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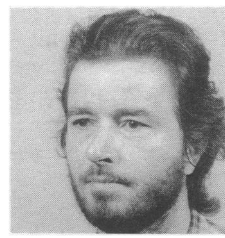
# An Alternative Proof of the Lindemann-Weierstrass Theorem

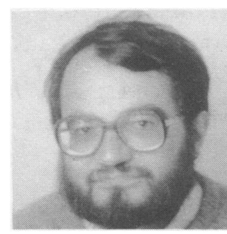
F. Beukers, J. P. Bézivin, P. Robba

*Dedicated to the Memory of Philippe Robba*

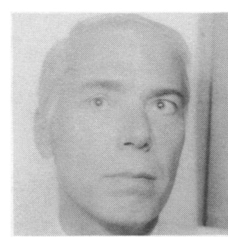
Frits Beukers: born in 1953. University education in Leiden, Netherlands. Currently teaching at the State University of Utrecht. Research interests are number theory and arithmetic properties of linear differential equations.



Jean-Paul Bézivin: educated at the École Normale Supérieure and the Universities of Paris VI and VII. He currently teaches at the University of Paris VI. Research interests are $p$-adic analysis, linear recurrent sequences and arithmetic properties of formal power series.



Philippe Robba: born in 1941, died on October 12, 1988. Doctorate degree at the University of Paris in 1972, and professor at the University Paris XI from 1975 until his untimely death. Domain of research: $p$-adic analysis, mainly $p$-adic differential equations with emphasis on $p$-adic cohomologies.



**Introduction.** In December 1987 J. P. Bézivin and Ph. Robba found a new proof of the Lindemann-Weierstrass theorem as a by-product of their criterion of rationality for solutions of differential equations. Let us recall the Lindemann-Weierstrass theorem, to which we shall refer as LW from now on.

*Let $\alpha_1, \dots, \alpha_t, b_1, \dots, b_t$ be algebraic numbers such that the $b_i$ are all nonzero and the $\alpha_i$ are mutually distinct. Then*

$$b_1 e^{\alpha_1} + b_2 e^{\alpha_2} + \cdots + b_t e^{\alpha_t} \neq 0.$$

It is well known that the transcendence of $\pi$ follows from LW in the following way. Suppose, on the contrary, that $\pi$ is algebraic. Then so is $\pi\sqrt{-1}$ and LW now implies that $e^{\pi\sqrt{-1}} + 1 \neq 0$, which is certainly not true. Thus we conclude that $\pi$ is transcendental.

The usual proof of LW is essentially due to Hilbert and has been polished by a number of authors. One such version can be found in [**2**, Ch. XI] or in [**4**, Ch. I], [**3**]. The new proof of Bézivin and Robba looks totally different. It can be considered as a direct consequence of their criterion of rationality for solutions of linear differential equations.

The Bézivin-Robba criterion is based on a theorem of Pólya-Bertrandias, which is far from easy and relies heavily on $p$- adic analysis. We refer the interested reader to [**1**]. In March 1988 F. Beukers found that the use of the Pólya-Bertrandias criterion is much too heavy and can be avoided in a very elementary way. The result is a new proof of LW which is elementary and can compete with the usual one in shortness and simplicity. There is always a possibility that the similarity between the proofs is stronger than one would expect at first sight. In fact, very soon after a first draft of this paper was written (May 1988) Yu. Nesterenko pointed out to us that the numbers $v_n(k)$ which we use are equal to the integrals

$$\sum_{j=1}^{t} b_j e^{\alpha_j} \int_{\alpha_j}^{\infty} e^{-x} x^{n-kt} (x-\alpha_1)^k \cdots (x-\alpha_t)^k \, dx$$

which are used in the Hilbert proof. In spite of such similarities we feel that the arguments of our proof are nice enough to present in front of a wider audience. We would like to thank the referee for several improvements upon our presentation.

THEOREM. *Let* $b_1, \ldots, b_t, \alpha_1, \ldots, \alpha_t \in \overline{\mathbb{Q}}$ *such that* $b_i \neq 0$ $\forall i$ *and the* $\alpha_i$ *are mutually distinct. Then*

$$b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} \neq 0.$$

*Proof.* Consider the Taylor series expansion

$$b_1 e^{\alpha_1 x} + \cdots + b_t e^{\alpha_t x} = \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n,$$

where, clearly,

$$u_n = \sum_{i=1}^{t} b_i \alpha_i^n. \tag{1}$$

Put $(X-\alpha_1)\cdots(X-\alpha_t) = X^t - a_1 X^{t-1} - \cdots - a_t$. Clearly, for any $i = 1, \ldots, t$ and any $n \in \mathbb{Z}_{\geqslant 0}$

$$\alpha_i^{t+n} = a_1 \alpha_i^{t+n-1} + \cdots + a_t \alpha_i^n.$$

By taking suitable linear combinations and using (1) it follows that

$$u_{n+t} = a_1 u_{n+t-1} + \cdots + a_t u_n. \tag{2}$$

Without loss of generality we may assume that $u_n \in \mathbb{Q}$, $\forall n$. If not, then consider the product

$$\prod_{\sigma} \left( \sigma(b_1) e^{\sigma(\alpha_1)x} + \cdots + \sigma(b_n) e^{\sigma(\alpha_t)x} \right)$$

taken over all $\sigma \in \mathrm{Gal}(\mathbb{Q}(b_1, \ldots, b_t, \alpha_1, \ldots, \alpha_t)/\mathbb{Q})$, which, after evaluation, again acquires the form

$$\sum_i b_i' e^{\alpha_i' x},$$

where now the sets $\{b_i'\}$, $\{\alpha_i'\}$ are Galois-stable. This implies that the corresponding numbers $a_i'$ and $u_n'$ are rational. So from now on we assume $u_n \in \mathbb{Q}$, $\forall n$ and

$a_i \in \mathbb{Q}$ $(i = 1, \ldots, t)$. Let $D$ be a common denominator of the $a_i$. Put $A = \max(1, |\alpha_i|)$. After multiplication with a suitable integer, if necessary, we may assume $u_0, \ldots, u_{t-1} \in \mathbb{Z}$. Hence, using (2) recursively,

$$D^n u_n \in \mathbb{Z}, \tag{3}$$

and by (1),

$$|u_n| \leqslant c_1 A^n$$

for some $c_1 > 0$ and all $n \geqslant 0$. Now suppose that $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$, or, equivalently,

$$\sum_{r=0}^{\infty} \frac{u_r}{r!} = 0.$$

Put

$$v_n = n! \sum_{r=0}^{n} \frac{u_r}{r!}$$

and notice

$$|v_n| = n! \left| \sum_{r=0}^{n} \frac{u_r}{r!} \right| = n! \left| \sum_{r=n+1}^{\infty} \frac{u_r}{r!} \right| \leqslant \frac{c_1}{n+1} \sum_{r=n+1}^{\infty} \frac{A^r}{(r = n+1)!} = c_2 \frac{a^{n+1}}{n+1}. \tag{4}$$

If we had $A = D = 1$, like in the high school proof of $e \notin \mathbb{Q}$, inequality (4) gives us a contradiction since we have both $v_n \in \mathbb{Z}$ and $|v_n| \leqslant c_2/(n+1)$, i.e. $v_n = 0$ for all sufficiently large $n$, in other words $\Sigma_0^\infty v_n X^n$ is a polynomial. In our general case a similar principle works.

*Claim:*

$$\sum_{n=0}^{\infty} v_n X^n \in \mathbb{Q}(X).$$

Assuming the claim we proceed with the proof. Define

$$v(X) = \sum_{n=0}^{\infty} v_n X^n.$$

Notice that

$$\frac{v_n}{n!} - \frac{v_{n-1}}{(n-1)!} = \frac{u_n}{n!} \quad \text{or} \quad v_n - n v_{n-1} = u_n.$$

So,

$$\sum_{n=0}^{\infty} (v_n - n v_{n-1}) X^n = \sum_{n=0}^{\infty} u_n X^n. \tag{5}$$

Using (1), the right-hand side of (5) is seen to be

$$\sum_{n=0}^{\infty} u_n X^n = \sum_{i=1}^{t} \frac{b_i}{1 - \alpha_i X},$$

whereas the left-hand side equals

$$v(X) - X \frac{d}{dX}(X v(X)) = (1 - X) v(X) - X^2 \frac{d}{dX} v(X).$$

So (5) becomes

$$\mathscr{L}v(X) = \sum_{i=1}^{t} \frac{b_i}{1-\alpha_i X}, \qquad \mathscr{L} = -X^2\frac{d}{dX} + (1-X). \tag{6}$$

By the claim we know that $v(X) \in \mathbb{Q}(X)$ and so the non-zero poles of $\mathscr{L}v(X)$ have order at least two. However, the right-hand side of (6) has only simple poles. This contradiction proves our theorem, since the assumption $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$ has turned out to be untenable. □

It now remains to prove our claim. We first observe that, as $v$ is a solution of the differential equation (6), if $v$ is a rational function then its poles must be at the points $1/\alpha_i$. Therefore we expect that there exists an integer $k$ such that $(1 - a_1 X - \cdots - a_t X^t)^k v(X)$ is a polynomial.

DEFINITION. *For any $k$, $n \in \mathbb{Z}_{\geqslant 0}$ we define $v_n(k)$ as coefficient in the formal power series*

$$\sum_{n=0}^{\infty} v_n(k) X^n = \left(1 - a_1 X - \cdots - a_t X^t\right)^k \sum_{n=0}^{\infty} v_n X^n.$$

*For later use we also note that*

$$v_n(k+1) = v_n(k) - a_1 v_{n-1}(k) - \cdots - a_t v_{n-t}(k) \quad \text{for all } n \geqslant t,\ k \geqslant 0. \tag{7}$$

LEMMA 1. *Let $C = 1 + |a_1| + \cdots + |a_t|$. For all $n \geqslant kt$ we have*

i) $|v_n(k)| \leqslant c_2 A^n C^k$

ii) $D^n v_n(k) \in \mathbb{Z}$

iii) $k!$ *divides* $D^n v_n(k)$.

*Proof.* The first two assertions follow easily by induction on $k$ from (3), (4) and (7). The third assertion can be shown as follows. Write

$$v_n = u_n + n u_{n-1} + \cdots + n(n-1)\cdots(n-k+2) u_{n-k+1} + w_n,$$

where $w_n = n! \sum_{r=0}^{\infty} u_r / r!$. Notice that $D^{n-k} w_n \in \mathbb{Z}$ and $k!$ divides $D^{n-k} w_n$. Consider the power series

$$v(X) = \sum_{n=0}^{\infty} v_n X^n \qquad w(X) = \sum_{n=0}^{\infty} w_n X^n.$$

Then

$$v(X) - w(X) = \sum_{n=0}^{\infty} \left\{u_n + n u_{n-1} + \cdots + n(n-1)\cdots(n-k+2) u_{n-k+1}\right\} X^n.$$

For any $0 \leqslant r \leqslant k-1$ we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} n(n-1)\cdots(n-r+1) u_{n-r} X^n &= \sum_{i=1}^{t} \sum_{n=0}^{\infty} r! \binom{n}{r} b_i \alpha_i^n X^n \\
&= \sum_{i=1}^{t} r!\, b_i \alpha_i^r X^r \frac{1}{(1-\alpha_i X)^{r+1}} \\
&= \frac{P_r(X)}{\left(1 - a_1 X - \cdots - a_t X^t\right)^{r+1}},
\end{aligned}$$

where $P_r(X)$ is a polynomial of degree $< t(r+1)$. Hence

$$v(X) - w(X) = \frac{P(X)}{\left(1 - a_1 X - \cdots - tX^t\right)^k},$$

where $P(X)$ is a polynomial of degree $< tk$. Hence $v_n(k) = w_n(k)\ \forall n \geqslant kt$, where

$$\sum_{n=0}^{\infty} w_n(k) X^n = \left(1 - a_1 X - \cdots - a_t X^t\right)^k w(X).$$

Since $k!$ divides $D^{n-k} w_n$, we see that $k!$ divides $D^n w_n(k) = D^n v_n(k)$ as asserted.

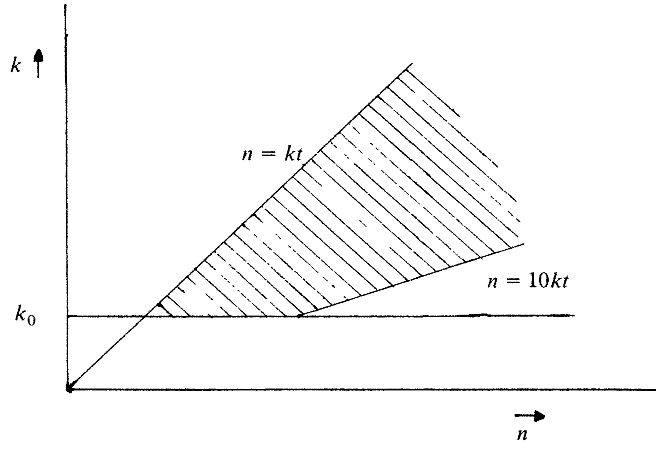
**Proof of the Claim.** It is sufficient to prove that $\sum_{n=0}^{\infty} v_n(k) X^n \in \mathbb{Q}[X]$ for some $k \in \mathbb{N}$. From the Lemma it follows that if $v_n(k) \neq 0$ and $n \geqslant kt$, then

$$k! \leqslant \left| D^n v_n(k) \right| \leqslant c_2 (AD)^n C^k.$$

Hence, if $k! > c_2(AD)^n C^k$ and $n \geqslant kt$ then $v_n(k) = 0$. Choose $k_0$ so large that $k! > c_2(AD)^{10kt} C^k\ \forall k \geqslant k_0$. Then

$$v_n(k) = 0 \quad \text{for all } k \geqslant k_0,\ kt \leqslant n \leqslant 10kt. \tag{8}$$

This situation can be pictured as follows: If a point $(n, k)$ falls in the shaded region, we have automatically $v_n(k) = 0$ according to (8). So, finally, by (7) and induction on $n - 10kt$, it follows that $v_n(k) = 0$ for all $(n, k)$ in the infinite triangular region $k_0 \leqslant k \leqslant n/10t$. Thus we conclude that $v_n(k_0) = 0\ \forall n \geqslant k_0 t$ and hence $\sum_{n=0}^{\infty} v_n(k) X^n \in \mathbb{Q}[X]$, which proves our claim.

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