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Remark on the Nature of the Spectrum of the Lamé Equation. A Problem From Transcendence Theory (*).

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Formulation of the results. – Let $\wp(z)$ be the Weierstrass elliptic function with algebraic invariants g_2, g_3. We consider the Sturm-Liouville problem with Lamé potential $u(x) = 2\wp(x)$:

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Psi = E\Psi . \tag{1}$$

We denote by 2ω, $2\omega'$ the periods of $\wp(x)$, where 2ω is the least real period of $\wp(x)$. Then it is natural to make the substitution

$$E = \wp(a) . \tag{2}$$

It is well known that the spectrum of (1) has the form (when $\Delta > 0$)

$$(e_1, e_2) \cup (e_3, +\infty) , \tag{3}$$

where

$$4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3) , \tag{4}$$

$$e_1 < e_2 < e_3 \tag{5}$$

(of course the e_i are real). Here $e_1 = \wp(\omega_i)$ are ends of lacunae-points of periodic

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(or anti-periodic) spectrum. In other words, let us consider such $E_1=\wp(a_i)$ that

(6) $\quad \psi(x, E_i)$—the solution of (1)—is periodic with period 2ω

or antiperiodic with period 2ω .

For an arbitrary potential $u(x)$ (not necessarily $2\wp(x)$) the points $E_1^{\pm}$ or the periodic and antiperiodic spectra are distributed as follows:

$$E_1^+ < E_2^- \leqslant E_3^- \leqslant E_4^+ \leqslant E_5^+ < \ldots < E_{4k+2}^- \leqslant E_{4k+3}^- < E_{4k+4}^+ \leqslant \ldots . \tag{7}$$

Here E_i^+ are the eigenvalues corresponding to the periodic spectrum

$$i = 1, 4, 5, 8, 9, \ldots, 4k+1, \ldots$$

and E_j^- are the eigenvalues corresponding to the antiperiodic spectrum

$$j = 2, 3, 6, 7, \ldots, 4k+2, 4k+3, \ldots$$

and

$$\begin{cases} E_{i_1}^+ < E_{j_1}^- < E_{i_2}^+ < E_{j_2}^- & \text{for } i_1 < j_1 < i_2 < j_2 , \\ E_{i_1}^+ \leqslant E_{i_2}^+ , \quad E_{j_1}^- \leqslant E_{j_2}^- & \text{for } i_1 \leqslant i_2,\ j_1 \leqslant j_2 \end{cases} \tag{8}$$

the sign $\pm$ of $E_i^{\pm}$ is determined by the rule

$$\text{for given } i,\ E_i^{\pm} = E_i^{\chi} \text{ have the form } \chi = (-1)^{[(i+1)/2]} . \tag{9}$$

All the equations from (7) to (9) are true [1] for an arbitrary potential $u(x)$ (with good analytical properties) periodic with period 2ω. For our potential $u(x) = 2\wp(x)$ by the classical theory of Lamé equation [2,3]

$$E_1^+ = e_1 , \qquad E_2^- = e_2 , \qquad E_3^- = e_3 \tag{10}$$

and

$$E_{4k+2}^- = E_{4k+3}^- , \qquad k = 1, 2, 3, \ldots ; \tag{11}$$

$$E_{4k}^+ = E_{4k+1}^+ , \qquad k = 1, 2, 3, \ldots . \tag{12}$$

The values $E_i^{\pm}$ for $i > 3$ are known as «transcendental eigenvalues» of the Lamé potential, as the continued fraction representing $E_i^{\pm}$ is not terminating [2,3].

We prove that $E_i^{\pm}$ for $i > 3$ are really transcendental numbers (of course, when g_2, g_3 are both algebraic).

Theorem 1. When $\wp(x)$ has algebraic invariants, for the potential $2\wp(x)$ the degenerate eigenvalues $E_i^{\pm}$ for $i > 3$ are transcendental numbers.

(1) D. V. Chudnovsky and G. V. Chudnovsky: preprint IHES/M/78/236.
(2) E. L. Ince: *Proc. R. Soc. Edinburg.* **60**, 47 (1940).
(3) E. L. Ince: *Proc. R. Soc. Edinburg*, **60**, 83 (1940).

This result can be generalized for any Lamé potential $n(n+1)\wp(x)$. In this case we have the following general

Theorem 2. Let $\wp(x)$ have algebraic invariants and let n be an integer, $n \geqslant 2$. Then for Lamé potential $u_n(x) = n(n+1)\wp(x)$ all degenerate eigenvalues $E_i^{\pm}$, $i > 2n+1$, are transcendental numbers.

As is well known, the nondegenerate values $E_i^{\pm}$, $i = 1, \ldots, 2n+1$, are algebraic numbers (2). Theorem 2 is interesting because the degenerate values $E_i^{\pm}$, $i > 2n+1$ of $u_n(x)$, were called « transcendental » without any reason for it. Theorems 1, 2 provide us with this reason.

Method of proof of theorem 1. – In the proof we use a recent result of one of the authors (4):

Proposition 3. Let $\wp(z)$ have algebraic invariants, $\zeta(z)$ be the corresponding ζ-function and (ω, η) be pair of period, quasi-period for $\wp(x)$: $\zeta(z+\omega) = \zeta(z) + \omega$. If u is an algebraic point for $\wp(z)$, $\wp(u) \in \bar{Q}$, and u is linearly independent over Q with w, then

(13) $\zeta(u) - \dfrac{\eta}{\omega} u$ and $\dfrac{\eta}{\omega}$ are algebraically independent; of course $\zeta(u) - \dfrac{\eta}{\omega} u$ is transcendental

If we now consider an arbitrary periodic potential $u(x)$ with period T and the Strum-Liouville problem

$$\left\{-\frac{d^2}{dx^2} + u(x) - E\right\}\psi = 0\,, \tag{14}$$

then there exists a monodromy operator $\hat{T}\psi(x) = \psi(x+T)$. As usual we determine Bloch eigenfunctions $\psi_{\pm}$ as eigenfunctions of (14) satisfying the condition

$$\hat{T}\psi_{\pm} = \exp\left[\pm iP(E)\right]\psi_{\pm} \tag{15}$$

(*i.e.* eigenfunctions of $\hat{T}$). The points in which

$$\cos P(E) = \pm 1 \tag{16}$$

are points of the periodic or the antiperiodic spectrum $E_i^{\pm}$ (7) of (14).

Let $\omega = \omega_1$ be the real half-period of $\wp(z)$, and ω' be the corresponding imaginary half-period of $\wp(z)$. For the potential $u_2(x) = 2\wp(x)$ the Bloch eigenfunctions have the form

$$\psi_{\pm}(x) = \frac{\sigma(x \pm \alpha)}{\sigma(x)} \exp\left[\mp \zeta(\alpha)x\right]\,, \tag{17}$$

where, as in (2),

$$E = \wp(\alpha)\,.$$

(4) G. CHUDNOVSKY: *Proceedings of the* ICM (Helsinki, 1978).

Then it is clear that for the quasi-impulse $P(E)$, where $T = 2\omega$, we have

$$\text{(18)} \qquad \text{for } u_2(x) = 2\wp(x)\,, \qquad P(E) = 2\eta\alpha - 2\zeta(\alpha)\,\omega\,.$$

As a consequence of (13)

$$\text{(19)} \qquad \text{for an algebraic } E,\ \frac{P(E)}{\omega} \text{ and } \frac{\eta}{\omega} \text{ are algebraic independent}\,.$$

Moreover, we have a corollary:

$$\text{(20)} \qquad \text{for an algebraic } E \text{ and } \alpha \text{ linearly independent of } \omega,\ P(E) \neq 0\,.$$

Taking into account the known periodic and antiperiodic points $\alpha = \omega_1, \omega_2, \omega_3$, we obtain from (20) theorem 1.

General integral of Lamé equation. – It is natural to present the precise description of the general integral of the Lamé equation by following classical studies. For this purpose, it is better to use Forsyth's exposition ((5), volume 4) (*).

Proposition 4. If n quantities $a_1, a_2, \dots, a_n$ are determined by the $n-1$ independent equations

$$\frac{\wp'(a_1) + \wp'(a_2)}{\wp(a_1) - \wp(a_2)} + \frac{\wp'(a_1) + \wp'(a_2)}{\wp(a_1) - \wp(a_2)} + \dots = 0\,,$$

$$\frac{\wp'(a_2) + \wp'(a_1)}{\wp(a_2) - \wp(a_1)} + \frac{\wp'(a_2) + \wp'(a_3)}{\wp(a_2) - \wp(a_3)} + \dots = 0\,,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\wp'(a_n) + \wp'(a_1)}{\wp(a_n) - \wp(a_1)} + \frac{\wp'(a_n) + \wp'(a_2)}{\wp(a_n) - \wp(a_2)} + \dots = 0$$

and by

$$(2n-1)\sum_{i=1}^{n} \wp(a_i) = E\,,$$

then the function

$$\Psi(z) = \frac{\sigma(z + a_1)\,\sigma(z + a_2)\,\dots\,\sigma(z + a_n)}{\sigma^n(z)} \exp\left[-z\sum_{i=1}^{n}\zeta(a_i)\right]$$

is the solution of Lamé equation

$$\frac{d^2\Psi}{dz^2} = \{n(n+1)\wp(z) + E\}\Psi\,.$$

(5) A. R. Forsyth: *Theory of Differential Equations* (New York, N. Y., 1959).

(*) It is not entirely clear (to the authors) to whom it is correct to attribute these results, to Lindemann, to Steiltjes, to Forsyth, ... etc.?

From this proposition it is possible to conclude that there is another representation for solutions of the Lamé equation in terms of only one function. This function was probably introduced into the general Lamé equation, by PICARD. However, the general exposition is again borrowed from Forsyth ((5) volume 4).

Therefore, according to (5) (volume 4, p. 151), we introduce the most important function $G(z)$ in our further investigations, for which we keep Forsyth notations:

$$G(z) = \frac{\sigma(z+a)}{\sigma(z)} \exp\left[\varrho - \zeta(a)s\right]. \tag{21}$$

Proposition 5. For the Lamé equation

$$\frac{d^2\Psi}{dz^2} = \{n(n+1)\wp(z) + E\}\Psi \tag{22}$$

the general integral $\Psi(z)$ can be expressed in the form

$$\Psi = b_n \frac{d^{n-1}G}{dz^{n-1}} + b_{n-1}\frac{d^{n-2}G}{dz^{n-2}} + \dots + b_1 G,$$

where $b_1, b_2, \dots, b_n, \varrho, \wp(a)$ are determined algebraically in terms of E (for a fixed integer n).

The precise form of propositions 4 and 5 for the case in which $n = 1, 2, 3$ only, can be bound in the classical literature.

The proof of Theorem 2. – In the proof of the main result of the Lamé equation we need the following simple result from the transcendental-number theory known as the Schneider-Lang theorem:

Theorem 6 (4). Let $f_1(z), f_2(z)$ be meromorphic functions of order of growth $\leqslant \zeta_1$, $\leqslant \zeta_2$, respectively. Let K be an algebraic number field and let $f_3(z), \dots, f_n(z)$ be functions such that the ring $K[f_1, \dots, f_n]$ is closed under differentiation d/dz. Then the set of points $z \in \mathbf{C}$ such that $f_i(z) \in K$ for all $i = 1, \dots, n$ has a cardinality of, at most, $[K:\mathbf{C}](\zeta_1 + \zeta_2)$.

In particular, if $f_1, \dots, f_n$ satisfy all the conditions of this theorem, and are periodic with common period T, then for any complex z one of the numbers $f_i(z)$, $i = 1, \dots, n$, is transcendental.

We use this theorem only for $\wp(z)$ and $\Psi(x)$, where $\Psi(x)$ is a Bloch eigenfunction of

$$\frac{d^2\Psi}{dz^2} = \{n(n+1)\wp(z) + E\}\Psi.$$

Corollary 7. If $\wp(z)$ has algebraic invariants g_2, g_3, then for algebraic E and $\Psi(z)$ as above having at least one nonzero period of $\wp(z)$ as a period, the two functions $\wp(z)$ and $\Psi(z)$ are algebraically independent.

Proof. Let E be an algebraic number. According to proposition 2 above, $\Psi(z)$ is algebraically dependent with $\wp(z)$, $G(z) = (\sigma(z+a)/\sigma(z)(\exp[\{\varrho - \zeta(a)\}z]$ with algebraic ϱ and $\wp(a)$. Now $G(z)$ and $\wp(z)$ satisfy algebraic differential equations over Q. If $\wp(z)$ and $\Psi(z)$ were algebraically independent, then, according to the corollary of the Schneider-Lang theorem, $\Psi(z)$ and $\wp(z)$ would have no common period.

From this corollary it is possible to deduce our main result. For example, by the Ince theorem (2), if E corresponds to points of the degenerate spectrum of the Sturm-Liouville problem for $n(n+1)\wp(x)$, then eigenfunctions corresponding to E are algebraically independent of $\wp(x)$. This result, proved in detail by INCE (3), belongs in fact to HERMITE, and one can find an old-fashioned proof of that in Poole (6).

Now let E be a point of a degenerate spectrum for the potential $n(n+1)\wp(x)$ with $\wp(x)$ having algebraic invariants. Then for Bloch eigenfunctions $\Psi(z)$ of (22) corresponding to E, we have $\Psi(x+T)=\exp[p(E)]\Psi(x)$ for T which is the smallest real period of $\wp(x)$. Since E is a degenerate eigenvalue, then $p(E)=0$ and Ψ is periodic with the period T. However, $\wp(x)$ and $\Psi(x)$ are algebraically independent, which contradicts the corollary of the Schneider-Lang theorem.

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(6) E. POOLE: *Introduction to the Theory of Linear Differential Equations* (New York, N. Y., 1960).