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CYCLIC VECTORS

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A differential structure on an n-dimensional vector space V over a field K is a pair (δ, D), where δ is a derivation on K and $D : V \to V$ is an additive mapping satisfying the Leibniz (product) rule. A cyclic vector is a vector v such that $(v, Dv, \ldots, D^{n-1}v)$ is a basis of V. In this note we offer two existence proofs for cyclic vectors under very mild hypotheses on K. The first proof is in the form of an algorithm; the second includes a simple description of the collection of all cyclic vectors. A MAPLE implementation of the algorithm for the case $(K, \delta) = (\mathbb{Q}(x), \frac{d}{dx})$ is detailed in the Appendix.

1 Introduction

Let K be a field and V a K-space (*i.e.*, a vector space over K). By a *differential structure* on V we mean a pair (δ, D), where $\delta : k \mapsto k'$ is a derivation on K, *i.e.*, K is a differential field, and $D : V \to V$ is an additive group homomorphism satisfying the *Leibniz rule*

$$D(kv) = k'v + kDv \qquad (k \in K,\, v \in V).$$

The triple (V, δ, D) is called a *differential vector space* or *differential system.* In this paper we restrict our attention to finite dimensional differential systems, $\dim V = n$.

A vector $v \in V$ is *cyclic* if $(v, Dv, \ldots, D^{n-1}v)$ is a basis for V. The Cyclic Vector Theorem asserts the existence of such a vector under quite mild assumptions on K.

The earliest proof known to these authors is found in [23], but is restricted to fields of meromorphic functions. The argument is by contradiction. The second proof appears in [10]. That treatment is constructive, in fact almost an algorithm, but is specific to the field of rational functions.

Many other proofs have since appeared, perhaps the most influential being that of Deligne [12]. His non-constructive approach apparently caused some consternation: Adjamagbo [1], in the abstract, remarks that the "*reductio ad absurdum* [argument of Deligne] ... caused much ink to flow".

Dabèche [11] has a constructive proof using special properties of the field of meromorphic functions (poles, evaluation, etc.). Other constructive arguments can be found in [6], [8], and [13] (using methods attributed to Ramis [25]).

Differential algebraists first learned of the Cyclic Vector Theorem, and of Loewy's proof, from [21]. A generalization to partial differential fields was presented in [28]. It is amusing to note that the "New York school" knew of Loewy but not of Cope. The "French school" knew of Cope but not of Loewy. And neither knew of the work of the other!

Using entirely different ideas, Katz [19] found cyclic vectors *locally* on $\mathrm{Spec}(R)$ for quite general differential rings. However, for the Cyclic Vector Theorem itself his proof requires strong hypothesis (see Remark 4 on page 68): K must be a differential field in which $(n-1)!$ is invertible and which is finitely generated over an algebraically closed field of constants (for example $K = \mathbb{C}(x)$).

Hilali [16] computes several invariants of a differential system avoiding use of the cyclic vector theorem, which he claims is "très coûteuse en pratique" (middle of page 405).

Adjamagbo [1] gives an algorithm for finding cyclic vectors over differential division rings. He uses ideas from the theory of modules over the non-commutative ring of linear differential operators (generalizations of Weyl algebras); in particular, left and right Euclidean division.

Adjamagbo [1] also sketches a novel (albeit non-trivial) proof of the Cyclic Vector Theorem in Remark 2, page 546. He refers to a paper in preparation that explains the connection between cyclic vectors and primitive elements. With those results, the Cyclic Vector Theorem follows from the existence of a primitive element [20, Proposition 9, p. 103]. The paper was published later as [2]; it relies on theorems of [26] and [27].

Barkatou [4] solves a related problem and, in Remark 3, p. 192, claims that the same method can be used to find a cyclic vector. §5 of that paper deals with the cyclic vector algorithm in more detail. Working over $\mathbb{C}(x)$ he states that it is always possible to find a cyclic vector whose components are polynomials of degree less than n, but does not give a specific reference. He also states that it is "well-known" that the probability that a random vector is cyclic is 1. His algorithm, at the bottom of page 194, is:

Algorithm 1.1 To find a cyclic vector:

Step 1. Choose a vector with random polynomial coefficients.

Step 2. If the vector is cyclic, we are done.

Step 3. Otherwise, go to Step 1.

Theoretically this "algorithm" could fail to terminate; in practice it generally provides a cyclic vector in just a few steps. The reason is easy to explain: such vectors form a non-empty open (and therefore dense) subset of affine space in the Kolchin topology (see Theorem 7.3).

Algorithm 1.1 seems to be the basis for the implementation in MAPLE. The `DETools` package (attributed to Mark van Hoeij) appears to first try $(1, 0, \ldots, 0)$, and then $(p_1, \ldots, p_n)$ where the p_i are random polynomials, with coefficients from the set $\{-1, 0, 1\}$, of degree one higher than in the previous trial.

In this note we offer two proofs of the Cyclic Vector Theorem with no restriction on the characteristic. The first (found in §3) is similar in spirit to that given in [12], but is effective and to these authors seems considerably more elementary than any of the constructive proofs referenced above. When K contains the real field the method enables one to produce a cyclic vector which can be viewed as an arbitrarily small perturbation $v + \sum_{j=2}^{n} \epsilon_j v_j$ of any preassigned $v \in V$.

We discuss the details of the algorithm, and its complexity, in §4. The Appendix contains a MAPLE implementation for the case $(K, \delta) = (\mathbb{Q}(x), \frac{d}{dx})$.

The second proof (found in §7) is influenced by [10], but is here cast in a more general setting. In §5 we remark on the hypotheses of our algorithm; in §6 we show they cannot be weakened.

We view a differential structure as a coordinate-free formulation of a first order (system of) homogeneous linear differential equation(s); proving the existence of a cyclic vector for the associated adjoint system is then seen to be equivalent to converting the given equation to an n-th order homogeneous form. The ideas are reviewed in §2 (or see [29], pp. 131–2).

The explicit construction of cyclic vectors is important for calculating invariants of differential systems at singular points. For example, Varadarajan [29, p. 152–3] explains how cyclic vectors can be used to compute principal levels ("Katz invariants") of connections. Levelt [22] makes essential use of cyclic vectors to find normal forms for differential operators. Additional applications can be found in [5], [7], [14], [16] and [17]. Hilali [15] contrasts the efficiency of particular algorithms with and without the use of cyclic vectors.

2 Linear Differential Equations

Throughout this paper we assume (V, δ, D) is a finite dimensional differential system. Thus $\delta : k \mapsto k'$ is a derivation on K, V is K-space of finite dimension $n > 0$, and $D : V \to V$ is a differential structure. We follow custom and when $v \in V$ write $D(v)$ as Dv unless confusion might otherwise result. The space (Lie algebra) of $n \times n$ matrices with entries in K is denoted by $\mathrm{gl}(n, K)$.

Fix a basis $\mathbf{e} = (e_1, \ldots, e_n)$ of V. We may use $\mathbf{e}$ to associate a matrix $A = (a_{ij}) \in \mathrm{gl}(n, K)$ with D by means of the formula

$$De_j = \sum_{i=1}^{n} a_{ij} e_i \,, \tag{2.1}$$

and the correspondence $D \mapsto A$ is easily seen to be bijective. We refer to A as the *defining* $\mathbf{e}$-*matrix* of D. If for any $v = \sum_j v_j e_j \in V$ we let $v_{\mathbf{e}}$ (resp. $v'_{\mathbf{e}}$) denote the column vector with j-th entry v_j (resp. v'_j) we then have

$$(Dv)_{\mathbf{e}} = v'_{\mathbf{e}} + A v_{\mathbf{e}} \,. \tag{2.2}$$

This last equality explains how Algorithm 1.1 in the Introduction can be implemented: choose any non-zero $v \in V$ and form the matrix

$$\begin{pmatrix} v_1 & (Dv)_1 & \cdots & (D^{n-1}v)_1 \\ \vdots & \vdots & & \vdots \\ v_n & (Dv)_n & \cdots & (D^{n-1}v)_n \end{pmatrix}; \tag{2.3}$$

if the determinant does not vanish the vector v is cyclic; otherwise choose a different non-zero v and try again; etc.

Vectors annihilated by D are said to be *horizontal* (with respect to D). Notice from (2.2) that a vector $v \in V$ is horizontal if and only if $v_{\mathbf{e}}$ is a solution of the first order (system of) homogeneous linear equation(s)

$$Y' + AY = 0 \,. \tag{2.4}$$

This is the *defining* $\mathbf{e}$-*equation* for D. We now see the relationship between differential structures and homogeneous linear differential equations mentioned in the introduction.

To understand the significance of cyclic vectors in the differential equation context first observe that to each differential structure D on V there is a naturally associated differential structure D^* on the dual space V^* defined by

$$(D^* u^*)v = \delta(u^* v) - u^*(Dv) \quad \text{for } u^* \in V^* \text{ and } v \in V \,. \tag{2.5}$$

It is easy to check that D^*u^* is K-linear and that D^* is a differential structure on V^*. The matrix associated with the basis $\mathbf{e}^*$ of V^* dual to $\mathbf{e}$ is $-A^\tau$, where the τ denotes the transpose, and as a consequence the corresponding differential equation is the (classical) adjoint equation

$$Y' - A^\tau Y = 0 \tag{2.6}$$

of (2.4). The natural identification $V^{**} \sim V$ induces an identification $D^{**} \sim D$; in particular, by regarding $\mathbf{e}$ as the dual basis of $\mathbf{e}^*$ we may view (2.4) as the adjoint equation of (2.6).

Suppose the basis $\mathbf{e}^*$ of V^* corresponds to a cyclic vector $v^* \in V^*$ for D^*, *i.e.*, suppose $\mathbf{e}^* = (v^*, D^*v^*, \ldots, D^{*n-1}v^*)$. Then $-A^\tau \in \mathrm{gl}(n, K)$ has the form

$$-A^\tau = \begin{pmatrix} 0 & \cdots & 0 & q_n \\ 1 & \ddots & \vdots & q_{n-1} \\ & \ddots & 0 & \vdots \\ & & 1 & q_2 \\ 0 & \cdots & 0 & q_1 \end{pmatrix}, \tag{2.7}$$

and the defining $\mathbf{e}$-matrix of D is therefore

$$A = \begin{pmatrix} 0 & -1 & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 \\ -q_n & -q_{n-1} & \cdots & -q_1 \end{pmatrix}. \tag{2.8}$$

Conversely, when a defining basis matrix of D has this form one sees directly from the companion matrix structures of A and $-A^\tau$ that the initial element of the dual basis is cyclic for D^*.

The relevance of (2.8) is completely standard: when A has this form a column vector $v = (v_1, \ldots, v_n)^\tau$ is a solution of the associated linear homogeneous system $Y' + AY = 0$ if and only if $y := v_1$ is a solution of the n-th-order homogeneous linear differential equation

$$y^{(n)} - q_1 y^{(n-1)} - \cdots - q_{n-1} y' - q_n y = 0\,. \tag{2.9}$$

We conclude that the existence of a cyclic vector for D^* is equivalent to the existence of a defining equation for D which can be expressed in n-th order form.

There is an alternate way to view this equivalence. Specifically, suppose $\mathbf{e} = (e_1, \ldots, e_n)$ and $\mathbf{f} = (f_1, \ldots, f_n)$ are bases of V, and A and $B \in \mathrm{gl}(n, K)$ are the defining $\mathbf{e}$ and $\mathbf{f}$-matrices of a differential structure D. Let $T = (t_{ij}) \in \mathrm{GL}(n, K)$ be the transition matrix between the given bases, *i.e.*, $e_j = \sum_i t_{ij} f_i$ for $j = 1, \ldots, n$, and write T' for the matrix $(t'_{ij}) \in \mathrm{gl}(n, K)$. Then from the Leibniz rule one sees that

$$A = T^{-1}BT + T^{-1}T' . \tag{2.10}$$

The mapping of $\mathrm{gl}(n, K) \times GL(n, K)$ into $\mathrm{gl}(n, K)$ defined by

$$(B, T) \mapsto T^{-1}BT + T^{-1}T'$$

is a right action of $GL(n, K)$ on $\mathrm{gl}(n, K)$, often referred to as the action by *gauge transformations*. From (2.10) and the preceding discussion, we see that by viewing B as a defining matrix for a differential structure D the existence of a cyclic vector for D^* is equivalent to the existence of an element A in the $\mathrm{GL}(n, K)$-orbit of B having the form displayed in (2.8).

In view of the duality discussed immediately following (2.6) we can work with either D^* or D for purposes of constructing cyclic vectors. We choose D to ease notation.

3 The Algorithm

Here $V = (V, \delta, D)$ is a differential K-space of finite dimension $n \geq 2$. (Any non-zero vector is cyclic when $n = 1$.) The field of constants of K (the kernel of δ), which always contains the prime field, is denoted K_C.

Hypotheses 3.1 We assume that:

1) the degree of the extension $K_C \subset K$ is at least n (possibly infinite); and

2) K_C contains at least n non-zero elements.

These hypotheses have been formulated so as to emphasize the finite nature of the algorithm. In fact 2) is a simple consequence of 1), as will be seen in Proposition 5.1(d). Assuming Hypotheses 3.1, we describe a simple algorithm for constructing a cyclic vector for D.

Suppose $v \in V$, $v \neq 0$, is *not* a cyclic vector, in which case

$$v \wedge Dv \wedge \cdots \wedge D^{m-1}v \neq 0 , \tag{3.2}$$

where $1 \leq m < n$, and

$$D^m v = \sum_{k=0}^{m-1} a_k D^k v \tag{3.3}$$

for some $a_0, \ldots, a_{m-1} \in K$. Choose any vector $u \in V$ not in the span of

$\{v, Dv, \ldots, D^{m-1}v\}$. We will indicate how one can select elements $\lambda_0 \in K_C$ and $k_0 \in K$ such that for

$$\overline{v} := v + \lambda_0 k_0 u,$$

one has

$$\overline{v} \wedge D\overline{v} \wedge \cdots \wedge D^{m-1}\overline{v} \wedge D^m\overline{v} \neq 0\,, \tag{3.4}$$

i.e., the differential vector space spanned by $\{\overline{v}, D\overline{v}, \ldots, D^{n-1}\overline{v}\}$ is strictly larger than that spanned by $\{v, Dv, \ldots, D^{n-1}v\}$.

To this end, set

$$v_0 := v, \quad v_i := D^i v, \qquad i = 1, \ldots, m-1,$$

and extend this K-linearly independent set to a basis of V by first adjoining u and then, if necessary, $e_1, \ldots, e_{n-m-1}$. For $0 \leq k \leq m$, write

$$D^k u = \sum_{i=0}^{m-1} \alpha_{ik} v_i + \beta_k u + \sum_{j=1}^{n-m-1} \gamma_{jk} e_j\,, \tag{3.5}$$

where α_{ik}, β_k, $\gamma_{jk} \in K$. (In particular, $\alpha_{i0} = \gamma_{j0} = 0$ and $\beta_0 = 1$.)

Define linear differential operators by the formulae

$$L_r(y) := \sum_{k=0}^{r} \binom{r}{k} \beta_k y^{(r-k)} = y^{(r)} + r\beta_1 y^{(r-1)} + \cdots + \beta_r y, \tag{3.6}$$

for $r = 0 \ldots, m$. Note that $L_0(y) = y$. Next define

$$L(y) := L_m(y) - \sum_{r=0}^{m-1} a_r L_r(y) = y^{(m)} + c_{m-1} y^{(m-1)} + \cdots + c_0 y\,, \tag{3.7}$$

where

$$c_i := \binom{m}{m-i} \beta_{m-i} - \sum_{r=i}^{m-1} \binom{r}{r-i} a_r \beta_{r-i}\,, \quad i = 0, \ldots, m-1.$$

Since the operator L in (3.7) is of order less than n, any collection of n distinct elements of K linearly independent over K_C will include a non-solution of $L(y) = 0$. This collection can be prespecified, *i.e.*, is independent of the differential system.

We choose a set $S_K \subset K$ containing at least n elements linearly independent over K_C. We also choose a set $S_{K_C} \subset K_C$ containing at least n non-zero elements.

Proposition 3.8 *Suppose $k_0 \in S_K$ is not a solution of (3.7). Then there exists a $\lambda_0 \in S_{K_C}$ for which $\overline{v} := v + \lambda_0 k_0 u$ satisfies $\overline{v} \wedge D\overline{v} \wedge \cdots \wedge D^m\overline{v} \neq 0$.*

Proof. Let λ be an indeterminate over K. Extend the derivation δ on K to $K(\lambda)$ by defining $\lambda' = \delta\lambda = 0$. Formally this is done by first defining δ on $K[\lambda] = K_C[\lambda] \otimes_{K_C} K$ by the formula

$$\delta(P \otimes k) = P \otimes k'$$

and then extending via the quotient rule.

The tensor product $\widehat{V} = K(\lambda) \otimes_K V$ has the natural structure of a differential $K(\lambda)$-space, obtained by defining

$$D(Q \otimes w) = \delta Q \otimes w + Q \otimes Dw$$

for any $w \in V$. To simplify notation we write this as

$$D(Qw) = Q'w + QDw.$$

Induction on the integer $r \geq 1$ gives the Leibniz rule

$$D^r(Qw) = \sum_{k=0}^{r} \binom{r}{k} Q^{(r-k)} D^k w.$$

Let $\widehat{v} := v + \lambda k_0 u$ $(= 1 \otimes v + k_0 \lambda \otimes u) \in \widehat{V}$. Then, for $0 \leq r < m$, we have

$$\begin{aligned}
D^r\widehat{v} &= D^r v + \lambda \sum_{k=0}^{r} \binom{r}{k} k_0^{(r-k)} D^k u \\
&= v_r + \lambda \sum_{k=0}^{r} \binom{r}{k} k_0^{(r-k)} \left(\sum_{i=0}^{m-1} \alpha_{ik} v_i + \beta_k u + \sum_{j=1}^{n-m-1} \gamma_{jk} e_j \right) \\
&= v_r + \lambda \sum_{i=0}^{m-1} \sum_{k=0}^{r} \binom{r}{k} k_0^{(r-k)} \alpha_{ik} v_i + \lambda L_r(k_0) u \\
&\quad + \lambda \sum_{j=1}^{n-m-1} \sum_{k=0}^{r} \binom{r}{k} k_0^{(r-k)} \gamma_{jk} e_j \\
&= v_r + \lambda \sum_{i=0}^{m-1} \theta_{ir} v_i + \lambda L_r(k_0) u + \lambda \sum_{j=1}^{n-m-1} \tau_{jr} e_j \,,
\end{aligned}$$

for some $\theta_{ir}, \tau_{jr} \in K$. A similar calculation, using (3.3), gives

$$\begin{aligned}
D^m\widehat{v} &= \sum_{i=0}^{m-1} a_i v_i + \lambda \sum_{i=0}^{m-1} \sum_{k=0}^{m} \binom{m}{k} \alpha_{ik} k_0^{(m-k)} v_i \\
&\quad + \lambda \sum_{k=0}^{m} \binom{m}{k} \beta_k k_0^{(m-k)} u + \lambda \sum_{j=1}^{n-m-1} \sum_{k=0}^{m} \binom{m}{k} \gamma_{jk} k_0^{(m-k)} e_j
\end{aligned}$$

$$= \sum_{i=0}^{m-1} a_i v_i + \lambda \sum_{i=0}^{m-1} \theta_{im} v_i + \lambda L_m(k_0)u + \lambda \sum_{j=1}^{n-m-1} \tau_{jm} e_j \,.$$

To summarize:

$$\begin{array}{rccccc}
\widehat{v} = & v_0 & + & 0 & + \lambda L_0(k_0)u & + & 0 \\
D\widehat{v} = & v_1 & + \lambda \sum_{i=0}^{m-1} \theta_{i1} v_i & + \lambda L_1(k_0)u & + \lambda \sum_{j=1}^{n-m-1} \tau_{j1} e_j \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
D^r\widehat{v} = & v_r & + \lambda \sum_{i=0}^{m-1} \theta_{ir} v_i & + \lambda L_r(k_0)u & + \lambda \sum_{j=1}^{n-m-1} \tau_{jr} e_j \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
D^m\widehat{v} = & \sum_{i=0}^{m-1} a_i v_i & + \lambda \sum_{i=0}^{m-1} \theta_{im} v_i & + \lambda L_m(k_0)u & + \lambda \sum_{j=1}^{n-m-1} \tau_{jm} e_j .
\end{array} \tag{3.9}$$

By the four "columns" of this array we mean the columns, delineated by the $+$ signs, appearing to the right of the equality signs, e.g., the first column has initial entry v_0; the second has initial entry 0; the third has initial entry $\lambda L_0(k_0)u = \lambda k_0 u$; and the fourth has initial entry 0.

Now consider the vector

$$\widehat{w} := \widehat{v} \wedge D\widehat{v} \wedge \cdots \wedge D^m\widehat{v} \in \bigwedge^{m+1} \widehat{V} \,.$$

We initially view $\widehat{w}$ as a polynomial in λ with "coefficients" in the K-space $\bigwedge^{m+1} V$. As such it has degree at most $m+1$ and the "constant term" is 0, as can be seen by substituting $\lambda = 0$ in the formulae above. Accordingly we write

$$\widehat{w} = \lambda w_1 + \lambda^2 w_2 \cdots + \lambda^{m+1} w_{m+1} \,, \tag{3.10}$$

where $w_i \in \bigwedge^{m+1} V$ for $i = 1, \ldots, m+1$.

We claim that the coefficient of $v_0 \wedge v_1 \wedge \cdots \wedge v_{m-1} \wedge u$ for w_1 is $L(k_0) \neq 0$. Assuming this is the case it follows that the coefficient of $v_0 \wedge v_1 \wedge \cdots \wedge v_{m-1} \wedge u$ for $\widehat{w}$ is a non-zero polynomial in $K[\lambda]$ whose degree is at most $m+1$ and whose constant term is 0. Since $S_{K_C} \subset K_C$ has at least $n > m$ non-zero elements there exists an element $\lambda_0 \in S_{K_C}$ which is not a zero of that polynomial, and the vector $\overline{v} := v + \lambda_0 k_0 u \in V$ will then satisfy the conclusion of the proposition.

To verify the claim, first note that, in computing this wedge product, the vector u can only result from a term in the third column of (3.9). But that term simultaneously contributes one λ, and since w_1 is the coefficient of λ in (3.10) we conclude that the remaining factors must be the terms from the first column in the m remaining rows. The coefficient in question is therefore determined from the calculation

$$\begin{aligned}
&\sum_{r=0}^{m-1} v_0 \wedge \cdots \wedge v_{r-1} \wedge L_r(k_0)u \wedge v_{r+1} \wedge \cdots \wedge v_{m-1} \wedge \sum_{j=0}^{m-1} a_j v_j \\
&\qquad + \; v_0 \wedge \cdots \wedge v_{m-1} \wedge L_m(k_0)u \\
&\qquad = \sum_{r=0}^{m-1} a_r L_r(k_0) v_0 \wedge \cdots \wedge v_{r-1} \wedge u \wedge v_{r+1} \wedge \cdots \wedge v_{m-1} \wedge v_r \\
&\qquad + \; L_m(k_0) v_0 \wedge \cdots \wedge v_{m-1} \wedge u \\
&\qquad = \Big(L_m(k_0) - \sum_{r=0}^{m-1} a_r L_r(k_0) \Big) v_0 \wedge \cdots \wedge v_{m-1} \wedge u \\
&\qquad = L(k_0)\, v_0 \wedge \cdots \wedge v_{m-1} \wedge u \,,
\end{aligned}$$

and the claim follows. □

Theorem 3.11 (The Cyclic Vector Theorem) *Suppose K_C contains at least n non-zero elements and that the extension $K_C \subset K$ is infinite or, if finite, of degree at least n. Then every n-dimensional differential K-space (V, δ, D) admits a cyclic vector.*

When K has characteristic 0 the degree n alternative on $K_C \subset K$ is impossible (see Proposition 5.1(a)).

Proof. First note that our hypotheses imply that K contains at least n elements linearly independent over constants. Therefore, for any $m < n$ the linear differential equation (3.7) admits a non-solution and Proposition 3.8 applies. Starting with an arbitrary non-zero vector $v \in V$ at most $n - 1$ applications of the proposition produce a cyclic vector. □

4 Remarks on the Algorithm

We fix a set $S_K \subset K$ containing at least n elements linearly independent over K_C, and a set $S_{K_C} \subset K_C$ containing at least n distinct non-zero elements. For example, if $K = \mathbb{Q}(x)$ it is quite natural to choose $S_K = \{1, x, x^2, \ldots, x^{n-1}\}$

and $S_{K_C} = S_{\mathbb{Q}} = \{1, 2, 3, \ldots, n\}$. We review the steps of the algorithm for purposes of reference. A MAPLE implementation of the case $(K, \delta) = (\mathbb{Q}(x), \frac{d}{dx})$ is presented in the Appendix.

Algorithm 4.1 To find a cyclic vector:

Step 1. Choose any non-zero vector $v \in V$.

Step 2. Compute the matrix M with columns $v, Dv, \ldots, D^{n-1}v$ as in (2.3).

Step 3. Compute the rank m of M, find $a_0, \ldots, a_{m-1} \in K$ as in (3.3), and a supplement to the subspace spanned by $v, Dv, \ldots, D^{m-1}v$.

Step 4. If $m = n$ the algorithm terminates and v is a cyclic vector.

Step 5. Choose any vector u not in the span of $\{v, Dv, \ldots, D^{m-1}v\}$.

Step 6. Compute the β_k as in (3.5).

Step 7. Find $k_0 \in S_K$ so that $L(k_0) \neq 0$ as in (3.7).

Step 8. Find $\lambda_0 \in S_{K_C}$ such that for $\overline{v} := v + \lambda_0 k_0 u$ the rank of the matrix in (2.3) is bigger than m.

Step 9. Go to Step 2.

In practice a differential structure $D : V \to V$ assumes the form of an $n \times n$ matrix A with coefficients in K. In particular, a basis $\mathbf{e}$ identifying V with K^n is implicit, and Dv, for any vector $v \in V$, is computed as in (2.2).

For Step 1 one might begin with a vector having a simple form, e.g. $(1, 0, \ldots, 0)$. When working over $K = \mathbb{Q}(x)$ this would increase the likelihood of constructing a cyclic vector not involving high powers of x or large coefficients. Alternatively, and more generally, one could choose a non-zero vector at random to increase the likelihood that the algorithm terminates upon first reaching Step 4. It is worth noting that when this random vector approach is used, replacing Step 5 with "Go to Step 1" results in the algorithm of the introduction.

To calculate the matrix M of Step 2 requires $O(n^2)$ differentiations and $O(n^3)$ multiplications in the field K. Assuming that differentiation is not too much more expensive than multiplication, we see that this step is $O(n^3)$.

Step 3 can be accomplished by applying Gaussian elimination as described in [9], Algorithm 2.3.6, p. 60, or see the procedure `CVRank` of the Appendix. This step requires $O(n^3)$ multiplications and divisions (see, for example, the discussion on pages 47-50 of [9]).

Each time Step 4 is reached the value of m increases; this step is involved at most $n - 1$ times.

For Step 6 we first compute the matrix with columns $u, Du, \ldots, D^m u$; this was seen in Step 2 to be $O(n^3)$. We then express the answer in the basis $v, Dv, \ldots, D^{m-1}, u, e_1, \ldots, e_{n-m-1}$. This requires matrix inversion and is $O(n^3)$.

In Step 7 the set S_K is known in advance and as a result all the derivatives of its elements are also known in advance. For $S_K = S_{\mathbb{Q}(x)} = \{1, x, x^2, \ldots, x^{n-1}\}$ this simply requires finding the smallest index i so that $c_i \neq 0$. The step is $O(n)$.

Step 8 involves computing the matrix of (2.3) and then its rank. The operations are $O(n^3)$, but may require repetition for each $\lambda_0 \in S_{K_C}$. The step is therefore $O(n^4)$; it is the most expensive of the algorithm.

An alternative approach would be to compute the polynomial for λ described in the proof of Proposition 3.8. Doing so would take $O(n^3)$ multiplications in $K[\lambda]$, and therefore at least $O(n^4)$ multiplications in K.

In Step 9 we loop back to Step 2. However the computations in Steps 2 and 3 repeat those done in Step 9. In the implementation in the Appendix we actually loop back to Step 4.

Since each step is at worst $O(n^4)$, and since we iterate up to n times the entire algorithm is $O(n^5)$. Of course this only counts multiplications and divisions; the actual complexity depends on the ground field K and the cost of multiplication and division within that field.

A more naive implementation might simply be to test all vectors having sums of products of elements of S_K and S_{K_C} as coefficients. There are n^3 such possibilities, and as a result the implementation would be $O(n^6)$.

The Appendix contains an implementation of the above algorithm. We also present a procedure that computes a companion matrix. The input is a matrix $A \in gl(n, \mathbb{Q}(x))$; the output consists of a matrix $T \in GL(n, \mathbb{Q}(x))$ and a matrix $C \in gl(n, \mathbb{Q}(x))$ such that $C = T^{-1}AT + T^{-1}T'$ has the companion matrix form

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & q_n \\ 1 & 0 & & \vdots & q_{n-1} \\ 0 & 1 & 0 & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 1 & 0 & q_2 \\ 0 & \cdots & \cdots & 0 & 1 & q_1 \end{pmatrix}; \tag{4.2}$$

a cyclic vector is provided by the first column of T.

We have also included the details of a command for converting a first order system $Y' + AY = 0$ of n homogeneous linear differential equations to

n-th order form. The input is a matrix $A \in gl(n, \mathbb{Q}(x))$; the output consists of a matrix $T \in GL(n, \mathbb{Q}(x))$ and a matrix $B \in gl(n, \mathbb{Q}(x))$ such that $B = T^{-1}AT + T^{-1}T'$ has the companion matrix form

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ \vdots & 0 & -1 & & \vdots \\ \vdots & & 0 & \ddots & \vdots \\ \vdots & & & \ddots & -1 & 0 \\ 0 & \cdots & \cdots & & 0 & -1 \\ -q_n & -q_{n-1} & \cdots & \cdots & -q_2 & -q_1 \end{pmatrix} . \tag{4.3}$$

The corresponding n-th order form is

$$y^{(n)} - q_1 y^{(n-1)} - \cdots - q_{n-1} y' - q_n y = 0 . \tag{4.4}$$

5 Remarks on the Hypotheses

These are based on a few elementary (and completely standard) observations on a differential field $K = (K, \delta)$ with field of constants K_C.

Proposition 5.1 (a) *When the characteristic of K is 0 any $k \in K \setminus K_C$ is transcendental* over K_C, i.e., *the subfield K_C is algebraically closed in K.*

(b) *The only derivation on a perfect field of positive characteristic is the trivial derivation. In particular, a finite field admits only the trivial derivation.*

(c) *When the characteristic of K is $p > 0$ any $k \in K \setminus K_C$ satisfies $k^p \in K_C$. In particular, the field extension $K_C \subset K$ is* (algebraic and) *purely inseparable.*

(d) *$K \neq K_C \Rightarrow K_C$ is infinite.*

It is immediate from (d) that Hypothesis 2) of §3 is redundant.

Proof. (a): Suppose k is algebraic over K_C with irreducible monic polynomial $P \in K_C[X]$. Then

$$0 = (P(k))' = \frac{dP}{dX}(k)\, k',$$

and since $dP/dX(k) \neq 0$ (the degree of dP/dX is too small) we conclude that $k' = 0$.

(b): When K is a perfect field of characteristic p each element of K has the form k^p for some $k \in K$. Therefore $(k^p)' = pk^{p-1}k' = 0$.

(c): $(k^p)' = 0$.

(d): For the characteristic 0 case this is trivial: K_C contains $\mathbb{Q}$.

When K has characteristic $p > 0$ the Frobenius mapping $k \mapsto k^p$ of K is an injection which by (c) has image in K_C. When K_C is finite the same then holds for K, which by the final assertion of (b) implies $K = K_C$. □

Corollary 5.2 *We have the following equivalences for* Hypotheses 3.1.

(a) *When K has characteristic* 0 Hypotheses 3.1 are *equivalent to $K \neq K_C$, i.e., to the assumption that the derivation on K is non-trivial.*

(b) *When K has characteristic $p > 0$ and $n \leq p$* Hypotheses 3.1 are *equivalent to $K \neq K_C$. In this case K is a purely inseparable extension of K_C of degree at least p.*

(c) *When K has characteristic $p > 0$ and $n > p$* Hypotheses 3.1 are *equivalent to the condition that K be a purely inseparable extension of K_C of degree at least n* (possibly infinite).

In all three cases the field of constants K_C must be infinite.

Proof. Hypothesis 1) obviously implies that $K \neq K_C$. Part (a) is immediate from Proposition 5.1(a). Parts (b) and (c) come from Proposition 5.1(c) and (d). □

6 Counterexamples

Here we investigate the necessity of Hypotheses 3.1. First we show, by example, that we cannot do with less than n elements of K linearly independent over K_C nor less than n non-zero elements of K_C.

Take $K = \mathbb{C}(x)$ $(x' = 1)$ and let V be an n-dimensional K-space $(n \geq 2)$ with basis $(e_1, \ldots, e_n)$. Now consider the differential structure on V given by

$$De_i = \begin{cases} e_{i+1} & \text{if } i = 1, \ldots, n-2, \\ 0 & \text{if } i = n-1, \\ x^{-n} e_1 & \text{if } i = n. \end{cases}$$

Following the notation of §3, we let

$$\begin{aligned} v_0 &= v = e_1, \\ v_i &= D^i v = e_{i+1} \quad \text{for } i = 1, \ldots, n-2, \\ u &= e_n. \end{aligned}$$

Then $D^{n-1}v = 0$, *i.e.*, the a_k of (3.3) are all 0. Next we compute the β_k of (3.5) (we do not care about the α_{ik}) and find that $\beta_0 = 1$ and $\beta_1 = \cdots = \beta_n = 0$. Using (3.7) and (3.6) we have $L(y) = y^{(n-1)} = 0$. Evidently $L(1) = L(x) = \cdots = L(x^{n-2}) = 0$, but $L(x^{n-1}) \neq 0$. This shows that the set $\{1, x, \ldots, x^{n-2}\}$, which consists of $n-1$ elements of K linearly independent over $K_C = \mathbb{Q}$, does not contain a *non*-solution of $L(y) = 0$. I.e. Hypothesis (1) is essential.

We choose a non-solution of $L(y) = 0$, namely $y = x^{n-1}$, and set

$$\overline{v} = v + \lambda x^{n-1} u,$$

where λ is an indeterminate constant, as in the proof of Proposition 3.8. We claim that for each $r = 0, \ldots, n-2$ we have

$$D^r \overline{v} = \sum_{i=1}^{r} \kappa_{ri} D^{i-1} \overline{v} + v_r + (-1)^r \lambda r! \binom{\lambda - n + r}{r} x^{n-r-1} u,$$

where $\kappa_{ri} \in K$. The case $r = 0$ is immediate from the definition of $\overline{v}$. Proceeding by induction, assume the formula holds for some $0 \leq r < n-2$ and apply D. The first two terms retain the desired form; the last becomes

$$(-1)^r \lambda r! \binom{\lambda - n + r}{r} \big((n-r-1)x^{n-r-2}u + x^{-r-1}v\big).$$

Split this in two by writing $v = \overline{v} - \lambda x^{n-1} u$. The resulting term involving $\overline{v}$ combines with the predecessors, the remaining term is

$$(-1)^r \lambda r! \binom{\lambda - n + r}{r} (n - r - 1 - \lambda) x^{n-r-2} u,$$

and the claim follows.

Applying D to the formula for $r = n-2$ gives

$$D^{n-1}\overline{v} = \sum_{i=1}^{n-1} \kappa_{n-1,i} D^{i-1} \overline{v} + (-1)^{n-1} \lambda(\lambda - 1) \cdots (\lambda - n + 1) u,$$

and therefore

$$\begin{aligned} &\overline{v} \wedge D\overline{v} \wedge \cdots \wedge D^{n-1}\overline{v} \\ &\qquad = (-1)^{n-1} \lambda(\lambda - 1) \cdots (\lambda - n + 1) \overline{v} \wedge D\overline{v} \wedge \cdots \wedge D^{n-2}\overline{v} \wedge u. \end{aligned}$$

We conclude that allowing $n+1$ possibilities for λ (counting 0) is also an essential requirement.

We now turn to examples in which our hypotheses fail, first considering the case $K = K_C$ (see (a) and (b) of Corollary 5.2). In that instance

any differential structure D on a finite-dimensional K-space V is a K-endomorphism of V, and as such admits a unique minimal monic polynomial $P \in K[X]$. When $\deg(P) = m$ the collection $v, Dv, \ldots, D^m v$ is linearly dependent for any vector $v \in V$. Therefore V admits a cyclic vector if and only if the degree of P is $n = \dim(V)$.

In particular, when some defining basis matrix of D is a scalar multiple of the identity and $\dim(V) \geq 2$ there can be no cyclic vector.

On the other hand, V can always be written as a direct sum of cyclic subspaces (see, e.g., [24, Theorem 8, Chapter XI, page 390]). For a more extensive discussion of this case see [3].

To complete the discussion it remains to consider the characteristic $p > 0$ case with $n := \dim(V) > p$ and $K_C \subset K$ purely inseparable of degree less than n (see Corollary 5.2(c)). Here we offer a example illustrating that cyclic vectors need not exist. (Alternatively, see [19, page 68].) Specifically, consider the field $K := \mathbb{F}_p(x)$ where x is transcendental over $\mathbb{F}_p$ and $x' = 1$. By Proposition 5.1(c) the element x has degree p over K_C and as a result the degree of the extension $K_C \subset K$ is $p < n$.

Note that $(x^m)^{(p)} = 0$ for all integers m, and from $k \in K \Rightarrow k' \in \mathbb{F}_p$ that $\mathbb{F}_p$ is a differential subfield of K. By Proposition 5.1(b) the restricted derivation must be trivial, and it follows that $k^{(p)} = 0$ for all $k \in K$.

Now suppose $V = (V, D)$ is a differential K-space of dimension $n > p$ and that the identity matrix is a defining basis matrix for D. The corresponding basis elements $e_1, \ldots, e_n$ then satisfy $De_j = e_j$ and for any $v = \sum_j k_j e_j \in V$ we have

$$D^p v = \sum_j \left(k_j^{(p)} e_j + k_j D^p e_j \right) = v \,.$$

We conclude there is no cyclic vector for D.

7 An Alternate Approach

Here we sketch a proof of the Cyclic Vector Theorem in the spirit of that of [10]. We begin with a purely algebraic lemma (*i.e.*, derivations are not involved).

Lemma 7.1 *Suppose that $A \subset K$ is a set containing at least $n+1$ distinct elements. Let $X_1, \ldots, X_r$ be indeterminates over K and let $Q(X_1, \ldots, X_r) \in K[X_1, \ldots, X_r]$, $Q \neq 0$, with $\deg Q \leq n$. Then there exist $a_1, \ldots, a_r \in A$ such that $Q(a_1, \ldots, a_r) \neq 0$.*

Proof. We use induction on the number s of indeterminates actually appearing in Q. If $s = 0$, *i.e.*, $Q \in K$, then $a_1, \ldots, a_r$ may be chosen arbitrarily.

Suppose that Q involves $s > 0$ variables and that the result is proved for polynomials involving fewer variables. Reordering the variables, if necessary, we may assume that Q only involves $X_1, \ldots, X_s$ and that the degree of Q in X_s is $d > 0$. Write

$$Q = Q_d X_s^d + Q_{d-1} X_s^{d-1} + \cdots + Q_0, \quad Q_i \in K[X_1, \ldots, X_{s-1}], \quad Q_d \neq 0 .$$

By induction there exist $a_1, \ldots, a_{s-1} \in A$ such that $Q_d(a_1, \ldots, a_{s-1}) \neq 0$. Choose $a_{s+1}, \ldots, a_r \in A$ arbitrarily and define

$$\overline{Q} = Q(a_1, \ldots, a_{s-1}, X_s, a_{s+1}, \ldots, a_r) \in K[X_s].$$

Then $\overline{Q}$ is a non-zero polynomial in one variable, X_s, with $\deg \overline{Q} \leq n$. Such a polynomial has at most n roots, so there exists $a_s \in A$ with $\overline{Q}(a_s) \neq 0$. □

Now let $y_1, \ldots, y_m$ be differential indeterminates over K. We let $K\{y_1, \ldots, y_m\}$ and $K\langle y_1, \ldots, y_n\rangle$ denote the differential ring of differential polynomials and the differential field of rational functions respectively. (For basic notions of differential algebra see [20] or [18].)

Proposition 7.2 *Suppose that $A \subset K$ is a set containing at least $n + 1$ elements. Let $P \in K\{y_1, \ldots, y_m\}$ be a non-zero differential polynomial of order $r - 1 \leq n - 1$ and degree $s \leq n$. Then for any $b_1, \ldots, b_r \in K$, linearly independent over K_C, there are elements $a_{ij} \in A$, $1 \leq i \leq m$, $1 \leq j \leq r$, such that for $k_i = a_{i1} b_1 + \cdots + a_{ir} b_r$ one has $P(k_1, \ldots, k_m) \neq 0$.*

Proof. Let X_{ij}, $1 \leq i \leq m$, $1 \leq j \leq r$, be indeterminates (not differential) over K. Because the $y_i^{(j-1)}$ are algebraically independent over K we may define a (non-differential) K-algebra homomorphism $\phi : K[y_i^{(j-1)}] \to K[X_{ij}]$ by

$$\phi : y_i^{(j-1)} \mapsto \sum_{k=1}^{r} b_k^{(j-1)} X_{ik} .$$

Since the $b_1, \ldots, b_r$ are linearly independent over K_C the Wronskian $\det(b_i^{(j-1)})$ does not vanish; it follows that ϕ is invertible, *i.e.*, is an isomorphism. The degree of $Q := \phi P$ is $s \leq n$. By Lemma 7.1 there are elements $a_{ij} \in A$, $1 \leq i \leq m$, $1 \leq j \leq r$, such that $Q(a_{ij}) \neq 0$. The desired result then follows from the observation that $P(k_1, \ldots, k_m) = Q(a_{ij}) \neq 0$ when the k_j are defined as in the statement of the proposition. □

For the remainder of the section we assume Hypotheses 3.1. The next result explains the success of the "algorithm" of the Introduction. By the Kolchin topology we mean the topology on affine space K^n having the zero

sets of differential polynomials as closed sets. We suppose that K satisfies Hypotheses 3.1.

Theorem 7.3 (The Cyclic Vector Theorem) *For any basis $(e_1, \ldots, e_n)$ of V the set of points $(k_1, \ldots, k_n) \in K^n$ for which $\sum_j k_j e_j$ is a cyclic vector is a non-empty Kolchin open (and therefore dense) subset of differential affine n-space K^n. In particular, V admits a cyclic vector for D.*

This formulation of the Cyclic Vector Theorem justifies the comments in Remarque 2.5, p. 133 of [2].

Proof. Let $y_1, \ldots, y_n$ be differential indeterminates over K. Then $W := K\langle y_1, \ldots, y_n\rangle \otimes_K V$ is a differential $K\langle y_1, \ldots, y_n\rangle$-space with differential structure $(\delta, \widehat{D})$, where

$$\widehat{D}(f \otimes v) = f' \otimes v + f \otimes Dv.$$

Choose any basis $\mathbf{e} = (e_1, \ldots, e_n)$ of V. We will show that the vector

$$w = \sum_{j=1}^{n} y_j \otimes e_j$$

is cyclic for $\widehat{D}$.

Define $p_{ij} \in K\langle y_1 \ldots, y_n\rangle$ by $\widehat{D}^{j-1} w = \sum_i p_{ij} \otimes e_i$. We claim that $p_{ij} = y_j^{(i-1)} + q_{ij}$, where q_{ij} is a linear differential polynomial in $y_1, \ldots, y_n$ of order strictly less than $i - 1$. The proof is a trivial induction on i: for $i = 1$ one has $p_{1j} = y_j + 0$; for $i > 1$ one uses $D^i w = D(D^{i-1})w$.

Recall (see (2.3)) that w is cyclic for $\widehat{D}$ if and only if $P = \det(p_{ij}) \neq 0$. To establish this nonvanishing first note that this differential polynomial has order less than n and degree at most n. Next observe that the matrix (p_{ij}) has a single entry involving $y_n^{(n-1)}$, namely p_{nn}, and that the coefficient of $y_n^{(n-1)}$ in the determinant is the minor $\det(p_{ij})_{1 \leq i,j \leq n}$. Arguing by induction we see that the coefficient of $y_1 y_2' \cdots y_n^{(n-1)}$ in P is 1, hence $P \neq 0$, and $w = \sum_j y_j \otimes e_j$ is therefore cyclic for $\widehat{D}$. The argument shows, in addition, that P has order $n - 1$ and degree n.

By Proposition 7.2 we can find $k_1, \ldots, k_n \in K$ such that $P(k_1, \ldots, k_n) \neq 0$. The result follows. ☐

As before, we can prespecify the set S_K of elements of K linearly independent over K_C and the set A of the proposition.

Suppose $K = \mathbb{Q}(x)$, $S_K = S_{\mathbb{Q}(x)} = \{1, x, x^2, \ldots, x^{n-1}\}$, and $A = \{0, 1, \ldots, n\}$. Then the k_i are polynomials in x of degree less than n whose coefficients are non-negative integers up to n. There are at most $(n+1)^2$ ways to choose each k_i.

This suggests an easy to implement algorithm that, unlike the "algorithm" of the introduction, is guaranteed to terminate: simply try all possibilities. However this algorithm is $O(n^6)$. Indeed, there are $O(n^2)$ possibilities for each k_i (and there are n of them), and the test for cyclicity, e.g. computing the determinant of (2.3), is $O(n^3)$.

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Appendix - A MAPLE implementation

CVvector - find a cyclic vector

Calling Sequence:

v := **CVvector**(A)
v := **CVvector**(A,v0)

Parameters:

A - square matrix defining a differential structure
v0 - (optional) initial trial vector

Synopsis:

CVvector returns a cyclic vector for the differential structure.
v0, if present, is used as the starting vector.

Examples:

```
> with(linalg);
> A := matrix([[0,0,1/x^3],[1,0,0],[0,0,0]]);
```

$$A := \begin{pmatrix} 0 & 0 & \frac{1}{x^3} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
> v := CVvector(A);
```

$$v := [1, 0, 3x^2]$$

```
> v := CVvector(A,[0,1,0]);
```

$$v := [x, 1, 1]$$

```
CVvector := proc (A, v0)
local n, v, M, a, B, m, m1, beta, c, i, r, u;
n := rowdim(A);

# Step 1 - choose any v in K^n
if nargs = 2 then
     v := v0;
else
     v := vector(n,0);
     v[1] := 1;
fi;

# Step 2 - compute the matrix with cols v, Dv, ...
M := CVdv (n, A, v, n);

# Step 3 - compute the rank, also a's and a supplement
m := CVrank(n, M, 'a', 'B');

# Step 4 - check if cyclic vector found,  come here from Step 8
while (m <> n) do

# Step 5 - compute u.  a's and e's have been computed by CVrank
u := col(B, m+1);

# Step 6 - compute the beta's
M := CVdv (n, A, u, m+1);
beta := row(evalm(inverse(B) &* M), m+1);

# Step 7 - find k_0 = x^ e so L(k_0) <> 0
for i from 0 to m-1 do
     c := binomial(m, m-i)*beta[m-i+1] -
          sum(binomial(r, r-i)*a[r+1]*beta[r-i+1], r=i..m-1);
     if c <> 0 then break; fi;
od;
if i <> 0 then u := evalm(x^i * u); fi;
```

```
# Step 8 - find lambda_0 = 1,2,...
for i from 0 to m do
     v := evalm(v + u);
     M := CVdv (n, A, v, n);
     m1 := CVrank(n, M, 'a', 'B');
     if m1 > m then break; fi;
od;
m := m1;

od;     # goto Step 4

RETURN(evalm(v));
end;
```

CVcompanion - compute a companion matrix

Calling Sequence:

C := **CVcompanion** (A)
C := **CVcompanion** (A, v0)
C := **CVcompanion** (A, 'T')
C := **CVcompanion** (A, v0, 'T')

Parameters:

A - square matrix defining a differential structure
v0 - (optional) initial trial vector
T - (optional) used to return the transition matrix

Synopsis:

CVcompanion returns a companion matrix C of the form (4.2) for the differential structure.
v0, if present, is used as the starting vector,
T, if present, is used to return the transition matrix.
The formula is: C := T^(-1) * A * T + T^(-1) * T', where T is the matrix with columns v, Dv, D^2v, ... and v is a cyclic vector.

Examples:

```
> with(linalg);
> A := matrix([[-x,1-x^2,x-x^3],[1,0,1],[0,1,x+1]]);
```

$$A := \begin{pmatrix} -x & 1-x^2 & x-x^3 \\ 1 & 0 & 1 \\ 0 & 1 & x+1 \end{pmatrix}$$

```
> C := CVcompanion (A, 'T');
```

$$C := \begin{pmatrix} 0 & 0 & x^2 \\ 1 & 0 & x \\ 0 & 1 & 1 \end{pmatrix}$$

```
> evalm(T);
```

$$\begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$$

```
CVcompanion := proc(A, v0, T)
local v, n, lT, lTi;

if nargs > 1 and type(args[2],{'vector','list'}) then
     v := CVvector(A, v0);
else
     v := CVvector(A);
fi;

n := rowdim(A);
lT := CVdv(n, A, v, n);

if nargs = 3 then
     T := evalm(lT);
elif nargs = 2 and not type(args[2],{'vector','list'}) then
     v0 := evalm(lT);
fi;
lTi := inverse(lT);
RETURN (evalm(lTi &* A &* lT + lTi &* map(diff, lT, x)));
end;
```

CVscalar - convert matrix equation to scalar

Calling Sequence:

C := **CVscalar** (A)
C := **CVscalar** (A, v0)
C := **CVscalar** (A, 'T')
C := **CVscalar** (A, v0, 'T')

Parameters:

A - a matrix
v0 - (optional) initial trial vector
T - (optional) used to return the transition matrix

Synopsis:

CVscalar returns a companion matrix C of the form (4.4)
for the scalar equation equivalent to $Y' + AY = 0$.
The equation is (D@@n)(y) + q[1]*(D@@(n-1))(y) + $\cdots$ + q[n]*y,
where q[i] = C[n,n-i].
v0, if present, is used as the starting vector when **CVcompanion**
is called,
T, if present, is used to return the transition matrix.

Examples:

```
> with(linalg);
> A := matrix([[-x,-1,-1],[x^2,x-1,2*x-2],[0,1,1-x]]);
```

$$A := \begin{pmatrix} -x & -1 & -1 \\ x^2 & x-1 & 2x-2 \\ 0 & 1 & 1-x \end{pmatrix}$$

```
> C := CVscalar(A);
```

$$C := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -x \end{pmatrix}$$

```
> q := row(C,3);
```

$$q := [0, 0, -x]$$

```
> # The corresponding linear differential equation is
> (D@@3)(y) - x * (D@@2)(y) = 0;
```

$$D^{(3)}(y) - xD^{(2)}(y) = 0$$

```
CVscalar := proc(A, v0, T)
local At, lT, C;
At := evalm(-1 * transpose(A));

if nargs > 1 and type(args[2],{'vector','list'}) then
     C := CVcompanion(At, v0, 'lT');
else
     C := CVcompanion(At, 'lT');
fi;

if nargs = 3 then
     T := evalm(transpose(inverse(lT)));
elif nargs = 2 and not type(args[2],{'vector','list'}) then
     v0 := evalm(transpose(inverse(lT)));
fi;

RETURN(evalm(-1 * transpose(C)));
end;
```

CVdv - internal subroutine to compute matrix v, Dv, ...

Calling Sequence:

M := CVdv(n, A, v, m)

Parameters:

n - the dimension of the vector space
A - the n by n matrix defining the differential structure
v - a vector
m - number of columns to be computed (may be <> n)

Synopsis:

CVdv returns the matrix with columns
v, Dv, D^2v, ... D^(m-1)v
This is used for Step 2 of CVvector.

```
CVdv := proc(n, A, v, m)
local i, j, k, w;
w := matrix(n,m);

for i from 1 to n do w[i,1] := v[i]; od;

for j from 2 to m do
     for i from 1 to n do
          w[i,j] := diff(w[i,j-1],x)
               + sum(A[i,k]* w[k,j-1],k=1..n);
     od;
od;
RETURN(evalm(w));
end;
```

CVrank - internal subroutine to compute rank

Calling Sequence:

r := CVrank (n, M, a, S)

Parameters:

n - the dimension of the vector space
M - the n by n matrix whose rank is sought,
usually M = CVdv (n, V, v, n);
a - returns a vector with
a[1] M[i,1] + $\cdots$ + a[r] M[i,r] = M[i,r+1]
S - returns an invertible matrix s whose first r columns
are M[i,1],...,M[i,r]

Synopsis:

CVrank returns the rank of the matrix M and other data.
The algorithm is as follows:
Let B = Id. Perform row reduction on M, and corresponding column operations on B. Thus the product BM is constant (= original M). On return, the first r columns of M are the identity and the last n-r rows are 0. It follows that the first r columns of B are the same as those of M. This is used for Step 3 of CVvector.

```
CVrank := proc(n, M, a, S)
local  B, i, j, k, d;
B := band([1], n);
```

```
for i from 1 to n do
     for j from i to n do
          if M[j,i] <> 0 then break; fi;
     od;
     if j = n+1 then
          a := evalm(col(M, i));
          S := evalm(B);
          break;
     fi;
     if i <> j then
          M := swaprow(M, i, j);
          B := swapcol(B, j, i);
     fi;
     d := M[i,i];
     if d <> 1 then
          M := mulrow(M, i, 1/d);
          B := mulcol(B, i, d);
     fi;
     for k from 1 to n do
          if k <> i then
          d := M[k,i];
          if d <> 0 then
          M := addrow(M, i, k, -d);
          B := addcol(B, k, i, d);
          fi;
          fi;
     od;
od;
RETURN(i-1);
end;
```

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