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VERY FUNCTORIAL, VERY FAST, AND VERY EASY RESOLUTION OF SINGULARITIES

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Abstract. The main proposition, Theorem 1.2, is the existence for excellent Deligne–Mumford champ of characteristic zero of a resolution functor independent of the resolution process itself. Received wisdom was that this was impossible, but the counterexamples overlooked the possibility of using weighted blow ups. The fundamental local calculations take place in complete local rings, and are elementary in nature, while being self contained and wholly independent of Hironaka's methods and all derivatives thereof, i.e. existing technology. Nevertheless Abramovich et al. (Functorial embedded resolution via weighted blowing ups, 2019. arXiv:1906.07106), have varied existing technology to obtain even shorter proofs of all the main theorems in the pure dimensional geometric case. Excellent patching is more technical than varieties over a field, and so easier geometric arguments are pointed out when they exist.

1 Introduction

Before beginning it must be noted that it was envisaged that something similar to Sections 2–4 would be the doctoral thesis of Gianluca Marzo. However, "the best laid schemes o' mice an' men gang aft agley", and, rather selfishly, I ended up writing these sections myself. Nevertheless, this didn't stop him from selflessly preparing the manuscript with care and attention, for which, along with his preliminary work on the aforesaid sections, he deserves substantive acknowledgement.

Otherwise: it is a known fact that resolution of singularities, already in characteristic zero, cannot be achieved in a way that is both étale local and independent of the resolution process itself while blowing up in smooth centres. An appropriately general category in which to work is that of reduced excellent algebraic spaces, which should be understood according to,

Remark/Definition 1.1. All Henselian local rings excellent together with an atlas which is Noetherian and J2. In particular an excellent algebraic space in this sense which is also a scheme is only quasi-excellent in standard parlance. Indeed the catenary condition is close to meaningless for algebraic spaces—its only interpretation is that every global irreducible component is everywhere étale locally equidimensional, [EGA-IV, 7.8.4 (iii)], and we won't use this—so it will be systematically eschewed globally while étale locally it is tautologically true.

Thus, a resolution process independent of its history, is a modification functor $U \mapsto M(U)$ from reduced excellent spaces to itself, together with an invariant $\mathrm{inv}(U) \in \Gamma_{\geqslant 0}$ in a (preferably discrete) ordered group such that
(M.1 bad) $M(U) \to U$ is a blow up in a smooth centre.
(M.2) $U = M(U)$ iff U is regular.
(M.3) M commutes with étale base change $U' \to U$, i.e. $M(U') = M(U) \times_U U'$ whenever U, U' are connected and $\mathrm{inv}(U') = \mathrm{inv}(U)$.
(M.4) For any $U' \to U$ étale, $\mathrm{inv}(U') \le \mathrm{inv}(U)$.
(M.5) $\mathrm{inv}\big(M(U)\big) < \mathrm{inv}(U)$ if U is not regular.
The impossibility of this is shown by the following example, cf. [Kol07, p. 142],

$$U : x^2 + y^2 + (zt)^2 = 0 \hookrightarrow \mathbb{A}^4_K \tag{1.1}$$

wherein the singular locus is the union of the two lines,

$$L_1 : x = y = z = 0 \ \&\ L_2 : x = y = t = 0.$$

On the other hand if $M(U) \to U$ were to exist then by (M.1 bad), (M.2) and (M.3) it must be a blow up in a smooth centre contained in the singular locus, so the only possibilities are L_1, L_2 or their intersection, i.e. the origin. Now the latter operation leaves (1.1) unchanged where the proper transform of either line meets the exceptional divisor, while a choice amongst L_1, L_2 is inadmissible because the process must respect, (M.3), the symmetry $z \leftrightarrow t$, and that's without even addressing the issue that (1.1) might only be valid after Henselisation, so that globally the L_i could be branches of the same curve.

The traditional get out from this difficulty is to change the problem, e.g. the argument of the modification functor becomes not just varieties but varieties with marked divisor, so, in particular, blowing up (1.1) in the origin creates a marked divisor and amongst the new singular lines one of them is marked. The point of view of this article is, however, to change the paradigm and adapt the modification to the problem, so that (M.1 bad) is replaced by,

(M.1 new) $M(U) \to U$ is a smoothed weighted blow up in a regular centre.

This operation is, briefly, defined in Revision/Definition 6.2 and in detail in [MP13, I.iv.3]. It should, however, be noted that $M(U)$ is by definition, op.cit., a (Deligne-Mumford) champ, albeit if we were to work with varieties over $\mathbb{C}$ the 2-category of orbifolds would be adequate for our current purposes, and in any case the 2-category of champs/orbifolds is just a categorical subterfuge which allows us to work with quotient singularities while doing linear algebra. With this in mind, the paradigm shift works, i.e.

Theorem 1.2 (Proposition 7.8, cf. [ATW19, 7.1.1] in the geometric case). In the 2-category of characteristic zero reduced excellent Deligne–Mumford champ (defined exactly as above for spaces so inter alia with no separation condition) there is a modification functor $U \mapsto M(U)$, Summary/Definition 7.7, satisfying (M.1 new), (M.2), (M.3), (M.4), (M.5), albeit inv takes values in $\mathbb{Q}^{\infty}_{\geqslant 0} = \varinjlim \mathbb{Q}^{N}_{\geqslant 0}$. Nevertheless, the invariant has self-bounding denominators, Definition 3.1.

Here self bounding denominators is a certain technical condition, Definition 3.1, which has all the effects, Fact 3.2, of defining the invariant in $\mathbb{Z}_{\geqslant 0}$ while allowing us to define the invariant and perform various constructions, e.g. Sub-Induction 3.13, where they naturally occur, i.e. $\mathbb{Q}_{\geqslant 0}$. More substantially Theorem 1.2 is the global manifestation of some much more basic local algebra. Specifically for I an ideal of a m-dimensional regular local ring, A, of characteristic zero, with maximal ideal $\mathfrak{m}$ we construct an invariant, Section 3, $\mathrm{inv}_A(I)$ with self bounding denominators in $\mathbb{Q}^{2m}_{\geqslant 0}$ ordered lexicographically.

Better there is a yoga for constructing inv that makes the resolution process more widely applicable to more difficult problems such as vector field singularities, which, essentially views the resolution process as a diagram chase, and manifests itself as follows,

(Y.1) Generically most things are regular, a.k.a. $I = \mathcal{O}$, so inv $= \underline{0}$ and there is nothing to do.

(Y.2) If (Y.1) didn't happen then generically most things have an isolated singularity at the closed point, and after a single blow up in the same the multiplicity will decrease,

(Y.3) If (Y.2) didn't happen then there is a proper sub-space of the tangent space where the multiplicity did not decrease and its annihilator in $\mathfrak{m}/\mathfrak{m}^2$ gives us the start of a filtration of A which depends only on I.

(Y.4) Construct inductively, Start of the Induction 3.6–Inductive Hypothesis 3.7, a sequence of filtrations, $F^{\bullet}_s(I)$, according to the dichotomy.

Case(A) 1.3. Something generic happens, Case(A) 3.16, then $s \mapsto s+1$;

Case(B) 1.4. Nothing generic happens, Case(B) 3.17, then at worst, $F^{\bullet}_s(I)$ converges $\mathfrak{m}$-adically.

Proceeding in this way leads to the key,

Fact 1.5. [cf. Fact 6.3] There is an invariant, $\mathrm{inv}_A(I) \in \mathbb{Q}^{2m}_{\geqslant 0}$, of regular m-dimensional characteristic zero local rings and their ideals with self bounding denominators such that if $\mathfrak{U}$ is the completion of its spectrum at the closed point, then there is a smoothed weighted blow up $\rho : \widetilde{\mathfrak{U}} \longrightarrow \mathfrak{U}$ such that at every closed point of $\mathfrak{U}$ the invariant strictly decreases provided $I \neq A$.

Rather plainly at this point the only remaining issue is whether the weighted centre defining ρ is well defined in A, or even just its strict Henselisation A^{h}, rather than its completion, $\widehat{A}$. It is, however, a genuine issue since both [EGA-IV, 7.9.3] and its proof are valid exactly as stated even on allowing resolutions either by algebraic spaces or Deligne–Mumford champs, i.e. excellent Henselian local rings (which is the same as quasi-excellent and Henselian) are a necessary condition for resolution of singularities. It is therefore pleasing to observe that (quasi-) excellence, cf. Remark/Definition 1.1, is just what's needed to establish.

Proposition 1.6. (cf. Alternative 6.10) If the centre in Fact 1.5 is of dimension 0 or, A is an excellent regular local ring then, Fact 6.9, the (canonically defined) smoothed weighted blow up of Fact 6.3 is the completion in the exceptional divisor of a smoothed weighted blow up of $\operatorname{Spec} A$. Similarly if A is an excellent reduced local ring, $\mathfrak{V}$ its completion in the closed point, and $\rho : \widetilde{\mathfrak{V}} \to \mathfrak{V}$ the proper transform of $\mathfrak{V}$ along ρ of Fact 1.5 after a choice of an embedding of $\mathfrak{V} \hookrightarrow \mathfrak{U}$ in a smooth formal scheme, Set Up/Construction 7.1, then there is a smoothed weighted blow up, Fact 7.5, of $\operatorname{Spec} A$ whose completion in the exceptional divisor is ρ.

Needless to say convergence is (much) easier when everything is of finite type over a field, and whence Alternative 6.10, resp. Alternative 7.6, are offered to the more general Fact 6.9, resp. Fact 7.5. Similarly to go from convergence to Theorem 1.2 one needs the upper semi-continuity of the invariant which is an attractive consequence, Alternative 7.3, of the properties peculiar to the diagonal in the geometric case. Otherwise, Fact 6.11 and Alternative 7.3, this adopts Dade's proof of the u.s.c. of the multiplicity in his un-published 1960 Princeton thesis (of which Villamayor's summary, [Vil14, 6.1.3], was invaluable) and leads to the wholly natural intervention of the (global) J-2 condition. It is also important not to lose sight of the wood for trees, and in particular the critical principalisation statement which in the geometric case has been obtained simultaneously, [ATW19, 5.1.1], by Abramovich, Temkin, and Wlodarczyk,

Theorem 1.7 [Proposition 6.14]. There is a modification functor from the 2-category whose objects, $(U, \mathcal{I})$, are ideals on regular excellent characteristic zero Deligne–Mumford champs whose value

$$M_{(U,\mathcal{I})} = (\widetilde{U}, \widetilde{\mathcal{I}}) \tag{1.2}$$

is the proper (rather than total) transform $\widetilde{\mathcal{I}}$ on a smoothed weighted blow up $\widetilde{U} \to U$, satisfying (in the obvious change of notation) (M.1 new), (M.2), (M.4), (M.5), while (M.3) becomes, $M_{(U,\mathcal{I})} = 0$ iff $\mathcal{I} = \mathcal{O}_U$, and, again, inv takes values in $\mathbb{Q}^{\infty}_{\geqslant 0} = \varinjlim \mathbb{Q}^{N}_{\geqslant 0}$ with self-bounding denominators.

which is equivalent to the more pleasing assertion that there is a fully étale local modification functor, Fact/Definition 6.13, by smoothed weighted blow ups which resolves any rational map. In any case, the paper is organised as follows,

§2 This contains some linear algebra about weighted projective spaces (technically champs because we want them to be regular) which describes the manifestation of item (Y.3) above in the generality necessary for the distinctions between generic and non-generic phenomena in item (Y.4).

§3 This is the inductive definition of the invariant as outlined in (Y.1)–(Y.4). The key step is the Sub-Induction 3.13 whose illustration by way of its Newton polyhedron,

Figure 1, should facilitate not only its understanding, but illustrates the pleasing fact, observed by the referee, that the method is a wholly natural generalisation of Newton's rotating ruler, cf. [Kol07, 1.1]. One should, however, note that the face of the Newton polyhedron defined by the invariant while extremal and functorial need be neither co-dimension 1 nor that, even for curves, which would lead to the quickest algorithm, cf. [You90, pg. 2]

§4 Calculates the invariant for ideals on weighted projective champs. It is the proof that the invariant goes down on blowing up in its weighted centre.

§5 This begins to address the aforesaid convergence issues, and related questions such as upper semi-continuity of the invariant by calculating it in a suitably general, Observation/Definition 6.1, relative setting. It does convergence and u.s.c. out of the box in the geometric case of A/K essentially of finite type over a field of characteristic zero on completing 2 copies of Spec A in the diagonal, while more generally, cf. Variant 6.6, systematically working with formal champs sidesteps thorny issues like the diagonal is an embedding iff the champ is a (separated) algebraic space.

§6 Is the final assembly of the preceeding into a modification functor, Proposition 6.14, for the (weak) principalisation (a.k.a. resolution of rational maps, Remark 6.15) of ideals on excellent regular champs. Unlike the preceeding sections it assumes a working familiarity with the rudiments of algebraic champs and is much less elementary than Sections 2–4 wherein any intervention of champs does not go much beyond linear algebra of graded rings.

§7 Pushes things into a resolution functor, Summary/Definition 7.7, for excellent champs. The geometric case is easy, Alternatives 7.3 and 7.6, for a geometric reason, cf. the summary of **§5** above, and otherwise it's an exercise in appreciating Grothendieck's excellent definition.

All of which, as the referee observed, may usefully be tidied up by way of a clarifying,

Remark 1.8. Although characteristic zero is a blanket assumption, Set Up/Definition 2.1, Set Up/Notation 3.3, Set Up 4.1, Set Up/Notation 5.1, Set Up/Construction 7.1, it is not equally necessary in all sections. Specifically Sections 2–3 could have, at the cost of a certain complication, been done in any characteristic. However the key fact, Proposition 4.3, of Section 4 fails in characteristic p, i.e. the invariant needn't decrease under blowing up. The first counterexample (for which I'm indebted to Dan Abramovich) is the characteristic p Whitney umbrella $x^p = y^p z$. As such, the extra complication required to carry out Sections 2–3 in characteristic p didn't appear to be merited.

Talks about the paper have been given at U.C.S.D., N.Y.U., and Valencia (in homage to the university's founder and his nephew, and in no way related to the Celtic game) but it was from an informal seminar in Paolo Cascini's office at Imperial, in February, 2019, from which Johannes Nicaise brought news to Oberwolfach, where, amongst the participants Dan Abramovich, Michael Temkin and Jaroslaw Wlodarczyk provided demonstrable proof that they were writing, and have now written, [ATW19], an algorithm satisfying (M.1 new), (M.2), (M.3) and (M.4). Similarly, credit must also go to Daniel Panazzolo who although he did not participate in the preparation of this manuscript introduced the majority of the key ideas in [Pan06]. Indeed, the only one he was missing was the functoriality yoga, cf. (Y.1)–(Y.4), which first appeared in [MP13]. Finally it is a pleasure to thank the referee for a pertinent and efficient examination of the manuscript.

2 Weighted Projective Champs

Set Up/Definition 2.1. Throughout this section, k is a ring of characteristic 0, and $A_k := \mathbb{A}_k^{N+1} \backslash \{\underline{0}\}$. For $n \leqslant N$, let $\underline{a} = (\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_n) \in \mathbb{Z}_{>0}^{N+1}$ with each $\underline{a}_i = \big(a_i, \ldots, a_i\big) \in \mathbb{Z}_{>0}^{c_i}$, $c_i \geqslant 1$ and $N+1 = c_0 + \cdots + c_n$. We denote the coordinates of $\mathbb{A}_k^{N+1}$ by x_{ij} for $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant c_i$, and we will call the set of variables with the same weight a_i, i.e. $\{x_{i\,1}, \ldots, x_{i\,c_i}\}$, a *block*, or a *block of weight* a_i, and often abbreviate it by X_i, similarly, consistent with this decomposition, we will abbreviate monomials $\prod x_{ij}^{e_{ij}}$ by $X_i^{E_i}$, where $|E_i| = \sum_j e_{ij}$ (i.e. the degree of the monomial in the relevant block); while $X_i = 0$ means $x_{ij} = 0$, $\forall\, 1 \leqslant j \leqslant c_i$.

Definition 2.2. The weighted projective champ $\mathbb{P}_k(\underline{a}) := \mathbb{P}(\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_n)$ is defined to be the classifying champ $[A_k/\mathbb{G}_m]$ of the action

$$\mathbb{G}_m \times A_k \overset{\lambda}{\underset{\mathrm{id}}{\rightrightarrows}} A_k,\ (\lambda^{a_0} X_0, ..., \lambda^{a_n} X_n) \xleftarrow{\lambda} (X_0, ..., X_n) \xmapsto{\mathrm{id}} (X_0, ..., X_n) \tag{2.1}$$

on which the tautological bundle $\mathscr{O}_{\mathbb{P}_k(\underline{a})}(1)$ corresponds to the character:

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m : \lambda \longmapsto \lambda^{-1}. \tag{2.2}$$

In particular, functions on $\mathbb{A}_k^{N+1}$ are naturally graded by the action, and we denote the grading of a $\mathbb{G}_m$-homogeneous equivariant function by $\mathbf{wt}$, i.e.

$$\mathbf{wt}(X_i) = a_i, \quad \mathbf{wt}(X_i^{E_i}) = a_i |E_i| \tag{2.3}$$

Finally if $\underline{r} = \big(\underline{r}_0, \ldots, \underline{r}_n\big) \in \mathbb{Q}_{>0}^{N+1}$, $\underline{r}_i := (r_i, \ldots, r_i) \in \mathbb{Q}_{>0}^{c_i}$, and $\underline{a} \in \mathbb{Z}_{>0}^{N+1}$ is the unique integer tuple parallel to $\underline{r}$ without common factors we define

$$\mathrm{P}_k(\underline{r}) := \mathbb{P}_k(\underline{a}) \tag{2.4}$$

to which we add the hypothesis specific to our situation i.e.

Hypothesis 2.3. Suppose $a_0 < a_1 < \cdots < a_n$ and let V_d be a k-submodule of $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(d)\big)$, $d \geqslant 0$, such that if $\mathbb{P}_k(\underline{a}') = \mathbb{P}_k(\underline{a}_1, \dots, \underline{a}_n)$ is the sub-weighted projective champ defined by the block of variables $X_0 = 0$ of weight a_0 and V'_d is the image of V_d in $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}'), \mathscr{O}_{\mathbb{P}_k(\underline{a}')}(d)\big)$ then for all quotients $k \twoheadrightarrow k'$, $-b < 0$ and $\partial \in \mathrm{H}^0\big(\mathbb{P}_{k'}(\underline{a}'), \mathrm{T}_{\mathbb{P}_{k'}(\underline{a}')}(-b)\big)$

$$\partial(f') = 0, \ \forall\, f' \in V'_d \otimes_k k' \iff \partial = 0. \tag{2.5}$$

In the presence of such a supposition we have,

Lemma 2.4. Let everything be as in Set Up/Definition 2.1–Hypothesis 2.3, and for $-b < 0$ a strictly negative integer define

$$\mathrm{L}_{-b}(V_d) := \big\{\, \partial \in \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}),\, \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-b)\big) \,\big|\, \partial(V_d) = 0 \,\big\} \tag{2.6}$$

the sub-module of global weighted vector fields of weight $-b$ which vanish on V_d. Then if $b \neq a_0$, $\mathrm{L}_{-b}(V_d) = 0$, otherwise there is a natural injective map,

$$\mathrm{L}_{-a_0}(V_d) \hookrightarrow \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0)\big)^{\vee} := \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)^{\oplus c_0}\big)^{\vee}. \tag{2.7}$$

Better still if for every quotient $k \twoheadrightarrow k'$,

$$\mathrm{L}_{-a_0} \otimes_k k' = \big\{\, \partial \in \mathrm{H}^0\big(\mathbb{P}_{k'}(\underline{a}),\, \mathrm{T}_{\mathbb{P}_{k'}(\underline{a})(-b)}\big) \,\big|\, \partial\big(V_d \otimes_k k'\big) = 0 \,\big\} \tag{2.8}$$

then (2.7) remains an injection on tensoring with k'.

Proof. Without loss of generality $\dim \mathbb{P}_k(\underline{a}) > 0$, so from the Euler Sequence,

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}_k(\underline{a})} \longrightarrow \textstyle\coprod_{i=0}^{n} \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_i)\big(:= \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_i)^{\oplus c_i}\big) \longrightarrow \mathrm{T}_{\mathbb{P}_k(\underline{a})} \longrightarrow 0, \tag{2.9}$$

tensored by $\mathscr{O}_{\mathbb{P}_k(\underline{a})}(-b)$ there is an isomorphism in co-homology

$$\textstyle\coprod_{i=0}^{n} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_i - b)\big) \xrightarrow{\ \sim\ } \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-b)\big) \tag{2.10}$$

unless $\dim \mathbb{P}_k(\underline{a}) = 1$ and $b = a_0 + a_1$. Indeed this is trivial for $\dim \mathbb{P}_k(\underline{a}) > 1$ by the analogue of Serre's explicit calculation for weighted projective champ, [McQ17, I.c.3], while by, op.cit., for $\dim \mathbb{P}_k(\underline{a}) = 1$, $\mathscr{O}_{\mathbb{P}_k(\underline{a})}(-b)$ has non-trivial h^1 if and only if $b \geqslant a_0 + a_1$ and $\mathrm{T}_{\mathbb{P}_k(\underline{a})} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0 + a_1)$, so again (2.10) follows unless $b = a_0 + a_1$. To avoid this fastidious exception observe that it only occurs if $\dim \mathbb{P}_k(\underline{a}) = 1$, $\mathbb{P}_k(\underline{a}) = \mathbb{P}(a_0, a_1)$ is defined by blocks X_0, X_1 of rank 1 and, by hypothesis, weights $a_0 < a_1$. As such if D were an element of $\mathrm{L}_{-a_1 - a_0}(V_d)$ then $X_0\mathrm{D} \in \mathrm{L}_{-a_1}(V_d)$, and $a_1 \neq a_0$, so Lemma 2.4 for $b = a_1$ implies the same for $b = a_1 + a_0$. Thus without loss of generality $b \neq a_1 + a_0$ if $\dim \mathbb{P}_k(\underline{a}) = 1$.

Now suppose $b \neq 0$ and $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big) \neq 0$, then $a_0 > b > 0$ which is equivalent to $a_0 > a_0 - b > 0$. However, for any $e > 0$, [McQ17, I.c.3],

$$\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big) = \coprod_{|E_0|a_0+\cdots+|E_n|a_n=e} k \cdot X_0^{E_0} \cdots X_n^{E_n} \tag{2.11}$$

so $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big) \neq 0$ implies $e = |E_0|a_0 + \cdots + |E_n|a_n \geqslant a_0$. Thus $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big) = 0$ for $a_0 > b > 0$ so by (2.10) both items in Lemma 2.4 will follow from the more general:

Claim 2.5. Let $b > 0$ (so $b = a_0$ is allowed) and pr the projection,

$$\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big) \xleftarrow{\mathrm{pr}} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-b)\big),$$

afforded by (2.10), then the submodule $\mathrm{L}^0_{-b} := \{\partial \in \mathrm{L}_{-b} \mid \mathrm{pr}(\partial) = 0\} \subset \mathrm{L}_{-b}$ consists only of the null derivation.

Proof. In order to emphasise their role say, by way of notation, that $\{y_1, \ldots, y_{c_0}\}$ is the (since any other is obtained via the action of $\mathrm{GL}_k(c_0)$) block $Y := X_0$ of weight a_0, and $\{x_{i\bullet}\}$, $i > 0$, are blocks X_i of weight $a_i > a_0$. Then $\partial \in \mathrm{H}^0(\mathbb{P}_k(\underline{a}), \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-b))$ can be written as

$$\partial = \sum\nolimits_I Y^I \partial_I, \qquad \text{with } \mathbf{wt}(\partial_I) = -b - |I|a_0 < 0. \tag{2.12}$$

where by hypothesis $\partial \mapsto 0$ in $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big)$, thus by (2.10) we have,

$$\partial \in \coprod\nolimits_{i \geqslant 1} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a_i} - b)\big)$$

and each ∂_I may be naturally identified to an element of $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}'), \mathrm{T}_{\mathbb{P}_k(\underline{a}')}(-b - |I|a_0)\big)$ via the Euler sequence and $\mathbb{G}_m$-equivariance. Now, suppose $\partial \in \mathrm{L}^0_{-b}$ different from 0 and let

$$i_0 = \min\{\, |I| \mid \partial_I \neq 0 \text{ for some } |I| \text{ as in (2.12)}\,\}.$$

Similarly, we can (again, wholly canonically because of the $\mathbb{G}_m$-equivariance) write each $f \in V_d$ as

$$f = f' + \sum\nolimits_{|J|>0} f_J Y^J, \quad \mathbf{wt}(f') = \mathbf{wt}(f_J Y^J) = a_0|J| + \mathbf{wt}(f_J) = d, \tag{2.13}$$

where f' and f_J are non-zero $\mathbb{G}_m$-homogeneous polynomials in the variables $X_1, \ldots, X_n$ (f' may be identified with its image in $V'_d \subseteq \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}'), \mathscr{O}_{\mathbb{P}_k(\underline{a}')}(d)\big)$) so, by hypothesis, $\partial(f) = 0$ and on the other hand

$$\partial(f) = \sum\nolimits_{|I|=i_0} \Big(Y^I \partial_I(f') \quad + \sum\nolimits_J Y^{I+J} \partial_I(f_J)\Big), \tag{2.14}$$

where $Y^{I+J}\partial_I(f_J)$ consists of monomials where Y is of degree $> i_0$, therefore $\partial(f) = 0$ only if $\sum_{|I|=i_0} Y^I \partial_I(f') = 0$. However, on identifying (as ever via the

$\mathbb{G}_m$-equivariance) V'_d with a subspace of V_d, $\partial_I(f')$ depends only on the blocks $X_{\geqslant 1}$, so $\partial_I(f') = 0$ for all $|I| = i_0$, which, by Hypothesis 2.3, implies the absurdity $\partial_I = 0$. $\square$

This certainly implies Lemma 2.4 when $b \neq a_0$, while for $b = a_0$ we have

$$\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-a_0)\big) \xrightarrow{\mathrm{pr}} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}(\underline{a}_0 - a_0)\big) \xrightarrow{\sim} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}(a_0)\big)^\vee$$

so in this case the claim is exactly (2.7). Better since by construction the Hypothesis 2.3 is stable under base change to an arbitrary quotient of k, our initial conclusions are too, so (2.7) is an injection on tensoring as soon as the definition of L_{-a_0} enjoys the stability under base change in (2.8) $\square$

To profit from the lemma, let us introduce,

Notation/Definition 2.6. Let $W := W_0 \amalg \cdots \amalg W_n$ be a k-module with a $\mathbb{G}_m$-action such that $\mathbb{G}_m$ acts on W_i by the character λ^{b_i}, $b_i \in \mathbb{Z}$, for $0 \leqslant i \leqslant n$, then for $q \in \mathbb{Z}$, $\underline{\mathrm{Sym}}^q(W)$ is the subspace of the symmetric algebra $\mathrm{Sym}(W)$ where $\mathbb{G}_m$ acts by the character λ^q. Similarly, given blocks X_i, $n \geqslant i \geqslant 0$, as in Set Up/Definition 2.1, with a slight abuse of notation, we define

$$\underline{\mathrm{Sym}}^q(X_0 \amalg \cdots \amalg X_n) := \coprod_{|E_0|a_0 + \cdots + |E_n|a_n = e} k \cdot X_0^{E_0} \cdots X_n^{E_n}$$

so, in this notation (2.11) is $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big) = \underline{\mathrm{Sym}}^e(X_0 \amalg \cdots \amalg X_n)$. Finally, as in (2.4), if the weights $r_0, \ldots, r_n \in \mathbb{Q}_{>0}$ were any rationals and $(a_0, \ldots, a_n) = D(r_0, \ldots, r_n)$ the unique parallel tuple of positive integers without common factors, we define for $q \in \mathbb{Q}_{\geqslant 0}$

$$\underline{\mathrm{Sym}}^q\big(X_0 \amalg \cdots \amalg X_n\big) := \coprod_{a_0|E_0| + \cdots a_n|E_n| = Dq} k \cdot X_0^{E_0} \cdots X_n^{E_n}. \tag{2.15}$$

In any case to apply the lemma, observe that,

$$\mathrm{L} := \coprod\nolimits_{b>0} \mathrm{L}_{-b} = \mathrm{L}_{-a_0} \tag{2.16}$$

is plainly a Lie algebra wherein by (2.7) the bracket is even trivial; thus

Corollary 2.7. Again let everything be as in Set Up/Definition 2.1–Hypothesis 2.3 and suppose further that (2.7) is an isomorphism onto a trivial (i.e. admitting a basis) free k-module. As such there is a block Z associated to the annihilator of L, i.e.

$$\bigcap\nolimits_{\partial \in \mathrm{L}} \ker(\partial) \subset \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0)\big), \text{ and,} \tag{2.17}$$

(i) there are *blocks*, i.e. weighted projective coordinates $X_1, \ldots, X_n$, of weight $a_1, \ldots, a_n$, generating a space of functions, X, such that

$$V_d \subset \underline{\mathrm{Sym}}^d\big(X \amalg Z\big) := \underline{\mathrm{Sym}}\big(X_0 \amalg \cdots \amalg X_n \amalg Z\big).$$

(ii) If $\widetilde{X}_i$, $1 \leqslant i \leqslant n$, is a system of coordinates with $\mathbf{wt}(\widetilde{X}_i) = a_i$, which generates a space of functions $\widetilde{X}$, and $\widetilde{Z} \subseteq \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)\big)$ such that (i) holds i.e. $V_d \subset \underline{\mathrm{Sym}}^d\big(\widetilde{X} \amalg \widetilde{Z}\big)$, then the k-module generated by $\widetilde{Z}$ contains Z.

(iii) If $\widetilde{X}_i$, $1 \leqslant i \leqslant n$, Z is any other system of coordinates such that Corollary 2.7.(i) holds, then $\widetilde{X}_i = \widetilde{X}_i(X, Z)$, $1 \leqslant i \leqslant n$, i.e. unused coordinates are not involved.

Proof. Item (i) of Corollary 2.7. is trivial if $\mathrm{L}_{-a_0} = 0$, so suppose the image of (2.7) is non-zero, and profit from the fact that the the image is a trivial k-module to choose $0 \neq \partial \in \mathrm{L}_{-a_0}$ along with coordinates Z, y_1 where the former is a basis of

$$\ker \partial \subset \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0)\big)$$

(thus empty if c_0 and the dimension of L_{-a_0} are 1), and $\partial y_1 = 1$. Again, let X_i, for $1 \leqslant i \leqslant n$, be the blocks of weight strictly greater than a_0; so Z, $\{y_1\}$, $X_i = \{x_{i\bullet}\}$, $1 \leqslant i \leqslant n$, is a basis for everything and in these coordinates ∂ takes the form

$$\partial = \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} \Bigg(\sum_{j=1}^{c_i} \lambda_{ij} \frac{\partial}{\partial x_{ij}} \Bigg), \qquad \mathbf{wt}(x_{ij}) = a_i > a_0, \tag{2.18}$$

where $\mathbf{wt}(\lambda_{ij}) - \mathbf{wt}(x_{ij}) = -a_0$, so $\mathbf{wt}(\lambda_{ij}) = \mathbf{wt}(x_{ij}) - a_0 < \mathbf{wt}(x_{ij})$, thus

$$\lambda_{ij} = \lambda_{ij}(Z, y_1, X_{<i}), \quad \text{where } \mathbf{wt}(X_{<i}) < a_i, \tag{2.19}$$

i.e. λ_{ij} only depends on variables of weight strictly less than a_i. To simplify the notation we'll write ∂_{y_1}, resp. $\partial_{x_{ij}}$, for $\frac{\partial}{\partial y_1}$, resp. $\frac{\partial}{\partial x_{ij}}$, and employ the summation convention so that (2.18) becomes:

$$\partial = \partial_{y_1} + \lambda_{ij}\, \partial_{x_{ij}} \; . \tag{2.20}$$

By increasing induction on $\mathbf{wt}(X_i)$ we will eliminate everything from (2.20), except ∂_{y_1}, by way of a global change of weighted projective coordinates. The starting point is $a_{i-1} = a_0$ which is a minor abuse of notation, but it is certainly true, so by induction we have

$$\partial = \partial_{y_1} + \lambda_{hj}\, \partial_{x_{hj}}, \quad \mathbf{wt}(x_{hj}) \geqslant a_i. \tag{2.21}$$

Thus in weight a_i we aim for a global change of coordinates of the form

$$x_{ij} \mapsto x_{ij} + G_{ij}(Z, y_1, X_{<i}), \quad \mathbf{wt}(X_{<i}) < a_i = \mathbf{wt}(G_{ij}) \tag{2.22}$$

and otherwise do nothing for weights strictly greater than a_i. Consequently we need to solve $\partial\big(x_{ij} + G_{ij}\big) = 0$, i.e.

$$\partial\big(x_{ij} + G_{ij}\big) = \lambda_{ij} + \partial_{y_1} G_{ij} = 0, \tag{2.23}$$

which is trivially solvable on any ring of characteristic 0 by (2.19) with $\mathbf{wt}(G_{ij}) = \mathbf{wt}(\lambda_{ij}) - \mathbf{wt}(\partial_{y_1}) = a_i$. As such for our given ∂ we have a system of coordinates

$\{ Z, y_1, X_i \}$ such that $\partial = \partial_{y_1}$ and, of course, any other $D \in \mathrm{L}_{-a_0}$ can be expressed in this basis as

$$D = \nu\, \partial_Z + \mu\, \partial_{y_1} + \lambda_i\, \partial_{X_i}, \tag{2.24}$$

with $\mu, \nu, \in k$ but not λ_i if $\lambda_i \neq 0$, where $\lambda_i \partial_{X_i} := \sum_{j=1}^{c_i} \lambda_{ij} \partial_{x_{ij}}$. By (2.7) if D is linearly independent of ∂_{y_1}, replacing D by $D - \mu\, \partial_{y_1}$, $\mu = 0$ and some $\nu\, \partial_Z \neq 0$. Further, from $[\partial, D] = 0$, D is canonically a derivation of the algebra $k[Z, X_1, \ldots, X_n]$, which in turn inherits a $\mathbb{G}_m-$action. Consequently we may repeat the first step for D and $\ker D$ to get coordinates y_1, y_2, X_i, $n \geqslant i > 0$, and Z, which, now, is a block of coordinates of $\ker \partial \cap \ker D$, in which

$$\partial = \partial_{y_1}, \quad D = \partial_{y_2}. \tag{2.25}$$

and whence, by induction we arrive at a $\mathbb{G}_m$-equivariant system of coordinates Z, $Y = \{y_1, \ldots, y_\ell\}$, X_i, $n \geqslant i > 0$ with the properties

$$\begin{aligned} &(1) \quad \partial_Y = \{\partial_{y_1}, \ldots, \partial_{y_\ell}\} \text{ is a basis of } \mathrm{L}_{-a_0},\ Y = \{y_1, \ldots, y_\ell\} \text{ its dual basis;} \\ &(2) \quad Z \text{ is a basis of ann(L) in } \mathrm{H}^0(P, \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)), \text{cf. (2.17);} \\ &(3) \quad X_i,\ 1 \leqslant i \leqslant n, \text{ are the other coordinates;} \\ &(4) \quad V_d \text{ is a submodule of weight } da_0 \text{ of the } \mathbb{G}_m\text{-algebra } k[Z, X_{\geqslant 1}]; \end{aligned} \tag{2.26}$$

which complete the proof of part (i) of Corollary 2.7.

In regard to part (ii) of Corollary 2.7, under the hypothesis of op.cit., L contains a subspace of fields, M, whose annihilator under the natural map of (2.7) is generated by $\widetilde{Z}$, while the annihilator of L is generated by Z, so from $M \subseteq \mathrm{L}$ we get Z is contained in the k-module generated by $\widetilde{Z}$.

Finally as to part (iii), by definition the $\widetilde{X}_i$'s and the X_i's, $1 \leqslant i \leqslant n$, modulo $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)\big)$, are systems of weighted projective coordinates of the sub-weighted projective champ $\mathbb{P}_k(\underline{a}') = \mathbb{P}(a_1, \ldots, a_n)$, cf. Hypothesis 2.3, so without loss of generality (i.e. after replacing say the X_i's by a weighted automorphism of themselves) with Y, Z as in (2.26)

$$\widetilde{X}_i = X_i \mod (Y, Z). \tag{2.27}$$

and we have:

$$\widetilde{X}_i = X_i + \widetilde{X}_i(Z, X_{\leqslant i}) + \sum\nolimits_{|E| = \alpha_i} Y^E \lambda_E(Z, X_{\leqslant i}) + \big(\text{higher order in } Y'\text{s}\big), \tag{2.28}$$

where $\mathbf{wt}(Y^E \lambda_E) = |E| a_0 + \mathbf{wt}(\lambda_E) = a_i$, and by definition $\alpha_i \leq \infty$ is of minimal weight amongst the monomials in Y, so $\alpha_i = \infty$ is equivalent to our goal of item (iii) in Corollary 2.7., i.e. there are no Y terms in (2.28). As such, we may, without loss of generality, replace $X_i + \widetilde{X}_i(Z, X_{\leqslant i})$ by X_i (which is an automorphism because it is so modulo Z) and, with a view to a contradiction, suppose $\beta = \min_i\{\alpha_i\} < \infty$, so that (2.28) becomes

$$\widetilde{X}_i = X_i + \sum\nolimits_{|E_i|=\beta} Y^{E_i}\lambda_{E_i}(Z,\underline{X}) + \big(\text{order } \geqslant \beta+1 \text{ in } Y'\text{s}\big) =: X_i + \eta_i, \tag{2.29}$$

while by hypothesis every $f = f(Z,\underline{X}) \in V_d$ can be written as $\varphi_f(Z,\underline{\widetilde{X}})$, where $\underline{X} := (X_1,\ldots,X_n)$ and $\underline{\widetilde{X}} = \underline{X} + \underline{\eta}$, (2.29). However from

$$\varphi_f(Z,\underline{X}+\underline{\eta}) = f(Z,\underline{X}) \tag{2.30}$$

we must have $\varphi_f = f$ and whence

$$\begin{aligned} f(Z,\underline{X}) &= f(Z,\underline{X}+\underline{\eta}) \\ &= f(Z,\underline{X}) + \eta_i(\partial_{X_i} f)(Z,\underline{X}) + \big(\text{order } \geqslant \beta+1 \text{ in } Y'\big) \\ &= f(Z,\underline{X}) + \sum\nolimits_{|E_i|=\beta} Y^{E_i}\lambda_{E_i}(\partial_{X_i} f) + \big(\text{order } \geqslant \beta+1 \text{ in } Y'\big) \end{aligned} \tag{2.31}$$

Thus for every E_i with $|E_i| = \beta$ the term $\sum_i \lambda_{E_i}(\partial_{X_i} f)$ must be equal to 0 and, as we have said, $|E_i|a_0 + \mathbf{wt}(\lambda_{E_i}) = \mathbf{wt}(X_i)$, so the operator $\lambda_{E_i}\partial_{X_i}$ has weight

$$\mathbf{wt}(\lambda_{E_i}\partial_{X_i}) = \mathbf{wt}(\lambda_{E_i}) + \mathbf{wt}(\partial_{X_i}) = a_i - |E_i|a_0 - a_i = -|E_i|a_0 < 0 \tag{2.32}$$

and it vanishes on all of V_d, so it belongs to L_{-a_0}. Therefore by Lemma 2.4 its image under (2.7) in $\mathrm{H}(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0))^\vee$ is, under the hypothesis $\beta < \infty$, non-zero, which is nonsense since $\lambda_{E_i}\partial_{X_i}$ has value 0 on both the Y's and the Z's. □

3 The Invariant

We are going to define an invariant of rings and their ideals which is most naturally expressed in an appropriate number of copies of $\mathbb{Q}_{\geqslant 0}$ with the lexicographic ordering. On the other hand this is not a discrete group, so to avoid fastidious statements about denominators we introduce,

Definition 3.1. Let $N \in \mathbb{Z}_{\geqslant 0}$; $\mathbb{Q}^{N+1}$ ordered lexicographically; and pr_i, resp. $\mathrm{pr}_{\leqslant i}$, the projection onto the i^{th} factor, resp. first i factors, $1 \leqslant i \leqslant N$, then a function $f : E \to \mathbb{Q}^{N+1}_{\geqslant 0}$, with domain, E, plausibly a proper class, is said to have self bounding denominators if,

(i) $f^*\mathrm{pr}_1 : E \to \mathbb{Q}_{\geqslant 0}$ takes values in $\mathbb{Z}_{\geqslant 0}$.

(ii) If $N \geqslant 1$, then for all $1 \leqslant i \leqslant N$ there are increasing (in the lexicographic order) functions $D_i : \mathbb{Q}^i_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ such that,

$$\big(f^*\mathrm{pr}^*_{\leqslant i} D_i\big) f^*\mathrm{pr}_{i+1} \in \mathbb{Z}_{\geqslant 0}. \tag{3.1}$$

The utility of the definition results from,

Fact 3.2. Let everything be as in Definition 3.1 with $f : E \to \mathbb{Q}_{\geqslant 0}^{N+1}$ a function enjoying self bounding denominators, and define a function $F : E \to Z_{\geqslant 0}^{N+1}$ whose first projection is that of f while its $(i+1)^{\text{th}}$ projection is (3.1) for $1 \leqslant i \leqslant N$, then in the lexicographic order,

$$f(x) \leqslant f(y) \quad \Longleftrightarrow \quad F(x) \leqslant F(y).$$

Proof. Manifestly Fact 3.2 is true if $N = 0$, so suppose $N \geqslant 1$ and $f(x) < f(y)$, then without loss of generality, $\mathrm{pr}_{\leqslant N} f(x) = \mathrm{pr}_{\leqslant N} f(y)$ but $\mathrm{pr}_{N+1} f(x) < \mathrm{pr}_{N+1} f(y)$. Consequently, $\big(f^* \mathrm{pr}^*_{\leqslant N} D_N\big)$ is the same at x and y, so: $\mathrm{pr}_{N+1} f(x) \leqslant \mathrm{pr}_{N+1} f(y)$ iff $\mathrm{pr}_{N+1} F(x) \leqslant \mathrm{pr}_{N+1} F(y)$. $\square$

Set Up/Notation 3.3. *A is a regular local ring of dimension m, with residue field k of characteristic 0, and $\mathfrak{m}$ its maximal ideal. We will employ,*

Definition 3.4. A regular weighted filtration (or simply a weighted filtration or even just filtration if there is no danger of confusion) on a ring A, is the filtration, $F^\bullet$, associated to a set of coordinates (i.e. modulo $\mathfrak{m}^2$ affords a subset of a basis of $\mathfrak{m}/\mathfrak{m}^2$) $\{\, x_1 , \dots , x_m \,\}$ and non-negative rational numbers, $r_1, \dots , r_m \in \mathbb{Q}^m_{\geqslant 0}$, by the ideals,

$$F^p A = \{\, x_1^{e_1} \cdot \dots \cdot x_m^{e_m} \;\big|\; r_1\, e_1 + \dots + r_m\, e_m \geqslant p \,\}, \quad p \in \mathbb{Q}_{>0}. \tag{3.2}$$

In addition, since in the string of rationals $(r_1, \dots , r_m) \in \mathbb{Q}^m_{\geqslant 0}$, repetitions are allowed, we define

Definition 3.5. A block of coordinates, X, is a set which may be extended to a system of coordinates and, which is maximal amongst such sets with the same weight. In particular any weighted filtration can always be expressed in terms of a system of blocks $X_0, \dots , X_s$, $s < m$, where each X_i has the same weight and $X_0 \amalg \dots \amalg X_s$ is a system of coordinates of A.

For I an ideal of A we will define inductively a weighted filtration $F^\bullet(I)$ which only depends on the pair (A, I) together with

$$\mathrm{inv}(I) = \mathrm{inv}_A(I) \in \mathbb{Q}^{2m}_{\geqslant 0} \tag{3.3}$$

where $\mathbb{Q}^{2m}_{\geqslant 0}$ is endowed with the lexicographic ordering. At each step $s \geqslant 0$ of the induction we will, actually, define two successive entries of $\mathrm{inv}(I)$, (g_s, ℓ_s), beginning with

Start of the Induction 3.6. Let A be as in Set Up/Notation 3.3, and $I \lhd A$ an ideal, then the multiplicity of I, is

$$\mathrm{mult}(I) := \begin{cases} \max\{\alpha \in \mathbb{Q}_{\geqslant 0} \,\big|\, I \subseteq \mathfrak{m}^\alpha\}, & I \neq 0 \\ \infty\ , & I = 0 \end{cases}$$

As such if $\mathrm{mult}(I) = d \in \mathbb{Z}_{>0}$,

$$V_d := I \mod \big(\mathfrak{m}^{d+1}\big) \longhookrightarrow \mathrm{Sym}^d\big(\mathfrak{m}/\mathfrak{m}^2\big),$$

and we apply Lemma 2.4 to

$$V_d \longhookrightarrow \mathrm{H}^0\big(\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2),\, \mathscr{O}_{\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)}(d)\big) \tag{3.4}$$

with $\ell_0(I) := \dim \mathrm{L}_{-1}(V_d)$, in notation of (2.6), then by Corollary 2.7.(i) there is a unique minimal subspace $Z = Z(I) \subseteq \mathfrak{m}/\mathfrak{m}^2$ of dimension $c_0 := m - \ell_0$ such that $V_d \subseteq \mathrm{Sym}^d(\,Z\,)$. We therefore start the induction by way of:

(S.0) The first two entries of $\mathrm{inv}(I)$ are equal to $\big(\mathrm{mult}(I), \ell_0(I)\big)$.

(S.1) If either of these entries of the invariant are zero, then so are all the subsequent ones, and the process terminates.

(S.2) The weighted filtration $F_0^\bullet(I)$ is the weighted filtration in which each x_i has weight 1, i.e. the powers of the maximal ideal $\mathfrak{m}^\bullet$.

(S.3) Under the hypothesis of (S.1), the definition of $F^\bullet(I)$ also terminates, $F^\bullet(I) = F_0^\bullet(I)$.

(S.4) The first block, X_0, of cardinality c_0 is a choice of basis of Z.

Inductive Hypothesis 3.7. For $s \geqslant 1$, there is a (weighted) filtration $F_{s-1}^\bullet(I)$ depending only on I (and for this reason we will write just $F_{s-1}^\bullet$ if there is no danger of confusion) defined by blocks of coordinates $X_{s-1}^0, \dots, X_{s-1}^{s-1}$, respectively Y of cardinality $c_0, c_1, \dots, c_{s-1}$, respectively ℓ_{s-1}, where, for $0 \leqslant i < s-1$,

$$\ell_i := m - (c_0 + \cdots + c_i) \ \text{ or equivalently } \ell_{i+1} := \ell_i - c_{i+1}, \tag{3.5}$$

and rationals weights $g_{s-1}^0 > g_{s-1}^1 > \cdots > g_{s-1}^{s-1} \in \mathbb{Q}_{>0}^s,\ g_{s-1}^i \geqslant 1$ such that:

(F.0) If Y is any block completing $X_{s-1}^0, \dots, X_{s-1}^{s-1}$ to a system of coordinates then $1 = \mathbf{wt}(Y) \leqslant g_{s-1}^{s-1}$.

(F.1) $I \subseteq F_{s-1}^{dg_{s-1}^0}$.

(F.2) For $V_{s-1}^{dg_{s-1}^0} := I \mod F_{s-1}^{>dg_{s-1}^0}$, $V_{s-1}^{dg_{s-1}^0} \subseteq \underline{\mathrm{Sym}}^{dg_{s-1}^0}\big(X_{s-1}^0 \amalg \cdots \amalg X_{s-1}^{s-1}\big)$, cf. Notation/Definition 2.6.

(F.3) There are no vector fields of negative weight on the $\mathrm{P}_k(\underline{g})$, cf. (2.4), associated to the quotient of the graded algebra

$$\mathrm{gr}_{s-1}A = \coprod\nolimits_{q\geqslant 0} F_{s-1}^q \Big/ F_{s-1}^{q+1} \tag{3.6}$$

by the ideal generated by Y leaving $V_{s-1}^{dg_{s-1}^0}$ invariant.

(F.4) There are strictly positive integers d_i^t, $0 \leqslant i \leqslant t \leqslant s-1$, $d_0^0 = d$ as in the Start of the Induction 3.6, such that the weights g_t^i are derived from $g_t \in \mathbb{Q}_{>0}$

according to the following rules: if given g_t, we define $g_t^i = g_{i+1}\dots g_t,\ g_t^t = 1$, then

$$\begin{aligned} g_0^0 = g_0 &= 1 \\ g_1^0 d_0^0 - (g_1^0 d_0^1) &= d_1^1 \\ g_2^0 d_0^0 - (g_2^0 d_0^2 + g_2^1 d_1^2) &= d_2^2 \\ \vdots \qquad\qquad & \qquad \vdots \\ g_{s-1}^0 d_0^0 - \big(g_{s-1}^0 d_0^{s-1} + g_{s-1}^1 d_1^{s-1} + \cdots + g_{s-1}^{s-2} d_{s-2}^{s-1}\big) &= d_{s-1}^{s-1}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \text{and, } g_t^0 d_0^{t+1} + g_t^1 d_1^{t+1} + \cdots + g_t^{t-1} d_{t-1}^{t+1} + d_t^{t+1} + d_{t+1}^{t+1} &> g_t^0 d_0^0, \\ \text{for every } \ 0 \leqslant t \leqslant s-2\,. \end{aligned} \tag{3.8}$$

Notice that by (3.7) and (3.8), $g_t > 1$ for every $1 \leqslant t \leqslant s-1$.

(F.5) The function $\underline{g} = (d, g_1, \dots, g_{s-1})$ of rings and their ideals has self bounding denominators, Definition 3.1.

Induction Defining 3.8. $F_s^\bullet$ from $F_{s-1}^\bullet$. The induction is divided as follows:

Step 3.9. If $c_0 + \cdots + c_{s-1} = m$, or equivalently, by (3.5), if $\ell_{s-1} = 0$, then stop and define $F^p(I) := F_{s-1}^p(I)$, together with the invariant:

$$\operatorname{inv}(I) := \begin{cases} (\, d,\ \ell_0, \underline{0}\,)\ , & s = 1; \\ (\, d,\ \ell_0,\ g_1,\ \ell_1, \dots,\ g_{s-1}, \ell_{s-1} = 0,\ \underline{0}\,)\ , & s \geqslant 2, \end{cases} \tag{3.9}$$

wherein ℓ_{s-1} and the last $2(m-s)$ entries are equal to 0.

Step 3.10. Otherwise $m - (c_0 + \cdots + c_{s-1}) = \ell_{s-1} > 0$, and define for $H \in \mathbb{Q}_{>1}$ a set $\Lambda_H := \{(\alpha_0, \dots, \alpha_{s-1}, \beta)\} \subseteq \mathbb{Z}_{\geqslant 0}^{s+1}$ by the rules:

(R.1) $H \cdot \big(g_{s-1}^0\, \alpha_0 + \cdots + g_{s-1}^{s-1}\, \alpha_{s-1}\big) + \beta \geqslant H \cdot \big(g_{s-1}^0\, d\big)$;

(R.2) $g_{s-1}^0\, \alpha_0 + \cdots + g_{s-1}^{s-1}\, \alpha_{s-1} < g_{s-1}^0\, d$.

Now observe that by (R.2) the possibilities for (α_i) are finite, so if (R.1) is an actual equality for some H then the denominator of H is bounded. It therefore makes sense to introduce.

Fact/Definition 3.11. The discrete set of sub-inductive parameters $\Theta_{s-1}(I, A)$, contained in $\mathbb{Q}_{>1}$, is the subset of $H \in \mathbb{Q}_{>1}$ where equality occurs in (R.1) for some tuple of integers satisfying (R.2), and its predecessor $h = h(H)$ is the minimum of $\Theta_{s-1} \cap \mathbb{Q}_{<H}$ or 1 if H is already the minimum of Θ_{s-1}.

Better still, observe,

Fact 3.12. Let $\underline{g} = \underline{g}(I, A)$ be as in the Inductive Hypothesis 3.7.(F.5), and $h_s = h_s(I, A)$ any function taking values in the set $\Theta_{s-1}(I, A)$ of sub-inductive parameters in Fact/Definition 3.11, then $\underline{g} \times h_s$ is a function of rings and their ideals with self bounding denominators.

Proof. By the definition of h_s there are non-negative integers α_i and a positive integer β such that Step 3.10.(R.1) is an equality. In addition there are $D_i : \mathbb{Q}^i_{\geqslant 0} \to Z_{\geqslant 0}$, $0 \leqslant i \leqslant s-1$ self bounding the denominators of $\underline{g}$ in the sense of Definition 3.1. Consequently we must have,

$$\big(D_0 \cdots D_{s-1}\big)(\underline{g})\beta = h_s N$$

where $N \in \mathbb{Z}_{>0}$ is an integer no greater than

$$dg^0_{s-1}\big(D_0 \cdots D_{s-1}\big)(\underline{g}) \tag{3.10}$$

so D_s the factorial of (3.10) will do. $\square$

Having cleared any scruples about denominators, consider the following,

Sub-Induction 3.13. ($H \in \Theta_{s-1}$) For $h = h(H)$ the predecessor of H, and $h^i_{s-1} = h \cdot g^i_{s-1}$, $0 \leqslant i \leqslant s-1$, there is a weighted filtration $F^\bullet_{s-1}(h)$ depending only on I, in which all of Inductive Hypothesis 3.7.(F.0)–(F.3) hold but with h^i_{s-1} instead of g^i_{s-1}.

Plainly the Sub-Induction 3.13 begins with $F^\bullet_{s-1}(1) = F^\bullet_{s-1}$, while by Corollary 2.7.(iii) each block X^i_{s-1}, $0 \leqslant i \leqslant s-1$, is (up to a weighted projective transformation in the X^t_{s-1}, $0 \leqslant t < i \leqslant s-1$) well defined modulo $F^{h^i_{s-1}}_{s-1}(h)$. As such if $\widetilde{X}^i_{s-1}$ and $\widehat{X}^i_{s-1}$ are any two liftings of the i-th block to A, then

$$\widetilde{X}^i_{s-1} = \widehat{X}^i_{s-1} \qquad \mathrm{mod}\ F^{>h^i_{s-1}}_{s-1}(h) \tag{3.11}$$

and we assert that for H as in the Sub-Induction 3.13,

Lemma 3.14. If $\widetilde{X}^i_{s-1}$, $0 \leqslant i \leqslant s-1$, is a lifting of the blocks from $\mathrm{gr}^{(h)}_{s-1}A$ (cf. (3.6)), and $\widetilde{X}^s_{s-1}$ some choice of completing this to a system of coordinates, then the new filtration, $F^\bullet_{s-1}(H)$ say, defined by the weights

$$\begin{aligned} &\mathbf{wt}_H(\widetilde{X}^i_{s-1}) = H \cdot g^i_{s-1},\ \text{for}\ \ 0 \leqslant i \leqslant s-1, \\ &\mathbf{wt}_H(\widetilde{X}^s_{s-1}) = 1\,, \end{aligned} \tag{3.12}$$

does not depend on the aforesaid choices.

Proof. To this end, by (3.11), it is sufficient to prove

Claim 3.15. $f \in F^{>hg^i_{s-1}}_{s-1}(h) \Longrightarrow \mathbf{wt}_H(f) \geqslant H \cdot g^i_{s-1}$, i.e. $f \in F^{Hg^i_{s-1}}_{s-1}(H)$.

Proof. By hypothesis f is contained in the ideal, $F^{>hg^i_{s-1}}_{s-1}(h)$, generated by monomials with total degrees α_i, resp. β, for the blocks X^i_{s-1}, $0 \leqslant i \leqslant s-1$, resp. X^s_{s-1}, such that:

$$h \cdot \big(g^0_{s-1}\,\alpha_0 + \cdots + g^{s-1}_{s-1}\,\alpha_{s-1}\big) + \beta \ >\ h \cdot g^i_{s-1}; \tag{3.13}$$

while from the definition of the integers d_i^i, Inductive Hypothesis 3.7-(F.4),

$$g_i^0\, d_0^i + g_i^1\, d_1^i + \cdots + g_i^{i-1}\, d_{i-1}^i + (d_i^i - 1) = g_i^0\, d_0^0 - 1 \tag{3.14}$$

so multiplying this by g_{s-1}^i we get

$$g_{s-1}^0\, d_0^i + g_{s-1}^1\, d_1^i + \cdots + g_{s-1}^{i-1}\, d_{i-1}^i + g_{s-1}^i\, (d_i^i - 1) = g_{s-1}^0\, d_0^0 - g_{s-1}^i \tag{3.15}$$

then multiplying (3.15) by h and adding it to (3.13) gives:

$$\begin{aligned} h \cdot \Big(& g_{s-1}^0\, (\alpha_0 + d_0^i) + \cdots + g_{s-1}^{i-1}\, (\alpha_{i-1} + d_{i-1}^i) + g_{s-1}^i\, (\alpha_i + d_i^i - 1) \\ & + g_{s-1}^{i+1}\, \alpha_{i+1} + \cdots + g_{s-1}^{s-1}\, \alpha_{s-1} \Big) + \beta > h \cdot g_{s-1}^0\, d_0^0 \end{aligned} \tag{3.16}$$

so from the definition of $h = h(H)$, Fact/Definition 3.11,

$$\begin{aligned} H \cdot \Big(& g_{s-1}^0\, (\alpha_0 + d_0^i) + \cdots + g_{s-1}^{i-1}\, (\alpha_{i-1} + d_{i-1}^i) + g_{s-1}^i\, (\alpha_i + d_i^i - 1) \\ & + g_{s-1}^{i+1}\, \alpha_{i+1} + \cdots + g_{s-1}^{s-1}\, \alpha_{s-1} \Big) + \beta \geqslant H \cdot g_{s-1}^0\, d_0^0. \end{aligned} \tag{3.17}$$

Now multiply (3.15) by H and subtract from (3.17) to get

$$H \cdot g_{s-1}^0\, \alpha_0 + \cdots + H \cdot g_{s-1}^{s-1}\, \alpha_{s-1} + \beta \ \geqslant\ H \cdot g_{s-1}^i\,, \tag{3.18}$$

wherein the left hand side is the monomial's weight in the new H-filtration. □

Which in turn complete the proof of Lemma 3.14. □

Now in the new filtration $F_{(s-1)}^{\bullet}(H)$, i.e. the filtration obtained from $F_{(s-1)}^{\bullet}(h)$ of (3.12) (and unambiguously by Lemma 3.14), define

$$V_{s-1}^d(H) := I \mod F_{s-1}^{>Hg_{s-1}^0\, d}(H), \tag{3.19}$$

then one of the following must occur,

Case(A) 3.16. $\mathrm{L}\big(V_{s-1}^d(H)\big)$ (cf. Lemma 2.4) does not have maximal dimension, i.e.

$$\dim \mathrm{L}\big(V_{s-1}^d(H)\big) = \ell_s < m_s := m - (c_0 + \cdots + c_{s-1}).$$

Then by Corollary 2.7 applied to $\mathrm{P}_k\big(Ha_{s-1}^0, \ldots, Ha_{s-1}^{s-1}, \underline{1}\big)$, (2.4), there is a filtration satisfying (F.1)–(F.4) of Inductive Hypothesis 3.7 but with blocks X_s^i, $0 \leqslant i < s$, respectively X_s^s, liftings of the blocks X_i, respectively Z, i.e. the annihilator of $\mathrm{L}\big(V_{s-1}^d(H)\big)$ in Corollary 2.7, and $c_s = m_s - \ell_s$ while $g_{s+1} = H$ in Inductive Hypothesis 3.7.(F.4), i.e. $g_s^i = H \cdot g_{s-1}^i$ with $g_s^s = 1$.

Case(B) 3.17. $\mathrm{L}\big(V_{s-1}^d(H)\big)$ has maximal dimension, so its annihilator in Corollary 2.7, Z, is the empty set. Nevertheless, op.cit. still applies to give new liftings, X_s^i, $0 \leqslant i \leqslant s-1$, of the blocks X_i (of op.cit. applied to $\mathrm{P}_k(Ha_{s-1}^0, \ldots, Ha_{s-1}^{s-1}, \underline{1})$), such that the sub-inductive hypothesis Sub-Induction 3.13 is now valid for the successor of H in Θ_{s-1}.

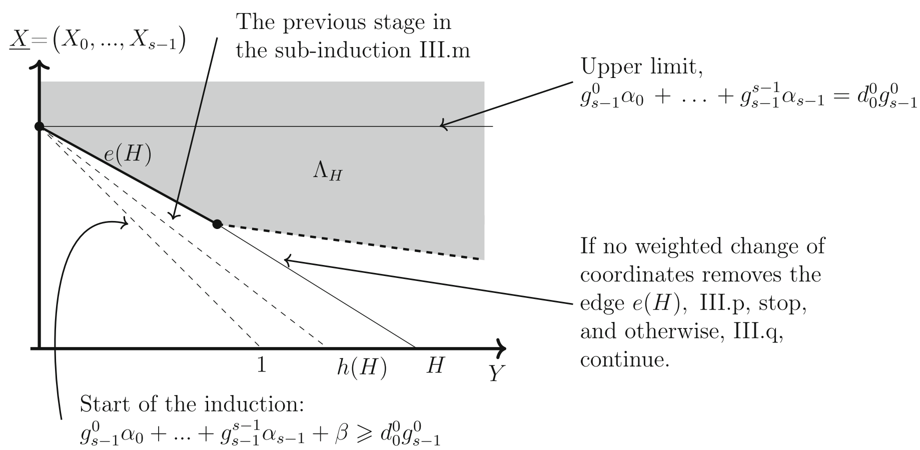

Figure 1: Newton Polyhedron for Sub-Induction 3.13.

Partial Finish 3.18. In Case(A) 3.16, the Sub-Induction 3.13 has terminated, and we have found our new filtration $F^\bullet_s$, to wit $F^\bullet_{s-1}(H)$, so that the induction now continues in s.

Otherwise in Case(B) 3.17, we either eventually fall into Case(A) 3.16, and, again, terminate the Sub-Induction 3.13, or we repeat Case(B) 3.17, ad infinitum. Suppose, therefore,

Hypothesis 3.19. Case(B) 3.17 occurs ad infinitum.

Such repetition is indexed by the possible h in Θ_{s-1} of Fact/Definition 3.11 and we continue to denote by H its successor. Our aim is to calculate the coordinates $X^i_{s-1}(H)$ of Corollary 2.7.(i) (whose liftings will be, again, the blocks $X^i_{s-1}(H)$) and, because we are in Case(B) 3.17, the relationship with the old coordinates $X^i_{s-1}(h)$ is given by:

$$X^i_{s-1}(H) - X^i_{s-1}(h) \in \underline{\mathrm{Sym}}^{hg^i_{s-1}}\big(X(h) \amalg \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}),\, \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)\big), \tag{3.20}$$

where $X(h)$ is the space of function generated by $X^i_{s-1}(h)$, for every $0 \leqslant i \leqslant s-1$, and $\mathbb{P}_k(\underline{a}) = \mathrm{P}_k\big(h\underline{g}^0_{s-1}, \ldots, h\underline{g}^{s-1}_{s-1}, \underline{1}\big)$, cf. (2.4) and (2.15). Now without loss of generality we have equality modulo $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}),\, \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)$, i.e. the projection of $X^i_{s-1}(H) - X^i_{s-1}(h)$ onto $\mathrm{Sym}^\bullet\big(X(h)\big)$ is always zero, thus, $X^i_{s-1}(H) - X^i_{s-1}(h)$ is a combination of monomials

$$X(h)^E \cdot Y^Q\,, \tag{3.21}$$

where $X(h)^E = \prod_i X_i(h)^{E_i}$, respectively $Y^Q = Y_1^{q_1} \ldots Y_{c_s}^{q_{c_s}}$, coming from $X(h)$ alone, respectively $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}),\, \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)$ alone, and by construction

$$h\, g^i_{s-1} = h\, \mathbf{wt}(E) + Q, \tag{3.22}$$

where $\mathbf{wt}(E) = g^0_{s-1}|E_0| + \cdots + g^{s-1}_{s-1}|E_{s-1}|$, $Q = q_1 + \cdots + q_{\ell_{s-1}}$. Therefore, $Q = h\,(g^i_{s-1} - \mathbf{wt}(E))$ while $e := \min_E\{g^i_{s-1} - \mathbf{wt}(E) > 0\}$ is attained since the weights of the $F^\bullet_{s-1}$ filtration are a discrete set. Thus $Q \geqslant h\,e$ and the right hand side of (3.22) tends to infinity. Consequently the $X^i_{s-1}(h)$ are a Cauchy sequence in the $\mathfrak{m}$-adic topology, so if Hypothesis 3.19 were to occur,

Fact/Proposition 3.20. The filtrations $F^\bullet_{s-1}(h)$, $h \in \Theta_{s-1}$ converges $\mathfrak{m}$-adically as $h \to \infty$ to a filtration $F^\bullet(I)$ determined uniquely by I consisting of blocks X^i_{s-1} of weights $\mathbf{wt}(X^i_{s-1}) = g^i_{s-1}$ and cardinality c_i, where $m_s = m - (c_0 + \cdots + c_{s-1}) > 0$.

Conclusion 3.21. Should the Sub-Induction 3.13, eventually not terminate. i.e. Hypothesis 3.19, then we arrive to a filtration $F^\bullet(I)$ of the completion $\widehat{A}$ of A in $\mathfrak{m}$ (depending only on I) with blocks X^i_{s-1} of cardinality $c_0, \ldots, c_{s-1}$ together with weights $g^0 > \cdots > g^{s-1}$, satisfying (F.1)–(F.4) of Inductive Hypothesis 3.7 and we define:

$$\mathrm{inv}(I) = \big(d, \ell_0, g_1, \ell_1, \ldots, g_{s-1}, \ell_{s-1}, \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0} \tag{3.23}$$

wherein the last block $\underline{0}$ has length $2(m-s)$. Otherwise, Case(A) 3.16, applies for all s and the invariant is eventually defined by (3.9).

Finally it is appropriate to explicitly observe the behaviour under regular maps beginning with.

Fact 3.22. The formation of the invariant is étale local, in fact better for $\widehat{A}$ the completion of our regular local ring A of Set Up/Notation 3.3, and $\widehat{I} := I \otimes_A \widehat{A}$ we have,

(i) $\mathrm{inv}_A(I\,) = \mathrm{inv}_{\widehat{A}}(\widehat{I}\,)$;

(ii) If $F^\bullet(I)$, resp. $F^\bullet(\widehat{I})$, is the filtration whether of A or $\widehat{A}$ resulting whether from the termination of the induction, 3.8, or the Sub-Induction 3.13, running ad infinitum, Hypothesis 3.19, then

$$F^\bullet(\widehat{I}) = \begin{cases} F^\bullet(I)\,, & \text{should Hypothesis 3.19 occur,} \\ F^\bullet(I) \otimes_A \widehat{A}\,, & \text{otherwise.} \end{cases} \tag{3.24}$$

Proof. In the situation of the Inductive Hypothesis 3.7,

$$\mathfrak{m}^N \subset F^N_{s-1} \quad \text{and} \quad F^p_{s-1} \subset \mathfrak{m}^{p/g^0_{s-1}}\,,$$

so if Hypothesis 3.19 never occurs everything is determined modulo a sufficiently large power of the maximal ideal, and both items (i) and (ii) are trivial. Otherwise if Hypothesis 3.19 occurs then the Conclusion 3.21 and the reasons for it (3.21)–(3.22) are $\mathfrak{m}$-adic by definition, so this is trivial too. ◻

In the same vein we may prepare for replacing étale by regular via,

Lemma 3.23. Suppose $B = A[\![z_1, \dots, z_\epsilon]\!]$ is a formal power series ring over A; J the pull-back of I to A with $\widehat{A}, \widehat{B}, \widehat{I}, \widehat{J}$ their completions in the maximal ideal of A, then:

(i) The odd entries of $\mathrm{inv}_B(J)$ and $\mathrm{inv}_A(I)$ agree.

(ii) Even entries where the invariant is zero agree, and otherwise the difference $\mathrm{inv}_B(J) - \mathrm{inv}_A(I)$ at an even entry is ϵ.

(iii) The filtrations (3.24) are related by, $F^\bullet(\widehat{J}) = F^\bullet(\widehat{I}) \otimes_{\widehat{A}} \widehat{B}$.

Proof. By induction in the parameter s, we assert that the relation between the graded rings $\mathrm{gr}_{s-1}A$, $\mathrm{gr}_{s-1}B$ of (3.6) is,

$$\mathrm{gr}_{s-1}B = \mathrm{gr}_{s-1}A \otimes_k k[z_1, \dots, z_\epsilon] \tag{3.25}$$

while in the Sub-Induction 3.13, the maximal contact spaces $\mathrm{L}_B\big(V^d_{s-1}(H)\big)$, resp. $\mathrm{L}_A\big(V^d_{s-1}(H)\big)$ are related by,

$$\begin{aligned}\mathrm{L}_B\big(V^d_{s-1}(H)\big) &= \mathrm{L}_A\big(V^d_{s-1}(H)\big) \amalg k \otimes_A \mathrm{Der}_A(B)\\ &= \mathrm{L}_A\big(V^d_{s-1}(H)\big) \coprod_{1\leqslant j\leqslant \epsilon} k\,\frac{\partial}{\partial z_j}\end{aligned} \tag{3.26}$$

Indeed for $s = 1$, (3.25) is obvious, while for any $s \geqslant 1$, (3.25) $\Rightarrow$ (3.26) since the $\frac{\partial}{\partial z_j}$ always vanish on generators of I so the right hand side of (3.26) is always contained in the left, while modulo the $\frac{\partial}{\partial z_j}$ they are plainly equal. Consequently in Case(A) 3.16 of the sub-induction, (3.26) implies (3.25) for s, while in Case(B) 3.17, the convergence is actually modulo the pull-back of the maximal ideal of A, equivalently the filtration is pulled back from $\widehat{A}$. $\square$

4 The Invariant on Weighted Projective Champs

Set Up 4.1. Let $\mathbb{P}_k(\underline{a}) = \mathbb{P}(\underline{a}^0, \dots, \underline{a}^s)$ be a $(m-1)$-dimensional weighted projective champ, with blocks of coordinates $X_0, \dots, X_s$ of weights $a^0 > \cdots > a^s$ and cardinality $c_0, \dots, c_s$ over a field k of characteristic zero. Suppose further that $d \in \mathbb{Z}_{>0}$ and $V \subset \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0)\big)$ is a space of sections such that:

Hypothesis 4.2. If for every $s \geqslant i > 0$, $\mathrm{P}_i \hookrightarrow \mathbb{P}_k(\underline{a})$ is the weighted projective sub-champ defined by $X_i = \cdots = X_s = 0$, with for convenience of notation $\mathrm{P}_{s+1} = \mathbb{P}_k(\underline{a})$, then

$$\mathrm{L}_i(V) := \coprod\nolimits_{-b<0} \Big\{\partial \in \mathrm{H}^0(\mathrm{P}_i, \mathrm{T}_{\mathrm{P}_i}(-b)) \,\Big|\, \partial(V_i) = 0\Big\} = 0 \tag{4.1}$$

where V_i is the image of V in $\mathrm{H}^0\big(\mathrm{P}_i, \mathscr{O}_{\mathrm{P}_i}(da^0)\big)$.

Now for consistency with Fact/Definition 3.11 and Inductive Hypothesis 3.7.(F.4), define $g_i := a^{i-1}/a^i$, $1 \leqslant i \leqslant s$, and $\ell_i = m - (c_0 + \cdots + c_i)$ then we assert,

Proposition 4.3. If I is the sheaf of ideals generated by V, under the non-degeneracy condition Hypothesis 4.2, then for every geometric point p of $\mathbb{P}_k(\underline{a})$ the value of the invariant $\mathrm{inv}_{\mathbb{P}_k(\underline{a})}(I)(p)$ at the stalk I_p is *strictly* less than

$$\big(\, d, \ell_0, g_1, \ell_1, \ldots, g_s, \ell_s, \underline{0}\,\big). \tag{4.2}$$

More precisely, if $\mathrm{inv}_{\mathbb{P}_k(\underline{a})}(I)(p) = \big(\mathrm{mult}_I(p), \ell_0(p), g_1(p), \ldots, \ell_s(p), \underline{0}\big)$ with $\ell_i(p) = m - (c_0(p) + \cdots + c_i(p))$ and $0 \leqslant \sigma \leqslant s$ is maximal such that $X_\sigma(p) \neq 0$, (i.e. there is some $1 \leqslant i \leqslant c_\sigma$, for which $x_{\sigma i}(p) \neq 0$) then:

(i) If $\sigma = 0$ the multiplicity of I at p is strictly less than d, unless $d = 0$.

(ii) If $\sigma > 0$ with, for immediate notational convenience, $g_0 = d$ and all of $g_i(p) \geqslant g_i$, $c_i(p) \leqslant c_i$, for any $0 \leqslant i \leqslant \sigma - 2$ then $g_i(p) = g_i$ and $c_i(p) = c_i$ for all $0 \leqslant i \leqslant \sigma - 2$.

(iii) If (ii) holds and $g_{\sigma-1}(p) \geqslant g_{\sigma-1}$, $c_{\sigma-1}(p) \leqslant c_{\sigma-1}$, then $g_{\sigma-1}(p) = g_{\sigma-1}$, $c_{\sigma-1}(p) = c_{\sigma-1}$, $c_\sigma \geqslant 2$, and $g_\sigma(p) < g_\sigma$; so in particular if $c_\sigma = 1$ then $g_{\sigma-1}(p) < g_{\sigma-1}$, i.e. $g_{\sigma-1}(p)$ goes down.

Observe that we can immediately reduce to $\sigma = s$ since,

Lemma 4.4. Let $\mathcal{Q}$ be a sub-champ of $\mathbb{P}_k(\underline{a})$ containing the geometric point p and such that Proposition 4.3.(i) holds, for $I\big|_{\mathcal{Q}}$, while denoting by a superscript $\mathcal{Q}$ the values of the blocks associated to the invariant of $I\big|_{\mathcal{Q}}$ calculated at p, items (ii) and (iii) of op.cit. hold, albeit, in the modified form:

(ii-bis) If $\sigma > 0$, $g_i(p) \geqslant g_i$, $c_i^{\mathcal{Q}}(p) \leqslant c_i$, for any $0 \leqslant i \leqslant \sigma - 2$, then $g_i(p) = g_i$, $c_i^{\mathcal{Q}}(p) = c_i$, for any $0 \leqslant i \leqslant \sigma - 2$.

(iii-bis) If (ii-bis) holds and $g_{\sigma-1}(p) \geqslant g_{\sigma-1}$, $c_{\sigma-1}^{\mathcal{Q}}(p) \leqslant c_{\sigma-1}$, then $g_{\sigma-1}(p) = g_{\sigma-1}$, $c_{\sigma-1}^{\mathcal{Q}}(p) = c_{\sigma-1}$, $c_\sigma \geqslant 2$, and $g_\sigma(p) < g_\sigma$; so in particular if $c_\sigma = 1$ then $g_{\sigma-1}(p) < g_{\sigma-1}$.

Proof. For the multiplicity $d = g_0$ this is clear, while c_0 is the minimum number of coordinates required to describe the ideal modulo $\mathfrak{m}^{d+1}(p)$, so its ambient value $c_0(p)$ is always at least that, $c_0^{\mathcal{Q}}(p)$, of a subspace whenever the multiplicity of the intersection coincides. Consequently if

$$c_0 \geqslant c_0(p) \text{ and } \big(\, c_0 \geqslant c_0^{\mathcal{Q}}(p) \Longrightarrow c_0^{\mathcal{Q}}(p) = c_0 \,\big) \text{ then } c_0(p) = c_0. \tag{4.3}$$

Similarly the presence of a non-zero gradient g_r, $1 \leqslant r \leqslant \sigma$ reflects the necessity, or otherwise, Corollary 2.7, of a new block of coordinates to describe the leading monomials in generators of the ideal, so if one needs a block after intersecting with a sub-widget one certainly needed it before hand, and should this occur $c_i^{\mathcal{Q}}(p) = c_i$ will imply $c_i(p) = c_i$ exactly as in (4.3). $\square$

In particular, therefore, after Lemma 4.4, and the definition of σ it is sufficient to prove Proposition 4.3 on the subspace $X_{\sigma+1} = \cdots = X_s = 0$, so without loss of generality $\sigma = s$.

Proof of Proposition 4.3. We proceed by induction on the number of blocks, s, starting from $s = \sigma = 0$. In this case by the action of PGL_{c_0} we may, without loss of generality suppose p is the point $[1 : 0 : \cdots : 0] \in \mathbb{P}_k^{m-1}$, in some basis $\{x_1, \ldots, x_m\}$. Consequently if the multiplicity does not go down Z of Corollary 2.7 is contained in the subspace generated by $x_2, \ldots, x_m$ which contradicts the definition of ℓ_0 (i.e. 0 under the present hypothesis) in the Start of the Induction 3.6 unless d were already 0.

Supposing, therefore, that $\sigma = s > 0$ let us adjust the notation accordingly by denoting the final block X_s as Y which in turn is a basis of $\mathrm{H}^0(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a^s))$, which we write as $Y = \{y\} \cup Z$ where

$$y(p) = 1, \qquad z(p) = 0, \ \forall\, z \in Z. \tag{4.4}$$

In particular, therefore, we have an étale neighbourhood U of p obtaine by slicing the groupoid $R := \mathbb{G}_m \times \mathbb{A}^m \backslash \{\underline{0}\} \rightrightarrows \mathbb{A}^m \backslash \{\underline{0}\}$ along the transversal $y = 1$, and we write the coordinate functions on U afforded by the elements of the blocks X_i as $x_{ij} + p_{ij}$, $0 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant c_i$, i.e.

$$U \ni p = \prod_{i=0}^{s-1} \underline{p_t} \times 1 \times \underline{0}, \quad \text{where } \underline{p_t} = p_{t1} \times \cdots \times p_{tc_t}. \tag{4.5}$$

In this notation the correspondence between a global section, $f(X_0, \ldots, X_{s-1}, Y)$ in $\underline{\mathrm{Sym}}^{da^0}\big(X_0 \amalg \cdots X_{s-1} \amalg Y\big) = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0)\big)$ and the associated function is simply

$$f \mapsto f(x_{ij} + p_{ij},\, 1,\, \underline{z}) \in \Gamma(U, \mathscr{O}_{\mathbb{P}_k(\underline{a})}), \quad \text{for } 0 \leqslant i \leqslant s-1 \text{ and } 1 \leqslant j \leqslant c_i. \tag{4.6}$$

Furthermore, and needless to say, U is an affine space with origin p via,

$$\left(\prod_{i=0}^{s-1} \prod_{j=1}^{c_{s-1}} x_{ij}\right) \times \underline{z} = \ U \longrightarrow \mathbb{A}^{m-1}. \tag{4.7}$$

so it makes perfect sense to talk about the maximal degree in the blocks of functions $\underline{x}_t := \{x_{ti} \,\big|\, 1 \leqslant i \leqslant c_t\}$, $0 \leqslant t \leqslant s-1$. With this in mind we assert,

Claim 4.5. The initial $2s$-part of the invariant $(g_0, \ell_0, g_1, \ell_1, \ldots, g_{s-1}, \ell_{s-1})$ cannot increase.

Proof. By induction in s. The starting point of the multiplicity $d = g_0$ is particular. Modulo the local functions x_{ij}, $i \geqslant 1$, $\underline{z}$, at p we have an affine space $\mathbb{A}^{c_0}$ on which the multiplicity is at most the degree in the block of functions $\underline{x}_0$ which is at most the degree in global block X_0, i.e. d. Furthermore were this bound to be achieved on U then the restriction I to $\mathbb{A}^{c_0}$ at p is, under the isomorphism afforded by: $X_{\bullet j} \mapsto \underline{x}_{\bullet j}$, exactly the ideal generated under,

$$\Gamma(\mathbb{A}^m \backslash \{\underline{0}\}) = \Gamma(\mathbb{A}^m) \xrightarrow{\bmod X_i} \Gamma(\mathbb{A}^{c_0}), \quad i \geqslant 1 \tag{4.8}$$

at the origin, so $c_0(p) \geqslant c_0$.

Now we put ourselves in the scenario of the Start of the Induction 3.6.(F.0)–(F.4), albeit with an inductive parameter $0 \leqslant t \leqslant s-1$, rather than $s-1$ of op.cit., and we add to the hypothesis:

(F.4 bis) The i^{th}-block, $0 \leqslant i \leqslant t$, is defined by the block of functions $\underline{x}_i$ and has weight $a^i/a^t = g_t^i$ (in notation of Inductive Hypothesis 3.7.(F.4)).

Quite possibly we arrive in Case(A) 3.16, for a value of $H < a^t/a^{t+1}$, but, plainly should this occur then the invariant strictly decreases. If, however, we were to continue in Case(B) 3.17, for every $H < a^t/a^{t+1}$ by way of changes of coordinates in the blocks $\underline{x}_i$, $0 \leqslant i \leqslant t$, then this in no way changes monomials of the form

$$\underline{x}_0^{D_0} \cdots \underline{x}_{t+1}^{D_{t+1}}, \quad a^0|D_0| + \cdots + a^{t+1}|D_{t+1}| = a^0 d \tag{4.9}$$

since the weight of the perturbation in $\underline{x}_i$ will be

$$H \cdot \left(a^i \big/ a^t\right) < a^i \big/ a^{t+1} . \tag{4.10}$$

Consequently were we to eliminate all $H < a^t/a^{t+1}$, modulo $\underline{x}_i$, $i > t+1$ we would find that, mod $\underline{x}_i$, $i > t+1$, the ideal at p is exactly that generated at the origin by the image of V in the origin obtained via the isomorphism

$$\Gamma(\mathbb{A}^m \backslash \{\underline{0}\}) = \Gamma(\mathbb{A}^m) \xrightarrow[\text{mod } X_i]{\sim} \Gamma(\mathbb{A}^{c_0+\cdots+c_{t+1}}), \quad i > t+1; \tag{4.11}$$

so the claim follows from Corollary 2.7, as employed in the definition of the invariant in Case(A) 3.16 . $\square$

Suppose therefore that the extremal situation of Claim 4.5 is attained (i.e. the invariant did not decrease), then from our original blocks of coordinates, $\underline{x}_i$, $0 \leqslant i \leqslant s-1$, $\underline{z}$ we will have performed a change of coordinates to blocks of the form

$$\begin{aligned} \underline{\xi}_0 &= \underline{x}_0 + \underline{\epsilon}_0(\underline{x}_1, \ldots, \underline{x}_{s-1}, \underline{z}), \\ \underline{\xi}_1 &= \underline{x}_1 + \underline{\epsilon}_1(\underline{x}_2, \ldots, \underline{x}_{s-1}, \underline{z}), \qquad \mathbf{wt}_{\underline{x}}(\underline{\epsilon}_i) < a^i, \\ &\ \ \vdots \qquad\qquad\qquad\qquad \text{for } \mathbf{wt}(x_i) = a^i; \\ \underline{\xi}_{s-1} &= \underline{x}_{s-1} + \underline{\epsilon}_{s-1}(\underline{z}); \end{aligned} \tag{4.12}$$

resulting in a filtration $F_{\underline{\xi}}^{\bullet}$ around p in which the blocks $\underline{\xi}_i$, $0 \leqslant i \leqslant s-1$ have weights a^i/a^s, $\underline{z}$ has weight 1, and around p the ideal generated by V belongs to $F_{\underline{\xi}}^{a^0 d/a^s}$. In particular

Warning 4.6. We allow the possibility that the Induction Defining 3.8 may still not have terminated in Case(A) 3.16 and whence the invariant might even go up.

To analyse this situation we replace the coordinates x_{ij} around p by the restriction to U of the $\mathbb{G}_m$-equivariant global coordinate functions X_{ij}, $0 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant c_i$ in the various blocks, so that (4.12) becomes,

$$\begin{aligned} \underline{\xi}_0 &= \left(\underline{X}_0 - \underline{\varepsilon}_0(X_1, \dots, X_{s-1}, Z)\right)\big|_U ,\\ \underline{\xi}_1 &= \left(\underline{X}_1 - \underline{\varepsilon}_1(X_2, \dots, X_{s-1}, Z)\right)\big|_U ,\\ &\ \vdots \qquad\qquad \vdots \\ \underline{\xi}_{s-1} &= \left(\underline{X}_{s-1} - \underline{\varepsilon}_{s-1}(Z)\right)\big|_U ; \end{aligned} \qquad \mathbf{wt}_X(\underline{\varepsilon}_i) < a^i, \tag{4.13}$$

and we assert.

Claim 4.7. In the above notation and under the hypothesis (cf. Claim 4.5) that the first $2s$ terms in the invariant at p are at least $\left(d, \ell_0, g_1, \ell_1, \dots, g_{s-1}, \ell_{s-1}\right)$ the coordinate change (4.13) is global, i.e. there are homogeneous functions $\underline{G}_i$ on $\mathbb{A}_k^{m-1}$ of weight a^i such that,

$$\underline{\varepsilon}_i(X_{i+1}, \dots, X_{s+1}, Z)\big|_U = \underline{G}_i(X_{i+1}, \dots, X_{s+1}, Z). \tag{4.14}$$

Proof. We have filtrations in which the blocks X_i, $0 \leqslant i \leqslant s-1$, $X_s = \{Z, Y\}$, respectively $\underline{\xi}_i$, $\underline{z}$, with weights a^i, $0 \leqslant i \leqslant s-1$, a^s, may a priori be different and so we will employ the notation $\mathbf{wt}_X$, resp. $\mathbf{wt}_\xi$, to avoid ambiguity. In any case for $f \in V_d$, we have from (4.13):

$$\begin{aligned} f\big|_U &= f(X_0, \dots, X_{s-1}, 1, Z)\big|_U = f\left(\underline{\xi}_0 + \underline{\varepsilon}_0, \dots, \underline{\xi}_{s-1} + \underline{\varepsilon}_{s-1}, \underline{1}, \underline{z}\right) \\ &= f\left(\underline{\xi}_0, \dots, \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) + \sum_{i=0}^{s-1} \left(\frac{\partial f}{\partial X_i}\, \underline{\varepsilon}_i\right)\left(\underline{\xi}_0, \dots, \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) + \text{ stuff}, \end{aligned} \tag{4.15}$$

wherein $\frac{\partial f}{\partial_{X_i}}\, \underline{\varepsilon}_i = \sum_{j=1}^{c_i} \frac{\partial f}{\partial_{x_{ij}}}\, \varepsilon_{ij}$, and stuff has smaller weight in the $\underline{\xi}$-filtration than the expected top weight in

$$\left(\sum_{i=0}^{s-1} \frac{\partial f}{\partial X_i} \underline{\varepsilon}_i^{\text{top}}\right)\left(\underline{\xi}_0, \dots, \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) \tag{4.16}$$

to wit: $(da^0) - \min_{0 \leqslant i \leqslant s-1}\left\{a^i - \mathbf{wt}_\xi\left(\underline{\varepsilon}_i^{\text{top}}\right)\right\}$, where $\underline{\varepsilon}_i^{\text{top}}$ are the monomials in $\underline{\xi}, \underline{z}$ in $\underline{\varepsilon}_i$ which have maximal $\underline{\xi}$-weight,

$$\underline{\varepsilon}_i^{\text{top}} := \sum_D \lambda_D\, \underline{\xi}_0^{D_0} \cdots \underline{\xi}_{s-1}^{D_{s-1}} \underline{z}^{D_s} + \text{ stuff}, \tag{4.17}$$

where, again, stuff is monomials with lower ξ-weight. Let us therefore define homogenous functions on the ambient space, $\mathbb{A}_k^{m-1}$ by way of the formula:

$$\Delta_i := \sum\nolimits_{D_i} \lambda_{D_i}\, X_0^{D_0} \cdots X_{s-1}^{D_{s-1}} Z^{D_s}, \tag{4.18}$$

and a homogeneous vector field,

$$\mathrm{D} = \sum_{i=0}^{s-1} \varDelta_i \frac{\partial}{\partial X_i} \qquad \text{of} \ \ \mathbf{wt}_X(\mathrm{D}) = - \min_{0\leqslant i\leqslant s-1} \left\{ a^i - \mathbf{wt}_\xi(\underline{\varepsilon}_i^{\mathrm{top}}) \right\}. \tag{4.19}$$

So that by construction and (4.13), (4.16) vanishes if and only if the top weight term in the grading of $\Gamma(\mathscr{O}_U)$ which assigns to $X_i\big|_U$ weight a^i, $0\leqslant i\leqslant s-1$, and to $Z\big|_U$ weight a^s of every $\mathrm{D}(f)\big|_U$ vanishes for every $f\in V_d$. Thus, a fortiori, on the weighted projective hypersurface $\mathcal{Q}$, defined by the function $Y=0$,

$$\mathrm{D}(f) = 0 \mod Y, \qquad \forall\, f\in V_d. \tag{4.20}$$

As such there are two cases: either $Z\neq\emptyset$, then since D acts trivially on $\mathrm{H}^0\big(\mathcal{Q}, \mathscr{O}_{\mathcal{Q}}(a^s)\big)$ by (4.19), $\mathrm{D}=0 \mod Y$ by Hypothesis 4.2 and Corollary 2.7.(ii); or $Z=\emptyset$ and $\mathrm{D}=0 \mod Y$ by the non-degeneracy Hypothesis 4.2 and Corollary 2.7.(ii). In either case $\mathrm{D}=0 \mod Y$, and whence all the $\varDelta_i\equiv 0$ by virtue of their definition (4.18), which in turn is nonsense (unless Claim 4.7 is true with $\underline{\varepsilon}_i=\underline{G}_i=0$). Thus the top weight term in (4.16) is not zero for some $f\in V_d$. However for such a f, according to our hypothesis that the invariant does not decrease, the top $\underline{\xi}$−weight term in (4.16) must cancel with the top $\underline{\xi}$−weight of

$$f\big(\underline{\xi}_0,\dots,\underline{\xi}_{s-1},\underline{1},\underline{z}\big) \mod F_{\underline{\xi}}^{a^0 d}, \tag{4.21}$$

which in turn has weight, $a^0 d - a^s n$, for some integer n. We therefore conclude,

$$a^0\, d - a^s\, n = a^0\, d - \min_{0\leqslant i\leqslant s-1} \{ a^i - \mathbf{wt}_X(\varDelta_i) \}, \tag{4.22}$$

i.e. for $0\leqslant i\leqslant s-1$ where the minimum in (4.22) is attained,

$$a^i = \mathbf{wt}_X(\varDelta_i) + a^s\, n\,. \tag{4.23}$$

Now consider the change of variables on $\mathbb{P}(\underline{a}^0,\dots,\underline{a}^s)$ defined by,

$$X_{i,\mathrm{new}} := \underline{X}_i + Y^n\, \varDelta_i(\underline{X}_{\geqslant i+1}, \underline{Z}), \quad 0\leqslant i\leqslant s-1, \tag{4.24}$$

then in the new coordinates the invariant, $\min_{0\leqslant i\leqslant s-1}\{ a^i - \mathbf{wt}_\xi(\underline{\varepsilon}_t^{\mathrm{top}}) \}$, of the coordinate change (4.13) has increased and since it is an integer which is at most a^0 (cf. Fact/Definition 3.11), this process eventually terminates establishing the claim.

□

The practical upshot of Claim 4.7 is when we come to compute the invariant at p we can suppose not only that all the p_{ij} are zero for $0\leqslant i\leqslant s-1$, but that the filtration defined by $\mathbf{wt}\big(X_{ij}\big|_U\big) = a^i/a^s$, $\mathbf{wt}(Z\big|_U)=1$ is exactly that defined by the inductive procedure 3.8, albeit for the moment we remain in the situation of Warning 4.6. However by Claim 4.7 we can now just read the invariant at p from the newton polyhedron, cf. Figure 1, calculated in the coordinates $X_{ij}\big|_U, Z\big|_U$. As such if $Z=\emptyset$ then at worst g_{s-1} goes down, whereas if $Z\neq\emptyset$ at worst g_s must go down. □

5 The Relative Invariant

We proceed to construct the invariant relatively in a generality which is adequate for applications but only coincides with Section 3 for complete local rings, to wit:

Set Up/Notation 5.1. Let $\pi : \mathfrak{U} = \operatorname{Spf} A \to B = \operatorname{Spec} k$ be a map from an affine formal scheme to a Noetherian affine scheme of characteristic zero, and suppose that the trace of $\mathfrak{U}$ is a regularly embedded section σ of π of co-dimension m. Furthermore if M is the ideal of σ, suppose M/M^2 is trivial, i.e. $M = (x_1, \dots, x_m)$ is the ideal of σ (so $A \xleftarrow{\sim} k[\![x_1, \dots, x_m]\!]$), and let I be an other ideal of $\mathfrak{U}$ (so M-adically separated by definition), while for objects, over B, denote by a subscript in b the fibre (as a formal scheme, i.e. M-adically complete tensor product) over $b \in B$.

Plainly we begin with the multiplicity, i.e.

Fact 5.2. For $b \in B$, define $d_b(I) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ by,

$$d_b(I) := \sup \left\{ \alpha \in \mathbb{Z}_{\geqslant 0} \,\middle|\, M_b^\alpha \supset I_b \right\};$$

then $b \mapsto d_b(I)$ is upper semi-continuous (often abbreviated to u.s.c.).

Proof. Since I is M-adically separated, it is either zero and $d_b(I)$ is identically ∞, or there is a smallest $e \in \mathbb{Z}_{\geqslant 0}$ such that $I \subset M^e$. The former case is trivial, while in the latter case we have a non-trivial quotient of a free module, i.e.

$$I \longrightarrow M^e \big/ M^{e+1} \longrightarrow Q \longrightarrow 0 \tag{5.1}$$

and the condition $d_b(I) \geqslant e + 1$ is equally the non-trivial closed condition,

$$\dim_{k(b)} Q_b \geqslant \operatorname{rank}\left(M^e \big/ M^{e+1}\right) \tag{5.2}$$

so we conclude by Noetherian induction. $\square$

Next we proceed to the maximal contact space by way of

Fact 5.3. Suppose the multiplicity d_b is identically $d \in \mathbb{Z}_{\geqslant 0}$ and define the sub-module V in M^d/M^{d+1} to be I modulo M^{d+1}, then the following is u.s.c. ,

$$b \mapsto \lambda_0(b) := \begin{cases} \dim_{k(b)} \left\{ \partial \in \left(M \big/ M^2 \otimes k(b)\right)^\vee \,\middle|\, \partial(V_b) = 0 \right\}, & d > 0, \\ 0\,, & d = 0, \end{cases} \tag{5.3}$$

Proof. Plainly, without loss of generality $d > 0$, while the action of $\left(M/M^2\right)^\vee$ by derivations affords a pairing,

$$V \otimes_k \left(M^{d-1} \big/ M^d\right)^\vee \longrightarrow M \big/ M^2 \quad : \quad F \otimes \varphi \longmapsto \{\partial \mapsto \varphi(\partial F)\} \tag{5.4}$$

whose image is a k-submodule,

$$\Lambda' \hookrightarrow M \big/ M^2 \tag{5.5}$$

such that the $k(b)$-vector spaces (5.3) are the annihilators of the image of Λ'_b, so equivalently,

$$\lambda_0(b) = \dim_{k(b)} \Lambda'' \tag{5.6}$$

where Λ'' is the quotient of (5.5). □

Prior to the inductive definition of the relative invariant let us make a,

Warning 5.4. In practice one wishes to take $\mathfrak{U}$ to be the completion in the diagonal of the product of B with itself whenever the latter is smooth over a field. In such a scenario if $b \in B$, then m in the sense of Sect. 3 for the local ring B_b will be its dimension, $m(b)$, which will only coincide with the ambient dimension m in the sense of Set Up/Notation 5.1 if b is closed.

In any case if in addition $b \mapsto \lambda_0(b)$ is constant on B then generalising Definition 3.5,

Fact/Definition 5.5. In the situation of the Set Up/Notation 5.1, a block of (relative, should there be danger of confusion) coordinates is a subset $X \subset M$ of regular parameters whose image modulo M^2 is a subset of a k-basis. In particular whenever $b \mapsto \lambda_0(b)$ is constant we have, possibly at the price of shrinking B to ensure that the implied free k-module is trivial, cf. hypothesis in Corollary 2.7, a block X_0 consisting of the lifting of (5.5), and of course, modulo the Warning 5.4,

$$\lambda_0(b) := m - c_0. \tag{5.7}$$

Inductive Hypothesis 5.6. Exactly as in the Inductive Hypothesis 3.7, with exactly the same notation up to the following minor observations consistent with the Warning 5.4,

(MO.1) I is to be understood in the sense of the Set Up/Notation 5.1.

(MO.2) The definition, cf. (5.7), of λ_i, $0 \leqslant i \leqslant s-1$, is exactly as for the ℓ_i in (3.5) but in light of the Warning 5.4 we will change the notation.

(MO.3) By the definition of a relative block the graded algebra of the filtration has graded pieces free k-modules, and after clearing denominators to integers $a^0 > \cdots > a^{s-1} > a^s$, without common factors, defines a family, in the notation of (2.4), $\mathrm{P}_k(\underline{g}, 1) := \mathbb{P}_k(\underline{a})$ of relative weighted projective champs.

(MO.4) The starting point/initial block is X_0 of Fact/Definition 5.5 under the hypothesis that the functions $d(b)$ and $\lambda_0(b)$ of the Inductive Hypothesis 3.7.(F.1)–(F.2) are identically constant and B is sufficiently small to guarantee the triviality of Λ'' in Fact 5.2

To which we must again adjoin.

Sub-Induction 5.7. Define the set of sub-inductive parameters Θ_{s-1} exactly as in Fact/Definition 3.11, and for $H \in \Theta_{s-1}$ we suppose the hypothesis of the Sub-Induction 3.13 under which we will say that $g_s(b) \geqslant H$, $\forall\, b \in B$.

With this in mind, we have.

Observation/Definition 5.8. We have a filtration $F^{\bullet}_{s-1}(H)$ defined as in (3.11) which for exactly the same reason, Lemma 3.14, is independent of any choices and $V^d_{s-1}(H)$ is defined exactly as in (3.19). Finally by way of notation let Δ be the global vector fields on the associated weighted projective champ, $\mathbb{P}_k(\underline{a})$, in the Inductive Hypothesis 5.6.(MO.3), i.e.

$$\Delta := \coprod_{-n<0} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathrm{T}_{\mathbb{P}_k(\underline{a})}(-n)\big), \tag{5.8}$$

which in turn is a free k-module by the generalisation, [McQ17, I.c.3], of Serre's explicit calculation.

At this juncture Fact 5.3 easily generalises to,

Fact 5.9. Let everything be as in the Sub-Induction 5.7 so in particular $m_s := m - (c_0 + \cdots + c_{s-1}) > 0$, then the following function is u.s.c. ,

$$b \longmapsto \lambda^H_s(b) := \dim_{k(b)} \Big\{ \partial \in \Delta_b \,\Big|\, \partial\big(V^d_{s-1}(H) \otimes k(b)\big) = 0 \Big\} \tag{5.9}$$

Proof. As in the proof of Fact 5.3, derivation gives a pairing,

$$V \otimes_k \coprod_{-n<0} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0 - n)\big)^{\vee} \to \Delta^{\vee} : F \otimes \varphi \mapsto \{\partial \mapsto \varphi(\partial F)\}, \tag{5.10}$$

whose image Λ' affords a short exact sequence of k-modules,

$$\Lambda' \longrightarrow \Delta^{\vee} \longrightarrow \Lambda'' \longrightarrow 0 \tag{5.11}$$

such that the $k(b)$-vector spaces in (5.9) are the annihilators of the image of Λ', while the fibre dimensions,

$$\lambda^H_s(b) = \dim_{k(b)} \Lambda'' \otimes k(b). \tag{5.12}$$

are plainly u.s.c. $\square$

From which we have the corollary.

Corollary 5.10. Under the hypothesis of the Sub-Induction 5.7, let $H' \in \Theta_{s-1}$ be the successor of H and define, $g_s(b) > H$ to mean $g_s(b) \geqslant H'$ and $g_s(b) = H$ its complement then,

(i) The conditions $g_s(b) = H$, resp. $g_s(b) > H$, are open, resp. closed.

(ii) On the open set of $b \in B$ such that $g_s(b) = H$ the function λ^H_s is u.s.c.

Equally we have the relative version of the termination of the sub-induction, i.e.

Case(A) 5.11. (Relative, cf. Case(A) 3.16) At $b \in B$, $g_s(b) = H$ (say B', by way of notation, for the open in Corollary 5.10.(ii)) then we define a function g_s to take the value H at b, and define, $\lambda_s(b)$ to be $\lambda_s^H(b)$ of (5.9). Now replace B' by the constructible subset of $b \in B'$ on which $g_s(b) = H$, and $\lambda_s(b)$ takes the constant value $m_s - c_s < m_s$; form the fibre of π, Set Up/Notation 5.1, over (the new) B'; apply Corollary 2.7 to get blocks $X_0, \dots, X_s$ of cardinality $c_0, \dots, c_s$ (thus around every $b \in B$ we replace B' by a sufficiently small Zariski neighbourhood); and proceed from $s-1$ to s in the Inductive Hypothesis 5.6.

Case(B) 5.12. (Relative, cf. Case(B) 3.17) The complimentary closed set B'', i.e. $g_s > H$, is non-empty, then at $b \in B''$ apply Corollary 2.7 to get a Zariski neighbourhood of b, in B'', on which there are blocks $X_0, \dots, X_{s-1}$ such that after taking the fibre of π over this open the hypothesis of the Sub-Induction 5.7 holds at the successor of H.

In so much as this procedure now involves multiple base changes to the initial base in the Set Up/Notation 5.1, we can usefully observe that if Case(B) 5.12, does not occur at $b \in B$ ad infinitum then a posteriori we can simply replace B in Set Up/Notation 5.1 by a Zariski open neighbourhood of b and drop the precision of restricting to an open neighbourhood of b in Case(A) 5.11. Necessarily we also want to be able to do this should Case(B) 5.12, occur ad infinitum, and this requires a little more care, to wit:

Fact 5.13. Suppose the hypothesis of the Sub-Induction 5.7 and let $B^\bullet \hookrightarrow B$ be the set of parameters where $g_s \geqslant H$ for all $H \in \Theta_{s-1}$ then

(i) $B^\bullet$ is closed.

(ii) Every $b \in B^\bullet$ admits a Zariski open neighbourhood $B \supset V_b \ni b$ such that on replacing B by V_b in the Set Up/Notation 5.1, the precision of shrinking to an open neighbourhood of b at every instance of Case(B) 5.12, as H varies in Θ_{s-1}, may be omitted.

(iii) After base change of π to the constructible set $B \cap V_b \ni b$ the blocks $X_0, \dots, X_{s-1}$ converge in the M-adic topology.

Proof. We have already proved in Corollary 5.10 that for any given H, $g_s \geqslant H$ is a closed condition so not only is $B^\bullet$ closed, it is equal to $g_s \geqslant h$ for h sufficiently large. As such by base change we may suppose, without loss of generality, that $B^\bullet = B$ and Case(A) 5.11, never occurs. Now the reason why we may have to restrict to an open neighbourhood of b is, in the notation of Fact 5.9, that the rank m_s k-modules,

$$\mathrm{D}(H) := \left\{ \partial \in \Delta \,\middle|\, \partial\big(V^d_{s-1}(H)\big) = 0 \right\} \subset \Delta \tag{5.13}$$

may not be trivial. On the other hand for any H we have a surjection,

$$M \big/ M^2 \longrightarrow\!\!\!\!\!\rightarrow F^1_{s-1}(H) \Big/ F^{>1}_{s-1}(H) \tag{5.14}$$

whose kernel (generated by the blocks X_i, $0 \leqslant i \leqslant s-1$) is by construction, (3.11), independent of H. Consequently the quotient (5.14) is a vector bundle independent of H, but by the better still in Lemma 2.4, $\mathrm{D}(H)$ is naturally isomorphic to its dual should Case(A) 5.11, never occur, so we get Fact 5.13.(ii) by Corollary 2.7. Once this is established, (iii) is exactly as in the absolute case (3.21)–(3.22). $\square$

Definition/Fact 5.14. Supposing the Set Up/Notation 5.1 define the relative invariant,

$$\mathrm{INV}_{\mathfrak{U}/B}(I) : B \longrightarrow \mathbb{Q}^{2m}_{\geqslant 0} \tag{5.15}$$

starting from the rules (S.0) and (S.1) of the Start of the Induction 3.6 albeit with d_b, $\lambda_0(b)$ as defined in Facts 5.2 and 5.3. Subsequently if at $b \in B$ in the inductive procedure in s, every sub-induction terminates at a finite H (i.e. Case(A) 5.11), then define

$$\mathrm{INV}_{\mathfrak{U}/B}(I)(b) := \big(d(b),\ \lambda_0(b),\ \dots,\ \lambda_{s-1}(b),\ g_s(b),\ \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0}; \tag{5.16}$$

where s is minimal for the property $\lambda_s(b) = 0$. Finally if Case(B) 5.12, occurs ad infinitum at some $s \geqslant 1$ put,

$$\mathrm{INV}_{\mathfrak{U}/B}(I)(b) := \big(d(b),\ \lambda_0(b),\ \dots,\ g_{s-1}(b),\ \lambda_{s-1}(b),\ \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0}. \tag{5.17}$$

Consequently for $m(b)$ as in Warning 5.4, $\epsilon = m - m_b$, and $\big(g_0 = d,\ \ell_0,\ \dots,\ \ell_t,\ g_t,\ \underline{0}\big)$ the value of the invariant, $\mathrm{inv}_{B_b}(I_b)$ of Sect. 3, with t minimal amongst even entries ℓ_{2i} such that $\ell_{2i} = 0$, is

$$\mathrm{INV}_{\mathfrak{U}/B}(I)(b) := \begin{cases} \big(g_0, \ell_0 + \epsilon, \dots, g_t, \ell_t + \epsilon, \underline{0}\big), & \text{if } g_t \neq 0\ , \\ \big(g_0, \ell_0 + \epsilon, \dots, \ell_{t-1} + \epsilon, \underline{0}\big), & \text{if } g_t = 0,\ t \geqslant 1\ , \\ \qquad\qquad \underline{0} & \text{if } t = 0,\ \text{and } g_0 = 0. \end{cases} \tag{5.18}$$

We have already encountered a similar difference in Lemma 3.23.(ii) and whence the difference merits a specific notation, to wit:

$$\mathrm{diff}(\epsilon) := \begin{cases} \big(0,\ \epsilon,\ \dots, 0,\ \underbrace{\epsilon}_{t^{\mathrm{th}}\text{-place}},\ \underline{0}\big), & \text{if } g_t \neq 0, t \geqslant 1, \\ \big(0,\ \epsilon,\ \dots, 0,\ \underbrace{\epsilon}_{(t-1)^{\mathrm{th}}\text{-place}},\ \underline{0}\big), & \text{if } g_t = 0, t \geqslant 1, \\ \big(0,\ \dots,\ 0,\ \underline{0}\big), & \text{if } t = 0, g_t = 0, \end{cases} \tag{5.19}$$

Plainly the difference, (5.19), between the invariants is minimal, but it is the relative invariant that has the good properties one would expect, for example:

Fact 5.15. Let $\mathrm{INV}_{\mathfrak{U}/B} : B \to \mathbb{Q}^{2m}_{\geqslant 0}$ be as per Definition/Fact 5.14, then

(i) As a function of formal neighbourhoods $\mathfrak{U}$, ideals on the same, and points on the base, $\mathrm{INV}_{\mathfrak{U}/B}$ has self bounding denominators in the sense of Definition 3.1.

(ii) The function $\mathrm{INV}_{\mathfrak{U}/B}$ is upper semi-continuous in the Zariski topology.

The proof will require some topological trivialities, to wit:

Lemma 5.16. Let X be a topological space,

$$\underline{F} := F_1 \times F_2 \,:\, X \longrightarrow \mathbb{Z}^{n_1}_{\geqslant 0} \times \mathbb{Z}^{n_2}_{\geqslant 0}$$

a function and equip each $\mathbb{Z}^{n_i}_{\geqslant 0}$, respectively the aforesaid product, with the the lexicographic order then for $\underline{f} := f_1 \times f_2 \in \mathbb{Z}^{n_1}_{\geqslant 0} \times \mathbb{Z}^{n_2}_{\geqslant 0}$, the set $X_{\geqslant \underline{f}}$, of those $x \in X$ such that $\underline{F}(x) \geqslant \underline{f}$, is closed if the followings hold:

(i) F_1 is upper semi-continuous on $Y_0 := X$;

(ii) $Y_1' := \left\{ x \in Y_1 \,\middle|\, \underline{F}_2(x) \geq \underline{f}_2 \right\}$ is closed in the constructible set $Y_1 := \{x \in Y_0 \,|\, F_1(x) = f_1\}$.

Proof. By item (i) Y_1 is an open subset of $Y := \{x \in X \,|\, F_1(x) \geq f_1\}$, so Y_1 is constructible. Now, by construction

$$X_{\geqslant \underline{f}} = Y_1' \cup \left\{ x \in X \,\middle|\, F_1(x) > f_1 \right\} = Y_1' \cup \left(Y \setminus Y_1 \right) \subseteq Y, \tag{5.20}$$

where the latter is closed in X, so it is sufficient to prove that $Y_1' \cup \left(Y \setminus Y_1 \right)$ is closed in Y. However its closure in Y is

$$\overline{Y_1'} \cup \left(Y \setminus Y_1 \right) = \left(\overline{Y_1'} \cap Y_1 \right) \cup \left(Y \setminus Y_1 \right) = Y_1' \cup \left(Y \setminus Y_1 \right), \tag{5.21}$$

where $\left(\overline{Y_1'} \cap Y_1 \right) = Y_1'$ by item (ii), and we conclude. □

We will apply this in the form:

Corollary 5.17. Let X be a topological space, $F_i \,:\, X \longrightarrow \mathbb{Z}^{n_i}_{\geqslant 0}$ functions, respectively $f_i \in \mathbb{Z}^{n_i}_{\geqslant 0}$, for $n_i \in \mathbb{Z}_{>0}$, $1 \leqslant i \leqslant N$, such that if $N > r \geqslant 0$, with $Y_r := \{x \in X \,\big|\, F_i(x) = f_i,\ 1 \leqslant i \leqslant r\}$, $Y_0 := X$, and for all $0 \leqslant t \leqslant r$ the function F_{t+1} is u.s.c. on the set Y_t, then Y_r is constructible while

$$\underline{F}_{r+1} := (F_1, ..., F_{r+1}) : X \longrightarrow Z^{n_1+...+n_{r+1}}_{\geqslant 0} \quad \text{is u.s.c.}$$

Proof. By induction on $r \in \mathbb{Z}_{\geqslant 0}$, with the case $r = 0$ being trivial. As such let $r \geqslant 1$, and suppose the proposition for $r - 1$, then we may apply Lemma 5.16 to

$$\underline{F}_r \times F_{r+1} : X \longrightarrow \mathbb{Z}^{n_1+...+n_r}_{\geqslant 0} \times \mathbb{Z}^{n_{r+1}}_{\geqslant 0}. \tag{5.22}$$

to conclude by induction. □

Proof of Fact 5.15 The difference between INV and inv is given by (5.18), so in particular their difference is integer valued, thus self bounding denominators for inv, Fact 3.12, implies self bounding denominators for INV while the pre-requisites for deducing the u.s.c. by way of Corollary 5.17 have already been done in Facts 5.2, 5.3, 5.9 and Corollary 5.10. □

The particular case of Fact 3.22.(ii) where $\mathfrak{U}$ is a formal neighbourhood of the diagonal in an algebraic variety, cf. Construction 6.4, suggests that the upper semi-continuity will demand a modified invariant, to wit:

Definition 5.18. Let I be an ideal of a regular ring A of characteristic zero with dimension m; $x \in \operatorname{Spec} A$, while A_x, I_x denote localisation of A, I at x; $\epsilon_x := \dim A - \dim A_x$; and, cf. (5.17) et seq., $(g_0 = d, \ell_0, \dots, \ell_{t-1}, g_t, \underline{0})$ the value of the invariant $\mathrm{inv}_{A_x}(I_x)$ of Section 3 wherein t is minimal among entries ℓ_{2i} with $\ell_{2i} = 0$, then the difference $\mathrm{inv}^!_A(I)(x) - \mathrm{inv}_{A_x}(I_x)$ is defined to be $\mathrm{diff}(\epsilon_x)$ of (5.19).

While it is premature to assert the upper semi-continuity of $\mathrm{inv}^!_A$ we do have,

Fact 5.19. Let everything be as in Definition 5.18 and $y \in \operatorname{Spec} A$, then the set,

$$\left\{ b \in \bar{y} \,\middle|\, \mathrm{inv}^!_A(I)(b) = \mathrm{inv}^!_A(I)(y) \right\}$$

contains a non-empty Zariski open subset of $\bar{y}$.

Unsurprisingly the key point in which we will abuse notation slightly in order to emphasise its relation to the preceeding definitions is:

Claim 5.20. Let everything be as in Fact 5.19, then there is an affine neighbourhood $V := \operatorname{Spec} A' \ni y$ such that if $\widehat{A}'$ is the completion of A' in y, and $A_{\{y\}}$ the completion of A_y in the maximal ideal then there is a regular weighted filtration $F^p(I)$ of $\widehat{A}$ such that if $F^p_{\{y\}}$ is the filtration of $A_{\{y\}}$ of (3.24) associated to the pair I_y, A_y then,

$$F^p_{\{y\}} := F^p(I) \widehat{\otimes}_{\widehat{A}'} A_{\{y\}}$$

Better still not only is the multiplicity d of I constant along $\bar{y} \cap V$, but if a^0 is the highest weight amongst the blocks of the filtration and,

$$V_d := \left(\frac{I + F^{>da^0}(I)}{F^{>da^0}(I)} \right) \otimes_{\widehat{A}'} k,$$

where $k := A'/\bar{y}$, then V_d enjoys the non-degeneracy condition (2.5) of Hypothesis 2.3 for the associated $\mathbb{P}_k(\underline{a})$.

Proof. For obvious reason we don't worry about the difference between A' and A and simply understand $\operatorname{Spec} A \ni y$ to be a sufficiently small Zariski neighbourhood of the same. Similarly we put $k = A/\bar{y}$, and, of course suppose that for M the maximal ideal of $\bar{y}$ in A, M/M^2 is the trivial k-module. Now while we may not be in the hypothesis of Set Up/Notation 5.1, i.e. k may not embed in A, (5.6) and (5.12) apply as stated to deduce that $\lambda_i(b)$, and whence implicitly the $g_i(b)$, are constant for b in a Zariski open subset of $\bar{y}$. A priori there remains the substantive difference between the Zariski localisation $\bar{A}_y$ and the formal localisation $A_{\{y\}}$ but this has been addressed in (5.13) et seq. in the proof of Fact 5.13 (iii). $\square$

We are now in position to give,

Proof of Fact 5.19. We may without loss of generality suppose that the conclusion of Claim 5.20 holds. Consequently, on replacing $\operatorname{Spec} A$ by $\operatorname{Spec} A'$, it will be sufficient to show the stranger statement that $\mathrm{inv}^!$ is constant. As such let $b \in \bar{y}$, and A_b, resp. $A_{\{b\}}$ the localisation, resp. formal localisation at b, then by Fact 3.22 we have the identities,

$$\mathrm{inv}_{A_y}(I_y) = \mathrm{inv}_{A_{\{y\}}}(I_{\{y\}}), \quad \mathrm{inv}_{A_b}(I_b) = \mathrm{inv}_{A_{\{b\}}}(I_{\{b\}}). \tag{5.23}$$

Furthermore since $A_{\{b\}}$ and $k_{\{b\}}$ are complete local rings we have (non canonically) a splitting,

$$\begin{array}{c} \mathfrak{U} = \operatorname{Spf} A_{\{b\}} \\ {\scriptstyle\pi}\big\downarrow \big\uparrow{\scriptstyle\sigma} \\ B = \operatorname{Spec} k_{\{b\}}, \end{array} \tag{5.24}$$

in which the trace of σ is the pull-back of $\bar{y}$. As such if K is the quotient field of $k_{\{b\}}$, then the fibre (qua formal scheme) $\mathfrak{U}_K$ is the (formal) base change

$$A_{\{b\}} \longmapsto A_{\{b\}} \widehat{\otimes}_{k(b)} K = A_{\{y\}}. \tag{5.25}$$

Now (unsurprisingly) Fact 5.21 the relative invariant is stable under base change, so from (5.23), (5.25), and Definition/Fact 5.14 we have,

$$\mathrm{inv}_{A_y}(I_y) = \mathrm{INV}_{\mathfrak{U}/B}(I)(K) \tag{5.26}$$

while by the better still in Claim 5.20, we have,

$$\mathrm{INV}_{\mathfrak{U}/B}(I)(K) = \mathrm{INV}_{\mathfrak{U}/B}(I)(b). \tag{5.27}$$

On the other hand by Definition/Fact 5.14 the latter is the invariant of I restricted to the special fibre $\mathfrak{U}_b$ in (5.24) which itself is a product $\mathfrak{U}_b \times_{k(b)} B$ so we have a second projection pr to $\mathfrak{U}_{k(b)}$ and a second ideal $\mathrm{pr}^* I\,\big|_{\mathfrak{U}_b}$, with

$$\mathrm{INV}_{\mathfrak{U}/B}\big(\mathrm{pr}^* I\,\big|_{\mathfrak{U}_b}\big) = \mathrm{inv}_{A_y}\big(I_y\big)$$

by (5.26), (5.27), and the base change formula Fact 5.21, while, Lemma 3.23.(ii), the difference between $\mathrm{inv}_{A_{\{b\}}}(\mathrm{pr}^* I\,\big|_{\mathfrak{U}_b})$ and $\mathrm{INV}_{\mathfrak{U}/B}(\mathrm{pr}^* I\,\big|_{\mathfrak{U}_b})$ is $\mathrm{diff}(\epsilon)$ of (5.19), with $\epsilon = \dim A_b - \dim A_y$. As such it will suffice to prove,

$$\mathrm{inv}_{A_{\{b\}}}\big(I\big) = \mathrm{inv}_{A_{\{b\}}}\big(\mathrm{pr}^* I\,\big|_{\mathfrak{U}_b}\big)$$

which in the presence of the better still in Claim 5.20, follows by induction in s in the definition of the invariant as encountered at the Start of the Induction 3.6, subsequently in the Inductive Hypothesis 3.7. $\square$

An equally useful property is stability under base change, i.e.

Fact 5.21. Let $\beta : B' \to B$ be a map of schemes, and $\pi : \mathfrak{U}' \to B'$ the base change of π of Set Up/Notation 5.1, qua formal scheme with I' the pull-back of I then,

$$\mathrm{INV}_{\mathfrak{U}'/B'} = \beta^* \mathrm{INV}_{\mathfrak{U}/B}$$

Proof. By way of notation let M' be the pull-back of M, then the condition $I \subset M^e$ plainly implies $I' \subset (M')^e$. At which point we just need to check that the conditions that the dimension of the modules (since the odd entries of INV are determined by whether this is maximal or not) (5.6) and (5.12) are stable under base change which is indeed the case since tensor products are right exact. □

Of which a particularly pertinent corollary is.

Corollary 5.22. Let $\pi : U \to B$ be a regular map of characteristic zero affine Noetherian schemes in which both $\Gamma(U)$ and $\Gamma(B)$ are regular Noetherian local rings, then if I is an ideal on B, $J = \pi^* I$ and $\epsilon = \dim \Gamma(U) - \dim \Gamma(B)$ the invariants $\mathrm{inv}_{\Gamma(U)}$, $\mathrm{inv}_{\Gamma(B)}$ enjoy exactly the same relation as enunciated in items (i)–(ii) of the particular case of Lemma 3.23 ($U = \operatorname{Spec} B$, $B = \operatorname{Spec} A$ in the notation of op.cit.) i.e. their difference is $\mathrm{diff}(\epsilon)$ of (5.19). Better still if there exists a filtration $F^\bullet(I)$ on B whose completion is (3.24), then the same is true on U and,

$$F^\bullet(\pi^* I) = \pi^* F^\bullet(I). \tag{5.28}$$

Proof. By Fact 3.22 we may suppose that $\Gamma(B)$ is a complete local ring, so, inter alia it has a coefficient field isomorphic to its residue field $k(b)$, and by Lemma 3.14, the invariant $\mathrm{inv}_{\Gamma(B)}(I)$ is equally the relative invariant for,

$$\begin{array}{c} \operatorname{Spf} \Gamma(B) \\ \downarrow \\ \operatorname{Spec} k(b). \end{array} \tag{5.29}$$

On the other hand if u is the closed point of U, then we can, Fact 5.21, base extend (5.28) to $k(u) \to \Gamma(B) \otimes_{k(b)} k(u)$ without changing the invariant. At the same time we can complete $\Gamma(U)$ in either u or $\pi^* b$, and since completion in u is equally completion in $\pi^* b$ subsequently completed in u, neither operation changes the invariant. We may thus suppose that $\Gamma(B)$ and $\Gamma(U)$ are complete local rings with the same residue field, and since we are in characteristic zero π remains regular (otherwise we'd need to suppose geometrically regular). In particular therefore, $\Gamma(U)$ is a power series ring over $\Gamma(B)$ and Lemma 3.23 applies to give the relation between the invariants while (5.28) follows from Lemma 3.23 and the fact that completion is faithfully flat. □

Notice that en passant we have proved

Fact 5.23. Let $\pi : U \to B$ be a regular map of regular affine schemes of dimension N, n respectively and I a sheaf of ideals on B then,

$$\mathrm{inv}^!_{\Gamma(U)}(\pi^* I) - \mathrm{inv}^!_{\Gamma(B)}(I) = \mathrm{diff}(N - n). \tag{5.30}$$

Proof. From the proof of Corollary 5.22 for any $u \in U$ over b,

$$\mathrm{inv}_{\mathscr{O}_{U,u}}(\pi^* I_u) - \mathrm{inv}_{\mathscr{O}_{B,b}}(I_b) = \mathrm{diff}\big(\dim \mathscr{O}_{U,u} - \dim \mathscr{O}_{B,b}\big)$$

while by Definition 5.18,

$$\mathrm{inv}^!_{\Gamma(B)}(I) - \mathrm{inv}_{\mathscr{O}_{B,b}}(I_b) = \mathrm{diff}\big(n - \dim \mathscr{O}_{B,b}\big)$$

and similarly for U, whence (5.30) by the additivity of diff. □

6 Principalisation

To begin with let us make

Observation/Definition 6.1. Let $\mathcal{U}$ be a regular Noetherian Deligne–Mumford, or indeed formal champ, of characteristic 0, and $\mathscr{I}$ a sheaf of ideals on $\mathcal{U}$ then for $x : \mathrm{Spec}\, K \to \mathcal{U}$ a geometric point (i.e. K is algebraically closed) the invariant $\mathrm{inv}_{\mathcal{U}}(\mathscr{I})(x)$ is defined to be $\mathrm{inv}_{\mathscr{O}_{\mathcal{U},x}}(\mathscr{I}_x)$ where $\mathscr{I}_x$ is the stalk of $\mathscr{I}$ in the strictly Henselian ring $\mathscr{O}_{\mathcal{U},x}$. In particular therefore by Fact 3.22 if,

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[dr, "x"'] & \mathrm{V} \arrow[d] \\ & \mathcal{U} \end{tikzcd}$$

is any factorisation through an affine étale neighbourhood, with y the image on V then,

$$\mathrm{inv}_{\mathcal{U}}(\mathscr{I})(x) = \mathrm{inv}_{\mathscr{O}_{\mathrm{V},y}}(\mathscr{I}_y), \tag{6.1}$$

and we will vary this construction in the obvious way for the variants $\mathrm{inv}^!$, resp. $\mathrm{inv}^\sharp$.

In addition for the convenience of the reader let us recall,

Revision/Definition 6.2. Any characteristic zero Deligne–Mumford champ $\mathcal{Y}$ with quotient singularities admits a unique finite surjective map $\nu : \widetilde{\mathcal{Y}} \to \mathcal{Y}$, the *Vistoli covering champ*, from a regular Deligne–Mumford champ which is an isomorphism in co-dimension 1, cf. [Vis89, 2.8] or [MP13, I.iv.2]. A priori the operation of weighted blowing up creates such singularities. Specifically if $F^\bullet \subseteq \mathcal{O}_{\mathcal{X}}$ are sheaves of a weighted filtration on a characteristic zero Deligne–Mumford champ $\mathcal{X}$, i.e. everywhere locally as per Definition 3.4, indexed by $p \in \mathbb{Z}_{\geq 0}$ so that F^p is identified with $F^{q(p)}$ of op. cit. where, inductively, $q(p+1) > q(p)$ is minimal in $\mathbb{Q}_{\geq 0}$ with the property that $F^{q(p+1)} \subset F^{q(p)}$ is a strict inclusion, $q(0) = 0$, then, [MP13, I.iv.3], we have the *weighted blow up*,

$$\pi : \mathcal{Y} := \mathrm{Proj}(\mathcal{O}_{\mathcal{X}} \oplus F^1 \oplus F^2 \oplus \cdots) \to \mathcal{X} \tag{6.2}$$

and we define the *smoothed weighted blow up* $\widetilde{\mathcal{X}} \to \mathcal{X}$ to be the Vistoli covering champ of (6.2). In particular if $\mathrm{Spec}\,(k) \to \mathcal{X}$ is a map from an affine scheme factoring through the support of $\mathcal{O}_{\mathcal{X}}/F^1$, then, in the notation of Definition 3.4 and (2.4), we have an isomorphism,

$$\widetilde{\mathcal{X}} \times_{\mathcal{X}} \mathrm{Spec}\,(k) \xrightarrow{\sim} \mathrm{P}_k(\underline{r}) \tag{6.3}$$

With this in mind we have the key,

Fact 6.3. Let $\mathfrak{U} = \mathrm{Spf} A$ be the formal spectrum of a complete regular ring of characteristic zero, I an ideal of A, $F^\bullet(I)$ as in Fact 3.22, and $\rho : \widetilde{\mathfrak{U}} \to \mathfrak{U}$ the smoothed weighted blow up of Revision/Definition 6.2 associated to the aforesaid weighted filtration, then for $\widetilde{I}$ the proper transform of I, at every closed geometric point x of $\widetilde{\mathfrak{U}}$,

$$\mathrm{inv}_{\widetilde{\mathfrak{U}}}(\widetilde{I})(x) < \mathrm{inv}_{\mathfrak{U}}(I).$$

Proof. Upon clearing denominators the blocks of the filtration have weights a^i, and we have an $\left(A\,/_{F^{>0}}\right)$-module,

$$\bar{I} := I \mod F^{>da^0}$$

such that if $\mathscr{I}$ is the resulting sheaf of ideals on the associated weighted projective champ, equivalently the exceptional divisor $\mathscr{E} \hookrightarrow \widetilde{\mathfrak{U}}$, then,

$$\widetilde{I}\,\big|_{\mathscr{E}} = \mathscr{I}.$$

Consequently we can conclude by Proposition 4.3 provided that

$$\mathrm{inv}_{\mathscr{E}}\big(\widetilde{I}\,\big|_{\mathscr{E}}\big)(x) \geq \mathrm{inv}_{\widetilde{\mathfrak{U}}}\big(\widetilde{I}\big)(x)$$

at closed geometric points x. As far as the odd entries of the invariant are concerned, cf. the proof of Proposition 4.3, this is clear. There is, however, need for caution at the even entries which is provided by items (ii-bis) and (iii-bis) of Lemma 4.4, which are satisfied for the inclusion $\mathscr{E} \hookrightarrow \mathfrak{U}$, i.e. replace $\mathcal{Q}$ by $\mathscr{E}$ in op.cit. and the values of c_i on the ambient space by their value on $\mathfrak{U}$. □

Plainly, therefore, the algebraicity or otherwise of the filtration $F^\bullet$ of $\widehat{A}$ of Fact 3.22 should Hypothesis 3.19 occur is the only obstruction to constructing a resolution of singularities for local rings from the invariant, and to address this problem we will proceed from varieties over a field to spectra of complete local rings by way of a particular instance of the relative invariant, to wit:

Construction 6.4. Let V/K be a smooth affine scheme of dimension m over a field K of characteristic 0 and let $\mathscr{P}^n_{\mathrm{V}/K}$ be the sheaf of n-jets of [EGA-IV, 16.7] then, for any map $\tau : T \to \mathrm{V}$ from a scheme T, we have a formal scheme equipped with a projection,

$$\begin{tikzcd} \mathrm{V} & \mathfrak{P}_T := \operatorname{Spf}\left(\varprojlim_n \tau^* \mathscr{P}^n_{V/K}\right) \arrow[l, "\mathrm{pr}"'] \arrow[d, "\pi"] \\ & T \arrow[u, bend right, "\sigma"'] \end{tikzcd} \tag{6.4}$$

whose trace is a regularly embedded section σ - in fact $\mathfrak{P}_T$ is the completion of the graph of τ. In particular if $T = \mathrm{V}$ and I is an ideal on V then we have

$$\mathrm{INV}_{\mathfrak{P}_{\mathrm{V}}/\mathrm{V}}(\mathrm{pr}^* I)(x) = \mathrm{inv}^!_{\Gamma(\mathrm{V})}(I)(x) \tag{6.5}$$

where the latter is the invariant of Definition 5.18, so, to reiterate, their difference with the invariant $\mathrm{inv}_{\mathrm{V}}(I)(x)$ of (3.3) is $\mathrm{diff}(\epsilon)$ of (5.19), where

$$\epsilon = \operatorname{Tr}\deg_K K(x) = \dim \mathrm{V} - \dim \mathscr{O}_{\mathrm{V},x}. \tag{6.6}$$

In light of Facts 5.15 and 5.21, we therefore make,

Fact/Definition 6.5. Let everything be as in Construction 6.4, then for $\tau : T \to \mathrm{V}$ a map from a Noetherian scheme T, we define,

$$\mathrm{inv}^!_T(I) : T \longrightarrow \mathbb{Q}^{2m}_{\geqslant 0} \ : \ t \longmapsto \mathrm{INV}_{\mathfrak{P}_T/T}(\mathrm{pr}^* I)(t) \tag{6.7}$$

so, by Fact 5.21, $\mathrm{inv}^!_T(I)$ is u.s.c. and equal to $\tau^*\mathrm{inv}^!_{\mathrm{V}}(I) = \mathrm{inv}^!_{\Gamma(\mathrm{V})}(I)$ of Definition 5.18.

The next step is complete local rings and weighted blow ups of their spectra, by way of,

Variant 6.6. (of Construction 6.4) Let W be the spectrum of a complete local regular Noetherian ring with residue field of characteristic 0 (e.g. the completion around a not necessarily closed point of V of Construction 6.4) then by [Mat86, 28.3] we can choose a coefficient field K and coordinates $x_1, \dots, x_m$ such that $W \xrightarrow{\sim} \operatorname{Spec} K[\![x_1, \dots, x_m]\!]$. Now consider the diagram of rings,

$$\begin{tikzcd} K[\![y_1, ..., y_m]\!] \arrow[r] & K[\![x_1, ..., x_m, y_1, ..., y_m]\!] \\ & K[\![x_1, ..., x_m]\!] \arrow[u] \end{tikzcd} \tag{6.8}$$

so, for example, in the situation that $W \to \mathrm{V}$ arises from completing V of Construction 6.4 in a K-point, the ring in the top right hand corner is $\Gamma\big(\mathfrak{P}_W\big)$ in the notation of (6.4). We are, however, at liberty to apply the functor Spec to (6.8) to get,

$$\begin{tikzcd} & & \mathfrak{P}_W \arrow[dl, hook', "i"'] \arrow[ddl, bend left, "\pi_W"] \\ W & P \arrow[l, "\mathrm{pr}"'] \arrow[d, "\pi"] & \\ & W \arrow[u, bend left, "\sigma=\Delta"] & \end{tikzcd} \tag{6.9}$$

wherein the distinctions with (6.4) in the case that W comes from V are:

(a) $\mathfrak{P}_W$ is the completion of P in the diagonal $(x_i - y_i | 1 \leqslant i \leqslant m)$.

(b) Nevertheless both the projections in (6.4) and (6.9) are projections (6.10) to schemes with pr of (6.9) the continuous extension of pr of (6.4).

Now in the first instance we can profit from these observations to extend the definition of $\mathrm{inv}^!$ to ideals J of W, i.e. exactly as in (6.7) but for $T \longrightarrow W$,

$$\mathrm{inv}^!_T(J) := \mathrm{INV}_{\mathfrak{P}_T/T}\big(i^*\mathrm{pr}^*J\big) \tag{6.11}$$

with i, pr as in (6.9), and, of course, this is compatible with Definition 5.18, resp. (6.7) by Fact 5.21 if $T = W$, resp. J were pulled back from V. Consequently Fact 5.15 applies and $\mathrm{inv}^!$ is u.s.c. on W irrespectively of whether J is pulled back from something of finite type or not. The risk, however, is that we lose the possibility of having a good construction of the relative invariant, and whence the u.s.c. once we start making weighted blow ups of W. To get around this suppose a weighted centre with blocks $X_0, \ldots, X_s$ and weights $a_0, \ldots, a_s$ is given to which we add (a possibly empty) block Y to obtain a system of coordinates on W and identify $\Gamma(W)$ with the completion in the origin of the ring of functions on,

$$A := \mathrm{Spec}\, K[X_0, ..., X_s, Y] \xrightarrow{\ \sim\ } \mathbb{A}^m_K$$

Now form the smoothed weighted blow up $\rho : \mathcal{A} \to A$ in the blocks X_i with weights a_i (so $\widetilde{\mathfrak{U}}$ of Fact 6.3 would be the completion of $\mathcal{A}$ in the exceptional divisor) to get a diagram,

$$\begin{tikzcd} \mathcal{A} & \mathfrak{P}_{\mathcal{A}} := \mathrm{Spf}\Big(\varprojlim_n \mathscr{P}^n_{\mathcal{A}/K}\Big) \arrow[l, "\mathrm{pr}"'] \arrow[d, "\pi"'] \\ & \mathcal{A} \arrow[u, bend right, "\sigma= \text{zero-section}"'] \end{tikzcd} \tag{6.12}$$

wherein it goes without saying that, even though the diagonal fails to be an embedding, the jets of [EGA-IV, 16.7] are well defined on any Deligne–Mumford champ because of their étale local nature. Finally observe that the smoothed weighted blow up $\rho : \mathcal{W} \to W$ in the said blocks is just the base change,

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\rho"'] & \mathcal{W} \arrow[l] \arrow[d, "\rho"] \\ A & W \arrow[l] \end{tikzcd} \tag{6.13}$$

so that on base changing the projection π of (6.12) we get a diagram,

$$\begin{tikzcd} \mathcal{W} \arrow[dd, "\rho"'] & & \mathfrak{P}_{\mathcal{W}} \arrow[ll, "\mathrm{pr}"'] \arrow[ddll, "\mathrm{pr}"] \arrow[dd, "\pi_{\mathcal{W}}"'] \\ & & \\ \mathcal{A} & & \mathcal{W} \arrow[uu, bend right, "\sigma"'] \end{tikzcd} \tag{6.14}$$

wherein the existence of the horizontal arrow results from the horizontal arrow in (6.9), equivalently (6.10).(a).

At this juncture, in line with Observation/Definition 6.1, the theory of the relative invariant applies étale locally to the (formally) representable map $\pi_{\mathcal{W}}$ and we conclude,

Fact/Definition 6.7. Let $\mathscr{I}$ be a sheaf of ideals on the weighted smoothed blow up $\rho : \mathcal{W} \longrightarrow W$ of (6.13) and define at a geometric (but not necessarily closed) point $w : \mathrm{pt} \longrightarrow \mathcal{W}$,

$$\mathrm{inv}^!_{\mathcal{W}}\big(\mathscr{I}\big)(w) := \mathrm{inv}_{\mathfrak{P}_{\mathcal{W}}/\mathcal{W}}\big(\mathrm{pr}^*\mathscr{I}\big)(w) \tag{6.15}$$

then in the moduli $|\mathcal{W}|$ of $\mathcal{W}$ (i.e. the projectivisation of the graded algebra, cf. (6.2), which defines the weighted rather than smoothed weighted blow up) $\mathrm{inv}^!_{\mathcal{W}}$ is u.s.c. .

Better still the discussion also reveals that we can make numerous improvements to Fact 6.3 to wit:

Corollary 6.8. Let A be a complete regular ring of characteristic zero, $W = \mathrm{Spec}\, A$, $I \neq A$ an ideal of A, $F^\bullet(I)$ the filtration of Fact 3.22, $\mathcal{W} \to W$ the associated smoothed weighted blow up of Revision/Definition 6.2 and x a closed point, then:

(i) For all $\mathfrak{p} \in W$,

$$\mathrm{inv}^!_W(I)(\mathfrak{p}) \leqslant \mathrm{inv}^!_W(I)(x).$$

(ii) If $\widetilde{I}$ is the proper transform of I on $\mathcal{W}$ then for all geometric points w of $\mathcal{W}$,

$$\mathrm{inv}^!_{\mathcal{W}}(\widetilde{I})(w) < \mathrm{inv}^!_W(I)(x).$$

(iii) If $X_0, \dots, X_s$ are the blocks defining the filtration $F^\bullet(I)$ then the sub-scheme $X_0 = \cdots = X_s = 0$ is exactly

$$\widehat{Z} = \left\{ \mathfrak{p} \in W \,\middle|\, \mathrm{inv}^!_W(I)(\mathfrak{p}) = \mathrm{inv}^!_{\mathcal{W}}(I)(x) \right\} \tag{6.16}$$

Proof. Since A is a local ring, Corollary 6.8.(i) is just the u.s.c. of $\mathrm{inv}^!$ in Fact/Definition 6.5 as extended to arbitrary ideals of A in (6.11) et seq. Similarly we already know Corollary 6.8.(ii) after completing in the exceptional divisor $\mathcal{E} \hookrightarrow \mathcal{W}$ by Fact 6.3, and since $\mathrm{inv}^!_{\mathcal{W}}(\widetilde{I})$ is also u.s.c. in the Zariski topology of the moduli by Fact/Definition 6.7, we have it everywhere since ρ is proper. Consequently if in Corollary 6.8.(iii) $\widehat{Z}$ were not contained in $X_0 = \cdots = X_s = 0$, then by the u.s.c. of Fact/Definition 6.7 we would have the absurdity that the invariant would not go down. Conversely the inclusion of $X_0 = \cdots = X_s = 0$ in $\widehat{Z}$ is essentially automatic from Fact/Proposition 3.20, Case(B) 5.12, and the lower semi-continuity of the c_i. □

All of which can be combined to establish,

Fact 6.9. Let A be an excellent regular local ring of characteristic zero, $\widehat{A}$ its completion in the maximal ideal then for every ideal I of A there exists a filtration $F^\bullet(I)$ whose completion is the filtration $F^\bullet(\widehat{I})$ of $\widehat{A}$ afforded, Fact 3.22, by $\widehat{I} = \widehat{A} \otimes_A I$.

Proof. Put $V = \operatorname{Spec} A$, $W = \operatorname{Spec} \widehat{A}$, $R := W \times_V W \overset{t}{\underset{s}{\rightrightarrows}} W$ the resulting groupoid, and $\rho : \mathcal{W} \to W$ the smoothed weighted blow up of Corollary 6.8 associated to $F^p(\widehat{I})$. Now the fibre of R over a point, which in turn is cut out by the pull-back to R of any system of coordinates on V, is a point thus although R may have many connected components their dimension is at most that of V, which is equally that of W, and only one has this maximal value. Furthermore by hypothesis $W \to$ V is regular so this is equally true of the source s and sink t, whence by Fact 5.23,

$$s^*\mathrm{inv}^!_W(\widehat{I}) = \mathrm{inv}^!_R(s^*\widehat{I}) = \mathrm{inv}^!_R(t^*\widehat{I}) = t^*\mathrm{inv}^!_W(\widehat{I}). \tag{6.17}$$

Similarly by Lemma 3.23, we have that $s^*F^p(\widehat{I})$, resp. $t^*F^p(\widehat{I})$, are (after completion) the filtration of Fact 3.22 at every point where $s^*\mathrm{inv}^!_W(\widehat{I})$, resp. $t^*\mathrm{inv}^!_W(\widehat{I})$, is maximal. Consequently by (6.17), and of course, as implied therein, $s^*\widehat{I} = t^*\widehat{I} = \widehat{I}\,\big|_R$, $s^*F^p(\widehat{I})$ and $t^*F^p(\widehat{I})$ are the filtration of $I\,\big|_R$ at every point where $\mathrm{inv}^!_R(I\,\big|_R)$ is maximal. Thus we have a canonical isomorphism between $s^*\mathcal{W}$ and $t^*\mathcal{W}$ which is uniquely determined by its value (e.g. the identity is standard birational parlance) where ρ is an isomorphism, whence a descent datum for $F^p(\widehat{I})$ with respect to the faithfully flat map $W \to V$ so we're done by [SGA-I, exposé VIII, 1.1]. □

An alternative in the geometric case is to appeal directly to the relative invariant in a formal neighbourhood of the diagonal, to wit:

Alternative 6.10. Let x be a not necessarily closed point of a smooth variety V/K over a field of characteristic zero, I an ideal of V and $F^{\bullet}(\widehat{I})$ the canonical filtration, Fact 3.22, of the completion of $\mathscr{O}_{\mathrm{V},x}$ in its maximal ideal determined by I, then $F^{\bullet}(\widehat{I})$ is algebraic, i.e. Fact 6.9 holds for $A = \mathscr{O}_{\mathrm{V},x}$.

Proof. By way of notation put $\mathrm{V}_{\mathrm{Zar}} = \mathrm{Spec}\, \mathscr{O}_{\mathrm{V},x}$, $W = \mathrm{Spec}\, \widehat{\mathscr{O}}_{\mathrm{V},x}$, and $\widehat{Z} \hookrightarrow W$ as in (6.16) then from the compatibility of Fact/Definition 6.5 with (6.11), the sub-scheme $\widehat{Z}$ is, by the former, the pre-image under $W \to \mathrm{V}$ of,

$$Z := \left\{ \mathfrak{p} \in \mathrm{V}_{\mathrm{Zar}} \,\middle|\, \mathrm{inv}^!_{\mathrm{V}}(I)(\mathfrak{p}) \geqslant \mathrm{inv}^!_{\mathrm{V}}(I)(x) \right\},$$

so, as the notation suggests, if I_Z is the ideal of the sub-scheme then,

$$I_{\widehat{Z}} = I_Z \otimes_{\mathscr{O}_{\mathrm{V},x}} \widehat{\mathscr{O}}_{\mathrm{V},x}.$$

It remains to find the blocks themselves rather than just the centre, Z, on which they are supported. To do this it is sufficient to do Fact/Proposition 3.20 I_Z-adically rather than $\mathfrak{m}(x)$-adically. If, however, we denote by the subscript ét strict Henselisation at x, then in $\mathrm{V}_{\text{ét}}$ we can choose a projection π and a section σ,

$$\begin{tikzcd} \mathrm{V}_{\text{ét}} \arrow[r, "\pi"'] & Z_{\text{ét}} \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

such that σ is the embedding of $Z_{\text{ét}} \xhookrightarrow{\sigma} \mathrm{V}_{et}$, so I_Z-adic convergence of the blocks follows from Fact 5.13.(iii). As such we have Fact 6.9 but for a filtration $F^{\bullet}_{\text{ét}}(I)$ of $\mathscr{O}_{\mathrm{V}_{\text{ét}}}$. Now make V sufficiently small in the Zariski topology such that,

(a) $\mathrm{inv}^!_{\mathrm{V}}(I)(x)$ is the maximum of the invariant over V.
(b) There is a filtration $F^{\bullet}_{\text{ét}}(I)$ on a geometrically irreducible étale neighbourhood $\mathrm{V}' \longrightarrow \mathrm{V}$ of x satisfying Fact 6.9 after completing in a point $x' \in \mathrm{V}'$ over x. (6.18)

In particular, therefore, the support of the graded algebra $\mathrm{gr}^{\bullet}_{F_{\text{ét}}}$ is the fibre Z' over Z, which in turn is the locus where $\mathrm{inv}^!_{\mathrm{V}'}(I)$ is maximal. Similarly if we consider the groupoid,

$$R := \mathrm{V}' \times_{\mathrm{V}} \mathrm{V}' \overset{t}{\underset{s}{\rightrightarrows}} \mathrm{V}' \tag{6.19}$$

$s^*F_{\text{ét}}$ and $t^*F_{\text{ét}}$ define the canonical filtration at every point in $s^*Z' = t^*Z'$, equivalently the locus where $\mathrm{inv}^!_R(I)$ is maximal, so by Fact 3.22 they are equal, and whence Fact 6.9 in the Zariski topology. ◻

In the geometric case, we already have upper semi-continuity of the invariant in Fact/Definition 6.5, while in general we appeal to:

Fact 6.11. Let I be an ideal in an excellent regular ring A, then on $\operatorname{Spec} A$, the function $x \mapsto \operatorname{inv}^!_A(I)(x)$ of Definition 5.18 is u.s.c. on $\operatorname{Spec} A$.

The strategy follows Villamayor's exposition, [Vil14, 6.13], of Dade's unpublished Princeton thesis, in the case of the multiplicity, from which we plagiarise,

Claim 6.12. Let $f : \operatorname{Spec} A \to \Gamma$ be a function to a discrete ordered group then f is u.s.c. iff,

(i) $\forall\, y \in \operatorname{Spec} A,\ x \in \bar{y} \Longrightarrow f(x) \geqslant f(y)$.

(ii) $\forall\, y \in \operatorname{Spec} A$, the set $\{x \in \bar{y} \big| f(x) \leqslant f(y)\}$ contains a non-empty Zariski open subset.

Proof. Plainly given Claim 6.12.(i) we can replace inequality by equality in Claim 6.12.(ii), while the conditions are clearly necessary, and we do the converse by induction on dimension of closed sub-spaces, Y, which without loss of generality are irreducible of positive dimension. However by Claim 6.12.(ii), $Y = Y' \amalg Z$ where f takes the value of its generic point, $f(y)$, on Y'. Furthermore by Claim 6.12.(i) $f\big|_Z \geqslant f(y)$ and $Z \hookrightarrow Y$ is a closed subset of smaller dimension, so $f(x) > f(y)$ is closed and everything left over takes the value $f(y)$. □

At which juncture we may proceed to,

Proof of Fact 6.11. Since the invariant has self bounding denominators, Definition 3.1–Fact 3.2, we plainly need only verify items Claim 6.12.(i)–(ii) with f replaced by $\operatorname{inv}^!_A$. Now (ii) is Fact 5.19 and is valid for any regular ring while (i) follows from Corollary 5.22 and the upper semi-continuity of $\operatorname{inv}^!$ for complete local rings, (6.11) et seq.. □

At which point we can move rapidly towards a conclusion by way of,

Fact/Definition 6.13. Let V be a connected regular excellent affine scheme of dimension m, I an ideal of V; $\underline{i} \in \mathbb{Q}^{2m}_{\geqslant 0}$ the maximum value of $\operatorname{inv}^!_{\mathrm{V}}(I)$ over V; $\mathcal{V} \to \mathrm{V}$ the smoothed weighted blow up (whose existence and uniqueness is guaranteed by Facts 6.9 and 3.22 associated to the canonical filtration $F^\bullet(I)$; while for $\underline{q} \in \mathbb{Q}^{2m}_{\geqslant 0}$ define a modification functor

$$M_{I,\underline{q}}(\mathrm{V}) := \begin{cases} \mathcal{V}, & \text{if } \underline{i} = \underline{q}, \\ \mathrm{V}, & \text{otherwise}; \end{cases} \tag{6.20}$$

and extend $M_{I,\underline{q}}$ to arbitrary smooth affine V, i.e. a disjoint union of connected components $\coprod_\alpha \mathrm{V}_\alpha$ by way of,

$$M_{I,\underline{q}}(\mathrm{V}) := \coprod_\alpha M_{I,\underline{q}}(\mathrm{V}_\alpha). \tag{6.21}$$

Then by Fact 3.22 the modification functor $M_{I,\underline{q}}$ is étale local, i.e. if $\mathrm{V}' \to \mathrm{V}$ is étale and I' the pull-back of I to V then there is a fibre square,

$$\begin{tikzcd} M_{I,\underline{q}}(\mathrm{V}) \arrow[d] & M_{I',\underline{q}}(\mathrm{V}') \arrow[l] \arrow[d] \\ \mathrm{V} \arrow[ur, phantom, "\square"] & \mathrm{V}' \arrow[l] \end{tikzcd} \tag{6.22}$$

In particular if $\mathcal{X}$ is a regular excellent Deligne–Mumford champ, $\mathscr{I}$ a sheaf of ideals on the same and $\underline{q}$ the maximum at geometric points of $\mathrm{inv}^!_{\mathcal{X}}(\mathscr{I})$, (6.1) et seq., then for $\mathrm{V} \to \mathcal{X}$ an étale atlas and $R = \mathrm{V} \times_{\mathcal{X}} \mathrm{V} \underset{s}{\overset{t}{\rightrightarrows}} V$ the implied groupoid,

$$\begin{tikzcd} M_{\mathscr{I}_R,\underline{q}}(R) \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[d] & M_{\mathscr{I}_{\mathrm{V}},\underline{q}}(\mathrm{V}) \arrow[d] \\ R \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathrm{V} \end{tikzcd} \tag{6.23}$$

is a map of groupoids in which $M_{\mathscr{I}_R,\underline{q}}(R)$ (which we may abusively consider unique since it's a modification) is equally the fibre of the rightmost vertical arrow over either s or t by (6.22), i.e. the $M_{\mathscr{I}_{\mathrm{V}},\underline{q}}$ patch to a smoothed weighted blow up,

$$M_{\mathscr{I}}(\mathcal{X}) \longrightarrow \mathcal{X} \tag{6.24}$$

depending only on $\mathscr{I}$. We therefore get our first global results, to wit:

Proposition 6.14. Let $\mathscr{I}$ be a (coherent) sheaf of ideals on a regular excellent Deligne–Mumford champ of characteristic zero, and define inductively a sequence of smoothed weighted blow ups in regular weighted centres by,

$$(\mathcal{X}_0, \mathscr{I}_0) := (\mathcal{X}, \mathscr{I}) \quad \text{and} \quad (\mathcal{X}_{p+1}, \mathscr{I}_{p+1}) := (M_{\mathscr{I}_p}\mathcal{X}_p, \widetilde{\mathscr{I}_p}),\ p > 0$$

where $\widetilde{\mathscr{I}_p}$ is the proper transform of $\mathscr{I}_p$, then for $p \gg 0$, $\mathscr{I}_p$ is trivial, i.e. equal to the structure sheaf $\mathcal{O}_{\mathcal{X}_p}$. In particular if $\mathscr{I}$ is the sheaf of ideals of an irreducible embedded sub-champ, $N+1$ the smallest p such that $\mathscr{I}_p$ is trivial and $\mathcal{Y}_p \hookrightarrow \mathcal{X}_p$ the sub-champ cut out by $\mathscr{I}_p$ then if at every closed point $\mathcal{X}$ has the same dimension the chain,

$$\mathcal{Y}_0 \longleftarrow \mathcal{Y}_1 \longleftarrow \cdots \longleftarrow \mathcal{Y}_{N-1} \longleftarrow \mathcal{Y}_N \tag{6.25}$$

is a sequence of smoothed weighted blow ups in regular centres Z_p contained in the singular locus of $\mathcal{Y}_p$, $p < N$, such that $\mathcal{Y}_N$ is regular. Otherwise, i.e. the dimension of $\mathcal{X}$ is not constant on closed points, the same conclusion (6.25) holds provided for each p one changes the invariant to

$$\widetilde{\mathrm{inv}}_{\mathcal{X}_p}(\mathscr{I}_p)(x) = \begin{cases} \mathrm{inv}^!_{\mathcal{X}}(\mathscr{I}_p)(x), & x \notin \mathcal{Y}_p \\ \mathrm{inv}^!_{\mathcal{X}}(\mathscr{I}_p)(x) + \mathrm{diff}\big(\dim \mathcal{Y} - d_p(x)\big) \end{cases} \tag{6.26}$$

where $d_p(x)$ is the dimension of the connected component of $\mathcal{Y}_p \ni x$; and then blows up in strata where $\widetilde{\mathrm{inv}}$ rather than $\mathrm{inv}^!$ is maximal.

Proof. Since $\mathrm{inv}^!$ goes down, Corollary 6.8.(ii), under the modification $M_{\mathscr{I}}\mathcal{X} \to \mathcal{X}$ while preserving excellence and the invariant has self bounding denominators, Fact 5.15.(i), the only thing to check is the in particular which in turn only requires checking that Z_p is contained in $\mathrm{Sing}(\mathcal{Y}_p)$, $p < N$. If, however, there were a geometric, so without loss of generality closed, point $y \in Z_p$ where $\mathcal{Y}_p$ was regular then the value of $\widetilde{\mathrm{inv}}_{\mathcal{X}}(\mathscr{I}_p)$ at y would be,

$$\big(1, \dim(\mathcal{Y}), \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0}$$

and since this is equally the minimum value of $\widetilde{\mathrm{inv}}_{\mathcal{X}}(\mathscr{I}_p)$, $\mathcal{Y}_p$ would be regular contradicting the choice of N. ☐

Arguably the good way to think about Proposition 6.14 is in terms of resolving rational maps, which merits:

REMARK 6.15. If in Proposition 6.14 $\mathcal{X}$ were a projective variety, X/K, over a field K and H an ample line bundle then given a sheaf of ideals $\mathscr{I}$ there is a $n \gg 0$, such that $H^n \otimes \mathscr{I}$ is generated by global sections, and whence $\mathscr{I}$ is the indeterminacy locus of rational map

$$\varphi_{\mathscr{I}} : X \dashrightarrow \mathbb{P}\big(\mathrm{H}^0(X, H^n \otimes \mathscr{I})\big) \tag{6.27}$$

while, since X is, by hypothesis smooth, every rational map $\varphi : X \dashrightarrow \mathbb{P}^N_K$ determines a unique line bundle, L_φ, which is equal to $\varphi^*\mathscr{O}(1)$ in codimension 2, and a space of sections,

$$\varphi^*\mathrm{H}^0\big(\mathbb{P}^N_K, \mathscr{O}(1)\big) \subset \mathrm{H}^0\big(X, L\big)$$

which generates the indeterminacy locus, i.e. $\mathscr{I}_\varphi L_\varphi$, for some sheaf of ideals $\mathscr{I}_\varphi$. Of course this relation between ideals and rational maps may fail even for X/K a scheme of finite type, albeit it suffices to replace (6.27) by $\varphi : \mathcal{X} \dashrightarrow \mathrm{Bl}_{\mathscr{I}} X$ to maintain it in absolute generality. In any case the relationship between ideals and rational maps is rather tight, so we can equally think about the modification functor $M_{\mathscr{I}}\mathcal{X}$ as a modification functor $M_\varphi\mathcal{X}$ for φ a rational map, so that Proposition 6.14 becomes the rather satisfactory statement:

Let φ be a rational map on a Deligne–Mumford champ $\mathcal{X}$, and define inductively a sequence of rational maps by,

$$\big(\mathcal{X}_0, \varphi_0\big) = \big(\mathcal{X}, \varphi\big), \qquad \big(\mathcal{X}_{p+1}, \varphi_{p+1}\big) = \big(M_{\varphi_p}\mathcal{X}_p, \widetilde{\varphi}_p\big),\ p \geq 0,$$

where $\widetilde{\varphi}_p$ is the proper transform of φ, then for $p \gg 0$, φ_p is everywhere defined.

7 Excellent Resolution

Of course the in particular in (6.25) gives a resolution of singularities of anything admitting an embedding in something regular, but this is not a very satisfactory hypothesis so we improve it by way of

Set Up/Construction 7.1. Again, all rings and their residue fields are of characteristic zero, and for Y a connected reduced excellent affine scheme of dimension n, with $y \in Y$ a not necessarily closed point, we denote by $\widehat{\mathscr{O}}_{Y,y}$ the completion of $\mathscr{O}_{Y,y}$ in the maximal ideal. As such, we may again, [Mat86, 28.3], choose a coefficient field L and a presentation,

$$0 \longrightarrow I \longrightarrow A := L[\![S_1, ..., S_e]\!] \longrightarrow \widehat{\mathscr{O}}_{Y,y} \longrightarrow 0 \tag{7.1}$$

where,

$$e := e_Y(y) = \dim_{k(y)} \mathfrak{m}(y) \Big/ \mathfrak{m}^2(y) \tag{7.2}$$

is the embedding dimension, and observe that any 2 such presentations are related by a commutative diagram of exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A := L[\![S_1, ..., S_e]\!] \arrow[r] & \widehat{\mathscr{O}}_{Y,y} \arrow[r] & 0 \\ 0 \arrow[r] & I_0 \arrow[r] \arrow[u] & A_0 := L_0[\![T_1, ..., T_e]\!] \arrow[r] \arrow[u, "\wr"] & \widehat{\mathscr{O}}_{Y,y} \arrow[r] \arrow[u, equal] & 0\,. \end{tikzcd} \tag{7.3}$$

As such $\mathrm{inv}_Y(y) := \mathrm{inv}_A(I)$ is well defined, and for m the maximum over all embedding dimensions we correct this to

$$\mathrm{inv}^!_Y(y) := \big(\mathrm{inv}_Y(y) + \mathrm{diff}(m - e_Y(y))\big) \times \underline{0} \in \mathbb{Q}^{2m} \tag{7.4}$$

with an implies block of zeroes whenever $e_Y(y) < m$.

At the same time in the complete local ring $\widehat{\mathscr{O}}_{Y,y}$, or better, and equivalently since Y is excellent, in the strict Henselisation $\mathscr{O}^{\mathrm{h}}_{Y,y}$, we have,

$$d_Y(y) := \min_{\mathfrak{q}} \dim \frac{\widehat{\mathscr{O}}_{Y,y}}{\mathfrak{q}} = \min_{\mathfrak{q}} \dim \frac{\mathscr{O}^{\mathrm{h}}_{Y,y}}{\mathfrak{q}}$$

where the minimum is taken over all the minimal primes in $\widehat{\mathscr{O}}_{Y,y}$, or equivalently $\mathscr{O}^{\mathrm{h}}_{Y,y}$, which in turn affords the invariant,

$$\mathrm{inv}^\sharp_Y(y) := \big(\delta_Y(y) := e_Y(y) - d_Y(y)\big) \times \mathrm{inv}^!_Y(y) \ \in \mathbb{Q}^{2m+1}_{\geqslant 0} \tag{7.5}$$

Similarly the smoothed weighted blow up,

$$\widehat{\mathcal{Y}} \longrightarrow \mathrm{Spec}\, \widehat{\mathscr{O}}_{Y,y} \tag{7.6}$$

associated to the canonical filtration $F^\bullet(I)$ of $\operatorname{Spec} A$ is not only independent of the presentation, but,

Fact 7.2. If $\mathscr{O}_{Y,y} \to B$ is a regular map of local rings and $\mathcal{W} \to \operatorname{Spec} \widehat{B}$ the result of performing (7.1)–(7.6) for $\widehat{B}$ then $\mathcal{W}$ is the pull-back along $\widehat{\mathscr{O}}_{Y,y} \to \widehat{B}$ of the modification (7.6).

Proof. Since the map is regular then at worst after a base extension of L we may, as in the proof of Corollary 5.22, suppose that $\widehat{B} = \widehat{\mathscr{O}}_{Y,y}[\![z_1, \dots, z_e]\!]$ is a formal power series ring with coefficients in $\widehat{\mathscr{O}}_{Y,y}$, so this follows by either Corollary 5.22 or Lemma 3.23. $\square$

Nevertheless to make everything fit together in this generality we need to descend $\widehat{\mathcal{Y}}$ of (7.5) to a modification of $\operatorname{Spec} \mathscr{O}_{Y,y}$ and establish the upper semi-continuity of $\mathrm{inv}^\sharp$. The latter is somewhat involved for arbitrary excellent rings so it seems useful to observe that the geometric case is rather trivial, to wit:

Alternative 7.3. Let Y/K be a reduced affine scheme of finite type over a field of characteristic zero with $\mathscr{O}^{\mathrm{h}}_{Y,y}$ the strict Henselisation at some point $y \in Y$ then at the minor price of base changing, by a separable extension of L, we can suppose the presentation (7.1) arises from an étale neighbourhood $Y' \to Y$ around a point $y' \mapsto y$ with $K(y')/K(y)$ étale, i.e. there is an embedding $Y' \hookrightarrow \mathrm{V}'$ into a smooth K-scheme of dimension $e + \operatorname{Tr deg}_K K(y) - n$ such that after completion in y',

$$0 \longrightarrow I' \longrightarrow \Gamma(\mathrm{V}') \longrightarrow \mathscr{O}_{Y',y'} \longrightarrow 0 \tag{7.7}$$

becomes (7.1) upon applying $K(y') \otimes_L -$. In particular after replacing Y' and Y by appropriately small affine neighbourhoods of themselves we recognise that (7.6) is algebraic. Indeed after base changing to $K(y')$ it is the formal fibre of the proper transform,

$$\mathcal{Y}' \longrightarrow Y', \tag{7.8}$$

of Y in the canonical modification $\mathcal{V} \to \mathrm{V}$ of Fact/Definition 6.13 associated to I'.

Now observe that the leading term δ_Y in $\mathrm{inv}^\sharp_Y$ is just,

$$\dim_{k(y)} \Omega_{Y/K} \otimes k(y) - \dim \bar{y} - \min_{Y_0} \dim \mathscr{O}^{\mathrm{h}}_{Y_0,y} \tag{7.9}$$

where the minimum is taken over all the components Y_0 of the Henselian local ring at y, so (7.9) is equally,

$$\dim_{k(y)} \Omega_{Y/K} \otimes k(y) - \min_{Y_0} \dim Y_0 \tag{7.10}$$

where now the minimum is taken over components $Y_0 \ni y$ on a sufficiently small étale neighbourhood of y. Consequently δ_Y is the difference of an upper semi-continuous function and a lower semi-continuous one so δ_Y is u.s.c. . To conclude from here that $\mathrm{inv}^\sharp_Y$ is u.s.c. we require by Lemma 5.16 to establish that $\mathrm{inv}^!_Y$ is u.s.c. where δ_Y is constant. To this end say $\delta_Y(x) = \delta_Y(z)$, then we may as well say that we're on an étale neighbourhood Y' of a constructible set Z, with generic point z, where $\Omega_{Y/K}$ has constant rank and around Z we have an embedding $Y' \hookrightarrow M$ into a smooth K-variety of dimension $e_Y(x)$ for x any geometric point of Z. Consequently for $x \in Z$, closed or otherwise,

$$\mathrm{inv}^!_Y(x) = \mathrm{inv}^!_M(I_{Y'}) + \mathrm{diff}(m - \dim M) \tag{7.11}$$

so it's upper semi-continuous by Fact/Definition 6.5. We now proceed to the general case via.

Claim 7.4. Let everything be as in Set Up/Construction 7.1 then $\mathrm{inv}^\sharp_Y$ satisfies Dade's conditions as enunciated in Claim 6.12. More precisely, for $z \in Y$:

(i) If $\mathscr{O}_{Y,x}$ is just a characteristic zero local ring and f is either δ_Y or $\mathrm{inv}^\sharp_Y$ then for $\bar{z} \ni x$, $f(x) \geqslant f(z)$.

(ii) If Y is universally catenary and J-2, i.e. every reduced closed sub-scheme $Z \hookrightarrow Y$ contains a non-empty Zariski open subset where it is regular then there is a non-empty Zariski open subset of points $x \in \bar{z}$ where $\delta_Y(x) \leqslant \delta_Y(z)$.

(iii) If Y is excellent then there is a non-empty Zariski open subset of points $x \in \bar{z}$ where $\mathrm{inv}^\sharp_Y(x) \leqslant \mathrm{inv}^\sharp_Y(z)$.

In particular if Y is excellent then both δ_Y and $\mathrm{inv}^\sharp_Y$ are u.s.c. .

Proof. Consider first the behaviour of δ_Y in Claim 7.4.(i), and observe, [SS72], that there is a good theory of the universal finitely generated module, $\Omega_{\widehat{Y}/k(x)}$, of $k(x)$-derivations, and:

$$e = e_Y(x) = \dim_{k(x)} \Omega_{\widehat{Y}/k(x)} \otimes k(x) = \dim \Omega_{X/k(x)} \otimes k(x) \tag{7.12}$$

where $\widehat{Y} \hookrightarrow X = \mathrm{Spec}\, k(x)[\![x_1, \dots, x_e]\!]$ is an embedding, afforded by (7.3), with ideal I_Y. Similarly for $\zeta \in \widehat{Y}$ any point lying over z and $Z = \bar{z}$ with $\widehat{Z}$ the formal fibre we have an exact sequence,

$$0 \longrightarrow I_Z/I_Z^2 \otimes k(\zeta) \longrightarrow \Omega_{\widehat{Y}/k(x)} \otimes k(\zeta) \longrightarrow \Omega_{\widehat{Z}/k(x)} \otimes k(\zeta) \longrightarrow 0 \tag{7.13}$$

from which we obtain,

$$e_Y(z) + \dim_x \bar{\zeta} = \dim_{k(\zeta)} \Omega_{\widehat{Y}/k(x)} \otimes k(\zeta). \tag{7.14}$$

Now appeal to the Henselian description of d_Y as in (7.9)–(7.10) to obtain,

$$\delta_Y(z) = \dim_{k(\zeta)} \Omega_{\widehat{Y}/k(x)} \otimes k(\zeta) - \min_{Y_0 \supset Z} \dim Y_0 \tag{7.15}$$

where the minimum is taken over components of $\mathscr{O}^{\mathrm{h}}_{Y,x}$ containing Z, so it's certainly the case that,

$$\min_{Y_0 \supset Z} \dim Y_0 \geqslant d_Y(x) \tag{7.16}$$

whence $\delta_Y(z) \leqslant \delta_Y(x)$ by (7.15) and the upper semi-continuity of the fibres of $\Omega_{\widehat{Y}/k(x)}$. Better still we've done Claim 7.4.(ii) unless $\delta_Y(z) = \delta_Y(x)$ which requires not only an identity in (7.16) but that $\Omega_{\widehat{Y}/k(x)}$ has constant rank $e = e_Y(x)$ along $\widehat{Z}$, so inter alia $\dim_x \bar{\zeta}$ is independent of ζ from (7.14). In any case if $\{\_\}$ denotes completion of a local ring in its maximal ideal then we have a presentation,

$$0 \longrightarrow I_{\widehat{Y}} \longrightarrow \mathscr{O}_{X,\{\zeta\}} \longrightarrow \mathscr{O}_{\widehat{Y},\{\zeta\}} \longrightarrow 0 \tag{7.17}$$

which by (7.14) is a base extension $k(z) \to k(\zeta)$ of a presentation of the form (7.1) of $\mathscr{O}_{Y,\{\zeta\}}$, so by Fact 5.21,

$$\mathrm{inv}_Y(z) = \mathrm{inv}_{\widehat{Y}}\big(\widehat{\zeta}\,\big) = \mathrm{inv}_X(I_{\widehat{Y}})(\zeta). \tag{7.18}$$

Consequently from the definitions Definition 5.18 and (7.4),

$$\mathrm{inv}^!_Y(z) - \mathrm{inv}^!_Y(x) = \mathrm{inv}^!_X\big(I_{\widehat{Y}}\big)(\zeta) - \mathrm{inv}^!_X\big(I_{\widehat{Y}}\big)(x) + \mathrm{diff}\big(\dim \mathscr{O}_{X,\zeta} - e_Y(z)\big) \tag{7.19}$$

while $\dim \mathscr{O}_{X,\zeta} = e_Y(z)$ from (7.14) under the hypothesis of $\delta_Y(z) = \delta_Y(x)$. Consequently $\mathrm{inv}^!_Y(x) \geqslant \mathrm{inv}_Y(z)$ from the u.s.c. of $\mathrm{inv}^!_X(I_{\widehat{Y}})$, i.e. (6.11) et seq., which in turn completes Claim 7.4.(i) by Lemma 5.16.

As to item (ii) we again begin with δ_Y, and without loss of generality we may suppose every point of $\bar{z}$ is regular. Now consider the co-normal sheaf, to $\bar{z}$, i.e.

$$\mathcal{C} := I_{\bar{z}}/I^2_{\bar{z}},$$

then, again, if we restrict to a small enough neighbourhood of z we may suppose that $\mathcal{C}$ is a bundle of rank c. Consequently for any $x \in \bar{z}$ with maximal ideal $\mathfrak{m}(x)$ in Y, and $\mathfrak{m}_z(x)$ along $\bar{z}$ we have an exact sequence

$$\mathcal{C} \otimes k(x) \longrightarrow \mathfrak{m}(x)/\mathfrak{m}(x)^2 \longrightarrow \mathfrak{m}_{\bar{z}}(x)/\mathfrak{m}_{\bar{z}}(x)^2 \longrightarrow 0$$

which for x arbitrary, resp. the particular choice of $x = z$, gives,

$$e_Y(x) \leqslant e_{\bar{z}}(x) + c, \quad \text{resp.} \quad e_Y(z) = c. \tag{7.20}$$

Similarly since $\mathscr{O}_Y$ is supposed universally catenary, there's no difficulty in taking $Y \ni z$ sufficiently small such that $\forall\, x \in \bar{z}$,

$$d_Y(x) = d_Y(z) + \dim_x \bar{z}, \tag{7.21}$$

so that from (7.20) and (7.21) we have,

$$\delta_Y(x) - \delta_Y(z) \leqslant \big(e_{\bar{z}}(x) - \dim_x \bar{z}\big), \tag{7.22}$$

and we've already cut things down so that the right hand side of (7.22) is zero, so we get Claim 7.4.(ii). Now to complete the proof of (ii) for $\mathrm{inv}_Y^\sharp$ requires a fact of independent utility, to wit:

Fact 7.5. Let $\widehat{\mathcal{Y}} \to \widehat{Y}$ be the modification (7.6), then if $\mathscr{O}_{Y,y}$ is excellent there is a smoothed weighted blow up $\mathcal{Y} \to \mathrm{Spec}\, \mathscr{O}_{Y,y}$ such that $\widehat{\mathcal{Y}}$ is the formal fibre of $\mathcal{Y}$.

Proof. Just as in Fact 6.9 we aim to descend $\widehat{\mathcal{Y}}$ to $V := \mathrm{Spec}\, \mathscr{O}_{Y,y}$, so let

$$R = \widehat{Y} \times_V \widehat{Y} \underset{s}{\overset{t}{\rightrightarrows}} \widehat{Y}$$

be the groupoid afforded by $\widehat{Y} \to V$. Now in a variant of Fact 6.9 choose a system of parameters at y, i.e. functions x_i, $1 \leqslant i \leqslant \dim V$, such that the sub-scheme, $\bullet$, $x_i = 0$ for all i, has dimension 0 at y then the fibre of R over $\bullet$ has dimension 0 and is cut out by $\dim V$ functions, so, again all of R, $\widehat{\mathcal{Y}}$ and V have the same dimension. Furthermore since $\widehat{Y}$ is a local ring we know by what we've already established in Claim 7.4.(i) that wherever $\mathrm{inv}_{\widehat{Y}}^\sharp$ is maximal, δ_Y is maximal, and for $\mathfrak{p} \in \widehat{Y}$ we can, (7.12)–(7.18), without loss of generality suppose that the presentation employed in calculating $\mathrm{inv}_{\widehat{Y}}^\sharp$ is just (7.1) completed at $\mathfrak{p}$. Thus by Fact 5.23, $\mathrm{inv}_R^\sharp = s^*\mathrm{inv}_Y^\sharp = t^*\mathrm{inv}_Y^\sharp$, and $s^*\widehat{\mathcal{Y}}$, resp. $t^*\widehat{\mathcal{Y}}$, have, by Fact 7.2, the same formal fibre at every point where $\mathrm{inv}_R^\sharp$ is maximal. Consequently profiting from the fact that each contains an everywhere dense subset of R we may reasonably identify them, to obtain a descent datum, and again conclude by [SGA-I, VIII.1.1]. $\square$

If not perhaps any easier, the geometric case offers an,

Alternative 7.6. Consider the following statement whose argument is reduced K-varieties over a field K of characteristic 0.

S(Y) : For $\underline{i} = \underline{i}(Y) \in \mathbb{Q}_{\geqslant 0}^{2m+1}$ the maximum value of $\mathrm{inv}_Y^\sharp$ and $y \in Y$ there is a Zariski open neighbourhood $N_y \ni y$ and a smoothed weighted blow up $\mathcal{N}_y \to N_y$ whose formal fibre is (7.6) if $\mathrm{inv}_Y^\sharp(y) = \underline{i}$ and the identity otherwise. In particular therefore if $Z \hookrightarrow Y$ is the locus $\mathrm{inv}_Y^\sharp = \underline{i}$, then,

(C.1) for $y \notin Z$ we can take $N_y = Y \setminus Z$ and $\mathrm{S}(Y)$ is trivially true.

(C.2) If $\mathcal{N}_y$ exists then it is, by definition, unique.

(C.3) By (7.7)–(7.8) there is an étale atlas $U \to Y$ such that $S(U)$ is true at every point of U, so without loss of generality, we have a smoothed weighted blow up $\mathcal{U} \to U$, which is everywhere the modification of $\mathrm{S}(Y)$.

As such we can argue exactly as in (6.19), i.e. for $R := U \times_Y U \underset{s}{\overset{t}{\rightrightarrows}} U$, $s^*\mathcal{U}$ is canonically isomorphic (even equal since its birational) to $t^*\mathcal{U}$ by item (C.2) deduced from the statement, $S(R)$ at R, and whence conclude $\mathrm{S}(Y)$.

Irrespectively, we can apply Fact 7.5 in the spirit of Claim 5.20 to complete the proof of Claim 7.4. Specifically throwing away irrelevant closed sets without comment: we have, without loss of generality, an everywhere regular irreducible closed subscheme $Z = \{\bar{z}\} \hookrightarrow Y$, and by Fact 7.5 a smoothed weighted blow up $\mathcal{Y} \hookrightarrow Y$ whose formal fibre is (7.6). Now if $x \in Z$, we may from the u.s.c. of δ_Y suppose $\delta_Y(x) = \delta_Y(z)$, and all of (7.17) et seq. holds. As such if the symbol $\widehat{\bullet}$ denotes the spectrum of completion in x (rather than the formal scheme completion), and $X = \operatorname{Spec} A$ after a choice of the presentation (7.1) at x, then we have embeddings,

$$\widehat{Z} \hookrightarrow \widehat{Y} \hookrightarrow X \tag{7.23}$$

together with the fibre $\widehat{\mathcal{Y}} := \mathcal{Y} \times_Y Y \to \widehat{Y}$ of our modification, which if it were trivial, i.e. Y is regular at Z, there is nothing to do. Otherwise from $\delta_Y(x) = \delta_Y(z)$ and the independence of (7.6) from the presentation as employed in (7.18), we may identify $\widehat{\mathcal{Y}} \to \widehat{Z}$ with the modification associated to the filtration $F^p(I_{\widehat{Y}})$ associated to the ideal $I_{\widehat{Y}}$ in the completed local ring of $\widehat{Y}$ at any component of $\widehat{Z}$. Similarly, without loss of generality, we may equally suppose that the blocks $X_0, \dots, X_s$ of the filtration defining $\mathcal{Y}$ are defined on $\mathcal{Y}$, and their proper transform $\widetilde{X}_i$, $0 \leqslant i \leqslant s$, cut out a decreasing chain,

$$\mathcal{Y} = \mathcal{Y}_s, \qquad \mathcal{Y}_{t-1} = \mathcal{Y}_t \cap (\widetilde{X}_t = 0), \tag{7.24}$$

with completions $\widehat{\mathcal{Y}}_t$ at x. However by (7.18) one recognises from Claim 5.20 that for any $\widehat{z} \in \widehat{Y}$ over z,

$$\operatorname{inv}^!_{\widehat{Y}}(x) = \operatorname{inv}^!_{\widehat{Y}}(z) \iff \dim \widehat{\mathcal{Y}}_t(x) = \dim \widehat{\mathcal{Y}}_t(z),\ \forall\, 0 \leqslant t \leqslant s, \tag{7.25}$$

which in turn is equivalent to the condition $\dim \mathcal{Y}_t(x) = \dim \mathcal{Y}_t(z)$, which is certainly true on a Zariski open subset of Z. $\square$

We can put all of this together to conclude,

Summary/Definition 7.7. Let Y be an excellent affine scheme of characteristic zero, $\underline{i}(Y) \in \mathbb{Q}^{2m+1}_{\geqslant 0}$ the maximum value of $\operatorname{inv}^\sharp_Y$ then,

(E.1) By Fact 7.5 every point $y \in Y$ has a Zariski open neighbourhood N_y and a smoothed weighted blow up $\mathcal{N}_y \to N_y$ whose formal fibre is (7.6).

(E.2) These patch to a smoothed weighted blow up $\mathcal{Y} \to Y$. Indeed in the notation of items (C.1)–(C.3) encountered in the Alternative 7.6 to the conclusion of the proof of Claim 7.4, if $x \in N_z$, $x \in \bar{z}$, and $\operatorname{inv}^\sharp_Y(x) = \operatorname{inv}^\sharp_Y(z)$, then the formal fibre of $\mathcal{N}_z$ at x is that of z.

(E.3) Better still if for $\underline{i}(Y) \leqslant \underline{q}$ in $\mathbb{Q}^{2m+1}_{\geqslant 0}$ and Y connected we define

$$M_{\underline{q}}(Y) := \begin{cases} \mathcal{Y}, & \text{if } \underline{i}(Y) = \underline{q} \\ Y, & \text{if } \underline{i}(Y) < \underline{q} \end{cases} \tag{7.26}$$

and extend to direct sums of connected components as in (6.21), then (since it is enough to check the formal fibres) the smoothed weighted blow up $M_{\underline{q}}(Y)$ commutes with étale maps, i.e. if $Y' \to Y$ is étale or even just regular then, by Fact 7.2 and Summary/Definition 7.7.(E.2) we have a fibre square

$$\begin{tikzcd} M_{\underline{q}}(Y) \arrow[d] & M_{\underline{q}}(Y') \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\ Y & Y' \arrow[l] \end{tikzcd} \quad . \tag{7.27}$$

In particular therefore, cf. (6.23)–(6.24), if $\mathcal{Y}$ is an excellent reduced Deligne–Mumford champ i.e. all Henselian local rings are excellent, and some (whence any) atlas is J2, then there is a smoothed weighted blow up,

$$M_{\underline{q}}(\mathcal{Y}) \longrightarrow \mathcal{Y} \tag{7.28}$$

supported in the singular locus whose fibre over an étale atlas is the blow up functor (7.26), and which itself commutes with étale maps, i.e. replace $Y \longrightarrow Y'$ by an étale map of champ $\mathcal{Y}' \to \mathcal{Y}$ in (7.27). Finally for $\mathrm{inv}^\sharp_{\mathcal{Y}}$ defined as in (7.5), let $\underline{i}(\mathcal{Y})$ be the maximum value of $\mathrm{inv}^\sharp_{\mathcal{Y}}$ and $M(\mathcal{Y}) := M_{\underline{i}(\mathcal{Y})}(\mathcal{Y})$ then by construction,

$$\underline{i}(\mathcal{Y}) = 0 \quad \iff \quad \mathcal{Y} \text{ is regular.} \quad \iff \quad M(\mathcal{Y}) = \mathcal{Y}.$$

All of which is easily assembled into a resolution algorithm, to wit:

Proposition 7.8. For $\mathcal{Y}$ a reduced excellent Deligne–Mumford champ, define a sequence of smoothed weighted blow ups by,

$$\mathcal{Y}_0 = \mathcal{Y}, \qquad \mathcal{Y}_{p+1} = M(\mathcal{Y}_p), \quad p \geqslant 0 \tag{7.29}$$

and let $N \geqslant 0$ be the smallest integer such that $\mathcal{Y}_{N+1} \to \mathcal{Y}_N$ is the identity then the chain of smoothed weighted blow ups,

$$\mathcal{Y} = \mathcal{Y}_0 \longleftarrow \mathcal{Y}_1 \longleftarrow \cdots \longleftarrow \mathcal{Y}_{N-1} \longleftarrow \mathcal{Y}_N \tag{7.30}$$

is a resolution of singularities in the 2-category of Deligne–Mumford champs enjoying the functorial resolution properties (E.1)–(E.3) of Summary/Definition 7.7, but for champ rather than just affine schemes.

Proof. From the definition (7.5) and Fact 5.15.(i), $\mathrm{inv}^\sharp_{\mathcal{Y}}$ has self bounding denominators, Definition 3.1, so it suffices to check,

$$\underline{i}(M_{\mathcal{Y}}) < \underline{i}(\mathcal{Y})$$

Plainly, however, the embedding dimension cannot increase under a smoothed weighted blow up and since (7.6) is the formal fibre around any point, this is immediate from the corresponding proposition, Corollary 6.8.(ii), for $\mathrm{inv}^!$. $\square$

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