the second order case is that we have to allow infinitely many "standard equations". It seems there is no obvious way to adapt the method of finding explicit formulas for pullback maps to the case of third order equations.

A possible approach in studying differential equations is to try to make families of differential equations which share certain properties. In Chapter 2 we make such families by prescribing the local behavior of the equations. We show that we actually obtain so-called fine moduli spaces (classifying spaces) in this way. To a differential equation one can associate its differential Galois group, which gives information on the complexity of the solutions.

- How does the differential Galois group vary over a family of differential equations?

This question is studied in chapter 3. More precisely, suppose we have a family of differential equations parametrized by a space X. Given a group G, we consider the subset $X(\subset G) := \{x \in X |$ the differential equation at x has Galois group $\subset G\}$. We prove that this is a closed subspace of X. Using this, we describe sufficient conditions on the group G such that the analogously defined subset $X(= G)$ is "constructible" (cf. Section 3.1). These results are motivated by earlier work of M. F. Singer ([S93]).

Chapter 4 is concerned with the concept of monodromy. Let a differential equation $L(y) = 0$, with rational functions over $\mathbb{C}$ as coefficients, be given. Write $S := \{s_1, \cdots, s_r\}$ for the set of singular points of L on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$. Roughly speaking, the monodromy gives information on how solutions of L change under analytic continuation. To be more precise, let b be a point in $\mathbb{P}^1_{\mathbb{C}} \setminus S$, and write V for the local solution space of L at b. For a loop λ in the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S, b)$ analytic continuation along λ defines a linear automorphism on V. The monodromy is the natural homomorphism $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S) \longrightarrow \mathrm{GL}(V)$ which arises in this way.

The classical question, posed by Hilbert as the 21^{st} problem of his famous list, asks whether for any homomorphism $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S) \longrightarrow \mathrm{GL}(V)$ there is a differential equation with monodromy given by ρ. For a more precise formulation, see Chapter 4. This problem is known as the Riemann-Hilbert problem. As stated here, the answer turns out to be 'yes'. In this thesis we extend the Riemann-Hilbert problem to families, i.e.,

- For a given family of monodromy maps, does there exist a family of differential equations, such that the induced family of monodromy maps is the given one?

Under some weak conditions on the monodromy maps (cf. Section 4.2), we prove by explicit construction of a family of differential equations that the answer is again positive. The families of differential equations used here involve so-called *vector bundles*. To a vector bundle on $\mathbb{P}^1$ one associates a *type*. It happens to be the case that the type of the vector bundles, obtained by the mentioned construction, may vary (over the space which parametrizes the family). This leads us to a study of differential equations (or actually connections) on non-trivial vector bundles in Section 4.3. An important example in this setting is the so-called Lamé-example, which concludes the thesis.

Chapter 1

Pullbacks of Differential Equations

This chapter discusses second and third order differential operators. We will define standard operators, and proof that every differential operator with finite differential Galois group is a so-called pullback of some standard operator. We will also give an algorithm concerning certain field extensions, associated with algebraic solutions of a Riccati equation.

1.1 Field extensions for Riccati solutions

In this section we consider second order linear differential equations of the form $L : y'' = ry,\ r \in k(x)$. Here $k(x)$ is a *differential field* of characteristic zero, with derivation $\frac{d}{dx}$. The field of constants k is not supposed to be algebraically closed. We will denote its algebraic closure by $\bar{k}$. The differential Galois theory gives us an extension $\bar{k}(x) \subset K$, with K the so called *Picard-Vessiot extension*, which is the minimal differential field extension of $\overline{k}(x)$ which contains a basis $\{y_1, y_2\}$ (over $\bar{k}$) of solutions of L. The solution space $\bar{k}\langle y_1, y_2\rangle := \overline{k}y_1 + \overline{k}y_2 \subset K$ will be denoted V. The automorphisms of $K/\bar{k}(x)$ which commute with the differentiation constitute the *differential Galois group* G.

An interesting class of solutions are the so called *Liouvillian solutions*. These are solutions which lie in a Liouvillian extension of $\bar{k}(x)$, which roughly means they can be written down quite explicitly. For a precise definition of a (generalized) Liouvillian extension, see [Ka76, p. 39]. Related to this is the *Riccati equation*, denoted R_L, which is an equation depending on L with as solutions elements of the form $u = \frac{y'}{y}$, with y a solution of L. In our case it is the equation $u^2 + u' = r$. We have the following facts (see [PS03, p. 35,104]).

Fact 1.1 *$u \in K$ is a solution of $R_L \iff u = \frac{y'}{y}$, for some $y \in V$.*

Fact 1.2 *$u = \frac{y'}{y}$ is a solution of R_L, algebraic of degree m over $\bar{k}(x) \iff$ The stabilisor in G of the line $\bar{k} \cdot y$ is a subgroup of index m.*

The next fact is concerned with Liouvillian solutions of L.

Fact 1.3 *L has a Liouvillian solution $\iff R_L$ has an algebraic solution.*

Let u be an algebraic solution of R_L of minimal degree over $\overline{k}(x)$. We define the field k' to be the minimal field in $\overline{k}$ such that the coefficients of the minimal polynomial of u over $\bar{k}(x)$ are elements of $k'(x)$. We want to determine k' as explicit as possible. In [HP95] bounds on the degree $[k' : k]$ are given, depending on the differential Galois group G of L. We consider G as a subgroup of $\mathrm{GL}_2(\bar{k})$ by its action on y_1, y_2. It is known that G is an algebraic subgroup of $\mathrm{GL}_2(\bar{k})$. Note that changing the basis $\{y_1, y_2\}$ changes G by conjugation. Because in our equation L there is no first order term, we actually have that G lies in $\mathrm{SL}_2(\bar{k})$, see [Ka76] p41. We have the following lemma, which is essentially Theorem 5.4 of [HP95].

Lemma 1.4 *There are only three cases, with respect to G, for which k' can be different from k. These are (on an appropriate basis):*

(1) *$G \subset \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \bar{k}^*\}, \#G > 2$, a subgroup of a torus.*

(2) *$G = D_2^{\mathrm{SL}_2}$, a group of order 8, with generators $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

(3) *$G = A_4^{\mathrm{SL}_2}$, a group of order 24.*

We remark that in [HP95], the group $D_2^{\mathrm{SL}_2}$ is mistakenly denoted by D_4. We have $D_4 \neq D_2^{\mathrm{SL}_2}$, and in fact $D_2^{\mathrm{SL}_2} \cong Q_8$, where Q_8 denotes the quaternion subgroup $\{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}^*$. The notations $D_2^{\mathrm{SL}_2}$ and $A_4^{\mathrm{SL}_2}$ can be explained as follows. Using the natural homomorphism $\mathrm{SL}_2 \to \mathrm{PSL}_2$, these groups are the inverse image of $D_2 \subset \mathrm{PSL}_2$ and $A_4 \subset \mathrm{PSL}_2$ respectively. We will treat these three cases separately.

1.1.1 Subgroups of a torus

In this section we consider case (1) of Lemma 1.4. There are exactly two G-invariant lines in V. These correspond to the two solutions of R_L in $\bar{k}(x)$. Such solutions are called *rational*.

For the next lemma we need to introduce the second *symmetric power* of a given differential equation. This is the differential equation with as solutions, all products of two solutions of the given equation. For example take $L : y'' = ry$, with as basis of solutions $\{y_1, y_2\}$. Then the second symmetric power of L, denoted $\mathrm{Sym}(L, 2)$ is the equation $y''' - 4ry' - 2r'y = 0$. It has $\{y_1^2, y_1y_2, y_2^2\}$ as a basis of solutions. Indeed, $\{y_1^2, y_1y_2, y_2^2\}$ are linearly independent over $\overline{k}$ (compare [SU93] Lemma 3.5). In a similar way one defines higher order symmetric powers $\mathrm{Sym}(L, n)$ (see [PS03] Definition 2.24), which we will use later on. We note that $\mathrm{Sym}(L, n)$ can have order smaller than $n+1$. In the proof of the next lemma, we will also use that there is an action of $\mathrm{Gal}(\bar{k}/k)$ on K, which induces an action on V. It acts in the standard way on $\bar{k}(x)$. For details see [HP95].

Lemma 1.5 *Assume we are in case* (1) *of Lemma 1.4. Then* $\mathrm{Sym}(L, 2)$ *has (up to constants) a unique non-zero solution* $H \in k(x)$*. If one of the two rational solutions of* R *does not lie in* $k(x)$*, then the rational solutions of* R *are* $\frac{H'}{2H} \pm cH^{-1}$*, for some* $c \in \overline{k} \setminus k, c^2 \in k$*.*

Proof. For the basis $\{y_1, y_2\}$ for which the representation of G in SL_2 is as in 1. we have that y_1y_2 is G invariant, so $y_1y_2 \in \bar{k}(x)$. It is easily seen that up to constants, this is the only G-invariant solution of $\mathrm{Sym}(L, 2)$. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have that $\sigma(y_1y_2)$ is another rational solution of the symmetric square, so it must be a multiple of y_1y_2. Therefore we have a $\mathrm{Gal}(\bar{k}/k)$-invariant line, and thus by Hilbert theorem 90 an invariant point on this

line. After multiplying y_1 by a constant, we may suppose $H := y_1 y_2 \in k(x)$. Then $\frac{H'}{H} = \frac{y_1'}{y_1} + \frac{y_2'}{y_2}$. The rational solutions of R are $\frac{y_1'}{y_1}$ and $\frac{y_2'}{y_2}$, and since $\mathrm{Gal}(\bar{k}/k)$ acts on the set of solutions of R, each one is fixed by a subgroup of $\mathrm{Gal}(\bar{k}/k)$ of index ≤ 2. Now assume this index is 2, then we can write $\frac{y_1'}{y_1} =: u =: u_0 + du_1,\ u_0, u_1 \in k(x), d^2 \in k, d \notin k$, and then $\frac{y_2'}{y_2} = u_0 - du_1$, so $\frac{H'}{H} = 2u_0$. From $u' + u^2 = r \in k(x)$ one deduces that $2u_0 = -\frac{u_1'}{u_1}$, so u_1 must be $\lambda H^{-1}, \lambda \in k^*$. Therefore we can take $c = d\lambda$, and clearly $\frac{y_2'}{y_2} = \frac{H'}{2H} - cH^{-1}$.
□

We note that this gives a way to find in case (1) the rational solutions of the Riccati equation. Indeed H can be found (for example using Maple), and c can be calculated by substituting $\frac{H'}{2H} + cH^{-1}$ into the Riccati equation.

1.1.2 Klein's theorem

In the remaining two cases of Lemma 1.4, the differential Galois groups are finite. This implies that the differential Galois group equals the ordinary Galois group. An important tool in studying these cases is *Klein's Theorem.* We present a version of it suggested by F. Beukers. For a different approach we refer to [BD79].

It will be convenient to use *differential operators.* These are elements of the skew polynomial ring $\bar{k}(x)[\partial_x]$. The multiplication is defined by $\partial_x x = x\partial_x + 1$. We will identify the linear differential equation $\Sigma_i a_i y^{(i)} = 0$ with the differential operator $\Sigma_i a_i \partial_x^i$.

We recall from [HP95] the following easy lemma.

Lemma 1.6 *The $\bar{k}$-algebra homomorphisms $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ are given by $\phi(t) = a$ and $\phi(\partial_t) = \frac{1}{a'}\partial_x + b$ with $a \in \bar{k}(x) \setminus \bar{k}$; $a' := \frac{d}{dx}a$ and $b \in \bar{k}(x)$.*

First we will discuss the process of normalization. A second order differential operator $L := a_2\partial_x^2 + a_1\partial_x + a_0$ is said to be in *normal form* if $a_2 = 1$ and $a_1 = 0$. We can put L into normal form, $\mathrm{Norm}(L)$, by dividing L by a_2, and then applying the 'shift' $\partial_x \mapsto \partial_x - \frac{a_1}{2a_2}$. Note that normalization transforms the old solution space V to $f \cdot V$, with $f' = \frac{a_1}{2a_2}f$. The operator remains defined over $k(x)$, but the associated Picard-Vessiot extension K changes if $f \notin K$.

Notation 1.7

- For $F \in \bar{k}(x) \backslash \bar{k}$ we define the $\bar{k}$-homomorphism $\phi_F : \bar{k}(t) \to \bar{k}(x)$, by $\phi_F(t) = F$.
- Let ϕ be an injective homomorphism $\phi : \bar{k}(t) \to \bar{k}(x)$. Then we also write ϕ for the extension of ϕ to the homomorphism of differential operators $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$, defined by $\phi(\partial_t) = \frac{1}{\phi(t)'}\partial_x$.
- For $F \in \overline{k}(x) \setminus \overline{k}, b \in \bar{k}(x)$, we define $\phi_{F,b} : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ by $\phi_{F,b}(t) = F$, $\phi_{F,b}(\partial_t) = \frac{1}{F'}(\partial_x + b)$.
- We will call an automorphism of $\overline{k}(t)[\partial_t]$, given by $t \mapsto t, \partial_t \mapsto \partial_t + b$ a *shift*.
- For a differential operator $L \in \overline{k}(t)[\partial_t]$ we define $\mathrm{Aut}(L)$ to be the group $\{\psi \in \mathrm{Aut}_{\bar{k}}\bar{k}(t) \mid \mathrm{Norm}(\psi(L)) = L\}$, where $\psi(\partial_t) = \frac{1}{\psi(t)'}\partial_t$.

Klein's theorem is concerned with differential operators $L := \partial_x^2 - r$ with finite non-cyclic differential Galois group $G \subset \mathrm{SL}_2(\bar{k})$. If we again use the notation H^{SL_2} for the inverse image in SL_2 of a group $H \subset \mathrm{PSL}_2$, the possibilities for such G are (up to conjugation): $\{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. In [BD79] we find for each such group G a *standard operator*, denoted St_G, which is in normal form, and has differential Galois group G. These are:

$$St_{D_n^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{3}{16}\frac{1}{(t-1)^2} - \frac{n^2+2}{8n^2}\frac{1}{t(t-1)},$$

$$St_{A_4^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{3}{16}\frac{1}{t(t-1)},$$

$$St_{S_4^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{101}{576}\frac{1}{t(t-1)},$$

$$St_{A_5^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{611}{3600}\frac{1}{t(t-1)}.$$

The so-called *local exponents* of these standard equations are given by the following table.

	0	1	∞
$St_{D_n^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{4},\frac{3}{4}$	$-\frac{n+1}{2n},-\frac{n-1}{2n}$
$St_{A_4^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{1}{3},-\frac{2}{3}$
$St_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{3}{8},-\frac{5}{8}$
$St_{A_5^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{2}{5},-\frac{3}{5}$

In the proof of Klein's Theorem we will need the following lemma.

Lemma 1.8 *Let L be a monic second order differential operator over $k(x)$, with finite differential Galois group G, and Picard-Vessiot extension K. Let $\{y_1, y_2\}$ be a basis of solutions of L, and write $s := \frac{y_1}{y_2}$.*
(1) Normalizing L does not change the field $K^p := \overline{k}(x)(s) \subset K$.
(2) Let $L_1 \in \overline{k}(x)[\partial_x]$ be a monic differential operator, which also has a basis of solutions in K of the form $\{sy, y\}$. Then L_1 can be obtained from L by the shift $\partial_x \mapsto \partial_x - (\frac{y}{y_1})'/(\frac{y}{y_1})$.
If moreover G is non-cyclic and $G \subset \mathrm{SL}_2(\overline{k})$, then also the following statements hold.
(3) $K^p = K^{\pm I}$, the fixed field of $-I$ in K.
(4) $K = K^p(\sqrt{s'})$.
(5) $\overline{k}(s)$ is G-invariant and $\exists\, t \in \overline{k}(x)$ such that $\overline{k}(s)^G = \overline{k}(t)$.

Proof.
(1) This follows immediately from the fact that the normalization of L has a basis of solutions $\{fy_1, fy_2\}$ (for some f with $\frac{f'}{f} \in \overline{k}(x)$).
(2) The monic differential operator $\phi_{x,-(\frac{y}{y_1})'/(\frac{y}{y_1})}$ clearly has $\{sy, y\}$ as a basis of solutions, and therefore is equal to L_1.
(3) Since $\bar{k}(x) \subset \overline{k}(x)(y_1, y_2)$ is a finite extension, we have $y_1', y_2' \in \overline{k}(x)(y_1, y_2)$, so $K = \overline{k}(x)(y_1, y_2)$. Because K^p is algebraic over $\bar{k}(x)$ the derivation on K induces a derivation on K^p. So $(\frac{y_1}{y_2})' = \frac{d}{y_2^2} \in \bar{k}(x)(\frac{y_1}{y_2})$, where $d = y_1'y_2 - y_2'y_1$. It is easily seen that $d' = 0$, and $d \neq 0$, so $d \in \overline{k}^*$. We find that $y_2^2 \in K^p$ and for a similar reason also $y_1^2 \in K^p$. So the only elements in G that fix $\bar{k}(x)(\frac{y_1}{y_2})$ are $\pm I$. By Galois correspondence K^p is the fixed field of $\{\pm I\}$.
(4) We have $K = K^p(y_2)$, and $y_2^2 = \frac{d}{s'}$, so $K = K^p(\sqrt{s'})$.

(5) From the G-action on $\overline{k}\langle y_1, y_2\rangle$ one immediately finds that $\overline{k}(s)$ is G-invariant. Since $\bar{k}(s)$ is a purely transcendental extension of $\bar{k}$ we get by Lüroth's theorem that the fixed field of G is also purely transcendental. So we can write $\overline{k}(s)^G = \overline{k}(t)$, and because $t \in K$ is invariant under G, we get $t \in \overline{k}(x)$. □

Theorem 1.9 (Klein)
Let L be a second order differential operator over $k(x)$ in normal form, with differential Galois group $G \in \{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. There exists an element $F \in \bar{k}(x)$ such that $\mathrm{Norm}(\phi_F(St_G)) = L$. Moreover $\phi_F : \bar{k}(t) \to \bar{k}(x)$ is unique up to composition with an automorphism $\psi \in \mathrm{Aut}(St_G)$.

Proof. We will use the notation of the above lemma. Write $G^p := G/\{\pm I\}$ for $\mathrm{Gal}(K^p/\overline{k}(x)) = \mathrm{Gal}(\overline{k}(s)/\overline{k}(t))$. The field extension $\bar{k}(t) \subset \bar{k}(s)$ corresponds to a covering of $\mathbb{P}^1_t$ by $\mathbb{P}^1_s$, with Galois group G^p. It is known that for the groups $G^p \subset \mathrm{PGL}(2)$ considered here, the map $\mathbb{P}^1_s \to \mathbb{P}^1_t$ is ramified above three points. If necessary replacing t by the image of t under a Möbius-transformation, these three points are $0, 1, \infty$. The list of ramification indices is (up to permutations of $0, 1, \infty$):

G^p	e_0	e_1	e_∞
D_n	2	2	n
A_4	2	3	3
S_4	2	3	4
A_5	2	3	5

We choose t such that we get precisely the above ramification indices for $0, 1, \infty$.

We now want to construct a differential operator in $\overline{k}(t)[\partial_t]$, with differential Galois group G, and with Picard-Vessiot extension some field K_1, such that $K_1^p = \overline{k}(s)$. As suggested by F. Beukers one takes $K_1 := \bar{k}(s, \sqrt{s'})$, where $'$ denotes the unique extension of the derivation $\frac{d}{dt}$ on $\overline{k}(t)$. We write V for the solution space of L in K, and we define $V_1 := \overline{k}\langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}}\rangle \subset K_1$.

Lemma 1.10
(1) *The field K_1 does not depend on the choice of t.*
(2) *K_1 is a Galois extension of $\overline{k}(t)$, and we can identify* $\mathrm{Gal}(K_1/\overline{k}(t))$ *with*

G. The vector space V_1 is G-invariant, and isomorphic to V as a G-module.
(3) V_1 does not depend on the choice of s.

Proof.
(1) For $t_1 = \frac{at+b}{ct+d}$, $ad - bc = 1$, we have $\frac{ds}{dt} = \frac{ds}{dt_1}\frac{dt_1}{dt} = \frac{ds}{dt_1}\frac{1}{(ct+d)^2}$, so $\overline{k}(s, \sqrt{\frac{ds}{dt}}) = \overline{k}(s, \frac{1}{ct+d}\sqrt{\frac{ds}{dt_1}}) = \overline{k}(s, \sqrt{\frac{ds}{dt_1}})$.
(2) We will show that K_1 is the splitting field over $\bar{k}(t)$ of P_1P_2, where P_1 is the minimal polynomial of s over $\overline{k}(t)$, and P_2 is the minimal polynomial of $\sqrt{s'}$ over $\overline{k}(t)$. By construction the extension $\bar{k}(t) \subset \bar{k}(s)$ is Galois, so all zeroes of P_1 lie in $\bar{k}(s)$. The only thing that remains to be shown is that all roots of P_2 lie in K_1. This minimal polynomial is a factor of $\prod_{\sigma \in G^p}(T^2 - \sigma(s'))$, and $\sigma(s') = \sigma(s)' = \frac{s'}{(cs+d)^2}$, for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So all zeroes of the minimal polynomial of $\sqrt{s'}$ are of the form $\frac{\pm\sqrt{s'}}{cs+d}$, and therefore lie in K_1.
We can define an isomorphism $V \to V_1$, by $y_1 \mapsto \frac{s}{\sqrt{s'}}, y_2 \mapsto \frac{1}{\sqrt{s'}}$. This induces a G-action on V_1. A direct computation shows that this action extends to a G-action on K_1, extending the existing G-action on $\bar{k}(s)$. The invariant field in K_1 under this action is $\overline{k}(t)$, as can be seen from the inclusions

$$\overline{k}(t) \overset{G^p}{\subset} \bar{k}(s) \subset K_1.$$

We also conclude from this that $G = \mathrm{Gal}(K_1/\overline{k}(t))$.
(3) We have $(\frac{as+b}{cs+d})' = s'\frac{ad-bc}{(cs+d)^2}$, and it immediately follows that V_1 does not change if we replace s by $\frac{as+b}{cs+d}$, $ad - bc = 1$. Note that changing t in general does change V_1. □

We continue the proof of Klein's Theorem. Since the 2-dimensional vector space V_1 is invariant under the Galois group of K_1 over $\overline{k}(t)$, it is the solution space of some monic second order differential operator M_G over $\bar{k}(t)$. Clearly K_1 is the corresponding Picard-Vessiot extension. Further $s = (\frac{s}{\sqrt{s'}})/(\frac{1}{\sqrt{s'}})$, so $\bar{k}(s)$ is the corresponding subfield.

Claim: $M_G = St_G$.

We note that a monic second order differential operator with three fixed singular points is completely determined by its *local exponents* (see [PU00]

Chapter 5). The singular points of the differential operator M_G are $\{0, 1, \infty\}$. So to prove the claim, it suffices to show that the local exponents of M_G and St_G coincide for every singular point. We can calculate the local exponents of M_G. We give the calculation for $t = 0$. After applying a Möbius-transformation to s (which is allowed), we can suppose that s is a local parameter of a point above $0 \in \mathbb{P}^1_t$. So we get an embedding of complete local rings $\bar{k}[[t]] \subset \bar{k}[[s]]$, and we have $t = s^{e_0} + *s^{e_0+1} + \cdots$, where again e_0 is the ramification index of the embedding $\bar{k}(t) \subset \bar{k}(s)$ at $t = 0$. We find $s = t^{\frac{1}{e_0}} + \cdots$, so the power series expansion of the basis of solutions of M_G looks like $\frac{1}{\sqrt{s'}} = t^{\frac{1}{2} - \frac{1}{2e_0}} + \cdots$, and $\frac{s}{\sqrt{s'}} = t^{\frac{1}{2} + \frac{1}{2e_0}} + \cdots$. Therefore the local exponents at $t = 0$ are $\frac{1}{2} \pm \frac{1}{2e_0}$. In the same way we find the local exponents at $t = 1, \infty$ to be $\frac{1}{2} \pm \frac{1}{2e_1}$ and $-\frac{1}{2} \pm \frac{1}{2e_\infty}$ respectively. These are precisely the local exponents of the standard operator, which proves our claim.

Since $t \in \overline{k}(x)$, we can write $t = F \in \overline{k}(x)$. We have that $\phi_F(St_G)$ is a differential operator with corresponding intermediate field $\bar{k}(x)(s)$. By Lemma 1.8 the differential operator $\mathrm{Norm}(\phi_F(St_G))$ also has $\bar{k}(x)(s)$ as corresponding intermediate field, and L can be obtained from $\mathrm{Norm}(\phi_F(St_G))$ by a shift. Since both operators are in normal form, we must have $L = \mathrm{Norm}(\phi_F(St_G))$. This proves the existence of F.

We now consider the unicity of F. First of all, note that the choice of ramification indices over $\{0, 1, \infty\}$ of the covering $\mathbb{P}^1_s \to \mathbb{P}^1_t$ still leaves us some choice for t. To be precise,

- if $G^p = D_2$ we can replace t by its image under an automorphism of the $\mathbb{P}^1_t$ which permutes $\{0, 1, \infty\}$.
- if $G^p = D_n,\ n \neq 2$ we can replace t by $1 - t$.
- if $G^p = A_4$ we can replace t by $\frac{t}{t-1}$.

Lemma 1.11 *Let $\psi \in \mathrm{Aut}_{\bar{k}}\bar{k}(t)$ be an automorphism of $\mathbb{P}^1_t$ respecting the ramification data of the covering $\mathbb{P}^1_s \to \mathbb{P}^1_t$. Then $\psi \in \mathrm{Aut}(St_G)$.*

Proof. Suppose we can replace t by $z, t = \frac{az+b}{cz+d}$, $ad - bc = 1$, without changing the ramification indices at $\{0, 1, \infty\}$ of the covering induced by the field extension $\overline{k}(t) \subset \overline{k}(s)$. The resulting vector space $\tilde{V}_1$ can be written as $\tilde{V}_1 = (cz+d)V_1$. Let $\tilde{M}_G$ be the monic differential operator in $\overline{k}(z)[\partial_z]$, with solution space $\tilde{V}_1$. We find that $\tilde{M}_G = \frac{1}{(cz+d)^4}\phi_{\frac{az+b}{cz+d},\frac{c}{cz+d}}(M_G)$. Indeed $\phi_{\frac{az+b}{cz+d}}(M_G)$ is a differential operator over $\overline{k}(z)$ with solution space V_1, and multiplying all solutions by $cz+d$ corresponds to the shift $\partial_z \mapsto \partial_z - \frac{c}{cz+d}$. Because $\tilde{M}_G$ is constructed in the same way as M_G, we have that $\phi_t(\tilde{M}_G) = St_G$, $\phi_t : \bar{k}(z)[\partial_z] \to \overline{k}(t)[\partial_t]$. We find that $\frac{1}{(ct+d)^4}\phi_{\frac{at+b}{ct+d},\frac{c}{ct+d}}(St_G) = St_G$, so $\frac{at+b}{ct+d} \in \mathrm{Aut}(St_G)$. □

We will now show that ϕ_F is unique up to composition with an element in $\mathrm{Aut}(St_G)$. Our constructions give rise to the following diagram,

$$\begin{array}{ccccc}
\bar{k}(x) & \subset & \bar{k}(x)(s) & \subset & \bar{k}(x)(y_1, y_2) \\
\cup & & \cup & & \\
\bar{k}(t) & \subset & \bar{k}(s) & \subset & \bar{k}(s, \sqrt{s'})
\end{array}$$

Now suppose we can write $L = \mathrm{Norm}(\phi_P(St_G))$ for some $P \in \bar{k}(x)$. Then we can make a diagram as above, where the image of t in $\overline{k}(x)$ is now P. As we proved above, t is almost unique up to composition with some $\psi \in \mathrm{Aut}(St_G)$. Therefore we must have $\phi_P = \phi_F \circ \psi$, for some $\psi \in \mathrm{Aut}(St_G)$. □

Remark 1.12 It is not essential to consider differential operators over $k(x)$. Klein's theorem remains valid over arbitrary differential fields with field of constants $\overline{k}$. We can still construct the field $\overline{k}(s)$, and the proof only involves this field. •

Remark 1.13 In this remark we want to explain the following phenomenon. Let $\mathbb{C}(x) \subset K_G$ be a Picard-Vessiot extension for St_G, $G \in \{S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. For each G, we find two normalized differential operators in [PU00] with Picard-Vessiot extension equal to K_G (and satisfying certain nice properties). They correspond to the two irreducible two-dimensional representations of G. One of these two operators is St_G. Write L_G for the other operator. By Klein's theorem, we have that L_G is a pullback of St_G. On the other hand we will show that St_G is not a pullback of L_G, so L_G cannot be used as "standard operator" in Klein's theorem.

We will now explain this phenomenon in detail. First we consider the case $G = S_4^{\mathrm{SL}_2}$. The two operators of interest are

$$St_{S_4^{\mathrm{SL}_2}} = \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{101}{576}\frac{1}{x(x-1)},$$

$$L_{S_4^{\mathrm{SL}_2}} = \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{173}{576}\frac{1}{x(x-1)}.$$

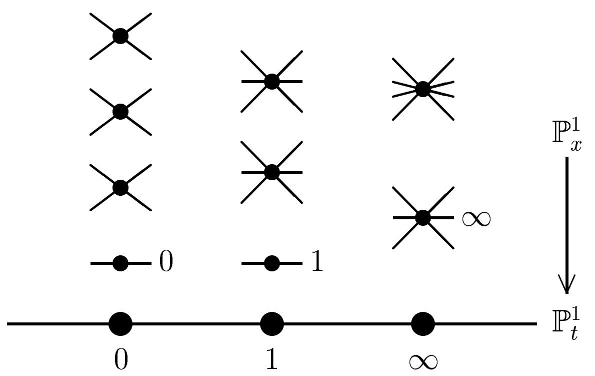
The local exponents of $St_{S_4^{\mathrm{SL}_2}}$ and $L_{S_4^{\mathrm{SL}_2}}$ are given by the following table.

	0	1	∞
$St_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{3}{8}, -\frac{5}{8}$
$L_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{1}{8}, -\frac{7}{8}$

Using the pullback formula of Theorem 1.31 we find that

$$L_{S_4^{\mathrm{SL}_2}} = \phi_{F,b}(St_{S_4^{\mathrm{SL}_2}}),\ F = \frac{(x-1)(144x^2-232x+81)^3}{(28x-27)^4} + 1,\ b = \frac{F''}{2F'}.$$

As we will see in Lemma 1.19, the difference of the local exponents of $L_{S_4^{\mathrm{SL}_2}}$ in a point a is equal to the ramification index of F at a times the difference of the local exponents of $St_{S_4^{\mathrm{SL}_2}}$ in $F(a)$. This is in accordance with the fact that the difference of the local exponents of $L_{S_4^{\mathrm{SL}_2}}$ at ∞ is $\frac{3}{4}$. Indeed, F has ramification index 3 at ∞, and the difference of the local exponents of $St_{S_4^{\mathrm{SL}_2}}$ at ∞ is $\frac{1}{4}$ (and $F(\infty) = \infty$). It also follows that $St_{S_4^{\mathrm{SL}_2}}$ cannot be written as a pullback of $L_{S_4^{\mathrm{SL}_2}}$. The complete ramification data of F are given by the following figure.



We note that the local exponents of $\phi_F(St_{S_4^{\mathrm{SL}_2}})$ at the ramified points ($\neq \infty$) above $0, \infty$ lie in $\frac{1}{2}\mathbb{Z}$ (see the proof of Lemma 1.19), but after applying the shift over $\frac{F''}{2F'}$, the local exponents become $\{0, 1\}$ at these points.

We will now explain how the representation of $S_4^{\mathrm{SL}_2}$ on the solution space changes by applying the pullback $\phi_{F,b}$. As in the proof of Klein's theorem (using the variables x, u instead of t, s), we can write $K = \mathbb{C}(u, \sqrt{u'})$, $' = \frac{d}{dx}$ for the Picard-Vessiot extension of $St_{S_4^{\mathrm{SL}_2}}$. The solution space of $St_{S_4^{\mathrm{SL}_2}}$ is $V := \langle \frac{u}{\sqrt{u'}}, \frac{1}{\sqrt{u'}} \rangle$, and $K^p := K^{\pm I} = \mathbb{C}(u)$. We can assume that the ramification data of $\mathbb{C}(x) \subset \mathbb{C}(u)$ is as in the proof of Klein's theorem. Let $W := \langle w_1, w_2 \rangle$ be the solution space of $L_{S_4^{\mathrm{SL}_2}}$, and define $s := \frac{w_1}{w_2}$. Then the group S_4 acts on $\mathbb{C}(s)$, and we define $\mathbb{C}(t) := \mathbb{C}(s)^{S_4}$, with the appropriate ramification data. These constructions give rise to the following diagram.

$$\begin{array}{ccccc} \mathbb{C}(x) & \subset & \mathbb{C}(x)(s) = \mathbb{C}(u) & \subset & \mathbb{C}(u, \sqrt{u'}) \\ \cup & & \cup & & \\ \mathbb{C}(t) & \subset & \mathbb{C}(s) & & \end{array}$$

We have $t = F \in \mathbb{C}(x)$, and s is some rational expression of degree 7 in u, say $s = g(u)$. We will now calculate g.

The extension $\mathbb{C}(x) \subset \mathbb{C}(u)$ has degree 24, and using [BD79], we find that we can write $x = h(u)$, where

$$h = -\frac{(u^8 + 14u^4 + 1)^3}{108u^4(u^4 - 1)^4} + 1.$$

We can also take $t = h(s)$, so $t = F(x) = F(h(u))$ and $t = h(s) = h(g(u))$. Therefore g satisfies $h(g(u)) = F(h(u))$. Using the ramification data of F and h, we can calculate the ramification data for g. Using these ramification data, together with some heuristics, we find

$$g = -\frac{u^3(u^4 + 7)}{7u^4 + 1}.$$

We can now express W in terms of u and $\sqrt{u'}$. We have $\frac{ds}{dx} = \frac{w_1' w_2 - w_1 w_2'}{w_2^2}$, and since the operator $L_{S_4^{\mathrm{SL}_2}}$ is in normal form $w_1' w_2 - w_1 w_2' \in \mathbb{C}$. So we find that

$W = \langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}} \rangle$, $' = \frac{d}{dx}$. Clearly $\frac{ds}{dx} = \frac{dg(u)}{du} \cdot \frac{du}{dx}$, and $\frac{dg(u)}{du} = -21(\frac{u(u^4-1)}{7u^4+1})^2$. So we find a basis for W in terms of u and $\sqrt{u'}$, namely

$$\{\frac{u^2(u^4+7)}{(u^4-1)\sqrt{u'}}, \frac{7u^4+1}{u(u^4-1)\sqrt{u'}}\}.$$

We will now examine the group $S_4^{\mathrm{SL}_2}$ in detail, and we will see how we can distinguish between the two irreducible representations ρ_1, ρ_2 of $S_4^{\mathrm{SL}_2}$ in $\mathrm{GL}_2(\mathbb{C})$. The abstract group $S_4^{\mathrm{SL}_2}$ is generated by two elements α, β, with image $(1234), (12)$ in S_4 respectively. For ρ_1 we take the representation $S_4^{\mathrm{SL}_2} \to \mathrm{GL}_2(\mathbb{C}), \alpha \mapsto \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}$, $\beta \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\zeta_8 = e^{\frac{2\pi i}{8}}$ (see [Ko86] p.30). Then for ρ_2 we can take the representation obtained by composition of ρ_1 with the automorphism of $\mathbb{Q}(\zeta_8)$ given by $\zeta_8 \mapsto \zeta_8^3$. We remark that the induced representations of S_4 in $\mathrm{PGL}(2, \mathbb{C})$ are conjugate. We can distinguish ρ_1 from ρ_2 by the eigenvalues of $\rho_i(\alpha)$. For ρ_1 these are $\{\zeta_8, \zeta_8^{-1}\}$ and for ρ_2 they are $\{\zeta_8^3, \zeta_8^{-3}\}$.

We fix an identification of $\mathrm{Gal}(K/\mathbb{C}(x))$ with $S_4^{\mathrm{SL}_2}$. We remark that since the group $\mathrm{Out}(S_4^{\mathrm{SL}_2})$ has two elements, there are essentially two ways to do this. We may assume that $S_4^{\mathrm{SL}_2}$ acts on V via the representation ρ_1. So $\alpha(\frac{u}{\sqrt{u'}}) = \zeta_8 \frac{u}{\sqrt{u'}}$ and $\alpha(\frac{1}{\sqrt{u'}}) = \zeta_8^{-1} \frac{1}{\sqrt{u'}}$. We will now calculate the action of α on W. We have $\alpha(u) = \zeta_8^2 u$, so $\alpha(\frac{u^2(u^4+7)}{(u^4-1)\sqrt{u'}}) = \zeta_8^3 \frac{u^2(u^4+7)}{(u^4-1)\sqrt{u'}}$ and $\alpha(\frac{7u^4+1}{u(u^4-1)\sqrt{u'}}) = \zeta_8^{-3} \frac{7u^4+1}{u(u^4-1)\sqrt{u'}}$. It immediately follows that the representation of $S_4^{\mathrm{SL}_2}$ in W is conjugate to ρ_2, which is what we wanted to show.

Now consider the case $G = A_5^{\mathrm{SL}_2}$. We will use the same terminology as in the $S_4^{\mathrm{SL}_2}$-case. The equations of interest are

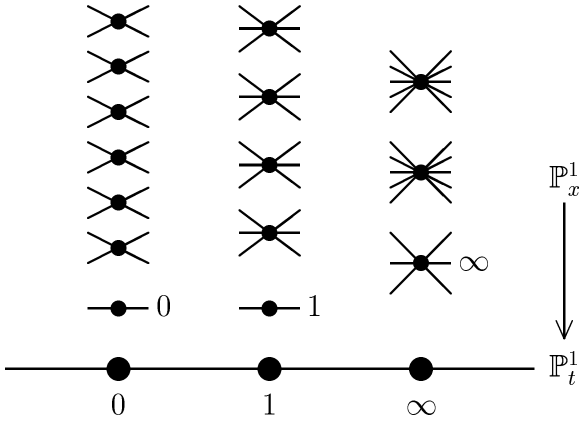
$$St_{A_5^{\mathrm{SL}_2}} := \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{611}{3600}\frac{1}{x(x-1)},$$

$$L_{A_5^{\mathrm{SL}_2}} := \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{899}{3600}\frac{1}{x(x-1)}.$$

We have that $L_{A_5^{\mathrm{SL}_2}}$ is a pullback of $St_{A_5^{\mathrm{SL}_2}}$, with pullback function

$$F = \frac{(1-x)(147456x^4 - 403456x^3 + 379296x^2 - 57591x - 59049)^3}{(1664x^2 - 2457x + 729)^5} + 1.$$

The ramification of F is given by the following diagram.



As in the $S_4^{\mathrm{SL}_2}$ case we have the following diagram.

$$\begin{array}{ccccc} \mathbb{C}(x) & \subset & \mathbb{C}(u) & \subset & \mathbb{C}(u, \sqrt{u'}) \\ \cup & & \cup & & \\ \mathbb{C}(t) & \subset & \mathbb{C}(s) & & \end{array}$$

Again write $x = h(u)$ and $s = g(u)$. In the same way as in the $S_4^{\mathrm{SL}_2}$-case, we find

$$h = \frac{(u^{20} - 228u^{15} + 494u^{10} + 228u^5 + 1)^3}{1728u^5(u^{10} + 11u^5 - 1)^5} + 1,$$

$$g = -\frac{u^3(u^{10} - 39u^5 - 26)}{26u^{10} - 39u^5 - 1}.$$

We have $W = \langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}} \rangle$, $' = \frac{d}{dx}$, and $\frac{ds}{dx} = \frac{dg(u)}{du} \cdot \frac{du}{dx}$. Using the fact that $\frac{dg(u)}{du} = -78(\frac{u(u^{10}+11u^5-1)}{26u^{10}-39u^5-1})^2$, we obtain the following basis for W

$$\{\frac{u^2(u^{10} - 39u^5 - 26)}{(u^{10} + 11u^5 - 1)\sqrt{u'}}, \frac{26u^{10} - 39u^5 - 1}{u(u^{10} + 11u^5 - 1)\sqrt{u'}}\}.$$

The group $A_5^{\mathrm{SL}_2}$ has two irreducible representations ρ_1, ρ_2 in $\mathrm{GL}_2(\mathbb{C})$. We have that $A_5^{\mathrm{SL}_2}$ is generated by two elements α, β, with image (12345) and $(12)(34)$ in A_5 respectively. We fix ρ_1 to be the representation of $A_5^{\mathrm{SL}_2}$ in $\mathrm{GL}_2(\mathbb{C})$ given by $\alpha \mapsto \begin{pmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{pmatrix}$, $\beta \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $\zeta_{10} = e^{\frac{2\pi i}{10}}$, $a = \frac{1}{5}(3\zeta_{10}^3 - \zeta_{10}^2 + 4\zeta_{10} - 2)$, $b = \frac{1}{5}(\zeta_{10}^3 + 3\zeta_{10}^2 - 2\zeta_{10} + 1)$. This explicit formulas come from [Ko86] p.30, note that we can also write $a = i\sqrt{\frac{1}{2}+\frac{1}{10}\sqrt{5}}, b = \frac{\sqrt{5}-1}{2}a$. Then ρ_2 is the representation obtained by composition of ρ_1 with the automorphism of $\mathbb{Q}(\zeta_{10})$ given by $\zeta_{10} \mapsto \zeta_{10}^3$. In contrast to the $S_4^{\mathrm{SL}_2}$-case, the induced representations of A_5 in $\mathrm{PGL}(2,\mathbb{C})$ are not isomorphic. As in the $S_4^{\mathrm{SL}_2}$-case, we can distinguish ρ_1 from ρ_2 by the eigenvalues of $\rho_i(\alpha)$. For ρ_1 these are $\{\zeta_{10}, \zeta_{10}^{-1}\}$ and for ρ_2 they are $\{\zeta_{10}^3, \zeta_{10}^{-3}\}$.

Fix an identification of $\mathrm{Gal}(K/\mathbb{C}(x))$ with $A_5^{\mathrm{SL}_2}$. Again there are essentially two ways to do this. We may assume that $A_5^{\mathrm{SL}_2}$ acts on V via the representation ρ_1. So $\alpha(\frac{u}{\sqrt{u'}}) = \zeta_{10}\frac{u}{\sqrt{u'}}$ and $\alpha(\frac{1}{\sqrt{u'}}) = \zeta_{10}^{-1}\frac{1}{\sqrt{u'}}$. Again we calculate the action of α on W. We have $\alpha(u) = \zeta_{10}^2 u$, so $\alpha(\frac{u^2(u^{10}-39u^5-26)}{(u^{10}+11u^5-1)\sqrt{u'}}) = \zeta_{10}^3\frac{u^2(u^{10}-39u^5-26)}{(u^{10}+11u^5-1)\sqrt{u'}}$ and $\alpha(\frac{26u^{10}-39u^5-1}{u(u^{10}+11u^5-1)\sqrt{u'}}) = \zeta_{10}^{-3}\frac{26u^{10}-39u^5-1}{u(u^{10}+11u^5-1)\sqrt{u'}}$. It follows that the representation of $A_5^{\mathrm{SL}_2}$ in W is conjugate to ρ_2. •

Only for some specific $F \in \bar{k}(x)$ the differential operator $\mathrm{Norm}(\phi_F(St_G))$ lies in $k(x)[\partial_x]$. The next corollary makes this precise.

Corollary 1.14

(1) $\mathrm{Norm}(\phi_F(St_G))$ *is defined over* $k \iff \forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)\ \exists\ S(\sigma) \in \bar{k}(t)$ *such that* $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$ *and* $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$.

(2) *Furthermore,* ϕ_F *satisfies the equivalent properties of* (1) *if and only if* $\phi_F = \phi_f \circ \phi_h$*, with* $f \in k(x)$*, and* ϕ_h *an automorphism of* $\bar{k}(t)$ *satisfying the equivalent properties of* (1).

Proof.
(1) '$\Longleftarrow$' $\forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$ we have $\sigma(\mathrm{Norm}(\phi_F(St_G))) = \mathrm{Norm}(\sigma(\phi_F(St_G))) = \mathrm{Norm}(\phi_{\sigma(F)}(St_G)) = \mathrm{Norm}(\phi_F \circ \phi_{S(\sigma)}(St_G)) = \mathrm{Norm}(\phi_F(St_G))$, so the operator is $\mathrm{Gal}(\bar{k}/k)$ invariant, hence has coefficients in $k(x)$.
'$\Longrightarrow$' Because $\mathrm{Norm}(\phi_F(St_G))$ is $\mathrm{Gal}(\bar{k}/k)$ invariant we get $\mathrm{Norm}(\phi_F(St_G)) = \sigma(\mathrm{Norm}(\phi_F(St_G))) = \mathrm{Norm}(\phi_{\sigma(F)}(St_G))\ \forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$. Hence Klein's theorem gives $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$, with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$. This proves (1).
(2) The if-part follows immediately from $\phi_{\sigma(F)} = \phi_{\sigma(f)} \circ \phi_{\sigma(h)} = \phi_f \circ \phi_{\sigma(h)}$. For the other implication write $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$, with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$ an automorphism of $\bar{k}(t)$ that permutes $0, 1, \infty$. Then there is also an automorphism ϕ_h of $\bar{k}(t)$, with $\phi_{\sigma(h)} = \phi_h \circ \phi_{S(\sigma)}\ \forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$. Namely, take

$h = \frac{a_1 - a_\infty}{a_1 - a_0}\frac{t-a_0}{t-a_\infty}$, where $a_0, a_1, a_\infty \in \bar{k}$ are elements which are permuted in the same way by every $\sigma \in \mathrm{Gal}(\bar{k}/k)$ as $0, 1, \infty$ by $\phi_{S(\sigma)}$. These elements are proven to exist in the lemma below. Note that for such a_0, a_1, a_∞ the extension $k(a_0, a_1, a_\infty)/k$ has degree at most 6. Define $f \in \bar{k}(x)$ by $\phi_f := \phi_F \circ \phi_h^{-1}$. Then $\phi_F = \phi_f \circ \phi_h$, and we only need to show that f is $\mathrm{Gal}(\bar{k}/k)$ invariant. But we have that $\phi_{\sigma(f)} = \phi_{\sigma(F)} \circ \phi_{\sigma(h)}^{-1} = \phi_F \circ \phi_{S(\sigma)} \circ (\phi_h \circ \phi_{S(\sigma)})^{-1} = \phi_f$ and therefore $f \in k(x)$. □

Remark 1.15 The above corollary states that every differential operator $\partial_x^2 - r$, with $r \in k(x)$ is the pullback of a differential operator over $k(x)$ with three singularities, and with the same local exponents as the corresponding standard operator (use $\mathrm{Norm}(\phi_h(St_G))$). So we can see this corollary as a "rational version" of Klein's theorem. •

In the proof above we used the following lemma. Its content is well known, and we prove it only for the sake of completeness.

Lemma 1.16 *Given an action of $G := Gal(\bar{k}/k)$ on the set $\{1, 2, 3\}$, there exists a Galois extension $k \subset k(a_1, a_2, a_3) \subset \bar{k}$, such that G permutes the set $\{a_1, a_2, a_3\}$ in the corresponding manner.*

Proof. We first assume G acts as S_3. Let H be the subgroup of G which fixes $\{1, 2, 3\}$. Then $F := \bar{k}^H$ is a Galois extension of k of degree 6. We have an action of $G/H \cong S_3$ on F. For some element σ of order two in S_3, write $k(a_1) = F^\sigma$. Then $k \subset k(a_1)$ is an extension of degree 3, which is not a Galois extension. Writing a_2, a_3 for the conjugates of a_1 in F, we have $F = k(a_1, a_2, a_3)$. Furthermore G acts as S_3 on the set $\{a_1, a_2, a_3\}$. We can rename the a_i, in such a way that G permutes the set $\{a_1, a_2, a_3\}$ in the desired manner. The remaining cases, where G acts as $1, C_2$ or C_3 are easy. □

Notation 1.17

- Let $L \in k(x)[\partial_x]$ be an arbitrary second order differential operator, with differential Galois group $G \subset \mathrm{GL}_2(\bar{k})$. We write G^p for the image of G in $\mathrm{PGL}(2)$, and call G^p the *projective differential Galois group* of L. This definition of G^p is consistent with the definition of G^p in the proof of Klein's theorem.

- For L as above, and $a \in \mathbb{P}^1(\overline{k})$, we have a set of local exponents $\{l_1, l_2\}$ at a. We will call $|l_1 - l_2|$ the *local exponent difference* at a.

Again let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p \in \{D_n, A_4, S_4, A_5\}$. We have that $\mathrm{Norm}(L)$ has the same projective differential Galois group. Indeed L and $\mathrm{Norm}(L)$ define the same field extension $\overline{k}(x) \subset \overline{k}(x)(s)$ (notation from the proof of Klein's theorem), and we can identify G^p with $\mathrm{Gal}(\overline{k}(x)(s)/\overline{k}(x))$, where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^p$ acts on s by $\sigma(s) = \frac{as+b}{cs+d}$. Consequently, the differential Galois group of $\mathrm{Norm}(L)$ is an element of $\{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. Using Klein's theorem we find that there exist elements $a, F, b \in \overline{k}(x)$, such that $L = a \cdot \phi_{F,b}(St_G)$.

1.1.3 Differential Galois group $D_2^{\mathrm{SL}_2}$

For generality, we formulate the following theorem for differential operators with projective differential Galois group D_2. This of course includes differential operators in normal form with differential Galois group $D_2^{\mathrm{SL}_2}$.

Theorem 1.18 *Let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p = D_2$. There exists a point $a \in \mathbb{P}^1(\bar{k})$ for which L has local exponent difference in $\frac{1}{2} + \mathbb{Z}$. For any such a there is an algebraic solution of minimal degree of the corresponding Riccati equation, with minimal polynomial in $k(a)[x]$.*

Proof. We will first show that we can assume L to be in normal form. We can write $\mathrm{Norm}(L) = a \cdot \phi_{x,b}(L)$, for some $a, b \in k(x)$. If u is a solution of the Riccati equation $R_{\mathrm{Norm}(L)}$, then $u + b$ is the corresponding solution of R_L. Writing f_u for the minimal polynomial of u over $\overline{k}(x)$, we clearly have $f_u \in k'(x)[T] \iff f_{u+b} \in k'(x)[T]$. Furthermore normalization does not affect the local exponent difference at a point.

Klein's theorem gives an $F \in \bar{k}(x)$ such that $L = \mathrm{Norm}(\phi_F(St_G))$, where $G := D_2^{\mathrm{SL}_2}$. We will use notations as in the proof of Klein's theorem. We have that $\{\frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}}\}$ is a basis of solutions of St_G. Then $\{\frac{s}{\sqrt{F's'}}, \frac{1}{\sqrt{F's'}}\}$ is a basis of solutions of L, where $'$ now denotes $\frac{d}{dx}$. We find that the solutions of R_L are precisely the elements $F'u - \frac{1}{2}\frac{F''}{F'}$, with u a solution of R_{St_G}. From the explicit description of $D_2^{\mathrm{SL}_2}$ in Lemma 1.4, we know that there are six solutions of R_{St_G} of degree two over $\overline{k}(t)$, which correspond to three minimal polynomials $\{P_1, P_2, P_3\}$. By [HP95] 6.5.3 we know that $P_i \in k(t)[T], i = 1, 2, 3$. Let u be one of the six solutions of R_{St_G} of degree 2 over $\overline{k}(t)$. Write $\tilde{u} := F'u - \frac{1}{2}\frac{F''}{F'}$ for the corresponding solution of R_L. If P_j is the minimal polynomial of u, then $F(P_j) := (F')^2 P_j(\frac{T}{F'} + \frac{1}{2}\frac{F''}{(F')^2}) \in \bar{k}(x)[T]$ is the minimal polynomial of $\tilde{u}$. Let $k \subset \tilde{k}$ be a minimal extension, such that $F \in \tilde{k}(x)$. Then $F(P_j) \in \tilde{k}(x)[T]$, so we can take $k' \subset \tilde{k}$, where k' is the field defined in the beginning of this section. Because $L \in k(x)[\partial_x]$, we have that F satisfies the properties stated in Corollary 1.14. Using notation as in the proof of this corollary, we see that we can take $\tilde{k}$ to be the extension of k generated by the coefficients of h, so $\tilde{k} = k(a_0, a_1, a_\infty)$. This is a field extension of k of degree at most 6.

Claim: for any $j \in \{0, 1, \infty\}$ there is a solution of R_L with minimal polynomial in $k(a_j)[T]$

The Galois group $\mathrm{Gal}(\overline{k}/k)$ acts as a group of permutations on the set $\{F(P_1), F(P_2), F(P_3)\}$. In fact $\sigma(F(P_i)) = \sigma(F)(P_i)$ for $\sigma \in \mathrm{Gal}(\overline{k}/k)$. By Corollary 1.14 we have $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$ with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$. We know the polynomials P_i explicitly, see Example 1.21. A calculation shows that all non-trivial automorphisms $\phi_S, S \in \{\frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t}{t-1}, \frac{t-1}{t}\}$ act non-trivially on the P_i. Using this we see that there exists a $\mathrm{Gal}(\overline{k}/k)$-equivariant bijection between $\{a_0, a_1, a_\infty\}$ and $\{F(P_1), F(P_2), F(P_3)\}$. This immediately proves the claim.

Let f be as in Corollary 1.14 (2). If $f(a) = a_i$, then $k(a_i) \subset k(a)$. So the only thing left to prove is that there exist points a with local exponent difference in $\frac{1}{2} + \mathbb{Z}$, and that any such point satisfies $f(a) \in \{a_0, a_1, a_\infty\}$. For this we need the following lemma.

Lemma 1.19 *With the above notation the following holds.*
The extension $\bar{k}(t) \subset \bar{k}(x)$ corresponds to a covering $\mathbb{P}^1_x \to \mathbb{P}^1_t$. Suppose that this covering is ramified with index e in a point $a \in \mathbb{P}^1_x(\bar{k})$ lying above some $b \in \mathbb{P}^1_t(\bar{k})$. The local exponent difference of $L = \mathrm{Norm}(\phi_F(St_G))$ at a is

$|e(l_1 - l_2)|$, where $\{l_1, l_2\}$ are the local exponents of St_G at b.

Proof. By a calculation as in the proof of Klein's theorem, we find that the local exponents of $\phi_F(St_G)$ at a are $\{el_1, el_2\}$. The lemma now follows from the fact that normalization does not change the local exponent difference at a point. $\square$

We continue the proof of Theorem 1.18. Using the above lemma, we see that if a point $a \in \mathbb{P}^1_x(\bar{k})$ does not lie above one of the points $0, 1, \infty$, then the local exponent difference of L at a lies in $\mathbb{Z}$. If a does lie above $b \in \{0, 1, \infty\}$, then the local exponents of St_G at b are $\{l_1, l_2\} = \pm\{\frac{1}{4}, \frac{3}{4}\}$, so the local exponent difference of L at a is in $\frac{1}{2} + \mathbb{Z}$ if e is odd, and in $\mathbb{Z}$ if e is even.

The only thing left to prove is that there exist points $a \in \mathbb{P}^1_x(\bar{k})$, such that L has local exponent difference in $\frac{1}{2} + \mathbb{Z}$ at a. By [PS03] Theorem 5.8, the differential Galois group of L is equal to the monodromy group, so there is a local monodromy matrix which has order 2 in $\mathrm{PGL}_2(\overline{k})$. It follows that the local exponents at the corresponding singular point have local exponent difference in $\frac{1}{2} + \mathbb{Z}$. $\square$

1.1.4 Differential Galois group $A_4^{\mathrm{SL}_2}$

Theorem 1.20 *Let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p = A_4$. There exists a point $a \in \mathbb{P}^1(\bar{k})$ for which L has local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$. For any such a there is an algebraic solution of minimal degree of the corresponding Riccati equation, with minimal polynomial in $k(a)[x]$.*

Proof. This case can be treated similar to the D_2-case above, now taking $G := A_4^{\mathrm{SL}_2}$. Again we will use notation of Corollary 1.14. We will only give the differences with the proof of Theorem 1.18.

The Riccati equation R_{St_G} has eight solutions of degree 4 over $\overline{k}(t)$, corresponding to two minimal polynomials $P_1, P_2, \in k(t)[T]$ (see Example 1.22, [HP95] 6.5.4, or [Ko86] 5.2). The group $\mathrm{Aut}(St_G)$ consists of two elements, namely $\{\phi_t, \phi_{\frac{t}{t-1}}\}$. Therefore an automorphism of St_G can only permute the singular points $\{1, \infty\}$. This implies $a_0 \in k$, $k(a_1) = k(a_\infty) = \tilde{k}$, and $[\tilde{k} : k] \leq 2$. So $F(P_i) := (F')^4 P_i(\frac{T}{F'} + \frac{1}{2}\frac{F''}{(F')^2}) \in k(a_1)[T] = k(a_\infty)[T]$. The only thing left to prove is that all points a with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$ satisfy $f(a) \in \{a_1, a_\infty\}$, and that there exists such a point.

The local exponents of St_G at the point 0 are $\{\frac{1}{4}, \frac{3}{4}\}$. At the point 1 the local exponents are $\{\frac{1}{3}, \frac{2}{3}\}$, and at the point ∞ they are $\{-\frac{1}{3}, -\frac{2}{3}\}$. Now Lemma 1.19 gives the following. The points with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$ are precisely the points $a \in \mathbb{P}^1_x(\bar{k})$ lying above $1, \infty$ with ramification index not divisible by 3. To prove that indeed there are such points a, we again use that the differential Galois group is equal to the monodromy group. We may assume that L is of the form $L = \phi_F(St_G)$. It follows that if the local exponent difference at a point lies in $\mathbb{Z}$, then the local exponents lie in $\frac{1}{2}\mathbb{Z}$. If all local exponents lie in $\frac{1}{2}\mathbb{Z}$, then the monodromy group is generated by elements of order ≤ 2. This contradicts the assumption that $G = A_4^{\mathrm{SL}_2}$, because $A_4^{\mathrm{SL}_2}$ is not generated by elements of order 2. $\square$

1.1.5 Examples

In the following examples we will give explicitly the minimal polynomials of solutions of R_{St_G} of minimal degree over $\overline{\mathbb{Q}}(t)$, for $G \in \{D_2^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}\}$. We will also calculate these minimal polynomials corresponding to pullbacks of standard equations.

Example 1.21 In the proof of Theorem 1.18 we showed that the Riccati equation R_{St_G}, $G := D_2^{\mathrm{SL}_2}$ has six algebraic solutions of degree two over $\overline{\mathbb{Q}}(t)$. Let $\{y_1, y_2\}$ be a basis of solutions of $St_{D_2^{\mathrm{SL}_2}}$, on which the differential Galois group G has the explicit form of Lemma 1.4. Then these six solutions of the Riccati equation are $\frac{y'}{y}, y \in \{y_1, y_2, y_1 + y_2, y_1 - y_2, y_1 + iy_2, y_1 - iy_2\}$, which are the solutions of the three polynomials

$$P_1 := T^2 - (\frac{1}{2}\frac{1}{t} + \frac{1}{t-1})T + \frac{1}{16}\frac{9t^2 - 7t + 1}{t^2(t-1)^2},$$

$$P_2 := T^2 - (\frac{1}{2}\frac{1}{t} + \frac{1}{2}\frac{1}{t-1})T + \frac{1}{16}\frac{3t^2 - 3t + 1}{t^2(t-1)^2},$$

$$P_3 := T^2 - (\frac{1}{t} + \frac{1}{2}\frac{1}{t-1})T + \frac{1}{16}\frac{9t^2 - 11t + 3}{t^2(t-1)^2}.$$

Now consider the function $F := \frac{2x}{x-\sqrt{2}}$, mapping $0, -\sqrt{2}, \sqrt{2}$ to $0, 1, \infty$ respectively. We have that ϕ_F satisfies the properties of Corollary 1.14 (1), so $L := \mathrm{Norm}(\phi_F(St_G))$ is defined over $\mathbb{Q}$. A calculation gives

$$L = \partial_x^2 + \frac{3}{8}\frac{3x^2+2}{x^2(x^2-1)^2}.$$

Using the formula in the proof of Theorem 1.18, we find that the six solutions of R_L of degree two over $\overline{\mathbb{Q}}(x)$ are the solutions of the polynomials

$$T^2 - \frac{1}{2}\frac{4x^2-\sqrt{2}x-2}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4-4\sqrt{2}x^3-9x^2+4\sqrt{2}x+2}{x^2(x^2-1)^2},$$

$$T^2 - \frac{1}{2}\frac{4x^2+\sqrt{2}x-2}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4+4\sqrt{2}x^3-9x^2-4\sqrt{2}x+2}{x^2(x^2-1)^2},$$

$$T^2 - \frac{2(x^2-1)}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4-15x^2+6}{x^2(x^2-1)^2}.$$

We remark that the local exponent difference is $\frac{1}{2}$ for each singular point of L. This is in accordance with Theorem 1.18. In [HP95] it is stated that $[k' : k] \in \{1, 3\}$ for $G = D_2^{\mathrm{SL}_2}$. This does not contradict the fact that we find $k' = \mathbb{Q}(\sqrt{2})$ for some of the solutions of R_L, because in [HP95] only fields k' of minimal degree over k are considered. $\diamond$

Example 1.22 We consider the standard equation St_G, $G := A_4^{\mathrm{SL}_2}$. In [Ko86] 5.2 one of the two minimal polynomials for solutions of R_{St_G} of degree 4 over $\overline{\mathbb{Q}}(t)$ is computed. It is the polynomial

$$P_1 := T^4 - \frac{7t-3}{3t(t-1)}T^3 + \frac{48t^2-41t+9}{24t^2(t-1)^2}T^2 - \frac{320t^3-409t^2+180t-27}{432t^3(t-1)^3}T + \frac{2048t^4-3484t^3-2313t^2-702t+81}{20736t^4(t-1)^4}.$$

The other minimal polynomial is $P_2 := S(P_1)$, $S = \frac{t}{t-1}$, where we use notation of the proof of Theorem 1.18. A calculation gives

$$P_2 = T^4 - \frac{8t-3}{3t(t-1)}T^3 + \frac{64t^2-49t+9}{24t^2(t-1)^2}T^2 - \frac{512t^3-598t^2+225t-27}{432t^3(t-1)^3}T +$$

$$\frac{-530t^4 + 2788t^3 - 909t^2 - 918t + 81}{20736t^4(t-1)^4}.$$

Let $a \in \mathbb{Q}$, and define $F := \frac{2x}{x-\sqrt{a}}$ which maps $0, -\sqrt{a}, \sqrt{a}$ to $0, 1, \infty$ respectively. Then $L := \mathrm{Norm}(\phi_F(St_G))$ is

$$\partial_x^2 + \frac{3}{16}\frac{1}{x^2} - \frac{3}{16}\frac{1}{x^2-a} + \frac{8}{9}\frac{a}{(x^2-a)^2}.$$

The local exponents at $0, -\sqrt{a}, \sqrt{a}$ are $\{\frac{1}{4}, \frac{3}{4}\}, \{\frac{1}{3}, \frac{2}{3}\}, \{\frac{1}{3}, \frac{2}{3}\}$ respectively. So theorem 1.20 states that there is a solution of R_L of degree 4 over $\overline{\mathbb{Q}}(x)$, such that the corresponding field k' lies in $\mathbb{Q}(\sqrt{a})$. A calculations shows that in fact for each solution of R_L of degree 4 over $\overline{\mathbb{Q}}(x)$, the corresponding field k' equals $\mathbb{Q}(\sqrt{a})$. $\diamond$

1.1.6 Algorithm

We will now give an algorithm to compute a field k' (as defined in the beginning of this chapter). We will also give some examples.

Let $L \in k(x)[\partial_x]$ be a second order differential operator in normal form with known differential Galois group G in $\{D_2^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}\}$. We can use theorems 1.18 and 1.20 to find a field k'. Write $L = \partial_x^2 - \frac{T}{N}, T, N \in k[x]$, where $gcd(T, N) = 1$, and N is monic. Because G is finite, all singularities of L are regular singular (see [PS03] Definition 3.9). Therefore, the zeros of N can at most have order two. So we can write $N = N_1 \cdot N_2^2$, such that N_1, N_2 have only zeros of order one, and are monic. We can make a decomposition $\frac{T}{N} = \frac{A}{N_2^2} + \frac{B}{N_1}$. Now the local exponents at some point $p \in \bar{k}$ are the solutions of the equation $\lambda(\lambda-1) = \frac{A\cdot(x-p)^2}{N_2^2}|_{x=p}$. So the local exponents λ satisfy $\lambda(\lambda-1) = \frac{A(p)}{N_2'(p)^2}$.

For the D_2-case we search for points with local exponent difference in $\frac{1}{2} + \mathbb{Z}$. Because L is in normal form, the local exponents of L at such a point are $\{\frac{2n+1}{4}, \frac{3-2n}{4}\}$, for some $n \in \mathbb{Z}$. Therefore we get the system of equations:

$$D_2\text{-case} : \begin{cases} (3 + 4n - 4n^2)N_2'(p)^2 + 16A(p) = 0 \\ N_2(p) = 0. \end{cases}$$

To solve this system we can calculate the resultant of $(3+4n-4n^2)N_2'(x)^2+16A(x)$ and $N_2(x)$ with respect to x. This gives a polynomial in n, for which it is easy to determine if it has integer solutions. If this resultant is zero for some n_0, then we can substitute $n = n_0$ into the system of equations. Then solutions of the system are given by $\gcd((3+4n_0-4n_0^2)(N_2')^2+16A, N_2) = 0$.

For the A_4-case we search for points with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$. At such a point the local exponents are $\{\frac{3+n}{6}, \frac{3-n}{6}\}$, for some $n \in \mathbb{Z}$. We find the system of equations:

$$A_4\text{-case}: \begin{cases} (9-n^2)N_2'(p)^2 + 36A(p) = 0 \\ N_2(p) = 0, \end{cases}$$

We search solutions, with $n \not\equiv 0 \bmod 3$. This system can be solved in the same way as in the D_2-case. We conclude that for a differential operator L satisfying our assumptions, we can find a corresponding field k'.

Example 1.23 We will demonstrate the algorithm for

$$L := \partial_x^2 + \frac{16x^{18} - 288x^{15} + 2160x^{12} - 8947x^9 + 20745x^6 - 25056x^3 + 13456}{8(x^9 - 9x^6 + 27x^3 - 29)^2 x^2}.$$

This is the operator obtained as the pullback of $St_{D_2^{\mathrm{SL}_2}}$ with $F = h \circ (x^3 - 3)$, where h is some automorphism of $\bar{k}(t)$ that sends the roots of $x^3 - 2$ to $\{0, 1, \infty\}$. With the notation of the algorithm we have
$T = \frac{1}{8}(16x^{18} - 288x^{15} + 2160x^{12} - 8947x^9 + 20745x^6 - 25056x^3 + 13456)$,
$N = (x^9 - 9x^6 + 27x^3 - 29)^2 x^2$.
We calculate N_2 by $N_2 = gcd(N, N')$ which obviously is $(x^9-9x^6+27x^3-29)x$ and furthermore $A = T$. Using for example Maple we find that the resultant of $(3+4n-4n^2)(N_2')^2 + 16A$ and N_2 over x is
$(457668486144n^3 - 1373005458432n^4 + 1373005458432n^5 - 457668486144n^6)^3$ $(29435 + 3364n - 3364n^2)$. This expression has as integer solutions $n = 0$ and $n = 1$, which both correspond to the same set of local exponents. Substituting $n = 0$ we get $gcd(3N_2' + 16A, N_2) = x^9 - 9x^6 + 27x^3 - 29$, a polynomial in x^3, with as a solution $a := (3 + 2^{\frac{1}{3}})^{\frac{1}{3}}$. So there is a field $k' \subset \mathbb{Q}(a)$. We know that $[k' : k] \leq 3$, so $k' = k$ or $[k' : k] = 3$. We can calculate all subfields of $\mathbb{Q}(a)$ of degree 3 over $\mathbb{Q}$ in Maple 7, with the command

```
evala(Subfields(x^9-9x^6+27x^3-29,3));
```

It turns out that the only such subfield is $\mathbb{Q}(\sqrt[3]{2})$. It follows that there is a field $k' \subset \mathbb{Q}(\sqrt[3]{2})$. $\diamond$

Example 1.24 Let h to be the automorphism of $\mathbb{P}^1$ sending $1, -\sqrt{2}, \sqrt{2}$ to $0, 1, \infty$ respectively. Let $f = x^2 - 3$, and $F = h \circ f$, then $\mathrm{Norm}(\phi_F(St_{A_4}))$ is

$$\partial_x^2 - \frac{27x^{12} - 540x^{10} + 4145x^8 - 16366x^6 + 37160x^4 - 46872x^2 + 21168}{(6x(x-2)(x+2)(x^4 - 6x^2 + 7)^2}.$$

As before we write this as $\partial_x^2 - (\frac{A}{N_2^2} + \frac{B}{N_1})$. The resultant of $(9-n^2)(N_2')^2 + 36A$ and N_2 is $-2^{58}3^{26}7^6(n-6)(n+6)(2n-3)^2(2n+3)^2(n-1)^4(n+1)^4$. The integer solutions for n are $n \in \{-6, -1, 1, 6\}$. So only $n = 1$ (which gives the same as $n = -1$) is of interest, for $-6, 6 \equiv 0 \bmod 3$. We now substitute $n = 1$ into $(9 - n^2)(N_2')^2 + 36A$, and calculate the greatest common divisor with N_2. This gives the polynomial $x^4 - 6x^2 + 7 = (x-3)^2 - 2$. A zero of this polynomial is $a = \sqrt{\sqrt{2} + 3}$, so there is a field k' of degree ≤ 2 over $\mathbb{Q}$ in $\mathbb{Q}(\sqrt{\sqrt{2}+3})$. By a calculation in Maple 7 we find that the only field extension of $\mathbb{Q}$ of order 2 in $\mathbb{Q}(\sqrt{\sqrt{2}+3})$ is $\mathbb{Q}(\sqrt{2})$. Therefore there is a field $k' \subset \mathbb{Q}(\sqrt{2})$. Note that in this example we can explicitly calculate k' from knowing only the operator and the differential Galois group. $\diamond$

1.2 Algorithms for finding the pullback function

The material in this section is joint work with Mark van Hoeij and Jacques-Arthur Weil.

Let $L = \partial_x^2 + a_1\partial_x + a_0 \in k(x)[\partial_x]$ be a monic order 2 differential operator. We suppose the differential Galois group G over $\bar{k}(x)$ is known and is a finite subgroup of $\mathrm{GL}(2, \bar{k})$. We will write G^p for the image of G in the $\mathrm{PGL}_2(\bar{k})$. The normalization $\mathrm{Norm}(L)$ of L is obtained by a shift $\partial_x \mapsto \partial_x - \frac{a_1}{2}$, and we write G^n for the differential Galois group of $\mathrm{Norm}(L)$. We assume G^n is non-cyclic, which implies $G^n \in \{A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}, D_n^{\mathrm{SL}_2}\}$. By Klein's theorem we have $\mathrm{Norm}(L) = \mathrm{Norm}(\phi_F(St_{G^n}))$, for some $F \in \overline{k}(x)$. Therefore $\exists\, b \in \overline{k}(x)$, such that for $\phi := \phi_{F,b}$ we have $L = (\phi(t)')^2\phi(St_{G^n})$.

In this section we will concentrate on finding $\phi(t)$, which we will do case by case with respect to G^p. We will define new standard equations St_{G^p}, with projective Galois group G^p, for $G^p \in \{A_4, S_4, A_5, D_n\}$. For this standard equations Klein's theorem still holds, and we are able to give an explicit formula for $\phi(t)$.

Notation 1.25

- Let $\phi : \overline{k}(t)[\partial_t] \to \overline{k}(x)[\partial_x], \phi(\partial_t) = \frac{1}{\phi(t)'}(\partial_x + b)$ be a homomorphism. Then we call $\phi(t)$ the *pullback function* corresponding to ϕ.

- Let $L_1 \in \overline{k}(t)[\partial_t], L_2 \in \overline{k}(x)[\partial_x]$, be differential operators, such that we can write $L_2 = a\phi_{F,b}(L_1), a, F, b \in \overline{k}(x)$. If $b = 0$, we call L_2 a *pullback* of L_1. If $b \neq 0$, we call L_2 a *weak pullback* of L_1.

1.2.1 Projective Galois group A_4

We define the following new standard equation:

$$St_{A_4} := \partial_t^2 + \frac{8t+3}{6t(t+1)}\partial_t + \frac{s}{t(t+1)^2},\ s = \frac{1}{48}.$$

This differential operator is obtained from $St_{A_4^{\mathrm{SL}_2}}$ by first making the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{1}{3(t-1)}$, and then applying the coordinate transformation $t \mapsto \frac{t}{t+1}$. So $St_{A_4} = \phi_{\frac{t}{t+1}, \frac{1}{4t} - \frac{7}{12(t+1)}}(St_{A_4^{\mathrm{SL}_2}})$. We will now motivate this new choice of a standard operator.

From the fact that the projective differential Galois group of $St_{A_4^{\mathrm{SL}_2}}$ is A_4, it follows, using some representation theory, that this operator has solutions $y_1, \cdots, y_4$ such that $\frac{(y_1 \cdots y_4)'}{y_1 \cdots y_4} \in \overline{k}(t)$. This translates into the existence of a degree one right-hand factor of $\mathrm{Sym}(St_{A_4^{\mathrm{SL}_2}}, 4)$. In fact there are precisely two such right-hand factors. By a direct computation, we find that these right-hand factors are $\partial_t - \frac{1}{t} - \frac{4}{3(t-1)}$ and $\partial_t - \frac{1}{t} - \frac{5}{3(t-1)}$. We constructed St_{A_4} such that $\mathrm{Sym}(St_{A_4}, 4)$ has a right-hand factor ∂_t. To see this, note that for any differential operator L, we have $\mathrm{Sym}(\phi_{f,b}(L), n) = \phi_{f,nb}(\mathrm{Sym}(L, n))$. So applying the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{1}{3(t-1)}$ to $St_{A_4^{\mathrm{SL}_2}}$ gives a differential operator $\tilde{St} = \partial_t^2 + \frac{7t-3}{6t(t-1)}\partial_t - \frac{1}{48t(t-1)}$ with the property that $\mathrm{Sym}(\tilde{St}, 4)$ has a right-hand factor ∂_t. The coordinate transformation $t \mapsto \frac{t}{t+1}$ does not changes this property, and will make the pullback formula in Theorem 1.27 somewhat nicer. Note that applying the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{5}{12(t-1)}$ to $St_{A_4^{\mathrm{SL}_2}}$ also results in a differential operator such that its fourth symmetric power has a right-hand factor ∂_t. This differential operator is different from $\tilde{St}$. There is a non-trivial automorphism of $\overline{k}(t)[\partial_t]$ mapping St_{A_4} to a multiple of itself, namely $\phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$. It follows immediately from the proof of Klein's theorem that this is the unique non-trivial automorphism of St_{A_4}.

Proposition 1.26 *The differential Galois group G of St_{A_4} is a central extension of A_4 by the cyclic group C_4.*

Proof. Let F be a fundamental matrix for St_{A_4}, i.e., a matrix $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, where $\{y_1, y_2\}$ is a basis of solutions of St_{A_4}. The determinant $\mathrm{Det}(F)$ of F, satisfies the differential operator $\partial_t + \frac{8t+3}{6t(t+1)} = \bigwedge^2 St_{A_4}$. For $g \in G$, we have $g(\mathrm{Det}(F)) = \mathrm{Det}(g) \cdot \mathrm{Det}(F)$, so the differential Galois group of $\partial_t + \frac{8t+3}{6t(t+1)}$ is precisely the image of G under the determinant map. The differential Galois group of $\partial_t + \frac{8t+3}{6t(t+1)}$ is easily seen to be the group μ_6 consisting of the sixth roots of unity. So $G \subset H := \{M \in \mathrm{GL}_2(\overline{\mathbb{Q}}) | \mathrm{Det}(M)^6 = 1, M^p \in A_4\}$, where M^p denotes the image of M in $\mathrm{PGL}_2(\overline{\mathbb{Q}})$.

By a calculation in Maple, we find a basis $\{y_1, y_2\}$ of solutions for St_{A_4}, with $y_1 y_2 = \sqrt{\frac{a+1}{a}}, y_1^4 = \frac{\sqrt{3}(a-1)+2\sqrt{a^2-a+1}}{a}, a^3 + t + 1 = 0$. From this we see that the Picard-Vessiot extension $\overline{\mathbb{Q}}(t)(y_1, y_2)$ lies in the degree 48 extension $K := \overline{\mathbb{Q}}(t)(a, \sqrt{a^2 - a + 1}, \sqrt{\frac{a+1}{a}}, \sqrt[4]{\frac{\sqrt{3}(a-1)+2\sqrt{a^2-a+1}}{a}})$ of $\overline{\mathbb{Q}}(t)$, where $a^3+t+1 = 0$. In order to determine G precisely, we will make use of the local exponents of St_{A_4}. Let E_p denote the set of local exponents at the point p. Then we have $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{1}{4}, \frac{-1}{12}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. Now Proposition 5.1 in [PU00], provides us with an element $g_{-1} \in G$, which is conjugated to $e^{2\pi i D}$, where D is the diagonal matrix with $\frac{1}{4}, \frac{-1}{12}$ on the diagonal. So the eigenvalues of g_{-1} are $\{e^{\frac{\pi i}{2}}, e^{\frac{-\pi i}{6}}\}$, and therefore $\mathrm{Det}(g_{-1}) = e^{\frac{\pi i}{3}}$. We have $g_{-1}^3 = -i \cdot Id$. So the kernel of the natural map $G \to G^p$ has at least order 4, and we find that G has at least order 48. We already found that G had maximally order 48, so G is a central extension of A_4 by C_4 of order 48, and K is a Picard-Vessiot extension for St_{A_4}. □

We will now give the pullback function for a second order differential operator $L \in k(x)[\partial_x]$ with projective Galois group A_4. After applying a shift, we can suppose (as in the case of $St_{A_4^{\mathrm{SL}_2}}$) that $\mathrm{Sym}(L, 4)$ has a right-hand factor ∂_x. This shift does not change the pullback function. This shift can be found in the following way. By representation theory it follows that the operator $\mathrm{Sym}(L, 4)$ has two degree one right-hand factors, say $(\partial_x + b_1)$ and $(\partial_x + b_2)$. The b_i are rational solutions of the Riccati equation corresponding to $\mathrm{Sym}(L, 4)$, and therefore the b_i can be computed. The group $\mathrm{Gal}(\overline{k}/k)$ acts on $\{b_1, b_2\}$, so we find that $b_1, b_2 \in k'(x)$ for some minimal field $k' \subset \overline{k}$ of degree ≤ 2 over k. In fact this field k' is the field defined the beginning of this chapter. To see this, let $u = \frac{y'}{y}$ be an algebraic solution of the Riccati equation R_L of degree 4 over $\overline{k}(x)$. Then the sum b of the conjugates of u under the differential Galois group of L is a rational solution of the Riccati equation corresponding to $\mathrm{Sym}(L, 4)$, and we see that $b \in k'(x)$ if and only if the minimal polynomial of u is defined over $k'(x)$.

Theorem 1.27 *Let $L = \partial_x^2 + a_1\partial_x + a_0$, with $a_0, a_1 \in k(x)$, be a differential operator with projective Galois group A_4 such that $\mathrm{Sym}(L, 4)$ has a right-hand factor ∂_x. Then L is the pullback of St_{A_4}, with pullback function $\phi(t) := \frac{9s}{a_0}(\frac{a_0'}{a_0} + 2a_1)^2, s = \frac{1}{48}$. The only other (weak) pullback is obtained by* *composition with the unique non-trivial automorphism of* St_{A_4}.

Proof. We will first show, that for the suitable choice of $\phi(t)$ no shift is needed. By Klein's theorem, there exists $\phi : \overline{k}(t)[\partial_t] \to \overline{k}(x)[\partial_x]$, with $L = (\phi(t)')^2\phi(St_{A_4})$. The $\mathrm{Sym}(St_{A_4}, 4)$ has a right-hand factor ∂_t, and because $\phi(\mathrm{Sym}(St_{A_4}, 4)) = \mathrm{Sym}(\phi(St_{A_4}), 4)$, the $\mathrm{Sym}(L, 4)$ has a right-hand factor $\phi(\partial_t)$. The $\mathrm{Sym}(L, 4)$ has two right-hand factors of degree one, ∂_x and $\partial_x - u$ for some $u \in \overline{k}(x)$, so $\phi(\partial_t) \in \{\frac{1}{\phi(t)'}\partial_x, \frac{1}{\phi(t)'}(\partial_x - u)\}$. If $\phi(\partial_t) = \frac{1}{\phi(t)'}\partial_x$ we are done, otherwise consider the automorphism $\psi := \phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$ of St_{A_4}. Then $\phi \circ \psi$ is the other possible pullback, with a different image for ∂_t, so this image must be $\frac{1}{\phi(t)'}\partial_x$. Therefore we may suppose that ϕ has no shift. We will now calculate $\phi(t)$.

The formula for $\phi(t)$ can be obtained using the following trick. Write St_{A_4} as $\partial_t^2 + s_1\partial_t + s_0$, so $s_0 = \frac{s}{t(t+1)^2}$ and $s_1 = \frac{8t+3}{6t(t+1)}$. In the following we will use $\phi(f)' = \phi(t)'\phi(f')$, where $f \in \overline{k}(t)$, and $'$ denotes $\frac{d}{dx}$ or $\frac{d}{dt}$. Applying ϕ to $t = \frac{s}{s_0}(\frac{1}{t+1})^2$, we get $\phi(t) = \frac{s}{\phi(s_0)}(\frac{1}{\phi(t+1)})^2 = \frac{s}{(\phi(t)')^2\phi(s_0)}(\frac{\phi(t)'}{\phi(t+1)})^2$. Furthermore $a_0 = (\phi(t)')^2\phi(s_0)$, so $\phi(t) = \frac{s}{a_0}(\frac{\phi(t)'}{\phi(t+1)})^2$. We are done if we can prove $\frac{\phi(t)'}{\phi(t+1)} = 3\frac{a_0'}{a_0} + 6a_1$. Using $a_1 = \phi(t)'\phi(s_1) - \frac{\phi(t)''}{\phi(t)'}$, we can write $3\frac{a_0'}{a_0} + 6a_1$ as $3(2\frac{\phi(t)''}{\phi(t)'} + \frac{\phi(s_0)'}{\phi(s_0)}) + 6(\phi(t)'\phi(s_1) - \frac{\phi(t)''}{\phi(t)'}) = 3\phi(t)'\phi(\frac{s_0'}{s_0}) + 6\phi(t)'\phi(s_1) = \phi(t)'\phi(-3(\frac{1}{t} + \frac{2}{t+1}) + \frac{8t+3}{t(t+1)}) = \frac{\phi(t)'}{\phi(t+1)}$, which finishes the proof. □

Remark 1.28 This pullback formula was found using *semi-invariants*. The representation of A_4 in the $\mathrm{PGL}(\overline{k}x_1 + \overline{k}x_2)$ induces an action of A_4 on $\overline{k}[x_1, x_2]$. A polynomial P in this ring is a semi-invariant if $\forall\, \sigma \in A_4\ \exists\, c_\sigma \in \overline{k}^*$ such that $\sigma(P) = c_\sigma P$. There are two semi-invariants $H_1(x_1, x_2), H_2(x_1, x_2)$ of degree 4, such that for a basis of solutions $\{y_1, y_2\}$ of St_{A_4} we have $\frac{H_1(y_1,y_2)^3}{H_2(y_1,y_2)^3} = t + 1$. Let $\{v_1, v_2\}$ be a basis of solutions of L. Then we find $\frac{H_1(v_1,v_2)^3}{H_2(v_1,v_2)^3} = \phi(t + 1)$. The expressions $H_1(v_1, v_2), H_2(v_1, v_2)$ are so-called *exponential solutions* of $\mathrm{Sym}(L, 4)$, i.e., $\frac{H_i(v_1,v_2)'}{H_i(v_1,v_2)} \in \overline{k}(x)$. These exponential solutions can be found (up to constants). We can also give a formula for one of these exponential solutions in terms of the other and the coefficients of L. So if we suppose $H_1(v_1, v_2) = 1$, we find a formula for the pullback function in terms of the coefficients of L. •

Corollary 1.29 *Let $L = \partial_x^2 + a_1\partial_x + a_0$ be a differential operator, with projective Galois group A_4. There are two differential operators $L_i, i = 1,2$ obtained from L by a shift $\partial_x \mapsto \partial_x + b_i$, such that* $\mathrm{Sym}(L_i, 4)$ *has a right-hand factor ∂_x. Let F_i be the pullback function of L_i as in Theorem 1.27. Then $F_2 = \frac{-F_1}{F_1+1}$ and $b_2 = b_1 - \frac{F_1'}{12(F_1+1)}$.*

Proof. We recall that the unique non-trivial automorphism of St_{A_4} is $\phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$. We have $F_1^2\phi_{F_1}(St_{A_4}) = L_1 = \phi_{t,b_1-b_2}(L_2) = F_2^2\phi_{F_2,b_1-b_2}(St_{A_4})$. Because $b_2 \neq b_1$ we must have $\phi_{F_2,b_1-b_2} = \phi_{F_1} \circ \phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$, so $F_2 = \frac{-F_1}{F_1+1}$ and $b_2 = b_1 - \frac{F_1'}{12(F_1+1)}$. □

1.2.2 Projective Galois group S_4 or A_5

These two cases can be treated in almost the same way as the A_4-case. We will only give the differences. The new standard equations we will use are:

$$St_{G^p} := \partial_t^2 + \frac{8t+3}{6t(t+1)}\partial_t + \frac{s}{t(t+1)^2}$$

with $s = \frac{5}{576}$ for $G^p = S_4$, and $s = \frac{11}{3600}$ for $G^p = A_5$. In both cases there are no automorphisms (i.e no automorphisms of $\overline{\mathbb{Q}}(t)[\partial_t]$ mapping St_{G^p} to a multiple of itself). Using representation theory we find that S_4 and A_5 have a unique semi-invariant of degree $m = 6, 12$, respectively. The new standard equations are chosen in such a way that $\mathrm{Sym}(St_{G^p}, m)$ has a right-hand factor ∂_t.

Proposition 1.30 *The Galois group of St_{G^p}, $G^p \in \{S_4, A_5\}$ is a central extension of G^p by the cyclic group C_6.*

Proof. We start by calculating G_1, the Galois group of St_{S_4}. The local exponents of St_{S_4} are given by $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{5}{24}, -\frac{1}{24}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. As in the A_4 case, we conclude that there is an element $g_{-1} \in G_1$ of order 24 with eigenvalues $\{e^{-\frac{1}{12}\pi i}, e^{\frac{5}{12}\pi i}\}$, so with $\mathrm{Det}(g_{-1}) = e^{\frac{1}{3}\pi i}$. We have that $g_{-1}^4 = e^{-\frac{1}{3}\pi i} \cdot Id$ is an element in the kernel of the map $G_1 \mapsto G_1^p = S_4$. We find that this kernel has at least order 6, so G_1 has order ≥ 144. Reasoning as in the A_4-case we find that G_1 has order 144, and it is a central extension of S_4 by C_6.

We will now calculate the Galois group G_2 of St_{A_5}. The local exponents of St_{A_5} are given by $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{11}{60}, -\frac{1}{60}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. So there is an element $h_{-1} \in G_2$ of order 60, with eigenvalues $e^{-\frac{1}{30}\pi i}, e^{\frac{11}{30}\pi i}$. We have $\mathrm{Det}(h_{-1}) = e^{\pi i \frac{1}{3}}$, so h_{-1}^5 is an element in the kernel of the map $G_2 \mapsto G_2^p = A_5$. Again reasoning as in the A_4-case we get that G_2 has order 360, and it is a central extension of A_5 by C_6. $\square$

Let a differential operator L with projective Galois group $G^p \in \{S_4, A_5\}$ be given. Then after applying a shift we can assume that $\mathrm{Sym}(L, m)$ has a right-hand factor ∂_x, where $m = 6$ if $G^p = S_4$ and $m = 12$ if $G^p = A_5$. The shift we have to apply is $\partial_x \mapsto \partial_x + \frac{b}{m}$, with b a rational solution of the Riccati equation corresponding to $\mathrm{Sym}(L, m)$. From the uniqueness of b it also follows that the field k' as defined in the beginning of this chapter is equal to k (compare the A_4-case).

It is now clear that we get the following generalization of Theorem 1.27.

Theorem 1.31 *Let $L = \partial_x^2 + a_1\partial_x + a_0$, with $a_0, a_1 \in k(x)$, be a differential operator, with projective Galois group $G^p \in \{A_4, S_4, A_5\}$. Set $m = 4, s = \frac{1}{48}$ if $G^p = A_4$, set $m = 6, s = \frac{5}{576}$ if $G^p = S_4$, and set $m = 12, s = \frac{11}{3600}$ if $G^p = A_5$. If $\mathrm{Sym}(L, m)$ has a right-hand factor ∂_x, then L is the pullback of St_{G^p}, with pullback function $\phi(t) := \frac{9s}{a_0}(\frac{a_0'}{a_0} + 2a_1)^2$.*

1.2.3 Projective Galois group D_n, $n \geq 2$

Let $\tilde{L} \in k(x)[\partial_x]$ be a second order differential operator with $G^p = D_n$. Then for $n \geq 3$, we have that $\mathrm{Sym}(\tilde{L}, 2)$ has precisely one right-hand factor of degree one over $\overline{k}(x)$, say $\partial_x + a$. The shift $\partial_x \mapsto \partial_x - \frac{a}{2}$ transforms $\tilde{L}$ into a differential operator L, such that $\mathrm{Sym}(L, 2)$ has a right-hand factor ∂_x. In the case $G^p = D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the operator $\mathrm{Sym}(\tilde{L}, 2)$ has three degree one right-hand factors. So there are three possible shifts transforming $\tilde{L}$ into a differential operator L such that $\mathrm{Sym}(L, 2)$ has a right-hand factor ∂_x. We note that from the above we can conclude that the field k' defined in the beginning of this chapter satisfies $k' = k$ for $G^p = D_n, n > 2$ and $[k' : k] \leq 3$ for $G^p = D_2$ (see the A_4-case for details).

A calculation shows that if $L = \partial_x^2 + a_1\partial_x + a_0$ satisfies $\mathrm{Sym}(L, 2) = * \cdot \partial_x$, then $\frac{a_0'}{a_0} = -2a_1$ and a basis of solutions is given by $\{y, \frac{1}{y}\}, y = e^{\int \sqrt{-a_0}dx}$. We will now calculate the possibilities for the differential Galois group G of an operator L with these properties, and moreover with $G^p = D_n, n \geq 2$. We have that the extension $\overline{k}(t) \subset K^p = \overline{k}(t)(y^2)$ is Galois with Galois group D_n. So $\overline{k}(t)(y^2)$ is a differential field, and consequently $K = \overline{k}(t)(y)$ is a differential field, too. Therefore it is a Picard-Vessiot extension for L. The extension $\overline{k}(t)(y^2) \subset \overline{k}(t)(y)$ has degree one or two.

A small calculation shows that on the basis $\{y, \frac{1}{y}\}$, the differential Galois group G lies in $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}$. The image of G in PGL_2 must be $G^p = D_n = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \right\rangle$, where ζ_{2n} is a $2n$-th root of unity. We have $|G|/|G^p| \leq 2$ and we find that $|G| = |G^p|$ can only occur when n is odd. Then $G^p \cong G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \right\rangle$ or $G = \left\langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \right\rangle$, where we can take $\zeta_n = -\zeta_{2n}$. In case $|G| = 2|G^p|$ we have $D_{2n} \cong G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \right\rangle$.

We will use the following new standard equations:

$$St_{D_n} := \partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{4n^2(t^2-1)}.$$

For $n > 2$ we have one non-trivial automorphism of St_{D_n}, namely $\psi = \phi_{-t}$. The group of automorphisms of St_{D_2} is isomorphic to S_3, and has generators ψ, ψ_1, with ψ as above, and $\psi_1 = \phi_{\frac{t+3}{t-1}, -\frac{1}{4(t-1)}}$. The operator $\mathrm{Sym}(St_{D_n}, 2)$ has a right-hand factor ∂_t. The differential Galois group of St_{D_n} is D_{2n}, which follows from the fact that $\{y, \frac{1}{y}\}$, $y = (t + \sqrt{t^2-1})^{\frac{1}{2n}}$ is a basis of solutions. Indeed, y satisfies the irreducible polynomial $T^{4n} - 2tT^{2n} + 1$.

Lemma 1.32 $\mathrm{Sym}(St_{D_n}, 2n)$ *has a basis of rational solutions* $\{1, t\}$. *Furthermore* t *is up to constants the unique rational solution of the right-hand factor* $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$ *of* $\mathrm{Sym}(St_{D_n}, 2n)$.

Proof. Write $St_{D_n} = \partial_t^2 + s_1\partial_t + s_0$. We have that St_{D_n} has a basis of solutions $\{y, \frac{1}{y}\}$. A direct calculation (or an examination of the explicit form of the solutions presented above) shows that for any non-zero integer k, the operator $\partial_t^2 + s_1\partial_t + k^2 s_0$ has as basis of solutions $\{y^k, y^{-k}\}$. Therefore $\partial_t^2 + s_1\partial_t + k^2 s_0$ is a right-hand factor of $\mathrm{Sym}(St_{D_n}, k)$. In particular $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$ is a right-hand factor of $\mathrm{Sym}(St_{D_n}, 2n)$. Now observe that t is a solution of $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$. We note that $1 = y^n \cdot y^{-n}$ is also a solution of $\mathrm{Sym}(St_{D_n}, 2n)$. It is easily seen that the space of rational solutions of $\mathrm{Sym}(St_{D_n}, 2n)$ is 2-dimensional. Indeed the differential Galois group D_{2n} of St_{D_n} is generated by σ, τ with $\sigma(y) = \zeta_{2n} y$, with ζ_{2n} a primitive $2n$-th root of unity, and $\tau(y) = y^{-1}$. A basis of solutions of $\mathrm{Sym}(St_{D_n}, 2n)$ is $\{y^{2n}, y^{2n-2}, \cdots, y^{-2n}$, and it immediately follows that $\{1, y^{2n} + y^{-2n}\}$ is a basis of the rational solutions of $\mathrm{Sym}(St_{D_n}, 2n)$. $\square$

Theorem 1.33 *Let $L = \partial_x^2 + a_1\partial_x + a_0$ be a differential operator, with projective differential Galois group $D_n, n \geq 3$, such that $\mathrm{Sym}(L,2)$ has a right-hand factor ∂_x. The right-hand factor $\partial_x^2 + a_1\partial_x + 4n^2 a_0$ of $\mathrm{Sym}(L, 2n)$ has, up to constants, a unique rational solution, say a. Write $b := \frac{a'}{a}$, then b is independent of the choice of a. Now L is the pullback of St_{D_n}, with pullback function $\phi(t) := (1 + \frac{b^2}{4n^2 a_0})^{-\frac{1}{2}}$. The only other pullback function is $-\phi(t)$.*

Proof. The proof is somewhat similar to the proof of Theorem 1.27. The fact that no shift is needed follows from the fact that $\mathrm{Sym}(L, 2)$ has a unique degree one right-hand factor. An argument as in the proof of Lemma 1.32 shows the existence and unicity of the rational solution a. As before, write $St_{D_n} = \partial_t^2 + s_1\partial_t + s_0$. From the expression $s_0 = \frac{-1}{4n^2(t^2-1)}$ it follows that $t = (1 + \frac{1}{4n^2 s_0 t^2})^{-\frac{1}{2}}$. The pullback map transforms this expression into $\phi(t) = (1 + \frac{\phi(t)'^2}{4n^2 a_0 \phi(t)^2})^{-\frac{1}{2}}$, since $a_0 = \phi(t)'^2\phi(s_0)$. By the previous lemma, t is a rational solution of $\partial_t^2 + s_1\partial_t + 4n^2 s_0$. Therefore $\phi(t)$ is a rational solution of $\phi(\partial_t^2 + s_1\partial_t + 4n^2 s_0) = (\frac{1}{\phi(t)'})^2(\partial_x^2 + a_1\partial_x + 4n^2 a_0)$. Consequently $b = \frac{\phi(t)'}{\phi(t)}$, and it follows that $\phi(t) = (1 + \frac{b^2}{4n^2 a_0})^{-\frac{1}{2}}$. We see that a different choice for the square root changes $\phi(t)$ into $-\phi(t)$. It also follows from Klein's Theorem (1.9) that $-\phi(t)$ is the only other possible pullback function. $\square$

Remark 1.34 In the above proof, we see that $\phi(t) = c \cdot a$ for some constant $c \in \overline{k}$, and a as in Theorem 1.33. $\bullet$

For the D_2 case, we get the following variant of Theorem 1.33.

Theorem 1.35 *Let $L = \partial_x^2 + a_1\partial_x + a_0$ be a differential operator, with projective differential Galois group D_2, such that* $\mathrm{Sym}(L, 2)$ *has a right-hand factor ∂_x. Then* $\mathrm{Sym}(L, 2)$ *has three right-hand factors of degree one, say $\partial_x, \partial_x + b_1, \partial_x + b_2$. Write $b := \frac{4b_1b_2}{b_1+b_2}$. Now L is the pullback of St_{D_2}, with pullback function $\phi(t) := (1 + \frac{b^2}{4n^2a_0})^{-\frac{1}{2}}$. The other (weak) pullbacks are obtained by composition with automorphisms of St_{D_2}.*

Proof. By the argument in the proof of Theorem 1.33, we only have to show that $b = \frac{\phi(t)'}{\phi(t)}$. We have that $\mathrm{Sym}(St_{D_2}, 2)$ has three right-hand factors of degree one $\partial_t, \partial_t + \frac{1}{2}\frac{1}{t+1}, \partial_t + \frac{1}{2}\frac{1}{t-1}$. So the three degree one right-hand factors of L are $\partial_x, \partial_x + \frac{1}{2}\frac{\phi(t)'}{\phi(t)+1}, \partial_x + \frac{1}{2}\frac{\phi(t)'}{\phi(t)-1}$. Therefore we can write $b_1 = \frac{1}{2}\frac{\phi(t)'}{\phi(t)+1}, b_2 = \frac{1}{2}\frac{\phi(t)'}{\phi(t)-1}$. It follows that $b = \frac{\phi(t)'}{\phi(t)}$. □

Algorithm 1.36 (Determining n)
In the above, we assumed that the projective differential Galois group was known. For a second order differential operator $L \in k(x)[\partial_x]$ it is not hard to determine whether or not the projective differential Galois group is a group $D_n, n \in \mathbb{N}_{\geq 2} \cup \infty$. For completeness, we give an algorithm to determine n in case k is a number field, and L is a second order differential operator with dihedral differential Galois group. We note that this is a known algorithm (see [BD79] Section 6).

As above we may assume that $\mathrm{Sym}(L, 2)$ has a right-hand factor ∂_x. So L has a solution $y = e^{\int \sqrt{-a_0}dx}$. Let $K = \overline{k}(x)(y, y')$ be a Picard-Vessiot extension for L. Consider the tower of fields $\overline{k}(x) \subset \overline{k}(x, \sqrt{-a_0}) \subset K$, where $\sqrt{-a_0} = \frac{y'}{y}$. Since we assume the projective Galois group to be dihedral, it follows that a_0 is not a square in $\overline{k}(x)$. The field extension $\overline{k}(x, \sqrt{-a_0}) \subset K = \overline{k}(x)(y)$ is infinite in case $n = \infty$, and otherwise cyclic of order n or $2n$. Consider the differential $\omega := 2\sqrt{-a_0}dx$ on the hyperelliptic curve H with function field $\overline{k}(H) := \overline{k}(x, \sqrt{-a_0})$. We want to find the degree over $\overline{k}(x, \sqrt{-a_0})$ of the solution y^2 of the equation $\omega = \frac{dy^2}{y^2}$.

Suppose y is algebraic over $\overline{k}(x)$. Then by the action of the differential Galois group we find that $y^{2n} \in \overline{k}(H)$. Let $D := \mathrm{Div}(y^{2n})$ be the divisor of

y^{2n}. If $D = \sum a_i[p_i]$, then the residue of $n\omega = \frac{dy^{2n}}{y^{2n}}$ in p_i is a_i. We also find that ω has only poles of order 1 and no zeroes. So a necessary condition for y to be algebraic is $\mathrm{ord}_h(\omega) \in \{-1, 0\}, \mathrm{res}_h(\omega) \in \mathbb{Q} \; \forall \; h \in H$. In the following we will assume ω to satisfy these easily verifiable conditions.

Let m_1 be the least common multiple of the denominators of all nonzero residues of ω. Then $D_1 := \sum \mathrm{res}_h(m_1\omega)[h]$ is a divisor on H, and we want to find the smallest integer m_2 such that $m_2 D_1$ is a principal divisor. If such an integer m_2 exists, then $n = m_1 m_2$ and otherwise $n = \infty$. Indeed if $m_2 D_1 = \mathrm{Div}(f), f \in \overline{k}(H)$, then $\frac{df}{f} = m_1 m_2 \omega$ and we can take $y^2 = f^{\frac{1}{m_1 m_2}}$. Because $m_1 m_2 \omega$ is defined over k, one finds using Hilbert theorem 90 that we may suppose f to be defined over k (compare the argument in the proof of Lemma 1.5).

We want to find the order m_2 of the element $D_1 \in \mathrm{Jac}(H)(k)$. We will use the following known result.

Lemma 1.37 *Let k be a number field, and A/k an abelian variety. Let $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}_k$ of k, extending the prime number p. Suppose:*

1. *A has good reduction at $\mathfrak{p}$,*
2. *the ramification index $e_\mathfrak{p}$ is smaller than $p - 1$.*

Then reduction modulo $\mathfrak{p}$ yields an injective homomorphism

$$A(k)_{tors} \to A \bmod \mathfrak{p}(\mathcal{O}_k/\mathfrak{p}).$$

Proof. Let $a \in A(k)_{\mathrm{tors}}$ be a point of prime order ℓ. The subgroup $\langle a \rangle \subset A$ defines a constant group scheme of order ℓ over k. Since $e_\mathfrak{p} < p - 1$, by Theorem 4.5.1 in [T97] this group scheme extends uniquely to the finite flat group scheme $\underline{\mathbb{Z}/\ell\mathbb{Z}}_{\mathcal{O}_\mathfrak{p}}$, where $\mathcal{O}_\mathfrak{p}$ denotes the completion of $\mathcal{O}$ at $\mathfrak{p}$. This shows that a reduces modulo $\mathfrak{p}$ to a point of again order ℓ. So the kernel of the reduction map $A(k)_{\mathrm{tors}} \to A \bmod \mathfrak{p}(\mathcal{O}_k/\mathfrak{p})$ contains no points of prime order, and therefore the map is injective. □

Now let $\mathfrak{p}$ be a prime ideal in the ring of integers of k, such that H has good reduction modulo $\mathfrak{p}$ and $e_\mathfrak{p} < p - 1$. Then we can apply the above lemma to the abelian variety $\mathrm{Jac}(H)$, and the prime ideal $\mathfrak{p}$. It follows that if $\mathrm{ord}(D_1) < \infty$ then this order equals the order of D_1 in $\mathrm{Jac}(H)$ mod $\mathfrak{p}$. We can calculate this order $\tilde{m_2}$ using the algorithm in [GH00] 3.2. Write $\tilde{n} = \tilde{m_2} m_1$. If $\partial_x^2 + a_1\partial_x + 4\tilde{n}^2 a_0$ has a rational solution then $n = \tilde{n}$, otherwise $n = \infty$.

Note that it is not strictly necessary to calculate the order of D_1 in a reduction of H. The Hasse-Weil bound gives an upper bound for this order. This produces a number N which is an upper bound for n in case n is finite. Now n is the smallest integer such that $\partial_x^2 + a_1\partial_x + 4n^2 a_0$ has a rational solution. If there is no such solution for $n < N$, then $n = \infty$. •

1.3 A generalization of Klein's theorem

In this section we will give a variant of Klein's theorem for third order operators. We will define a notion of standard operator, such that each differential operator L with finite irreducible differential Galois group $G \subset \mathrm{SL}_3$ is a weak pullback of a standard operator for G. We start by giving an alternative construction of standard operators of order 2, more in line with our construction of order 3 standard operators, which we will give subsequently. In this section we will work over an algebraically closed field of characteristic zero, denoted by C.

1.3.1 Standard operators of order 2 revisited

Let V be a 2-dimensional vector space over C, and let $G \subset \mathrm{SL}(V)$ be an irreducible finite group.

Notation 1.38

- $Z(G)$ denotes the center of G. We have $G^p \cong G/Z(G)$ (with G^p the image of G in $\mathrm{PGL}(V)$).
- $\mathbb{P}(V) := \mathrm{Proj}\ C[V]$, where $C[V]$ is the *symmetric algebra* of V.
- $K^p := C(\mathbb{P}(V))$, the function field of $\mathbb{P}(V)$. Note that $K^p = C[V]_{((0))}$, i.e. K^p consists of quotients of homogeneous elements of $C[V]$ of the same degree.

There is an action of G^p on K^p, and by Lüroth's theorem we can write $(K^p)^{G^p}$ as $C(t)$, where t is unique up to a Möbius-transformation. We will construct a Galois extension $K^p \subset K$, such that $\mathrm{Gal}(K/C(t)) \cong G$, and a G-invariant C-vector space $W \subset K$ that is G-isomorphic to V. The corresponding monic differential operator with solution space W will be called a standard operator for G.

Construction 1.39 (Second order standard operators)
For $0 \neq \ell \in V$, we can see $\frac{V}{\ell}$ as a set of functions on $\mathbb{P}(V)$. This gives an injection $\frac{V}{\ell} \hookrightarrow K^p$. For $\sigma \in G$ we have $\sigma(\frac{V}{\ell}) = \frac{\ell}{\sigma(\ell)}\frac{V}{\ell}$. The set $\frac{V}{\ell}$ is not

G-invariant, for $\sigma(\frac{V}{\ell}) = \frac{V}{\ell}\ \forall\, \sigma \in G$ would imply $\frac{\ell}{\sigma(\ell)} \in C\ \forall\, \sigma \in G$, but there are no G-invariant lines in V. Roughly spoken, we want to construct some f in an extension of K^p such that $f\frac{V}{\ell}$ is a G-invariant vector space.

The map $c : G \to (K^p)^*,\ \sigma \mapsto \frac{\ell}{\sigma(\ell)}$ is a 1-cocyle in $H^1(G,(K^p)^*)$. We want to use Hilbert theorem 90 to construct a G-invariant space, but the problem is that G is not the Galois group of $K^p/C(t)$, which is G^p. We can avoid this problem by considering the map $d : G^p \to (K^p)^*, \tau \mapsto c(\sigma)^2$, where $\sigma \in G$ is some lift of $\tau \in G^p$. The value $c(\sigma)^2$ is independent of the chosen lift. So d is an element of $H^1(G^p,(K^p)^*)$, and therefore Hilbert theorem 90 implies that there exists an $f \in (K^p)^*$, with $d(\tau) = \frac{f}{\tau(f)}\ \forall\, \tau \in G^p$. In other words, $\frac{f}{\bar\sigma(f)} = \frac{\ell^2}{\sigma(\ell)^2}\ \forall\, \sigma \in G$, where $\bar\sigma \in G^p$ denotes the image of σ. This f is unique up to multiplication by an element in $C(t)^*$. We define $K := K^p(f_2)$, by $f_2^2 = f$. We have $f_2 \notin K^p$, for otherwise $\tilde V := f_2\frac{V}{\ell}$ would have a G^p-action, which is impossible because $G \twoheadrightarrow G^p$ has no section. The field extension $C(t) \subset K$ is a Galois extension. This follows from the fact that for $\tau \in G^p$ with lift $\sigma \in G$, we have $\tau(f) = c(\sigma)^2 f$, so the square roots of the conjugates of f are present. Note that the choice of $\ell \in V$ is irrelevant, because for an other choice ℓ', we can take $f_2' = f_2\frac{\ell'}{\ell}$, which leaves $\tilde V$ unchanged.

Lemma 1.40 *Using the above notations, there is a natural isomorphism* $\mathrm{Gal}(K/C(t)) \cong G$, *and* $\tilde V := f_2\frac{V}{\ell}$ *is* G*-invariant and* G*-isomorphic to* V.

Proof. We will extend the G^p-action on K^p to a G-action on K. Write $K = K^p + K^p f_2$. We define a G-action on K by $\sigma(\alpha+\beta f_2) = \bar\sigma(\alpha)+\bar\sigma(\beta)\frac{\sigma(\ell)}{\ell}f_2$. Using $\frac{f}{\bar\sigma(f)} = \frac{\ell^2}{\sigma(\ell)^2}$ it is clear that G acts by automorphisms. A counting argument shows $G = \mathrm{Gal}(K/C(t))$, and clearly $\tilde V$ is G-isomorphic to V. □

To $\tilde V$ corresponds a differential operator $\tilde L = \partial_t^2 + a\partial_t + b \in C(t)[\partial_t]$ with solution space $\tilde V$. We will now show that the normalization L of $\tilde L$ corresponds to a different choice for f. Let $\{y_1, y_2\}$ be a basis of solutions of $\tilde L$. Then $q := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ lies in $C(t)$ because the differential Galois group is unimodular. We have $q' = -aq$. We can normalize $\tilde L$ by making the shift $\partial_t \mapsto \partial_t - \frac{1}{2}a$ which changes the solution space $\tilde V$ into $q_2^{-1}\tilde V$, with $q_2^2 = q$. This corresponds to replacing the f above by $q^{-1}f$. This is allowed, because f was defined up to multiplication by elements of $C(t)$. We have that K^p

is a purely transcendental extension of C. The Galois extension $C(t) \subset K^p$, with Galois group G^p is ramified in three points, which we can suppose to be $\{0, 1, \infty\}$ by making an appropriate choice for t. For any such t we call the constructed operator L a standard operator for G. •

With the appropriate choice for t, the constructed differential operator L is equal to the differential operator St_G defined in Section 1.1. This follows from Theorem 1.46, which we prove for third order operators, but which is also valid for second order operators.

1.3.2 Standard operators of order 3

Now let V be a 3-dimensional vector space over C, and again let $G \subset \mathrm{SL}(V)$ be an irreducible finite group. We will now give a construction of third order standard operators, with projective differential Galois group isomorphic to G^p. This construction is to some extend a copy of the construction in the previous section.

Definition 1.41 Let $Z \subset \mathbb{P}(V)$ be a G^p-invariant irreducible curve, such that $Z/G^p \cong \mathbb{P}^1_C$. Note that by Remark 1.45 such a curve always exists. We write $C(t) := C(Z/G^p)$. We define a standard operator corresponding to Z and G^p, to be a differential operator $L_Z \in C(t)[\partial_t]$ given by the construction below. •

Construction 1.42 (Standard operator corresponding to Z)
For the construction of standard operators, we must consider two different cases for G^p. Let $\pi : \mathrm{SL}(V) \to \mathrm{PGL}(V)$ be the canonical map. We have that the center of G is trivial, or a cyclic group of order three. We will write C_3 for a cyclic group of order three. The cases we have to consider are:

1. the natural map $\pi^{-1}(G^p) \to G^p$ has no section, so $G = \pi^{-1}(G^p)$,
2. the natural map $\pi^{-1}(G^p) \to G^p$ has a section, so $\pi^{-1}(G^p) \cong C_3 \times G^p$. In this case $G = \pi^{-1}(G^p)$ or $G \cong G^p$.

We will now give the construction of standard operators case by case.

case 1.
For $0 \neq \ell \in V$, we regard $\frac{V}{\ell}$ as a set of functions on $\mathbb{P}(V)$, which induce functions on Z. This gives a map $\frac{V}{\ell} \to C(Z)$, which is an injection, for otherwise Z would be a line in $\mathbb{P}(V)$. This is impossible because G is irreducible, and therefore has no G-invariant planes.

As in the construction of second order standard operators, we consider the cocycle $d : G^p \to C(Z)^*, \sigma \mapsto (\frac{\ell}{\sigma(\ell)})^3$. Then $d \in H^1(G^p, C(Z)^*)$, and therefore Hilbert theorem 90 implies that there exists an $f \in C(Z)^*$, with $d(\tau) = \frac{f}{\tau(f)}\ \forall\ \tau \in G^p$. Now take f_3, with $f_3^3 = f$. We have $f_3 \notin C(Z)$, for otherwise $f_3\frac{V}{\ell}$ would be G^p-invariant, which is impossible because $G \twoheadrightarrow G^p$ has no section. So we consider the degree 3 extension $C(Z) \subset C(Z)(f_3)$. We have that $C(t) \subset C(Z)(f_3)$ is a Galois extension. As in the second order case, we have $\mathrm{Gal}(C(Z)(f_3)/C(t)) = G$, and $\tilde{V} := f_3\frac{V}{\ell}$ is G-invariant.

To $\tilde{V}$ corresponds a unique monic differential operator $\tilde{L}$. As in the second order case, normalizing $\tilde{L}$ corresponds to making a different choice for f. This normalization L_Z of $\tilde{L}$ is now uniquely determined and will be called the standard differential operator corresponding to Z. Note that the standard operator depends on the choice of t. By construction, the differential Galois group of L_Z is G.

case 2.
Let H be a lift of G^p in $\mathrm{SL}(V)$ that is isomorphic to G^p. Now Hilbert theorem 90, applied to $H^1(H, C(Z)^*)$, implies the existence of an $f \in C(Z)^*$ with $\frac{\ell}{h(\ell)} = \frac{f}{h(f)}\ \forall\ h \in H$. So $\tilde{V} := f\frac{V}{\ell}$ is H-invariant. This defines an operator $\tilde{L}$ which in general is not in normal form. We call the normalization L_Z of $\tilde{L}$ a standard operator for Z. The projective differential Galois group of L_Z is G^p, but the differential Galois group can be different from G! •

From the construction above, we get the following properties for a standard operator L_Z with solution space V_Z and Picard-Vessiot extension K_Z.

1. L_Z is uniquely defined, up to a Möbius-transformation of t.

2. The projective differential Galois group of L_Z is isomorphic to G^p, and $\mathbb{P}(V_Z)$ is G^p-isomorphic to $\mathbb{P}(V)$.

3. There is a G^p-equivariant isomorphism $K_Z^{Z(G)} \cong C(Z)$.

1.3.3 A Klein-like theorem for order 3 operators

Let $G \subset \mathrm{SL}_3$ be a finite irreducible group. Let L be a monic third order differential operator over $C(x)$ with Picard-Vessiot extension K, and solution space $V \subset K$. We assume that the representation of the differential Galois group in V is isomorphic to G.

Remark 1.43 Let $L_1 = \partial_x^3 + a\partial_x^2 + \cdots$ be a differential operator with finite differential Galois group in GL_3. Then $a = -\frac{q'}{q}$, $q = \mathrm{Det}(F)$, where F is a "fundamental matrix" as in 1.3.1. In particular q is algebraic. Applying the shift $\partial_x \mapsto \partial_x - \frac{1}{3}a$ to L_1 produces a differential operator L with differential Galois group in SL_3. Writing V_1 for the solution space of L_1, the solution space of L is $q_3^{-1}V_1$, $q_3^3 = q$. So the solutions of L are also algebraic, and therefore the differential Galois group of L is finite. We will prove that L is the pullback of some standard equation, and therefore L_1 is a pullback of this standard equation, too. So the restriction to the case $G \subset \mathrm{SL}(3)$ is no real restriction. •

We will start by constructing an irreducible curve $Z \subset \mathbb{P}(V)$ corresponding to L. This Z will be G^p invariant, and satisfy $Z/G^p \cong \mathbb{P}^1_C$.

Construction 1.44
The map $V \to K$ extends to a map $\varphi : C[V] \to K$. This map is G-equivariant. Now take some $v_0 \in V \setminus \{0\}$. For $f := \prod_{\sigma \in G} \sigma(v_0)$, we can consider the ring $(C[V][\frac{1}{f}])_0$ of homogeneous elements of degree zero in $C[V]_f$. We can extend φ to a map $\psi : (C[V][\frac{1}{f}])_0 \to K$. Write $I := \ker(\psi)$. We have that $(C[V][\frac{1}{f}])_0 = \mathcal{O}(\mathbb{P}(V) \setminus Z(f))$, where $Z(f)$ is the variety given by $f = 0$. Now I defines a subset $Z_1 \subset \mathbb{P}(V) \setminus Z(f)$, and we write Z for its closure in $\mathbb{P}(V)$. Note that Z is independent of the choice of v_0. The function field of Z is $C(Z) = \mathrm{frac}((C[V][\frac{1}{f}])_0/I)$. Furthermore ψ induces a G^p-equivariant injection of $C(Z)$ in $K^p := K^{Z(G)}$. The fixed field $C(Z)^{G^p}$ is a subfield of $C(x)$ of transcendence degree 1 over C, so it can be written as $C(t)$, for some $t \in C(Z)$, where t is unique up to a Möbius transformation. We conclude that L defines a G^p-invariant irreducible curve $Z \subset \mathbb{P}(V)$, with $Z/G^p \cong \mathbb{P}^1_C$. •

Remark 1.45 We can use the above construction to show that for every finite group $G \subset \mathrm{SL}(3)$ there exists a curve Z as in Definition 1.41. Indeed, let $G \subset \mathrm{SL}(3)$ be a finite group. We can make a Galois extension $C(z) \subset K$ with Galois group G, by realizing G as a quotient of the fundamental group of $\mathbb{P}^1_C$ minus a finite number of points. As in [PU00], we can construct a third order differential operator over $C(z)$, with Picard-Vessiot extension K, such that the Galois action on the solution space equals $G \subset \mathrm{SL}(3)$. Now the construction above gives the desired curve Z. •

We can now state an equivalent of Klein's theorem, for third order operators.

Theorem 1.46 *Let L and G be as above. These data define a G^p-invariant projective curve $Z \subset \mathbb{P}(V)$ with $Z/G^p \cong \mathbb{P}^1_C$. If L_Z is a corresponding standard differential operator, then L is a weak pullback of L_Z.*

Proof. From the construction of Z above, we get the following diagram:

$$\begin{array}{ccccc} C(x) & \subset & K^p & \subset & K \\ \cup & & \cup & & \\ C(t) & \subset & C(Z) & & \end{array}$$

We have that the G^p-action on $C(Z)$ corresponds with the G^p-action on K^p. Let K_Z be the Picard-Vessiot extension of L_Z. By the definition of a standard operator we can write $K_Z^{Z(G)} = C(Z)$. Let $V_Z \subset K_Z$ be the solution space of L_Z. In the compositum $\mathcal{K}$ of K and K_Z over $C(Z)$, we have the identity $V_Z = \frac{f}{\ell}V$, for the appropriate f, ℓ as defined in the construction of L_Z. We will now use Notation 1.7. If F is the image of t in $C(x)$, then the pullback $\phi_F(L_Z)$ again has solution space V_Z. The derivation $\frac{d}{dt}$ extends uniquely to $\mathcal{K}$. Also $\frac{d}{dx}$ extends uniquely to $\mathcal{K}$, and $\frac{d}{dx}(a) := \frac{dF}{dx}\frac{d}{dt}(a)$, $a \in \mathcal{K}$. So we can define $b := \frac{d}{dx}(\frac{f}{\ell})/(\frac{f}{\ell})$. Applying the shift $\partial_x \mapsto \partial_x + b$ to $\phi_F(L_Z)$ defines a differential operator with solution space V. So $(F')^2\phi_{F,b}(L_Z) = L$, and therefore L is a weak pullback of L_Z. □

1.3.4 Examples with Galois group A_5 or G_{168}

In this subsection we will give, for the cases $G = A_5$ and $G = G_{168}$, all possible non-singular curves Z, as in Definition 1.41. We will also give an example of a standard operator with projective differential Galois group A_5.

Consider $A_5 \subset \mathrm{SL}(3,\mathbb{C})$, with generators

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^{-1} \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta_5^2+\zeta_5^{-2} & \zeta_5+\zeta_5^{-1} \\ 1 & \zeta_5+\zeta_5^{-1} & \zeta_5^2+\zeta_5^{-2} \end{pmatrix}, \zeta_5^5 = 1$$

Now A_5 acts on the polynomial ring $\mathbb{C}[x,y,z]$. It acts on linear terms by $g(ax+by+cz) = \langle g \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rangle$, and this action is extended to all of $\mathbb{C}[x,y,z]$. In [Be96] we find the basic invariants:

$$f_n := c_n 5^{\frac{n}{2}-1} \left(x^n + \sum_{i=0}^{4} \left(\frac{x+\zeta_5^i y + \zeta_5^{-i} z}{\sqrt{5}} \right)^n \right), n \in \{2,6,10\}$$

We take as constants $c_2 = \frac{1}{2}, c_6 = 1, c_{10} = 3$.
There is one more basic invariant f_{15} which is the determinant of the Jacobian matrix of (f_2, f_6, f_{10}). So we have $\mathbb{C}[x,y,z]^{A_5} = \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$, where

$$\begin{aligned}
f_2 &= x^2 + yz, \\
f_6 &= 2(13x^6 + 3xy^5 + 15x^4yz + 45x^2y^2z^2 + 10y^3z^3 + 3xz^5), \\
f_{10} &= 3(626x^{10} + y^{10} + 90x^8yz + 1260x^6y^2z^2 + 4200x^4y^3z^3 + 3150x^2y^4z^4 + \\
&\quad 252y^5z^5 + z^{10} + (252x^5(y^5+z^5) + 840x^3yz + 360xy^2z^2)(y^5+z^5)), \\
f_{15} &= 180(2x + (\zeta_5+\zeta_5^4)(y+z))(2x + (\zeta_5^2+\zeta_5^3)(y+z)) \\
&\quad (2x + (1+\zeta_5)y + (1+\zeta_5^4)z)(2x + (1+\zeta_5^2)y + (1+\zeta_5^3)z) \\
&\quad (2x + (1+\zeta_5^3)y + (1+\zeta_5^2)z)(2x + (1+\zeta_5^4)y + (1+\zeta_5)z) \\
&\quad (2x + (\zeta_5+\zeta_5^2)y + (\zeta_5^3+\zeta_5^4)z)(2x + (\zeta_5^2+\zeta_5^4)y + (\zeta_5+\zeta_5^3)z) \\
&\quad (2x + (\zeta_5^3+\zeta_5^4)y + (\zeta_5+\zeta_5^2)z)(2x + (\zeta_5+\zeta_5^3)y + (\zeta_5^2+\zeta_5^4)z) \\
&\quad (y-z)(y-\zeta_5 z)(y-\zeta_5^2 z)(y-\zeta_5^3 z)(y-\zeta_5^4 z).
\end{aligned}$$

We have the relation

$\frac{1}{400}f_{15}^2 = 3f_{10}^3 - 1590f_{10}^2f_6f_2^2 + 25014f_{10}^2f_2^5 - 90f_{10}f_6^3f_2 + 285840f_{10}f_6^2f_2^4 -$

$8928000f_{10}f_6f_2^7 + 70060500f_{10}f_2^{10} + 18f_6^5 + 14860f_6^4f_2^3 - 17651900f_6^3f_2^6 +$

$810582000f_6^2f_2^9 - 12634745000f_6f_2^{12} + 65956225000f_2^{15}.$

We want to find all irreducible A_5-invariant plane curves Z, with $Z/A_5 \cong \mathbb{P}^1_{\mathbb{C}}$. Such a curve Z is given by $f = 0$, for some $f \in \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$. We get a Galois covering $Z \to Z/A_5$. The ramification points of this covering are points in $\mathbb{P}^2_{\mathbb{C}}$ which are fixed by a cyclic subgroup of A_5. So to calculate the genus of Z/A_5 and the ramification data, we need information on the points in $\mathbb{P}^2_{\mathbb{C}}$ fixed by a cyclic subgroup of A_5. From [We96] we get the following table. The first column gives the type of cyclic subgroup of A_5. The second column gives the number of subgroups of that type. The third column gives the number of points in $\mathbb{P}^2_{\mathbb{C}}$, which have a stabilizer of the type given by the first column.

H	#	*pts.*
C_2	15	∞
C_3	10	20
C_5	6	12

There are 15 lines in $\mathbb{P}^2_{\mathbb{C}}$, given by $f_{15} = 0$, and the points with stabilizer C_2 are the points that lie on precisely one of these lines. Note that each line as a whole is invariant under a group $C_2 \times C_2 \subset A_5$.

From this data we get the following information. For a branch point of the covering $Z \to Z/A_5$, the ramification index e must be in $\{2, 3, 5\}$. Then the stabilizer of a ramification point above this branch point is C_e. Above a branch point with ramification index 3, there lie 20 points. So there is at most one branch point with ramification index 3. In the same way we see that there is at most one branch point with ramification index 5.

The ramification points with ramification index 2 are intersection points of Z with $Z(f_{15}) := \{p \in \mathbb{P}^2_{\mathbb{C}} | f_{15}(p) = 0\}$. If $Z = Z(f)$, and f has degree d, $f \not| f_{15}$, then for a fixed line $l \subset Z(f_{15})$, $Z \cap l$ consists of at most d points. Such a line l is fixed by a group D_2, so a point $p \in Z \cap l$ with stabilizer C_2 has a conjugate in $Z \cap l$ different from p. All lines in $Z(f_{15})$ are images under G of l, so all branch points with ramification index 2 are given by the images of $Z \cap l$ in Z/A_5. We see that there are at most $\frac{d}{2}$ such branch points.

Now we can use Hurwitz's formula to calculate all possibilities for non-singular Z. Hurwitz's formula states $2g - 2 = 60(2g_0 - 2) + 60\sum_i \frac{e_i - 1}{e_i}$. Here i runs over the branch points of the covering $Z \to Z/A_5$, and e_i are the corresponding ramification indices. Further, g denotes the genus of Z, and g_0 denotes the genus of Z/A_5. We write n_i for the number of branch points with ramification index i, and d for the degree of $f, Z = Z(f)$. Then for non-singular Z, we have $g = \frac{(d-1)(d-2)}{2}$. So we can rewrite Hurwitz's formula as $d^2 - 3d + 120 = 120g_0 + 30n_2 + 40n_3 + 48n_5$.

If $g_0 = 0$, the restrictions $n_3, n_5 \leq 1, n_2 \leq \frac{d}{2}$ give the bound $d \leq 15$. A computation in Maple shows that the homogeneous irreducible nonsingular $f \in \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$ of degree ≤ 15 are given by

$$\{f_2, \lambda f_2^3 + f_6, \lambda f_2^5 + \mu f_2^2 f_6 + f_{10}, \lambda f_2^6 + \mu f_2^3 f_6 + \nu f_6^2 + f_2 f_{10} | \lambda, \mu \in \mathbb{C}, \nu \in \mathbb{C}^*\}.$$

For such a polynomial f of degree d, the values of n_2, n_3, n_5 can be computed using Hurwitz's formula. For $d \in \{10, 12\}$, there is the possibility that $g_0 = 1$, but an explicit calculation of the number of intersection points of f_{10} and $f_6^2 + f_2 f_{10}$ with the invariant line $y = z$ rules out this possibility. We find the following table for the possibilities for d, n_2, n_3, n_5, for non-singular Z.

d	n_2	n_3	n_5
2	1	1	1
6	3	0	1
10	5	1	0
12	6	0	1

From this table, we can see that the 12 points with stabilizer C_5 lie on $Z(f_2)$ and on $Z(f_6)$. Therefore by Bezout's theorem, they are the points $Z(f_2) \cap Z(f_6)$. So we have a complete list of all non-singular curves Z satisfying definition 1.41, and we see that there are infinitely many such curves Z.

Unfortunately we are not able to give a complete list of all singular curves Z, with $Z/A_5 \cong \mathbb{P}^1_{\mathbb{C}}$. We can give the list up to a certain degree. By the previous, we see that the singular curves of degree 10 are given by $f_2^5 + \lambda f_{10}, \lambda \in \mathbb{C}^*$, and the genus is 36. For degree 12 we find the family

$f_2^6 + \lambda f_2^3 f_6 + \mu f_6^2, \lambda \in \mathbb{C}, \mu \in \mathbb{C}^*$ of genus 19. For degree 16, all irreducible curves in our family are non-singular, so here $g_0 \geq 1$.

Example 1.47 We can calculate a standard operator for the curve Z given by $x^2 + yz = 0$. Since the group A_5 is simple, we are in case 2. of Construction 1.42. Writing $u := \frac{x}{z}, v := \frac{y}{z}$, we have that $\mathbb{C}(Z)$ is the quotient field of $\mathbb{C}[u,v]/(u^2+v)$. So $\mathbb{C}(Z) \cong \mathbb{C}(u)$. There is a G-invariant element $t := \frac{f_6^5}{f_{10}^3} = \frac{u^5(u^{10}-11u^5-1)^5}{(u^{20}+228u^{15}+494u^{10}-228u^5+1)^3}$ in $\mathbb{C}(Z)$. We see that u has degree 60 over $\mathbb{C}(t)$, so indeed $\mathbb{C}(t) = \mathbb{C}(Z)^G$. For the vector space $\frac{V}{z} = \mathbb{C}\langle u, -u^2, 1\rangle$ in $\mathbb{C}(Z)$ (taking $\ell = z$ in the construction), we want to find an element $f \neq 0$ such that $f\frac{V}{z}$ is G-invariant. Consider $c : G \to \mathbb{C}(Z)^*, \sigma \mapsto \frac{z}{\sigma(z)}$. By [S62], p158, Proposition 2, there exists an element $a \in \mathbb{C}(Z)$, such that $f := \sum_{\sigma \in G} c(\sigma) \cdot \sigma(a) \neq 0$. Then $f\frac{V}{z}$ is G-invariant. We can take $a := \frac{z}{x+y}$, and consequently $f = \sum_{\sigma\in G} \frac{z}{\sigma(x+y)}$.

The vector space $f\frac{V}{z}$ is the solution space of an operator who's normalization is the desired standard operator L_Z. This determines L_Z as an element of $\mathbb{C}(u)[\partial_t]$. We know that this operator lies in $\mathbb{C}(t)[\partial_t]$, and we can compute the coefficients of L_Z as rational functions in t. After an appropriate shift and Möbius-transformation we find

$$L_Z = \partial_t^3 + \frac{800t^3 - 989t + 864}{900t^2(t-1)^2}\partial_t - \frac{1600t^3 - 2967t^2 + 4445t - 1728}{1800t^3(t-1)^3}.$$

This turns out to be the second symmetric power of $St_{A_5^{\mathrm{SL}(2)}}$. $\diamond$

Remark 1.48 Let W be a 2-dimensional vector space over $\mathbb{C}$. There is a faithful representation $\rho : A_5^{\mathrm{SL}_2} \to \mathrm{GL}(W)$. This induces a representation of A_5 in $\mathrm{Sym}(W, 2)$, which is isomorphic to the representation $A_5 \to \mathrm{SL}(V)$ defined above. We can take ρ to be the representation of $A_5 \times C_3$ induced by the differential operator $St_{A_5^{\mathrm{SL}_2}}$. Now by Proposition 5.12 of [P99] (the Tannakian approach to differential Galois theory) we have an isomorphism of differential modules $\mathrm{Sym}(\mathbb{C}(t)[\partial_t]/St_{A_5^{\mathrm{SL}_2}}\mathbb{C}(t)[\partial_t], 2) \cong \mathbb{C}(t)[\partial_t]/L_Z\mathbb{C}(t)[\partial_t]$, with L_Z as defined in the example above. This does not imply that L_Z is the second symmetric power of a second order differential operator. Nevertheless this is the case. $\bullet$

We note that applying a shift $\partial_t \mapsto \partial_t - \frac{1}{3}\frac{f'}{f}$ to the operator L_Z in general gives a differential operator with differential Galois group $A_5 \times C_3$.

The group G_{168}

We take $G_{168} \subset \mathrm{SL}(3,\mathbb{C})$, with generators:

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}, \frac{1}{\sqrt{-7}} \begin{pmatrix} \zeta^5-\zeta^2 & \zeta^6-\zeta & \zeta^3-\zeta^4 \\ \zeta^6-\zeta & \zeta^3-\zeta^4 & \zeta^5-\zeta^2 \\ \zeta^3-\zeta^4 & \zeta^5-\zeta^2 & \zeta^6-\zeta \end{pmatrix}.$$

Here ζ is a primitive 7-th root of unity.
In [Be96] we find that the ring of invariants for G_{168} is $C[f_4, f_6, f_{14}, f_{21}]$.

$$\begin{aligned}
f_4 &= 2(xy^3+yz^3+zx^3), \\
f_6 &= \tfrac{1}{216}\mathrm{Det}(\mathrm{Hes}(f_4)), \\
f_{14} &= \tfrac{1}{144}\mathrm{Det}\begin{pmatrix} \frac{\partial^2 f_4}{\partial x^2} & \frac{\partial^2 f_4}{\partial x \partial y} & \frac{\partial^2 f_4}{\partial x \partial z} & \frac{\partial f_6}{\partial x} \\ \frac{\partial^2 f_4}{\partial y \partial x} & \frac{\partial^2 f_4}{\partial y^2} & \frac{\partial^2 f_4}{\partial y \partial z} & \frac{\partial f_6}{\partial y} \\ \frac{\partial^2 f_4}{\partial z \partial x} & \frac{\partial^2 f_4}{\partial z \partial y} & \frac{\partial^2 f_4}{\partial z^2} & \frac{\partial f_6}{\partial z} \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial f_6}{\partial z} & 0 \end{pmatrix}, \\
f_{21} &= \tfrac{1}{28}\mathrm{Det}(\mathrm{Jac}(f_4, f_6, f_{14})).
\end{aligned}$$

For completeness, we give f_6, f_{14} explicitly.

$$\begin{aligned}
f_6 &= 2(5z^2x^2y^2 - z^5x - y^5z - x^5y), \\
f_{14} &= z^{14}+x^{14}+y^{14}+18y^7x^7+18y^7z^7+18z^7x^7-126z^3x^6y^5-250y^4x^9z- \\
&\quad 34y^2z^{11}x-34z^2x^{11}y+375z^4x^8y^2-250z^4xy^9+375z^8x^2y^4-34zx^2y^{11}- \\
&\quad 126z^5x^3y^6-250z^9x^4y+375z^2x^4y^8-126z^6x^5y^3.
\end{aligned}$$

f_{21} factors as a product of linear terms over $\mathbb{Q}(\zeta_7)$, where ζ_7 is a primitive 7-th root of unity. In fact f_{21} has as linear factors:

$x - y(1+\zeta_7^5+\zeta_7^6) + z(\zeta_7^3+\zeta_7^5)$ and its 5 conjugates,
$x - y(\zeta_7+\zeta_7^4+\zeta_7^5) + z(\zeta_7+\zeta_7^6)$ and its 5 conjugates,
$x - y(\zeta_7^3+\zeta_7^4+\zeta_7^5) - z(\zeta_7^4+\zeta_7^6)$ and its 5 conjugates, and
$x - y(1+\zeta_7^2+\zeta_7^5) + z(\zeta_7^2+\zeta_7^5)$ and its 2 conjugates.

There is one relation between the f_i:

$$f_{21}^2 = 4f_{14}^3 - 8f_{14}f_4^7 - 44f_{14}^2f_4^2f_6 - 8f_4^9f_6 + 68f_{14}f_4^4f_6^2 + 172f_4^6f_6^3 + 126f_{14}f_4f_6^4 - 938f_4^3f_6^5 + 54f_6^7$$

According to [We96], we have the following table of points in $\mathbb{P}^2_{\mathbb{C}}$, fixed by some subgroup of G_{168} (for details, see the A_5-case):

H	#	*pts.*
C_2	21	∞
C_3	28	56
C_4	21	42
C_7	8	24

There are 21 lines with stabilizer C_2. To be precise, each point which is on precisely one line is fixed by a group C_2, and each line as a whole is invariant under a group $C_2 \times C_2$.

For Z a G_{168}-invariant curve of degree d, the covering $Z \to Z/G_{168}$ can have ramification indices in $\{2, 3, 4, 7\}$. For a non-singular curve Z such that the quotient has genus 0, the Hurwitz formula writes:

$$d^2 - 3d + 336 = 84n_2 + 112n_3 + 126n_4 + 144n_7.$$

By calculating the number of ramification points, we find $n_3, n_4, n_7 \in \{0, 1\}$, and $n_2 \leq \frac{d}{4}$. This gives the following possibilities for d, n_2, n_3, n_4, n_7:

d	n_2	n_3	n_4	n_7
4	1	1	0	1
6	1	0	1	1
14	3	1	1	0
18	4	0	1	1
20	5	1	0	1

There are infinitely many G_{168}-invariant non-singular irreducible curves Z in $\mathbb{P}^2_{\mathbb{C}}$, with $Z/G_{168} \cong \mathbb{P}^1_{\mathbb{C}}$. In fact all such curves are of the form $Z(f)$, with $f \in \{f_4, f_6, f_{14}, \lambda f_6^3 + f_{14}f_4, \lambda f_4^5 + f_{14}f_6 | \lambda \in \mathbb{C}^*\}$.

Chapter 2

Moduli Spaces

Thie contents of this chapter has been published in the book "Differential Equations and the Stokes Phenomenon", [ST02].

2.1 Introduction

In §2 of this chapter we will define moduli spaces of linear differential equations, associated to given local data. These linear differential equations are defined on $\mathbb{P}^1(C)$, the projective line over an algebraically closed field C of characteristic 0. The data prescribe the position of the singular points and their formal equivalence type. In §3 we will prove that the constructed moduli spaces are affine varieties, i.e. are of the form $\mathrm{Spec}(U)$, with U a finitely generated C-algebra.

Every closed point m of the moduli space represents a differential equation, and we will associate a differential Galois group $Gal(m)$ to m. In §4 we will prove that for any fixed linear algebraic group G, the condition $Gal(m) \subset G$ defines a Zariski closed set in the moduli space. In Chapter 3 we will prove that the condition $Gal(m) = G$ defines a Zariski constructible subset of the moduli space if G satisfies a certain condition.

2.2 Definition of the moduli space

The definition of the moduli space that we will give here, is a slight variation on the one given in Chapter 12 of [PS03]. This section contains results from [PS03]. For completeness, and since many of these results will be slightly altered or generalized, we present them here completely.

The definition of the moduli space uses the concept of a *connection* on a *vector bundle* on an irreducible, non-singular, projective curve X over an algebraically closed field C of characteristic zero. One writes Ω_X for the sheaf of holomorphic differentials. For an effective divisor S, one writes $\Omega_X(S)$ for the sheaf of meromorphic differentials with poles prescribed by S. A connection ∇ on a vector bundle $\mathcal{M}$ on X with poles prescribed by the divisor S is a morphism of abelian sheaves $\nabla : \mathcal{M} \to \Omega_X(S) \otimes \mathcal{M}$ satisfying $\nabla(fm) = df \otimes m + f\nabla(m)$ for each open $U \subset X, f \in \mathcal{O}_X(U), m \in \mathcal{M}(U)$. In the sequel we will consider $X = \mathbb{P}^1_C$. Now we will explain what the local data are and define the functor $\mathcal{F}$, associated to these local data. This functor will be shown to be representable and thus induces a (fine) moduli space.

Definition 2.1 The *data* on $\mathbb{P} := \mathbb{P}^1_C$ that we consider are:

- a vector space V over C of dimension m;
- distinct points $s_1, \ldots, s_r \in \mathbb{P}$;
- for each $i = 1, \ldots, r$ a formal connection ∇_i on $N_i := C[[t_i]] \otimes V$ having the form $\nabla_i : N_i \to C[[t_i]]t_i^{-k_i} dt_i \otimes N_i$, where $k_i > 0$. Here t_i is "the" *local parameter* at s_i i.e. $t_i = z - s_i$ for $s_i \neq \infty$ and $t_i = \frac{1}{z}$ for $s_i = \infty$.

In the sequel we suppose that $\sum k_i \geq 2$, since $\sum k_i \leq 1$ turns out to be uninteresting.

One associates to such data a covariant functor $\mathcal{F}$, sometimes called the *moduli functor*, from the category of C-algebras to the category of sets. One could also see this as a contravariant functor on affine schemes over C and extend this to a functor defined on all C-schemes. We will show that $\mathcal{F}$ is representable by a certain C-algebra U, so this point of view does not yield new information. However, we call $\mathrm{Spec}(U)$ the *moduli space* of the moduli functor $\mathcal{F}$ and denote this affine scheme by $\mathbb{M}$.

First we define $\mathcal{F}(C)$. For any C-algebra R a similar definition of $\mathcal{F}(R)$ will be stated later on.

$\mathcal{F}(C)$ consists of the equivalence classes of the tuples $(\mathcal{M}, \nabla, \{\phi_i\})$, where:

(a) $\mathcal{M}$ is a free vector bundle of rank m on $\mathbb{P}$, with a connection $\nabla : \mathcal{M} \to \Omega(\sum k_i[s_i]) \otimes \mathcal{M}$,

(b) $\phi_i : \widehat{\mathcal{M}}_{s_i} \to N_i$ are isomorphisms, such that $\phi_i \circ \widehat{\nabla}_i = \nabla_i \circ \phi_i$.

The above notation needs some explanation. $\widehat{\mathcal{M}}_{s_i}$ denotes the completion of the stalk $\mathcal{M}_{s_i}$. Thus $\widehat{\mathcal{M}}_{s_i}$ is equal to $C[[t_i]] \otimes_{C[t_i]_{(t_i)}} \mathcal{M}_{s_i}$, where $\mathcal{M}_{s_i}$ is a free $C[t_i]_{(t_i)}$-module of rank m. For each point s_i the connection ∇ induces a connection $\widehat{\nabla}_i : \widehat{\mathcal{M}}_{s_i} \to t_i^{-k_i} C[[t_i]] dt_i \otimes \widehat{\mathcal{M}}_{s_i}$. We also write ϕ_i for the extension of ϕ_i to a map $t_i^{-k_i} C[[t_i]] dt_i \otimes \widehat{\mathcal{M}}_{s_i} \to C[[t_i]] t_i^{-k_i} dt_i \otimes N_i$. Now condition (b) reads: $\nabla_i = \phi_i \circ \widehat{\nabla}_i \circ \phi_i^{-1} : N_i \to C[[t_i]] t_i^{-k_i} dt_i \otimes N_i$. We say that $(\mathcal{M}, \nabla, \{\phi_i\})$ is equivalent to $(\mathcal{M}', \nabla', \{\phi_i'\})$ if there exists an isomorphism $f : \mathcal{M} \to \mathcal{M}'$ of the free vector bundles, which is compatible with the ∇'s and the ϕ's.

For any C-algebra R, the elements of $\mathcal{F}(R)$ are the equivalence classes of tuples $(\mathcal{M}, \nabla, \{\phi_i\})$, consisting of:

(a') a free vector bundle $\mathcal{M}$ of rank m on $\mathbb{P}^1_R$ with a connection $\nabla : \mathcal{M} \to \Omega(\sum k_i[s_i]) \otimes \mathcal{M}$,

(b') isomorphisms $\phi_i : \widehat{\mathcal{M}}_{s_i} \to R[[z]] \otimes N$, such that $\phi_i \circ \widehat{\nabla}_i = \nabla_i \circ \phi_i$. •

Remark 2.2 A more explicit definition of $\mathcal{F}(C)$ (which extends to $\mathcal{F}(R)$) can be given as follows. Let W denote the vector space $H^0(\mathbb{P}, \mathcal{M})$. Then ∇ is determined by its restriction to W. This restriction is a linear map $L : W \to H^0(\mathbb{P}, \Omega(\sum k_i[s_i])) \otimes W$. Furthermore we have that the maps $\phi_i : \widehat{\mathcal{M}}_{s_i} = C[[t_i]] \otimes W \to N_i = C[[t_i]] \otimes V$ are determined by its restrictions to W. The latter is given by a sequence of linear maps $\phi_i(n) : W \to V$, for $n \geq 0$, such that $\phi_i(w) = \sum_{n \geq 0} \phi_i(n)(w) t_i^n$ holds for $w \in W$. The conditions in part (b) are equivalent to $\phi_i(0)$ is an isomorphism for all i, and certain relations between the linear map L and the sequence of linear maps $\{\phi_i(n)\}$. These relations can be made explicit if the ∇_i are given explicitly. In other words, (a) and (b) are equivalent to giving a vector space W of dimension m and a set of linear maps $L, \{\phi_i(n)\}$ having certain relations.

An object equivalent to the given $(\mathcal{M}, \nabla, \{\phi_i\})$ is, in terms of vector spaces and linear maps, given by a vector space W' and an isomorphism $W' \to W$ compatible with the other data. If we use the map $\phi_1(0)$ to identify W and V, then we have taken a representative in each equivalence class and the elements of $\mathcal{F}(C)$ can be described by tuples $(\nabla, \{\phi_i\})$, where:

(a') $\nabla : \mathcal{M} \to \Omega(\sum k_i[s_i]) \otimes \mathcal{M}$ is a connection on the free vector bundle $\mathcal{M} := O_{\mathbf{P}^1_C} \otimes V$,

(b') the ϕ_i are isomorphisms $\widehat{\mathcal{M}}_{s_i} \to N$ such that $\phi_i \circ \widehat{\nabla}_i = \nabla_i \circ \phi_i$ and such that $\phi_1(0)$ is the identity from V to itself. •

One observes that applying an automorphism of $\mathbb{P}$ changes the position of the singular points $s_1, \ldots, s_r$, but leaves the moduli functor essentially unchanged. Therefore we may, for notational convenience, suppose that ∞ is not a singular point. The moduli functor can be made even more explicit by considering $\nabla_{\frac{d}{dz}}$ instead of ∇. Before giving this variant of $\mathcal{F}$ we need to introduce some notation.

Notation 2.3

- For $\psi \in \mathrm{M}_m(C((z)))$, we define $\mathrm{ord}(\psi)$ to be the minimum of the orders of the coefficients of ψ.
- For a differential operator $\frac{d}{dz} + A$, $A \in \mathrm{End}(C((z)) \otimes V)$, or for A itself, we define the order to be the order of a matrix representing A on a basis of V. This definition is independent of the choice of basis.
- $O(n) := \{\psi \in \mathrm{End}(C((z)) \otimes V) |\ \mathrm{ord}(\psi) \geq n\}$.

We now give the explicit variant of $\mathcal{F}$, which will be the form used for our computations. The *data* are:

1. a vector space V over C of dimension m,

2. singular points $s_1, \dots, s_r \in C$, and for each point a differential operator $\frac{d}{dz} + B_i, B_i \in \mathrm{End}(C((t_i)) \otimes V)$ of order $\geq -k_i$.

The set $\mathcal{F}(R)$ consists of the tuples $(\frac{d}{dz} + A, \{\phi_i\}_{i=1}^r)$, where:

- $A = \sum_{i=1}^r \sum_{j=1}^{k_i} \frac{A(i,j)}{(z-s_i)^j}$, $A(i,j) \in \mathrm{End}(R \otimes V)$ satisfies $\sum_{i=1}^r A(i,1) = 0$ (this condition is equivalent to ∞ being nonsingular);
- the $\phi_i = \sum_{j=0}^\infty \phi_i(j)(t_i)^j$, $t_i = z - s_i$ are automorphisms of $R[[t_i]] \otimes V$ and $\phi_1(0) = I = Id_V$;
- $\phi_i(\frac{d}{dz} + A)\phi_i^{-1} = \frac{d}{dz} + B_i$, $i = 1, \cdots, s$, where we see A, ϕ_i as elements of $\mathrm{End}(R[[t_i]][t_i^{-1}] \otimes V)$. This can be restated as $\phi_i' = \phi_i A - B_i \phi_i$.

Remark 2.4 One observes that A can be recovered from the endomorphisms $\{\phi_i(j), 0 \leq j < k_i, 1 \leq i \leq r\}$. Therefore we will sometimes omit the A. •

Theorem 2.5 *The functor $\mathcal{F}$ is representable.*

Proof. Fix a basis of V over C. One introduces a collection of variables $\{X_j\}_{j \in J}$, representing the entries of the matrices of A and all $\phi_i(n)$, with respect to this basis. We have a set of relations in these variables, induced by the equations $\phi_i' = \phi_i A - B_i \phi_i$. Let S be the ideal generated by this relations, then it is easily seen that the C-algebra $U := C[X_j]_{j \in J}/S$ represents $\mathcal{F}$. □

Remark 2.6 *Other moduli functors.*
(1) A useful variation $\mathcal{F}^*$ on the moduli functor $\mathcal{F}$ considered above is the following. Suppose that the singular points $s_1, \dots, s_r$ are distinct from ∞. The elements of $\mathcal{F}^*(C)$ are equivalence classes of tuples $(\mathcal{M}, \nabla, \{\phi_i\})$, where again $\mathcal{M}$ is a free vector bundle, but now ∇ is a connection from $\mathcal{M}$ to $\Omega(\infty + \sum k_i[s_i]) \otimes \mathcal{M}$, again satisfying condition (b) from the definition of $\mathcal{F}(C)$. The definition of $\mathcal{F}^*(R)$ for any C-algebra R is similar. This amounts to admitting for the objects of $\mathcal{F}^*(R)$ a regular singularity at ∞ for which the formal local structure is not prescribed.

(2) In his thesis ([Bo99]) P. Boalch defines moduli spaces for the case of “nice” regular singularities. His moduli spaces are quotients of the moduli

spaces defined here (they are obtained by forgetting the local isomorphisms ϕ_i), and have even dimension and a natural symplectic structure. The moduli spaces defined here need not have an even dimension.

(3) There are still other interesting variations of the moduli functor. One can replace the condition that $\mathcal{M}$ is a free vector bundle of rank m by the condition that $\mathcal{M}$ is a vector bundle of rank m of a given type. Another possibility would be to allow moving singular points. It seems possible to prove that all these moduli functors are represented by a finitely generated C-algebra. •

2.3 The moduli space is of finite type over C

Proving that the algebra U representing the moduli functor $\mathcal{F}$ is finitely generated amounts to showing that a finite part of the $\{\phi_i\}_{i=1}^r$ determines the (universal) element $(\frac{d}{dz}+A, \{\phi_i\}_{i=1}^r)$ in $\mathcal{F}(U)$. Before proving that U is finitely generated, we first give an example.

Example 2.7 Consider the moduli problem, with the following data:

- singular points $\{s_1=0, s_2=-1, s_3=1, s_4=\infty\}$,
- local connections $\frac{d}{dz}+\frac{1}{z-s_i}C_i$, $i\in\{1,\cdots,4\}$, with

$$C_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C_2=C_3=\begin{pmatrix}\frac{1}{2}&0\\0&0\end{pmatrix},\ C_4=\begin{pmatrix}-1&1\\0&-1\end{pmatrix}.$$

The global differential operators are written $\frac{d}{dz}+A=\frac{d}{dz}+\sum_{i=1}^2\frac{A_i}{t_i}$, $t_i=z-s_i$. An element $(\frac{d}{dz}+A, \{\phi_i\}_{i=1}^4)$ belongs to $\mathcal{F}(C)$ if it satisfies $\phi_1(0)=I$, $\frac{d}{dt_i}(\phi_i)=\phi_iA-\frac{1}{t_i}C_i\phi_i$, $i=1,\cdots,4$. We find $A_1=C_1,\ A_i=\phi_i(0)^{-1}C_i\phi_i(0)$, $i=2,3$. The condition at ∞ writes $-(A_1+A_2+A_3)=\phi_4(0)^{-1}C_4\phi_4(0)$. It can be seen that the representing C-algebra U is generated by the entries of the matrices $\phi_1(1),\ \phi_i(0),\ i=2,3,4$ and $\frac{1}{\mathrm{Det}(\phi_i(0))}$, $i=2,3,4$. Indeed, the equations $\phi_i'=\phi_iA-\frac{1}{t_i}C_i\phi_i$ determine all entries of $\phi_1(j),\ j\geq 2$, $\phi_i(j),\ j\geq 1,\ i=2,3,4$ as polynomial expressions in the entries of the $\phi_1(1),\ \phi_i(0),\ i=2,3,4$. We have one more equation, namely $(A_2-A_3)_{21}=0$.

This equation comes from the degree 0 term of $\phi_1' = \phi_1 A - \frac{1}{z}C_1\phi_1$. From this one also deduces that there are no relations on $(\phi_1(1))_{21}$, and that the other coefficients of $\phi_1(1)$ are determined by A_2, A_3, so by $\phi_2(0), \phi_3(0)$. This computations show that the moduli space corresponding to this moduli problem is 8-dimensional.

We will now examine what the possibilities are for the global differential operator $\frac{d}{dz} + A$. We have $A_1 = C_1$, $A_2, A_3 \sim C_2$, $-(A_1 + A_2 + A_3) \sim C_4$ and $(A_2 - A_3)_{21} = 0$, here $\sim$ denotes similiarity of matrices. These conditions translate into

$$\mathrm{Det}(A_2) = \mathrm{Det}(A_3) = 0,\ \mathrm{Det}(A_1 + A_2 + A_3) = 1,\ \mathrm{Tr}(A_2) = \mathrm{Tr}(A_3) = \frac{1}{2}$$

$$\mathrm{Tr}(A_1 + A_2 + A_3) = 2,\ (A_2 - A_3)_{21} = 0,\ A_1 + A_2 + A_3 \neq I.$$

(see the proof of lemma 4.6 for details). The condition $\mathrm{Tr}(A_1+A_2+A_3) = -2$ is superfluous, and we see that the space of connections is a 2-dimensional Zariski-constructible set. To be more precise, we can write the matrices $A_i,\ i = 2, 3$ as $\begin{pmatrix} a_i & b_i \\ c_i & \frac{1}{2} - a_i \end{pmatrix}$. We have $c_2 = c_3$, and the determinant conditions give $a_2^2 - \frac{1}{2}a_2 + b_2c_2 = 0$, $a_3^2 - \frac{1}{2}a_3 + b_3c_2 = 0$, $(a_2+a_3)^2 + 2c_2(b_2+b_3) = 0$. We find that the space of connections consists of a 2-dimensional pointed plane $a_2 = a_3 = c_2 = 0$, $(b_2, b_3) \neq (0, 0)$ and a 2-dimensional part given by $(a_2 - a_3)^2 = a_2 + a_3$, $b_2 = \frac{\frac{1}{2}a_2 - a_2^2}{c_2}$, $b_3 = \frac{\frac{1}{2}a_3 - a_3^2}{c_2}$, $c_2 \neq 0$. ◇

Definition 2.8 Let some data $(V, \{s_1, \cdots, s_r\}, \{\frac{d}{dt_i} + B_i\}_{i=1}^r)$ be given. We define a functor $\mathcal{F}_n$ from C-algebras to sets as follows. $\mathcal{F}_n(R)$ consists of tuples $(\{A(i,j)|\ 1 \leq j \leq k_i,\ 1 \leq i \leq r\}, \{\phi_i(j))|0 \leq j < n + k_i,\ 1 \leq i \leq r\})$, with $k_i := -\mathrm{ord}(B_i)$ and $A(i,j), \phi_i(j) \in \mathrm{End}(R \otimes V)$ satisfying:

- $\phi_i(0)$ is invertible for $1 < i \leq r$ and $\phi_1(0) = I$,
- $\sum_{i=1}^r A(i, 1) = 0$,
- $A := \sum_{i=1}^r \sum_{j=1}^{k_i} \frac{A(i,j)}{(z-s_i)^j}$ satisfies $\phi_i' \equiv \phi_i A - B_i\phi_i \bmod t_i^n$ for $1 \leq i \leq r$. •

The functor $\mathcal{F}_n$ is represented by a finitely generated C-algebra U_n. There is a canonical forgetful natural transformation $\mathcal{F} \to \mathcal{F}_n$, which induces a C-algebra homomorphism $h_n : U_n \to U$. In the same way we can define forgetful natural transformations $\mathcal{F}_m \to \mathcal{F}_n,\ m \geq n$, with associated $h_{m,n} : U_n \to U_m$.

In the proof and the formulation of theorem 2.9, we will use the following *assertions*.

(1) There exists an integer $N \in \mathbb{N}$, such that for $n \geq N$ the natural maps $\mathcal{F}(R) \to \mathcal{F}_n(R)$ are injective.

(2) There exists an integer $c \in \mathbb{N}$, such that for $n \geq N$, with N as above, the maps $\mathcal{F}(R) \to \mathcal{F}_n(R)$ and $\mathcal{F}_{n+c}(R) \to \mathcal{F}_n(R)$ have the same image.

Theorem 2.9 *U is finitely generated. More precisely, for N, c as above and $n \geq N + c$ one has:*

(a) *the homomorphism $h_n : U_n \to U$ is surjective,*

(b) $\ker(h_n) = \ker(h_{n+c,n} : U_n \to U_{n+c})$.

Proof. We will show that the theorem follows from the two assertions. Take $n \geq N$. Assertion (2) can be restated as follows. Let R be a C-algebra and let $\alpha : U_n \to R$ be a homomorphism such that there exists a homomorphism $\beta : U_{n+c} \to R$ with $\beta \circ h_{n+c,n} = \alpha$. Then there exists a homomorphism $\gamma : U \to R$ such that $\gamma \circ h_n = \alpha$. Moreover assertion (1) states that γ is unique.

$$\begin{tikzcd}
U_n \arrow[r, "h_{n+c,n}"] \arrow[dr, "\alpha"'] & U_{n+c} \arrow[r, "h_{n+c}"] \arrow[d, "\beta"] & U \arrow[dl, dotted, "\gamma"] \\
 & R &
\end{tikzcd}$$

In the above diagram we can take $R = U_{n+c}, \alpha = h_{n+c,n}, \beta = id$. This yields a γ, with $\gamma \circ h_{n+c} \circ h_{n+c,n} = h_{n+c,n}$. From the fact that $h_{n+c} \circ h_{n+c,n} = h_n$ it immediately follows that $\ker(h_n) = \ker(h_{n+c,n})$, which proves part (b) even for $n \geq N$.

From $\gamma \circ h_n = h_{n+c,n}$ we also get $h_{n+c} \circ \gamma \circ h_n = h_{n+c} \circ h_{n+c,n} = h_n$. We see that $h_{n+c} \circ \gamma : U \to U$ and id_U induce the same element $h_n \in \mathcal{F}_n(U)$ under the map $\mathcal{F}(U) \to \mathcal{F}_n(U)$, so by assertion (1) we find $h_{n+c} \circ \gamma = id_U$. This immediately implies that $h_{n+c} : U_{n+c} \to U$ is surjective, which proves part (a). $\square$

Proof of the assertions.
In the following we will use the standard form for a differential operator $z\frac{d}{dz} + B, B \in \mathrm{End}(C((z)) \otimes V)$. We recall some facts from the classification of formal differential equations (see [PS03] Chapter 3).

There is a finite extension $C((z)) \subset C((t)),\ t^e = z,\ e \in \mathbb{N}$, such that $z\frac{d}{dz} + B$ is equivalent over $C((t))$ to an operator of the form $z\frac{d}{dz} + St$, where St satisfies:

- there is a decomposition $V = \oplus V_i$, such that St acts on V_i, by $St(v_i) = q_i v_i + l_i(v_i)$, for some $q_i \in t^{-1}C[t^{-1}]$ and for some linear map $l_i : V_i \to V_i$,
- any two distinct eigenvalues of a map l_i do not differ by integer multiples of $\frac{1}{e}$.

In this section we will use the term *standard operator* for a differential operator satisfying these requirements. We now come to the proofs of assertions (1) and (2).

Proof. [Proof of assertion (1)] Let $(\frac{d}{dz} + A, \{\phi_i\})$ and $(\frac{d}{dz} + \tilde{A}, \{\tilde{\phi_i}\})$ be two elements in $\mathcal{F}(R)$ with the same image in $\mathcal{F}_n$, for some $n \in \mathbb{N}$. For a fixed i we find $\phi_i^{-1}(\frac{d}{dz} + B_i)\phi_i = \frac{d}{dz} + A = \tilde{\phi}_i^{-1}(\frac{d}{dz} + B_i)\tilde{\phi}_i$. Hence $\phi := \phi_i \tilde{\phi}_i^{-1}$ satisfies $\phi(\frac{d}{dz} + B_i) = (\frac{d}{dz} + B_i)\phi$. Thus $\phi \in \mathrm{GL}(R[[z]] \otimes V)$ is a solution of the differential equation $\phi' = \phi B_i - B_i \phi$. This differential equation has a finite dimensional solution space in $\mathrm{End}(C((z)) \otimes V)$. Therefore there is an $N_i \geq 0$, such that if $\phi - I \in O(n), n \geq N_i$, implies $\phi = I$ (here we use notation 2.3). We conclude that $N = \max\{0, N_i - k_i\}$ has the required property. □

Remark 2.10 The numbers N_i in the proof of assertion (1) can be made explicit as follows. For $B \in \mathrm{End}(C((z)) \otimes V)$, let $\mathrm{Aut}(z\frac{d}{dz} + B)$ denote the group of automorphisms of the differential operator $z\frac{d}{dz} + B$. To be explicit, $\mathrm{Aut}(z\frac{d}{dz} + B)$ is given by $\{\psi \in \mathrm{GL}(C((z)) \otimes V) |\ \psi(z\frac{d}{dz} + B)\psi^{-1} = (z\frac{d}{dz} + B)\}$. From Lemma 12.13 in [PS03] it easily follows that for a standard operator $z\frac{d}{dz} + St$, the group $\mathrm{Aut}(z\frac{d}{dz} + St)$ is contained in $\mathrm{GL}(V)$. An arbitrary differential operator $z\frac{d}{dz} + B$ can be written in the form $\gamma(z\frac{d}{dz} + St)\gamma^{-1}$, for some $\gamma \in \mathrm{GL}(C((t)) \otimes V),\ t^e = z$. The automorphism group of $(z\frac{d}{dz} + B)$ in $\mathrm{GL}(C((t)) \otimes V)$ is $\gamma \mathrm{Aut}(z\frac{d}{dz} + St)\gamma^{-1}$. It follows that the properties

$\phi \in \mathrm{Aut}(z\frac{d}{dz} + B)$ and $\phi - I \in O(g+1)$, with $g := \lfloor \frac{ord(\gamma)+ord(\gamma^{-1})}{e} \rfloor$, imply that ϕ is the identity. In the situation of the proof of assertion (1), one has $z\frac{d}{dz} + zB_i = \gamma_i(z\frac{d}{dz} + St_i)\gamma_i^{-1}$, for some $\gamma_i \in \mathrm{GL}(C((t_i)) \otimes V), t_i^{e_i} = z$. Thus one can take for N_i the integer $\lfloor \frac{g_i}{e_i} \rfloor + 1$, where $g_i = ord(\gamma_i) + ord(\gamma_i^{-1})$. •

For the proof of assertion (2), we need the following two lemmas.

Lemma 2.11 *Given is a standard differential operator $z\frac{d}{dz} + St$ and a differential operator $z\frac{d}{dz} + A$, $A \in \mathrm{End}(C((z)) \otimes V)$ such that $A - St \in O(n)$ with $n \geq 1$. Then there is a unique $h \in \mathrm{GL}(C[[z]] \otimes V)$ of the form $h = I + k$ and $k \in O(n)$ such that $h(z\frac{d}{dz} + St)h^{-1} = (z\frac{d}{dz} + A)$.*

Proof. Write $A = St - R$ with $R \in O(n)$. Then we have to produce a $k \in O(n)$, with $k' + St \cdot k - k \cdot St = R + R \cdot k$. The differentiation is here $' = z\frac{d}{dz}$ instead of $\frac{d}{dz}$. Put $L(k) := k' + St \cdot k - k \cdot St$. Suppose that one can solve, for any $T \in O(n)$, $n \geq 1$, the equation $L(k) = T$ with a $k \in O(n)$. Then the equation $L(k) = R + R \cdot k$ has a solution $k \in O(n)$. Indeed, define a sequence of elements $k_1, k_2, k_3, \ldots$ by $L(k_1) = R$, $L(k_2) = R \cdot k_1$, $L(k_3) = R \cdot k_2$, etc. Then the orders of the k_n tend to ∞ and the sum $k := \sum_{i=1}^{\infty} k_i$ converges. This k satisfies $L(k) = R + R \cdot k$.

For solving $L(k) = T$ with $T \in O(n)$ one needs the structure of the standard differential operator $z\frac{d}{dz} + St$. Let $V = \oplus_{i=1}^{r} V_i$ be the decomposition corresponding to St. Every $T \in \mathrm{End}(C((t)) \otimes V)$ is given as a "block matrix" $T = (T_{i,j})_{1 \leq i,j \leq r}$, where $T_{i,j} \in L_{i,j}$, where $L_{i,j}$ denotes the space of $C((t))$-linear maps from $C((t)) \otimes V_i$ to $C((t)) \otimes V_j$. By definition $St = (St_{i,j})$ with $St_{i,j} = 0$ for $i \neq j$ and $St_{i,i} = q_i I + l_i$. For $k = (k_{i,j})$ one has $L(k) = (L(k)_{i,j})$, with $L(k)_{i,j} = k'_{i,j} + (q_i I + l_i)k_{i,j} - k_{i,j}(q_j I + l_j) = k'_{i,j} + (q_i - q_j)k_{i,j} + l_i k_{i,j} - k_{i,j} l_j$. Let $T = (T_{i,j}) \in O(n)$ be given. Then we have to solve for every i, j the equation $k'_{i,j} + (q_i - q_j)k_{i,j} + l_i k_{i,j} - k_{i,j} l_j = T_{i,j}$, with $k_{i,j} \in O(n)$.

Define an operator $M_{i,j}$ on $L_{i,j}$ by $M_{i,j}(A) = A' + (q_i - q_j)A + l_i A - A l_j$ for $A \in L_{i,j}$. For $i \neq j$, one observes that the terms of lowest order in $M_{i,j}(A)$ apear in the term $(q_i - q_j)A$. With the obvious definition for $O(m)$, this implies that $M_{i,j}(O(m)) = O(m-r)$, where r is the degree of $q_i - q_j$ in t^{-1}. This solves the problem for $i \neq j$. For $i = j$, one has that $M_{i,i}(O(m)) = O(m)$, because no non-zero difference of the eigenvalues of the constant matrix l_i is an integer multiple of $\frac{1}{e}$. Here we used the fact that the endomorphism on $\mathrm{GL}_n(C)$ given by $X \mapsto AX - XB,\ A, B \in \mathrm{GL}_n(C)$ is an automorphism if A, B have no eigenvalues in common. This solves the problem for $i = j$. The existence of h has now been shown. One easily sees that the steps in the proof also imply that h is unique. □

Lemma 2.12 *Fix an operator $\frac{d}{dz}+B$ with $B \in End(C((z))\otimes V)$. There exist an integer $c \geq 1$ (depending only on B), such that for any differential operator $\frac{d}{dz}+A$ with $A-B \in O(k),\ k \geq c$, there is a unique $h \in \mathrm{GL}(C[[z]]\otimes V)$, with $h(z\frac{d}{dz}+B)h^{-1} = z\frac{d}{dz}+A$, and $h - I \in O(k-c)$.*

Proof. There is a $\gamma \in \mathrm{GL}(C((t)) \otimes V)$, with $t^e = z,\ e \in \mathbb{N}$ such that $\gamma(z\frac{d}{dz}+zB)\gamma^{-1} =: z\frac{d}{dz}+St$ is a standard operator. Write $z\frac{d}{dz}+\tilde{A}$ for the operator $\gamma(z\frac{d}{dz}+zA)\gamma^{-1}$. If A satisfies $A-B \in O(k)$, then $St-\tilde{A} \in O(n)$ with respect to t, where $n = e(k+1)+\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1})$. If $k \geq \frac{1-\mathrm{ord}(\gamma)-\mathrm{ord}(\gamma^{-1})}{e}-1$, we can apply Lemma 2.11 to obtaine a $\tilde{h} \in \mathrm{GL}(C[[t]]\otimes V)$ with the properties $\tilde{h} - I \in O(n)$ and $\tilde{h}(t\frac{d}{dt}+eSt)\tilde{h}^{-1} = t\frac{d}{dt}+e\tilde{A}$. This can be restated as $\tilde{h}(z\frac{d}{dz}+St)\tilde{h}^{-1} = z\frac{d}{dz}+\tilde{A}$. Now $h := \gamma^{-1}\tilde{h}\gamma$ satisfies $h(z\frac{d}{dz}+B)h^{-1} = z\frac{d}{dz}+A$, and furthermore $h - I \in O(m),\ m = n+\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1})$. A priori, h has entries in $C((t))$, but the uniqueness of h implies that the entries of h are actually in $C((z))$. We have that $h \in O(k+1+\frac{2(\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1})}{e})$. Concluding, we find that for c to satisfy the properties of the lemma, we must have $c \geq \frac{1-\mathrm{ord}(\gamma)-\mathrm{ord}(\gamma^{-1})}{e}-1$ and $c \geq -\frac{2(\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1})}{e}-1$, and since $\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1}) \leq 0$, we can take $c = \max(0, -\frac{2(\mathrm{ord}(\gamma)+\mathrm{ord}(\gamma^{-1})}{e}-1)$. □

Proof. [Proof of assertion (2)] The local data for $\mathcal{F}$ are given by operators $\frac{d}{dt_i}+B_i$ for $i = 1,\ldots,s$. Lemma 2.12 attaches an integer $c_i \geq 1$ to each operator. Take $c := \max_i\{c_i\}$. We want to show that the maps $\mathcal{F}(R) \to \mathcal{F}_n(R)$ and $\mathcal{F}_{n+c}(R) \to \mathcal{F}_n(R)$ have the same image for $n \geq N$, with N as above. One inclusion is obvious.

Consider a tuple $\xi := \{\phi_i(j))|0 \leq j < n+k_i,\ 1 \leq i \leq r\} \in \mathcal{F}_n(R)$. By definition, $\phi_i(0)$ is invertible, $\phi_1(0) = I$ and the $\phi_i := \sum_{j=0}^{n+k_i-1}\phi_i(j)t_i^j$ satisfy the equations $\phi_i' \equiv \phi_i A - B_i\phi_i \mod t_i^n$ for all i. We recall that the map A is determined by a first part of the ϕ_i's. Suppose that ξ is the image of $\{\psi_i(j))|0 \leq j < n+c+k_i,\ 1 \leq i \leq r\} \in \mathcal{F}_{n+c}(R)$ under the map $\mathcal{F}_{n+c}(R) \to \mathcal{F}_n(R)$. The differential equation associated with

$\psi_i := \sum_{j=0}^{n+c+k_i} \psi_i(j)t_i^j$ is $\psi_i(\frac{d}{dt_i}+A)\psi_i^{-1}$, which we can write as $\frac{d}{dt_i}+B_i+R$, with $R \in O(n+c)$. According to Lemma 2.11 (and since $c \geq c_i$) there is an h_i with $h_i - I \in O(n+c-c) = O(n)$ such that $h_i(\frac{d}{dz}+B_i)h_i^{-1} = \psi_i(\frac{d}{dt_i}+A)\psi_i^{-1}$. Define $\tau_i = h_i^{-1}\psi_i$. Then $\{\tau_i\}_{i=1}^r$ is an element of $\mathcal{F}(R)$ which maps to the given ξ. □

2.4 Closed subspaces of moduli spaces

In his article [S93], Michael Singer proves a result on the variation of the differential Galois group in certain families of scalar differential equations. In our setting this would translate into the following. For a given moduli functor $\mathcal{F}$ with moduli space $\mathbb{M}$ and a given linear algebraic group G, the set of the closed points m of $\mathbb{M}$ such that the differential Galois group $Gal(m)$ is equal to G is Zariski constructible if G satisfies a certain group theoretic condition (the condition in Definition 3.1). We will prove this statement in a more general setting in Chapter 3. In our current setting, we will prove that the set of closed points m of $\mathbb{M}$ such that the differential Galois group $Gal(m)$ is contained in G is Zariski closed.

For convenience we consider a moduli problem and the corresponding moduli functor $\mathcal{F}$ and moduli space $\mathbb{M}$, for which 0 and ∞ are nonsingular. Write $S = \sum k_i[s_i]$ for the divisor corresponding to the data for $\mathcal{F}$. A closed point (or C-valued point) of $\mathbb{M}$ is an element $m = (\frac{d}{dz}+A, \{\phi_i\}) \in \mathcal{F}(C)$. We recall that this notation for elements of $\mathcal{F}(C)$ is derived from the notation $(\mathcal{M}, \nabla, \{\phi_i\})$ by an identification of $H^0(\mathbb{P}, \mathcal{M})$ with the vector space V and $\frac{d}{dz}+A = \nabla_{\frac{d}{dz}}$.

Definition 2.13 *$Gal(m)$ as algebraic subgroup of* $\mathrm{GL}(V)$.
Let $m = (\frac{d}{dz}+A, \{\phi_i\}) \in \mathbb{M}(C)$. A priori, the solution space, the Picard-Vessiot ring and the differential Galois group of $\frac{d}{dz}+A$ are only given up to isomorphism or up to conjugation. Using the regular point 0 for $\frac{d}{dz}+A$ we will fix these objects. The entries of A belong to the subring $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$ of $C[[z]]$. Therefore $W := \ker(\frac{d}{dz}+A, C[[z]]\otimes V)$ is a solution space. The canonical map $C[[z]]\otimes V \to V$, given by $f\otimes v \mapsto f(0)\cdot v$, induces a bijection $W \to V$. In this way we will identify V with the solution space W of $\frac{d}{dz}+A$. The ring generated over $C[z, \frac{1}{(z-s_1)\dots(z-s_r)}]$ by the coordinates of all elements

of W is a differential subring of $C[[z]]$. This ring will be denoted by $\mathrm{PVR}(m)$. It is easily seen that $\mathrm{PVR}(m)$ is a Picard-Vessiot ring for $\frac{d}{dz}+A$. The group of the differential automorphisms of $\mathrm{PVR}(m)$ over $C[z, \frac{1}{(z-s_1)\dots(z-s_r)}]$ is the differential Galois group $Gal(m)$ of $\frac{d}{dz}+A$. The natural C-linear action of $Gal(m)$ on W, which makes $Gal(m)$ into a Zariski closed subgroup of $\mathrm{GL}(W)$, is transported via the identification of W with V, to an identification of $Gal(m)$ as Zariski closed subgroup of $\mathrm{GL}(V)$. Depending on the context we will also write $Gal(A)$ or $Gal(\nabla)$ instead of $Gal(m)$. •

Remark 2.14 *In case of moving singularities, one still wants to fix a "base point". A good choice would be to replace the point* 0 *by an infinitesimal "base sector" at the point* 0.

Let G be some linear algebraic group $G \subset \mathrm{GL}(V)$. We want to prove that $\{m \in \mathbb{M}(C) |\ Gal(m) \subset G\}$ is a Zariski closed set. The first step is a reduction to the case where $G = \{g \in \mathrm{GL}(V) | g(L) = L\}$ for some line $L \subset V$. Chevalley's theorem states that there is a vector space V', obtained from V by some construction of linear algebra, and a line $L' \subset V'$, such that $G = \{g \in \mathrm{GL}(V) | \psi(g)(L') = L'\}$. Here $\psi : \mathrm{GL}(V) \to \mathrm{GL}(V')$ is the natural map induced by the construction of linear algebra on V. We need to describe this construction more explicitly.

Construction 2.15 Let $I \subset C[\mathrm{GL}(V)] := C[\{X_{i,j}\}, \frac{1}{Det((X_{i,j}))}]$ be the ideal defining G. The group $\mathrm{GL}(V)$ acts on the left on $C[\{X_{i,j}\}, \frac{1}{Det((X_{i,j}))}]$. Let $H(d)$ denote the vector space of the homogeneous polynomials in the variables $X_{i,j}$ of degree d. One can show that there exists an integer $k \geq 1$ such that I is generated by $I_k := I \cap \oplus_{d=1}^{k} H(d)$ and moreover that $g \in \mathrm{GL}(V)$ leaves I_k invariant if and only if $g \in G$ (see [Br69] §5). Let D be the dimension of I_k and consider the line $L := \bigwedge^D I_k$ in $V' := \bigwedge^D(\oplus_{d=1}^{k} H(d))$. Then $g \in \mathrm{GL}(V)$ belongs to G if and only if $\psi(g)(L') = L'$, with ψ as defined above. The vector space $H(1)$ with its $\mathrm{GL}(V)$-action can be identified with $V^n := V \oplus \cdots \oplus V$, where n is the dimension of V. The vector space $H(d)$ with its $\mathrm{GL}(V)$-action is then identified with $\mathrm{Sym}(V^n, d)$, the d-th symmetric power of V^n. Finally V' is identified as a $\mathrm{GL}(V)$-module with $\bigwedge^D(\oplus_{d=1}^{k}\mathrm{Sym}(V^n, d)$. •

This explicit construction of the V' is now copied for the local data defining the moduli functor $\mathcal{F}$ and its moduli space $\mathbb{M} = \mathrm{Spec}(U)$. The new moduli functor is denoted by $\mathcal{F}'$ and the new moduli space by $\mathbb{M}' = \mathrm{Spec}(U')$. The above construction applied to the universal element in $\mathcal{F}(U)$ of $\mathbb{M}$ yields an element of $\mathcal{F}'(U)$. There results a morphism of C-algebras $U' \to U$ and a C-morphism $f : \mathbb{M} \to \mathbb{M}'$. Put $H := \{g \in \mathrm{GL}(V')|\ gL' = L'\}$. We assume now that $Z := \{m' \in \mathbb{M}'(C)|\ Gal(m') \subset H\}$ is Zariski closed. Then also $f^{-1}Z = \{m \in \mathbb{M}(C)|\ Gal(m) \subset G\}$ is Zariski closed.

We will need the following lemma.

Lemma 2.16 *Let $G_L := \{g \in \mathrm{GL}(V)|g(L) = L\}$, where $L \subset V$ is a line. For any element $m = (\frac{d}{dz} + A, \{\phi_i\}) \in \mathbb{M}(C)$, with corresponding connection ∇, the condition $Gal(m) \subset G_L$ is equivalent to the existence of a line bundle $\mathcal{L} \subset \mathcal{M}$, with the properties:*

(1) *$\mathcal{L}$ is ∇ invariant, i.e. $\nabla(\mathcal{L}) \subset \Omega(S) \otimes \mathcal{L}$.*

(2) *$\mathcal{M}/\mathcal{L}$ is a vector bundle.*

(3) *$\mathcal{L}_0/z\mathcal{L}_0 = L$, where $\mathcal{L}_0$ is the stalk of $\mathcal{L}$ at 0.*

Proof. Suppose that a line bundle $\mathcal{L} \subset \mathcal{M}$ verifies the properties (1)-(3). Then clearly $L := \mathcal{L}_0/z\mathcal{L}_0$ is a $Gal(m)$-invariant line in V, so $\mathrm{Gal}(m) \subset G_L$.

For the other implication, assume that $Gal(m)$ leaves the line $L \subset V$ invariant. By Definition 2.13, the solution space $W \subset \mathrm{PVR}(m) \otimes V \subset C[[z]] \otimes V$ is identified with V. We write T for the $Gal(m)$-invariant line in W, corresponding to L under this identification. The Picard-Vessiot field $\mathrm{PV}(m)$ is by definition the field of fractions of $\mathrm{PVR}(m)$ Then $\mathrm{PV}(m) \otimes_C T$ is a $Gal(m)$-invariant line (over the field $\mathrm{PV}(m)$) in $\mathrm{PV}(m) \otimes_C W$. The natural isomorphism $\mathrm{PV}(m) \otimes_C W \to \mathrm{PV}(m) \otimes_{C(z)} M$, where $M = C(z) \otimes_C V$, yields an isomorphism $(\mathrm{PV}(m) \otimes W)^{Gal(m)} \to M$. It is well known that $N := (\mathrm{PV}(m) \otimes_C T)^{Gal(m)}$ is a 1-dimensional submodule of M. An explicit reference for this is Proposition 5.12 in [P99]. We observe that M is the stalk $\mathcal{M}_\xi$ of $\mathcal{M}$ at the generic point ξ of $\mathbb{P}^1_C$. One associates to $N \subset M$ the line bundle $\mathcal{L} \subset \mathcal{M}$ given by $\mathcal{L}(O) = \{l \in \mathcal{M}(O)|$ the stalk l_ξ of l belongs to $N\}$ for any open O of $\mathbb{P}^1_C$. It is easily seen that $\mathcal{L}$ has the required properties. $\square$

Theorem 2.17 *Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup, then $\{m \in \mathbb{M}(C) | Gal(m) \subset G\}$ is a Zariski closed set.*

Proof. We have already shown that the theorem follows from the special case where a line $L \subset V$ is given and $G = \{g \in \mathrm{GL}(V) | g(L) = L\}$. By the previous lemma, this is equivalent to the existence of a line bundle $\mathcal{L}$, with the stated properties. Write $d = -\deg(\mathcal{L})$. Since $\mathcal{L} \subset \mathcal{M}$, we have $d \geq 0$. The line bundle $\mathcal{L}(d \cdot [\infty]) = \mathcal{L} \otimes \mathcal{O}(d \cdot [\infty])$ is free and generated by an element $w := \sum_{i=0}^{d} v_i z^i \in H^0(\mathbb{P}, \mathcal{M}(d \cdot [\infty]))$, where $v_0, \ldots, v_d \in V = H^0(\mathbb{P}, \mathcal{M})$. The term v_0 is a non-zero element of $L \subset V$ and will be fixed. We note that the condition that $\mathcal{M}/\mathcal{L}$ is a vector bundle implies that $v_d \neq 0$. The same condition implies that w evaluated at any point s_j, i.e. $\sum_i v_i s_j^i$, is non-zero.

The ∇-invariance of $\mathcal{L}$ implies that

$$\nabla : \mathcal{L}(d \cdot [\infty]) \to \Omega(S + [\infty]) \otimes \mathcal{L}(d \cdot [\infty]).$$

Thus $\nabla w \in H^0(\mathbb{P}, \Omega(S + [\infty])) \otimes w$. The vector space $H^0(\mathbb{P}, \Omega(S + [\infty]))$ consists of the expressions $(\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{i,j}}{z - s_i}) dz$ with all $g_{i,j} \in C$. After replacing ∇ by $\nabla_{\frac{d}{dz}}$ the differential equation that we obtain is

$$\left(\frac{d}{dz} + \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(z - s_i)^j}\right)\left(\sum_{i=0}^{d} v_i z^i\right) = \left(\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{i,j}}{(z - s_i)^j}\right)\left(\sum_{i=0}^{d} v_i z^i\right).$$

The entries of the matrices $A_{i,j}$ are in fact regular functions on the moduli space $\mathbb{M}$. In other words, these entries are in U, where $\mathbb{M} = \mathrm{Spec}(U)$. We will show in the sequel that there are finitely many possibilities for the term $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{ij}}{(z - s_i)^j}$. Expanding the equation at ∞, and comparing the coefficient of z^{d-1} yields $d v_d = (\sum_{i=1}^{r} g_{i,1}) v_d$. Indeed, $\sum_{i=1}^{r} A_{i,1} = 0$ since ∇ has no singularity at ∞. Since $v_d \neq 0$, one has $d = \sum_{i=1}^{r} g_{i,1}$. We note that this puts a condition on the $g_{i,1}$, namely $\sum_i g_{i,1}$ is a non-negative integer.

For a fixed choice of $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{ij}}{(z - s_i)^j}$, the above differential equation is regular at $z = 0$ and has a unique formal solution $\sum_{i=0}^{\infty} v_i z^i$ with prescribed v_0. The coefficients v_i, $i \geq 1$ of this formal solution are, like the entries of the $A_{i,j}$, regular vector functions on the moduli space $\mathbb{M}$. In other words, $v_i \in U \otimes V$ for $i \geq 1$.

Consider for $i > d$ the condition $v_i = 0$. Fix a basis $e_1, \ldots, e_n$ of V over C and write $v_i = u_1 \otimes e_1 + \cdots + u_n \otimes e_n$ with $u_1, \ldots, u_n \in U$. Then $v_i = 0$ defines the closed subset $\{m \in \mathbb{M}(C) |\ u_j(m) = 0 \text{ for } j = 1, \ldots, n\}$. Thus, for a given $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{ij}}{(z-s_i)^j}$, the existence of a solution $\sum_{i=0}^{d} v_i z^i$ (with $v_0 \in L$ fixed), defines a Zariski closed subset of $\mathbb{M}(C)$. Since we do not insist on $v_d \neq 0$, this closed subset describes the points $m \in \mathbb{M}(C)$ such that there is a ∇-invariant line bundle $\mathcal{L} \subset \mathcal{M}$ of degree $\geq -d$ corresponding to the line $L \subset V$ and with prescribed choice of $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{ij}}{(z-s_i)^j}$. Further $\{m \in \mathbb{M}(C) |\ Gal(m) \subset G\}$ is a finite union of these subsets and therefore a Zariski closed set.

For the proof of the fact that there are only finitely many possibilities for $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{g_{i,j}}{(z-s_i)^j}$ we will need the following lemma.

Lemma 2.18 *Consider a local connection $\nabla : N \to C[[z]]z^{-k}dz \otimes N$, with N a free $C[[z]]$-module of rank r and $k > 0$. There are finitely many elements $\lambda_1, \ldots, \lambda_s \in z^{-1}C[z^{-1}]$, $s \leq r$, such that $\nabla w = f dz \otimes w$ with $w \in N \setminus zN$ and $f \in C((z))$ implies that f has the form $\lambda_i + R$ with $1 \leq i \leq s$ and $R \in C[[z]]$.*

Proof. It suffices to prove the following statement:

suppose there exist elements $e_1 \ldots, e_s \in N \setminus zN$ that satisfy $\nabla e_i = f_i dz \otimes e_i$ for some $f_1, \ldots, f_s \in C((z))$ with $f_i - f_j \notin C[[z]]$ for $i \neq j$. Then $e_1, \ldots, e_s$ are independent over $C((z))$.

We prove this by induction on s. For $s = 1$ there is nothing to prove. Suppose that the statement holds for $s - 1$ and let $e_1, \cdots, e_s$ be elements satisfying the conditions of the statement. Suppose that there is a non-trivial relation $\sum_i h_i e_i = 0,\ h_i \in C((z))$. We may assume that $h_1 = 1$ and that all $h_i \in C[[z]]$. The induction hypothesis implies that all $h_i \neq 0$ and that this is the only relation up to multiplication by an element in $C((z))^*$. Now $\nabla_{\frac{d}{dz}}$ applied to the relation $e_1 + \sum_{i=2}^{r} h_i e_i$ yields a new relation, namely $f_1 e_1 + \sum_{i=2}^{r} (h_i' + f_i h_i) e_i = 0$. Hence $h_i' + f_i h_i = f_1 h_i$ for $i = 2, \ldots, s$, so $f_1 - f_i = \frac{h_i'}{h_i}$ for $i = 2, \ldots, s$. Therefore we must have $h_2, \ldots, h_s \in zC[[z]]$ and this leads to the contradiction $e_1 \in zN$. $\square$

We now continue the proof of the theorem. Let us write g for the expression $\sum_{i=1}^{r}\sum_{j=1}^{k_i}\frac{g_{i,j}}{(z-s_i)^j}$ that we are considering. We will use the terminology of Definition 2.1. A non-zero solution w for $\nabla w = gdz\otimes w$ induces a non-zero solution $\phi_i w \in N_i$, $\phi_i w \notin (z-s_i)N_i$ for the equation $\nabla_i(\phi_i w) = gd(z-s_i)\otimes\phi_i w$. According to the above lemma, there are only finitely many possibilities for the principal part of g at each point s_i. Hence there are only finitely many possibilities for g. $\square$

Chapter 3

Singer's Theorem for families of differential equations

This chapter is joint work with Marius van der Put.

3.1 Introduction

In this chapter we will define *families of differential equations* on the projective line $\mathbb{P}^1_C$, parametrized by a scheme of finite type X. As before we suppose C to be algebraically closed and of characteristic zero. These families are of a more general nature than the moduli spaces, defined in Chapter 2. Theorem 2.17 is extended to a family of differential equations of dimension n, parametrized by some X. Thus the condition "$\mathrm{Gal}(x) \subset G$" for closed points x of X (i.e., $x \in X(C)$) defines a closed subset of X. The aim is to show that the set of closed points $x \in X$ for which the differential Galois group $\mathrm{Gal}(x)$ of the corresponding equation is equal to G is a *constructible* subset of X i.e., of the form $\cup_{i=1}^n (O_i \cap F_i)$ for open sets O_i and closed sets F_i. This statement (and the earlier one) has to be made more precise by providing a suitable definition of "family of differential equations" and a meaning for the expression $\mathrm{Gal}(x) \subset G$. Moreover, a condition on the group G is essential.

In his paper [S93], M.F. Singer defines a set of differential operators, by giving some local data. He proves that under a certain condition on G, the subset of differential equation with Galois group equal to G is constructible. This condition on G will be called the *Singer condition.* We will consider the same problem, in our context of families of differential equations parametrized by a scheme X. We will construct for any group G that does not satisfy the "Singer condition" an example of a moduli family $\mathbb{M}$ such that $\{x \in \mathbb{M} | \mathrm{Gal}(x) = G\}$ is not constructible. Finally, from these constructions one deduces an alternative description of the Singer condition.

3.2 The Singer condition

Let G be a linear algebraic group over C. First we will recall the Singer condition on G, as given in [S93]. A character χ of G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$, where $\mathbb{G}_m$ stands for the multiplicative group C^*. The set $X(G)$ of all characters is a finitely generated abelian group. Let ker $X(G)$ denote the intersection of the kernels of all $\chi \in X(G)$. This intersection is a characteristic (closed) subgroup of G. As usual, G^o denotes the connected component of the identity of G. The group ker $X(G^o)$ is a normal, closed subgroup of G^o and G. Let $\chi_1, \dots, \chi_r$ generate $X(G^o)$. Then ker $X(G^o)$ is equal to the intersection of the kernels of $\chi_1, \dots, \chi_r$. In other words ker $X(G^o)$ is the kernel of the morphism $G^o \to \mathbb{G}_m^s$, given by $g \mapsto (\chi_1(g), \dots, \chi_r(g))$. The image is a connected subgroup of $\mathbb{G}_m^s$ and therefore a torus T. Hence $G^o/\ker X(G^o)$ is isomorphic to T. Moreover, by definition, T is the largest torus factor group of G^o. One considers the exact sequence:

$$1 \to G^0/\ker X(G^0) \to G/\ker X(G^0) \to G/G^0 \to 1.$$

Since $G^0/\ker X(G^0)$ is abelian, this sequence induces an action of G/G^0 on $G^0/\ker X(G^0)$ by conjugation.

Definition 3.1 A linear algebraic group G satisfies the *Singer Condition* if the action of G/G^0 on $G^0/\ker X(G^0)$ is trivial. •

The Singer condition can be stated somewhat simpler, using $U(G) \subset G$, the subgroup generated by all unipotent elements in G.

Lemma 3.2 *$U(G) = U(G^o)$ is equal to* ker $X(G^o)$ *and the Singer condition is equivalent to "$G^o/U(G)$ lies in the center of $G/U(G)$".*

Proof. Fix an embedding $G \subset \mathrm{GL}(V)$, where V is a finite dimensional vector space over C. First we prove that $U(G)$ is a closed connected normal subgroup of G. Let $I+B$, $B \neq 0$ be a unipotent element of G. Then $I+B = e^D$, for some nilpotent element $D = \sum(-1)^{i-1}i^{-1}B^i \in \mathrm{End}(V)$. The Zariski closure $\overline{\{(I+B)^n | n \in \mathbb{Z}\}}$ of the group generated by $I+B$ lies in G and is equal to the group $\{e^{tD} | t \in C\}$, which is isomorphic to the additive group $\mathbb{G}_a$ over C. Hence $U(G)$ is generated by this connected subgroups of G and by Proposition 2.2.6 of [Sp98] the group $U(G)$ is closed and connected. Further $U(G)$ is a normal subgroup and even a characteristic subgroup, since the set of unipotent elements of G is stable under any automorphism of G. The connectedness of $U(G)$ implies $U(G) = U(G^o)$.

Now we will show that $G^o/U(G^o)$ is a torus. Since the *unipotent radical* $R_u(G^o)$ lies in $U(G^o)$, we may divide G^o by $R_u(G^o)$ and assume G^o to be *reductive*. Then by [Sp98] Corollary 8.1.6, we have $G^o = R(G^o) \cdot (G^o, G^o)$, where $R(G^o)$ is the radical of G^o, and where (G^o, G^o) is the commutator subgroup of G^o. The latter group is a semi simple subgroup, according to the same corollary. By [Sp98] Theorem 8.1.5, we get that (G^o, G^o) is generated by unipotent elements, so $(G^o, G^o) \subset U(G^o)$. Since $R(G^o)$ is a torus, its image $G^o/U(G^o)$ is a torus, too. This proves $U(G^o) \supset \ker X(G^o)$. The other inclusion follows from the observation that every unipotent element lies in the kernel of every character. Finally, the triviality of the action of G/G^o on $G^o/U(G^o)$ is clearly equivalent to $G^o/U(G^o)$ lies in the center of $G/U(G^o)$.
□

Assumptions 3.3 Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group. For the moment we assume the following items (see Definition 3.13, Remark 3.14, Proposition 3.16 and Corollary 3.17).

- The definition of a family of differential equations, parametrized by X.
- The meaning of $\mathrm{Gal}(x) \subset G$ for $x \in X(C)$.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset G\}$ is closed.
- $\{x \in X(C) \, |\mathrm{Gal}(x) \subset hGh^{-1}$ for some $h \in \mathrm{GL}(V)\}$ is constructible. •

Lemma 3.4 *Let G, X be as in the above assumptions. If G has finitely many proper closed subgroups $H_1, \ldots, H_r$, such that every proper closed subgroup is* *contained in a conjugate of one of the* H_i*, then* $\{x \in X(C) \mid \mathrm{Gal}(x) = G\}$ *is constructible.*

The proof is easy.

Remarks 3.5
(1) If G satisfies the group-condition of the lemma, then G satisfies the Singer condition, too. This follows from the fact that $G/U(G)$ is a finite group. Indeed, if $T := G^o/U(G) \neq \{1\}$, then one can produce many proper normal subgroups of $G/U(G)$. For example, for any integer $m > 1$ the subgroup $T[m]$, consisting of the m-torsion elements of T. By lifting this subgroups, we obtain a contradiction.

(2) Consider $G := \mathrm{SL}_2(C)$. The classification of the proper closed subgroups H of G states that H is either contained in a Borel subgroup or in a conjugate of the infinite dihedral group $D_\infty^{\mathrm{SL}_2}$ or is conjugated to one of the special finite groups: the tetrahedral group, the octahedral group, the icosahedral group. Thus G satisfies the conditions of the lemma and moreover, $G/U(G) = \{1\}$.

(3) The infinite dihedral group $G = D_\infty^{\mathrm{SL}_2}$ has the properties: $G^o = \mathbb{G}_m$, $U(G^o) = 1$ and G/G^o acts non-trivially on G^o. Thus G does not satisfy the Singer condition. For this group one can produce moduli spaces $\mathbb{M}$ such that $\{x \in \mathbb{M}(C) \mid \mathrm{Gal}(x) = G\}$ is not constructible (see example 3.8).

(4) For the following two examples, namely moduli spaces and the groups $\mathbb{G}_a^3$ and $\mathbb{G}_m^n$, the Singer condition is valid, but G does not satisfy the condition of the lemma. We will show explicitly that these groups define constructible subsets. •

Example 3.6 *A moduli space with differential Galois groups in* $\mathbb{G}_a^3$.
V is a 4-dimensional vector space over C with basis $e_1, \ldots, e_4$. $N \in \mathrm{End}(V)$ is given by $N(e_i) = 0$ for $i = 1, 2, 3$ and $N(e_4) = e_1$. The data for the moduli problem are.

- Three distinct singular points $s_1, s_2, s_3 \in C^*$. The point ∞ is allowed to have a, non prescribed, regular singularity.
- For each singular point s_i, the differential operator $\frac{d}{d(z-s_i)} + \frac{N}{z-s_i}$.

Some calculations lead to an identification $\mathrm{GL}(4,C)\times\mathrm{GL}(4,C)\to\mathbb{M}$, where $\mathbb{M}$ is the moduli space of the problem. Let $m := (\phi_2,\phi_3)$ denote a closed point of the first space, then the corresponding universal differential operator is

$$\frac{d}{dz}+\frac{N}{z-s_1}+\frac{\phi_2^{-1}N\phi_2}{z-s_2}+\frac{\phi_3^{-1}N\phi_3}{z-s_3}.$$

Let $G := \mathbb{G}_a^3$ the subgroup of $\mathrm{GL}(V)$ consisting of the maps of the form $I+B$, $Be_1=0$ and $Be_i\in Ce_1$ for $i=2,3,4$. The condition $\mathrm{Gal}(m)\subset\mathbb{G}_a^3$ can be seen to be equivalent to $\phi_2(e_1),\phi_3(e_1)\in Ce_1$. This describes the set $\{m\in\mathbb{M}\mid \mathrm{Gal}(m)\subset G\}$ completely. The above differential operator evaluated at a point of $\{m\in\mathbb{M}\mid\mathrm{Gal}(m)\subset G\}$ has the form

$$\frac{d}{dz}+\begin{pmatrix}0&h_1&h_2&h_3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

where $(h_1,h_2,h_3)=\frac{1}{z-s_1}(0,0,1)+\frac{1}{z-s_2}(f_1,f_2,f_3)+\frac{1}{z-s_3}(g_1,g_2,g_3)$. Moreover, f_1,f_2,f_3 are polynomials of degree ≤ 2 in the entries of ϕ_2 and g_1,g_2,g_3 are polynomials of degree ≤ 2 in the entries of ϕ_3.

G has many (non-conjugated) maximal proper closed subgroups and there is no obvious reason why $\{m\in\mathbb{M}\mid\mathrm{Gal}(m)=G\}$ should be constructible. We continue the calculation. The differential Galois group $\mathrm{Gal}(m)$, with m such that $\mathrm{Gal}(m)\subset G$, is in fact the differential Galois group for the three inhomogeneous equations $y_i'=h_i,\ i=1,2,3$ over $C(z)$. Thus $\mathrm{Gal}(m)$ is a proper subgroup of G if and only if there is a non trivial linear combination $c_1h_1+c_2h_2+c_3h_3$ with $c_1,c_2,c_3\in C$ such that $y'=c_1h_1+c_2h_2+c_3h_3$ has a solution in $C(z)$. Now y exists if and only if $c_1h_1+c_2h_2+c_3h_3$ has residue 0 at the points s_1,s_2,s_3. The existence of such a linear combination translates into a linear dependence and the explicit equation $f_1(s_2)g_2(s_3)-f_2(s_2)g_1(s_3)=0$. This defines a closed subset of $\{m\in\mathbb{M}\mid\mathrm{Gal}(m)\subset G\}$ and therefore $\{m\in\mathbb{M}\mid\mathrm{Gal}(m)=G\}$ is constructible. $\diamond$

Example 3.7 *A moduli space with differential Galois groups in* $\mathbb{G}_m^n$.
The data for the moduli problem are.

- A vector space V of dimension n over C, with basis $e_1,\dots,e_n$.

- Singular points $s_1, \dots, s_r \in C^*$. We allow ∞ to have a non-prescribed regular singularity.
- Local differential operators $\frac{d}{d(z-s_i)} + \frac{C_i}{z-s_i}$, where $e_1, \dots, e_n$ are eigenvectors for all $C_i \in \mathrm{End}(V)$.

The moduli space $\mathbb{M}$ can be identified with $\mathrm{GL}(V)^{s-1}$. At a closed point $m = (\phi_2, \dots, \phi_r) \in \mathrm{GL}(V)^{s-1}$ the universal differential operator reads

$$\frac{d}{dz} + \sum_{i=1}^{s} \frac{\phi_i^{-1} C_i \phi_i}{z - s_i},$$

where $\phi_1 = I$. The group $\mathbb{G}_m^n \cong G \subset \mathrm{GL}(V)$ consists of the maps for which each e_i is an eigenvector. Above the closed subset $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ the differential operator has the form

$$L := \frac{d}{dz} + \sum_{i=1}^{s} \frac{A_i}{z - s_i},$$

with $A_1 = C_1$ and each A_i is a diagonal matrix w.r.t. the basis $e_1, \dots, e_n$ and having the same eigenvalues as C_i. The space $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ has a positive dimension if there is at least one C_i with $i > 1$ having an eigenvalue with multiplicity > 1. However the number of differential operators L is finite! Thus only a finite number of algebraic subgroups of $G \cong \mathbb{G}_m^n$ occur as differential Galois group $\mathrm{Gal}(m)$. One concludes that for every algebraic subgroup $H \subset G$, the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = H\}$ is constructible. $\diamond$

The above example is the general pattern for "families" with differential Galois groups contained in some commutative algebraic group G. Again, there are only finitely many distinct differential operators L possible above the moduli family. Hence there are only finitely many possibilities for the differential Galois groups. This implies that for every algebraic subgroup $H \subset G$ the set of the points with differential Galois group equal to H is constructible.

Example 3.8 *A moduli space with differential Galois groups in $D_\infty^{\mathrm{SL}_2}$.*
Let $V = Ce_1 + Ce_2$. By $D_\infty^{\mathrm{SL}_2}$ we will denote the subgroup of $\mathrm{SL}(V)$ consisting of the maps which permute the lines Ce_1, Ce_2. The data for the moduli problem are.

- Singular points $s_1, \ldots, s_4 \in C^*$, and ∞ is supposed to be regular.
- For each point s_i the differential operator $\frac{d}{d(z-s_i)} + \frac{1}{z-s_i} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$ with respect to the basis e_1, e_2.

The moduli space $\mathbb{M}$ for this problem can be made explicit. The universal differential equation has 4 regular singular points with local exponents $1/4$ and $-1/4$. This is essentially the Lamé equation. It has a closed subset $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset D_\infty^{\mathrm{SL}_2}\}$. Let $D_n^{\mathrm{SL}_2} \subset D_\infty^{\mathrm{SL}_2}$ denote the dihedral subgroup (of order $4n$). It turns out that $D_\infty^{\mathrm{SL}_2}$ and $D_n^{\mathrm{SL}_2}$ for $n \geq 1$ occur as differential Galois groups $\mathrm{Gal}(m)$ for closed points. The conclusion is that $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = D_\infty^{\mathrm{SL}_2}\}$ is not constructible! One way to explain this is to consider the case where C is the field of the complex numbers. Since the s_i are regular singular points, the differential Galois group is the algebraic closure of the monodromy group. This monodromy group is generated by four elements $A_1, \ldots, A_4 \in \mathrm{SL}_2(C)$ having product 1 and such that each $A_i^2 = -I$. Above the moduli space $\mathbb{M}$ essentially all groups with these generators and relations do occur. Therefore each $D_n^{\mathrm{SL}_2}$ occurs as differential Galois group. A detailed study of this moduli space will be given in Chapter 4. $\diamond$

3.3 Families of differential equations

We will now come to the definition of families of differential equations on $\mathbb{P}^1$, parametrized by a scheme X. We will first recall some facts on local differential modules.

3.3.1 Formal connections and semi-simple modules

The usual differentiation on the field of formal Laurent series $C((u))$ is given by the formula $\sum a_n u^n \mapsto \frac{d}{du}(\sum a_n u^n) := \sum a_n n u^{n-1}$. For notational convenience we will use (in this section) the differentiation $f \mapsto \delta(f) := u\frac{d}{du}f$. A differential module M over $C((u))$ is a finite dimensional vector space over $C((u))$ provided with an additive map $\delta = \delta_M : M \to M$ satisfying $\delta(fm) = f\delta(m) + \delta(f)m$. Put $\mathcal{Q} := \bigcup_{m\geq 1} u^{-1/m}C[u^{-1/m}]$. The Galois group of the algebraic closure of $C((u))$ acts on $\mathcal{Q}$. Take $q \in \mathcal{Q}$ and let $m \geq 1$

be minimal such that $q \in u^{-1/m}C[u^{-1/m}]$. The differential module $E(q)$ over $C((u^{1/m}))$ is defined by $E(q) = C((u^{1/m}))e$ and $\delta(e) = qe$. This module can also be viewed as a differential module of dimension m over $C((u))$. As such, it depends only on the Galois orbit oq of q in $\mathcal{Q}$. We write $E(oq)$ for $E(q)$ considered as a differential module over $C((u))$. We note that $E(oq)$ is an irreducible differential module. The classification of differential modules over $C((u))$ can be formulated as follows:

Every differential module M over $C((u))$ can be written uniquely as $\oplus_{i=1}^{s} E(oq_i) \otimes M_i$, where the $oq_1, \ldots, oq_r$ are distinct Galois orbits in $\mathcal{Q}$ and where the M_i are regular singular differential modules.

We recall that a differential module N is regular singular if there exists a basis $b_1, \ldots, b_r$ of N over $C((u))$, with the property that the free $C[[u]]$-module $\Lambda := C[[u]]b_1 + \cdots + C[[u]]b_r$ is invariant under δ. One associates to a regular singular N a semi-simple regular singular differential module N_{ss} by the following construction. (compare [Levelt, Jordan decomposition for a class of singular differential operators. Arkiv för matematik, 13 (1): 1-27, may 1975]). The operator δ leaves $u^m\Lambda$ invariant for each $m \geq 0$. Thus δ induces a C-linear endomorphism δ_m on $\Lambda/u^m\Lambda$. The additive Jordan decomposition of δ_m is written as $\delta_m = \delta_{m,ss} + \delta_{m,nilp}$. Here ss denotes the semi-simple part and $nilp$ denotes the nilpotent part. It is easily seen that the families of endomorphisms $\{\delta_{m,ss}\}$ and $\{\delta_{m,nilp}\}$ form projective systems. Now we write δ_{ss} and δ_{nilp} for the induced maps on Λ. One verifies that δ_{nilp} is $C[[u]]$-linear and that $\delta_{ss}(fm) = f\delta_{ss}(m) + \delta(f)m$ for $f \in C[[u]]$ and $m \in \Lambda$. Both operators are extended to N. The vector space N provided with δ_{ss} is denoted by N_{ss}. It is a differential module over $C((u))$ and it is semi-simple in the sense that every submodule of N_{ss} has a complement.

In terms of matrix differential equations this construction has an easy translation. One knows that N contains a basis such that the corresponding matrix differential equation has the form $u\frac{d}{du} + A$, where A is a constant matrix (i.e., has entries in C). Then N_{ss} corresponds (on the same basis) with the matrix differential equation $u\frac{d}{du} + A_{ss}$, where $A = A_{ss} + A_{nilp}$ is the usual Jordan decomposition of A. We note that the "classical" solution space for the matrix differential equation $u\frac{d}{du} + A$ contains logarithmic terms if $A_{nilp} \neq 0$.

Let M be a differential module over $C((u))$, with canonical decomposition $\oplus_{i=1}^{s} E(oq_i) \otimes M_i$. Then we define $M_{ss} := \oplus_{i=1}^{s} E(oq_i) \otimes M_{i,ss}$. Thus M_{ss} is equal to M as vector space over $C((u))$. One has $\delta_M = \delta_{M_{ss}} + \nu$ where ν is a nilpotent endomorphism of M commuting with $\delta_{M_{ss}}$ and δ_M. In particular, every submodule of M is also a submodule of M_{ss}. Moreover, the differential module M_{ss} is semi-simple.

A formal connection is a connection $\nabla : N \to C[[u]]u^{-k}du \otimes N$, where N is a free $C[[u]]$-module of finite rank. One associates to N the differential module $M = C((u)) \otimes N$ (with δ_M induced by $\nabla_{u\frac{d}{du}}$). The formal connection N_{ss} is now defined as the connection on N induced by the $\delta_{M_{ss}}$ on M_{ss}. We will call N_{ss} and M_{ss} the *semi-simplifications* of N and M. Suppose that $R \subset N$ is a $C[[u]]$-submodule such that N/R is free and $\nabla R \subset C[[u]]u^{-k}du \otimes R$. Then also $\nabla_{ss} R \subset C[[u]]u^{-k}du \otimes R$.

3.3.2 Defining families

The statement that we want to prove concerns the closed points of X and therefore we may suppose that X is reduced. For the same reason we may suppose (at every stage of the proof) that X is irreducible and affine. Assume that $X = \mathrm{Spec}(R)$ with R reduced and finitely generated over C. In order to avoid technical complications we will consider families for which the singular points (apparent or not) lie in a fixed subset $\{s_1, \ldots, s_r\}$ of $\mathbb{P}^1_C$. For convenience we suppose that $0, \infty \notin \{s_1, \ldots, s_r\}$.

A *first attempt* to define a family parametrized by $X = \mathrm{Spec}(R)$, is to consider a matrix differential equation $\frac{d}{dz} + A$, where A is an $R[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$-linear endomorphism of $R[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_C V$. More explicitely, A has the form $\sum_{j=1}^{s} \sum_{i=1}^{k} \frac{A(i,j)}{(z-s_j)^i}$ where each $A(i,j)$ is an R-linear endomorphism of $R \otimes V$. For every closed point x of X, i.e., $x \in X(C)$, one writes $A(x)$ for the $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$-linear endomorphism of $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes V$, obtained by applying $x : R \to C$ to A. In this way, $\frac{d}{dz} + A$ is a family of differential equations on the projective line over C. The equation $\frac{d}{dz} + A$ is regular at $z = 0$. One considers $R[[z]] \otimes_C V$ and the canonical map

$$\mathrm{Mod}_z : R[[z]] \otimes_C V \to R[[z]] \otimes_C V/(z) \stackrel{\cong}{\to} R \otimes_C V.$$

Lemma 3.9 *Consider the kernels:*

$$S = \ker(\frac{d}{dz} + A, R[[z]] \otimes_C V) \text{ and } S(x) = \ker(\frac{d}{dz} + A(x), C[[z]] \otimes_C V).$$

The maps $\mathrm{Mod}_z : S \to R \otimes_C V$ *and* $\mathrm{Mod}_z : S(x) \to V$ *are bijections. Moreover, the image of* S *under the map* $R[[z]] \otimes_C V \to C[[z]] \otimes_C V$*, induced by* $x : R \to C$*, is equal to* $S(x)$*.*

Proof. One considers an endomophism $F = F_0 + zF_1 + \cdots$ of $R[[z]] \otimes_C V$ (i.e., each F_i is an endomorphism of $R \otimes_C V$) with $F_0 = 1$. One requires that F is a "fundamental matrix", which means that $F' + AF = 0$. Put $A = A_0 + A_1 z + \cdots$. This leads to equations

$$(n+1)F_{n+1} + A_0 F_n + A_1 F_{n-1} + \cdots + A_n F_0 = 0 \text{ for all } n \geq 0.$$

Clearly F exists and is unique. This implies that $\mathrm{Mod}_z : S \to R \otimes_C V$ is a bijection. Let $F(x)$, for a closed point x, be obtained from F by the map $x : R \to C$; then $F(x)$ is a fundamental matrix for $\frac{d}{dz} + A(x)$. The other two statements of the lemma follow from this. $\square$

$\frac{d}{dz} + A(x)$ is viewed as a differential equation over the ring $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$. Let $PVR(x)$ denote the subring of $C[[z]]$ generated over $C[z, \frac{1}{(z-s_1)\cdots,(z-s_r)}]$ by all the entries of $F(x)$ and the inverse of the determinant of $F(x)$. Then $PVR(x)$ is a Picard-Vessiot ring for $\frac{d}{dz} + A(x)$. Let $\mathrm{Gal}(x)$ denote the group of the differential automorphisms of $PVR(x)$ over $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$. By construction $S(x) = \ker(\frac{d}{dz} + A(x), PVR(x) \otimes_C V)$ and $\mathrm{Gal}(x)$ acts faithfully on $S(x)$. Using the isomorphism $\mathrm{Mod}_z : S(x) \to V$, one finds a faithful action of $\mathrm{Gal}(x)$ on V. We conclude that the above constructions provide a canonical way to embed every $\mathrm{Gal}(x)$ into $\mathrm{GL}(V)$.

The next lemma will not be used in the proof of the main result. However, its contents and the ideas behind it are closely related to our main theme. In what follows we will prove a converse of this lemma.

Lemma 3.10 (Specialization of the differential Galois group) *We use the above notation. Suppose that* R *is a domain with field of fractions* K*. We can consider* $\frac{d}{dz} + A$ *as a differential equation over* $K[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$*. Let* $\overline{K}$ *denote an algebraic closure of* K*. Then the following holds:*

(a) *the differential Galois group $G_{\overline{K}}$ over the field of constants $\overline{K}$ descends to an algebraic subgroup G of* $\mathrm{GL}(K \otimes V)$,

(b) *the schematic closure G_R of G as algebraic subgroup of* $\mathrm{GL}(R \otimes V)$ *has the property: for every closed point x, with corresponding maximal ideal m_x, there is an inclusion* $\mathrm{Gal}(x) \subset (G_R \otimes R/m_x)$.

We note that this lemma and its proof are rather close to a result of O. Gabber (see [Kt90], Theorem 2.4.1 on page 39).

Proof.
(a) The solution space $\ker(\frac{d}{dz} + A, K[[z]] \otimes_C V)$ is equal to $K \otimes_R S$. Let PVR denote the subring of $K[[z]]$, generated over $K[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$ by the entries of F and the inverse of the determinant of F. Then $\overline{K} \otimes_K PVR$ is a Picard-Vessiot ring and we write $G_{\overline{K}}$ for its differential Galois group. The latter is characterized as the group of the $\overline{K}[z, \frac{1}{(z-s_1)\cdots(z-s_r)}]$-linear differential automorphisms of $\overline{K} \otimes PVR$. The group $G_{\overline{K}}$ acts faithfully on $\overline{K} \otimes_C V$. Choose a basis of V over C. The affine ring of $\mathrm{GL}(\overline{K} \otimes_C V)$ can be written as $\overline{K}[\{X_{i,j}\}_{i,j=1}^n, \frac{1}{det}]$, where det denotes the determinant of the matrix $(X_{i,j})$. The ideal J defining $G_{\overline{K}}$ is the kernel of the $\overline{K}$-homomorphism $\phi : \overline{K}[\{X_{i,j}\}_{i,j=1}^n, \frac{1}{det}] \to \overline{K} \otimes PVR$, given by $\phi(X_{i,j})$ is equal to $F_{i,j}$ (the (i,j)-entry of the matrix F). Since ϕ "descends" to K, the ideal J descends to an ideal I of $K[\{X_{i,j}\}_{i,j=1}^n, \frac{1}{det}]$. The latter defines an algebraic subgroup G of $\mathrm{GL}(K \otimes_C V)$ satisfying $G \otimes_K \overline{K} = G_{\overline{K}}$.

(b) The schematic closure G_R of G is the group scheme over R given by the ideal $I_R := I \cap R[\{X_{i,j}\}_{i,j=1}^n, \frac{1}{det}]$. The inclusion $\mathrm{Gal}(x) \subset G_R \otimes R/m_x$ follows from a combination of Chevalley's theorem and some properties of matrix differential operators (or connections). The expression $\frac{d}{dz} + A$ is seen as a regular differential operator on $\mathrm{Spec}(R) \times (\mathbb{P}^1_C \setminus \{s_1, \dots, s_r\})$. Let V_b^a denote the tensor product $V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$ (of a copies of the dual V^* of V and b copies of V). There is a K-subspace W of some finite direct sum $K \otimes_C \oplus_i V_{b_i}^{a_i}$ such that G is the stabilizer of W. The differential operator $\frac{d}{dz} + A$ on $R[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_C V$ induces a differential operator $\frac{d}{dz} + B$ on $R[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_C \oplus_i V_{b_i}^{a_i}$. By differential Galois theory, $K[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_K W$ is invariant under $\frac{d}{dz} + B$. Put $\tilde{W} := W \cap (R \otimes_C \oplus_i V_{b_i}^{a_i})$. Then $\tilde{W}$ is invariant under G_R and moreover

$R[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_R \tilde{W}$ is invariant under $\frac{d}{dz} + B$. The regularity of this differential operator implies that $\tilde{W}$ is a projective R-module (see loc.cit. for more details). Let $x \in X(C)$. The group $R/m_x \otimes G_R$ is defined by the invariance of the subspace $R/m_x \otimes_R \tilde{W}$ of $R/m_x \otimes_C \oplus_i V_{b_i}^{a_i} = \oplus_i V_{b_i}^{a_i}$. Furthermore, the space $C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \otimes_C (R/m_x \otimes_R \tilde{W})$ is invariant under $\frac{d}{dz} + B(x)$. By differential Galois theory, the group $\mathrm{Gal}(x)$ leaves $R/m_x \otimes_R \tilde{W}$ invariant. Hence $\mathrm{Gal}(x) \subset (R/m_x \otimes G_R)$. □

In our present setup the constructibility result that we want to prove is not valid. This is illustrated by the rather obvious example: $R = C[t]$ and the differential operator $\frac{d}{dz} + \frac{t}{z-s_1}$. If the value of t is rational number of the form $\frac{p}{q}$ with $q \geq 1$ and $(p,q) = 1$, then the differential Galois group is a cyclic group of order q. For other values of t in C, the differential Galois group is the multiplicative group $\mathbb{G}_m$. However, the group $\mathbb{G}_m$ satisfies the "Singer condition".

In order to avoid this and other examples of this sort we will suppose that there are only finitely many possibilities for the formal local structure of $\frac{d}{dz} + A(x)$ at any of the singular points $s_1, \ldots, s_r$. Again this is not sufficient for our goal, namely the statement that the set of closed points x with $\mathrm{Gal}(x) = G$ is constructible. The new problem is that the formal isomorphism between $\frac{d}{dz} + A(x)$ at s_j and one of the prescribed formal connections can have a pole at s_j of arbitrarily high order. A remedy for this is the introduction of connections on the projective line over C. In order to work out this idea the following (probably known) result on vector bundles on $\mathbb{P}^1_X := X \times \mathbb{P}^1_C$ is needed. We introduce some notation. Let $pr_1 : X \times \mathbb{P}^1_C \to X$ and $pr_2 : X \times \mathbb{P}^1_C \to \mathbb{P}^1_C$ denote the two projections. For vector bundles $\mathcal{A}$ and $\mathcal{B}$ on X and $\mathbb{P}^1_C$ we write $\mathcal{A} \otimes \mathcal{B}$ for the vector bundle $pr_1^*\mathcal{A} \otimes pr_2^*\mathcal{B}$. The line bundle of degree d on $\mathbb{P}^1_C$ is denoted by $\mathcal{O}(d)$. For $\mathcal{O}_X \otimes \mathcal{O}(d) = pr_2^*\mathcal{O}(d)$ we also write $\mathcal{O}_X(d)$. We recall that any vector bundle of rank n on $\mathbb{P}^1_C$ has the form $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$ with unique $a_1 \geq a_2 \geq \cdots \geq a_n$. We call the sequence $a_1 \geq \cdots \geq a_n$ the *type of the vector bundle.*

Proposition 3.11 *Let X be a scheme of finite type over C and let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$ of rank n. Let $x \in X$ be a closed point. Suppose that the induced vector bundle $\mathcal{M}(x)$ on $\mathbb{P}_C$ is free. Then there exists an open neighbourhood U of x such that the restriction of $\mathcal{M}$ to $\mathbb{P}^1_U$ is free.*

Proof. We remark that $\mathcal{M}(x)$ denotes the vector bundle on $\mathbb{P}^1_C$ obtained by evaluating $\mathcal{M}$ at x. More precisely, write $j_x : \mathrm{Spec}(C) \to X$ for the morphism corresponding to x and write $g_x = j_x \times id : \mathbb{P}^1_C = \mathrm{Spec}(C) \times \mathbb{P}^1_C \to X \times \mathbb{P}^1_C$. Then $\mathcal{M}(x)$ is defined as $g_x^*\mathcal{M}$.

One may suppose that X is affine. Let D_0 and D_∞ denote the divisors $X \times \{0\}$ and $X \times \{\infty\}$. Define the sheaf $\mathcal{N} = \mathcal{O}(-D_\infty) \otimes \mathcal{M}$ and consider the covering of $\mathbb{P}^1_X$ by the affine sets $U_0 = \mathbb{P}^1_X - D_\infty$ and $U_\infty = \mathbb{P}^1_X - D_0$. Put $U_{0,\infty} = U_0 \cap U_\infty$. The following sequence

$$0 \to H^0(\mathcal{N}) \to \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \to H^1(\mathcal{N}) \to 0$$

is exact. The two $O(X)$-modules $H^0(\mathcal{N})$ and $H^1(\mathcal{N})$ are finitely generated. Indeed, since the natural projection $pr : \mathbb{P}^1_X \to X$ is proper one has that $pr_*\mathcal{N}$ and $R^1pr_*\mathcal{N}$ are coherent. Moreover, $H^0(\mathcal{N}) = H^0(X, pr_*\mathcal{N})$ and $H^1(\mathcal{N}) = H^0(X, R^1pr_*\mathcal{N})$ (by Leray's spectral sequence). Let m_x denote the maximal ideal of $O(X)$ corresponding to the closed point x. The assumption that $\mathcal{M}(x)$ is free implies that $H^0(\mathcal{N}(x)) = H^1(\mathcal{N}(x)) = 0$. This implies that the map $\alpha \otimes_{O(X)} O(X)/m_x$ is a bijection. Hence x does not lie in the support of the $O(X)$-module $H^1(\mathcal{N})$. After shrinking X, we may assume that $H^1(\mathcal{N}) = 0$ and that α is surjective. The $O(U_{0,\infty})$-module $\mathcal{N}(U_{0,\infty})$ is projective. Therefore $\mathcal{N}(U_{0,\infty})$ is also a projective module over the ring $O(X)$. Hence the exact sequence of $O(X)$-modules

$$0 \to H^0(\mathcal{N}) \to \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \to 0$$

splits. The bijectivity of the map $\alpha \otimes_{O(X)} O(X)/m_x$ implies that the module $H^0(\mathcal{N}) \otimes_{O(X)} O(X)/m_x = 0$. After shrinking X, we may suppose that $H^0(\mathcal{N}) = 0$. Define the sheaf $\mathcal{Q}$ by the exactness of

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0.$$

Then $\mathcal{Q} = \mathcal{M}/(\mathcal{O}(-D_\infty) \otimes \mathcal{M})$ and therefore $\mathcal{Q}$ is a vector bundle on $X \cong X \times \{\infty\}$. The rank n of $\mathcal{Q}$ is the same as the rank of $\mathcal{M}$. After shrinking X, we may suppose that $\mathcal{Q}$ is a free vector bundle on X. The above exact sequence of sheaves yields: $H^0(\mathcal{M}) = H^0(X, \mathcal{Q}) = O(X)^n$ and $H^1(\mathcal{M}) = 0$. It suffices now to show that $\mathcal{M}$ is generated at every closed point w of $\mathbb{P}^1_X$ by its group of global sections $H^0(\mathcal{M})$. This property is equivalent to the surjectivity of the map $H^0(\mathcal{M}) \to \mathcal{M}_w/m_w\mathcal{M}_w$, where m_w

denotes the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^1_X,w}$. The point w lies on a divisor $D = X \times \{p\}$ for some closed point p of $\mathbb{P}^1_C$. Put $w = (q,p)$. Define the sheaf S by the exact sequence

$$0 \to \mathcal{O}(-D) \otimes \mathcal{M} \to \mathcal{M} \to S \to 0.$$

We note that $\mathcal{O}(-D)$ is isomorphic to $\mathcal{O}(-D_\infty)$. As before one concludes that S is a vector bundle on $X \cong X \times \{p\}$ and that $H^0(\mathcal{M}) \to H^0(X \times \{p\}, S)$ is surjective. Since $X \times \{p\}$ is affine, the map $H^0(X \times \{p\}, S) \to S_q/m_q S$ (where m_q denotes the maximal ideal corresponding to the point $w = (q,p)$) is surjective. Finally, $\mathcal{M}_w/m_w\mathcal{M}_w \to S_q/m_q S_q$ is an isomorphism. □

Remarks 3.12 *More on vector bundles on* $\mathbb{P}^1_X$.
(1) We start with an example showing that the type of a vector bundle on $\mathbb{P}^1_X$ is not locally constant, i.e., the type of $\mathcal{M}(x)$ is not locally constant in X. Take $X = \mathrm{Spec}(C[t])$ and consider a vector bundle $\mathcal{M}$ of rank 2 on $\mathbb{P}^1_X$. Let z denote the usual global parameter on $\mathbb{P}^1_C$. Write again $D_0 = X \times \{0\}$, $D_\infty = X \times \{\infty\}$, $U_0 = \mathbb{P}^1_X - D_\infty$ and $U_\infty = \mathbb{P}^1_X - D_0$. The restriction of $\mathcal{M}$ to the two affine sets U_0, U_∞ is free (since every projective module over a polynomial ring is free). Hence $\mathcal{M}$ is given by a matrix $A \in \mathrm{GL}(2, C[t][z, z^{-1}])$. This matrix defines a unique double coset $\mathrm{GL}(2, C[t][z]) \cdot A \cdot \mathrm{GL}(2, C[t][z^{-1}])$. On the other hand each double coset, as above, defines a vector bundle of rank 2 on $\mathbb{P}^1_X$. We consider now the vector bundle associated to

$$A = \begin{pmatrix} z & 0 \\ t & z^{-1} \end{pmatrix}.$$

For $t = 0$, this defines the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1_C$. For $t \neq 0$, this defines the free vector bundle on $\mathbb{P}^1_C$. Indeed,

$$A = \begin{pmatrix} 1 & t^{-1}z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -t^{-1} \\ t & z^{-1} \end{pmatrix}.$$

(2) Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$ of rank n. Then the set of closed points $x \in X(C)$, such that $\mathcal{M}(x)$ has type $a_1 \geq a_2 \geq \cdots \geq a_n$, is a constructible subset. We sketch the proof of this result. It suffices to consider the case where X is affine and connected. For a point $x \in X(C)$, the type $a_1 \geq \cdots \geq a_n$ of the vector bundle $\mathcal{M}(x)$ is determined by the dimensions $h^i(k, x)$, $i = 0, 1$ of the cohomology groups $H^i(\mathbb{P}^1_C, \mathcal{M}(x) \otimes \mathcal{O}(k))$, for all

$k \in \mathbf{Z}$. The degree D of $\mathcal{M}(x)$ is independent of $x \in X(C)$. By Riemann-Roch, $h^0(k,x) - h^1(k,x) = D + n$ for all k. There exists an integer N, depending on $\mathcal{M}$, such that for $k \geq N$ one has $h^1(k,x) = 0$ and for $k \leq -N$ one has $h^0(k,x) = 0$. Hence the type of $\mathcal{M}(x)$ is determined by the values of $h^1(k,x)$ for $-N < k < N$. Therefore we have to investigate the dependence of $h^1(k,x)$ on x. For convenience we consider $h^1(0,x)$. The proof of Proposition 3.11 asserts that $H^1(\mathbb{P}^1_C, \mathcal{M}(x)) = O(X)/m_x \otimes H^1(\mathcal{M})$, where m_x denotes the maximal ideal corresponding to x. This implies that for any integer $d \geq 0$ the set $\{x \in X(C) |\ h^1(0,x) \leq d\}$ is open. From this observation the above statement follows.

(3) The *defect* of a vector bundle on $\mathbb{P}^1_C$ of type $a_1 \geq \cdots \geq a_n$ is defined as $a_1 - a_n$. The reasoning in (2) above implies that for any integer $d \geq 0$ the set $\{x \in X(C) |$ the defect of $\mathcal{M}(x)$ is $\leq d\}$ is open. This generalizes the statement of Proposition 3.11.

(4) *Suppose that X is a reduced, irreducible scheme of finite type over C. Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$. Suppose that there exists a closed point $x_0 \in X$ such that $\mathcal{M}(x_0)$ is free. Then the set of closed points x such that $\mathcal{M}(x)$ is not free is either empty or equal to a divisor on X.*

Sketch of the proof. We may suppose that $X = \mathrm{Spec}(R)$ with R a finitely generated C-algebra having no zero-divisors. We will use the notation of the proof of Proposition 3.11. The statement that we want to prove is equivalent to: the R-module $H^1(\mathcal{N})$ is either 0, or its support is a divisor on X. Consider again the exact sequence

$$0 \to H^0(\mathcal{N}) \to \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \to H^1(\mathcal{N}) \to 0$$

The assumption that $\mathcal{M}(x_0)$ is free implies that α becomes an isomorphism after localizing R at a suitable non-zero element. Thus $H^0(\mathcal{N}) = 0$ (since R has no zero-divisors) and $H^1(\mathcal{N})$ is a finitely generated torsion module over R. The modules $\mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty)$ and $\mathcal{N}(U_{0,\infty})$ are projective R-modules of infinite rank. The above exact sequence is therefore a resolution of $H^1(\mathcal{N})$ by projective modules of infinite rank. Consider an exact sequence

$$0 \to V_1 \xrightarrow{f} V_0 \to H^1(\mathcal{N}) \to 0$$

with V_0 a finitely generated free R-module. Then V_1 is a projective R-module (of finite rank). After replacing $\mathrm{Spec}(R)$ by the elements of an open affine covering, we may suppose that V_1 is a free R-module, too. Furthermore, V_1 and V_0 have the same rank. The support of $H^1(\mathcal{N})$ is equal to the closed subset defined by $\mathrm{Det}(f) = 0$. This finishes the proof.

The above result is also valid in the complex analytic case. A proof is given by B. Malgrange in [M83], Section 4.

(5) In trying to classify the vector bundles on $X \times \mathbb{P}^1_C$, one encounters the question whether a vector bundle $\mathcal{M}$ of rank n on $X \times \mathbb{P}^1_C$ has, at least locally with respect to X, the property that the restrictions of $\mathcal{M}$ to the affine sets $\mathrm{Spec}(R) \times (\mathbb{P}^1_C - \{\infty\})$ and $\mathrm{Spec}(R) \times (\mathbb{P}^1_C - \{0\})$ are free. If the answer is positive, then $\mathcal{M}$ is (locally with respect to X) defined by a double coset $\mathrm{GL}(n, R[z]) \cdot A \cdot \mathrm{GL}(n, R[z^{-1}])$ with $A \in \mathrm{GL}(n, R[z, z^{-1}])$. This seems a useful way to present $\mathcal{M}$. The above question is directly related to the following question posed by H. Bass and D. Quillen:

Let R be a regular noetherian ring. Does every finitely generated projective module over $R[z]$ come from a finitely generated projective module over R?

There are partial answers to this question (see [L78]). It seems that the general problem remains unsolved. •

Definition 3.13
A family of differential equations on $\mathbb{P}^1$, parametrized by X
Distinct points $\{s_1, \ldots, s_r\} \subset \mathbb{P}^1_C \setminus \{0, \infty\}$ are given. Moreover, a finite set I of semi-simple formal connections $\nabla_i : C[[u]]^n \to C[[u]]u^{-k}du \otimes C[[u]]^n$ (with $i \in I$) is given. This collection will be called the *local data*. The next items are $X, \mathcal{M}, \nabla, V$ where:

(i) X is a reduced scheme of finite type over C.

(ii) $\mathcal{M}$ is a vector bundle on $\mathbb{P}^1_X$ of the form $\mathcal{O}_X \otimes \mathcal{N}$, where $\mathcal{N}$ is a vector bundle on $\mathbb{P}^1_C$.

(iii) V is a vector space of dimension n over C, and there is given an isomorphism $\mathcal{N}_0/(z) \xrightarrow{\sim} V$.

(iv) A connection $\nabla : \mathcal{M} \to \Omega(k[s_1] + \cdots + k[s_r]) \otimes \mathcal{M}$.

For every point $x \in X(C)$, we find a vector bundle $\mathcal{M}(x)$ on $\mathbb{P}^1_C$, where $\mathcal{M}(x) = j_x^*(\mathcal{M})$, $j_x : \{x\} \times \mathbb{P}^1_C \to \mathbb{P}^1_X$. The above data induce a connection $\nabla(x) : \mathcal{M}(x) \to \Omega(k[s_1] + \cdots + k[s_r]) \otimes \mathcal{M}(x)$. For every point $x \in X(C)$ and every j, we write u for the local parameter $z - s_j$. We require that the semi-simplification of the connection $\widehat{\nabla(x)}_j : \widehat{\mathcal{M}(x)}_{s_j} \to C[[u]]u^{-k}du \otimes \widehat{\mathcal{M}(x)}_{s_j}$ is isomorphic to ∇_i for some $i \in I$. More precisely, there exists a $C[[u]]$-linear isomorphism $\widehat{\mathcal{M}(x)}_{s_j,ss} \to C[[u]]^n$ that is compatible with the connections.
•

Remarks 3.14
(1) A more precise formulation of part (iv) is:

$$\nabla : \mathcal{M} \to \Omega_{\mathbb{P}^1_X/X}(\sum k[X \times \{s_i\}]) \otimes \mathcal{M},$$

where the $[X \times \{s_i\}]$ are divisors on $\mathbb{P}^1_X$. Moreover, the integer k occurring here can be replaced by any integer $\ell \geq k$ without changing the family.

(2) A moduli space $\mathbb{M}$, as defined in Chapter 2, is a special case of a family. Such a moduli problem yields a universal family, parametrized by $\mathbb{M}$. The corresponding family is given by $X = \mathbb{M}$, $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1_X} \otimes V$, and ∇, such that the universal family is $(\mathcal{M}, \nabla, \{\phi_i\})$.

(3) Let a family, parametrized by, say, $X = \mathrm{Spec}(R)$ be given. For every $x \in X$, we have a full solution space $W(x)$ of $\nabla(x)$ in $\widehat{\mathcal{M}(x)}_0$. We want to make an identification of $W(x)$ with V. By (iii) of the definition, we have an isomorphism $\mathcal{N}_0/(z) \xrightarrow{\sim} V$. This isomorphism can be lifted to an isomorphism $\mathcal{N}_0 \to C[z]_{(z)} \otimes V$. The latter is unique up to a $C[z]_{(z)}$-linear automorphism of $C[z]_{(z)} \otimes V$ that is the identity modulo the ideal (z). The isomorphism $\mathcal{N}_0 \to C[z]_{(z)} \otimes V$ can be extended to an isomorphism $\widehat{\mathcal{N}_0} \to C[[z]] \otimes V$, which is unique up to an element $h \in \mathrm{GL}(C[z]_{(z)} \otimes V)$ that is the identity modulo (z). We have that $\mathcal{M}(x)$ is canonically isomorphic to $\mathcal{N}$, so the above map gives a canonical way to identify $\ker(\nabla, \widehat{\mathcal{M}(x)}_0)$ with V (via mod z). So the differential Galois group $\mathrm{Gal}(x)$ is canonically embedded into $\mathrm{GL}(V)$.

(4) Let a family, parametrized by, say, $X = \mathrm{Spec}(R)$, be given. We will make some changes to this family. The isomorphism $V \to \mathcal{N}_0/(z)$ can be

lifted to a map $V \to \mathcal{N}_0$. Now V can be considered as a subspace of the generic fiber $\mathcal{N}_\xi$, using the canonical map $\mathcal{N}_0 \to \mathcal{N}_\xi$. Now $\mathcal{N}$ is replaced by $\mathcal{N}_1 := \mathcal{N}(\ell[b_1] + \cdots + \ell[b_r])$ for suitable $\ell > 0$ and points $b_1, \cdots, b_r \neq 0$, such that $V \subset H^0(\mathcal{N}_1)$. Then we consider the free vector bundle $\mathcal{F} := \mathcal{O}_{\mathbb{P}^1_C} \otimes V$, subbundle of $\mathcal{N}_1$, and the free vector bundle $\mathcal{F}_X = \mathcal{O}_X \otimes \mathcal{F}$.

In general, $\nabla(\mathcal{F}_X) \subset \Omega(\sum_{i=1}^s k[s_i]) \otimes \mathcal{F}_X$ does no hold. At the cost of introducing some points $\{s_{s+1}, \ldots, s_t\}$ as new (apparent) singularities and adding finitely many new items to the local data, one obtains a new family, parametrized by X, with

$$\nabla : \mathcal{F}_X \to \Omega(\sum_{i=1}^{t} k[s_i]) \otimes \mathcal{F}_X \text{ (for a suitable, large enough } k > 0).$$

One of the new singular points s_j might be the point ∞. An automorphism of $\mathbb{P}^1_C$, which fixes 0, takes care of that. The original family is closely related to this new family. In particular, $\mathrm{Gal}(x) \subset \mathrm{GL}(V)$ remains unchanged for every $x \in X$. So for the constructibility result that we want to prove, we may replace the original family by the new one. In what follows we may therefore (at any stage of the proof) assume that the vector bundle $\mathcal{M}$ on $\mathbb{P}^1_X$ is equal to $\mathcal{O}_X \otimes \mathcal{N}$ with $\mathcal{N}$ a free vector bundle on $\mathbb{P}^1_C$. Moreover V is identified with $H^0(\mathcal{N})$. In other terms $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$.

(5) For an algebraic subgroup H of $\mathrm{GL}(V)$ we write $X(\subset H)$ (resp. $X(H \subset)$) for the set of closed points $x \in X$ such that $\mathrm{Gal}(x) \subset H$ (resp. $H \subset \mathrm{Gal}(x)$). For two algebraic subgroups $H_1 \subset H_2$ we will write $X(H_1 \subset, \subset H_2)$ for $X(H_1 \subset) \cap X(\subset H_2)$. Furthermore, $X(= H) := X(H \subset, \subset H)$. The main result of this chapter is the following. •

Theorem 3.15 *Suppose that the linear algebraic subgroup* $G \subset \mathrm{GL}(V)$ *satisfies the "Singer condition". Let a family of differential equations on* $\mathbb{P}^1$, *parametrized by* X *be given. Then* $X(= G)$ *is a constructible subset of* X.

In the proof we follow some of the steps of the proof given in [S93]. However, we like to point out some important differences. In our setup, the differential Galois group $\mathrm{Gal}(x)$ is given as a subgroup of $\mathrm{GL}(V)$, whereas in [S93] this group is only determined up to conjugacy in $\mathrm{GL}(V)$. The bounds B and real algebraic subspaces $\mathcal{L}(n, m, B)$ of $\mathcal{L}(n, m)$ are not present in our proof.

The prescribed local connections and the type of the vector bundle $\mathcal{M}$ provide the necessary bounds on the degrees of ∇-invariant line bundles. The "constructions of linear algebra", needed in the proof, are rather involved for differential operators (especially when one has to produce another "cyclic vector"). Here the constructions are the natural ones known for differential modules. Our proof can be adapted to the case where the singular points are not fixed. However we prefer to avoid the technical complications introduced by "moving singularities". Finally, Singer's result applies to certain sets of differential equations. It seems possible to make a translation between those sets of differential equations, and our families of differential equations on $\mathbb{P}^1$, but now with moving singularities.

3.4 Proof of Singer's theorem for families

Throughout this section we will mainly consider families of differential equations $(\mathcal{M}, \nabla, V, \{\nabla_i\}_{i\in I})$, with $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$. We will write such a family as $(\nabla, V, \{\nabla_i\})$.

3.4.1 The set $X(\subset G)$ is closed

We denote by V_b^a the tensor product of a copies of the dual V^* of V and of b copies of V. One considers a subspace W of dimension d of a finite sum $\oplus_i V_{b_i}^{a_i}$. Then $G := \{g \in \mathrm{GL}(V)|\ gW = W\}$ is an algebraic subgroup of $\mathrm{GL}(V)$. According to Chevalley's theorem, every algebraic subgroup of $\mathrm{GL}(V)$ has this form. Put $Z := \bigwedge^d(\oplus_i V_{b_i}^{a_i})$ and $L := \bigwedge^d W$. Then G is equal to $\{g \in \mathrm{GL}(V)|\ gL = L\}$, too. The subgroups of $\mathrm{GL}(V)$, conjugated to G, are the stabilizers of the lines $hL \subset Z$ with $h \in \mathrm{GL}(V)$. This family of lines in Z is a constructible subset of $\mathbb{P}(Z)$. Write $L = Cz_0$. Then the set $\{hz_0|\ h \in \mathrm{GL}(V)\}$ is also constructible. Indeed, the action of $\mathrm{GL}(V)$ on Z and $\mathbb{P}(Z)$ is algebraic.

Proposition 3.16 *Let be given a family of differential equations on the projective line, parametrized by X. Then $X(\subset G)$ is closed.*

Proof. We have to extend the proofs of Chapter 2 to the present more general situation. We suppose that X is reduced, irreducible and affine. Let

G be given as above as the stabilizer of a (special) line L in a construction Z. Each step in the construction of Z can be supplemented by a new family of differential equations parametrized by the same X. Indeed, for the dual V^* one constructs from the given family, a new family obtained by taking everywhere duals. This works well since the free vector bundle $\mathcal{O}_{\mathbb{P}^1_C} \otimes V$ has an obvious dual $\mathcal{O}_{\mathbb{P}^1_C} \otimes V^*$. For a tensor product, like V_b^a, one can form the tensor product of the corresponding vector bundles (including their connections and the local data). Direct sums and exterior powers are treated in the obvious way. Thus we find a family, parametrized by X and corresponding to Z, consisting of a free vector bundle $\mathcal{N}$, identified with $\mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes Z)$, a connection ∇ on $\mathcal{N}$ and a new finite set of prescribed formal connections over $C[[u]]$. Then, according to Lemma 2.16, the set $X(\subset G)$ consists of the closed points x such that there is a line bundle $\mathcal{L}$, contained in $\mathcal{N}(x)$ and satisfying:

(i) $\mathcal{L}$ is invariant under $\nabla(x)$,

(ii) $\mathcal{N}(x)/\mathcal{L}$ is again a vector bundle,

(iii) $\mathcal{L}_0/z\mathcal{L}_0$ is equal to L.

We follow closely the proof of Theorem 2.17. Write $L = Cv_0$ and let $-d \leq 0$ denote the degree of a putative $\mathcal{L}$. Then one finds an equation for the generator $v_0 + v_1 z + \cdots + v_d z^d$ of $\mathcal{L}(d \cdot [\infty])$ (see the proof of Theorem 2.17). This equation has the form

$$(\frac{d}{dz} + \sum_{i=1}^{s} \sum_{j=1}^{k} \frac{B_{i,j}(x)}{(z-s_i)^j} - T)(\sum_{i=0}^{d} v_i z^i) = 0,$$

where the $B_{i,j}$ are endomorphisms of $O(X) \otimes Z$; $B_{i,j}(x)$ is the evaluation of $B_{i,j}$ at x, and $T := \sum \frac{g_{i,j}}{(z-s_i)^j}$ with $g_{i,j} \in C$. We note that T does not depend on $x \in X$. There are finitely many possibilities for T, according to Lemma 2.18 and Definition 3.13. Each possibility yields at most one value for d. Now we consider a fixed choice for the term T. The equation

$$(\frac{d}{dz} + \sum_{i=1}^{s} \sum_{j=1}^{k} \frac{B_{i,j}}{(z-s_i)^j} - T)(\sum_{i \geq 0} v_i z^i) = 0,$$

with the prescribed $v_0 \in Z$ and $v_i \in O(X) \otimes Z$ for $i \geq 1$ has a unique solution (which is denoted by the same symbols). One can see v_i, for $i \geq 1$, as a morphism from X to Z. This determines a closed subset, say $X(T)$ of X, defined by $v_i(x) = 0$ for $i > d$. In other words, $X(T)$ is the intersection $\cap_{i>d} v_i^{-1}(0)$. Finally, $X(\subset G)$ is the union of the finitely many closed sets $X(T)$. □

Corollary 3.17 *Let a family* $(\nabla, V, \{\nabla_i\}_{i \in I})$ *of differential equations, parametrized by* X*, be given.*

(1) *Consider a vector space* Z *of the form* $\bigwedge^d(\oplus_i V_{b_i}^{a_i})$ *and a constructible subset* S *of* $Z \setminus \{0\}$*. The set of the closed points* $x \in X(C)$ *such that* $\mathrm{Gal}(x) \subset \mathrm{GL}(V)$ *fixes a line* $Cs \subset Z$ *with* $s \in S$ *(for the induced action of* $\mathrm{Gal}(x)$ *on* Z*), is constructible.*

(2) *Let* G *be an algebraic subgroup of* $\mathrm{GL}(V)$*. The set of the closed points* $x \in X(C)$*, such that* $\mathrm{Gal}(x)$ *lies in a conjugate of* G*, is constructible.*

Proof.
(1) As in the proof of Proposition 3.16, one supposes that $\mathcal{M}$ is equal to $\mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$. There is an induced family $(\nabla, Z,$ local data $)$. As in that proof, a fixed choice for the term T is made. The element v_0 is not fixed but lies in a given constructible subset S of $Z \setminus \{0\}$. The elements v_i with $i \geq 1$ are now viewed as morphisms $S \times X \to Z$. The set $\cap_{i>d} v_i^{-1}(0)$ is a closed subset of $S \times X$. Its image $X(T, S)$, under the projection $S \times X \to X$, is constructible. The union of the finitely many $X(T, S)$ is the set of the closed points x such that $\mathrm{Gal}(x) \subset \mathrm{GL}(V)$ fixes, for its action on Z, a line L of the form $L = Cs$ with $s \in S$.

(2) Take Z as in (1) and a line $L \subset Z$ such that $G = \{g \in \mathrm{GL}(V) |\ gL = L\}$. Write $L = Cv_0$. Then (1), applied to the constructible set $S = \{hv_0 |\ h \in \mathrm{GL}(V)\}$, yields (2). □

3.4.2 Galois invariant subspaces and subbundles

Let a family of differential equations $(\nabla, V, \{\nabla_i\})$, parametrized by a reduced, irreducible, affine X be given. Let W be a subspace of V such that W is invariant under all $\mathrm{Gal}(x)$. Our aim is to prove that there is a subbundle of $\mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$, invariant under ∇, corresponding to W. We start by discussing the special case where $W = Ce$ (with $e \neq 0$). We can give $\nabla_{\frac{d}{dz}}$ in the explicit form

$$\frac{d}{dz} + A = \frac{d}{dz} + \sum_{i,j} \frac{A_{i,j}}{(z - s_j)^i},$$

where the $A_{i,j}$ are $O(X)$-linear endomorphisms of $O(X) \otimes V$ and where $\sum_j A_{1,j} = 0$. We return to the proof of Theorem 2.17 and its terminology. For a fixed $x \in X(C)$, there is a term $T = \sum_{i,j} \frac{g_{i,j}}{(z-s_j)^i}$ with all $g_{i,j} \in C$ such that $\sum_j g_{1,j}$ is an integer $d \geq 0$ and there is a solution $v_0 + v_1 z + \cdots + v_d z^d$ of $(\frac{d}{dz} + A(x))(v_0 + v_1 z + \cdots + v_d z^d) = T(v_0 + v_1 z + \cdots + v_d z^d)$, such that $v_0 = e$ and $v_d \neq 0$. Moreover, there are only finitely many possibilities for T. Now we fix T and consider the equation $(\frac{d}{dz} + A)(\sum_{i \geq 0} v_i z^i) = T(\sum_{i \geq 0} v_i z^i)$ with $v_0 = e$ and $v_i \in O(X) \otimes V$ for $i \geq 1$. This equation has a unique solution. The closed subset of X given by $v_i(x) = 0$ for $i > d$, is denoted by $X(T)$. By assumption X is the union of the finitely many sets $X(T)$. Since X is irreducible, X is equal to a single $X(T)$. We continue with this T.

Let $v_0 + v_1 z + \cdots + v_d z^d$ denote the solution corresponding to this T (with again $v_0 = e$ and $v_i \in O(X) \otimes V$ for $i \geq 1$). It is, a priori, possible that v_d is identically zero. Let ℓ be maximal such that v_ℓ is not identical zero. It is also possible that $v_0 + v_1 z + \cdots + v_\ell z^\ell$ is divisible by some $(z - s_j)$. We divide $v_0 + v_1 z + \cdots + v_\ell z^\ell$ by $(z - s_1)^{m_1} \cdots (z - s_r)^{m_r}$ with $m_1, \ldots, m_r \geq 0$ as large as possible (this changes the T as well). The result is a section, say $v_0 + w_1 z + \cdots + w_q z^q$, of $\mathcal{M}(q \cdot [\infty])$ such that none of the expressions w_q and $v_0 + w_1 s_j + \cdots + w_q s_j^q$ for $j = 1, \ldots, r$, is identical zero. Let X' be the open, non-empty, subset of X given by $w_q(x) \neq 0$ and the $v_0 + w_1(x) s_j + \cdots + w_r(x) s_j^q \neq 0$ for $j = 1, \ldots, r$. We claim that the section $v_0 + w_1 z + \cdots + w_q z^q$ of $\mathcal{M}(q \cdot [\infty])$ does not vanish on $X' \times \mathbb{P}^1_C$. For points (x, ∞) or (x, s_j) with x a closed point of X', this is obvious. For a point (x, s) with $s \notin \{s_1, \ldots, s_r, \infty\}$ and $x \in X'(C)$, the expression $v_0 + w_1(x) z + \cdots + w_q(x) z^q$ is a solution of the differential operator $\frac{d}{dz} + A(x) - T$. Since this operator is regular at s, the vanishing of $v_0 + w_1(x) s + \cdots + w_q(x) s^q$ implies that $v_0 + w_1(x) z + \cdots + w_q(x) z^q$ is identical zero. This contradicts $w_q(x) \neq 0$.

In what follows, X is already replaced by the non-empty open subset X'.

In the next steps, we will shrink X even further. Let $\mathcal{F} = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1_C}$. The line bundle $\mathcal{F}$ is embedded into $\mathcal{M}(q \cdot [\infty])$ by sending the global section 1 of $\mathcal{O}_{\mathbb{P}^1_C}$ to $v_0 + w_1 z + \cdots + w_q z^q$. This induces a connection on $\mathcal{F}$ and local data for $\mathcal{F}$. Moreover, we identify $(\mathcal{O}_{\mathbb{P}^1_C})_0/(z)$ with Cv_0, by sending 1 to $v_0 = e$. Now we consider $\mathcal{L} := \mathcal{F}(-q \cdot [\infty]) = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1_C}(-q \cdot [\infty])$. The above data make $(\mathcal{L}, \nabla, Cv_0,$ local data $)$ into a family, parametrized by X.
The quotient $\mathcal{Q} := \mathcal{M}/\mathcal{L}$ is again a vector bundle on $\mathbb{P}^1_X$ with an induced connection and induced local data. After shrinking X, there exists a vector bundle $\mathcal{N}$ on $\mathbb{P}^1_C$ such that $\mathcal{Q} = \mathcal{O}_X \otimes \mathcal{N}$. A choice of an isomorphism $\lambda : \mathcal{N}_0/(z) \to V/Ce$ induces an isomorphism $\mathcal{Q}_0/(z) \to \mathcal{O}_X \otimes (V/Ce)$. We require that this map is induced by the given isomorphism $\mathcal{M}_0/(z) \to \mathcal{O}_X \otimes V$. For every closed point x of X, there is an induced exact sequence of connections $0 \to \mathcal{L}(x) \to \mathcal{M}(x) \to \mathcal{Q}(x) \to 0$ on $\mathbb{P}^1_C$. The action of $\mathrm{Gal}(x)$ on V induces the actions on Ce and V/Ce for the connections $\mathcal{L}(x)$ and $\mathcal{Q}(x)$. We come now to the general result.

Proposition 3.18 *Let $(\nabla, V, \{\nabla_i\})$ be a family, parametrized by a reduced, irreducible scheme X of finite type over C. Let $W \subset V$ be a proper subspace such that W is invariant under $\mathrm{Gal}(x)$ for all $x \in X(C)$. After replacing X by a non-empty open subset, there exists a family $(\mathcal{N}, \nabla^*, W,$ local data $)$ parametrized by X such that:*

(i) *$\mathcal{N}$ is a subbundle of $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$, invariant under ∇. Moreover, ∇^*, the local data of $\mathcal{N}$ and the isomorphism $\mathcal{N}_0/(z) \to \mathcal{O}_X \otimes W$ are induced by those of $\mathcal{M}$.*

(ii) *The sheaf $\mathcal{Q} := \mathcal{M}/\mathcal{N}$ is a vector bundle on $\mathbb{P}^1_X$, isomorphic to $\mathcal{O}_X \otimes \mathcal{S}$ for a suitable vector bundle $\mathcal{S}$ on $\mathbb{P}^1_C$. Moreover, $\mathcal{Q}$ can be made into a family, parametrized by X, with connection, local data, and isomorphism $\mathcal{Q}_0/(z) \to \mathcal{O}_X \otimes (V/W)$, induced by those of the family $\mathcal{M}$.*

(iii) *For every closed point $x \in X(C)$, the exact sequence*

$$0 \to \mathcal{N}(x) \to \mathcal{M}(x) \to \mathcal{Q}(x) \to 0 \text{ of connections on } \mathbb{P}^1_C ,$$

has the property that the action of $\mathrm{Gal}(x)$ on V induces the actions of the differential Galois groups on W and V/W, that are produced by $\mathcal{N}(x)$ and $\mathcal{Q}(x)$.

Proof. Put $d = \dim W$. The case $d = 1$ is discussed above. For the general case one considers $L = \bigwedge^d W \subset \bigwedge^d V$ and the family $(\bigwedge^d \mathcal{M}, \dots)$ associated to $\bigwedge^d V$. One finds a line bundle $\mathcal{L} \subset \bigwedge^d \mathcal{M}$ (above a suitable open subset of X) with the required properties. This line bundle is decomposable since the line $L \subset \bigwedge^d V$ is decomposable. Thus there exists a vector bundle $\mathcal{N} \subset \mathcal{M}$ (above a suitable open subset of X) with $\bigwedge^d \mathcal{N} = \mathcal{L}$ and $\mathcal{N}$ has the required properties. In particular, $\mathcal{Q}$ is a connection on $\mathbb{P}^1_X$. It is not difficult to provide $\mathcal{N}$ and $\mathcal{Q}$ with the additional structure, which makes them into families, parametrized by X. This proves (i) and (ii). Part (iii) follows from the explicit construction. □

Proposition 3.18 is a sort of converse of Lemma 3.10. Indeed, let K denote the function field of X. The assumption that W is invariant under all $\mathrm{Gal}(x)$ implies that the differential Galois group $H \subset \mathrm{GL}(K \otimes V)$ of the generic differential equation on $\mathrm{Spec}(K) \otimes \mathbb{P}^1_C$ leaves the subspace $K \otimes_C W$ invariant.

Proposition 5.1 and Corollary 5.3 of Hrushovski's paper [H02] is also related to Proposition 3.18 above.

3.4.3 Constructions of linear algebra

Let H be an algebraic subgroup of $\mathrm{GL}(V)$. In other words, V is a faithful H-module. Let W be another H-module. It is well known that W can be obtained from V by a "construction of linear algebra". Explicitly, $W \cong W_2/W_1$, where $W_1 \subset W_2$ are H-invariant subspaces of a finite direct sum $\oplus_i V_{n_i}^{m_i}$.

Proposition 3.19 *Let a family $(\nabla, V, \{\nabla_i\})$, parametrized by a reduced, irreducible scheme X of finite type over C, be given. Let H be an algebraic subgroup of $\mathrm{GL}(V)$ and suppose that $\mathrm{Gal}(x) \subset H$ for every closed point $x \in X$. For any construction of linear algebra $W := W_2/W_1$, as above, there exists a family $(\mathcal{N}, \nabla, W,$ local data $)$, parametrized by a non-empty open subset U of X such that:*

(i) *For every closed point $x \in U(C)$, the connection $(\mathcal{N}(x), \nabla(x))$ on $\mathbb{P}^1_C$ is obtained by the same construction.*

(ii) *The action of* Gal(x) *on* W*, induced by the construction of linear algebra, coincides with the action of the differential Galois group of the connection* $\mathcal{N}(x)$ *on* W.

Proof. For an H-module of the form $\tilde{V} = \oplus_i V_{b_i}^{a_i}$ the construction of the new family, parametrized by X, is discussed in the proof of Proposition 3.16. For an H-submodule W_2 we apply Proposition 3.18 and we have to replace X with an open subset of X. For a H-submodule W_1 of W_2 one applies Proposition 3.18 again. The result is a family, parametrized by an open subset of X, corresponding to the H-module W_2/W_1. The construction of $(\mathcal{N}, W, \dots)$ implies at once the properties (i) and (ii). □

3.4.4 The set $X(U(G^o) \subset)$ is constructible

We introduce some notation. Let H be a linear algebraic group over C acting upon a finite dimensional vector space W over C. For every character $\chi : H \to \mathbb{G}_m = C^*$ one defines $W_\chi := \{w \in W \mid hw = \chi(h)w \text{ for all } h \in H\}$. This is a subspace of W. Let $\chi_1, \dots, \chi_r$ denote the distinct characters of H such that $W_{\chi_i} \neq 0$. Then $\sum_{i=1}^r W_{\chi_i} \subset W$ is in fact a direct sum $\oplus_{i=1}^r W_{\chi_i}$. This space is denoted by $\mathrm{Ch}_H(W)$. As before, an algebraic subgroup $G \subset \mathrm{GL}(V)$ is given. The group $U(G^o) = U(G)$ denotes the algebraic subgroup of G generated by all the unipotent elements of G. Any character of G^o is trivial on $U(G^o)$ and $G^o/U(G^o)$ is a torus. It easily follows that for any G-module W one has $\mathrm{Ch}_{G^o}(W) = W^{U(G^o)}$ (i.e., the set of $U(G^o)$-invariant elements $w \in W$). An essential result is the following.

Theorem 3.20 (M. F. Singer)
There exists a faithful G-module W such that for every subgroup H of G the following statements are equivalent.

(1) $U(G^o) \subset H$.

(2) $\mathrm{Ch}_{G^o}(W) = \mathrm{Ch}_{H \cap G^o}(W)$.

We note that the inclusion $\mathrm{Ch}_{G^o}(W) \subset \mathrm{Ch}_{H \cap G^o}(W)$ is valid for any G-module W. Moreover, for any G-module W, the implication (1)⇒(2) holds. Indeed,

$U(G^o) \subset H$ implies that $U(G^o) \subset H^o \subset H \cap G^o$. One has $U(G^o) = U(H^o)$, hence

$$\mathrm{Ch}_{H \cap G^o}(W) \subset \mathrm{Ch}_{H^o}(W) = W^{U(H^o)} = W^{U(G^o)} = \mathrm{Ch}_{G^o}(W).$$

For the rather involved proof of the existence of a faithful G-module W for which the implication (2)$\Rightarrow$(1) holds, we refer to [S93].

Corollary 3.21 *Put $m := [G : G^o]$. There exists a faithful G-module W such that for every subgroup H of G the following statements are equivalent.*

(i) *$U(G^o) \subset H$.*

(ii) *For every $r \leq m^m$ and for every H-invariant decomposable line $L = Cu_1 \otimes u_2 \otimes \cdots \otimes u_r \subset \mathrm{Sym}(W, r)$, the elements $u_1, \ldots, u_r$ belong to $\mathrm{Ch}_{G^o}(W)$.*

Proof. W will denote the G-module of Theorem 3.20.
(i)$\Rightarrow$(ii). As remarked above, the implication (1)$\Rightarrow$(2) in Theorem 3.20 holds for every G-module. We have $u_1 \otimes \cdots \otimes u_r \in \mathrm{Ch}_{H \cap G^o}(\mathrm{Sym}(W, r))$, so $u_1 \otimes \cdots \otimes u_r \in \mathrm{Ch}_{G^o}(\mathrm{Sym}(W, r)) = \mathrm{Sym}(W, r)^{U(G^o)}$. Let $x_1, \ldots, x_n$ denote a basis of W over C. The algebra $\oplus_{m \geq 0} \mathrm{Sym}(W, m)$ is identified with $C[x_1, \ldots, x_n]$. The group G acts linearly on $C[x_1, \ldots, x_n]$ and the element $u := u_1 \otimes \cdots \otimes u_r$ is a homogeneous polynomial which is a product of homogeneous linear terms. From the $U(G^o)$-invariance of u, the connectedness of $U(G^o)$ and the unicity of the decomposition of u (up to scalars and order), one deduces that $g(u_i)$ is a C^*-multiple of u_i for every $g \in U(G^o)$ and every i. We find that $u_i \in \mathrm{Ch}_{U(G^o)}(W) = W^{U(G^o)} = \mathrm{Ch}_{G^o}(W)$ for all i.

(ii)$\Rightarrow$(i). We will show that (ii) implies condition (2) of Theorem 3.20. It suffices to show that any $H \cap G^o$-invariant line $Cu \subset W$ belongs to $\mathrm{Ch}_{G^o}(W)$. The group $H \cap G^o$ is a subgroup of H of index at most $m := [G : G^o]$. There is a normal subgroup $\tilde{H}$ of H contained in $H \cap G^o$, such that $[H : \tilde{H}] \leq m^m$. Let $h_1, \ldots, h_r$ denote representatives of $H/\tilde{H}$. Then the line spanned by $h_1 u \otimes h_2 u \otimes \cdots \otimes h_r u \in \mathrm{Sym}(W, r)$ is decomposable and invariant under H. By (ii), $h_1 u \in \mathrm{Ch}_{G^o}(W)$ and so $u \in \mathrm{Ch}_{G^o}(W)$. $\square$

Proposition 3.22 *Let a family $(\nabla, V, \{\nabla_i\})$, parametrized by an irreducible, reduced X, be given. Let G be an algebraic subgroup of $\mathrm{GL}(V)$. Suppose that*

$\mathrm{Gal}(x) \subset G$ *holds for every closed point* x *of* X. *There exists an open non-empty subset* X' *such that the set* $X'(U(G^o) \subset)$ *is constructible.*

Proof. Let W be the G-module having the properties of Theorem 3.20 and Corollary 3.21. By Proposition 3.19, there corresponds to W a family $(\mathcal{N}, \nabla, W, \dots)$, parametrized by an open non-empty subset X' of X. Again we may suppose that $\mathcal{N}$ is free. Consider some integer r with $1 \leq r \leq m^m$, where $m := [G : G^o]$. The set $S(r)$ of elements $u = u_1 \otimes \cdots \otimes u_r \in \mathrm{Sym}(W, r)$ with all $u_i \neq 0$, and not all u_i belonging to $\mathrm{Ch}_{G^o}(W)$, is constructible. By part (1) of Corollary 3.17, the set $X'(r)$, consisting of the closed points $x \in X'(C)$ such that $\mathrm{Gal}(x)$ fixes a line $Cu \subset \mathrm{Sym}(W, r)$ with $u \in S(r)$, is constructible. $X'(U(G^o) \subset)$ is constructible since it is, by Corollary 3.21, the complement in X' of $\bigcup_{1 \leq r \leq m^m} X'(r)$. □

3.4.5 The final step, involving the Singer condition

As before, an algebraic subgroup $G \subset \mathrm{GL}(V)$ is given. We suppose that G satisfies the "Singer condition". Let a family $\mathcal{F} := (\nabla, V, \text{local data})$, parametrized by an irreducible, reduced X, be given. We will show, by induction on the dimension of X, that $X(= G)$ is constructible.

We have shown that there exists an open non-empty $X' \subset X$ such that $X'(U(G^o) \subset, \subset G)$ is constructible. By induction, $\{x \in X \setminus X' \mid \mathrm{Gal}(x) = G\}$ is constructible. After replacing X by an irreducible component of the set $X'(U(G^o) \subset, \subset G)$, one has $U(G^o) \subset \mathrm{Gal}(x) \subset G$ for all $x \in X$.

Consider a faithful $G/U(G^o)$-module W. The family $\mathcal{F}$ induces a family $\mathcal{G} := (\mathcal{N}, \nabla, W, \text{ local data})$, parametrized by X. For every $x \in X(C)$, one has $\mathrm{Gal}(x) \subset G/U(G^o)$. For the family $\mathcal{G}$, we have to prove that $X(= G/U(G^o))$ is constructible. We change the notation and write G for $G/U(G^o)$ and V for W. If G is finite, then an application of Proposition 3.16 finishes the proof. If G is infinite, then G^o is a torus and G^o lies in the center of G (this is precisely the Singer condition).

We continue the proof. For a closed point x and a singular point s_j one obtains a differential module $\mathcal{M}(x, s_j) := C((z - s_j)) \otimes \widehat{\mathcal{M}(x)}_{s_j}$ over the differential field $C((z - s_j))$. Let $PVF(x, s_j)$ denote a Picard-Vessiot field for this differential module. The formal local Galois group $\mathrm{Gal}(x, s_j)$ is the group of the differential automorphisms of $PVF(x, s_j)/C((z - s_j))$. Let $PVF \supset C(z)$ denote the Picard-Vessiot field for the generic differential module $\mathcal{M}(x)_\xi$ over $C(z)$. The differential Galois group $\mathrm{Gal}(x)$ is the group of the differential automorphisms of $PVF/C(z)$. This group is canonical embedded into $\mathrm{GL}(V)$ by our constructions. There exists a $C(z)$-linear embedding $PVF \subset PVF(x, s_j)$. This induces an injective algebraic homomorphism $\mathrm{Gal}(x, s_j) \to \mathrm{Gal}(x)$. Another embedding changes this homomorphisms by conjugation (with an element in $\mathrm{Gal}(x)$). The connected component of the identity $\mathrm{Gal}(x, s_j)^o$ is mapped to a subgroup of $\mathrm{Gal}(x)^o \subset G^o$, and lies therefore in the center of G and $\mathrm{Gal}(x)$. In particular, the image of $\mathrm{Gal}(x, s_j)^o$ in G does not depend on the chosen embedding $PVF \to PVF(x, s_j)$.

We note that the local connection $\mathcal{M}(x, s_j)$ is semi-simple since the formal local differential Galois group does not contain $\mathbb{G}_a$. Indeed, by construction $U(G^o) = \{1\}$, so $\mathrm{Gal}(x)$ does not contain a copy of $\mathbb{G}_a$. Now there are finitely many possibilities for the equivalence class of $\mathcal{M}(x, s_j)$. It is easily seen that this equivalence class depends in a constructible way on x. Therefore there exists an open non-empty subset of X, where the equivalence classes of $\mathcal{M}(x, s_j)$ does not depend on x. After restricting to this open subset, all the differential modules $\mathcal{M}(x, s_j)$ are isomorphic. In particular, $PVF(x, s_j)$ and $\mathrm{Gal}(x, s_j)$ do not depend on x. We will write $PVF(s_j)$ and $\mathrm{Gal}(s_j)$ for these objects. For a fixed embedding $PVF \to PVF(s_j)$, one has a fixed image of the groups $\mathrm{Gal}(x, s_j) = \mathrm{Gal}(s_j)$ into $\mathrm{Gal}(x)$. Moreover, the image of $\mathrm{Gal}(x, s_j)^o$ into $\mathrm{Gal}(x)$ does not depend on any choice and is independent of x.

Let $H \subset G^o$ denote the subgroup, generated by the images of all $\mathrm{Gal}(s_j)^o$. Then H does not depend on x and H is a connected normal subgroup of G. Now we take a faithful G/H-module W and its corresponding family, parametrized by a non-empty open subset X' of X. For notational convenience, we replace G with G/H. For this new family, parametrized by X', one has:

(i) the differential Galois groups are contained in G,

(ii) the formal local differential Galois groups are finite,

(iii) the singularities are regular singular,

(iv) the group $\mathrm{Gal}(x)$ is generated (as an algebraic group) by the finite local differential Galois groups.

We have to show that $X'(=G)$ is constructible. By [BS64] Lemme 5.11 (also known as Platonov's Theorem), there is a finite subgroup $E \subset G$ that maps surjectively to G/G^o. The surjective map $\tilde{G} := G^o \times E \to G$ has a finite kernel. The group $\tilde{G}$ has the property: any subgroup generated by s subgroups, each one of order bounded by some D, is finite (and in fact contained in $G^o[m] \times E$ for a suitable m depending in D). Thus the same statement holds for G. It follows that all $\mathrm{Gal}(x)$ are finite. If $G^o \neq \{1\}$, then $X'(=G) = \emptyset$. If $G^o = \{1\}$, then G is finite and therefore $X'(=G)$ is constructible.

3.5 Non-constructible sets $X(=G)$

The aim of this section is to produce for any linear algebraic G that does not satisfy the "Singer condition", a family of differential equations, parametrized by some X, such that $X(=G)$ is not constructible. We start by investigating a rather special case namely, G is a semi-direct product $G = T \rtimes E$. Here E is a finite group and T is a torus. Furthermore, there is given a homomorphism of groups $\psi : E \to \mathrm{Aut}(T)$. The group structure of G is then defined by the formula $ete^{-1} = \psi(e)(t)$. The induced action ϕ of E on the character group $X(T)$ of T, is given by the formula $(\phi(e)(\chi))(t) = \chi(e^{-1}te)$.

Lemma 3.23 *The following properties of $G = T \rtimes E$ are equivalent.*

(i) $\sum_{e\in E} \mathrm{im}(\phi(e) - 1)$ *has finite index in* $X(T)$.

(ii) $\bigcap_{e\in E} \ker(\phi(e) - 1) = 0$.

(iii) *The E-module $X(T) \otimes \mathbb{Q}$ does not contain the trivial representation.*

(iv) *The center of G is finite.*

Proof. The vector space $X(T) \otimes \mathbb{Q}$ is an E-module and can be written as a direct sum of irreducible E-modules $I_1, \ldots, I_r$. Consider a non-trivial irreducible representation $\rho : E \to \mathrm{GL}(W)$ over $\mathbb{Q}$. Then the submodule $\sum_{e\in E} \mathrm{im}(\rho(e) - 1)$ of W is not zero and hence equal to W. Moreover,

$\bigcap_{e\in E}\ker(\rho(e)-1)$ is a proper submodule of W and hence equal to $\{0\}$. For the trivial, 1-dimensional representation $\rho : E \to \mathrm{GL}(\mathbb{Q})$, one has that $\sum_{e\in E}\mathrm{im}(\rho(e)-1) = 0$ and $\bigcap_{e\in E}\ker(\rho(e)-1) = \mathbb{Q}$. This proves the equivalence of (i),(ii) and (iii). The elements of T can be considered as group homomorphisms $t : X(T) \to C^*$. Now t lies in the center of G if and only if $\chi(e^{-1}te) = \chi(t)$ for every χ and every $e \in E$. This translates into: t is equal to 1 on the submodule $\sum_{e\in E}\mathrm{im}(\phi(e)-1)$. This proves the equivalence of (i) and (iv). $\square$

Lemma 3.24 *As above $G = T \rtimes E$. Suppose that $X(T)\otimes\mathbb{Q}$ is a non-trivial irreducible E-module. Let H be an algebraic subgroup of G which maps surjectively to E. Then:*

(i) *If $H \neq G$, then there exists an integer $n \geq 1$ such that $H \subset T[n] \rtimes E$. Here $T[n]$ denotes the subgroup of T consisting of the elements with order dividing n.*

(ii) *Let $e \in E$ have order $m > 1$ and let $t \in T$ be given as a homomorphism $t : X(T) \to X(T)/\ker(\phi(e)-1) \to C^*$. Then $(te)^m = 1$.*

(iii) *There exist integers $N, M \geq 1$ and subgroups $G_n \subset T[n]\rtimes E$ for infinitly many $n \geq 1$ such that the following holds.*

  (a) *The index of G_n in $T[n] \rtimes E$ is bounded by a constant independent of n.*

  (b) *G and every G_n is generated, as an algebraic subgroup, by N elements of order $\leq M$.*

Proof.
(i) The subtorus $(H\cap T)^o$ of T is invariant under the action of E on T. For let $t \in H \cap T$, then for any $e \in E$ there exists an element $s \in T$ such that $se \in H$, so $ete^{-1} = sete^{-1}s^{-1} \in H$. We find that $H \cap T$ is invariant under the action of E, so the same holds for $(H \cap T)^o$. Let N denote the kernel of the surjective homomorphism $X(T) \to X((H\cap T)^o)$, then $X(T)/N$ has no torsion and $(H \cap T)^o$ consists of the homomorphisms $t : X(T) \to C^*$ which are 1 on N. If $N = X(T)$, then H is finite and clearly contained in $T[n] \rtimes E$ for some $n \geq 1$. If $N \neq X(T)$, then $N = 0$ and $H = G$.

(ii) One verifies that

$$(te)^m = t \cdot \psi(e)(t) \cdot \psi(e^2)(t) \cdots \psi(e^{m-1})(t).$$

For any character χ one finds

$$\chi((te)^m) = \chi(t) \cdot (\phi(e^{-1})\chi)(t) \cdots (\phi(e^{-m+1}\chi)(t).$$

Therefore the only thing we have to show is that t has value 1 on the submodule $(1 + \phi(e^{-1}) + \cdots + \phi(e^{-m+1}))X(T)$ of $X(T)$. Since this submodule is contained in $\ker(\phi(e^{-1}) - 1) = \ker(\phi(e) - 1)$, one concludes that $(te)^m = 1$.

(iii) For G one takes as set of generators E and an element te, with $e \in E$ of order m, $t \in T$ of infinite order and te of order m. It follows from (i) that G is generated, as an algebraic subgroup, by this set. Consider an integer $n > 1$. Let G_n be the subgroup of $T[n] \rtimes E$ generated by E and for every $e \in E$ a collection of products te, with $t \in T$, that we now describe. Let $e \in E$ have order $m > 1$. Take a $\mathbb{Z}$-basis $b_1, \ldots, b_r$ of $X(T)/\ker(\phi(e) - 1)$ and define the homomorphisms $h_1, \ldots, h_r : X(T)/\ker(\phi(e) - 1) \to C^*$ by $h_i(b_j) = 1$ if $i \neq j$ and $h_i(b_i) = \zeta_n$ for $i = 1, \ldots, r$ and with ζ_n a fixed nth root of unity. The $t_i e$ that we use as generators of G_n are $t_i : X(T) \to X(T)/\ker(\phi(e) - 1) \xrightarrow{h_i} C^*$. Part (b) is clear. For the proof of part (a) we consider the obvious map $\alpha : X(T) \to M := \oplus_{e \in E} X(T)/\ker(\phi(e) - 1)$. This map is injective by Lemma 3.23. For every homomorphism $h : M \to \mu_n$, (here μ_n denotes the group of the nth roots of unity), the element $t = h \circ \alpha$ belongs to G_n. Let N denote the smallest submodule of M such that $\text{im } \alpha \subset N$ and M/N has no torsion. Then N is a direct summand of M, so the image of $\text{Hom}(N, \mu_n) \to \text{Hom}(X(T), \mu_n) = T[n]$ is the same as the image of $\text{Hom}(M, \mu_n) \to \text{Hom}(X(T), \mu_n) = T[n]$, and is therefore contained in G_n. For infinitely many of the n, we have $\text{Hom}(X(T), \mu_n) \subset \text{Hom}(N, \mu_n)$, and $[\text{Hom}(X(T), \mu_n) : \text{Hom}(N, \mu_n)] = [N : \text{im } \alpha]$. For these n we have $[T[n] \rtimes E : G_n] = [T[n] : G_n \cap T[n]] \leq [\text{Hom}(X(T), \mu_n) : \text{Hom}(N, \mu_n)]$, so $[T[n] \rtimes E : G_n] \leq [N : \text{im } \alpha] < \infty$ and (a) follows. □

Proposition 3.25 *Suppose that C is the field of the complex numbers $\mathbb{C}$. Let $G = T \rtimes E$ and suppose that $X(T) \otimes \mathbb{Q}$ is an irreducible E-module. There is a moduli space $\mathbb{M}$ such that $\mathbb{M}(= G)$ is not constructible.*

Proof. Let $G \subset \mathrm{GL}(V)$ be a faithful irreducible representation. Fix a finite subset $\{s_1, \ldots, s_r\}$ of $\mathbb{C}^*$ and integers $d_i > 1$ for $i = 1, \ldots, r$. Let π_1 denote the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \{s_1, \ldots, s_r\}$ with base point 0. Take loops $\lambda_1, \ldots, \lambda_r \in \pi_1$ around the s points such that π_1 is generated by $\lambda_1, \ldots, \lambda_r$ and such that the only relation between these generators is $\lambda_1 \cdots \lambda_r = 1$. Let $G_n,\ n \in I \subset \mathbb{Z}$ be subgroups of G as given by Lemma 3.24. By the previous lemma we get that for a suitable choice of r and the d_i, and an infinite subset $I_1 \subset I$, there exist homomorphisms $\rho, \rho_n : \pi_1 \to G \subset \mathrm{GL}(V),\ n \in I_1$ with the following properties:

(a) $\rho(\lambda_i)$ and the $\rho_n(\lambda_i)$ have order d_i (for $i = 1, \ldots, r$),

(b) the image of ρ is Zariski dense in G and $G_n = \mathrm{im}\ \rho_n$ for every $n \in I_1$.

Let te be the element used as a topological generator of G, as in the proof of Lemma 3.24. Some continuity argument shows that the eigenvalues of te and e are the same. It follows that there is an infinite set $I_2 \subset I_1$ such that for each i the set of eigenvalues of $\rho_n(\lambda_i)$ and $\rho(\lambda_i)$ are the same for all $n \in I_2$. The Riemann-Hilbert correspondence attaches to each $\rho_n,\ n \in I_2$ a differential module $M_n \cong \mathbb{C}(z) \otimes V$ over $\mathbb{C}(z)$ (unique up to conjugation, see [PS03], Theorem 6.15). For each M_n and each i, there is a unique lattice $\Lambda_{n,i} \subset C((z - s_i)) \otimes M_n$, with the following property. $\Lambda_{n,i}$ has a basis, on which the differential is given by $\frac{d}{d(z-s_i)} + \frac{A_{n,i}}{z-s_i}$, where $A_{n,i}$ is a diagonal map with diagonal entries in $[0, 1) \cap \mathbb{Q}$. We can take $A_{n,i}$ independent of n. By [PS03] Lemma 6.18, these data define a unique connection $(\mathcal{M}_n, \nabla)$ with generic differential module M_n. Now $\mathcal{M}_n$ is in general not free, but has the form $O(a_1) \oplus \cdots \oplus O(a_v)$ with $a_1 \geq \cdots \geq a_v$ and $v := \dim V$. The sum $a_1 + \cdots + a_v$ is fixed since the local exponents of $\Lambda^v \mathcal{M}_n$ are given. Since ρ_n is irreducible the defect of $\mathcal{M}_n$ is uniformly bounded (see [PS03], Proposition 6.21). It follows that there is an infinite subset $I_3 \subset I_2$ such that $\mathcal{M}_n$ is of type $a_1 \geq \cdots \geq a_v$ for all $n \in I_3$. The embedding of V in M_n and the regularity of M_n at the point $z = 0$ yield a canonical isomorphism $\mathbb{C}[z]_{(z)} \otimes V \to (\mathcal{M}_n)_0$. One defines now a moduli problem by fixing the type of the vector bundle $\mathcal{M}$ (namely $a_1 \geq \cdots \geq a_v$), an identification $\mathbb{C}[z]_{(z)} \to \mathcal{M}_0$ and the above local data. There is a universal family, parametrized by a variety $\mathbb{M}$. Then $\mathbb{M}(= G_n)$ is not empty for $n \in I_3$. We remove from $\mathbb{M}(\subset G)$ the union of the finitely many closed subsets $\mathbb{M}(\subset T \rtimes E')$ with E' a proper subgroup of E. For notational convenience we call the result again $\mathbb{M}(\subset G)$. The set

$\mathbb{M}(=G)$ is the complement in $\mathbb{M}(\subset G)$ of the sets $Z_n := \mathbb{M}(\subset T[n!] \rtimes E)$ for $n \geq 1$. It suffices now to show that $\cup_{n\geq 1} Z_n$ is not constructible. Indeed, $\mathbb{M}(=G)$ is the complement in the closed set $\mathbb{M}(\subset G)$ of the non-constructible set $\cup_{n\geq 1} Z_n$.

By construction, $\{Z_n\}$ is an increasing sequence of closed sets, i.e., $Z_n \subseteq Z_{n+1}$ and $Z_n \neq \cup_{i\in\mathbb{N}} Z_i \ \forall\ n \in \mathbb{N}$. Suppose that this union is equal to $\cup_{i=1}^d O_i \cap F_i$ with open sets O_i and closed sets F_i. For some i the sets $Z_n \cap (O_i \cap F_i)$ again form an increasing sequence of closed subsets. After replacing $O_i \cap F_i$ by a suitable irreducible component, say Y, we have an increasing sequence of closed subsets $Y_n = Z_n \cap Y$ with union Y and such that each $Y_n \neq Y$. This is not possible because the field $\mathbb{C}$ is uncountable. $\square$

Remarks 3.26

(1) The moduli space m occurring in the proof of Proposition 3.25 is in general not the one studied in detail in Chapter 2, since the vector bundle $\mathcal{M}$ is not free. Suppose that one of the local data $\frac{d}{d(z-s_i)} + \frac{A_i}{z-s_i}$ is such that the eigenvalues of A_i have multiplicity 1, then one can change each $\mathcal{M}_n$ (with $n \in I$) into a free vector bundle by shifting the eigenvalues of A_i over integers. There are only finitely many ways to do this. Thus for some infinite subset $I' \subset I$ one single change of A_i will make all $\mathcal{M}_n$ with $n \in I'$ into a free vector bundle. Now one can define the moduli space $\mathbb{M}$ by a free vector bundle $\mathcal{M}$ with $H^0(\mathbf{P}^1_{\mathbf{C}}, \mathcal{M})$ identified with V and with the prescribed local data.

(2) The proof of Propostion 3.25 extends to the case where C is any algebraically closed field, not algebraic over $\mathbb{Q}$. Indeed, it suffices to consider a field C of finite transcendence degree ≥ 1. This field is embedded into $\mathbb{C}$. The moduli space $\mathbb{M}$ of the proof descends to C, i.e., $\mathbb{M} = \mathbb{M}_C \otimes_C \mathbb{C}$ for a suitable space $\mathbb{M}_C$.
The group G is given as an algebraic subgroup of $\mathrm{GL}(V)$ where V is a vector space over C. One easily verifies that $\mathbb{M}(\subset G \otimes_C \mathbb{C}) = \mathbb{M}_C(\subset G) \otimes_C \mathbb{C}$. The same statement is valid for the groups G_n. It follows that $\mathbb{M}_C(=G)$ is not constructible. $\bullet$

We now give the proof of the general result, omitting some of the more obvious details.

Theorem 3.27 *Let C be the field of the complex numbers $\mathbb{C}$. Suppose that the linear algebraic group G does not satisfy the Singer condition. Then there is a moduli space $\mathbb{M}$ such that $\mathbb{M}(= G)$ is not constructible.*

Proof. As we will show, it suffices to prove this theorem for a linear algebraic group G' for which there exists a surjective morphism $G' \to G$ with finite kernel. By [BS64] Lemme 5.11, there exists a finite subgroup E of G such that $E \to G/G^o$ is surjective. Thus we may replace G with $G^o \rtimes E$. The group $G^o/U(G^o)$ is a torus.

Lemma 3.28 *(We use the above notations) There is a torus $T \subset G^o$, invariant under conjugation with the elements of E, such that $T \to G^o/U(G^o)$ is surjective and has a finite kernel.*

Proof. First we will assume G^o to be reductive. Then by [Sp98] Corollary 8.1.6 (G^o, G^o) is semi-simple and $G^o = (G^o, G^o) \cdot R(G^o)$, where $R(G^o)$ denotes the radical of G^o. By [Sp98] Proposition 7.3.1, $R(G^o)$ is a central torus of G^o and $R(G^o) \cap (G^o, G^o)$ is finite. Furthermore, by [Sp98] Theorem 8.1.5, we have $(G^o, G^o) \subset U(G^o)$. We have a surjective map $R(G^o) \twoheadrightarrow G^o/(G^o, G^o)$, so $G^o/(G^o, G^o)$ is a torus, and we find $(G^o, G^o) = U(G^o)$. The subgroup $R(G^o) \subset G^o$ is a characteristic subgroup, so in particular $eR(G^o)e^{-1} = R(G^o)$ for all $e \in E$. We find that we can take $T = R(G^o)$.

We now consider the general case. We define T to be a maximal torus in $R(G^o)$. We have $R(G^o) = T \times R_u(G^o)$, where $R_u(G^o)$ is the unipotent radical of G^o. The image of $R(G^o)$ under the map $\pi : G^o \to G^o/R(G^o)$ is the radical of $G^o/R_u(G^o)$, and clearly $\pi(R(G^o)) = \pi(T)$. We find that π defines an isomorphism of T with the radical of $G^o/R_u(G^o)$, so the cannonical map $T \to G^o/U(G^o)$ is surjective and has a finite kernel. The only thing left to show is that T is invariant under conjugation with the elements of E. For $e \in E$, we have that eTe^{-1} is again a maximal torus in $R(G^o)$, so we can write $eTe^{-1} = rTr^{-1}$ for some $r \in R(G^o)$. Because $R(G^o) = T \times R_u(G^o)$, we can take $r \in R_u(G^o)$. Let $N := \{u \in R_u(G^o) | uTu^{-1} = T\}$, then N is a normal subgroup of $R_u(G^o)$ and we find a map $c : E \to R_u(G^o)/N, e \mapsto r$. Let $e \in E$, then e is semi-simple, so $c(e)$ is semisimple, but also unipotent, because $c(e) \in R_u(G^o)/N$. Therefore $c(e) = 1$, $\forall e \in E$, so indeed T is invariant under conjugation with the elements of E. $\square$

Thus we may replace G^o with $U(G^o) \rtimes T$. After replacing T with a torus T' such that $T' \to T$ is surjective and has a finite kernel, one can write T as a product of two tori T_1 and T_2, both invariant under conjugation by E and such that the group $T_2 \rtimes E$ satisfies the assumptions of Lemma 3.24. To be precise, Let $X(T)$ be the character group of T, which has a structure of E-module. Then we can write $X(T) \otimes \mathbb{Q}$ as a direct sum of irreducible E-modules, say $X(T) \otimes \mathbb{Q} = D_1 \oplus \cdots \oplus D_r$. Since G does not satisfy the Singer condition, we may assume that D_1 is a non-trivial E-module. Let X_2 be the projection of $X(T)$ on D_1 and X_1 the projection of $X(T)$ on $D_2 \oplus \cdots \oplus D_r$. Then $X(T) \subset X_1 \oplus X_2$ has finite index. Now let T_i be a torus with $X(T_i) = X_i,\ i = 1, 2$, then $T' = T_1 \times T2$, has the desired properties. The result after these changes is a group G' of the form

$$(U(G^o) \rtimes T_1) \rtimes (T_2 \rtimes E)$$

which maps surjectively to G and has a finite kernel. We will construct a moduli space $\mathbb{M}$ such that $\mathbb{M}(= G)$ is not constructible.

One takes a finite subset $\{b_1, \ldots, b_t, s_1, \ldots, s_r\}$ in $\mathbb{C}^*$. The fundamental group π_1 of the complement of this set in $\mathbb{P}^1_{\mathbb{C}}$, with base point 0, is given generators $\mu_1, \ldots, \mu_t, \lambda_1, \ldots, \lambda_r$ according to loops around these points. The only relation is $\mu_1 \cdots \mu_t \lambda_1 \cdots \lambda_r = 1$. We will consider homomorphisms $\rho : \pi_1 \to G'$ by assigning images for these $t + s$ generators. For notational convenience we will ignore the relation between the generators of π_1. The trick which allows us to do so is the following. One doubles the finite set by adding new points $s_r^*, \ldots, s_1^*, b_t^*, \ldots, b_1^*$. The fundamental group has now generators $\mu_1, \ldots, \mu_t, \lambda_1, \ldots, \lambda_r, \lambda_r^*, \ldots, \lambda_1^*, \mu_t^*, \ldots, \mu_1^*$. The only relation is their product being 1. Suppose that we want to assign elements $g_1, \ldots, g_t, h_1, \ldots, h_r \in G'$ to $\mu_1, \ldots, \lambda_r$. Then for the larger fundamental group, we complete this by assigning $h_r^{-1}, \ldots, h_1^{-1}, g_t^{-1}, \ldots, g_1^{-1}$ to the generators $\lambda_r^*, \ldots, \mu_1^*$. The homomorphisms $\rho'_n : \pi_1 \to G'$ that interest us are given by:

(a) $\rho'_n(\mu_1), \ldots, \rho_n(\mu_{t-1}) \in U(G^o)$; these elements are unipotent, $\neq 1$ and they generate $U(G^o)$ as an algebraic group. Moreover, these elements will not depend on n.

(b) $\rho'_n(\mu_t) \in T_1$ which generates T_1 as an algebraic group. Moreover, this element will not depend on n.

(c) $\rho'_n(\lambda_1), \ldots, \rho'_n(\lambda_r) \in T_2 \rtimes E$ are chosen as in the proof of Proposition 3.25.

As above this is completed by assigning values to $\mu_t^*, \ldots, \lambda_1^*$. The homomorphism $\rho_n : \pi_1 \to G$ are obtained by composing ρ'_n with $G' \to G$. We take an irreducible faithful G-module V. Riemann-Hilbert (see [PS03] Theorem 6.15) produces a differential module $M_n = \mathbb{C}(z) \otimes V$ with singularities in $\{b_1, \ldots, s_1, \ldots, a_r^*, \ldots, b_t^*, \ldots, b_1^*\}$. The local monodromies at the points $b_1, \ldots, b_t$ are fixed and we choose local connections for these singular points. For the local connections at the regular singular points $s_1, \ldots, s_r$ we make a choice which fits infinitely many of the ρ_n. The local data at the other points $a_r^*, \ldots, b_1^*$ are just the negatives of the corresponding points in $\{b_1, \ldots, s_r\}$. As in the proof of Proposition 3.25, there exists an infinite subset I of $\mathbb{N}$, such that the corresponding vector bundles $\mathcal{M}_n$ have the same type. This defines the moduli problem and the moduli family, parametrized by some space $\mathbb{M}$. According to Proposition 3.22, $\mathbb{M}(U(G^o) \subset, \subset G)$ is constructible. Let H denote the image of the group $U(G^o) \rtimes T_1$ in G. Then it can be seen that $\mathbb{M}(H \subset, \subset G)$ is also constructible. The final part of the proof of Proposition 3.25 applies here as well and the result is that $\mathbb{M}(= G)$ is not constructible. $\square$

Remarks 3.29 *Another formulation of the Singer condition.*
(1) The constructions in Lemma 3.23, Lemma 3.24, Proposition 3.25 and Theorem 3.27 lead to the following observation.

A linear algebraic group G does not satisfy the Singer–condition if and only if it has a factor group H of dim ≥ 1, with the following property: There exist integers $N, M, I > 1$ such that every algebraic subgroup $K \subset H$ which is mapped surjectively to H/H^o contains an algebraic subgroup of index $\leq I$ which is, as algebraic group, generated by N elements of order $\leq M$.

(2) Theorem 3.27 remains valid for an algebraically closed field C that is not algebraic over $\mathbb{Q}$ (See Remarks 3.26).

(3) For Theorem 3.15 to hold, it is essential to consider families of differential equations on $\mathbb{P}^1$. For example on an elliptic curve E over $\mathbb{C}$, one can construct a family of differential equations parametrized by some X, such that $X(= \mathbb{C}^*)$ is not constructible (see [S93] p.384). If this family is pushed

down to $\mathbb{P}^1_{\mathbb{C}}$ then after a shift one obtains the Lamé family we considered in Example 3.8. $\bullet$

Chapter 4

The Riemann-Hilbert problem and examples

In this chapter we will give some examples of moduli spaces of differential equations, and we describe the connection with the Riemann-Hilbert problem.

4.1 The classical Riemann-Hilbert problem

We start by briefly recalling the classical Riemann-Hilbert problem as described in [PS03] Chapter 6.

Let (M, ∇) be a regular singular connection over $\mathbb{C}(z)$ with singular locus equal to $S = \{s_1, \cdots, s_r\} \subset \mathbb{P}^1_{\mathbb{C}}$. This means that M is a finite dimensional $\mathbb{C}(z)$-vector space, and $\nabla : M \to \mathbb{C}(z)dz \otimes M$ is a regular singular connection (see [PS03] 6.4.2). One defines a monodromy map associated to (M, ∇) in the following manner. Write $V := \ker(\mathbb{C}((z-b)) \otimes_{\mathbb{C}(z)} M, \nabla)$ for the local solution space at a regular point $b \in \mathbb{P}^1_{\mathbb{C}} \setminus S$ and define $\pi_1 := \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S, b)$. Let $\lambda \in \pi_1$ be a loop, then we can make an analytic continuation of the local solutions V along λ. This analytic continuation defines a linear map on V, so we can associate an element of $\mathrm{GL}(V)$ to λ. This process results in a map $\pi_1 \to \mathrm{GL}(V)$, called the *monodromy map*. The question if for any given representation $\rho : \pi_1 \to \mathrm{GL}(V)$ there is a regular singular connection with monodromy representation equivalent to ρ is known as the weak Riemann-Hilbert problem. This question has a positive answer, which is precisely formulated in [PS03], Theorem 6.15.

The strong Riemann-Hilbert problem asks whether for a given representation $\rho : \pi_1 \to \mathrm{GL}(V)$ there is a *Fuchsian* connection over $\mathbb{C}(z)$ with monodromy representation equivalent to ρ. A Fuchsian connection is a connection which for the differentiation $\frac{d}{dz}$ can be written in the form $\frac{d}{dz} + \sum_{i=1}^{r} \frac{A_i}{z-s_i}$, with $A_i \in M_n(\mathbb{C})$. In general, for a given ρ there is no such Fuchsian connection. However under some conditions on ρ the strong Riemann-Hilbert problem has a positive answer, see sections 6.4 and 6.5 of [PS03] for details.

The strong Riemann-Hilbert problem can be restated in terms of connections on vector bundles. For a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ (where $\mathcal{M}$ is not necessarily free), we get an induced connection (M_η, ∇_η) over $\mathbb{C}(z)$ by localization at the generic fibre. Therefore we can associate a monodromy map to $(\mathcal{M}, \nabla)$. It is easily seen that the strong Riemann-Hilbert problem precisely asks whether there is a connection on a *free* vector bundle with some given monodromy map.

For a representation $\rho : \pi_1 \to \mathrm{GL}(V)$, by [PS03] Theorem 6.15, we find an associated connection (M, ∇) over $\mathbb{C}(z)$. The following lemma states how we can associate a connection over $\mathbb{P}^1_{\mathbb{C}}$ to (M, ∇).

Lemma 4.1 (Lemma 6.18 of [PS03]) *Let (M, ∇) be a regular singular connection over $\mathbb{C}(z)$ with singular locus S. For every $s \in S$ we choose a local parameter t_s. For every $s \in S$ let $\Lambda_s \subset \widehat{M_s} := \mathbb{C}((t_s)) \otimes M$ be a lattice that satisfies $\nabla(\Lambda_s) \subset \frac{dt_s}{t_s} \otimes \Lambda_s$ (the existence of such a lattice is equivalent to (M, ∇) being regular singular at s). Then there is a unique regular singular connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ with singular locus in S such that:*

1. *For every open $U \subset \mathbb{P}^1_{\mathbb{C}}$, one has $\mathcal{M}(U) \subset M$.*
2. *The generic fibre of $(\mathcal{M}, \nabla)$ is (M, ∇).*
3. *$\widehat{\mathcal{M}}_s = \Lambda_s$ for all $s \in S$.*

In the case where (M, ∇) is irreducible, one can make a choice for the lattices Λ_s in such a way that the corresponding vector bundle $\mathcal{M}$ is free (see

Theorem 6.22 of [PS03]). For general (M, ∇) this is not always the case.

We will conclude by briefly describing how a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ with a prescribed monodromy representation can be constructed. We will use a generalization of this construction in the next section.

Write $U := \mathbb{P}^1_{\mathbb{C}} \setminus S$, so $\pi_1 := \pi_1(U, 0)$. We start by constructing a regular connection on U with the prescribed monodromy. For this consider the universal covering $u : \widetilde{U} \to U$ of U. Define a connection $(\mathcal{N}, \nabla)$ on $\widetilde{U}$ by $\mathcal{N} := \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{U}}$, and $\nabla(v \otimes f) = v \otimes f'$ for all $v \in \mathbb{C}^n,\ f \in \mathcal{O}_{\widetilde{U}}$. Furthermore we define a π_1-action on $\mathcal{N}$ by $\lambda(v \otimes f) = \rho(\lambda)(v) \otimes (f \circ \lambda^{-1})\ \forall\ \lambda \in \pi_1$. The vector bundle $\mathcal{N}$ corresponds to the geometric vector bundle $\mathbb{C}^n \times \widetilde{U}$, and the corresponding π_1-action is given by $\lambda(v, \tilde{u}) = (\rho(\lambda)(v), \lambda(\tilde{u})),\ v \in \mathbb{C}^n,\ \tilde{u} \in \widetilde{U}$. Indeed in this way we get for a section $h \times id : \widetilde{U} \to \mathbb{C}^n \times \widetilde{U},\ h \in \mathcal{N}(\widetilde{U})$, that $(\lambda(h) \times id)(\tilde{u}) = \lambda(h \times id(\lambda^{-1}(\tilde{u})))$. It is clear that the quotient $\pi_1 \setminus (\mathbb{C}^n \times \widetilde{U})$ defines a geometric vector bundle on $U = \pi_1 \setminus \widetilde{U}$, with corresponding vector bundle $\mathcal{M}_U := \mathcal{N}^{\pi_1}$. The π_1-action on $\mathcal{N}$ commutes with ∇. So we find an induced connection ∇_U on $\mathcal{M}_U$. Write $\mathcal{L} := \ker(\nabla_U, \mathcal{M}_U)$. The only thing left to show is that $\mathcal{L}$ is the local system corresponding to ρ. There is a one to one correspondence between local systems on U and (trivial) local systems on $\widetilde{U}$ with a π_1-action. Under this correspondence $\mathcal{L}$ clearly corresponds to $\mathbb{C}^n$ with the defined π_1-action which is given by ρ. This proves that $(\mathcal{M}_U, \nabla_U)$ has monodromy given by ρ.

We now want to extend this connection $(\mathcal{M}_U, \nabla_U)$ to a connection on $\mathbb{P}^1_{\mathbb{C}}$. Let $s \in S$, and consider the pointed disk $U_s^* := 0 < |z - s| < \varepsilon$. We will construct a connection on $U_s := |z - s| < \varepsilon$ that glues to the restriction of $(\mathcal{M}_U, \nabla_U)$ to U_s^*. For this we consider the local solution space V_s at $s + \frac{\varepsilon}{2}$. The circle around s through $s + \frac{\varepsilon}{2}$ induces a monodromy map $B \in \mathrm{GL}(V_s)$. Choose $A \in \mathrm{End}(V)$ such that $e^{2\pi i A} = B$, then we define the connection ∇_s on the vector bundle $\mathcal{M}_s := \mathcal{O}|_{U_s} \otimes V_s$ by $\nabla_s(f \otimes v) = df \otimes v + z^{-1} \otimes A(v)$. The restriction of $(\mathcal{M}_s, \nabla_s)$ to U_s^* clearly has local monodromy B. By [PS03] 6.6-3 this restriction is isomorphic to the restriction of $(\mathcal{M}_U, \nabla_U)$ to U_s^*. Therefore we can glue the connection $(\mathcal{M}_s, \nabla_s)$ to $(\mathcal{M}_U, \nabla_U)$. In this way we obtain the desired connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ extending $(\mathcal{M}_U, \nabla_U)$.

4.2 The Riemann-Hilbert problem for families

We will now consider the Riemann-Hilbert problem for families of differential equations. Let Y be an analytic manifold, and let $S := \{s_1, \cdots, s_r\}$ be a set of points in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$. Suppose that $(\mathcal{M}, \nabla)$ is an analytic family of differential equations on $\mathbb{P}^1$, parametrized by Y (the definition of an analytic family is a straightforward variation of Definition 3.13). We suppose that S is the set of singular points of ∇; more precisely, for every $y \in Y$ the set of singular points of $\nabla(y)$ is S.

We write $pr_1 : Y \times \mathbb{P}^1 \to Y$, $pr_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ for the two projection maps. Let $U := \mathbb{P}^1 \setminus S$, and $\pi_1 := \pi_1(U, 0)$. We will also write pr_1, pr_2 for the restrictions to $Y \times U$ of the two projection maps. The kernel $\mathcal{L} := \ker(\nabla)$ of $\nabla|_{Y \times U}$ is a locally free $pr_1^*(\mathcal{O}_Y)$-module of rank n, where $n = \dim(V)$. For any $a \in U$ the embedding $j_a : Y \cong Y \times \{a\} \hookrightarrow Y \times U$ defines a vector bundle $\mathcal{L}_a := j_a^*(\mathcal{L})$ on Y .

We will now define a monodromy map $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$. Let $\lambda : [0,1] \to U$ be a path in U. Then $(id \times \lambda)^*(\mathcal{L})$ is a $pr^*(\mathcal{O}_Y)$-module on $Y \times [0,1]$, where $pr : Y \times [0,1] \to Y$ is the projection map. Since $(id \times \lambda)^*(\mathcal{L})$ is a locally free sheaf, we find a canonical isomorphism $\mathcal{L}_{\lambda(0)} \xrightarrow{\sim} \mathcal{L}_{\lambda(1)}$. In particular, a closed path λ with $\lambda(0) = 0 \in \mathbb{P}^1_{\mathbb{C}}$ yields an automorphism of $\mathcal{L}_0$ and this defines a homomorphism $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$.

Definition 4.2 We call the map $\pi_1 \to \mathrm{Aut}(\mathcal{L}_0)$ constructed above, the monodromy map associated to $(\mathcal{M}, \nabla)$. •

We now present a converse construction, which we interpret as a solution to the Riemann-Hilbert problem for families.

Theorem 4.3 *Let Y be an analytic irreducible reduced manifold with a vector bundle $\mathcal{L}$ on it. Suppose we are given a set $S := \{s_1, \cdots, s_r\} \subset \mathbb{P}^1_{\mathbb{C}}$ and a representation $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S) \to \mathrm{Aut}(\mathcal{L})$ satisfying the following properties.*

- *Let $\lambda_i \in \pi_1$ be loops around the points s_i with $\prod_{i=1}^{s} \lambda_i = 1$. For every $y \in Y$ we have that $\rho(\lambda_i)(y) \sim e^{2\pi i A_i}$ for some fixed $A_i \in \mathrm{M}_n(\mathbb{C})$. Here*

$\rho(\lambda_i)(y)$ denotes the automorphism on $\mathcal{L}_y/(m_y\mathcal{L}_y) \cong \mathbb{C}^n$ induced by $\rho(\lambda_i)$.

- *None of the differences of the eigenvalues of A_i is in $\mathbb{Z} \setminus \{0\}$.*

Then there exists an analytic connection $(\mathcal{M}, \nabla)$ on $Y \times \mathbb{P}^1$, with singular points in S and monodromy map given by ρ.

Proof. We can cover Y by Stein-manifolds Y_i such that $\mathcal{L}|_{Y_i}$ is free for all i. We will now construct a solution to the Riemann-Hilbert problem for families over Y_i. From the construction it can be seen that the connections on the Y_i glue to a connection on Y with the appropriate monodromy map, hence this also solves the Riemann-Hilbert problem for families over Y. From now on we assume $\mathcal{L}$ to be free and Y to be a Stein-manifold, and ρ is given as a homomorphism $\rho : \pi_1 \to \mathrm{GL}_n(\mathcal{O}(Y))$.

Let $\widetilde{U}$ be the universal covering of U. We can identify π_1 with $\mathrm{Aut}(\widetilde{U}/U)$. Write $pr_1 : Y \times \widetilde{U} \to Y$, $pr_2 : Y \times \widetilde{U} \to \widetilde{U}$ for the two projection maps. The vector bundle $\mathcal{N} := \mathcal{O}^n_{Y\times\widetilde{U}}$ can be written as $pr_1^{-1}(\mathcal{O}^n_Y)\widetilde{\otimes} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$, where $\widetilde{\otimes}$ is an "analytic tensor product", as defined in [GR71] p.179.

Remark 4.4 We note that $pr_1^{-1}(\mathcal{O}_Y)$ are the analytic functions on $Y \times \widetilde{U}$ which are constant with respect to $\widetilde{U}$. The sheaf $pr_1^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$ (the usual tensor product over $\mathbb{C}$) consists of functions of the form $\sum_{i=1}^m f_i \cdot g_i$, where the f_i are constant with respect to $\widetilde{U}$ and the g_i are constant with respect to Y. Therefore the sheaf $pr_1^{-1}(\mathcal{O}^n_Y) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$ consisting of n-tuples of such functions is much smaller then $\mathcal{N}$. •

We will now define a connection $(\mathcal{M}_U, \nabla_U)$ on $Y \times U$ by a construction similar to the one in the previous section. Define a π_1-action on $\mathcal{N}$ given by the formula $\lambda(v\widetilde{\otimes}f) = \rho(\lambda)v\widetilde{\otimes}(f \circ \lambda^{-1})$ for $v \in pr_1^{-1}(\mathcal{O}^n_Y)$, $f \in pr_2^{-1}(\mathcal{O}_{\widetilde{U}})$, and $\lambda \in \pi_1 \cong \mathrm{Aut}(\widetilde{U}/U)$. Let $\mathcal{M}_U := \mathcal{N}^{\pi_1}$, then $\mathcal{M}_U$ defines a vector bundle on $Y \times U$. Let $\nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega_{(Y\times\widetilde{U})/Y}$ be given by $\nabla(v \otimes f) = v \otimes df$. The connection ∇ commutes with the π_1-action, and we get an induced connection $(\mathcal{M}_U, \nabla_U)$ on $Y \times U$ with monodromy representation given by ρ.

Now we will extend $(\mathcal{M}_U, \nabla_U)$ to a connection on $Y \times \mathbb{P}^1$. Let $s \in S$, and let $O_s^* := \{z \in U | 0 < |z - s| < \varepsilon\}$ be a small neighborhood of a. The inverse image of O_s^* under the natural map $u : \widetilde{U} \to U$ consists of a number of connected components. Let $\widetilde{O}_s$ be one of them, then $u : \widetilde{O}_s \to O_s^*$ is a universal covering ([F77] Section 31.4). Let $\lambda \in \pi_1$ be a loop around s. The subgroup of π_1 mapping $\widetilde{O}_s$ to itself is cyclic with generator λ.

Lemma 4.5 *Let Y be an irreducible Stein manifold over $\mathbb{C}$ and $A \in \mathrm{M}_n(\mathbb{C})$ a matrix with the property that the differences of the eigenvalues of A are not in $\mathbb{Z} \setminus \{0\}$. If $M \in \mathrm{GL}_n(\mathcal{O}(Y))$ satisfies $M(y) \sim e^{2\pi i A}\ \forall\ y \in Y$, then there exists $B \in \mathrm{M}_n(\mathcal{O}(Y))$ with $M = e^{2\pi i B}$ and $B(y) \sim A\ \forall\ y \in Y$.*

Proof. Let $K := \mathrm{Frac}(\mathcal{O}(Y))$, and let $\mu_1, \cdots, \mu_p$ be the distinct eigenvalues of A. Write $\nu_j := e^{2\pi i \mu_j}$, then $\nu_1, \cdots, \nu_p$ are the distinct eigenvalues of M. We can make a decomposition $M = M_{ss}M_u$, with M_{ss} semi-simple and M_u unipotent. One can write M_{ss} and M_{ss}^{-1} as polynomials in M with coefficients in $\mathbb{C}$, so $M_{ss}, M_u \in \mathrm{GL}_n(\mathcal{O}(Y))$. Let $V_i := \ker(M_{ss} - \nu_i I, K^n)$, then $K^n = V_1 \oplus \cdots \oplus V_p$. For $w \in \mathcal{O}(Y)^n$ we can write $w = w_1 + \cdots + w_p$, with $w_i \in V_i$. Now $M_{ss}^m(w) = \nu_1^m w_1 + \cdots + \nu_p^m w_p \in \mathcal{O}(Y)^n, m \geq 0$. Using the fact that the Vandermonde matrix $\begin{pmatrix} 1 & \nu_1 & \cdots & \nu_1^p \\ \vdots & & & \vdots \\ 1 & \nu_p & \cdots & \nu_p^p \end{pmatrix}$ is invertible, we see that all $w_i \in \mathcal{O}(Y)^n$, so we can write $\mathcal{O}(Y)^n = \oplus W_i,\ W_i := \ker(M_{ss} - \nu_i I, \mathcal{O}(Y)^n)$.

Let $B_{ss} \in \mathrm{M}_n(\mathcal{O}(Y))$ be the linear map that acts as multiplication by μ_i on W_i, and let B_n be defined as the finite sum $\frac{1}{2\pi i}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(M_u - I)^j$. We will show that $B := B_{ss} + B_n$ satisfies the lemma. We have that $e^{2\pi i B_{ss}} = M_{ss}$ and $e^{2\pi i B_n} = M_u$ Since B_{ss} and B_n commute it is clear that $M = e^{2\pi i B}$. Furthermore $e^{2\pi i B(y)} \sim e^{2\pi i A}\ \forall\ y \in Y$, and the eigenvalues of $B(y)$ and A correspond. By construction we have $B(y) \sim A\ \forall\ y \in Y$. $\square$

We find that we can write $\rho(\lambda_i) = e^{2\pi i B_i},\ B_i \in \mathrm{M}_n(\mathcal{O}(Y))$, with $B_i(y) \sim A_i$ for all $y \in Y$. Let $s = s_i$ be the singular point we fixed, then we write B for B_i.

For notational convenience we replace the covering $u : \widetilde{O}_s \to O_s^*$ by the covering $exp : \mathbb{C} \to \mathbb{C}^*,\ z \mapsto e^{2\pi i z}$. The group $\mathrm{Aut}(\mathbb{C}/\mathbb{C}^*)$ is generated by $t : z \mapsto z + 1$. The restriction of $\mathcal{N}$ to $Y \times \mathbb{C}$ is $pr_1^{-1}(\mathcal{O}_Y^n)\widetilde{\otimes} pr_2^{-1}(\mathcal{O}_{\mathbb{C}})$, and we want to calculate $(\mathcal{N}^{\langle t\rangle}, \nabla)$ explicitly. So we have to calculate the action of $t \in \mathrm{Aut}(\mathbb{C}/\mathbb{C}^*)$ on $\mathcal{N}|_{Y\times\mathbb{C}}$. Let $v(y,z)$ be a section of $\mathcal{N}|_{Y\times\mathbb{C}}$. Using the explicit description of the π_1-action on $\mathcal{N}$ given in the beginning of this construction we find $t(v(y,z)) = e^{2\pi i B} v(y, z-1)$. Write $v(y,z) = e^{2\pi i Bz} w(y,z)$, then the condition $t(v) = v$ is equivalent to $w(y,z) = w(y, z-1)$. So $t(v) = v \Longleftrightarrow w(y,z) = \tilde{w}(y, e^{2\pi i z})$ for some section $\tilde{w}$ of $\mathcal{O}^n_{Y\times\mathbb{C}}$.

We find that $(\mathcal{N}|_{Y\times\mathbb{C}})^{\langle t\rangle} \cong \mathcal{O}^n_{Y\times\mathbb{C}^*}$ is a free vector bundle on $Y\times\mathbb{C}^*$ with generators $\{f_1,\cdots,f_n\}$, $f_i = e^{2\pi i Bz} e_i$, where $\{e_1,\cdots,e_n\}$ is the standard free basis for $\mathcal{O}^n_{Y\times\mathbb{C}^*}$. Furthermore ∇ is given by $\nabla(f_i) = 2\pi i B f_i dz$. We have that $u := e^{2\pi i z}$ is a parameter on $\mathbb{C}^*$, and we find $\nabla(f_i) = Bf_i\frac{du}{u}$. Using this formula, we can extend the connection $((\mathcal{N}|_{Y\times\mathbb{C}})^{\langle t\rangle}, \nabla)$ on $\mathbb{C}^*$ to a connection on $\mathbb{C}\supset\mathbb{C}^*$. In this way we can make an extension of $(\mathcal{M}_U, \nabla_U)$ to a connection on $Y\times\mathbb{P}^1$. □

In the following we want to construct a family of differential equations parametrized by a certain space of monodromy representations. Suppose we are given regular singular moduli problem in the sense of Chapter 2, with data $(V, \{s_1,\cdots,s_r\}, \{\frac{d}{dt_j}+\frac{C_j}{t_j}\}_{j=1}^r)$, $C_j\in\mathrm{GL}(V)$. Consider the set of corresponding monodromy representations $\mathbf{M} := \{\rho\in\mathrm{Repr}(\pi_1,V)|\rho(\lambda_j)\sim e^{-2\pi i C_j}\}$. We can identify $\mathbf{M}$ with the set $\{(M_1,\cdots,M_r)|M_j\sim e^{-2\pi i C_j}, \prod_{j=1}^r M_j = I\}$ by identifying ρ with $\{\rho(\lambda_1),\cdots,\rho(\lambda_r)\}$.

Lemma 4.6 *The set* $\mathbf{M}$ *is a Zariski constructible subset of* $\mathrm{GL}(V)^r$ *and, if all matrices* C_i *are diagonalizable, even Zariski closed. Furthermore the subset of* $\mathbf{M}$ *consisting of irreducible representations is also Zariski constructible.*

Proof. It clearly is sufficient to prove corresponding statements for the set $\mathbf{M}'$ obtained by dropping the condition $\prod_{i=1}^r M_i = I$. In proving the first statement, we may suppose $r=1$. For a diagonal matrix $C\in\mathrm{GL}(V)$ with characteristic polynomial $P_C = \prod(T-\mu_i)^{m_i}$, the set $\{ACA^{-1}|A\in\mathrm{GL}(V)\}$ is given by $\{B\in\mathrm{GL}(V)|P_B = P_C, \mathrm{rank}(B-\mu_i I) = n-m_i\ \forall\ i\}$. The latter condition is equivalent to the condition that the determinant of all $l\times l$-submatrices of $B-\mu_i I$, with $l>n-m_i$, is zero. This clearly defines a closed set. For an arbitrary matrix $C\in\mathrm{GL}(V)$, we have that B is similar to C if and only if $P_B = P_C$, and $\mathrm{rank}((B-\mu I)^m) = \mathrm{rank}((C-\mu I)^m), m = 1,\cdots,n$, for every eigenvalue μ. This defines a constructible set. To be precise,

$\mathrm{rank}(A) \leq m$ defines a closed subset of $\mathrm{GL}(V)$, so $\mathrm{rank}(A) = m$ defines a constructible subset.

We will now prove the second statement. Note that the set of matrices in $\mathrm{GL}(V)$ that leave a line $\mathbb{C} \cdot v,\ v \in V$ invariant is given by $\{M | Mv \wedge v = 0\}$, where $\wedge$ denotes the exterior product. So the set of tuples $(M_1, \cdots, M_s)$ that leave a line invariant is obtained by first taking the kernel of the map $V \setminus \{0\} \times \mathrm{GL}(V)^s \to \mathbb{C}^s,\ (v, M_1, \cdots, M_s) \mapsto (M_1 v \wedge v, \cdots, M_s v \wedge v)$ and then taking the projection of this kernel onto $\mathrm{GL}(V)^s$. This clearly defines a constructible set. The matrices that leave a subspace of dimension $l < n$ invariant, are the matrices that leave a decomposable line in $\bigwedge^l V$ invariant. This also defines a constructible set. Since the complement of a constructible set is constructible, this proves the lemma. $\square$

The family $\mathbf{M}$ of representations gives rise to a family of differential equations parametrized by $\mathbf{M}$, according to Theorem 4.3. In more detail, given $\lambda \in \pi_1$, a representation $m \in \mathbf{M}$ yields an element $m(\lambda) \in \mathrm{GL}(V)$. This defines a morphism $\rho(\lambda) : \mathbf{M} \to \mathrm{GL}(V)$ which we regard as an element $\rho(\lambda) \in \mathrm{GL}(\mathcal{O}(\mathbf{M}) \otimes V)$. We obtain a representation $\rho : \pi_1 \to \mathrm{GL}(\mathcal{O}(\mathbf{M}) \otimes V)$. By Theorem 4.3 the representation ρ gives rise to a family of differential equations $(\mathcal{M}, \nabla, V, \frac{d}{dt_i} + \frac{C_i}{t_i})$ parametrized by $\mathbf{M}$. For $m \in \mathbf{M}$, the monodromy representation of $(\mathcal{M}(m), \nabla(m))$ is clearly congruent to m. By the classical Riemann-Hilbert theorem, and Lemma 4.1 the connection $(\mathcal{M}(m), \nabla(m))$ is unique up to isomorphism.

We conclude this section by a lemma on the local invertibility of the exponential map. It states that under more general conditions than in Lemma 4.5 one can still locally construct a logarithm.

Lemma 4.7 *The map $E : M_d(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C}), A \mapsto e^{2\pi i A}$ is locally invertible in A if and only if $\lambda_i - \lambda_j \notin \mathbb{Z} \setminus \{0\}$ for all couples of eigenvalues λ_i, λ_j of A.*

Proof. We start by proving that if there are two eigenvalues λ_1, λ_2 of A with $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$, then E is not locally invertible. Write $A = SJS^{-1}$, with J in Jordan normal form. We will show that there exists a matrix $B \neq 0$, with $E(J + \varepsilon B) = E(J), \varepsilon^2 = 0$ (where we use the extension of E to a map on $M_n(\mathbb{C}[\varepsilon])$). If $E(J + \varepsilon B) = E(J)$ then also $E(A + \varepsilon SBS^{-1}) = E(A)$ holds. We can suppose that J has only two eigenvalues $\lambda, \lambda + m, m \in \mathbb{Z} \setminus \{0\}$ and only two Jordan blocks of size j and $d-j$ respectively. Subtracting $\lambda \cdot Id$ from J, we may assume that J has eigenvalues $0, m$. Define B by $B_{j+1,j} = 1$, and zeros everywhere else. Then $JB = mB$, $BJ = 0$. It follows that

$$E(J+\varepsilon B)-E(J) = (\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p})\varepsilon = (\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} m^{n-1} B)\varepsilon =$$

$$\frac{1}{m}(e^{2\pi i m} - 1)B\varepsilon = 0.$$

To proof the converse, again write $A = SJS^{-1}$. We will first consider the case where J is a diagonal matrix. For a matrix B with only one nonzero entry $B_{ij} = 1$, we have that $E(J+\varepsilon B)-E(J)$ also has (at most) one nonzero entry at the same place. The fact that the remaining coefficient is nonzero follows from an explicit calculation in the case $J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \notin \mathbb{Z}$. We conclude that the derivative of E at A is bijective. For the general case, write $J = D + N$, where D is diagonal, N is nilpotent, and $ND = DN$. We will use the matrix norm $\|A\| = \max\{|A_{ij}|, 1 \leq i, j \leq d\}$, which has the property $\|AB\| \leq d\|A\|\|B\|$. The idea of the proof is as follows. Local invertibility at J is equivalent to local invertibility at a conjugate SJS^{-1}. We can pick S such that $\|SNS^{-1}\|$ becomes arbitrary small. An estimate then shows that local invertibility at SDS^{-1} implies local invertibility at SJS^{-1}.

We have

$$E(J+\varepsilon B) - E(J) = (\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p})\varepsilon =$$

$$\varepsilon \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p)BJ^{n-1-p} + D^p B(J^{n-1-p} - D^{n-1-p} +$$

$$\varepsilon \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p}.$$

Write $a := \| \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p} \|$. Then $a > 0$ by the argument above. We can make the following estimate:

$$\| \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \| \geq$$

$$a - \|\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p) B J^{n-1-p} + D^p B (J^{n-1-p} - D^{n-1-p})\|.$$

Write $\delta := \|N\|, s := \|D\|, t := s + \delta$. We will now use the estimate

$$\|J^p - D^p\| = \|N \sum_{k=1}^{p} \binom{p}{k} D^{p-k} N^{k-1}\| \leq d^p \delta \sum_{k=1}^{p} \binom{p}{k} s^{p-k} \delta^{k-1} \leq d^p p t^{p-1} \delta.$$

Writing $b := \|B\|$, we find that

$$\|\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p}\| \geq$$

$$a - \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (p t^{p-1} b t^{n-1-p} \delta + (n-1-p) t^p b t^{n-p-2} \delta) =$$

$$a - \sum_{n\geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (n-1) t^{n-2} b\delta = a - (\sum_{n\geq 1} \frac{(2\pi i)^n}{n!} n(n-1)(dt)^{n-2}) db\delta =$$

$$a - (2\pi i)^2 \sum_{n\geq 0} \frac{(2\pi i)^n}{n!} (dt)^n) db\delta = a - (2\pi i)^2 e^{2\pi i dt} db\varepsilon.$$

So for any matrix B, we can pick a basis (and therefore a small δ), such that $\|E(J + \varepsilon B) - E(J)\| > 0$, which shows that E is locally invertible at J, and therefore at A. $\square$

For a vector bundle $\mathcal{M}$ obtained by Theorem 4.3, there can be points $y \in Y$ such that the induced vector bundle $\mathcal{M}(y)$ on $\mathbb{P}^1_{\mathbb{C}}$ is not free. This situation already appears in the Lamé example as we will see later on. Before we get to the Lamé example, we will first study connections on non-free vector bundles in detail.

4.3 Connections on non-free vector bundles

We will now give a precise description of connections on non-free vector bundles, and construct a fine moduli space for such connections.

Let $\mathcal{M} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, $a_1 \geq \cdots \geq a_n$ be a vector bundle, and $D = \sum_{i=1}^r k_i[s_i]$ a divisor of degree $k := \sum_{l=1}^r k_l$, with all $s_i \neq \infty$. We can write $\mathcal{O}(a_i)(U_0) = \mathbb{C}[z]e_i$, $\mathcal{O}(a_i)(U_\infty) = \mathbb{C}[z^{-1}]f_i$, with $f_i = z^{a_i}e_i$. A connection $\nabla : \mathcal{M} \to \Omega(D) \otimes \mathcal{M}$ is given by two connections on the free vector bundles $\mathcal{M}(U_0), \mathcal{M}(U_\infty)$, say $\nabla_0 : \mathcal{M}(U_0) \to \Omega(D)(U_0) \otimes \mathcal{M}(U_0)$ and $\nabla_\infty : \mathcal{M}(U_\infty) \to \Omega(D)(U_\infty) \otimes \mathcal{M}(U_\infty)$ that glue on $U_0 \cap U_\infty$. We have that $\Omega(D)(U_0) = \mathbb{C}[z]\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$ (where as always $t_l = z - s_l$), so ∇_0 is given by a $C[z]$-linear map A on $C[z]\langle e_1, \cdots, e_n\rangle$, taking $\nabla_0(e_i) = A(e_i)\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$. We wil also write A for the matrix of A on the basis $\{e_1, \cdots, e_n\}$. In the same way the connection ∇_∞ is defined by $\nabla_\infty(f_i) = B(f_i)\frac{dz}{\prod_{l=1}^r t_l^{k_l}}$, with B given by a matrix $B \in M_n(z^{k-2}\mathbb{C}[z^{-1}])$. For the connections ∇_0 and ∇_∞ to glue, we must have $\nabla_0(z^{a_i}e_i) = \nabla_\infty(f_i)$. This translates into $\prod_{l=1}^r t_l^{k_l} a_i z^{a_i-1} + z^{a_i}A_{ii} = z_i^a B_{ii}$ for $i = 1, \cdots, n$ and $z^{a_j}A_{ij} = z^{a_i}B_{ij}$ for $i, j = 1 \cdots n$, $i \neq j$. From this we obtain the following properties for A:

- $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$,
- $\deg(A_{ii}) = k - 1$,
- A_{ii} has as highest order coefficient $-a_i$.

Conversely, a matrix $A \in \mathrm{M}_n(\mathbb{C}[z])$ satisfying these properties defines a connection on $\mathcal{M}$.

In the following we will use the group of automorphisms of $\mathcal{M}$, so we give an explicit description of it. An automorphism ψ of $\mathcal{M}$ is given by a $C[z]$-linear automorphism of $\mathcal{M}(U_0)$ and a $C[z^{-1}]$-linear automorphism of $\mathcal{M}(U_\infty)$ that glue. So $\psi(U_0)$ is given on the basis $\{e_1, \cdots, e_n\}$ by a matrix $A \in \mathrm{GL}_n(C[z])$. Furthermore $\psi(U_\infty)$ is given on the basis $\{f_1, \cdots, f_n\}$ by a matrix $B \in \mathrm{GL}_n(C[z^{-1}])$ with $B = Z^{-1}AZ$, where Z is the diagonal matrix with $Z_{ii} = z^{a_i}$. Let $a_{n_1}, \cdots, a_{n_p}$ be the subsequence of $a_1, \cdots, a_n$ consisting of a_1 and the a_i with $a_i - a_{i-1} < 0$. Then we can write A in block form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & A_{pp} \end{pmatrix}.$$

Here the $A_{ii} \in \mathrm{GL}_{n_{i+1}-n_i}(C)$ (where we take $n_{p+1} = n+1$) and the coefficients of $A_{ij},\ i > j$ are polynomials of degree $\leq a_{n_i} - a_{n_j}$. Conversely any such matrix A defines an automorphism of $\mathcal{M}$.

4.3.1 Moduli spaces of non-free connections

We will define a moduli space of connections on a vector bundle of some fixed type associated to a data set $(V, \{s_1, \cdots, s_r\}, \{\frac{d}{dz} + B_i\}_{i=1}^r)$ as in Chapter 2. We fix an ordered basis for V, say $\{e_1, \cdots, e_n\}$. Define a vector bundle $\mathcal{M}$ of type $(a_1, \cdots, a_n),\ a_1 \geq \cdots \geq a_n$ by $\mathcal{M}(U_0) = \mathbb{C}[z] \otimes V$, $\mathcal{M}(U_\infty) = \mathbb{C}[z^{-1}] \otimes (\oplus \mathbb{C} z^{a_i} e_i)$. We fix a type $(a_1, \cdots, a_n)$ with $a_1 - a_n \leq r-1$, and we will only consider connections on the corresponding vector bundle $\mathcal{M}$. Note that in case $\mathcal{M}$ has rank 2, and there exists an irreducible connection on $\mathcal{M}$, then by [PS03] Proposition 6.21 we get $a_2 - a_1 \leq r - 2$.

We start by defining a functor $\mathcal{F}^+$ in a similar way to the definition of $\mathcal{F}$ in Chapter 2, but now we do not divide out equivalence.

Definition 4.8 The functor $\mathcal{F}^+ : \{\mathbb{C}\text{-algebras}\} \to \{\text{sets}\}$ is defined as follows. For any $\mathbb{C}$-algebra R, the set $\mathcal{F}^+(R)$ consists of the tuples $(A, \{\phi_i\}_{i=1}^r)$, where:

- $A \in M_n(R[z])$ satisfies $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$ and $\deg(A_{ii}) = k - 1$. Furthermore A_{ii} has as highest order coefficient $-a_i$.
- the $\phi_i = \sum_{j=0}^{\infty} \phi_i(j)(t_i)^j$, $\phi_i(j) \in M_n(R)$ are automorphisms of $R[[t_i]]^n$.
- $\phi_i(\frac{d}{dz} + \frac{A}{\prod_{l=1}^r t_l^{k_l}})\phi_i^{-1} = \frac{d}{dz} + B_i,\ i = 1, \cdots, s$, where we see $\frac{A}{\prod_{l=1}^r t_l^{k_l}}$ and ϕ_i as elements of $\mathrm{End}(R[[t_i]][t_i^{-1}]^n)$. This condition can be restated as $\phi_i' = \phi_i \frac{A}{\prod_{l=1}^r t_l^{k_l}} - B_i \phi_i$. •

This functor $\mathcal{F}^+$ is represented by a $\mathbb{C}$-algebra of finite type U, as can be shown in a way similar to the proof of Theorem 2.9. We can also consider $\mathcal{F}^+$ as a contravariant functor on schemes of finite type over $\mathbb{C}$. In this setting $\mathcal{F}^+$ is represented by $\mathbb{M} := \mathrm{Spec}(U)$.

We say that two tuples $(A_1, \{\phi_i^1\}), (A_2, \{\phi_i^2\}) \in \mathcal{F}^+(R)$ are equivalent if there exists an automorphism ψ of $\mathcal{M}\otimes R$ such that $\frac{d}{dz}+\frac{A_2}{\prod_{l=1}^r t_l^{k_l}} = \psi^{-1}(\frac{d}{dz}+\frac{A_1}{\prod_{l=1}^r t_l^{k_l}})\psi$ and $\phi_i^2 = \phi_i^1 \circ \psi,\ i = 1, \cdots, r$ where we consider ψ as an element of $\mathrm{GL}_n(R[z])$ and $\mathrm{GL}_n(R[[t_i]])$ respectively. We define a functor $\mathcal{F}$ by $\mathcal{F}(R) = \mathcal{F}^+(R)/\sim$.

Theorem 4.9 *There is a coarse moduli scheme for the functor $\mathcal{F}$ defined above, which is in fact a quasi projective variety.*

Proof. Consider the group $G := \mathrm{Aut}(\mathcal{M})$. this group acts on $\mathbb{M}(\mathbb{C})$ and we want to make a quotient. We can make an embedding $G \subset \mathrm{GL}_n(\mathbb{C}[z])$. From the description of G above we see that the degree of the coefficients of elements of G is bounded by $\max_{i=1\cdots n-1}(a_i - a_{i+1})$. By our assumption on $\mathcal{M}$ this bound is less or equal to $r-2$. Therefore the map $\psi : G \to \mathrm{GL}_n(\mathbb{C})^r$ given by $A(z) \mapsto (A(s_1), \cdots, A(s_r))$ is injective. In this way we can consider G as a linear algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})^r$. By [Br69] Theorem 6.8, the quotient $\mathrm{GL}_n(\mathbb{C})^r/G$ exists and is given by $(Q, \pi),\ \pi : \mathrm{GL}_n(\mathbb{C})^r \to Q$, with Q a quasi-projective variety. Let $\phi : \mathbb{M} \to \mathrm{GL}_n(\mathbb{C})^r,\ (A, \{\phi_i\}) \mapsto (\phi_1(0), \cdots, \phi_r(0))$ be the projection map. We want to use the following proposition to prove that a geometric quotient $\mathbb{M}/G$ exists and is quasi-projective.

Proposition 4.10 (Proposition 7.1 of [MFK94])
Let G be a group scheme, flat and of finite type over S. Let X and Y be schemes of finite type over S, let σ and τ be actions of G on X and Y, and let $\phi : X \to Y$ be a G-linear morphism. Assume that Y is a principal fibre bundle over an S-scheme Q, with group G, and with projection $\pi : Y \to Q$. Assume that there exists an $L \in Pic^G(X)$ which is relatively ample for ϕ, and that Q is quasi-projective over S. Then there is a scheme P, quasi-projective over S, and an S-morphism $\omega : X \to P$ such that X becomes a principal fibre bundle over P with group G, and projection ω.

This needs some explanation. A principal fibre bundle is defined as follows: let $\sigma : G \times_S X \to X$ be an action, with a geometric quotient (Q, π), then X is a principal fibre bundle over Q with group G if

- π is a flat morphism of finite type,
- the map $(\sigma, pr_2) : G \times_S X \xrightarrow{\sim} X \times_Q X \subset X \times_S X$ is an isomorphism.

By Proposition 0.9 of [MFK94] for a free action of an algebraic group G on an algebraic scheme X with geometric quotient (Q, π), the scheme X always is a principal fibre bundle over Q with group G.

We further remark that $Pic^G(X)$ is the group of G-linearized line bundles on X. For details see [MFK94].

We want to apply this proposition with $S = \mathrm{Spec}(\mathbb{C})$, $X = \mathbb{M}$, $Y = \mathrm{GL}_n^r$, and G, ϕ, Q, π as above. There are a number of conditions to be checked.

(1) ϕ is G-linear.

(2) GL_n^r is a principal fibre bundle over Q with group $\mathrm{Aut}(M)$.

(3) There exists an L as in the proposition.

Condition (1) is clearly fulfilled. For the line bundle L in (3) we can take the trivial line bundle since $\mathbb{M}$ is affine. By Proposition 0.9 of [MFK94] for a free action of an algebraic group G on an algebraic scheme Y with geometric quotient (Q, π), the scheme Y always is a principal fibre bundle over Q with group G. So to prove (3) it suffices to show that the action of G on GL_n^r is free, and that Q is a geometric quotient. The action being free means that $(\sigma, pr_2) : G \times \mathrm{GL}_n^r \to \mathrm{GL}_n^r \times_Q \mathrm{GL}_n^r$ is a closed immersion, which is the case. The fact that (Q, π) is a geometric quotient follows from the definition of a quotient in [Br69].

We will now proof that P is a coarse moduli scheme for $\mathcal{F}$ by an argument as in the proof of Proposition 5.4 of [MFK94]. There is a natural isomorphism $\phi^+ : \mathcal{F}^+ \to \mathrm{Hom}(*, P)$, which induces a natural isomorphism $\phi : \mathcal{F} \to \mathrm{Hom}(*, P)$. For (P, ϕ) to be a coarse moduli space, the folowing conditions have to be verified.

- for every algebraically closed field k, the map

  $$\phi(\mathrm{Spec}\ k) : \mathcal{F}(\mathrm{Spec}\ k) \to \mathrm{Hom}(\mathrm{Spec}\ k, P)$$

  is an isomorphism.

- given a scheme N and a morphism $\psi : \mathcal{F} \to \mathrm{Hom}(*, N)$, there is a unique morphism $\chi : \mathrm{Hom}(*, P) \to \mathrm{Hom}(*, N)$, such that $\psi = \chi \circ \phi$.

The first condition is verified since (P, ω) is a geometric quotient. To proof that the second condition is verified, consider the element $\overline{id} \in \mathcal{F}(\mathbb{M})$ induced by $id \in \mathcal{F}^+(\mathbb{M}) \cong \mathrm{Hom}(\mathbb{M}, \mathbb{M})$. To a morphism $\psi : \mathcal{F} \to \mathrm{Hom}(*, N)$, we associate the morphism $f := \psi_{\mathbb{M}}(\overline{id}) : \mathbb{M} \to N$. This induces a bijection of the set of morphisms from $\mathcal{F}$ to representable functors and the set of morphisms $f : \mathbb{M} \to N$ with N a scheme. It follows that the second condition is verified, and therefore (P, ϕ) is a coarse moduli space. $\square$

4.4 The Lamé equation

We will now consider the moduli problem with data

$$(\{s_1, \cdots, s_4\}, \{\frac{d}{dt_i} + \frac{1}{t_i}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\}_{i=1}^4).$$

The corresponding set of "monodromy representations" $\mathbf{M}$ defined above is given by $\mathbf{M} = \{(M_1, \cdots, M_4) | M_i \sim \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \prod_{i=1}^4 M_i = 1\}$. Write $M_i := \left(\begin{smallmatrix} p_i & q_i \\ r_i & -p_i \end{smallmatrix}\right)$, $i = 1, 2, 3$, then he coordinate ring of $\mathbf{M}$ is given by $R := \mathbb{C}[p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3]/I$,

$$I = \langle p_1^2 + q_1 r_1 + 1, p_2^2 + q_2 r_2 + 1, p_3^2 + q_3 r_3 + 1, f \rangle$$

$$f := -p_3 q_2 r_1 + p_2 q_3 r_1 + p_3 q_1 r_2 - p_1 q_3 r_2 - p_2 q_1 r_3 + p_1 q_2 r_3.$$

The following properties of $\mathbf{M}$ are known (but also easily verified).

- $\mathbf{M}$ is a five dimensional variety.
- The group $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbf{M}$ (by conjugation) and on its coordinate ring R. The ring $R^{\mathrm{PGL}_2} = \mathbb{C}[t_1, t_2, t_3]/\langle t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 - 4 \rangle$ is the ring of invariants, where $t_1 := \mathrm{Tr}(M_2 M_3)$, $t_2 := \mathrm{Tr}(M_1 M_3)$, $t_3 := \mathrm{Tr}(M_1 M_2)$. This follows immediately from [Bo03] Section 2.
- The variety $\mathbf{M}^{\mathrm{PGL}_2} := \mathrm{Spec}(R^{\mathrm{PGL}_2})$ has 4 singular points, namely $(t_1, t_2, t_3) \in \{(-2, 2, 2), (2, -2, 2), (2, 2, -2), (-2, -2, -2)\}$. Each one of these points corresponds to multiple $\mathrm{PGL}_2(\mathbb{C})$-orbits. After deleting the 4 singular points and their preimages in $\mathbf{M}$ we obtain a good quotient under $\mathrm{PGL}_2(\mathbb{C})$. In particular $\mathbf{M}$ is reduced and irreducible.

- The preimage of the 4 singular points of $\mathbf{M}^{\mathrm{PGL}_2}$ in $\mathbf{M}$ precisely consists of all reducible representations in $\mathbf{M}$.
- The complement $\mathbf{M}_{\mathrm{ir}}$ is a smooth connected variety.

By Theorem 4.3 we obtain a family of differential equations parametrized by $\mathbf{M}$, say $(\mathcal{M}, \nabla, \mathbb{C}^2, \text{ local data })$. For every irreducible representation $m \in \mathbf{M}$, the following lemma shows that $\mathcal{M}(m)$ is either free, or of type $(1, -1)$.

Lemma 4.11 *Let $(\mathcal{M}, \nabla)$ be an irreducible connection of rank 2 on $\mathbb{P}^1_{\mathbb{C}}$ with four singular points such that the sum of the local exponents at each singular point is 0. Then the vector bundle $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \in \{0, 1\}$.*

Proof. Because the sum of the local exponents of $(\mathcal{M}, \nabla)$ is zero at each singular point, the induced connection $\bigwedge^2 \nabla$ on $\bigwedge^2 \mathcal{M}$ is everywhere regular. Since $\mathbb{P}^1_{\mathbb{C}}$ is simply connected, $\bigwedge^2 \mathcal{M}$ is the trivial line bundle, and $\bigwedge^2 \nabla$ is the trivial connection. So $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \geq 0$. By [PS03] Proposition 6.21, the defect of $\mathcal{M}$ is ≤ 2. This proves the lemma. $\square$

We will now show that the set $\mathbf{M}^{(1,-1)} := \{m \in \mathbf{M}_{\mathrm{ir}} | \mathcal{M}(m) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)\}$ is nonempty.

Proposition 4.12 $\mathbf{M}^{(1,-1)}$ *is a non-trivial divisor in* $\mathbf{M}_{\mathrm{ir}}$.

Proof. By Remark 3.12 (4) we have that $\mathbf{M}^{(1,-1)}$ is a divisor. So we only need to show that $\mathbf{M}^{(1,-1)}$ is non-empty. As we saw in Section 4.2 the connection $(\mathcal{M}(m), \nabla(m))$ is uniquely determined for every $m \in \mathbf{M}$. Therefore we only have to construct a connection on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ with the correct local behavior. By the description of connections on non-free vector bundles above, we get that such a connection is given by a matrix A of the form

$$A = \begin{pmatrix} a_0 + a_1 z + a_2 z^2 - z^3 & b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 \\ c_0 & d_0 + d_1 z + d_2 z^2 + z^3 \end{pmatrix}.$$

We want that the connection is locally at the points s_i formally isomorphic to $\frac{d}{dz} + \frac{1}{t_i}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$, so the Laurent series expansion of $\frac{A}{\prod_{l=1}^{r} t_l^{k_l}}$ at a point s_i has to be of the form $\frac{A_i}{t_i}$ + h.o.t., with A_i similar to $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$. The

A_i are of the form $\begin{pmatrix} p_i & q_i \\ r_i & -p_i \end{pmatrix}$, $p_i^2 + q_i r_i = \frac{1}{16}$ for all i, and we can write $\frac{A}{\prod_{l=1}^r t_l^{k_l}} = \sum_{i=1}^4 \frac{A_i}{t_i} + \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix}$. We find that $p_1 + p_2 + p_3 + p_4 + 1 = 0$, and r_2, r_3, r_4 are multiples of r_1. So we get a 5-dimensional family of tuples $(A_1, \cdots, A_4, b_4)$, and hence a 5-dimensional family X of connections on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

The automorphism group G of $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is $\{\begin{pmatrix} a & b + cz + dz^2 \\ 0 & e \end{pmatrix}, \ a, e \neq 0\}$. A connection given by a matrix A is equivalent to the one given by the matrix $\tilde{A} = \Phi^{-1}\Phi' + \Phi^{-1}A\Phi$, with $\Phi \in G$. We can construct a one dimensional subfamily of X consisting of matrices of the form $\begin{pmatrix} -z^3 & * \\ 1 & z^3 \end{pmatrix}$ parameterized by b_4 with no equivalent elements. In case $s_i = i, \ i = 1, \cdots, 4$ this family is

$$\{\begin{pmatrix} -z^3 & \frac{6265}{4} - 3015z + 1800z^2 - 350z^3 + b_4(24 - 50z + 35z^2 - 10z^3 + z^4) \\ 1 & z^3 \end{pmatrix}, \ b_4 \in \mathbb{C}\}.$$

$\square$

We remark that the above does not imply that $\mathbf{M}^{(1,-1)}$ is 1-dimensional. Indeed, let $\mathcal{M}$ be the vector bundle on $\mathbb{P}^1_{\mathbf{M}}$ given by the Riemann-Hilbert construction. The type of $\mathcal{M}(m), \ m \in \mathbf{M}$ is $(0,0)$ or $(1,-1)$, and since $\mathbf{M}$ is irreducible we find by Remarks 3.12 that $\mathbf{M}^{(1,-1)}$ is an analytic divisor on $\mathbf{M}$ (since $\mathbf{M}^{(1,-1)}$ is nonempty). We find that $\mathbf{M}^{(1,-1)}$ is 4-dimensional, and so there are isomorphic connections in $\mathbf{M}^{(1,-1)}$.

We can make similar calculations (which are in fact simpler) for the case of a free vector bundle. We get a 5-dimensional space of connections, with an action of the group SL_2. There exists a categorical quotient, which maps all reducible connections to four points. This 2-dimensional quotient is actually a geometric quotient on the space of irreducible connections.

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List of Notation

Index

Samenvatting

In deze samenvatting zal ik proberen op, zeg maar, "middelbare school niveau" uit te leggen waar dit proefschrift over gaat. Voor een nauwkeuriger beschrijving, die echter wel wat meer voorkennis vereist, verwijs ik naar de *Introduction*. Maar om echt te begrijpen wat er gebeurt zal men toch de hoofdtekst in moeten duiken.

Dit proefschrift heet "Algoritmen en moduliruimten voor differentiaalvergelijkingen". Ik zal de woorden in deze titel duidelijk proberen te maken.

Om een differentiaalvergelijking te kunnen maken moet je kunnen differentiëren (ook wel "afgeleide nemen" genoemd). Het meest eenvoudige voorbeeld hiervan is het differentiëren van een functie op de "getallenlijn", die door wiskundigen wordt genoteerd als $\mathbb{R}$. De afgeleide functie geeft aan hoe steil de grafiek van zo'n functie is. Het is wel duidelijk dat je niet van elke functie de afgeleide kunt nemen. Bijvoorbeeld de functie die breuken naar 1 en andere getallen naar 0 stuurt, springt wild op en neer. Je kunt de grafiek niet eens met één pennenstreek tekenen, laat staan dat je kunt zeggen hoe steil deze functie is in een punt. Zelfs van functies die je wel met één pennenstreek kunt tekenen, is het begrip "steilheid" niet altijd gedefinieerd (denk aan functies met een knik, je kunt niet zeggen hoe steil de functie in zo'n knik is).

Als je een verzameling functies hebt, dan kun je je afvragen of een gegeven functie in die verzameling de afgeleide is van een andere functie in die verzameling. Bijvoorbeeld de constante functie met waarde 1 is de afgeleide van de functie $f(x) = x$. Als je de afgeleide van en functie f schrijft als f' dan vinden we dus dat de *differentiaalvergelijking* $f' = 1$ als oplossing x heeft.

Een eenvoudige verzameling van functies op $\mathbb{R}$ zijn de zogenaamde polynoomfuncties. Dit zijn functies als x^2, x^4+x+1, etc. De afgeleide van zo'n functie is ook altijd weer een polynoomfunctie. We kunnen ook de afgeleide van een afgeleide nemen, en dit schrijven we als f'', dus $(x^2)'' = (2x)' = 2$. De differentiaalvergelijking $f'' = 2$ heeft dus als oplossing $f = x^2$, maar ook alle andere polynomen van de vorm $x^2 + a \cdot x + b$ met b, c constantes, zijn een oplossing. Het aardige is dat bijvoorbeeld de differentiaalvergelijking $f' = f$ geen oplossing (ongelijk 0) heeft binnen de polynoomfuncties. Dit kun je zien als je weet hoe je afgeleides van polynomen moet nemen. Er blijkt dat de graad (ofwel de hoogste macht van x) bij het afgeleide nemen met 1 afneemt, dus de afgeleide van een functie kan nooit gelijk zijn aan zichzelf (op de functie 0 na). Er bestaat wel zo'n functie, namelijk e^x, maar dat is geen polynoom functie. We zien dus dat om differentiaalvergelijkingen op te kunnen lossen het nodig kan zijn om je verzameling van functies te vergroten. Hoofdstuk 1 gaat erover hoe je dit moet doen voor differentiaalvergelijkingen van de vorm $f'' + a \cdot f' + b \cdot f = 0$, met a en b bepaalde functies. Zulke differentiaalvergelijkingen noemen we lineaire tweede orde vergelijkingen.

Er zijn een aantal simpele differentiaalvergelijkingen, die we standaard noemen met de volgende eigenschap. Elke lineaire tweede orde vergelijking die voldoet aan een bepaalde voorwaarde, is te schrijven als een variant van een standaard vergelijking (dit is Kleins stelling). Hierdoor zijn ook de oplossingen van zo'n vergelijking te schrijven als varianten van de oplossingen van die standaard vergelijking. Deze methode (ofwel dit *algoritme*) is door Mark van Hoeij geprogrammeerd voor het wiskunde softwarepakket "Maple".

In hoofdstuk 2 wordt gekeken naar families van differentiaalvergelijkingen. Deze families bestaan uit vergelijkingen die op een of andere manier wat met elkaar gemeen hebben. Als de elementen van zo'n familie allemaal speciale gevallen zijn van een soort "moedervergelijking" dan noemen we deze familie een *moduliruimte*. We kunnen voorwaarden opstellen zodanig dat de verzameling van alle differentiaalvergelijkingen die voldoen aan die voorwaarden een moduliruimte vormen.

Bij een differentiaalvergelijking hoort iets wat we een groep noemen, en die groep geeft informatie over hoe ingewikkeld de oplossingen van de vergelijking zijn. Deze groep heet de differentiaal Galois groep. Binnen een familie van differentiaalvergelijkingen kunnen we kijken naar de verzameling vergelijkingen waarvoor deze groep hetzelfde is. In hoofdstuk 3 wordt iets gezegd over hoe dit soort verzamelingen eruit zien.

Het laatste hoofdstuk behandeld een oud probleem, namelijk het zogenaamde Riemann-Hilbert probleem. Hilbert was een beroemd wiskundige, die in 1900 een lijst opstelde met problemen waarvan hij dacht dat ze bepalend zouden zijn voor de ontwikkeling van de wiskunde. Een van deze problemen, om precies te zijn nummer 21, is bekend komen te staan als het Riemann-Hilbert probleem (Riemann is zo mogelijk een nog vooraanstaander wiskundige). Er is een sterke en een zwakke versie van dit probleem. De sterke versie is alleen waar in bepaalde gevallen, en de zwakke versie is gewoon waar. Dit Riemann-Hilbert probleem gaat over het vinden van een differentiaalvergelijking bij een zekere "monodromie" afbeelding. Ik zal dit hier niet verder uitleggen, maar één van de nieuwe resultaten in dit proefschrift is dat het ook mogelijk is om voor een familie van monodromie afbeeldingen een bijpassende familie van differentiaalvergelijkingen te maken.

Dankwoord

Om te beginnen, Marius, ontzettend bedankt. Ik zal het niet zo gek maken als Richard in zijn afstudeerwerk, die je zo ongeveer trouw tot in den dood beloofde, maar niettemin is mijn dank groot. Ik heb ontzettend veel van je geleerd, en jouw ideeën zijn van grote invloed geweest op dit proefschrift. Door mijn gefiets heeft mijn promotie wel iets langer geduurd dan eerst gepland, en ik neem aan dat je wel eens gewild zult hebben dat ik wat meer wiskunde en wat minder wielrennen deed. Ik vind echter dat je hierover zeer begripvol bent geweest, en het heeft in ieder geval een zeer plezierige omgang niet in de weg gestaan.
Jaap, Ook jouw ben ik heel veel dank verschuldigd. Toen ik tijdens mijn VWO opleiding alvast wat wiskunde wilde gaan doen, ben jij daarbij zeer motiverend en behulpzaam geweest. Mede dankzij jouw heb ik een jaar in Frankrijk gestudeerd (waar ik wel besmet ben geraakt met het "virus du vélo"). Tijdens mijn AIO-tijd heb jij veel tijd besteed aan het verbeteren van mijn schrijfstijl. Ik hoop dat je tevreden bent met het uiteindelijke resultaat. Verder hebben we grote stukken van het proefschrift nog eens doorgenomen, en op veel punten verbeterd. Je bent ook altijd bereid geweest me te helpen bij het oplossen van wiskundige problemen van allerhande aard. Dank hiervoor.
De tijd als AIO in Groningen is me zeer goed bevallen, en dat is voor een groot deel te danken aan de goede sfeer binnen de AIO groep. De lunches en koffiepauzes waren vaak erg gezellig. Arie, Barteld, Conny, Dirk-Jan, Ena, Erwin, Geert, Gert-Jan, Henk, Irene, Jasper, Jeroen, Joost, Jun, Lenny, Marc, Martijn, Remke, Renato, Richard, Rick, Robert & Robert, Simon en Theresa hiervoor hartelijk bedankt. Barteld en Remke moet ik ook nog bedanken voor het verschaffen van riante woonruimte, en Richard voor zijn last-minute mechaniekerschap. Dankzij Geert kon ik ook op het instituut af

en toe nog eens over wielrennen praten. Robert, bedankt voor het zijn van een leuke kamergenoot. Ik heb vast ook nog een paar woorden Duits van je opgestoken, wat momenteel erg handig is.
Tot slot, Karst en Saakje, bedankt voor jullie interesse, ook al valt het niet altijd mee om wat zinnigs te zeggen over mijn onderzoek.
Hier zal ik het bij laten, ik ben natuurlijk ook nog dank verschuldigt aan veel andere mensen die mijn AIO-tijd veraangenaamd hebben, maar een opsomming daarvan wordt wat te gortig. Het moet wel leuk blijven.