Funkcialaj Ekvacioj, **51** (2008) 221–243

Algebraic Transformations of ${}_3F_2$

By

Mitsuo KATO

(University of the Ryukyus, Japan)

Abstract. Let $E(x)$ be the third order differential equation ${}_3E_2$ satisfied by a generalized hypergeometric function ${}_3F_2(a_0, a_1, a_2; b_1, b_2; x)$. Under some assumptions which guarantee that $E(x)$ is irreducible and has no logarithmic solutions, we determine all the algebraic transformations $E(x) = \theta(x)E'(\varphi(x))$, where $E'(z)$ is a Fuchsian differential equation on $\boldsymbol{P}^1$, $\varphi(x)$ a rational function, and $\theta(x)$ a finite product of complex powers of rational functions. We find $E'(z)$ also turns out to be a ${}_3E_2$.

Key Words and Phrases. Hypergeometric function, Fuchsian differential equation, Local exponent, Algebraic transformation.

2000 *Mathematics Subject Classification Numbers.* Primary 33C20.

1. Introduction

Let $E(x)$ be a Fuchsian (linear) differential equation on $\boldsymbol{P}^1$ of order three with the independent variable x. Let λ_1, λ_2, λ_3 be the local exponents of $E(x)$ at a point x_0. We call x_0 a non-logarithmic point if there are linearly independent solutions of $E(x)$ of the form $(x - x_0)^{\lambda_j} u_j(x)$, $j = 1, 2, 3$, where $u_j(x)$ are holomorphic and non-vanishing at x_0. We call x_0 a regular point of $E(x)$ if it is non-logarithmic, and if $\{\lambda_1, \lambda_2, \lambda_3\} = \{\alpha, \alpha + 1, \alpha + 2\}$ for some $\alpha \in \boldsymbol{C}$. A non-regular point is called a singular point.

Let a_0, a_1, a_2, b_1, b_2 be complex numbers satisfying

(1.1) 1 (no-log condition) $a_i - a_j$ $(i \neq j)$, b_i, $b_1 - b_2$, $\sum a_i - \sum b_i \notin \boldsymbol{Z}$,

2 (irreducibility condition) a_i, $a_i - b_j \notin \boldsymbol{Z}$.

Under these assumptions, the generalized hypergeometric function

$$ {}_3F_2(a_0, a_1, a_2; b_1, b_2; x) = \sum_{m=0}^{\infty} \frac{(a_0, m)(a_1, m)(a_2, m)}{(b_1, m)(b_2, m)} \frac{x^m}{(1, m)}, $$

where $(\alpha, m) = \Gamma(\alpha + m)/\Gamma(\alpha)$, satisfies a Fuchsian differential equation on $\boldsymbol{P}^1$ defined by

$$ [\vartheta(\vartheta + b_1 - 1)(\vartheta + b_2 - 1) - x(\vartheta + a_0)(\vartheta + a_1)(\vartheta + a_2)]u = 0, \tag{1.2} $$

where $\vartheta = x\,d/dx$. We denote the equation (1.2) by ${}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$. This is a generalization of Gauss' hypergeometric differential equation ${}_2E_1(a, b; c; x)$ defined by

$$[\vartheta(\vartheta + c - 1) - x(\vartheta + a)(\vartheta + b)]u = 0,$$

which has the local exponents a, b at $x = \infty$; 0, $1 - c$ at $x = 0$; 0, $c - a - b$ at $x = 1$; and 0, 1 at any other regular point. When we want to emphasize the differences of local exponents $\lambda_\infty = b - a$, $\lambda_0 = 1 - c$, $\lambda_1 = c - a - b$ at the three singular points, we denote ${}_2E_1(a, b; c; x)$ by $HD(\lambda_\infty, \lambda_0, \lambda_1; x)$, that is,

$$HD(\lambda_\infty, \lambda_0, \lambda_1; x) = {}_2E_1(a, a + \lambda_\infty; 1 - \lambda_0; x), \qquad a = (1 - \lambda_\infty - \lambda_0 - \lambda_1)/2.$$

The differential equation ${}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ has also only three singular points ∞, 0, 1. It has the local exponents

$$\begin{array}{lll} (1.3) \qquad & a_0,\ a_1,\ a_2 & \text{at } x = \infty, \\ & 0,\ 1 - b_1,\ 1 - b_2 & \text{at } x = 0, \\ & 0,\ 1,\ \sum b_j - \sum a_j & \text{at } x = 1, \\ & 0,\ 1,\ 2 & \text{elsewhere.} \end{array}$$

The no-log condition 1 in (1.1) means that there is only one singular point (that is $x = 1$), where local exponents differ by an integer ($= 1$). This condition is a sufficient one in order that (1.2) has no logarithmic solutions, that is, all the points are non-logarithmic. Under the no-log condition 1, (1.2) has an irreducible monodromy group if and only if the irreducibility condition 2 in (1.1) holds.

In general, a third order Fuchsian differential equation $E(x)$ on $\boldsymbol{P}^1$ which has three singular points ∞, 0, 1, and has the local exponents (1.3) is not uniquely determined, but has one accessory parameter. If, moreover, $E(x)$ has no logarithmic solution at $x = 1$, then the value of the accessory parameter is fixed so that $E(x)$ is given by (1.2).

Let $\varphi(x)$ and $\theta_j(x)$ be rational functions, and $\theta(x) = \prod_{j=1}^{m} (\theta_j(x))^{\alpha_j}$ for some $\alpha_j \in \boldsymbol{C}$ which we call a finite product of complex powers of rational functions. For Fuchsian differential equations $E(x)$ and $E'(z)$ on $\boldsymbol{P}^1$ of order three, the equation

$$(1.4) \qquad E(x) = \theta(x)E'(\varphi(x))$$

means that if $f(z)$ is a solution of $E'(z)$, then $\theta(x)f(\varphi(x))$ is a solution of $E(x)$. If (1.4) holds, we say that $E(x)$ is a pull-back of $E'(z)$ by an algebraic transformation $z = \varphi(x)$, or that $E(x)$ reduces to $E'(z)$ by the algebraic transformation.

In this paper, except in Appendix, $E(x)$ always denotes (1.2) satisfying two conditions in (1.1), and we determine all the algebraic transformations (1.4) (see Theorem 2.2). Because the choice of $\theta(x)$ in (1.4) has an ambiguity (see Section 2), the choice of $E'(z)$ is not unique. Theorem 2.2 asserts that $E'(z)$ can be chosen to be a ${}_3E_2(a_0', a_1', a_2'; b_1', b_2'; z)$ for some parameters a_j', b_j' if we choose a suitable $\theta(x)$.

2. Algebraic transformations

First, we give five examples of algebraic transformations of ${}_3E_2$, where we assume that all ${}_3E_2$ on the left hand sides satisfy the two conditions in (1.1). The equalities are verified by showing that the differential equations on the right hand sides have no logarithmic solutions at $x = 1$, and have the same local exponents as ${}_3E_2$ on the left hand sides at every point.

Quadratic transformation.

$$(2.1)\qquad {}_3E_2(2\alpha, 2\alpha+\lambda, 2\alpha+\mu; 1-\lambda, 1-\mu; x)$$
$$= (1-x)^{-2\alpha}{}_3E_2(\alpha, \alpha+1/2, 1-2\alpha-\lambda-\mu; 1-\lambda, 1-\mu; \varphi_2(x)),$$

where

$$\varphi_2(x) = -\frac{4x}{(1-x)^2} = 1 - \frac{(1+x)^2}{(1-x)^2}.$$

Cubic transformation.

$$(2.2)\qquad {}_3E_2(3\alpha, 3\alpha+2\lambda, 3\alpha+2\mu; 1-\lambda, 1-\mu; x)$$
$$= (1-4x)^{-3\alpha}{}_3E_2(\alpha, \alpha+1/3, \alpha+2/3; 1-\lambda, 1-\mu; \varphi_3(x)),$$

where

$$\varphi_3(x) = -\frac{27x}{(1-4x)^3} = 1 - \frac{(1-x)(1+8x)^2}{(1-4x)^3}, \qquad 3\alpha+\lambda+\mu = 1/2.$$

Quartic transformation.

$$(2.3)\qquad {}_3E_2(4\alpha, 4\alpha+1/3, 4\alpha+\lambda; 2/3, 1/3; x)$$
$$= (1-x)^{-3\alpha}{}_3E_2(\alpha, \alpha+1/3, \alpha+\lambda; 2/3, 1/3; \varphi_4(x)),$$

where

$$\varphi_4(x) = \frac{x(x+8)^3}{64(x-1)^3} = 1 + \frac{(x^2-20x-8)^2}{64(x-1)^3}, \qquad \lambda = 1/6 - 3\alpha.$$

Composite transformations.

(2.4) $\quad {}_3E_2(4\alpha, 4\alpha + 1/2, 4\alpha + \lambda; 1/2, 1 - \lambda; x)$

$$= (1+x)^{-4\alpha} {}_3E_2(\alpha, \alpha + 1/2, \alpha + 1/4; 1/2, 1 - \lambda; (\varphi_2 \circ \varphi_2)(x)),$$

where

$$(\varphi_2 \circ \varphi_2)(x) = \frac{16x(1-x)^2}{(1+x)^4} = 1 - \frac{(x^2 - 6x + 1)^2}{(1+x)^4}, \qquad \lambda = 1/4 - 3\alpha,$$

(2.5) $\quad {}_3E_2(6\alpha, 6\alpha + 1/4, 6\alpha + \lambda; 3/4, 1 - \lambda; x)$

$$= (1 + 14x + x^2)^{-3\alpha} {}_3E_2(\alpha, \alpha + 1/3, \alpha + 2/3; 3/4, 1 - \lambda; (\varphi_3 \circ \varphi_2)(x)),$$

where

$$(\varphi_3 \circ \varphi_2)(x) = \frac{108x(1-x)^4}{(1 + 14x + x^2)^3} = 1 - \frac{(x+1)^2(x^2 - 34x + 1)^2}{(1 + 14x + x^2)^3},$$

$$\lambda = 1/4 - 3\alpha.$$

The following lemma asserts that there are algebraic transformations of ${}_3E_2$ derived from those of Gauss' hypergeometric differential equations ${}_2E_1$.

Lemma 2.1. *Let*

(2.6) $$\quad {}_2E_1(a, b; a + b + 1/2; x) = \theta(x) {}_2E_1(a', b'; a' + b' + 1/2; \varphi(x))$$

be an algebraic transformation of ${}_2E_1$*, where* $\theta(x)$*,* $\varphi(x)$ *are the same type of functions as in* (1.4)*. Then we have*

(2.7) $\quad {}_3E_2(2a, a + b, 2b; a + b + 1/2, 2a + 2b; x)$

$$= \theta(x)^2 {}_3E_2(2a', a' + b', 2b'; a' + b' + 1/2, 2a' + 2b'; \varphi(x)).$$

Proof. The lemma follows from Clausen's formula (see [Bly, p. 86])

$${}_3E_2(2a, a + b, 2b; a + b + 1/2, 2a + 2b; x) = ({}_2E_1(a, b; a + b + 1/2; x))^2. \qquad \square$$

The transformation (2.7) induced from (2.6) is called a Clausen type transformation.

Remark. If

$$HD(\lambda_\infty, \lambda_0, 1/2; x) = \theta(x) HD(\lambda'_\infty, \lambda'_0, 1/2; \varphi(x)),$$

then we have

$$
\begin{aligned}
&{}_3E_2(1/2-\lambda_0-\lambda_\infty, 1/2-\lambda_0, 1/2-\lambda_0+\lambda_\infty; 1-\lambda_0, 1-2\lambda_0; x) \\
&\quad = \theta(x)^2 {}_3E_2(1/2-\lambda_0'-\lambda_\infty', 1/2-\lambda_0', 1/2-\lambda_0'+\lambda_\infty'; 1-\lambda_0', 1-2\lambda_0'; \varphi(x)).
\end{aligned}
$$

As is well known, $E(x) = {}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ has six algebraic transformations (including the identity transformation) with $\varphi(x) = x$ or $\varphi(x) = 1/x$ such as

$$E(x) = x^{1-b_1} {}_3E_2(a_0+1-b_1, a_2+1-b_1, a_2+1-b_1; 2-b_1, b_2+1-b_1; x),$$

or

$$E(x) = x^{-a_0} {}_3E_2(a_0, a_0+1-b_1, a_0+1-b_2; a_0+1-a_1, a_0+1-a_2; 1/x).$$

Let $E(x) = x^s \tilde{E}(L(x))$ be one of these six transformations. Let (1.4) be another algebraic transformation of $E(x)$. If $E'(z)$ in (1.4) is equal to $\theta'(z)E''(L'(z))$, where $L'(z)$ is a fractional linear transformation of z, and $\theta'(z)$ is a finite product of complex powers of rational functions of z, then (1.4) changes to

$$\tilde{E}(x) = \tilde{\theta}(x) E''(\tilde{\varphi}(x)),$$

where $\tilde{\theta}(x) = L(x)^{-s}\theta(L(x))\theta'(\varphi(L(x)))$, and $\tilde{\varphi}(x) = L'(\varphi(L(x)))$. This new algebraic transformation is said to be projectively equivalent to the original transformation (1.4). Now we give the main theorem.

Theorem 2.2. *Assume $E(x) = {}_3E_2(a_0, a_1, a_2; b_1, b_2; x)$ satisfies the two conditions in* (1.1). *Then any algebraic transformation* (1.4) *is projectively equivalent to one of* (2.1)–(2.5), *or to a Clausen type transformation induced from one of the following transformations of Gauss' hypergeometric differential equations $HD(\lambda_\infty, \lambda_0, \lambda_1; x)$:*

(1) *Quadratic transformations*

$$HD(1/s, 1/s, 1/2; x) = \theta(x) HD(1/4, 1/s, 1/2; \varphi_2(x)), \tag{2.8}$$

where φ_2 is the function at (2.1), *and $\theta(x) = (1-x)^{1/s-1/4}$.*

(2) *Cubic transformations*

$$HD(2/s, 1/s, 1/2; x) = \theta(x) HD(1/3, 1/s, 1/2; \varphi_3(x)), \tag{2.9}$$

where φ_3 is the function at (2.2), *and $\theta(x) = (1-4x)^{3/(2s)-1/4}$.*

(3) *A composite sextic transformation*

$$HD(1/8, 1/8, 1/2; x) = \theta(x) HD(1/3, 1/8, 1/2; (\varphi_3 \circ \varphi_2)(x)), \tag{2.10}$$

where $(\varphi_3 \circ \varphi_2)(x)$ was given in (2.5), *and $\theta(x) = (1+14x+x^2)^{-1/16}$.*

(4) *A ninth-degree transformation*

(2.11) $$HD(1/7,1/7,1/2;x)=\theta(x)HD(1/7,1/3,1/2;\varphi_9(x)),$$

where

$$\varphi_9(x)=\frac{(-\alpha^2x^3-\beta x^2+\bar{\beta}x+\bar{\alpha}^2)^3}{216x(\alpha x-\bar{\alpha})^7}$$

$$=1-\frac{(x-1)(-\alpha^3x^4+\gamma x^3-420\sqrt{-7}x^2-\bar{\gamma}x+\bar{\alpha}^3)^2}{216x(\alpha x-\bar{\alpha})^7},$$

$$\theta(x)=x^{-1/84}(x-\bar{\alpha}/\alpha)^{-1/12},$$

$$\alpha=(13+7\sqrt{-7})/16,\qquad \beta=(319-35\sqrt{-7})/4,\qquad \gamma=(3139+1001\sqrt{-7})/16.$$

(5) *Tetrahedral transformations*

(2.12) $$HD(m_\infty/3,m_0/3,1/2;x)=\theta(x)HD(1/3,1/3,1/2;\varphi(x)),$$

for some $\theta(x)$ *and some n-th degree rational function* $\varphi(x)$, *where* $m_\infty, m_0\not\equiv 0$ *mod* 3, *and* $n=2(m_\infty+m_0)-3$. *If* $m_\infty+m_0\equiv 0$ *mod* 3, *then* $\varphi(\infty)=\varphi(0)=\infty$, $\varphi(1)=1$, *otherwise* $\varphi(\xi)=\xi$ *for* $\xi=\infty,0,1$.

(6) *Octahedral transformations*

(2.13) $$HD(m_\infty/4,m_0/3,1/2;x)=\theta(x)HD(1/4,1/3,1/2;\varphi(x)),$$

for some $\theta(x)$ *and some n-th degree rational function* $\varphi(x)$, *where* $m_\infty\not\equiv 0$ *mod* 2, $m_0\not\equiv 0$ *mod* 3, $n=3m_\infty+4m_0-6$, *and* $\varphi(\xi)=\xi$ *for* $\xi=\infty,0,1$.

(7) *Icosahedral transformations I*

(2.14) $$HD(m_\infty/5,m_0/3,1/2;x)=\theta(x)HD(1/5,1/3,1/2;\varphi(x)),$$

for some $\theta(x)$ *and some n-th degree rational function* $\varphi(x)$, *where* $m_\infty\not\equiv 0$ *mod* 5, $m_0\not\equiv 0$ *mod* 3, $n=6m_\infty+10m_0-15$, *and* $\varphi(\xi)=\xi$ *for* $\xi=\infty,0,1$.

(8) *Icosahedral transformations II*

(2.15) $$HD(m_\infty/5,m_0/5,1/2;x)=\theta(x)HD(1/5,1/3,1/2;\varphi(x)),$$

for some $\theta(x)$ *and some n-th degree rational function* $\varphi(x)$, *where* $\{m_\infty,m_0\}$ *is equivalent to one of* $\{1,2\}$, $\{1,3\}$, $\{4,2\}$, $\{4,3\}$, *modulo* 5, $n=6(m_\infty+m_0)-15$, *and* $\varphi(\infty)=\varphi(0)=\infty$, $\varphi(1)=1$.

(9) *Composite transformations of* (2.12) *and a quadratic transformation*

$$HD(1/3,1/3,1/2;z)=(1-z)^{1/12}HD(1/4,1/3,1/2;\varphi_2(z)),$$

where φ_2 *is the function at* (2.1).

(10) *Composite transformations of* (2.12) *and a fifth-degree transformation*

(2.16) $$HD(1/3,1/3,1/2;z)=\theta(z)HD(1/5,1/3,1/2;\varphi_5(z)),$$

where

$$\varphi_5(z) = 50\sqrt{-15}\frac{z(\alpha^3 z + \bar{\alpha}^3)^3}{(\alpha^2 z - \bar{\alpha}^2)^5} = 1 - \frac{(z-1)(\alpha^5 z^2 + 72|\alpha|^4 z + \bar{\alpha}^5)^2}{(\alpha^2 z - \bar{\alpha}^2)^5},$$

$$\theta(z) = (z - \bar{\alpha}^2/\alpha^2)^{1/12}, \qquad \alpha = 9 + \sqrt{-15}.$$

Remark. The Clausen type transformations induced from (2.8), (2.9) and (2.10) are special cases of the transformations (2.1), (2.2) and (2.5) respectively. It is easily verified that the transformations (2.3) or (2.4) cannot be of Clausen type.

The functions $\varphi_5(z)$ and $\varphi_9(x)$ in this theorem appear in [Vdn2] and [Vdn1]. That is, $\varphi_5(z)$ is equal to the function

$$50(5 + 3\sqrt{-15})z(1024z - 781 - 171\sqrt{-15})^3/(128z + 7 + 33\sqrt{-15})^5$$

which is given in [Vdn2, (50)], and $1/\varphi_9(x/(x-1))$ is equal to the function

$$\frac{27x(x-1)(49x - 31 - 13\xi)^7}{49(7203x^3 + (9947\xi - 5831)x^2 - (9947\xi + 2009)x + 275 - 87\xi)^3}$$

with $\xi = (-1 + \sqrt{-7})/2$, which is given in [Vdn1, (24)].

If $\varphi(x) = \varphi_5(x)$ or $\varphi_9(x)$, and if $c \in \boldsymbol{R} \cup \{\infty\}$, then $\varphi^{-1}(\{c\})$ is invariant under the reflection $x \mapsto 1/\bar{x}$ with respect to the unit circle $\{x \in \boldsymbol{C} \mid |x| = 1\}$.

The aim of this paper is to prove Theorem 2.2.

3. Preliminary facts

In this section, we give some notation and preliminary facts concerning algebraic transformations (1.4).

A singular point ζ of $E'(z)$ is called a regularizable singular point with the ramification index r if $E'(z)$ has, at ζ, the local exponents of the following form:

$$\alpha,\ \alpha + 1/r,\ \alpha + 2/r, \qquad \alpha \in \boldsymbol{C},\ r\ (\geq 2) \in \boldsymbol{Z}.$$

From now on, we use the following notation:

n	the degree of the rational function φ,
$r_\varphi(\xi)$	the ramification index of φ at $x = \xi$,
z_j	the singular points of $E'(z)$,
n_j	the sum of $r_\varphi(\xi)$ for $\xi \in \varphi^{-1}(\{z_j\}) \cap \{0, 1, \infty\}$,
r_j	the ramification index at z_j if z_j is regularizable, 2 otherwise,
l	the number of non-regularizable singular points of $E'(z)$.

Let $\theta(x) = c\prod_{\xi\in \boldsymbol{C}}(x-\xi)^{r(\xi)}$, where $r(\xi)\in \boldsymbol{C}$ is zero except finite number of ξ in $\boldsymbol{C}$. Put $r(\infty) = -\sum_{\xi\in\boldsymbol{C}} r(\xi)$. We call $r(\xi)$ the order of $\theta(x)$ at ξ, and denote by $\operatorname{ord}_\xi \theta(x)$.

Lemma 3.1. *Let $\varphi(\xi) = \zeta$, and $r_\varphi(\xi) = m$. If the local exponents of $E'(z)$ at $z = \zeta$ are α', $\alpha' + \lambda_1$, $\alpha' + \lambda_2$ for some α', then those of $E(x)$ at $x = \xi$ are α, $\alpha + m\lambda_1$, $\alpha + m\lambda_2$, where $\alpha = m\alpha' + \operatorname{ord}_\xi \theta(x)$.*

Lemma 3.2. *The function $z = \varphi(x)$ does not ramify over a regular point.*

Proof. Let ζ be a regular point of $E'(z)$, and $\xi\in\varphi^{-1}(\{\zeta\})$. Let $m = r_\varphi(\xi)$. Then, from the previous lemma, $E(x)$ has the local exponents α, $\alpha + m$, $\alpha + 2m$ for some α, at $x = \xi$. If $m > 1$, then ξ is a singular point, and $E(x)$ does not satisfy the no-log condition in (1.1). □

Lemma 3.3. (1) *If z_j is not regularizable, then there is no regular point of $E(x)$ in*

(2) *If z_j is regularizable, and if $\xi\in\varphi^{-1}(\{z_j\})$ is a regular point of $E(x)$, then we have $r_\varphi(\xi) = r_j$.*

(3) *If z_j is regularizable with $r_j = 2$, and if $\xi\in\varphi^{-1}(\{z_j\})$ is a singular point of $E(x)$, then $\xi = 1$, and $r_\varphi(\xi) = 1$.*

(4) *If z_j is regularizable, and if $1\in\varphi^{-1}(\{z_j\})$, then $r_j = 2r_\varphi(1)$.*

Proof. If $\xi\in\varphi^{-1}(\{z_j\})$ is a regular point of $E(x)$, and $r_\varphi(\xi) = m$, then, from Lemma 3.1, $E'(z)$ has, at z_j, the local exponents α', $\alpha' + 1/m$, $\alpha' + 2/m$ for some α'. This proves (1) and (2).

If z_j is regularizable, and $\xi\in\varphi^{-1}(\{z_j\})$ is a singular point of $E(x)$ with $r_\varphi(\xi) = m$, then the local exponents of $E(x)$ at ξ are α, $\alpha + m/r_j$, $\alpha + 2m/r_j$ for some α. If $r_j = 2$ and $\xi \neq 1$, then $E(x)$ does not satisfy no-log condition. If $\xi = 1$, we have $2m/r_j = 1$. This proves (3) and (4). □

Corollary 3.4. (1) *If z_j is not regularizable, then $n_j = n$.*

(2) *The number of regular points of $E(x)$ in $\varphi^{-1}(\{z_j\})$ is $(n - n_j)/r_j$.*

Proof. Since

$$
\begin{aligned}
n &= \sum\{r_\varphi(\xi) \mid \xi\in\varphi^{-1}(\{z_j\})\}\\
&= \sum\{r_\varphi(\xi) \mid \text{singular } \xi\in\varphi^{-1}(\{z_j\})\} + \sum\{r_\varphi(\xi) \mid \text{regular } \xi\in\varphi^{-1}(\{z_j\})\}\\
&= n_j + \sum\{r_\varphi(\xi) \mid \text{regular } \xi\in\varphi^{-1}(\{z_j\})\},
\end{aligned}
$$

(1) (resp. (2)) of the previous lemma implies (1) (resp. (2)) of this corollary. □

Proposition 3.5. *The number of the singular points of $E'(z)$ is three, and we have*

$$n = 1 + \sum_{j=1}^{3} (n - n_j)/r_j. \tag{3.1}$$

Proof. Let z_j, $1 \le j \le k$ be all the singular points of $E'(z)$. From Riemann-Hurwitz formula applied to $\varphi : \boldsymbol{P}^1 \to \boldsymbol{P}^1$, we have

$$\chi(\boldsymbol{P}^1) = n\chi(\boldsymbol{P}^1) - \sum_{j=1}^{k} (n - \#\varphi^{-1}(\{z_j\})) = n\chi(\boldsymbol{P}^1) - nk + \sum_{j=1}^{k} \#\varphi^{-1}(\{z_j\}),$$

where χ denotes the Euler number. Since $\sum_{j=1}^{k} \#\varphi^{-1}(\{z_j\})$ is the sum of the number of singular points ($= 3$) and that of regular points ($= \sum_{j=1}^{k} (n - n_j)/r_j$), we have $2 = 2n - kn + 3 + \sum_{j=1}^{k} (n - n_j)/r_j$, that is

$$n(k-2) = 1 + \sum\nolimits_{j=1}^{k} (n - n_j)/r_j.$$

Since we have

$$\sum\nolimits_{j=1}^{k} (n - n_j)/r_j \le \sum\nolimits_{j=1}^{k} (n - n_j)/2 = nk/2 - \sum\nolimits_{j=1}^{k} n_j/2 \le nk/2 - 3/2,$$

we have $k \le 3$. Since $E(x)$ is irreducible, $k = 3$. Consequently we have (3.1). $\square$

Lemma 3.6. *If $E(x)$ has two algebraic transformations $E(x) = \theta(x)E'(\varphi(x))$ and $E(x) = \tilde{\theta}(x){}_3E_2(\tilde{a}_0, \tilde{a}_1, \tilde{a}_2; \tilde{b}_1, \tilde{b}_2; \varphi(x))$ with a common rational function $\varphi(x)$, and if the parameters $\tilde{a}_j$, $\tilde{b}_j$ satisfy the two conditions in* (1.1)*, then these two transformations are projectively equivalent to each other.*

Proof. Let $\varphi(\xi) = \zeta$ with $m = r_\varphi(\xi)$. Let the local exponents of $E'(z)$ at ζ be $\lambda'_j(\zeta)$, $j = 1, 2, 3$. Then $\lambda_j(\xi)$ defined by

$$\lambda'_j(\zeta) = (\lambda_j(\xi) - \mathrm{ord}_\xi\, \theta(x))/m, \qquad j = 1, 2, 3$$

are the local exponents of $E(x)$ at $x = \xi$. The local exponents of $\tilde{E}(z)$ at $z = \zeta$ are given by

$$\tilde{\lambda}_j(\zeta) = (\lambda_j(\xi) - \mathrm{ord}_\xi\, \tilde{\theta}(x))/m, \qquad j = 1, 2, 3.$$

The quantities

$$\tilde{\lambda}_j(\zeta) - \lambda'_j(\zeta) = (\mathrm{ord}_\xi\, \theta(x) - \mathrm{ord}_\xi\, \tilde{\theta}(x))/m, \qquad j = 1, 2, 3$$

do not depend on j or ξ $(\in \varphi^{-1}(\zeta))$. Put

$$\theta'(z) = \prod\nolimits_{\zeta \in \boldsymbol{C}} (z-\zeta)^{\tilde{\lambda}_j(\zeta)-\lambda'_j(\zeta)}.$$

We note that $\mathrm{ord}_\infty\, \theta'(z) = \tilde{\lambda}_j(\infty) - \lambda'_j(\infty)$, because $\sum_{\zeta \in \boldsymbol{P}^1} \tilde{\lambda}_j(\zeta) - \lambda'_j(\zeta) = 0$. Since

$$\mathrm{ord}_\xi\, \theta'(\varphi(x)) = m\, \mathrm{ord}_\zeta\, \theta'(z) = m(\tilde{\lambda}_j(\zeta) - \lambda'_j(\zeta)) = \mathrm{ord}_\xi\, \theta(x) - \mathrm{ord}_\xi\, \tilde{\theta}(x),$$

for any $\xi \in \boldsymbol{P}^1$, we have $\theta(x) = c\tilde{\theta}(x)\theta'(\varphi(x))$ for some constant c. Since $\theta'(z)E'(z)$ has no logarithmic solutions, and has the same local exponents as $\tilde{E}(z)$ at any point ζ in $\boldsymbol{P}^1$, we have $\theta'(z)E'(z) = \tilde{E}(z)$. This completes the proof. $\square$

Let $\#\varphi^{-1}(\{z_j\}) = k_j$, and $\varphi^{-1}(\{z_j\}) = \{x_{j,k} \mid 1 \le k \le k_j\}$. We call the formal sum (or a partition of n) $\sum_{k=1}^{k_j} r_\varphi(x_{j,k})$ the branching pattern of φ over z_j. After R. Vidūnas [Vdn2], we call

$$BP(\varphi) = \left(\sum_{k=1}^{k_1} r_\varphi(x_{1,k}), \sum_{k=1}^{k_2} r_\varphi(x_{2,k}), \sum_{k=1}^{k_3} r_\varphi(x_{3,k})\right)$$

the branching pattern of φ.

Lemma 3.7. *Assume* $(z_1, z_2, z_3) = (0, \infty, 1)$. *In the following five cases, the values of* $BP(\varphi)$, $(\varphi(0), \varphi(\infty), \varphi(1))$ *and* $(r_\varphi(0), r_\varphi(\infty), r_\varphi(1))$ *determine the function* $\varphi(x)$, *where* $\varphi_2(x)$, $\varphi_3(x)$, $\varphi_4(x)$ *are functions in* (2.1), (2.2), (2.3), *respectively.*

	$BP(\varphi)$	$(\varphi(0), \varphi(\infty), \varphi(1))$	$(r_\varphi(0), r_\varphi(\infty), r_\varphi(1))$	φ
(1)	$(1+1, 2, 2)$	$(0, 0, \infty)$	$(1, 1, 2)$	φ_2
(2)	$(1+2, 3, 1+2)$	$(0, 0, 1)$	$(1, 2, 1)$	φ_3
(3)	$(1+3, 1+3, 2+2)$	$(0, \infty, \infty)$	$(1, 1, 3)$	φ_4
(4)	$(1+1+2, 4, 2+2)$	$(0, 0, 0)$	$(1, 1, 2)$	$\varphi_2 \circ \varphi_2$
(5)	$(1+1+4, 3+3, 2+2+2)$	$(0, 0, 0)$	$(1, 1, 4)$	$\varphi_3 \circ \varphi_2$

Proof. (1) We have

$$(1-x)^{1/2} {}_2E_1(-1/4, 1/4; 1/2; \varphi(x)) = {}_2E_1(0, -1/2; 1/2; x).$$

It follows that ${}_2F_1(-1/4, 1/4; 1/2; \varphi(x)) = (1-x)^{-1/2}$, which determines $\varphi(x)$ uniquely from the inverse function theorem. This proves $\varphi(x) = \varphi_2(x)$ in (2.1).

(2) Put $\psi(x) = 1 - 1/\varphi(1/(1-x))$. Then $(\psi(0), \psi(\infty), \psi(1)) = (0, \infty, \infty)$ with $(r_\psi(0), r_\psi(\infty), r_\psi(1)) = (1, 1, 2)$. Hence we have

$$(1-x)^{1/3} {}_2E_1(-1/6, 1/3; 1/2; \psi(x)) = {}_2E_1(0, -1/2; 1/2; x),$$

which implies ${}_2E_1(-1/6, 1/3; 1/2; \psi(x)) = (1-x)^{-1/3}$. This equality determines $\psi(x)$, hence $\varphi(x)$ uniquely. This proves $\varphi(x) = \varphi_3(x)$ in (2.2).

(3) We have

$$(1-x)^{1/4}{}_2E_1(-1/12, 1/4; 2/3; \varphi(x)) = {}_2E_1(0, -1/3; 2/3; x),$$

which proves (3) as the previous cases.

(4) In this case we have

$$x^{1/8}(1-x)^{1/4}{}_2E_1(-1/8, 3/8; 3/4; 1/\varphi(x)) = {}_2E_1(0, -1/2; 1/2; x),$$

which implies

$$x^{1/8}(1-x)^{1/4}\varphi(x)^{-1/8}{}_2F_1(-1/8, 1/8; 1/2; \varphi(x)) = c_1,$$

$$x^{1/8}(1-x)^{1/4}\varphi(x)^{3/8}{}_2F_1(3/8, 5/8; 3/2; \varphi(x)) = c_2 \cdot x^{1/2},$$

for some constant numbers c_1, c_2. From these equations, we have

$${}_2F_1(-1/8, 1/8; 1/2; \varphi(x))^3{}_2F_1(3/8, 5/8; 3/2; \varphi(x)) = c_1^3c_2(1-x)^{-1}.$$

Since $\varphi(x) = 0$, we have $c_1^3c_2 = 1$. Consequently $\varphi(x)$ is determined uniquely from the inverse function theorem. Hence we have $\varphi(x) = (\varphi_2 \circ \varphi_2)(x)$ in (2.4).

(5) We have

$$x^{1/24}(1-x)^{1/6}{}_2E_1(-1/24, 5/24; 2/3; 1/\varphi(x)) = {}_2E_1(0, -1/4; 3/4; x),$$

which proves (5) as the previous case (4). □

Lemma 3.8. (1) *If $BP(\varphi) = (1+1, 2, 2)$, then* (1.4) *is projectively equivalent to* (2.1).

(2) *If $BP(\varphi) = (1+2, 3, 1+2)$, then* (1.4) *is projectively equivalent to* (2.2).

(3) *If $BP(\varphi) = (1+3, 1+3, 2+2)$, then* (1.4) *is projectively equivalent to* (2.3).

(4) *If $BP(\varphi) = (1+1+2, 4, 2+2)$, then* (1.4) *is projectively equivalent to* (2.4).

(5) *If $BP(\varphi) = (1+1+4, 3+3, 2+2+2)$, then* (1.4) *is projectively equivalent to* (2.5).

Proof. Before the proof we give two easy facts. First, if $\varphi(x_0) = z_j$ with $r_\varphi(x_0) = 1$, then x_0 is a singular point of $E(x)$. Second, if $\varphi(x_1) = \varphi(x_2)$ with $r_\varphi(x_1) = r_\varphi(x_2)$, then x_1 and x_2 are both regular or both singular at one time.

(1) Since the two points in $\varphi^{-1}(\{z_1\})$ are singular, we may assume the point in $\varphi^{-1}(\{z_3\})$ is regular. Since $E(x)$ satisfies the no-log condition in (1.1), we have $\varphi^{-1}(\{z_1\}) = \{\infty, 0\}$ and $\varphi^{-1}(\{z_2\}) = \{1\}$. Let the local exponents of $E(x)$ at $x = 0$ be 0, λ, μ. Then those at $x = \infty$ are a, $a+\lambda$, $a+\mu$ for some a, and we have $E(x) = {}_3E_2(a, a+\lambda, a+\mu; 1-\lambda, 1-\mu; x)$. By a fractional linear

transformation of z, we may assume $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$. Then, from Lemma 3.7, $\varphi(x)$ is equal to $\varphi_2(x)$ in (2.1), and (1.4) is projectively equivalent to (2.1) from Lemma 3.6.

(2) Let $\varphi^{-1}(\{z_1\}) = \{x_{1,1}, x_{1,2}\}$, $\varphi^{-1}(\{z_2\}) = \{x_2\}$, and $\varphi^{-1}(\{z_3\}) = \{x_{3,1}, x_{3,2}\}$, with $r_\varphi(x_{1,1}) = r_\varphi(x_{3,1}) = 1$, $r_\varphi(x_{1,2}) = r_\varphi(x_{3,2}) = 2$, and $r_\varphi(x_2) = 3$. If $x_{1,2}$ and $x_{3,2}$ are regular points, then $E(x)$ has the local exponents α_j, $\alpha_j + 1/2$, $\alpha_j + 1$ for some α_j at $x_{j,1}$. Hence $E(x)$ does not satisfy the no-log condition. Consequently, by the symmetry of z_1 and z_3, we may assume $x_{1,2}$ is singular, and $x_{3,2}$, x_2 are regular. By the above consideration, $E(x)$ has the local exponents α, $\alpha + 1/2$, $\alpha + 1$ for some α at $x_{3,1}$. This means that $x_{3,1} = 1$, and hence $\{x_{1,1}, x_{1,2}\} = \{0, \infty\}$. By fractional linear transformations of x and z, we may assume $x_{1,1} = 0$, $x_{1,2} = \infty$, $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$. Let the local exponents of $E(x)$ at $x = 0$ be 0, λ, μ. Then those at $x = \infty$ are a, $a + 2\lambda$, $a + 2\mu$ for some a, and hence we have $E(x) = {}_3E_2(a, a + 2\lambda, a + 2\mu; 1 - \lambda, 1 - \mu; x)$. Since the local exponents of $E(x)$ at $x = 1$ are 0, $1/2$, 1, we have $a + \lambda + \mu = 1/2$. Then, by the same way as the previous case, we find that (1.4) is projectively equivalent to (2.2).

(3) Let $\varphi^{-1}(\{z_j\}) = \{x_{j,1}, x_{j,2}\}$, $j = 1, 2, 3$, and $r_\varphi(x_{1,1}) = r_\varphi(x_{2,1}) = 1$, $r_\varphi(x_{1,2}) = r_\varphi(x_{2,2}) = 3$, $r_\varphi(x_{3,1}) = r_\varphi(x_{3,2}) = 2$. Since $x_{1,1}$ and $x_{2,1}$ are singular points, $x_{3,1}$ and $x_{3,2}$ must be regular points. Consequently we may assume that $x_{1,2}$ is singular, and $x_{2,2}$ is regular.

Since $x_{2,2}$ is regular, the local exponents at $x_{2,1}$ are α, $\alpha + 1/3$, $\alpha + 2/3$ for some α. This means that $x_{2,1} \neq 1$. If $x_{1,1} = 1$, then the local exponents at $x_{1,2}$ are β, $\beta + 3$, $\beta + \gamma$ for some β and γ. Then $E(x)$ cannot satisfy the no-log condition. Hence we have $x_{1,2} = 1$, and the local exponents at $x_{1,1}$ are β, $\beta + 1/3$, $\beta + \gamma$ for some β and γ. By fractional linear transformations of x and z, we may assume $x_{1,1} = \infty$, $x_{2,1} = 0$, $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$. Then (1.4) is projectively equivalent to (2.3), by the same way as as the previous cases.

(4) Let $\varphi^{-1}(\{z_1\}) = \{x_{1,1}, x_{1,2}, x_{1,3}\}$, with $r_\varphi(x_{1,1}) = r_\varphi(x_{1,2}) = 1$, $r_\varphi(x_{1,3}) = 2$, $\varphi^{-1}(\{z_2\}) = \{x_2\}$, with $r_\varphi(x_2) = 4$, and $\varphi^{-1}(\{z_3\}) = \{x_{3,1}, x_{3,2}\}$, with $r_\varphi(x_{3,1}) = r_\varphi(x_{3,2}) = 2$. By the same reason as above, $x_{1,1}$, $x_{1,2}$, $x_{1,3}$ are singular points and $x_{1,3} = 1$. If the local exponents at $x = 1$ are 0, 1, ν, then those at $x = 0$ are α_0, $\alpha_0 + 1/2$, $\alpha_0 + \nu/2$, and those at $x = \infty$ are α_∞, $\alpha_\infty + 1/2$, $\alpha_\infty + \nu/2$ for some α_0 and α_∞. By a fractional linear transformation of z, we may assume $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$. Then (1.4) is projectively equivalent to (2.4), by the same way as the previous cases.

(5) Let $\varphi^{-1}(\{z_1\}) = \{x_{1,1}, x_{1,2}, x_{1,3}\}$, with $r_\varphi(x_{1,1}) = r_\varphi(x_{1,2}) = 1$, $\varphi^{-1}(\{z_2\}) = \{x_{2,1}, x_{2,2}\}$, with $r_\varphi(x_{2,1}) = r_\varphi(x_{2,2}) = 3$, $\varphi^{-1}(\{z_3\}) = \{x_{3,1}, x_{3,2}, x_{3,3}\}$, with $r_\varphi(x_{3,1}) = r_\varphi(x_{3,2}) = r_\varphi(x_{3,3}) = 2$. By the same reason as above, $x_{1,1}$, $x_{1,2}$, $x_{1,3}$ are singular points and $x_{1,3} = 1$. By the same way as the previous cases, (1.4) is projectively equivalent to (2.5). $\square$

4. Algebraic transformations with $l > 0$

In this section, we determine the algebraic transformations (1.4) with $l > 0$. We assume that z_j is regularizable if and only if $j > l$.

4.1. $l = 3$

In this case, none of z_j is regularizable. From (1) of Corollary 3.4, we have $n_j = n$. We have $n = 1$ from (3.1).

4.2. $l = 2$

In this case, z_1, z_2 are not regularizable. Hence we have $n_j = n$ for $j = 1, 2$, and $\varphi^{-1}(\{z_1, z_2\}) \subset \{\infty, 0, 1\}$ from (1) of Lemma 3.3.

If $\#\varphi^{-1}(\{z_1, z_2\}) = 2$, then we have $n_3 > 0$ and $n = 1 + (n - n_3)/r_3$, $r_3 \geq 2$. This implies $n = 1$.

If $\#\varphi^{-1}(\{z_1, z_2\}) = 3$, then we have $n_3 = 0$, and $n = 1 + n/r_3$, $r_3 \geq 2$ from (3.1). This means $n = 2$, $r_3 = 2$. By the symmetry of z_1 and z_2, we may conclude $BP(\varphi) = (1 + 1, 2, 2)$. Hence, from Lemma 3.8, (1.4) is projectively equivalent to (2.1).

4.3. $l = 1$

In this case, $n - n_1 = 0$ in (3.1), and we have

$$n = 1 + \sum_{j=2}^{3} (n - n_j)/r_j. \tag{4.1}$$

Since $\{0, 1, \infty\} = \varphi^{-1}(\{z_1\}) \cup (\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) \cup (\varphi^{-1}(\{z_3\}) \cap \{0, 1, \infty\})$, it is enough to consider the following four cases:

$\#\varphi^{-1}(\{z_1\}) = 3$,
$\#\varphi^{-1}(\{z_1\}) = 2$ and $\#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 1$,
$\#\varphi^{-1}(\{z_1\}) = 1$ and $\#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 2$,
$\#(\varphi^{-1}(\{z_j\}) \cap \{0, 1, \infty\}) = 1$, $j = 1, 2, 3$.

4.3.1. $\#\varphi^{-1}(\{z_1\}) = 3$

We have $\varphi^{-1}(\{z_1\}) = \{0, 1, \infty\}$, $n_2 = n_3 = 0$, and have $n = 1 + n/r_2 + n/r_3$, which implies $1/n + 1/r_2 + 1/r_3 = 1$. Hence (n, r_2, r_3) is one of $(3, 3, 3)$, $(4, 4, 2)$, $(6, 3, 2)$.

If $(n, r_2, r_3) = (3, 3, 3)$, then the differences of local exponents of $E(x)$ at $x = 0, 1, \infty$ are the same. This contradicts the no-log condition in (1.1).

If $(n, r_2, r_3) = (4, 4, 2)$, we have, from (2) of Corollary 3.4, $\#\varphi^{-1}(z_2) = 1$, $\#\varphi^{-1}(z_3) = 2$. Put $\varphi^{-1}(\{z_3\}) = \{x_{3,1}, x_{3,2}\}$. From (2) of Lemma 3.3, we have $r_\varphi(x_{3,1}) = r_\varphi(x_{3,2}) = 2$. Thus we have $BP(\varphi) = (1+1+2, 4, 2+2)$, and (1.4) is projectively equivalent to (2.4) from Lemma 3.8.

If $(n, r_2, r_3) = (6, 3, 2)$, we have $\varphi^{-1}(\{z_2\}) = \{x_{2,1}, x_{2,2}\}$, $\varphi^{-1}(\{z_3\}) = \{x_{3,1}, x_{3,2}, x_{3,3}\}$ for some $x_{2,j}$, $x_{3,j}$, and we have, from (2) of Lemma 3.3,

$$r_\varphi(x_{2,j}) = 3, \quad j = 1, 2; \qquad r_\varphi(x_{3,j}) = 2, \quad j = 1, 2, 3.$$

As for $\varphi^{-1}(\{z_1\}) = \{0, \infty, 1\}$, there may be three cases:

$$r_\varphi(0) + r_\varphi(\infty) + r_\varphi(1) = 1 + 1 + 4, \text{ or } 1 + 2 + 3, \text{ or } 2 + 2 + 2.$$

If $\varphi(x)$ is of the first type, then we have $BP(\varphi) = (1+1+4, 3+3, 2+2+2)$. Hence (1.4) is projectively equivalent to (2.5) from Lemma 3.8.

If $\varphi(x)$ is of the second or third type, we may assume $(r_\varphi(\infty), r_\varphi(0), r_\varphi(1)) = (1, 2, 3)$, and $(z_1, z_2, z_3) = (\infty, 0, 1)$ by fractional linear transformations of x and z. Then we have

$$x^{1/6}(1-x)^{1/4}{}_2E_1(-1/12, 1/4; 2/3; \varphi(x)) = {}_2E_1(-1/2, -1/6; 1/3; x).$$

The right hand side of the this equation has a logarithmic solution, while the left hand one does not. This implies that there is no $\varphi(x)$ of the second type.

If $\varphi(x)$ is of the last type, that is, if $r_\varphi(0) = r_\varphi(\infty) = r_\varphi(1) = 2$, then $E(x)$ does not satisfy the no-log condition.

4.3.2. $\#\varphi^{-1}(\{z_1\}) = 2$, $\#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 1$

We have $n_3 = 0$. From (4.1), we have

$$1/n + 1/r_2 + 1/r_3 = 1 + n_2/(nr_2) > 1. \tag{4.2}$$

Since $(n - n_2)/r_2, n/r_3 \in \boldsymbol{Z}$, (n, r_2, r_3) is one of $(n, 2, 2)$ with even n, $(2, r_2, 2)$ with $n_2 = 2$, $(3, 2, 3)$ with $n_2 = 1$, $(4, 3, 2)$ with $n_2 = 1$.

If $(n, r_2, r_3) = (n, 2, 2)$, we have $\varphi(1) = z_2$, and $n_2 = r_\varphi(1) = 1$, from (3) of Lemma 3.3. This contradicts $(n - n_2)/r_2, n/r_3 \in \boldsymbol{Z}$.

If $(n, r_2, r_3) = (2, r_2, 2)$ with $n_2 = 2$, (1.4) is projectively equivalent to (2.1).

If $(n, r_2, r_3) = (3, 2, 3)$ with $n_2 = 1$, we have $BP(\varphi) = (1+2, 1+2, 3)$, and (1.4) is projectively equivalent to (2.2) from Lemma 3.8.

If $(n, r_2, r_3) = (4, 3, 2)$ with $n_2 = 1$, we have $\varphi(1) = z_1$, from (4) of Lemma 3.3. From the no-log condition in (1.1), we have $r_\varphi(1) = 3$. From (2) of Corollary 3.4, there is one regular point in $\varphi^{-1}(\{z_2\})$, and two regular points in $\varphi^{-1}(\{z_3\})$. From (2) of Lemma 3.3, we have $BP(\varphi) = (1+3, 1+3, 2+2)$. Hence (1.4) is projectively equivalent to (2.3) from Lemma 3.8.

4.3.3. $\#\varphi^{-1}(\{z_1\}) = 1,\ \#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 2$

We also have the equality $n_3 = 0$ and the inequality (4.2). From (3) of Lemma 3.3, $r_2 \neq 2$. We have $n_2 \geq 2$, and $(n - n_2)/r_2, n/r_3 \in \boldsymbol{Z}$. Consequently (n, r_2, r_3) is equal to $(2, r_2, 2)$. Hence (1.4) is projectively equivalent to (2.1).

4.3.4. $\#(\varphi^{-1}(\{z_j\}) \cap \{0, 1, \infty\}) = 1,\ j = 1, 2, 3$

In (4.1), $n_2 > 0$, $n_3 > 0$. If $n_2 = n$, we have $n = 1 + (n - n_3)/r_3 \leq 1 + (n - 1)/2$. Hence we have $n - n_2 > 0$, and by the same reason, we also have $n - n_3 > 0$. From (4.1), we have

$$1/n + 1/r_2 + 1/r_3 = 1 + n_2/(nr_2) + n_3/(nr_3) > 1, \tag{4.3}$$

where $n_2 > 0$, $n_3 > 0$. Since $n - n_j$ (> 0) is a multiple of r_j, we have $n \geq n_j + r_j$ for $j = 2, 3$. From (3) of Lemma 3.3, we have $(r_2, r_3) \neq (2, 2)$. Consequently, (n, r_2, r_3) is either $(4, 3, 2)$ or $(5, 3, 2)$, both with $n_3 = r_\varphi(1) = 1$. In the former case, $(n - n_3)$ is not a multiple of r_3. In the latter case, we have $n_2 = 2$, and (4.3) does not hold.

5. Algebraic transformations with $l = 0$

In this section, we assume $l = 0$, that is, z_j, $j = 1, 2, 3$ are all regularizable. Let ξ be a singular point of $E(x)$. Then $\varphi(\xi) \in \{z_1, z_2, z_3\}$. Put

$$s_\xi = r_j \qquad \text{if } \varphi(\xi) = z_j.$$

Then, at $\xi = 0, 1, \infty$, $E(x)$ has the local exponents

$$\alpha_\xi,\ \alpha_\xi + r_\varphi(\xi)/s_\xi,\ \alpha_\xi + 2r_\varphi(\xi)/s_\xi,$$

where $\alpha_0 = \alpha_1 = 0$, $2r_\varphi(1)/s_1 = 1$, and $\alpha_\infty = 1/2 - r_\varphi(\infty)/s_\infty - r_\varphi(0)/s_0$. Hence we have, from Clausen's formula,

$$\begin{aligned} E(x) &= {}_3E_2(\alpha_\infty, \alpha_\infty + r_\varphi(\infty)/s_\infty, \alpha_\infty + 2r_\varphi(\infty)/s_\infty; \\ &\qquad 1 - r_\varphi(0)/s_0, 1 - 2r_\varphi(0)/s_0; x) \\ &= ({}_2E_1(\alpha_\infty/2, \alpha_\infty/2 + r_\varphi(\infty)/s_\infty; 1 - r_\varphi(0)/s_0; x))^2 \\ &= (HD(r_\varphi(\infty)/s_\infty, r_\varphi(0)/s_0, 1/2; x))^2. \end{aligned} \tag{5.1}$$

Lemma 5.1. *$E(x)$ given by* (5.1) *does not satisfy the irreducibility condition in* (1.1) *if and only if*

$$1/2 - \varepsilon_\infty r_\varphi(\infty)/s_\infty - \varepsilon_0 r_\varphi(0)/s_0 \in \boldsymbol{Z} \qquad \textit{for some } \varepsilon_\infty, \varepsilon_0 \in \{0, 1, -1\}.$$

We prove that any algebraic transformation (1.4) for $E(x)$ given by (5.1) is projectively equivalent to a Clausen type transformation induced from one of the transformations (1)–(10) in Theorem 2.2. Thanks to Lemma 3.6, in order to show that (1.4) is projectively equivalent to the Clausen type transformation induced from (2.8), for example, it is enough to show both that $E(x) = (HD(1/s, 1/s, 1/2; x))^2$, and that $\varphi(x) = \varphi_2(x)$. In such a case, we simply say that (1.4) is induced from (1) of Theorem 2.2, for short.

Without loss of generality, it is enough to consider the following three cases:

$\#(\varphi^{-1}(\{z_1\}) \cap \{0, 1, \infty\}) = 3$,

$\#(\varphi^{-1}(\{z_1\}) \cap \{0, 1, \infty\}) = 2$, $\#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 1$,

$\#(\varphi^{-1}(\{z_j\}) \cap \{0, 1, \infty\}) = 1$, $j = 1, 2, 3$.

5.1. $\#(\varphi^{-1}(\{z_1\}) \cap \{0, 1, \infty\}) = 3$

We have $n_2 = n_3 = 0$, $s_0 = s_1 = s_\infty = r_1$, and n is divisible by r_2 and r_3. From (4) of Lemma 3.3, $r_1 = 2r_\varphi(1)$. From (3) of the same lemma, $r_1 \neq 2$, that is $r_1 \geq 4$. We have $n_1 = r_\varphi(\infty) + r_\varphi(0) + r_\varphi(1) = r_\varphi(\infty) + r_\varphi(0) + r_1/2$, and, from (3.1),

$$n = 1/2 + (n - r_\varphi(\infty) - r_\varphi(0))/r_1 + n/r_2 + n/r_3. \tag{5.2}$$

We have $(r_2, r_3) \neq (2, 2)$ from this equation.

If $\sum_{j=1}^{3} 1/r_j = 1$, then $1/2 - (r_\varphi(\infty) + r_\varphi(0))/r_1 = 0$ from (5.2). If $\sum_{j=1}^{3} 1/r_j > 1$, then $r_1 = 4$ (and $\{r_2, r_3\} = \{2, 3\}$), and $r_\varphi(0)$, $r_\varphi(\infty)$ are odd. In both cases, $E(x)$ is reducible from Lemma 5.1.

Assume $\sum_{j=1}^{3} 1/r_j < 1$. Then, from (5.2), we have

$$\begin{aligned} n &= 1/2 + (n - r_\varphi(\infty) - r_\varphi(0)) \sum 1/r_j + (r_\varphi(\infty) + r_\varphi(0))(1/r_2 + 1/r_3) \\ &< 1/2 + (n - r_\varphi(\infty) - r_\varphi(0)) + (r_\varphi(\infty) + r_\varphi(0))(1/r_2 + 1/r_3), \end{aligned}$$

which implies $r_\varphi(0) = r_\varphi(\infty) = 1$, and $\{r_2, r_3\} = \{2, 3\}$. Then, from (5.2) again, we have $(1/6 - 1/r_1)n = 1/2 - 2/r_1$. Since $n \geq r_\varphi(\infty) + r_\varphi(0) + r_1/2 = 2 + r_1/2$, we have $r_1 \leq 8$. Hence $r_1 = 8$ (and $n = 6$) because r_1 is even and $r_1 > 6$. By the symmetry of z_2 and z_3, we may assume $(r_2, r_3) = (3, 2)$. Then we have $BP(\varphi) = (1 + 1 + 4, 3 + 3, 2 + 2 + 2)$. From (5.1), we have $E(x) = (HD(1/8, 1/8, 1/2; x))^2$. If we normalize so that $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$, by a fractional linear transformation of z, we have $\varphi(x) = \varphi_3 \circ \varphi_2(x)$ from Lemma 3.7. Consequently (1.4) is induced from (3) of Theorem 2.2.

5.2. $\#(\varphi^{-1}(\{z_1\}) \cap \{0, 1, \infty\}) = 2$**,** $\#(\varphi^{-1}(\{z_2\}) \cap \{0, 1, \infty\}) = 1$

We have $n_1 \geq 2$, $n_2 \geq 1$, $n_3 = 0$, and n is a multiple of r_3 in this case. From (3) of Lemma 3.3, $r_1 \neq 2$. Hence we have $r_1 \geq 3$.

5.2.1. $\sum_{j=1}^{3} 1/r_j < 1$

If $1 \in \varphi^{-1}(\{z_1\})$, then $n_1 \geq r_\varphi(1) + 1 = r_1/2 + 1$. Hence we have, from (3.1),

$$\begin{aligned} n &\leq 1 + (n - r_1/2 - 1)/r_1 + (n-1)/r_2 + n/r_3 \\ &= 1/2 + (n-1)\sum 1/r_j + 1/r_3 < 1/2 + (n-1) + 1/r_3, \end{aligned}$$

which cannot happen. Consequently we have $1 \in \varphi^{-1}(\{z_2\})$, and $n_2 = r_\varphi(1) = r_2/2$. From (3.1), we have

$$\begin{aligned} n &= 1 + (n - n_1)/r_1 + (n - r_2/2)/r_2 + n/r_3 \\ &= 1/2 + (n - n_1)\sum 1/r_j + n_1(1/r_2 + 1/r_3) \\ &\leq 1/2 + (n - n_1) + n_1(1/r_2 + 1/r_3), \end{aligned}$$

which implies

$$1 \leq 1/(2n_1) + 1/r_2 + 1/r_3,$$

where the equality holds only when $n - n_1 = 0$. There are three cases when the above inequality holds:

(i) $n = n_1 = 2,\ r_2 = 4,\ r_3 = 2$, (ii) $n = n_1 = 3,\ r_2 = 2,\ r_3 = 3$,
(iii) $n_1 = 2,\ r_2 = 2,\ r_3 = 3$.

(i) If $n = n_1 = 2$, $r_2 = 4$, $r_3 = 2$, then $BP(\varphi) = (1+1, 2, 2)$. Since $r_\varphi(0) = r_\varphi(\infty) = 1$, $s_0 = s_\infty = r_1$, we have $E(x) = (HD(1/r_1, 1/r_1, 1/2; x))^2$ from (5.1). If $(z_1, z_2, z_3) = (0, \infty, 1)$, $\varphi(x) = \varphi_2(x)$ in (2.1). Consequently (1.4) is induced from (1) of Theorem 2.2.

(ii) If $n = n_1 = 3$, $r_2 = 2$, $r_3 = 3$, then $BP(\varphi) = (1+2, 1+2, 3)$. By fractional linear transformations of x and z, we may assume $r_\varphi(0) = 1$, $r_\varphi(\infty) = 2$, $r_\varphi(1) = 1$, $s_0 = s_\infty = r_1$, and $(z_1, z_2, z_3) = (0, 1, \infty)$. Then $\varphi(x) = \varphi_3(x)$ in (2.2), from Lemma 3.7, and $E(x) = (HD(2/r_1, 1/r_1, 1/2; x))^2$ from (5.1). Consequently (1.4) is induced from (2) of Theorem 2.2.

(iii) Assume $n_1 = 2$, $r_2 = 2$, $r_3 = 3$ (and $n_2 = r_\varphi(1) = 1$). Then, from (3.1), we have $n(1/6 - 1/r_1) = 1/2 - 2/r_1$. Since $n \geq n_1 + r_1 = 2 + r_1$, we have $r_1 = 7$. Now since $r_\varphi(0) = r_\varphi(\infty) = 1$, and $s_0 = s_\infty = r_1 = 7$, we have $E(x) = (HD(1/7, 1/7, 1/2; x))^2$ from (5.1). The branching pattern of φ is given by

$$BP(\varphi) = (1 + 1 + 7, 1 + 2 + 2 + 2, 3 + 3 + 3).$$

If $(z_1, z_2, z_3) = (\infty, 1, 0)$, then $\varphi(x)$ has the form of $\varphi_9(x)$ in (4), and satisfies the algebraic transformation in (4). The existence of a rational function of this type was shown by R. Vidūnas (the equation (24) in [Vdn1]). Consequently (1.4) is induced from (4) of Theorem 2.2.

5.2.2. $\sum_{j=1}^{3} 1/r_j = 1$

From (3.1), we have $n_1/r_1 + n_2/r_2 = 1$. Since $n_1 \geq 2$, $r_1 \geq 3$, and $r_1 r_2$ must be even, one of the following cases holds.

$$(r_1, r_2, r_3) = (4, 4, 2) \quad \text{with } (n_1, n_2) = (2, 2) \text{ or } (3, 1), \tag{5.3}$$

$$(r_1, r_2, r_3) = (4, 2, 4) \quad \text{with } (n_1, n_2) = (2, 1), \tag{5.4}$$

$$(r_1, r_2, r_3) = (6, 3, 2) \quad \text{with } (n_1, n_2) = (2, 2) \text{ or } (4, 1), \tag{5.5}$$

$$(r_1, r_2, r_3) = (6, 2, 3) \quad \text{with } (n_1, n_2) = (3, 1), \tag{5.6}$$

$$(r_1, r_2, r_3) = (3, 6, 2) \quad \text{with } (n_1, n_2) = (2, 2). \tag{5.7}$$

In case of (5.3) and (5.4), we have $s_0 = s_\infty = 4$, and $r_\varphi(0) = r_\varphi(\infty) = 1$. In case of (5.5) with $(n_1, n_2) = (4, 1)$, we have $\{s_0, s_\infty\} = \{6, 3\}$, and $r_\varphi(0) = r_\varphi(\infty) = 1$. In case of (5.6), we have $s_0 = s_\infty = 6$, and $\{r_\varphi(0), r_\varphi(\infty)\} = \{1, 2\}$. In any of these cases, $E(x)$ is reducible from Lemma 5.1. In case of (5.5) with $(n_1, n_2) = (2, 2)$ and of (5.7), $E(x)$ cannot be ${}_3E_2$.

5.2.3. $\sum_{j=1}^{3} 1/r_j > 1$

From (3) and (4) of Lemma 3.3 (and from the equality $r_1 \geq 3$), one of the following cases holds.

$$(r_1, r_2, r_3) = (r_1, 2, 2) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1, \tag{5.8}$$

$$(r_1, r_2, r_3) = (3, 2, 3) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1, \tag{5.9}$$

$$(r_1, r_2, r_3) = (4, 2, 3) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1, \tag{5.10}$$

$$(r_1, r_2, r_3) = (4, 3, 2) \quad \text{with } \varphi(1) = z_1,\ r_\varphi(1) = 2, \tag{5.11}$$

$$(r_1, r_2, r_3) = (3, 2, 4) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1, \tag{5.12}$$

$$(r_1, r_2, r_3) = (3, 4, 2) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 2, \tag{5.13}$$

$$(r_1, r_2, r_3) = (5, 2, 3) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1, \tag{5.14}$$

$$(r_1, r_2, r_3) = (3, 2, 5) \quad \text{with } \varphi(1) = z_2,\ n_2 = r_\varphi(1) = 1. \tag{5.15}$$

Since $(n - n_2)/r_2, n/r_3 \in \boldsymbol{Z}$, neither (5.8) nor (5.12) can happen.

In case of (5.9), we have $n = 2n_1 - 3$ from (3.1), and

$$(5.16)\quad BP(\varphi) = (r_\varphi(\infty) + r_\varphi(0) + 3 + \cdots + 3, 1 + 2 + \cdots + 2, 3 + \cdots + 3).$$

From the no-log condition, we have $r_\varphi(0), r_\varphi(\infty) \not\equiv 0 \mod 3$, and $n_1 = r_\varphi(0) + r_\varphi(\infty) \equiv 0 \mod 3$. Since $s_0 = s_\infty = r_1 = 3$, we have $E(x) = (HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x))^2$ from (5.1). Assume $(z_1, z_2, z_3) = (\infty, 1, 0)$. Then $\varphi(x)$ satisfies (2.12) for some $\theta(x)$.

The existence of $\varphi(x)$ satisfying (2.12) is shown from Theorem A. Such a function $\varphi(x)$ must have the branching pattern (5.16), provided that $r_\varphi(0) + r_\varphi(\infty) \equiv 0 \mod 3$. Then $n = 2(r_\varphi(\infty) + r_\varphi(0)) - 3$ from Riemann-Hurwitz formula. Consequently (1.4) is induced from (5) of Theorem 2.2 in this case.

In case of (5.10), $n = 3n_1 - 6$ from (3.1). Since $(n-1)/r_2 \in \boldsymbol{Z}$, n_1 $(= r_\varphi(0) + r_\varphi(\infty))$ is odd. Assume $r_\varphi(0)$ is even, for example. Then, at $x = 0$, $E(x)$ has the local exponents α, $\alpha + r_\varphi(0)/4$, $\alpha + r_\varphi(0)/2$ for some α. Consequently $E(x)$ does not satisfy the no-log condition.

In case of (5.11), we have $0, 1 \in \varphi^{-1}(\{z_1\})$ or $0, \infty \in \varphi^{-1}(\{z_1\})$. Assume $0, 1 \in \varphi^{-1}(\{z_1\})$, for example. Since $(n - n_1)/r_1, n/r_3 \in \boldsymbol{Z}$, n_1 $(= r_\varphi(0) + r_\varphi(1) = r_\varphi(0) + 2)$ is even. Hence $r_\varphi(0)$ is even. Then $E(x)$ does not satisfy the no-log condition, by the same reason as in the previous case.

In case of (5.13), $n = 4n_1 - 6$ from (3.1), and

$$(5.17)\quad BP(\varphi) = (r_\varphi(\infty) + r_\varphi(0) + 3 + \cdots + 3, 2 + 4 + \cdots + 4, 2 + \cdots + 2),$$

with $r_\varphi(0), r_\varphi(\infty) \not\equiv 0 \mod 3$. We have $E(x) = (HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x))^2$ from (5.1). Let $E_O(z) = HD(1/4, 1/3, 1/2; z)$, $E_T(z) = HD(1/3, 1/3, 1/2; z)$. Assume $(z_1, z_2, z_3) = (0, \infty, 1)$. Then $\varphi(x)$ satisfies the algebraic transformation

$$(5.18)\qquad HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x) = \theta_1(x) E_O(\varphi(x)),$$

for some $\theta_1(x)$. From Theorem 6.1, there is an algebraic transformation

$$(5.19)\qquad HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x) = \theta_2(x) E_T(R(x)),$$

for some $\theta_2(x)$. The equation (5.19) determines the branching pattern of R as follows: that over 1 is $1 + 2 + \cdots + 2$, and those over ∞ and 0 are $r_\varphi(\infty) + r_\varphi(0) + 3 \cdots + 3$ and $3 + \cdots + 3$ if $r_\varphi(\infty) + r_\varphi(0) \equiv 0 \mod 3$, and $r_\varphi(\infty) + 3 + \cdots + 3$ and $r_\varphi(0) + 3 + \cdots + 3$ if $r_\varphi(\infty) \equiv r_\varphi(0) \mod 3$. In both cases, $\deg R = 2(r_\varphi(\infty) + r_\varphi(0)) - 3$ from Riemann-Hurwitz formula.

Let $s_T(z)$ (resp. $s_O(z)$) be a Schwarz map of $E_T(z)$ (resp. $E_O(z)$) defined by a quotient of linearly independent solutions. Then $s(x) := s_T(R(x))$ is a Schwarz map of (5.19). Let $\{s(\xi)\}$ (resp. $\{s_T(\zeta)\}$) be the set of values of $s(x)$ at $x = \xi$ (resp. $s_T(z)$ at $z = \zeta$) consisting of twelve points, in general. If $\zeta = R(\xi)$, then $\{s(\xi)\} = \{s_T(\zeta)\}$. Consequently we have

$$R(\xi_1) = R(\xi_2) \Leftrightarrow \{s(\xi_1)\} = \{s(\xi_2)\}.$$

The equality (5.18) implies that

$$\{s(\xi_1)\} = \{s(\xi_2)\} \Rightarrow \varphi(\xi_1) = \varphi(\xi_2).$$

Since $\#\{s_O(\zeta)\} = 24$, in general, there is a quadratic rational function $R_2(z)$ satisfying $\varphi(x) = (R_2 \circ R)(x)$.

The equation (5.17) determines the branching pattern of R_2 so that we have $R_2(z) = \varphi_2(z)$ in (2.1). Consequently we have $\varphi = \varphi_2 \circ R$, and (1.4) is induced from (9) of Theorem 2.2.

In case of (5.14), $n = 6n_1 - 15$ from (3.1), and

$$(5.20)\qquad BP(\varphi) = (r_\varphi(\infty) + r_\varphi(0) + 5 + \cdots + 5, 1 + 2 + \cdots + 2, 3 + \cdots + 3),$$

with $r_\varphi(0), r_\varphi(\infty) \not\equiv 0 \mod 5$. We have $E(x) = (HD(r_\varphi(\infty)/5, r_\varphi(0)/5, 1/2; x))^2$. Assume $(z_1, z_2, z_3) = (\infty, 1, 0)$. Then $\varphi(x)$ satisfies the algebraic transformation (2.15) for some $\theta(x)$. The differential equation on the left hand side of (2.15) has a finite projective monodromy. Hence $\{r_\varphi(0), r_\varphi(\infty)\}$ is equivalent to one of $\{1, 2\}$, $\{1, 3\}$, $\{4, 2\}$, $\{4, 3\}$ mod 5. Conversely if $\{r_\varphi(0), r_\varphi(\infty)\}$ is equivalent to one of these, we find, from Theorem A, that there exists a rational function $\varphi(x)$ satisfying (2.15) with the branching pattern (5.20), and with $n = 6(r_\varphi(\infty) + r_\varphi(0)) - 15$. Consequently we conclude that (1.4) is induced from (8) of Theorem 2.2.

In case of (5.15), $n = 10n_1 - 15$ from (3.1), and

$$BP(\varphi) = (r_\varphi(\infty) + r_\varphi(0) + 3 + \cdots + 3, 1 + 2 + \cdots + 2, 5 + \cdots + 5),$$

with $r_\varphi(0), r_\varphi(\infty) \not\equiv 0 \mod 3$. We have $E(x) = (HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x))^2$. By the same way as in the case of (5.13), there is a fifth degree rational function $\varphi_5(z)$ satisfying $\varphi(x) = (\varphi_5 \circ R)(x)$, where $R(x)$ is a function in (5.19). The branching pattern of $\varphi(x)$ determines that of $\varphi_5(z)$ so that (1.4) is induced from (10) of Theorem 2.2.

5.3. $\#(\varphi^{-1}(\{z_j\}) \cap \{0, 1, \infty\}) = 1$, $j = 1, 2, 3$

We assume $\varphi(1) = z_3$, $n_3 = r_\varphi(1) = r_3/2$. From (3) of Lemma 3.3, we have $r_1, r_2 \geq 3$. From (3.1), we have

$$n\Big(1 - \sum\nolimits_{j=1}^{3} 1/r_j\Big) = 1 - \sum\nolimits_{j=1}^{3} n_j/r_j.$$

If $\sum_{j=1}^{3} 1/r_j < 1$, then $n = n_j = 1$.

If $\sum_{j=1}^{3} 1/r_j = 1$, then we have $n_j = 1$, and (r_1, r_2, r_3) is one of $(4, 4, 2)$, $(6, 3, 2)$ and $(3, 6, 2)$. In any case, $E(x)$ is reducible from Lemma 5.1.

If $\sum_{j=1}^{3} 1/r_j > 1$, then it is enough to consider the following three cases: $(r_1, r_2, r_3) = (3,3,2), (4,3,2), (5,3,2)$. In each case, $(n_j, r_j) = 1$, $j = 1,2$, and $n_3 = r_\varphi(1) = 1$.

If $(r_1, r_2, r_3) = (3,3,2)$, then $n = 2(r_\varphi(\infty) + r_\varphi(0)) - 3$ from (3.1), and

$$BP(\varphi) = (r_\varphi(\infty) + 3 + \cdots + 3, r_\varphi(0) + 3 + \cdots + 3, 1 + 2 + \cdots + 2),$$

with $r_\varphi(\infty), r_\varphi(0) \not\equiv 0 \mod 3$. We have $E(x) = (HD(r_\varphi(\infty)/3, r_\varphi(0)/3, 1/2; x))^2$ from (5.1). Assume $(z_1, z_2, z_3) = (\infty, 0, 1)$. Then $\varphi(x)$ satisfies (2.12) for some $\theta(x)$. Theorem A guarantee the existence of such $\varphi(x)$ with the same branching pattern, provided that $r_\varphi(\infty) \equiv r_\varphi(0) \mod 3$. Hence, by the same way as in the previous subsubsection, we find that (1.4) is induced from (5) of Theorem 2.2.

Let $(r_1, r_2, r_3) = (4,3,2)$, and assume $\varphi(\infty) = \infty = z_1$, $\varphi(0) = 0 = z_2$, and $\varphi(1) = 1 = z_3$. Then $r_\varphi(\infty)$ is odd, $r_\varphi(0) \not\equiv 0 \mod 3$ and $n = 3r_\varphi(\infty) + 4r_\varphi(0) - 6$. In this case, $E(x) = (HD(r_\varphi(\infty)/4, r_\varphi(0)/3, 1/2; x))^2$ from (5.1), and $\varphi(x)$ satisfies (2.13) for some $\theta(x)$. Consequently (1.4) is induced from (6) of Theorem 2.2 as the above case.

Let $(r_1, r_2, r_3) = (5,3,2)$, and assume $\varphi(\infty) = \infty = z_1$, $\varphi(0) = 0 = z_2$, and $\varphi(1) = 1 = z_3$ as above. Then $r_\varphi(\infty) \not\equiv 0 \mod 5$, $r_\varphi(0) \not\equiv 0 \mod 3$ and $n = 12r_\varphi(\infty) + 20r_\varphi(0) - 30$. In this case, $E(x) = (HD(r_\varphi(\infty)/5, r_\varphi(0)/3, 1/2; x))^2$ from (5.1), and $\varphi(x)$ satisfies (2.14) for some $\theta(x)$. Consequently (1.4) is induced from (7) of Theorem 2.2 as the above cases.

This completes the proof of Theorem 2.2.

Appendix

Let G be an irreducible finite subgroup of $GL(2, \boldsymbol{C})$, and g' be the order of $G/(\text{the center of } G)$. G is one of a dihedral, tetrahedral, octahedral, icosahedral group. There are three G-relative invariant homogeneous polynomials $I_{d_j}(v_0, v_1)$ of degree d_j, $j = 1,2,3$ satisfying

$$I_{d_3}^{g'/d_3} = I_{d_1}^{g'/d_1} - I_{d_2}^{g'/d_2},$$

where $(d_1, d_2, d_3) = (2, g'/2, g'/2)$ (g' is even) if G is dihedral, $(d_1, d_2, d_3) = (4,4,6)$ ($g' = 12$) if G is tetrahedral, $(d_1, d_2, d_3) = (6,8,12)$ ($g' = 24$) if G is octahedral, $(d_1, d_2, d_3) = (12,20,30)$ ($g' = 60$) if G is icosahedral. In each case, $I_{d_j}(v_0, v_1)^{g'/d_j} / I_{d_k}(v_0, v_1)^{g'/d_k}$ is G-invariant.

The following theorem was proved by F. Klein [Kl]. Here we give our proof (after Klein's original one) for the self-containedness.

Theorem A. *Let $E(x)$ be a Fuchsian differential equation on $\boldsymbol{P}^1$ of order two with a finite irreducible monodromy group G. Then there is an algebraic transformation $E(x) = \theta(x)E'(R(x))$, where $E'(z) = {}_2E_1(-1/g', 1/g'; 1/2; z)$,*

${}_2E_1(-1/12, 1/4; 2/3; z)$, ${}_2E_1(-1/24, 5/24; 2/3; z)$ *or* ${}_2E_1(-1/60, 11/60; 2/3; z)$, *according as* G *is a dihedral, tetrahedral, octahedral, or icosahedral group*.

Proof. We prove the theorem in an icosahedral case. Let $S(E)$ and $S(E')$ $(= \{0, 1, \infty\})$ be the set of singular points of $E(x)$ and $E'(z)$, respectively. We fix a fundamental system $v_0(x)$, $v_1(x)$ of solutions of $E(x)$ at a base point $x_0 \notin S(E)$, such that the monodromy representation of $\pi_1(\boldsymbol{P}^1 - S(E), x_0)$ into $GL(2, \boldsymbol{C})$ with respect to this system has the image G.

Put

$$R(x) = I_{20}(v_0(x), v_1(x))^3 / I_{12}(v_0(x), v_1(x))^5, \tag{A.1}$$

which is a rational function in x. We may assume that $z_0 := R(x_0) \notin S(E')$, by a sufficiently small shift of x_0, if necessary.

Let $u_0(z)$, $u_1(z)$, be a fundamental system of solutions of $E'(z)$ at z_0, such that the image of the monodromy representation of $\pi_1(\boldsymbol{P}^1 - S(E'), z_0)$ is equal to G up to the center. It is known that

$$z = I_{20}(u_0(z), u_1(z))^3 / I_{12}(u_0(z), u_1(z))^5,$$

which implies that the function $\pi(w_0, w_1) = I_{20}(w_0, w_1)^3 / I_{12}(w_0, w_1)^5$ defines a quotient map $\boldsymbol{P}^1 \to \boldsymbol{P}^1/G$. Since

$$\pi(v_0(x), v_0(x)) = \pi(u_0(R(x)), u_1(R(x))),$$

$u_1(R(x))/u_0(R(x))$ and $v_1(x)/v_0(x)$ have the same G-orbit in $\boldsymbol{P}^1$. Hence we may assume that

$$u_1(R(x))/u_0(R(x)) = v_1(x)/v_0(x), \tag{A.2}$$

by exchanging the branches of $v_j(x)$, if necessary.

Let $S = S(E) \cup R^{-1}(S(E'))$. Put

$$\theta(x) = v_0(x)/u_0(R(x)) = v_1(x)/u_1(R(x)),$$

which is holomorphic at $x = x_0$, and can be analyticaly continued along any curve in $\boldsymbol{P}^1 - S$. Let $\gamma \in \pi_1(\boldsymbol{P}^1 - S, x_0)$. Then the analytic continuation γ_* along γ defines linear maps

$$\gamma_* \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \qquad \text{and} \qquad \gamma_* \begin{pmatrix} u_0 \circ R \\ u_1 \circ R \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} u_0 \circ R \\ u_1 \circ R \end{pmatrix}.$$

The equality (A.2) implies $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \alpha(\gamma) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\alpha(\gamma) \in \boldsymbol{C}$. Then we have

$$\gamma_* \theta(x) = \gamma_* \frac{v_0}{u_0 \circ R} = \frac{a v_0 + b v_1}{\alpha(\gamma)(a u_0 \circ R + b u_1 \circ R)} = \alpha(\gamma)^{-1} \theta(x)$$

Since $\alpha(\gamma)^m = 1$ for some integer m, we have $\theta(x) = U(x) \prod_{\xi \in S, \xi \neq \infty} (x - \xi)^{\lambda_\xi}$, for some rational function $U(x)$ and some rational numbers λ_ξ. This proves the theorem. $\square$

References

[Bly] Bailey, W. N., *Generalized Hypergeometric Series* (Cambridge Tracts in Mathematics and Mathematical Physics **32**), 1935.

[Kl] Klein, F., Ueber lineare Differentialgleichungen, Math. Ann., **11** (1877), 115–118.

[Vdn1] Vidūnas, R., Transformations of some Gauss hypergeometric functions, J. Comput. Appl. Math., **178** (2005), 473–487.

[Vdn2] Vidūnas, R., Algebraic transformations of Gauss hypergeometric functions, preprint 2004.

nuna adreso:
Department of Mathematics
College of Education
University of the Ryukyus
Okinawa 903-0213
Japan
E-mail: mkato@edu.u-ryukyu.ac.jp

(Ricevita la 27-an de novembro, 2006)
(Reviziita la 31-an de oktobro, 2007)