ON THE ENUMERATION OF PLANAR MAPS

BY W. T. TUTTE[1]

A *planar map* is determined by a finite connected nonnull graph embedded in the 2-sphere or closed plane. It is permissible for the graph to have loops or multiple joins. It separates the remainder of the surface into a finite number of simply-connected regions called the *faces* of the map. We refer to the vertices and edges of the graph as the *vertices* and *edges* of the map, respectively. The *valency* of a vertex is the number of incident edges, loops being counted twice.

A *vertex-map* is a planar map having exactly one vertex and no edges. Clearly a vertex-map has only one face. A map with exactly one edge is called a *link-map* or a *loop-map* according as the two ends of the edge are distinct or coincident. Thus a link-map has exactly one face and a loop-map exactly two.

Two planar maps are *combinatorially equivalent* if there is a homeomorphism of the surface which transforms one into the other. To within a combinatorial equivalence there is only one vertex-map, one link-map and one loop-map. But the vertex-map, link-map and loop-map are combinatorially distinct from one another.

Consider a planar map M which is not a vertex-map. Each face of M has an associated *bounding path* in the graph. We can consider this to be the path traced out by a point moving along the edges of the graph in accordance with the following rules. Normally in any small interval of time the point traces out a simple arc. On one side of this directed arc, let us say the right side, there is locally nothing but points of the face. Having started along one edge, the point continues along it, without reversing direction, until it comes to the far end. If this end is monovalent, the point then proceeds back along the same edge. This behaviour at a monovalent vertex constitutes the only exception to the rule of simple arcs in short intervals of time.

The bounding path is the cyclic sequence of positions of the moving point, some of which may be repeated. We restrict it to a single cycle by the rule that it may not traverse any edge twice in the same direction. When the distinction between right and left has been made for a map, the above rules determine the bounding path of each face

An address delivered to the New York Meeting of the Society on April 8, 1967, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors September 22, 1967.

[1] The author did some of this work as a Consultant to the RAND Corporation. (August, 1966.)

uniquely. Figure I shows the bounding path in a map of one moderately complicated face F.

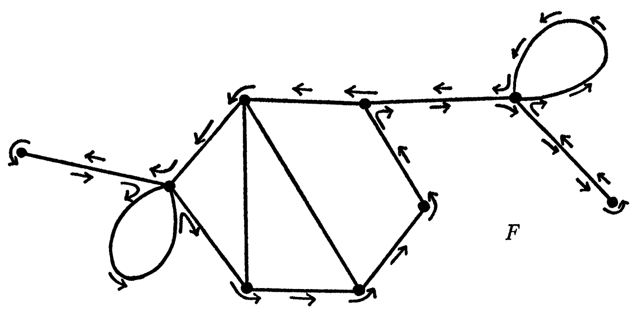

FIGURE I

An edge A can occur at most twice in the bounding path of a face F. If it does occur twice it does so in opposite directions, and it must be an isthmus of the graph. A vertex can appear more than once, but this happens only for cut-vertices.

The edges traversed by the bounding path of a face F are said to be *incident* with F. The *valency* of F is the number of edges incident with F, isthmuses being counted twice. A face of valency m is called an *m-gon.*

It is natural to ask how many combinatorially distinct planar maps there are with n edges, or with k vertices and l faces, or with some other specification. Such problems are difficult. It is easier to study the analogous problems for the "rooted" planar maps which we now proceed to describe.

A planar map is said to be *rooted* when one edge is specified as the *root,* a direction is assigned to the root, and the two sides of the root are distinguished as "left" and "right." Two rooted maps are regarded as combinatorially equivalent if there is a homeomorphism of the surface which transforms one into the other but preserves the root, its direction and its right side. Since there are homeomorphisms of the closed plane that interchange right and left, the choice of the "left" side of a root is arbitrary.

We now state and discuss a very general enumerative problem on rooted planar maps. We shall solve it however only in special cases. In any rooted planar map let us distinguish the face on the right of the root as the *outer* face. The other faces are the *inner* ones. Let

$$A(m;\, n_1, n_2, n_3, \cdots)$$

denote the number of combinatorially distinct rooted planar maps in which the outer face is an m-gon and the number of inner i-gons is n_i $(i=1, 2, 3, \cdots)$.

It is convenient to say that a vertex-map has exactly one rooting, though the foregoing definition of a rooted planar map is not applicable to this case. We express this convention by writing

$$A(0; \ ) = 1.$$

We now introduce an indeterminate x and write

$$a(n_1, n_2, n_3, \cdots) = \sum_{m=0}^{\infty} A(m; n_1, n_2, n_3, \cdots) x^m.$$

Next we introduce infinitely many more independent indeterminates $y_1, y_2, y_3, \cdots$, and define the generating function

$$= \sum a(n_1, n_2, n_3, \cdots) y_1^{n_1} y_2^{n_2} y_3^{n_3} \cdots.$$

The formal sum is over all possible sets of values of the n_i, with only a finite number nonzero. The formula has terms representing maps with no inner faces at all, including a unit term for the vertex-map. We proceed to obtain an equation for f.

The terms of f fall naturally into three classes. The first class consists solely of the constant term 1.

The second class consists of the terms representing maps, having at least one edge, in which the root is an isthmus. Figure II represents such a rooted planar map. A is the root and F is the outer face.

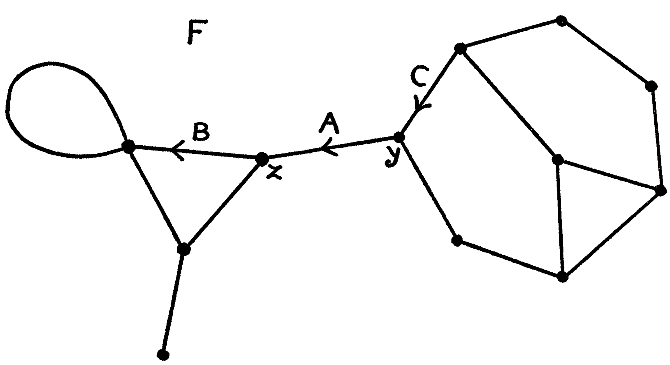

Figure II

If we delete the root, the graph separates into two distinct components, either or both of which may determine vertex-maps.

Suppose the component containing the positive end z of A has at least one edge. It determines a map M_z which has a face F_z containing F, A and the other component of the graph. The bounding path of F_z is that part of the bounding path of F which extends from the occurrence of A directed to z to the occurrence of A directed from z. We can convert M_z into a rooted planar map with outer face F_z by choosing as root the edge B which is traversed by the bounding path of F immediately after that path traverses A to z, and by assigning B the direction in which the path traverses it on this occasion.

Let the negative end of A be y. The second component contains y and determines a planar map M_y which can be rooted in much the same way as M_z, if it is not a vertex-map. We define the root as the edge C traversed by the bounding path of F immediately before it traverses A from y to z, assigning to the root the direction of this traversal.

In virtue of the above construction, each rooted planar map corresponding to a term of the second class decomposes uniquely into a pair of rooted planar maps M_y and M_z, either or both of which may be vertex-maps. The inner faces of the original map M are those of M_y, together with those of M_z, and the valency of the outer face of M exceeds by two the sum of the valencies of the outer faces of M_y and M_z.

Conversely, given two rooted planar maps M_y and M_z, we can combine them with an extra edge A to form a rooted planar map M from which they can be recovered by the above construction. We deduce that the formal sum of the terms of f of the second class is the power series x^2f^2.

The terms of f of the third class correspond to maps, having at least one edge, in which the root is not an isthmus. Consider such a map M. Let its outer face F be an m-gon, and let the inner face F' incident with the root be an i-gon. The root is an edge A directed from y to z. (See Figures III and IV.)

When we delete the root A, we obtain from M a new planar map M_1 in which F and F' are united with A to form a new face F_1. The remaining faces of M_1 are the inner faces of M other than F'.

We can convert M_1 into a rooted map with outer face F_1 in the following way. If $m>1$, as in Figure III, we take the root of M_1 to be the edge B which is traversed by the bounding path of F immediately after it traverses A, and we give B the direction of this traversal.

If $m=1$ but $i>1$, as in Figure IV, we take the root to be the edge B which is traversed by the bounding path of F' immediately before it traverses A. In the remaining case M_1 is a vertex-map and we may consider it rooted by convention.

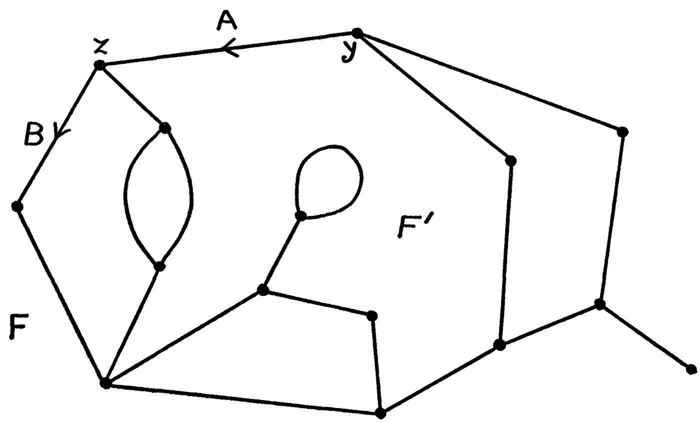

FIGURE III

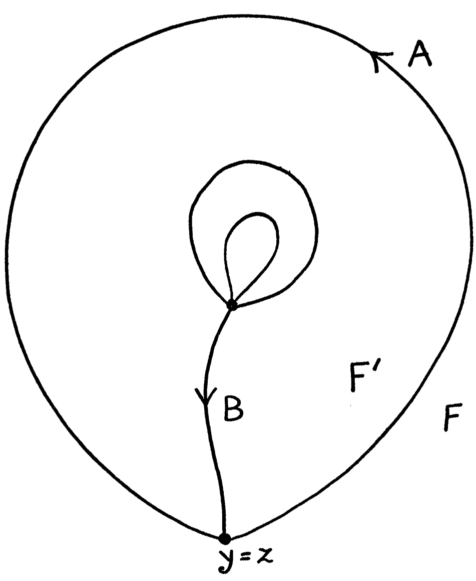

FIGURE IV

Conversely, given a rooted map M_1 with an outer m_1-gon, we can construct a map of the third class from it by adding a new edge A

separating the outer face into an m-gon and an i-gon, where $m_1 = m+i-2$. The new edge is to be directed to the negative end of the old root and to have the m-gon on its right. We then take A as the new root and the m-gon as the new outer face. Note however that this construction is possible, for a given $i \geqq 1$, if and only if $m_1 \geqq i-1$.

The maps which correspond to terms of the third class, and which have a given value of i, are thus enumerated by the function

$$x^{2-i}y_i[f]_{i-1},$$

where $[f]_{i-1}$ denotes the sum of those terms of f in which x occurs at least to the $(i-1)$th power.

Combining the above results we obtain the identity

$$f = 1 + x^2f^2 + \sum_{i=1}^{\infty} x^{2-i}y_i[f]_{i-1}. \tag{1}$$

As a very simple exercise on this result, let us calculate the number $T(n)$ of rooted planar maps with n edges, but with no faces other than the outer one. These maps are the "rooted plane trees" of n edges [2]. If we write f_0 for the sum of the terms of f not involving any of the y_i, then $T(n)$ is the coefficient of x^{2n} in f_0. But, by (1), f_0 is given by the quadratic equation

$$f_0 = 1 + x^2f_0^2. \tag{2}$$

It follows that $T(n) = (2n)!/n!(n+1)!$.

We can now equate the coefficients of y_i in (1) and so obtain an expression for the coefficient of y_i in f, as a function of x. Next we can find the coefficients of y_i^2 and y_iy_j, and so on. We can in fact regard (1) as a recursion formula by which the functions $a(n_1, n_2, n_3, \cdots)$ can be computed.

The special case of (1) in which each y_i with an odd suffix i is set equal to zero has a solution of simple form. This special case, though with a difference of terminology, is discussed in [4]. The "Eulerian maps" of that paper are the duals of the maps whose faces have even valencies.

The general solution of (1) is not yet known. It is possible however to obtain a parametric solution for the special case in which y_i is replaced by yz^i, where x, y and z are independent indeterminates. We denote the function f, after this substitution, by

$$f(x, y, z) = \sum f_{pqr}x^py^qz^r.$$

Thus f_{pqr} is the number of rooted planar maps with an outer p-gon, with q inner faces, and with $(p+r)/2$ edges. It is clear that only a finite number of terms of $f(x, y, z)$ correspond to a given value of $p+r$. Hence we obtain a well-defined power series $f(z, y, z)$ in y and z when we replace x by z in $f(x, y, z)$.

Let us write $f_p x^p$ for the formal sum of those terms of $f(x, y, z)$ in which the index of x is p. Then, by (1),

$$
\begin{aligned}
f(x, y, z) - 1 - x^2 f^2(x, y, z) &= \sum_{i=1}^{\infty} x^{2-i} y z^i \sum_{n=i-1}^{\infty} f_n x^n \\
&= \sum_{n=0}^{\infty} f_n \sum_{i=1}^{n+1} x^{2+n-i} y z^i \\
&= xyz \sum_{n=0}^{\infty} f_n (x^{n+1} - z^{n+1})/(x - z) \\
&= xyz(xf(x, y, z) - zf(z, y, z))/(x - z).
\end{aligned} \tag{3}
$$

Let us denote the numbers of vertices, edges and faces of a planar map by α_0, α_1 and α_2, respectively. Then

$$q = \alpha_2 - 1, \qquad p + r = 2\alpha_1 = 2\alpha_0 + 2\alpha_2 - 4,$$

by the Euler polyhedron formula. Thus

$$f(x, y, z) = \sum f_{pqr} (x/z)^p (yz^2)^{\alpha_2 - 1} (z^2)^{\alpha_0 - 1}.$$

It therefore seems convenient to transform to new variables u, v and w as follows.

$$
\begin{aligned}
u &= x/z, & v &= yz^2, & w &= z^2. \\
x &= uw^{1/2}, & y &= v/w, & z &= w^{1/2}.
\end{aligned}
$$

We write

$$f(x, y, z) = F = F(u, v, w) = \sum F_{pij} u^p v^i w^j.$$

Thus F_{pij} is the number of combinatorially distinct rooted planar maps with an outer p-gon, with $i+1$ faces and with $j+1$ vertices. We note that in each term of F the index of u does not exceed the sum of the indices of v and w. Thus for given values of i and j only a finite number of the coefficients F_{pij} are nonzero. For any real or complex number λ we define $F(\lambda, v, w)$ as the formal power series in v and w in which the coefficient of $v^i v^j$ is $\sum_p F_{pij} \lambda^p$. By the observation just made this sum is a polynomial in λ.

The difference equation (3) can be rewritten as

(4) $$F - 1 - u^2wF^2 = uv(uF - F(1, v, w))/(u - 1).$$

This equation determines F uniquely in the sense of the following theorem.

THEOREM 1. *Let $S(u, v, w)$ be a formal power series in v and w whose coefficients are real functions of a real variable u, all defined in the interval $1-\epsilon<u<1+\epsilon$ for some fixed positive ϵ, and all continuous at $u=1$. Suppose further that $S(u, v, w)$ satisfies the identity*

$$(S(u, v, w) - 1 - u^2wS^2(u, v, w)) \times (u - 1)$$
$$= uv(uS(u, v, w) - S(1, v, w))$$

for all values of u in the interval. Then $S(u, v, w)=F(u, v, w)$ for all such values of u.

PROOF. We have $S(u, v, w)=S_0+S_1v+S_2v^2+ \cdots$ where the S_i are independent of v. They are formal power series in w whose coefficients are real functions of u continuous at $u=1$. Let T_i denote the series obtained by putting $u=1$ in S_i.

From the given identity we have, when $u\neq 1$, $S_0-1-u^2wS_0^2=0$, whence

$$S_0 = (1 - (1 - 4u^2w)^{1/2})/(2u^2w).$$

Thus S_0 is uniquely determined, by continuity when $u=1$. But suppose the S_i have been found as far as $i=m$. For any value of u, other than 1, in the relevant interval we can equate coefficients of v^{m+1} in the given identity and so obtain an expression for

$$S_{m+1}(1 - 2u^2wS_0), \quad \text{i.e.} \quad S_{m+1}(1 - 4u^2w)^{1/2},$$

in terms of the S_i and T_i with $i\leqq m$. Then S_{m+1} is determined for all values of u, other than 1, in the given interval. The result can be extended to the case $u=1$ by continuity.

We deduce that $S(u, v, w)$ is uniquely determined by the given conditions. Since $F(u, v, w)$ satisfies these conditions the theorem follows.

We can obtain a parametric solution of (4) in the following way. From now on we interpret v and w as complex variables. We define independent parameters s and t in terms of them by the following equations

(5) $$v = s(s + 2)/4(s + t + 1)^2, \qquad w = t(t + 2)/4(s + t + 1)^2.$$

The parameters s and t are functions of v and w, analytic in a sufficiently small neighbourhood of (0, 0) [1, I, § 188, and II, Part I, § 104].

We write

$$h = 4(s + t + 1)/(s + 2)(t + 2). \tag{6}$$

THEOREM 2. $h = F(1, v, w)$.

PROOF. If this theorem holds, it follows from (4) that $F(u, v, w)$ is one solution for X of the quadratic

$$AX^2 + BX + C = 0, \tag{7}$$

where

$$\begin{aligned} A &= u^2(u - 1)w, \\ B &= u^2v - u + 1, \\ C &= u - 1 - uvh. \end{aligned}$$

Putting $U = u/2(s+t+1)$ and using the above definitions of A, B and C we find that

$$\begin{aligned} B^2 - 4AC = {} & U^4(s^4 + 4s^3 + (-16t^2 - 16t + 4)s^2 \\ & + (-32t^3 - 80t^2 - 48t)s \\ & \qquad + (-16t^4 - 64t^3 - 80t^2 - 32t)) \\ & + U^3(-4s^3 + (-4t - 12)s^2 + (16t^2 + 16t - 8)s \\ & \qquad + (16t^3 + 48t^2 + 32t)) \\ & + U^2(6s^2 + (8t + 12)s + 4) \\ & + U(-4s + (-4t - 4)) \\ & + 1 \\ = {} & (U(s + 2t + 2) - 1)^2(1 - 2sU + (s^2 - 4st - 4t(t + 2))U^2) \\ = {} & (2s + 2t + 2)^{-4}((u - 1)(2s + 2t + 2) - us)^2Q^2, \end{aligned}$$

where

$$Q^2 = u^2(s + 2)^2 - 4(u - 1)(s + t + 1)(u(t + 1) + s + t + 1)$$

and $Q = 2$ when $s = t = 0$.

When Q^2 is expanded as a polynomial in s and t, the term not involving s or t is 4. We deduce that for all real values of u the function Q is an analytic function of s and t in the neighbourhood of (0, 0). The coefficient of s^1t^0 in Q is found to be $2 - u$, and that of s^0t^1 is $1 - u^2$. Thus when $u = 1$ we have $Q = s + 2$.

We may now write

$$4(s+t+1)^2(-B-(B^2-4AC)^{1/2})=4(u-1)(s+t+1)^2-u^2s(s+2)$$
$$-2(u-1)(s+t+1)Q+usQ.$$

It can be verified that when the expression on the right is expanded in powers of s and t the terms of zero degree in t have zero coefficients. Hence for each positive u, other than 1, there is a function X_u of s and t which is analytic in the neighbourhood of (0, 0), which satisfies

$$AX_u^2 + BX_u + C = 0, \tag{8}$$

and which is given by the following equation

$$2u^2(u-1)t(t+2)X_u=4(u-1)(s+t+1)^2-2(u-1)(s+t+1)Q$$
$$-4us(u(s+2)+Q)^{-1}(u-1)(s+t+1)(u(t+1)+s+t+1),$$

that is,

$$X_u=\frac{(s+t+1)}{u^2t(t+2)}\left\{2(s+t+1)-Q-\frac{2us(u(t+1)+s+t+1)}{u(s+2)+Q}\right\}. \tag{9}$$

We use this formula to define X_u when $u=1$. We observe that X_1 is an analytic function of s and t in the neighbourhood of (0, 0) and that, by continuity, it satisfies (8). But when we put $u=1$, Q is transformed into $s+2$, and it follows from the above formula, together with (6), that $X_1=h$.

An application of Theorem 1 now shows that the function X_u given by (9) is identical with $F(u, v, w)$ for $0<u<2$. Theorem 2 follows. We deduce also that $F(u, v, w)$, for complex variables u, v and w, is the expansion of the expression on the right of (9) in powers of u, v and w.

Another formulation of the definition of h runs as follows. The parameter $\mu=s+t+1$ is given by the equation

$$\mu = (1 + 4v\mu^2)^{1/2} + (1 + 4w\mu^2)^{1/2} - 1,$$

by (5). We then have

$$\begin{aligned} h &= ((1 + 4v\mu^2)^{1/2} - 1)((1 + 4w\mu^2)^{1/2} - 1)/4vw\mu^3 \\ &= ((1 - 4v - 4w)\mu^{-1} - \mu^{-3})/(8vw). \end{aligned}$$

The coefficient h_{ij} of v^iw^j in h is the number of rooted planar maps with $i+1$ faces and $j+1$ vertices. By topological duality we have $h_{ij}=h_{ji}$. This is related to the definition of h as a function symmetrical in v and w.

The author has calculated the first few terms of h, with the following result.

$$\begin{aligned}
h = {} & 1 + v + 2v^2 + 5v^3 + 14v^4 + 42v^5 + 132v^6 \\
& \qquad + 429v^7 + 1430v^8 + 4862v^9 + 16796v^{10} + \cdots \\
& + w(1 + 5v + 22v^2 + 93v^3 + 386v^4 + 1586v^5 \\
& \qquad + 6476v^6 + 26333v^7 + 106762v^8 + 431910v^9 + \cdots) \\
& + w^2(2 + 22v + 164v^2 + 1030v^3 + 5868v^4 + 31388v^5 \\
& \qquad + 160648v^6 + 795846v^7 + 3845020v^8 + \cdots) \\
& + w^3(5 + 93v + 1030v^2 + 8885v^3 + 65954v^4 + 442610v^5 \\
& \qquad + 2762412v^6 + 16322085v^7 + \cdots) \\
& + w^4(14 + 386v + 5868v^2 + 65954v^3 + 614404v^4 \\
& \qquad + 5030004v^5 + 37460376v^6 + \cdots) \\
& + w^5(42 + 1586v + 31388v^2 + 442610v^3 + 5030004v^4 \\
& \qquad + 49145460v^5 + \cdots) + \cdots .
\end{aligned}$$

The calculations were checked by the use of the formula

$$\sum_{i+j=n} h_{ij} = 2(2n)!3^n/(n!(n+2)!)$$

which is a consequence of [3, § 5]. It can also be derived from equations (5) and (6) by putting $s=t$ and $v=w$, and then using Lagrange's expansion.

References

1. E. Goursat, *A course in mathematical analysis.* I and II, Part I, Ginn, Boston, 1904 and 1916.
2. F. Harary, G. Prins and W. T. Tutte, *The number of plane trees*, Indag. Math. **26** (1964), 319–329.
3. W. T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271.
4. ———, *A census of slicings*, Canad. J. Math. **14** (1962), 708–722.

University of Waterloo