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Introduction

In the vast literature on differential equations, one can distinguish two very different perspectives. The first and most widespread one is directed toward solving the equations and/or studying the properties of the solutions, in various contexts. The second one is directed toward studying the structure and properties of the equations themselves.

The present book fits into the second perspective. It focuses on linear differential equations with polynomial coefficients and, more generally, on algebraic differential modules in several variables, and on some of their relations with analytic differential modules.

In this (second) perspective, solutions are on equal footing with "cosolutions", i.e., elements of the cokernel of a differential operator, and their study becomes the study of cohomological properties of algebraic differential modules.

0.1. Let us explain the source of this book. The idea of computing the cohomology of a manifold, in particular its Betti numbers, by means of differential forms goes back to E. Cartan and G. de Rham. In the case of a smooth complex algebraic variety X, this comes in three variants:

(i) using the de Rham complex of algebraic differential forms on X;

(ii) using the de Rham complex of holomorphic differential forms on the analytic manifold X^{an} underlying X;

(iii) using the de Rham complex of C^∞ complex differential forms on the differentiable manifold X^{dif} underlying X^{an}.

Somewhat surprisingly, these variants turn out to lead to the same result: one has canonical isomorphisms of hypercohomology: $H_{\mathrm{DR}}(X) \cong H_{\mathrm{DR}}(X^{\mathrm{an}}) \cong H_{\mathrm{DR}}(X^{\mathrm{dif}})$.

While the second isomorphism is a simple sheaf-theoretic consequence of the Poincaré lemma (in the wake of A. Weil's sheaf-theoretic proof of de Rham's theorem), which identifies both vector spaces with the complex cohomology $H(X^{\mathrm{top}}, \mathbb{C})$ of the topological space underlying X, the first isomorphism is a deeper result of A. Grothendieck which shows in particular that the Betti numbers can be computed algebraically.

0.2. The latter result was generalized by P. Deligne to the case of nonconstant coefficients: for any algebraic vector bundle $\mathcal{M}$ on X endowed with an integrable *regular connection*, one has canonical isomorphisms $H_{\mathrm{DR}}(X, \mathcal{M}) \cong H_{\mathrm{DR}}(X^{\mathrm{an}}, \mathcal{M}^{\mathrm{an}})$. The notion of a regular connection is a higher-dimensional generalization of the classical notion of fuchsian differential equations (those with only regular singularities). These results were to have a significant influence on the later development of the theory of differential modules. The crucial point in their proof is a comparison between a meromorphic de Rham complex and a de Rham complex with essential singularities, carried out using Hironaka's resolution of singularities. This also turns out to be the main point in the proof of the non-archimedean counterparts of these comparison theorems, established by R. Kiehl and by the second author (under the assumption that the exponents of $\mathcal{M}$ are algebraic numbers).

This point became better understood when Z. Mebkhout, generalizing Grothendieck's comparison technique in the complex case, showed that the gap between meromorphic and essentially singular de Rham complexes is measured by a certain perverse sheaf ("positivity of the irregularity"); this also allowed to bypass resolution of singularities.

On the other hand, the situation is not so nice in the non-archimedean setting, where no analogue of the "positivity theorem" is known. This motivated us to pose the following problem, which was the starting point of this work:

($*$) *give an "elementary" proof of the comparison theorems, which is "formal" enough to apply both in the complex-analytic and the rigid-analytic situations.*

Such a proof should not only avoid Hironaka's theorem, but also any monodromy argument.

0.3. Our solution to ($*$) is inspired by M. Artin's proof of the comparison theorem between algebraic and complex-analytic étale cohomology. It relies on three new tools:

(i) an algebraic construction of Deligne's canonical extensions (Ch. IV),

(ii) a dévissage, inspired by Artin's technique of elementary fibrations, which permits the reduction of many problems on direct images to the relative one-dimensional case (Ch. VIII),

(iii) an abstract model of "comparison theorem" in the setting of differential algebra (Ch. IX).

Settling this problem led us to revisit some fundamental results on direct images of regular differential modules by a smooth morphism. For instance, we found *elementary and algebraic proofs of the so-called finiteness, base change, regularity and monodromy theorems* (Ch. VIII). As a by-product of these techniques, we present an *elementary proof of the generalized Riemann existence theorem for coverings* which uses neither resolution, nor any extension theorem for analytic coherent sheaves.

The final result of this book is a *proof of the general non-archimedean comparison theorem for de Rham cohomology* with coefficients in any integrable, possibly irregular, connection (under the assumption that the exponents of $\mathcal{M}$ are algebraic numbers), as was conjectured by the second author.

0.4. In the literature on algebraic differential modules, two different languages/viewpoints are in use. The first and more widespread is the language of $\mathcal{D}$-modules, the other is the language of integrable connections. They are equivalent in theory, but not in practice: the connection viewpoint casts the singularities to infinity (outside X), whereas the $\mathcal{D}$-module viewpoint includes them at "finite distance".

If the language of $\mathcal{D}$-modules dates back to E. Noether[1], the theory has been founded by J. Bernstein and M. Kashiwara independently in the 70's, and systematically developed by the latter (and his school) afterwards. It is not the purpose of this introduction to give an overview. But a reader familiar with this theory might legitimately ask: how is it possible to write a book on algebraic differential modules in the XXI'st century, without hardly mentioning characteristic varieties and holonomic $\mathcal{D}$-modules (for instance)?

Here is a short answer. By including singularities in its workspace instead of rejecting them at infinity, the $\mathcal{D}$-module viewpoint is much more precise than the connection viewpoint; in this perspective, the ($\mathcal{O}$-coherent) connection viewpoint becomes a coarse, "generic" view on holonomic $\mathcal{D}$-modules. Still and somewhat paradoxically, many problems on $\mathcal{D}$-modules, even when they deal with singularities, are of "generic" nature, and may be easily formulated and clearly treated in the language of connections.

0.5. In this book, we adopt the language of connections (Ch. II), in the wake of Deligne's memoir [35]. More precisely, we consider integrable algebraic connections with singularities along a divisor D "at infinity".

Chapters IV and VI present an algebraic theory of regularity, resp. irregularity along D. In particular, we give an *algebraic proof of Deligne's regularity criterion by restriction to curves, and a stratification of the polar divisor D according to Newton polygons which measure irregularity.* It turns out that the most advanced parts of regularity theory require a bit of irregularity theory in their proofs.

Our techniques, which have an ultrametric flavor (completion along D), are developed in Chapters III and V respectively. They originated in the works of Dwork, Katz, Gérard, Levelt *et al.* and recently got their full strength in the first author's proof of Malgrange conjecture on irregularity and (subsequently) in K. Kedlaya's proof of Sabbah's conjecture on good formal structures (T. Mochizuki's proof is of different nature). The book can also serve as a systematic introduction to these techniques.

[1] cf. [85], quoted in [36].

Chapter VII discusses the Gauss-Manin connection, and proposes an algebraic proof of Deligne's global index theorem.

Acknowledgments. We are very grateful to J. Bernstein for pointing out a mistake in the proof of a statement in the first edition of this book, and for offering an illuminating example in this context; the questioned statement has become the main theorem of Chapter VI, with a completely new proof.

This new, entirely rewritten edition has been completed during a visit at the Mathematisches Forschungsinstitut Oberwolfach, supported through the program "Research in Pairs" in February 2019. We are very grateful to this institution for excellent working conditions which have enabled us to finalize this project.

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Chapter I

Differential algebra

Introduction

We start by introducing the Gauss and Kummer hypergeometric differential equations. These classical equations will reenter the stage at the beginning of several later chapters, serving as guiding examples to motivate and illustrate their content.

We then present a few basic notions of differential algebra: differential rings and fields, differential modules and operations on them. We investigate in some detail the relation between differential modules and differential operators: properties and construction of cyclic vectors.

1 Hypergeometric origins

1.1 Gauss hypergeometric differential equation

Let a, b, c be positive rational numbers. The classical *hypergeometric series*, introduced by Euler in 1769[1], is

$$ {}_2F_1(a,b,c;x) = \sum_0^\infty \frac{(a)_n(b)_n}{(c)_n n!} x^n, $$

where we use the Pochhammer's symbol $(a)_n = a(a+1)\cdots(a+n-1)$.

For $c > a$, Euler obtained the integral representation

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt. $$

In connection with his "arithmetic-geometric mean", Gauss (1813) investigated this series systematically, proved its convergence in the open unit disk, noted

[1] or earlier, with the same name, in Wallis' 1655 book *Arithmetica Infinitorum*.

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(after Euler) that it satisfies the differential equation

$$x(1-x)\,\frac{\partial^2}{\partial x^2}\,y + \big(c-(a+b+1)x\big)\,\frac{\partial}{\partial x}\,y - ab\,y = 0. \tag{1.1.1}$$

A systematic way to find differential equations satisfied by a series $\sum R(n)x^n$ whose coefficients $R(n)$ are rational functions of n consists in using the fact that for any polynomial P with constant coefficients, $P\big(x\frac{\partial}{\partial x}\big)(x^n) = P(n)x^n$; one immediately finds that the differential operator $\big(a+x\frac{\partial}{\partial x}\big)\big(b+x\frac{\partial}{\partial x}\big) - \big(c+x\frac{\partial}{\partial x}\big)\big(1+x\frac{\partial}{\partial x}\big)\frac{1}{x}$ kills ${}_2F_1(a,b,c;x)$, which gives (1.1.1).

But one can also derive (1.1.1) by differentiating under the integral sign in the integral representation: this is the starting point of the theory of the Gauss-Manin connection, which will be the matter of Chapter VII.

1.2 Kummer confluent hypergeometric differential equation

In 1836, Kummer introduced the phenomenon of *confluence* on replacing x by x/b and letting b tend to ∞, which leads to the so-called *confluent hypergeometric series*

$${}_1F_1(a,c;x) = \sum_0^\infty \frac{(a)_n}{(c)_n n!}x^n.$$

Noting that $(1-t\frac{x}{b})^{-b}$ tends to e^{tx} (as b goes to ∞), Kummer obtained the integral representation

$${}_1F_1(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tx}dt,$$

and proved that it satisfies the confluent hypergeometric differential equation

$$x\,\partial_x^2\,y + (c-x)\,\partial_x\,y - a\,y = 0. \tag{1.2.1}$$

The classical hypergeometric differential equation has three singular points, 0, 1 and ∞. At 0, another solution (which is linearly independent if $c \neq 1$) is given by

$$x^{1-c}{}_2F_1(a+1-c, b+1-c, 2-c; x).$$

More generally, apart from fractional powers (and logarithms when $c = 1$ or $c = a+b$ or else $a = b$), solutions at $0, 1, \infty$ involve only convergent power series. After Fuchs and Frobenius, such singularities are called *regular*.

In contrast, the confluent hypergeometric differential equation has two singular points, 0 and ∞. At ∞, a solution is given by the divergent power series

$$\frac{1}{x^a}{}_2F_0(a, a-c+1; 1/x) = \frac{1}{x^a}\sum_0^\infty \frac{(a)_n(a-c+1)_n}{n!}\frac{1}{x^n}.$$

The occurrence of a divergence is a characteristic feature of an irregular singular point (Fabry, Poincaré). The discussion of regular versus irregular singularities in one or several variables, and the way in which regular singularities become irregular ones by confluence, will occupy the next chapters.

2 From differential equations to differential systems and differential modules

In this section we introduce the basic definitions of differential algebra: differential rings, differential systems and their equivalence, and we study the relations between them. We further introduce other useful constructions, such as duality and tensor product.

2.1 Derivations and differentials

Let $A \to C$ be an extension of unitary commutative rings. For any C-module M, the A-module $\mathrm{End}_A(M)$ of A-linear endomorphisms of M is a C-bimodule, the left C-module structure being defined by $(r\phi)(m) = r\phi(m)$ and the right C-module structure by $(\phi s)(m) = \phi(sm)$ (for any $\phi \in \mathrm{End}_A(M)$, $r, s \in C$, $m \in M$). The Lie bracket $[\phi, \psi] = \phi \circ \psi - \psi \circ \phi$, equips $\mathrm{End}_A(M)$ with the additional structure of an A-Lie algebra.

Definition 2.1.1 (Derivations). *An A-*linear derivation *of C with values in a C-module M is an A-linear map $D : C \to M$ such that $D(xy) = xD(y) + yD(x)$, for any $x, y \in C$.*

(In particular, $D(1) = 0$.) We denote by $\mathrm{Der}_{C/A} = \mathrm{Der}_A(C) \subseteq \mathrm{End}_A(C)$ the (left) C-submodule and A-Lie subalgebra of A-linear derivations of C.

Similarly, $\mathrm{Hom}_A(C, M)$ is equipped with a C-bimodule structure, and we define more generally $\mathrm{Der}_A(C, M) \subseteq \mathrm{Hom}_A(C, M)$ to be the (left) C-submodule of A-linear derivations of C with values in M.

Definition 2.1.2 (Differentials). *The module of differentials of C over A is $\Omega^1_{C/A} = I/I^2$, where $I = \mathrm{Ker}(C \otimes_A C \to C)$ (the product map). It is endowed with the canonical derivation $d_{C/A} : C \to \Omega^1_{C/A}$ induced by $x \mapsto 1 \otimes x - x \otimes 1$, for any $x \in C$.*

The ideal I is generated by the elements of the form $1 \otimes x - x \otimes 1$, for $x \in C$ (in fact, if $\sum_i x_i y_i = 0$, then $\sum_i y_i \otimes x_i = \sum_i (y_i \otimes 1)(1 \otimes x_i - x_i \otimes 1)$). Notice that the left and right structures of C-module of $C \otimes_A C$ coincide on $\Omega^1_{C/A}$: for any $m \in I$, and $x \in C$, $mx - xm = (1 \otimes x - x \otimes 1)m \in I^2$.

Composition with the canonical derivation $d_{C/A}$ induces a canonical and functorial identification

$$\mathrm{Hom}_C(\Omega^1_{C/A}, M) \cong \mathrm{Der}_A(C, M),$$

that is, the C-module $\Omega^1_{C/A}$ endowed with the derivation $d_{C/A}$ represents the endofunctor $M \mapsto \mathrm{Der}_A(C, M)$ of the category of C-modules.

In particular, $\mathrm{Der}_{C/A} = \mathrm{Hom}_C(\Omega^1_{C/A}, C) = \Omega^{1\,\vee}_{C/A}$ is the dual C-module of $\Omega^1_{C/A}$.

Remark 2.1.3. Conversely, it is not true, in general, that $\Omega^1_{C/A} = \mathrm{Der}^\vee_{C/A}$. For example, if A is an integrally closed domain, and C is a separable extension, the C-module $\Omega^1_{C/A}$ is a torsion module, not necessarily 0 (see [48, 0_{IV}, 20.4.13]).

2.2 Differential rings

From now on, K denotes a field of characteristic zero.

Definition 2.2.1 (Differential rings). *A* differential ring (F, ∂) *over K is a commutative K-algebra F equipped with a K-linear derivation $\partial \in \mathrm{Der}_{F/K}$. A* morphism of differential rings $\iota : (F, \partial) \to (F', \partial')$ *is a K-algebra homomorphism which commutes with the derivations (that is, $\partial'\iota = \iota\partial$).*

We sometimes drop ∂ to lighten notation.

2.2.2. When ι is injective, we say that $(F', \partial')/(F, \partial)$ is an *extension* of differential rings (or, abusively, that F'/F is a differential extension; moreover we write ∂ for ∂').

2.2.3 (Ring of constants). The kernel of ∂ is a sub-K-algebra of F, the *ring of constants*, denoted by F^∂.

If F is a domain, then F^∂ is integrally closed in F; indeed, if $f \in F$ satisfies a monic polynomial equation with coefficients in F^∂, which we may assume of minimal degree, applying ∂ we deduce that $\partial(f) = 0$, that is $f \in F^\partial$.

2.2.4. When F is a field, we say that (F, ∂) is a *differential field*; the ring of constants F^∂ is then a field extension of K, and it is algebraically closed in F.

Examples 2.2.5. F will often be a ring of "functions" in the variable x, such as

(1) a localization of the polynomial ring $K[x]$, e.g., the field $K(x)$ of rational functions,

(2) the ring $K[[x]]$ of formal power series, or its fraction field $K((x)) = K[[x]][\frac{1}{x}]$,

(3) for $K = \mathbb{C}$, the ring $\mathbb{C}\{x\}$ of convergent power series or its fraction field of meromorphic power series $\mathbb{C}(\{x\}) = \mathbb{C}\{x\}[\frac{1}{x}]$,

all equipped with the derivation $\partial_x = \frac{\partial}{\partial x}$ or $\vartheta_x = x\frac{\partial}{\partial x}$. In such situations, K is the field of constants of F.

2.2.6. Since these derivations $\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}$ and their iterates will be very important in the sequel, let us summarize their relations. There exists an upper triangular matrix $T_\mu \in M_\mu(\mathbb{Z})$, with 1's on the diagonal, such that

$$(x\partial_x, x^2\partial_x^2, \ldots, x^{\mu-1}\partial_x^{\mu-1}) = (\vartheta_x, \vartheta_x^2, \ldots, \vartheta_x^{\mu-1})T_\mu.$$

Precisely, if we write

$$(x\partial_x)^s = \sum_{i=0}^{s} c_{s,i} x^i \partial_x^i \quad \text{and} \quad x^s \partial_x^s = \sum_{i=0}^{s} d_{s,i} (x\partial_x)^i$$

then the coefficients $c_{s,i}$ and $d_{s,i}$ are solutions of the linear systems

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 2! & 0 & \cdots & 0 \\ 1 & 3 & 3_2 & 3! & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s & s_2 & s_3 & \cdots & s! \end{pmatrix} \begin{pmatrix} c_{s,0} \\ c_{s,1} \\ c_{s,2} \\ c_{s,3} \\ \vdots \\ c_{s,s} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2^s \\ 3^s \\ \vdots \\ s^s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & 2^3 & \cdots & 2^s \\ 1 & 3 & 3^2 & 3^3 & \cdots & 3^s \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s & s^2 & s^3 & \cdots & s^s \end{pmatrix} \begin{pmatrix} d_{s,0} \\ d_{s,1} \\ d_{s,2} \\ d_{s,3} \\ \vdots \\ d_{s,s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ s! \end{pmatrix}$$

(where $s_i = s(s-1)\cdots(s-i+1)$, see also [30, 1.1]). The coefficients $d_{s,i}$ can be also obtained using the factorization $x^s\partial_x^s = x\partial_x(x\partial_x - 1)\cdots(x\partial_x - s + 1)$.

For future reference, we record the following elementary lemma.

Lemma 2.2.7. *Let $F = R[[x]]$ (resp. $F = R((x))$), where R is a commutative domain. Any derivation of F which preserves xF is an F-linear combination of derivations that preserve xF and commute with ϑ_x.*

Proof. In fact, let D be a derivation as in the lemma. Then $D(x) = xf$ with $f \in F$, and the derivation $D' = D - f\vartheta_x$ has the property that $D'(x) = 0$, and therefore it commutes with ϑ_x. So $D = f\vartheta_x + D'$, as asserted. $\square$

2.3 Equivalence of differential systems

Let us consider a linear differential equation of order μ

$$\partial^\mu y + a_{\mu-1}\,\partial^{\mu-1} y + \cdots + a_1\,\partial y + a_0\, y = 0 \tag{2.3.1}$$

with coefficients a_i in the differential ring F. One usually looks for solutions y of this equation not only in F, but in various differential extensions F'/F with $(F')^\partial = K$. Such solutions form a K-vector space of dimension $\leqslant \mu$: this follows from the wronskian lemma, which asserts that elements $y_1, y_2, \ldots, y_\mu$ of F' are linearly independent over K if and only if the wronskian determinant

$$w(y_1, y_2 \ldots, y_\mu) := \det W(y_1, y_2, \ldots, y_\mu) \tag{2.3.2}$$

is non-zero, where

$$W(y_1, y_2, \ldots, y_\mu) := \begin{pmatrix} y_1 & y_2 & \cdots & y_\mu \\ \partial y_1 & \partial y_2 & \cdots & \partial y_\mu \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{\mu-1} y_1 & \partial^{\mu-1} y_2 & \cdots & \partial^{\mu-1} y_\mu \end{pmatrix}.$$

When the dimension is μ, a basis of solutions is called a *fundamental set of solutions of* (2.3.1).

2.3.3. It is standard to transform (2.3.1) into a linear differential system of order one

$$\partial \vec{y} = G \vec{y}, \tag{2.3.4}$$

where

$$\vec{y} = \begin{pmatrix} y \\ \partial y \\ \vdots \\ \partial^{\mu-1} y \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{\mu-1} \end{pmatrix}. \tag{2.3.5}$$

2.3.6. Let P be an invertible $\mu \times \mu$ matrix with coefficients in F. Then $\vec{y}$ satisfies the linear differential system (2.3.4) if and only if $P\vec{y}$ satisfies the "equivalent" linear differential system

$$\partial \vec{y} = G_P \vec{y}, \tag{2.3.7}$$

where

$$G_P = (\partial P)P^{-1} + PGP^{-1}. \tag{2.3.8}$$

Definition 2.3.9 (Equivalence of differential systems). *The differential systems $\partial \vec{y} = G \vec{y}$ and $\partial \vec{y} = H \vec{y}$ are equivalent over the differential ring (F, ∂) if there exists an invertible matrix $P \in \mathrm{GL}_\mu(F)$ such that $H = (\partial P)P^{-1} + PGP^{-1}$.*

If (2.3.1) admits a fundamental set of solutions $\{y_1, \dots, y_\mu\}$ in some differential extension F'/F, it gives rise to a fundamental set of solutions of the associated differential system, i.e., to an invertible solution matrix Y with entries in F' of the differential system

$$\partial Y = G Y. \tag{2.3.10}$$

The wronskian determinant $w = \det(Y)$ is a solution of

$$\partial w = \mathrm{Tr}(G)\, w. \tag{2.3.11}$$

2.4 Differential modules

We now introduce a more intrinsic way of dealing with equivalence classes of differential systems. Let again K be a field of characteristic zero, and $F = (F, \partial)$ be a differential ring over K.

Definition 2.4.1 (Differential modules over a differential ring). *A* differential module (M, ∇_∂) *over (F, ∂) is a projective F-module M of finite rank, equipped with a K-linear endomorphism ∇_∂ which satisfies the Leibniz rule:*

$$\forall f \in F,\ \forall m \in M,\ \ \nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m).$$

A morphism $(M, \nabla_\partial) \to (M', \nabla'_\partial)$ of differential modules over (F, ∂) *is a F-linear morphism $M \to M'$ making a commutative square with ∇_∂ and ∇'_∂ (one also says that such a morphism is* horizontal*).*

2.4.2. The rank of a differential module over (F, ∂) is the rank μ of the underlying module, which is well defined if F has no zero-divisor. The kernel of ∇_∂ is a K-subspace of M (and even a F^∂-submodule) often denoted by M^{∇_∂}. Following an old differential-geometric tradition (Cartan-Ehresmann), the elements of M^{∇_∂} are called *horizontal*.

We sometimes write ∂ instead of ∇_∂, or simply write M instead of (M, ∇_∂) or (M, ∇) in order to lighten notation.

Remark 2.4.3. If $\partial = 0$, differential modules over (F, ∂) are nothing but finite projective F-modules endowed with an F-linear endomorphism. This case is usually ruled out in the sequel.

For future reference, we also introduce the related notion of F/K-differential module, which takes into account the action of all derivations of F over K.

Definition 2.4.4. *An F/K-*differential module *is a projective F-module of finite rank endowed with an F-linear map*

$$\nabla : \mathrm{Der}_{F/K} \longrightarrow \mathrm{End}_K(M)$$

(using the left structure on the r.h.s.) such that for any $\partial \in \mathrm{Der}_{F/K}$ the pair (M, ∇_∂) is a differential module over (F, ∂). We say that the F/K-differential module (M, ∇) is integrable if ∇ is a Lie-algebra homomorphism, i.e., if $\nabla_{[\partial,\partial']} = [\nabla_\partial, \nabla_{\partial'}]$ for any $\partial, \partial' \in \mathrm{Der}_{F/K}$.

A morphism of F/K-differential modules $(M, \nabla) \to (M', \nabla')$ *is an F-linear morphism $M \to M'$ such that for any $\partial \in \mathrm{Der}_{F/K}$ the morphism induces a morphism of differential modules over (F, ∂).*

2.4.5. Let F'/F be a differential extension of differential rings over K. Any differential module (M, ∇_∂) over (F, ∂) gives rise to a differential module $(M_{F'}, \nabla_\partial)$ over (F', ∂) by setting

$$M_{F'} = M \otimes_F F', \quad \nabla_\partial(m \otimes f') = m \otimes \partial f' + \nabla_\partial(m) \otimes f'. \tag{2.4.6}$$

On the other hand, any differential module (M', ∇'_∂) over (F', ∂) can be considered as a differential module over (F, ∂), which we denote by ${}_F(M', \nabla'_\partial)$, $({}_FM', \nabla'_\partial)$ or simply ${}_FM'$.

2.5 Solutions in a differential extension. Duality

Definition 2.5.1 (Solvability). *A differential module is* trivial *if it is isomorphic to the direct sum of finitely many copies of (F, ∂). A differential module (M, ∇_∂) is said to be* solvable *in the differential extension F'/F, or* trivialized over *F', if $(M_{F'}, \nabla_\partial)$ is trivial.*

Remark 2.5.2. A differential module (M, ∇_∂) over (F, ∂) is solvable in the differential extension F'/F if and only if $(M \otimes_F F')^{\nabla_\partial}$ is a free module over F'^∂ and the canonical morphism

$$(M \otimes_F F')^{\nabla_\partial} \otimes_{F'^\partial} F' \longrightarrow M \otimes_F F'$$

is an isomorphism. If F'^∂ is a field, the first condition is guaranteed since $(M \otimes_F F')^{\nabla_\partial}$ is a vector space over F'^∂.

Moreover, if F' is a differential *field* (so that F'^∂ is a field), the second condition is equivalent to the equality $\dim_{F'^\partial}((M \otimes_F F')^{\nabla_\partial}) = \operatorname{rk} M$ (both equal to $\dim_{F'} M_{F'}$). Indeed, let $m_1, \ldots, m_n \in (M \otimes_F F')^{\nabla_\partial}$ and $a_1, \ldots, a_n \in F'$ satisfy $a_1 m_1 + \cdots + a_n m_n = 0$, and assume n is the minimal length of such a relation (in particular, $a_1 \cdots a_n \neq 0$, and "$n = 0$" means that no such relation exists). If $n > 0$, we may assume that $a_1 = 1$, and applying ∇_∂ to the relation we find a shorter relation, which contradicts the choice of n. We conclude that the canonical morphism is injective, hence an isomorphism.

Example 2.5.3. Differential modules over the differential ring $(K[[x]], \partial_x)$ or $(\mathbb{C}\{x\}, \partial_x)$ are trivial. *In constrast*, differential modules over the differential fields $(K((x)), \partial_x)$ or $(\mathbb{C}(\{x\}), \partial_x)$ are highly non-trivial, and the study of their structure will occupy much of the next five chapters.

2.5.4 (Duals). The dual $(M^\vee, \nabla_\partial^\vee)$ of a differential module (M, ∇_∂) over (F, ∂) is defined by $M^\vee = \operatorname{Hom}_F(M, F)$ (which is also a projective F-module), and $\nabla_\partial^\vee(s) = \partial \circ s - s \circ \nabla_\partial$.

If M is free and ∇_∂ is represented by H in a basis $\mathbf{m}$, then $\nabla_\partial^\vee$ is represented by $-{}^tH$ in the dual basis $\mathbf{m}^\vee$. In fact, one has

$$\nabla_\partial \, \mathbf{m} = \mathbf{m}\, H \quad \text{if and only if} \quad \nabla_\partial^\vee \, \mathbf{m}^\vee = -\mathbf{m}^\vee \, {}^tH,$$

since

$$(\nabla_\partial^\vee m_i^\vee)(m_j) = -m_i^\vee(\nabla_\partial m_j) = -m_i^\vee(\textstyle\sum_\ell h_{j\ell} m_\ell) = -\textstyle\sum_\ell h_{j\ell} m_i^\vee(m_\ell) = -h_{ji}$$

that is, $\nabla_\partial^\vee m_i^\vee = \sum_j (-h_{ji}) m_j^\vee$ (where h_{ij} are the entries of the matrix H).

Definition 2.5.5 (Solutions). *A* solution *of M in a differential extension F'/F is a F-linear morphism $s : M \to F'$ such that*

$$\partial(s(m)) = s(\nabla_\partial(m)), \ \ \forall\, m \in M.$$

Equivalently, this is a horizontal element of $(M^\vee)_{F'} = (M_{F'})^\vee$.

There is a subtlety here which deserves clarification: a differential module of the form (F, ∇_∂) which has a non-zero solution in F need not be trivial in F. For instance, for $F = \mathbb{C}[x, e^x]$, the differential module $(F, \partial_x - 1)$ has the solution e^x, but is non-trivial (because of the lack of e^{-x}). More generally, *the existence of a "full set" of solutions in F' is not enough to guarantee that M is solvable in F': one needs in addition that the wronskian be invertible* in F'.

It is clear that $M^\vee_{F'}$ is trivial if and only if $M_{F'}$ is trivial, i.e., M is solvable in F'.

2.6 Relation between differential modules and differential systems

Let us assume that M is free with ∇_∂ given, in a fixed basis $\mathbf{m} = (m_1, \ldots, m_\mu)$, by the matrix H, i.e., $\nabla_\partial(\mathbf{m}) = \mathbf{m}H = (\sum_i H_{i1}m_i, \ldots, \sum_i H_{i\mu}m_i),$. We set

$$G = {}^tH. \tag{2.6.1}$$

Let $\mathbf{m}^\vee$ be the dual basis. The differential system (2.3.4)

$$\partial\,\vec{y} = G\,\vec{y}$$

is then equivalent to the equation

$$\nabla_\partial^\vee\Big(\sum y_i m_i^\vee\Big) = 0, \tag{2.6.2}$$

i.e., $\vec{y}$ defines a *solution* of M in some differential extension F'/F. If $(F')^\partial = K$, the system (2.3.10): $\partial\,Y = G\,Y$, with $Y \in \mathrm{GL}_\mu(F')$ amounts to: $\mathbf{m}^\vee Y = \big(\sum_i Y_{i1}m_i^\vee, \ldots, \sum_i Y_{i\mu}m_i^\vee\big)$ is a K-basis of solutions of $M_{F'}$.

In another basis of M, say $\mathbf{m}Q = \big(\sum_i Q_{i1}m_i, \ldots, \sum_i Q_{i\mu}m_i\big)$, (where Q is an invertible matrix with entries in F), the matrix of ∇_∂ is given by

$$H_{[Q]} = Q^{-1}\partial(Q) + Q^{-1}HQ\;. \tag{2.6.3}$$

In terms of the invertible matrix $P := {}^tQ$, and of $G = {}^tH$, we have

$$H_P = {}^t(G_{[Q]}). \tag{2.6.4}$$

Indeed,

$$\begin{aligned} H_P = ({}^tG)_P &= (\partial P)P^{-1} + P\,({}^tG)\,P^{-1} \\ &= {}^t(\partial Q)\,{}^tQ^{-1} + {}^tQ\,{}^tG\,{}^tQ^{-1} \\ &= {}^t(Q^{-1}\,\partial Q) + {}^t(Q^{-1}\,G\,Q) = {}^t(G_{[Q]}). \end{aligned}$$

The same formula, with H and G interchanged (but not P and Q), becomes

$${}^t(G_P) = H_{[Q]}\,, \tag{2.6.5}$$

which can also be seen directly as

$${}^t(G_P) = {}^tP^{-1}\,{}^t(\partial P) + {}^tP^{-1}\,{}^tG\,{}^tP = Q^{-1}\,\partial Q + Q^{-1}\,{}^tG\,Q = H_{[Q]}\,.$$

At the level of differential systems, since $(\mathbf{m}Q)^\vee = \mathbf{m}^\vee\;{}^tQ^{-1} = \mathbf{m}^\vee P^{-1}$, this corresponds to the change $\vec{y} \mapsto P\vec{y}$ and fits with formula (2.3.8).

2.6.6. We summarize the relation between differential modules and differential systems as follows. Suppose that the differential module M over the differential ring (F, ∂) is free on the basis $\mathbf{m}$, and so its dual is free on the dual basis $\mathbf{m}^\vee$. Then

$$\nabla_\partial\mathbf{m} = \mathbf{m}H \quad\text{if and only if}\quad \nabla_\partial^\vee\mathbf{m}^\vee = \mathbf{m}^\vee(-{}^tH)\;.$$

A column vector $\vec{y}$ is a solution of M (in some differential extension F'/F) if and only if $\mathbf{m}^\vee\vec{y}$ is a horizontal section of $M_{F'}^\vee$, that is, if and only if $\partial\vec{y} = {}^tH\vec{y}$.

Example 2.6.7. A fundamental solution of the differential system attached to $(M^\vee, \nabla_\partial^\vee)$ and $\mathbf{m}^\vee$ is ${}^tY^{-1}$ (if Y is a fundamental solution of the differential system attached to (M, ∇_∂) and $\mathbf{m}$). In fact, we have

$$\partial Y = GY \quad \text{if and only if} \quad \partial({}^tY^{-1}) = -{}^tG\,{}^tY^{-1}$$

(by 2.5.4, the matrix of $\nabla_\partial^\vee$ is $-{}^tG$).

Example 2.6.8. The hypergeometric differential equation (1.1.1) corresponds to the differential system

$$\partial_x \vec{y} = \begin{pmatrix} 0 & 1 \\ \dfrac{ab}{x(1-x)} & \dfrac{(a+b+1)x-c}{x(1-x)} \end{pmatrix} \vec{y}, \tag{2.6.9}$$

hence to the differential module $M = F^2$ for which ∇_{∂_x} is given, in the canonical basis, by the matrix $H = \begin{pmatrix} 0 & \frac{ab}{x(1-x)} \\ 1 & \frac{(a+b+1)x-c}{x(1-x)} \end{pmatrix}$.

Remark 2.6.10. The advantage of (the more intrinsic notion of) differential modules over differential systems may be illustrated by the following question.

Let F_1 and F_2 be differential subrings of F. Consider a differential system

$$\partial\vec{y} = G_1\vec{y}, \quad G_1 \in M_\mu(F_1),$$

and assume that there exists $P_2 \in \mathrm{GL}_\mu(F)$ such that $(G_1)_{P_2} \in M_\mu(F_2)$. Does there exist $P \in \mathrm{GL}_\mu(F_1)$ such that $(G_1)_P \in M_\mu(F_1 \cap F_2)$?

The answer is *yes* under various natural assumptions, e.g., if F is a field, F_1 is the fraction field of $F_1 \cap F_2$, and F is the composition of F_1 and F_2. These assumptions ensure that $F_1 \otimes_{F_1\cap F_2} F_1 = F_1$ and $F_1 \otimes_{F_1\cap F_2} F_2 = F$. Let $M_1 = F_1^\mu$ be the differential module over F_1 associated to G_1 in the canonical basis, and set $M = (M_1)_F$. By assumption, there is a free sub-F_2-module M_2 of M, of rank μ, stable under ∇_∂, such that $M = (M_2)_F$.

Then $M_1 \cap M_2$ is a $(F_1 \cap F_2)$-module stable under ∇_∂, and the natural morphism $(M_1 \cap M_2) \otimes_{F_1\cap F_2} F_1 \to M_1$ is an isomorphism (cf. e.g., [24, I.2.6, prop. 7] which tells us that $(M_1 \cap M_2) \otimes_{F_1\cap F_2} F_1 = (M_1 \otimes_{F_1\cap F_2} F_1) \cap (M_2 \otimes_{F_1\cap F_2} F_1)$). Thus, if F is a field, there is a μ-uple $\mathbf{m}$ of elements of $M_1 \cap M_2$ which forms a basis of M. It then suffices to take for P the transpose of the matrix of passage from the canonical basis of M to $\mathbf{m}$. In fact, with this choice, $P \in M_\mu(F_1 \cap F_2) \cap \mathrm{GL}_\mu(F) \subseteq \mathrm{GL}_\mu(F_1)$.

2.7 Tensor product and related operations

2.7.1. The tensor product $(M, \nabla_\partial) \otimes (M', \nabla'_\partial)$ of two differential modules over (F, ∂) is defined as $M \otimes_F M'$ equipped with $\nabla_\partial \otimes 1_{M'} + 1_M \otimes \nabla'_\partial$. The fundamental solution matrix of the differential system attached to it, in a basis $\mathbf{m} \otimes \mathbf{m}'$, is the Kronecker tensor product of fundamental solution matrices $Y \otimes Y'$.

This allows to define the n^{th} tensor power $(M, \partial)^{\otimes n}$, and, taking appropriate submodules with respect to the symmetric group, the n^{th} symmetric power $S^n(M, \partial)$ and exterior power $\bigwedge^n(M, \partial)$, respectively. A fundamental solution of the differential system attached to $S^n(M, \partial)$ is

$$Y_{IJ}^{(n)} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{h=1}^{n} Y_{i_h, j_{\sigma(h)}}$$

(indexed by non-increasing sequences $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_n)$ of integers between 1 and μ)[2].

2.7.2. Notice that the "classical" definition of tensor product of differential operators (as the differential operator of minimal order whose space of solutions is the tensor product of the spaces of solutions of the given differential operators) does not "commute" with the corresponding notion for differential modules, because in general the tensor product of cyclic vectors is not a cyclic vector (see below). For example the tensor product of ∂^2 with itself is ∂^3, while the dimension of the tensor product of the corresponding differential modules is 4. A similar remark holds for the symmetric product. For example, the symmetric product of ∂^3 with itself is ∂^5, while the dimension of the corresponding symmetric product of differential modules is 4.

2.7.3. We write $\mathrm{Hom}((M, \nabla_\partial), (M', \nabla'_\partial))$ (resp. $\mathrm{End}((M, \nabla_\partial))$) instead of $(M^\vee, \nabla^\vee_\partial) \otimes (M', \nabla'_\partial)$ (resp. $(M^\vee, \nabla^\vee_\partial) \otimes (M, \nabla_\partial)$); its underlying F-module is $\mathrm{Hom}_F(M, M')$ (resp. $\mathrm{End}_F(M)$).

2.8 Trace morphism

2.8.1. Let (F', ∂) be a differential extension of (F, ∂) such that F' is a projective F-module of finite rank ν (so that (F', ∂) is a differential module over (F, ∂) of rank ν). The trace morphism $\mathrm{Tr} = \mathrm{Tr}_{F'/F} : F' \to F$ extends to a morphism $\mathrm{Tr} : (F', \partial) \to (F, \partial)$ of differential modules over (F, ∂). To check this, we first localize F so that we may assume that the F-module F' is freely generated by elements $(x_1, \ldots, x_\nu)$ of F'. We write $\partial(x_1, \ldots, x_\nu) = (x_1, \ldots, x_\nu)A_\partial$, for a matrix

[2] The matrix $G^{(n)} = \partial(Y_{IJ}^{(n)})(Y_{IJ}^{(n)})^{-1}$ of this system is related to the matrix $G = \partial(Y)Y^{-1}$ by the rule

$$G_{IJ}^{(n)} = \begin{cases} \sum_{h=1}^{n} G_{i_h, j_h}, & \text{if } I = J, \\ n_i G_{i,j}, & \text{if } I \text{ and } J \text{ differ only by one coordinate } (i \in I, j \in J), \\ & \quad \text{denoting by } n_i \text{ the number of times } i \text{ occurs in } I, \\ 0, & \text{otherwise.} \end{cases}$$

$A_\partial \in M_\nu(F)$, and, for any $x \in F'$, $(xx_1, \ldots, xx_\nu) = (x_1, \ldots, x_\nu)P_x$, with $P_x \in M_\nu(F)$. Then $\mathrm{Tr}(x) = \mathrm{Tr}(P_x)$. We have

$$\partial(xx_1, \ldots, xx_\nu) = \partial((x_1, \ldots, x_\nu)P_x) = (x_1, \ldots, x_\nu)(A_\partial P_x + \partial(P_x)),$$

but, on the other hand,

$$\begin{aligned}\partial(xx_1, \ldots, xx_\nu) &= ((\partial x)x_1, \ldots, (\partial x)x_\nu) + x(x_1, \ldots, x_\nu)A_\partial \\ &= (x_1, \ldots, x_\nu)(P_{\partial x} + P_x A_\partial).\end{aligned}$$

We conclude that

$$\partial(P_x) = P_{\partial x} + P_x A_\partial - A_\partial P_x$$

and therefore that

$$\partial\,\mathrm{Tr}(x) = \partial\,\mathrm{Tr}(P_x) = \mathrm{Tr}\,\partial(P_x) = \mathrm{Tr}(P_{\partial x}) = \mathrm{Tr}(\partial x).$$

The composition of the morphisms of differential modules over (F, ∂)

$$(F, \partial) \longrightarrow (F', \partial) \xrightarrow{\mathrm{Tr}} (F, \partial)$$

is multiplication by ν.

2.8.2. Any differential module (M', ∇'_∂) of rank μ over (F', ∂) can be considered as a differential module of rank $\mu\nu$ over (F, ∂); it that capacity, we denote it by ${}_F(M', \nabla'_\partial)$. In particular, for $(M', \nabla'_\partial) = (M, \nabla_\partial)_{F'}$, there are morphisms of differential modules

$$(M, \nabla_\partial) \longrightarrow {}_F((M, \nabla_\partial)_{F'}) = (M, \nabla_\partial) \otimes (F', \partial) \xrightarrow{1\otimes\mathrm{Tr}} (M, \nabla_\partial)$$

whose composition is multiplication by ν.

2.8.3 (**Algebraic extensions as differential extensions).** Any finite extension F' of a differential field F (of characteristic 0) is in a canonical way a differential ring, i.e., ∂ extends in a unique way. Given an element $y \in F'$ of degree μ, with minimal polynomial $P \in F[Y]$, one may compute explicitly the differential equation $\partial^{\mu-1}y + a_{\mu-2}\partial^{\mu-2}y + \cdots + a_1\partial y + a_0 y = a_{-1}$ satisfied by y using a combinatorial argument due to Comtet [33], as follows.

We differentiate the identity $P(y) = 0$, get an expression $\partial y = R_1(y)$ for some rational function, and rewrite the expression

$$\partial y - a_{1,0} - a_{1,1}y = a_{1,2}y^2 + \cdots + a_{1,\mu-1}y^{\mu-1}, \quad a_{0,i} \in F$$

(using $F(y) = F[y]$).

We differentiate this expression, and rewrite it in the form

$$\partial^2 y - a_{2,0} - a_{2,1}y = a_{2,2}y^2 + \cdots + a_{2,\mu-1}y^{\mu-1}, \quad a_{1,i} \in F.$$

(using $F(y) = F[y]$ and the previous expression for ∂y). We continue this process for $\mu - 1$ steps in the same way:

$$\partial^{\mu-1} y - a_{\mu-1,0} - a_{\mu-1,1} y = a_{\mu-1,2} y^2 + \cdots + a_{\mu-1,\mu-1} y^{\mu-1}, \quad \text{with } a_{\mu-1,i} \in F.$$

From the $\mu - 1$ equalities we can now eliminate the terms $y^2, \ldots, y^{\mu-1}$ using the determinantal equation

$$\det \begin{pmatrix} \partial y - a_{1,0} - a_{1,1} y & a_{1,2} & \cdots & a_{1,\mu-1} \\ \partial^2 y - a_{2,0} - a_{2,1} y & a_{2,2} & \cdots & a_{2,\mu-1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{\mu-1} y - a_{\mu-1,0} - a_{\mu-1,1} y & a_{\mu-1,2} & \cdots & a_{\mu-1,\mu-1} \end{pmatrix} = 0.$$

The Laplace expansion with respect to the first column gives the linear differential equation of the assertion. For an analysis of this and other similar algorithms, see [22]; and for an earlier algorithm, see [52].

3 Back to differential equations: cyclic vectors

A differential operator gives rise to a differential module naturally endowed with a cyclic vector, that is, a vector with the property that it generates a basis by iterated application of the derivation. In this section, we investigate the inverse problem: given a differential module, do there exists a cyclic vector, so that the module is associated to some differential operator?

The main theorem asserts the existence of a cyclic vector under some condition on the differential ring. We include here also some technical result about adjoint operators, which will be used in the next chapters.

We recall that the base field K has characteristic zero and $F = (F, \partial)$ is a differential ring over K.

3.1 Differential operators

3.1.1. The K-algebra of *differential operators* attached to the differential ring (F, ∂) is the non-commutative polynomial algebra $F\langle\partial\rangle$ with the commutation rule

$$[\partial, f] = \partial(f).$$

Notice that $F\langle\partial\rangle$ has a structure of F-bimodule (using the natural left and right product with elements of F). Every differential operator can be written in a unique way as $L = \sum_{i=0}^{\mu} a_i \partial^i$ (its degree in ∂ is called the *order* of L).

3.1.2. There is a natural K-linear action of $F\langle\partial\rangle$ on F. More generally, any differential module (M, ∇_∂) over (F, ∂) gives rise to a K-linear action of $F\langle\partial\rangle$ on M.

3.1.3. If $\partial \neq 0$ (e.g., as in examples 2.2.5) this action is faithful, that is, the natural map $F\langle\partial\rangle \to \mathrm{End}_K(F)$ is injective. In fact, there exists $x \in F$ such that $\partial(x) \neq 0$ (this element is transcendental over F^∂) and we may suppose $\partial(x) = 1$ (using the derivation $\partial_x = \frac{1}{\partial x}\partial$), since there is a canonical identification between $F\langle\partial\rangle$ and $F\langle\partial_x\rangle$). Then the coefficients of a differential operator $L = \sum_{i=0}^{\mu} a_i\partial^i$ are completely determined by the values of L as endomorphism of F on the elements x^i (for $i = 0, \dots, \mu$) by the (triangular) linear system

$$L(x^j) = \sum_{i=0}^{j} a_i (j!/i!) x^{j-i}.$$

In that case ($\partial \neq 0$), any K-linear action of $F\langle\partial\rangle$ on a projective F-module M of finite rank gives rise, conversely, to a differential module over (F, ∂).

3.1.4. There is a more intrinsic notion of differential operators for an extension $A \to C$ of commutative rings with unit (a general and extended discussion can be found in [48] and [18]). Restricting ourselves to the case of a C-module M, the C/A-differential operators of M can be defined in the following three equivalent ways:

(recursive definition) an element $\phi \in \mathrm{End}_A(M)$ belongs to $\mathrm{Diff}^n_{C/A}(M)$ if for any $c \in C$ we have $[\phi, c] \in \mathrm{Diff}^{n-1}_{C/A}(M)$, starting with $\mathrm{Diff}^0_{C/A}(M) = \mathrm{End}_C(M)$.

(combinatorial definition) an element $\phi \in \mathrm{End}_A(M)$ belongs to $\mathrm{Diff}^n_{C/A}(M)$ if for any $c_1, \dots, c_{n+1} \in C$ we have

$$\sum_{H \subseteq \{1,\dots,n+1\}} (-1)^{\mathrm{card}(H)} c_H \phi(c_{\complement H} m) = 0,$$

where $c_H = \prod_{i\in H} c_i$ (and $c_{\complement H} = \prod_{i \notin H} c_i$).

(Grothendieck linearization) an element $\phi \in \mathrm{End}_A(M)$ belongs to $\mathrm{Diff}^n_{C/A}(M)$ if it admits a factorization through a C-linear map $\overline{\phi}$

$$M \xrightarrow{\mathrm{j}^n_{C/A} \otimes_C \mathrm{id}_M} \mathrm{P}^n_{C/A} \otimes_C M \xrightarrow{\overline{\phi}} M$$

where $\mathrm{P}^n_{C/A} = (C \otimes_A C)/I^{n+1}$, for $I = \mathrm{Ker}(C \otimes_A C \to C)$, is the ring and C-bimodule of *A-linear jets of order n of C* (or of *n-th order principal parts of C/A*) and the right C-linear map $\mathrm{j}^n_{C/A} : C \to \mathrm{P}^n_{C/A}$ induced by $x \mapsto 1 \otimes x$ (for any $x \in C$) is called the *universal A-linear differential operator of order n on C*.

The equivalence of these definitions can be understood using the following two equalities:

$$\sum_{H \subseteq \{1,\dots,n+1\}} (-1)^{\mathrm{card}(H)} c_H \phi(c_{\complement H} m) = \sum_{K \subseteq \{1,\dots,n\}} (-1)^{\mathrm{card}(K)} c_K [\phi, c_{n+1}] \phi(c_{\complement K} m)$$

(this shows the relation between the recursive definition and the combinatorial one) and

$$\sum_{H\subseteq\{1,\dots,n+1\}} (-1)^{\mathrm{card}(H)} c_H \phi(c_{\complement H} m) = (\mathrm{id}_C \otimes \phi)\left(\prod_{i=1}^{n+1} (1\otimes c_i - c_i \otimes 1)m\right)$$

(this shows the relation between the combinatorial definition and the Grothendieck linearization).

3.1.5. The increasing union $\mathrm{Diff}_{C/A}(M)$ of the $\mathrm{Diff}^n_{C/A}(M)$ admits a structure of filtered ring by composition, with commutative associated graded ring. This can be seen directly with the recursive definition: we reason by induction on $n+m$, the case $n=m=0$ being trivial. So, let $\phi \in \mathrm{Diff}^n_{C/A}(M)$, $\psi \in \mathrm{Diff}^m_{C/A}(M)$, and $c \in C$. Then $[\phi\circ\psi, c] = \phi\circ[\psi, c] + [\phi, c]\circ\psi$, from which the statement follows. The construction of the ring structure using the definition via Grothendieck linearization is more subtle, and will be not needed in the sequel.

In the case $M = C$ we will use the notations $\mathrm{Diff}^n_{C/A}$ and $\mathrm{Diff}_{C/A}$. In particular $\mathrm{Diff}^n_{C/A} = \mathrm{P}^{n\,\vee}_{C/A}$ (C-dual of principal parts of order n).

In general, if (F,∂) is a differential field over K, the ring $\mathrm{Diff}_{F/F^\partial}$ does not coincide with $F\langle\partial\rangle$. For example, if $F = \mathbb{C}(x, e^x)$ endowed with $\partial = \partial_x$, we have $F^\partial = \mathbb{C}$, so that $\mathrm{Diff}_{F/F^\partial}$ is much bigger that $F\langle\partial\rangle$.

Proposition 3.1.6. *Let (F,∂) be a differential field of characteristic 0 and assume that F is of transcendence degree 1 over F^∂. Then the canonical morphism $F\langle\partial\rangle \to \mathrm{Diff}_{F/F^\partial}$ is an isomorphism of rings.*

Proof. As usual, let $x \in F$ such that $\partial(x) = 1$. Then x is a transcendental element over $\mathbb{F}^\partial$ and by hypothesis F is an algebraic extension of $F^\partial(x)$. In particular, every differential operator is completely determined by its values on $F^\partial(x)$. More precisely, from the recursive or the combinatorial definition it follows that $\phi \in \mathrm{Diff}^n_{F/F^\partial}$ is determined by its values on x^i for $i = 0, 1, \dots, n$.

In order to prove the surjectivity of the map, let $\phi \in \mathrm{Diff}^n_{F/F^\partial}$ be a differential operator of order $\leqslant n$. We may construct a differential polynomial $L = \sum_{i=1}^n a_i\partial^i$ satisfying the system

$$\phi(x^i) = \sum_{i=1}^{j} a_i (j!/i!) x^{j-i}$$

for $i = 0, 1, \dots, n$. Since ϕ and L coincide on $F^\partial(x)$, they coincide of F, so that ϕ belongs to (the image of) $F\langle\partial\rangle$. □

3.2 Cyclic vectors

3.2.1. In the sequel, we assume that the action of $F\langle\partial\rangle$ on F is faithful. Let $L = \sum_{i=0}^{\mu} a_i\partial^i$ be a differential operator of order μ, with leading coefficient $a_\mu = 1$.

Then

$$M = F\langle\partial\rangle / F\langle\partial\rangle L \tag{3.2.2}$$

is a differential module of rank μ. Its canonical basis is

$$(m := [1],\, [\partial] = \nabla_\partial(m), \dots,\, [\partial^{\mu-1}] = \nabla_\partial^{\mu-1}(m)). \tag{3.2.3}$$

The matrix of ∇_∂ in this basis is

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{\mu-1} \end{pmatrix}, \tag{3.2.4}$$

i.e., the transpose of the matrix of the differential system (2.3.5) attached to the differential equation $Ly = 0$.

Definition 3.2.5 (**Cyclic vectors and basis**). *Let (M, ∇_∂) be a differential module over the differential ring (F, ∂). A* cyclic element (*or* cyclic vector) *of M is an element $m \in M$ such that*

$$\mathbf{m} = (m,\, \nabla_\partial m, \dots,\, \nabla_\partial^{\mu-1} m) \tag{3.2.6}$$

is an F-basis of M. Such a basis is called a cyclic basis*. In this situation we have*

$$\nabla_\partial^{\mu} m = -a_{\mu-1}\nabla_\partial^{\mu-1} m - \cdots - a_1 \nabla_\partial m - a_0 m$$

for uniquely defined $a_0, a_1, \dots, a_{\mu-1} \in F$. We say that the differential operator

$$L = \partial^{\mu} + a_{\mu-1}\partial^{\mu-1} + \cdots + a_1\partial + a_0 \in F\langle\partial\rangle \tag{3.2.7}$$

is the monic differential operator attached to (M, ∇_∂) via its cyclic vector m*.*

In the situation of (3.2.2), L is the monic differential operator attached to (M, ∇_∂) via its cyclic vector [1]. Notice that the action of ∇_∂ on the cyclic basis (3.2.3) is given by the matrix H (3.2.4).

Remark 3.2.8. In the setting of the definition, the canonical map $F\langle\partial\rangle/F\langle\partial\rangle L \to M$ sending the generator [1] to the cyclic vector m is an isomorphism of differential modules over (F, ∂). Moreover, the action of ∇_∂ on M corresponds to the left multiplication by ∂ sending [1] to $[\partial]$.

Remark 3.2.9. If F is a field, then $m \in M$ is a cyclic vector if and only if the family $\nabla_\partial^i(m)$ (for $i \geqslant 0$) generates M. Notice that this condition is *not sufficient* if F is only a differential ring. For example, let $F = K[x]$, with the derivation $\partial = \frac{\partial}{\partial x}$.

Then, for the trivial differential module F a vector $f \neq 0$ is a cyclic vector if and only if $f \in K$, but the family $\partial^i(f)$ $(i \geqslant 0)$ generates F for any f $(\neq 0)$.

Here is an example where there is no cyclic vector. Let us set, as before, $F = K[x]$, and set $M = F.1 \oplus F.e^{x^2}$ (generated by two elements s_1, s_2 such that $\nabla_\partial(s_1) = 0$ and $\nabla_\partial(s_2) = 2xs_2$). Then any vector $m = f.1 + g.e^{x^2}$ with $fg \neq 0$ has the property that the family $\nabla^i_\partial(m)$ generates M. But M does not admit cyclic vectors, since the determinant of $\left(\begin{smallmatrix} f & g \\ \partial f & \partial g + 2xg \end{smallmatrix}\right)$, while being not zero, is never invertible in F.

For future reference, we insert the following lemma (taken from [13] or [77, III.1.6]).

Lemma 3.2.10. *Let (F, ∂) be a differential field, and let*

$$0 \longrightarrow M_1 \xrightarrow{\ i\ } M \xrightarrow{\ p\ } M_2 \longrightarrow 0$$

be an exact sequence of differential modules over (F, ∂). Suppose that M_1, M, M_2 admit cyclic vectors and let $L \in F\langle\partial\rangle$ be the differential operator associated to a cyclic vector of M. Then there exist differential operators $L_1, L_2 \in F\langle\partial\rangle$ such that $L = L_1L_2$ and M_i is isomorphic to $F\langle\partial\rangle/F\langle\partial\rangle L_i$ for $i = 1, 2$ (as differential modules over (F, ∂)).

Proof. The lemma follows from the fact that the ring of differential operators over a differential field admits a right (resp. left) division algorithm w.r.t. the order as differential operator, so that the left (resp. right) ideals are principal (see [88, 2.1]). Let v be a cyclic vector for M with associated differential operator L, and let $v_2 = p(v)$ its image in M_2, which is a cyclic vector for M_2. Let define $L_2 \in F\langle\partial\rangle$ as the differential operator associated to v_2. Then $L = L_1L_2$ for some $L_1 \in F\langle\partial\rangle$. From the exact sequence

$$0 \longrightarrow F\langle\partial\rangle/F\langle\partial\rangle L_1 \xrightarrow{\ L_2\ } F\langle\partial\rangle/F\langle\partial\rangle L \longrightarrow F\langle\partial\rangle/F\langle\partial\rangle L_2 \longrightarrow 0$$

(where the first map is left multiplication by L_2) we have $M_1 \cong F\langle\partial\rangle/F\langle\partial\rangle L_1$. □

Lemma 3.2.11. *Let $\mathbf{n} = (n_0, n_1, \ldots, n_{\mu-1})$ be the dual basis of the cyclic basis* $\mathbf{m}$ *in (3.2.6), so that $\langle n_i, \nabla^j_\partial m\rangle = \delta_{i,j}$ (for all $i, j = 0, 1, \ldots, \mu-1$). The last vector $n_{\mu-1}$ of the dual basis is a cyclic vector for $(M^\vee, \nabla^\vee_\partial)$. The differential operator attached to $(M^\vee, \nabla^\vee_\partial)$ via this cyclic vector is*

$$L^\vee = (-1)^\mu \sum_{i=0}^{\mu} (-1)^i \partial^i a_i \in F\langle\partial\rangle\,, \tag{3.2.12}$$

and in particular we have an isomorphism $F\langle\partial\rangle/F\langle\partial\rangle L^\vee \to M^\vee$. The monic differential operator $L^\vee$ is called the adjoint of L w.r.t. the derivation ∂*.*

Proof (*after* [63]). The action of $\nabla_\partial^\vee$ on the dual basis is given by the matrix $-{}^tH$ (see 2.5.4), namely

$$\nabla_\partial^\vee(n_0, n_1, \dots, n_{\mu-1}) = (n_0, n_1, \dots, n_{\mu-1}) \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_0 & a_1 & a_2 & \cdots & a_{\mu-1} \end{pmatrix}.$$

We prove that the vectors $(\nabla_\partial^\vee)^i n_{\mu-1}$ generate $M^\vee$. From the explicit expressions (for $i = 1, \dots, \mu-1$)

$$\nabla_\partial^\vee(n_0) = a_0 n_{\mu-1} \quad \text{and} \quad \nabla_\partial^\vee(n_i) = -n_{i-1} + a_i n_{\mu-1} \tag{3.2.13}$$

we get

$$n_{\mu-i-1} = (-1)^i \Big((\nabla_\partial^\vee)^i - (\nabla_\partial^\vee)^{i-1} a_{\mu-1} + \cdots + (-1)^i a_{\mu-i} \Big) n_{\mu-1}$$

and conclude that $n_{\mu-1}$ is a cyclic vector for $M^\vee$. For $i = \mu - 1$ the last formula gives

$$n_0 = (-1)^{\mu-1} \Big((\nabla_\partial^\vee)^{\mu-1} - (\nabla_\partial^\vee)^{\mu-2} a_{\mu-1} + \cdots + (-1)^{\mu-1} a_1 \Big) n_{\mu-1}$$

so that $0 = \nabla_\partial^\vee n_0 - a_0 n_{\mu-1} = (-1)^{\mu-1} \Big(\sum_{i=0}^{\mu} (-1)^i \nabla_\partial^i a_i \Big) n_{\mu-1}$. □

We also record, for future reference in Chapter VIII, the following remark (from [78]).

Lemma 3.2.14. *Let (M, ∇_∂) be a differential module over the differential ring (F, ∂) with a cyclic vector m and attached monic differential operator L as in (3.2.7). Let $(M^\vee, \nabla_\partial^\vee)$ be the dual differential module. We consider the cyclic basis $\mathbf{m} = (m, \nabla_\partial m, \dots, \nabla_\partial^{\mu-1} m)$ of (3.2.6) and the dual basis $\mathbf{n} = (n_0, n_1, \dots, n_{\mu-1})$ of $(M^\vee, \nabla_\partial^\vee)$. The diagram of F-modules and F^∂-linear maps*

$$\begin{tikzcd} F \arrow[r, "L"] \arrow[d] & F \arrow[d] \\ M^\vee = \bigoplus_{k=0}^{\mu-1} F n_k \arrow[r, "\nabla_\partial^\vee"'] & \bigoplus_{k=0}^{\mu-1} F n_k = M^\vee \end{tikzcd} \tag{3.2.15}$$

where the vertical arrows are

$$\eta \longmapsto \sum_{k=0}^{\mu-1} \partial^k(\eta)\, n_k = \mathbf{n} \begin{pmatrix} \eta \\ \partial\eta \\ \vdots \\ \partial^{\mu-1}\eta \end{pmatrix} \quad \text{and} \quad \eta \longmapsto \eta\, n_{\mu-1} = \mathbf{n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta \end{pmatrix},$$

respectively, commutes. Moreover, the diagram (3.2.15) *induces an isomorphism between the kernel* (*resp. cokernel*) *of* Λ *and the kernel* (*resp. cokernel*) *of* $\nabla^\vee_\partial$.

Proof. Using the expressions 3.2.13 in the proof of the previous lemma, we have

$$\nabla^\vee_\partial\Big(\sum_{k=0}^{\mu-1}\partial^k(\eta)\,n_k\Big)=\sum_{k=0}^{\mu-1}\partial^{k+1}(\eta)\,n_k+\sum_{k=0}^{\mu-1}\partial^k(\eta)\,\nabla^\vee_\partial(n_k)=L(\eta)n_{\mu-1}$$

for any $\eta\in F$, so that the diagram commutes. More generally, using the same formulas, we have

$$\nabla^\vee_\partial\Big(\sum_{k=0}^{\mu-1}\xi_k\,n_k\Big)=\sum_{k=0}^{\mu-2}\big(\partial(\xi_k)-\xi_{k+1}\big)\,n_k+\Big(\partial(\xi_{\mu-1})+\sum_{k=0}^{\mu-1}\xi_k a_k\Big)n_{\mu-1}$$

for all $\xi_0,\dots,\xi_{\mu-1}\in F$, from which we deduce that the diagram is cartesian and cocartesian in the category of abelian groups, and the assertion on kernels and cokernels follows. $\square$

Corollary 3.2.16. *Let* $M=F\langle\partial\rangle/F\langle\partial\rangle L$ *be the cyclic differential module of Definition* 3.2.2. *Then*

$$\mathrm{Ker}(\nabla_\partial)\cong\mathrm{Ker}(L^\vee:F\to F)\ \ \textit{and}\ \ \mathrm{Coker}(\nabla_\partial)\cong\mathrm{Coker}(L^\vee:F\to F)=F/L^\vee F\ ,$$

where $L^\vee$ *denotes the adjoint differential operator of* L *w.r.t.* ∂ (*see* 3.2.11).

3.3 Construction of cyclic vectors

The following classical lemma expresses the fact that linear differential systems are equivalent to linear differential equations, at the cost of introducing some extra “apparent” singularities, i.e., of further localizing F. We assume that $\partial\neq 0$.

Lemma 3.3.1. *Any differential module over a differential* field *has a cyclic element.*

We will deduce this result from a more precise variant: over a local algebra F of functions such as $K[[x]]$ or $\mathbb{C}\{x\}$, it turns out that linear differential systems are equivalent to linear differential equations *without inverting* x, *provided* $x=0$ *is not a singularity of the system.* This is formalized by the following (effective) lemma of the cyclic vector, due to N. Katz (see [64]).

Lemma 3.3.2. *Let* (F,∂) *be a differential ring over* K. *We assume that* F *is* local, *with maximal ideal containing an element* $x\in F$ *such that* ∂x *is invertible* (*in* F). *Then any differential module over* (F,∂) *has a cyclic element.*

Proof. Using $\partial(x)^{-1}\partial$ instead of ∂ we may assume that $\partial(x)=1$ Since F is local, the projective module M is free. Let us choose a basis $\mathbf{m}=(m_1,\dots,m_\mu)$ of M, and show that

$$m=m(\mathbf{m},x):=\sum_{i=0}^{\mu-1}\frac{x^i}{i!}\sum_{j\leqslant i}(-1)^j\tbinom{i}{j}\nabla^j_\partial(m_{i-j+1})$$

is a cyclic vector. For any $\nu < \mu$, one then has

$$\nabla_\partial^\nu m = \sum_{i=0}^{\mu-1} \frac{x^i}{i!} \sum_{j \leqslant i} (-1)^j \tbinom{i}{j} \nabla_\partial^j (m_{i-j+\nu+1})$$

as is seen by induction:

$$\begin{aligned}\nabla_\partial^{\nu+1} m &= \sum_{i=0}^{\mu-1} \nabla_\partial \Big[\frac{x^i}{i!} \sum_{j \leqslant i} (-1)^j \tbinom{i}{j} \nabla_\partial^j (m_{i-j+\nu+1})\Big] \\ &= \sum_{i=0}^{\mu-1} \frac{x^i}{i!} \Big[m_{i+\nu+2} + \sum_{j=1}^{i+1} \Big((-1)^j \tbinom{i+1}{j} + (-1)^{j-1} \tbinom{i}{j-1}\Big) \nabla_\partial^j (m_{i-j+\nu+2})\Big] \\ &= \sum_{i=0}^{\mu-1} \frac{x^i}{i!} \sum_{j \leqslant i} (-1)^j \tbinom{i}{j} \nabla_\partial^j (m_{i-j+\nu+2}).\end{aligned}$$

It follows that $\nabla_\partial^\nu m \equiv m_{\nu+1} \mod x$, hence that $(m, \nabla_\partial(m), \dots, \nabla_\partial^{\mu-1}(m))$ is a basis of M by Nakayama's lemma. □

Remark 3.3.3. In the same paper, Katz proposes a second "lemma of the cyclic vector" with slightly different assumptions: one drops the assumption that F is local, but assumes that F^∂ contains a field of cardinality $> \mu(\mu-1)$; one fixes elements $\alpha_0, \dots, \alpha_{\mu(\mu-1)} \in F^\partial$ such that $\alpha_i - \alpha_j$ is a unit whenever $i \neq j$; the statement is that locally on $\operatorname{Spec} F$, one of the vectors $m(\mathbf{m}, x - \alpha_i)$ is cyclic.

However, this statement is false: indeed, take $F = k[[y]]((x)) = k[[x,y]][\frac{1}{x}]$, $\partial = \frac{d}{dx}$, and $M = k[[x,y]][\frac{1}{x}] \oplus k[[x,y]][\frac{1}{x}]e^{y/x}$. Then M admits no cyclic vector around the closed point $y = 0$ of F, because for any $n = (a, be^{y/x})$, one has $n \wedge \partial n = [a\partial b - b\partial a - \frac{aby}{x^2}]e^{y/x}$, but $a\partial b - b\partial a - \frac{aby}{x^2}$ is not a unit at $y = 0$ since its term of lowest x-order is $\frac{aby}{x^2}$, which is divisible by y; cf. also 18.1.

To deduce Lemma 3.3.1 from Lemma 3.3.2 we will use the following result (see [70, Ch.II,11]).

Lemma 3.3.4 (Differential Zariski lemma). *Let F/E be a differential extension of differential fields over K. If F is finitely generated as a differential ring over E, then it is an algebraic extension of E. In particular, if F is a differential field finitely generated as a differential ring over F^∂, then $F = F^\partial$.*

Proof. If we suppose that the cardinality of the field E is more than countable, there is an elementary proof comparing the dimensions as E-vector spaces: since F is finitely generated as a differential ring over E, then $\dim_E F$ is at most countable. On the other hand, if F contains a transcendental element over E, the family $(X - \alpha)^{-1}$ for $\alpha \in E$ is in F and it is linearly independent over E, so that $\dim_E F$ is strictly bigger than countable, a contradiction.

To have a proof without additional assumptions, we follow an argument of Quillen using the generic flatness lemma. Since F is finitely generated as a differential ring over E, we have that F is finitely generated as module over $E\langle\partial\rangle$, with generators $u_1,\dots,u_r$. For any $u\in F$, let $E[\partial^{\cdot}u]$ be the differential ring generated by E and u (it is exactly the ring generated over E by $\partial^i u$ for $i=0,1,2,\dots$) and denote by $E(\partial^{\cdot}u)$ its fraction field. We have that F is a finitely generated as $E[\partial^{\cdot}u]\langle\partial\rangle$-module. Now we may filter $E[\partial^{\cdot}u]\langle\partial\rangle$ by the order of differential operators, and F by the induced filtration, starting with $E[u_1,\dots,u_r]$ in degree 0. Then $\mathrm{gr}F$ is a finitely generated $\mathrm{gr}E[\partial^{\cdot}u]\langle\partial\rangle$-module (which is a commutative ring), and $\mathrm{gr}E[\partial^{\cdot}u]\langle\partial\rangle$ is a finitely generated $E[\partial^{\cdot}u]$-algebra. Therefore, by the generic flatness lemma, there exists $P\in E[\partial^{\cdot}u]$ such that $(\mathrm{gr}F)_P=\mathrm{gr}(F_P)$ is a free $E[\partial^{\cdot}u]_P$-module. As a consequence, F_P is a free $E[\partial^{\cdot}u]_P$-module. Since F is a field, so an $E(\partial^{\cdot}u)$-vector space, we have that $F=F_P$ is an $E(\partial^{\cdot}u)$-vector space, hence $E[\partial^{\cdot}u]_P=E(\partial^{\cdot}u)$. From the last equality it follows that u satisfies a differential equation over E (let us consider $\partial^m u$ of order bigger than any order of derivation appearing in the polynomial P: then the multiplicative inverse of $\partial^m u$ belongs to $E(\partial^{\cdot}u)$, hence a non-trivial equation $P^s=\partial^m uQ$), so that the transcendence degree of $E[\partial^{\cdot}u]$ over E is finite.

Applying the previous argument for all (differential) generators u_i of F over E, we have that the transcendence degree of F over E is finite, and the classical Zariski lemma applies. □

Proof of 3.3.1. We deduce Lemma 3.3.1 from Lemma 3.3.2 as follows. Let $H\in M_\mu(F)$ be a matrix of ∇_∂ in some basis of M. Let F_H be the smallest differential sub-F^∂-algebra of F which contains the entries of H (and their derivatives). Notice that $F_H\neq F$ by the differential Zariski lemma, since F is a field ($\neq F^\partial$).

Let F_0 be a differential subring of F which contains F_H and a non-constant element $x\in F$ which is non-invertible in F_0 (such a pair (F_0,x) exists since $\partial\neq 0$, so that F contains transcendental elements w.r.t. F^∂, and $F_H\neq F$).

We may freely replace ∂ by its multiple $\partial_x=\frac{1}{\partial x}\partial$: since F is a field, it is easy to see that a cyclic vector for M with respect to ∂_x is also a cyclic vector with respect to ∂. Then the localization of F_0 at x is a differential ring F_1 over K with respect to ∂_x. Moreover, $M=M_1\otimes_{F_1}F$, where M_1 is the differential module over F_1 defined by the matrix H. Lemma 3.3.2 then applies to M_1: we get a cyclic vector for M_1, hence also for M. □

Remark 3.3.5. For the proof of 3.3.1 (from 3.3.2) we may also use the differential Zariski lemma under the assumption that F is uncountable, in which case its proof is elementary. In fact, we can extend F with (arbitrary many) differentially independent variables. The constructions of the proof will depend only on a finite number of added variables, and a generic specialization permits to conclude (for example, to be a cyclic vector is a linearly independence condition, equivalent to the non-vanishing of a certain determinant, which depends on a finite number of variables and is different from zero for a generic specialization of those variables).

3.3.6. In the literature there are many other proofs of the cyclic vector theorem, either by contradiction (for example Deligne [35], Dwork [38]), or constructive, or algorithmic (see [31] for a review, and also [21]).

Chapter II

Connections on algebraic varieties

Introduction

Most of this book deals with differential systems in a geometric context, in several variables. In order to give such systems an intrinsic meaning, two languages are at disposal, which are in principle equivalent: integrable connections and $\mathcal{D}$-modules. In this book, we use integrable connections, and refer to [16] and [19] for some translations into the framework of $\mathcal{D}$-modules.

4 Connections

In this section, we globalize the notions of differential algebra in the geometric context of algebraic varieties. We introduce various equivalent definitions of connection on a vector bundle, and establish the existence of cyclic vectors locally in a neighborhood of a point in the easiest case (of a smooth variety).

The following general conventions are in order. Let k be an algebraically closed field (of characteristic zero, as usual). An *algebraic k-variety* X is a separated reduced k-scheme of finite type. A *curve* C (resp. a *curve* C on X) is an irreducible algebraic variety (resp. a locally closed subvariety) of dimension one. A *prime divisor* of X is an irreducible closed subvariety of codimension one. A *divisor* D on X is a Weil divisor, i.e., an integral linear combination of prime divisors.

4.1 Differential forms and jets

4.1.1. We globalize the definitions of section 3.1.4, following again [48, 16] and [18]. Let $f : X \to S$ be a morphism of schemes and let $\Delta : X \to X \times_S X$ be the

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corresponding diagonal morphism, which is an immersion. The product $X \times_S X$ over S is equipped with the two projections p_1, p_2 with target X,

$$X \xrightarrow{\Delta} X \times_S X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X,$$

which provide $\mathcal{O}_{X\times_S X} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$ with two $\mathcal{O}_X$-module structures, $p_1^*(x) = x \otimes 1$ and $p_2^*(x) = 1 \otimes x$, for any section x of $\mathcal{O}_X$. We call the former (resp. the latter) the *left* (resp. the *right*) $\mathcal{O}_X$-module structure on $\mathcal{O}_{X\times_S X}$ and on a plethora of sheaves on $X \times_S X$ which we now introduce.

4.1.2. Let $\mathcal{I}$ be the ideal sheaf of Δ in $X \times_S X$. We define the sheaf of *principal parts*, or of *jets, of order n on X/S* as the sheaf of rings

$$\mathcal{P}^n_{X/S} := \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1})$$

on X. We regard $\mathcal{P}^n_{X/S}$ as an $\mathcal{O}_X$-module via the left structure, so that it is coherent as an $\mathcal{O}_X$-module, and define the n^{th} *jet map of X/S* as the right $\mathcal{O}_X$-linear map

$$\mathrm{j}^n_{X/S} : \mathcal{O}_X \longrightarrow \mathcal{P}^n_{X/S},$$

induced by p_2^*. The sheaves $\mathcal{P}^n_{X/S}$, for $n = 0, 1, \dots,$ form a projective system via the natural morphisms

$$\mathcal{P}^n_{X/S} \longrightarrow \mathcal{P}^m_{X/S}, \quad n \geqslant m.$$

4.1.3. We are interested in the kernel

$$\Omega^1_{X/S} := \mathcal{I}/\mathcal{I}^2$$

of the projection of lowest degree $p_{X/S} : \mathcal{P}^1_{X/S} \to \mathcal{P}^0_{X/S} = \mathcal{O}_X$, which is the sheaf of *relative differentials* or of *relative* 1-*differential forms, on X/S*. The left and right structures of $\mathcal{O}_X$-module coincide on $\Omega^1_{X/S}$. Moreover, and for the same reason, the map $p_{X/S} \circ (p_2^* - p_1^*) = p_{X/S} \circ (\mathrm{j}^1_{X/S} - p_1^*) = 0$, so that $\mathrm{j}^1_{X/S} - p_1^* : \mathcal{O}_X \to \mathcal{P}^1_{X/S}$ factors through an $\mathcal{O}_S$-linear map

$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S},$$

which is a derivation of $\mathcal{O}_X$ with values in $\Omega^1_{X/S}$. This is clearly the globalization of the map described in 2.1 in the affine case.

For any n, we consider also the $\mathcal{O}_X$-module of relative differential n-forms $\Omega^n_{X/S} = \bigwedge^n \Omega^1_{X/S}$.

The relative tangent sheaf $\mathcal{T}_{X/S}$ is the $\mathcal{O}_X$-dual of $\Omega^1_{X/S}$: equivalently, the sheaf of $\mathcal{O}_S$-linear derivations of $\mathcal{O}_X$, that is $\mathcal{T}_{X/S} = \Omega^{1\vee}_{X/S} = \mathcal{D}er_{X/S}$.

4.1.4. The sheaf $\mathcal{D}iff^n_{X/S}$ of differential operators of order $\leqslant n$ of X/S is the subsheaf of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$ whose sections locally factor as $h = \overline{h} \circ \mathrm{j}^n_{X/S}$, where $\overline{h}$:

$\mathcal{P}^n_{X/S} \to \mathcal{O}_X$ is $\mathcal{O}_X$-linear. The map $\overline{h} \mapsto \overline{h} \circ \mathrm{j}^n_{X/S}$ identifies $\mathcal{D}\mathit{iff}^n_{X/S}$ with the $\mathcal{O}_X$-dual $\mathcal{H}\mathit{om}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X)$ of $\mathcal{P}^n_{X/S}$, equipped with the further right $\mathcal{O}_X$-module structure coming from the right structure of $\mathcal{P}^n_{X/S}$. As explained above in 3.1.4 in the affine case, the union $\mathcal{D}\mathit{iff}_{X/S}$ of the subsheaves $\mathcal{D}\mathit{iff}^n_{X/S}$ of $\mathcal{E}\mathit{nd}_{\mathcal{O}_S}(\mathcal{O}_X)$ is a sheaf of subrings and sub-$\mathcal{O}_X$-bimodules. It is moreover a filtered sheaf of rings, whose associated graded ring is commutative.

When S is the point $\operatorname{Spec} k$, we drop the subscript $/S$ from the notation.

4.1.5. More generally, let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then one defines the differential operators of order n from $\mathcal{F}$ to $\mathcal{G}$, denoted $\mathcal{D}\mathit{iff}^n_{X/S}(\mathcal{F}, \mathcal{G})$, as the $\mathcal{O}_X$-module $\mathcal{H}\mathit{om}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G})$. Any differential operator can be identified with a $f^{-1}\mathcal{O}_S$-linear morphism by composition with the canonical map $\mathrm{j}^n_{X/S}$. This notion is useful in order to justify some functoriality construction involving $\mathcal{O}_X$-modules and maps which are not $\mathcal{O}_X$-linear, but just differential operators (of order 1, as in the case of de Rham complexes).

4.2 Connections

We assume here that $f : X \to S$ is a smooth morphism of schemes. Then $\Omega^1_{X/S}$ is locally free of rank equal to the relative dimension of f, and it is dual to the tangent bundle $\mathcal{T}_{X/S} = \mathcal{D}\mathit{er}_{X/S}$. The notion of X/S-connection comes in four different (but equivalent) ways.

Definition 4.2.1. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. A* connection *on $\mathcal{M}$ relative to S is one of the following equivalent data:*

– *A left-$\mathcal{O}_X$-linear map*

$$\nabla : \begin{array}{ccc} \mathcal{D}\mathit{er}_{X/S} & \longrightarrow & \mathcal{E}\mathit{nd}(\mathcal{M}) \\ \partial & \longmapsto & \nabla_\partial \end{array} \tag{4.2.2}$$

such that, for any open $U \subseteq X$ and $\partial \in \Gamma(U, \mathcal{D}\mathit{er}_{X/S})$, $(\mathcal{M}(U), \nabla_\partial)$ is a differential module over the differential ring $(\mathcal{O}_X(U), \partial)$, that is the Leibniz rule $\nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m)$ holds for any sections $f \in \mathcal{O}_X(U)$ and $m \in \mathcal{M}(U)$.

– **(Koszul's variant)** *An $\mathcal{O}_S$-linear map*

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_{X/S}(\mathcal{M}) = \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \tag{4.2.3}$$

satisfying the Leibniz rule, that is, $\nabla(fm) = d_{X/S}(f) \otimes m + f\nabla(m)$ for any local sections f of $\mathcal{O}_X$ and m of $\mathcal{M}$.

– **(Atiyah's variant)** *An $\mathcal{O}_X$-linear section*

$$J^1 : \mathcal{M} \longrightarrow \mathcal{P}^1_{X/S}(\mathcal{M}) = \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \tag{4.2.4}$$

of the projection $p_{\mathcal{M}} := p_{X/S} \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{M}} : \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$.

– **(Grothendieck's variant)** *An isomorphism of $\mathcal{P}^1_{X/S}$-modules*

$$\varepsilon : \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{P}^1_{X/S} \longrightarrow \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{M}, \tag{4.2.5}$$

which is the identity modulo $\Omega^1_{X/S}$.

Let us check the equivalence of these definitions.

- 4.2.2 ⇒ 4.2.3. Let $U \subseteq X$ be a Zariski open subset and let $(x_1, \dots, x_n)$ be étale coordinates on U/S. So, $\Omega^1_{U/S}$ is freely generated by $dx_1, \dots, dx_n$, and $\mathcal{D}er_{X/S}(U)$ is free on the dual basis $\partial_1, \dots, \partial_n$. We define the map $\nabla : \mathcal{M} \to \Omega^1_{X/S}(\mathcal{M})$ on U as $m \mapsto \sum_{i=1}^n dx_i \otimes \nabla_{\partial_i}(m)$, for any section m of $\mathcal{M}_{|U}$. This defines ∇ globally, and this map clearly satisfies the Leibniz property.
- 4.2.3 ⇒ 4.2.2. One obtains the map ∇_∂ by contraction: $\nabla_\partial = (\partial \otimes_{\mathcal{O}_X} \mathrm{id}_{\mathcal{M}}) \circ \nabla$.
- 4.2.3 ⇒ 4.2.4 We define J^1 as $\mathrm{j}^1_{\mathcal{M}} - \nabla$. We check $\mathcal{O}_X$-linearity. We have

$$\begin{aligned} J^1(am) &= \mathrm{j}^1_{X/S}(a) \otimes_{\mathcal{O}_X} m - \nabla(am) \\ &= \mathrm{j}^1_{X/S}(a) \otimes_{\mathcal{O}_X} m - a\nabla(m) - d_{X/S}(a) \otimes_{\mathcal{O}_X} m \\ &= (\mathrm{j}^1_{X/S}(a) - d_{X/S}(a)) \otimes_{\mathcal{O}_X} m - a\nabla(m) \\ &= a \otimes_{\mathcal{O}_S} m - a\nabla(m) = aJ^1(m), \end{aligned} \tag{4.2.6}$$

for sections a of $\mathcal{O}_X$ and m of $\mathcal{M}$. The projection $p_{\mathcal{M}} : \mathcal{P}^1_{X/S}(\mathcal{M}) \to \mathcal{M}$ kills $\Omega^1_{X/S}(\mathcal{M})$; therefore,

$$p_{\mathcal{M}}(J^1(m)) = p_{\mathcal{M}}(\mathrm{j}^1_{\mathcal{M}}(m) - \nabla(m)) = p_{\mathcal{M}}(\mathrm{j}^1_{\mathcal{M}}(m)) = m,$$

and J^1 is indeed a section of $p_{\mathcal{M}}$.

- 4.2.4 ⇒ 4.2.3 This follows from the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_{X/S}(\mathcal{M}) \arrow[r] & \mathcal{P}^1_{X/S}(\mathcal{M}) \arrow[r, "p_{\mathcal{M}}"] & \mathcal{M} \arrow[r] & 0 \\ & & \mathcal{M} \arrow[ul, dashed, "\nabla"] \arrow[u, "\mathrm{j}^1_{\mathcal{M}} - J^1"'] \arrow[ur, "0"'] & & \end{tikzcd}$$

where $p_{\mathcal{M}} \circ (\mathrm{j}^1_{\mathcal{M}} - J^1) = 0$, so that $\mathrm{j}^1_{\mathcal{M}} - J^1$ factors through ∇. Working backward (4.2.6), we see that $\nabla = \mathrm{j}^1_{\mathcal{M}} - J^1$ satisfies the Leibniz rule:

$$\begin{aligned} \nabla(am) &= \mathrm{j}^1_{\mathcal{M}}(am) - aJ^1(m) = \mathrm{j}^1_{\mathcal{M}}(m)a - a(\mathrm{j}^1_{\mathcal{M}}(m) - \nabla(m)) \\ &= (1 \otimes a - a \otimes 1)\mathrm{j}^1_{\mathcal{M}}(m) + a\nabla(m) = d_{X/S}(a) \otimes m + a\nabla(m). \end{aligned}$$

- 4.2.4 ⇒ 4.2.5 We extend scalars in the $\mathcal{O}_X$-linear map J^1 to obtain a $\mathcal{P}^1_{X/S}$-linear map ε as in (4.2.5), which reduces to the identity on $\mathcal{M}$ modulo $\Omega^1_{X/S}$.

To see that ε is an isomorphism, we follow the method of [18, 2.9]. There is an involution $\tau : \mathcal{P}^1_{X/S} \to \mathcal{P}^1_{X/S}$, $x \otimes y \mapsto y \otimes x$, and a τ-semilinear map $\sigma : \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}$, $x \otimes m \mapsto m \otimes \tau(x)$. The endomorphism $\varepsilon \circ \sigma$ of $\mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M}$ is τ-semilinear, so that $(\varepsilon \circ \sigma)^2$ is $\mathcal{P}^1_{X/S}$-linear. It reduces to the identity of $\mathcal{M}$ modulo the square zero ideal $\Omega^1_{X/S}$. Therefore, $(\varepsilon \circ \sigma)^2$ is an isomorphism, so that $\varepsilon \circ \sigma$ is bijective, and finally so is ε.

- 4.2.5 $\Rightarrow$ 4.2.4 We set $J^1 = \varepsilon \circ (\mathrm{id}_{\mathcal{M}} \otimes \mathrm{j}^1_{X/S})$, which clearly is $\mathcal{O}_X$-linear. Then $p_{\mathcal{M}} \circ J^1 = p_{\mathcal{M}} \circ \varepsilon \circ (\mathrm{id}_{\mathcal{M}} \otimes \mathrm{j}^1_{X/S}) = \mathrm{id}_{\mathcal{M}}$, as required. □

4.2.7 (Trivial connections). The differential $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ is called *the trivial connection on* $\mathcal{O}_X$. In this case, we have $J^1 = \mathrm{j}^1_{X/S} - d_{X/S} = p_1^*$, while $\varepsilon = \mathrm{id}_{\mathcal{P}^1_{X/S}}$. In fact, for any section a of $\mathcal{O}_X$, $J^1(a) = 1 \otimes a - (1 \otimes a - a \otimes 1) = a \otimes 1$.

4.2.8. A *morphism* between $\mathcal{O}_X$-modules with relative connection is an $\mathcal{O}_X$-linear homomorphism which is *horizontal*, i.e., compatible with the connections. The category of $\mathcal{O}_X$-modules with relative connection is a k-linear category denoted by $\mathbf{MC}(f)$ or $\mathbf{MC}(X/S)$ (or $\mathbf{MC}(X)$ if $f : X \to \operatorname{Spec} k$ is the structural map).

4.3 Integrable connections and de Rham complexes

4.3.1. Any (relative) connection ∇ "propagates" to a sequence of k-linear maps

$$\nabla_n : \Omega^n_{X/S} \otimes \mathcal{M} \longrightarrow \Omega^{n+1}_{X/S} \otimes \mathcal{M}$$

by setting

$$\nabla_n(\omega \otimes m) = d\omega \otimes m + (-1)^n \omega \wedge \nabla(m) \tag{4.3.2}$$

and checking that $\nabla_n(\omega \otimes fm) = \nabla_n(f\omega \otimes m)$ for local sections f, m of $\mathcal{O}_X$ and $\mathcal{M}$, respectively.

Definition 4.3.3 (Integrable connections). *A connection ∇ relative to S is said to be* integrable *if the composite map ("curvature") $\nabla_2 \circ \nabla$ is* 0.

4.3.4 (de Rham complex). For an integrable connection, one may thus construct the relative *de Rham complex* of $(\mathcal{M}, \nabla)$

$$\mathrm{DR}_{X/S}(\mathcal{M}, \nabla) := [\mathcal{M} \longrightarrow \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots] \tag{4.3.5}$$

In fact, the composition $\nabla_{n+1} \circ \nabla_n$ is given by

$$\begin{aligned}
\nabla_{n+1} \circ \nabla_n(\omega \otimes m) &= \nabla_{n+1}(d\omega \otimes m + (-1)^n \omega \wedge \nabla(m)) \\
&= \nabla_{n+1}(d\omega \otimes m) + (-1)^n \nabla_{n+1}(\omega \wedge \nabla(m)) \\
&= d^2\omega \otimes m + (-1)^{n+1} d\omega \otimes \nabla(m) + (-1)^n d\omega \otimes \nabla(m) \\
&\quad + \omega \wedge \nabla_2 \circ \nabla(m) \\
&= \omega \wedge \nabla_2 \circ \nabla(m),
\end{aligned}$$

and therefore is 0 if $\nabla_2 \circ \nabla$ is.

4.3.6. The integrability condition is equivalent to the compatibility of the map ∇ of 4.2.2 with the Lie bracket of the two terms (that is, that ∇ is a Lie-algebra morphism). This follows from the well-known formula

$$[\nabla_\partial, \nabla_\delta] - \nabla_{[\partial,\delta]} = (\partial \wedge \delta)(\nabla_2 \circ \nabla)$$

for any $\partial, \delta \in \mathcal{D}er_{X/S}$.

In terms of Atiyah's or Grothendieck's variants, the integrability condition is equivalent to the existence of a "stratification" which extends ε: a reformulation in the language of algebraic geometry of the classical notion of resolvent (we refer the interested reader to [18] for the notion of stratification, a view which is not used in this book).

4.3.7. $\mathcal{O}_X$-modules with relative integrable connection form a k-linear category denoted by $\mathbf{MIC}(f)$ or $\mathbf{MIC}(X/S)$ (or $\mathbf{MIC}(X)$ if $f : X \to \operatorname{Spec} k$ is the structural map).

4.3.8. In the general setting (X smooth algebraic variety over k, but no assumptions on the underlying $\mathcal{O}_X$-module), $\mathbf{MIC}(X)$ is *isomorphic* to the category of left $\mathcal{D}_X$-modules (cf. [16], [19]). The category $\mathbf{MIC}(X)$ is thus abelian and has enough injective objects (because the category of $\mathcal{D}_X$-modules has; notice that an injective $\mathcal{D}_X$-module is injective as $\mathcal{O}_X$-module, since $\mathcal{D}_X$ is flat over $\mathcal{O}_X$). It has tensor products and internal Homs: in terms of ∇_∂, they are defined by the same formulas as in the setting of differential modules.

4.4 Relation to differential modules and differential systems

4.4.1. Let $U \subseteq X$ be an affine open subset which is smooth over S. Then $F = \mathcal{O}_X(U)$ is a differential ring with respect to any $\partial \in \mathcal{T}_{X/S}(U)$. Let $(\mathcal{M}, \nabla)$ be a locally free $\mathcal{O}_X$-module of finite rank with connection. Then ∇ gives rise, by contraction, to an $\mathcal{O}(S)$-linear endomorphism ∇_∂ of $\mathcal{M}(U)$ which satisfies the Leibniz rule, so that $(\mathcal{M}(U), \nabla_\partial)$ is thus a differential module.

4.4.2. Similarly, the generic fiber of $(\mathcal{M}, \nabla)$ gives rise to differential modules $(\mathcal{M}_{\eta_X}, \nabla_\partial)$ over the function field $\kappa(X)$. In this situation, S plays the role of a space of parameters, and one can only differentiate with respect to vertical coordinates; we are free to take for K one of the three fields $k \subseteq \kappa(S) \subseteq \kappa(X)^\partial$. In particular, $(\mathcal{M}_{\eta_X}, \nabla)$ is a $\kappa(X)/\kappa(S)$-differential module.

4.4.3. Moreover, if ∇ is integrable, and if Θ is a set of derivations $\partial \in \mathcal{T}_{X/S}(U)$ which mutually commute, then the ∇_∂ mutually commute. If $\mathcal{M}(U)$ is free, with basis $\mathbf{m}$, and if ∇_∂ is given by the matrix H_∂ in the basis $\mathbf{m}$, the commutation of the ∇_∂'s amounts to

$$[H_\partial, H_\delta] = \delta(H_\partial) - \partial(H_\delta), \quad \forall\ \partial, \delta \in \Theta. \tag{4.4.4}$$

For instance, if X is étale over an open subset of $\mathbb{A}^d$, with global vertical coordinates $x_1, \ldots, x_d$, one may take $\Theta = \{\partial_{x_1}, \ldots, \partial_{x_d}\}$. For the differential system attached to $\mathcal{M}$ and $\mathbf{m}$

$$\partial_{x_i}\vec{y} = G_i\,\vec{y}, \quad i = 1, \ldots, d, \tag{4.4.5}$$

with

$$G_i = {}^tH_{\partial_{x_i}},$$

the integrability condition (4.4.4) reads

$$[G_i, G_j] = \partial_{x_i}(G_j) - \partial_{x_j}(G_i), \quad \forall\ i, j. \tag{4.4.6}$$

Conversely, assume that X is affine and smooth over S, and that $\mathcal{T}_{X/S}$ is free and generated by a set Θ of global sections. Let M be a projective module of finite rank over $\mathcal{O}(X)$. Assume that for any $\partial \in \Theta$, M is equipped with a $\mathcal{O}(S)$-linear endomorphism ∇_∂ satisfying the Leibniz rule with respect to ∂ (so that (M, ∇_∂) is a differential module over $(\mathcal{O}(X), \partial)$). Then there is a unique connection ∇ on the locally free $\mathcal{O}_X$-module $\mathcal{M}$ attached to M, which gives rise to the differential modules (M, ∇_∂) by the above process.

If moreover the elements $\partial \in \Theta$ mutually commute, as well as the endomorphisms ∇_∂, then ∇ is integrable.

4.5 Connections on vector bundles

4.5.1. Let X be a smooth algebraic variety over $S = \operatorname{Spec} k$. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and carries a (not necessarily integrable) connection, then it is locally free (cf. [8]).

In the integrable case (which is the only one which we shall treat in this book), this classical result may be proved as follows. By localization and completion, using the fact that $\widehat{\mathcal{O}}_{X,x}$ is faithfully flat over $\mathcal{O}_{X,x}$, one reduces to the fact that the differential module $\widehat{M} = \mathcal{M} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{X,x}$ is locally free over $\widehat{\mathcal{O}}_{X,x} \cong \widehat{F} := k[[x_1, \ldots, x_d]]$. Let us consider the K-endomorphism

$$\Pi = \sum_{n_1,\ldots,n_d} (-1)^{\sum n_i} \prod_{i=1}^d \frac{x_i^{n_i}}{n_i!} \prod_{i=1}^d \nabla_{\partial_{x_i}}^{n_i}.$$

One checks that $\Pi^2 = \Pi$, $\Pi(\widehat{M}) = \widehat{M}^\nabla$, and that $\forall\, f \in \widehat{F}$, $m \in \widehat{M}$, $\Pi(fm) = f(0)\Pi(m)$ and deduces that the natural morphism $\Pi(\widehat{M}) \otimes_k \widehat{F} \to \widehat{M}$ is an isomorphism (for details, see [61, 8.8]).

4.6 Cyclic vectors

Proposition 4.6.1. *Let (M, ∇) be a vector bundle with integrable connection on a smooth affine variety U. Let P be a point of U. Then for any derivation ∂ of $\mathcal{O}(U)$*

such that $\partial \mathfrak{m}_P \not\subseteq \mathfrak{m}_P$, and any sufficiently small affine neighborhood V of P in U, the differential module $(M_{\mathcal{O}(V)}, \nabla_\partial)$ over $(\mathcal{O}(V), \partial)$ is cyclic.

Proof. This is a consequence of 3.3.2 applied to the localization of M at the point P, since the assumption on ∂ implies the existence of an element $x \in \mathfrak{m}_P$ such that $\partial(x)$ is invertible in $\mathcal{O}_{U,P}$, hence in any sufficient small affine neighborhood V of P. □

Remark 4.6.2. More generally and more precisely, let $\underline{b}$ a basis of M_P. Then for any finite set of derivations ∂_i of $\mathcal{O}_U$ such that $\partial_i \mathfrak{m}_P \not\subseteq \mathfrak{m}_P$, in any sufficiently small affine neighborhood V of P in U, there is a cyclic basis $(v_i, \ldots, \partial_i^{\mu-1} v_i)$ for $(M_{\mathcal{O}_V}, \nabla_{\partial_i})$ which induces $\underline{b}$ at P.

By 3.3.2, it suffices to check that there exists $x \in \mathfrak{m}_P$ such that no $\partial_i x$ is in $\mathfrak{m}_P$. By Leibniz' rule, the preimage I_i of ∂_i in $\mathfrak{m}_P$ is an ideal of $\mathcal{O}_{U,P}$ which contains $\mathfrak{m}_P^2$. Since $I_i \neq \mathfrak{m}_P$ by the assumption on ∂_i, the image of $I_i/\mathfrak{m}_P^2$ is a proper k-subspace of $\mathfrak{m}_P/\mathfrak{m}_P^2$ by Nakayama. Since a finite-dimensional k-vector space cannot be a finite union of proper subspaces, we conclude that $\mathfrak{m}_P \neq \bigcup_i I_i$. □

Remark 4.6.3. Assume now that X is smooth affine, and D is a smooth prime divisor in X, and let (M, ∇) be a vector bundle with integrable connection on $U = X \setminus D$. In constrast with the previous proposition, if $P \in D$, it may happen that for any open affine neighborhood V of P, the differential module $(M_{\mathcal{O}_{V \cap U}}, \nabla_\partial)$ over $(\mathcal{O}_{V \cap U}, \partial)$ is *not cyclic*.

For instance, on the affine plane X deprived of the axis $D = V(x)$, consider the (integrable) module with connection

$$\mathcal{M} = \mathcal{O}_U \ \oplus \ \mathcal{O}_U e^{y/x}.$$

Then $\mathcal{M}$ admits no ∂-cyclic vector in $V \cap U$, for any open affine neighborhood V of the origin. In other words, the differential operator attached to any ∂-cyclic vector of $\mathcal{M}_{\eta_X}$ has apparent singularities along some divisor which passes through the origin; cf. Remark 3.3.3.

5 Inverse and direct images

We introduce here two fundamental operations on connections, and their relations with the extension/restriction of scalars for differential modules: the inverse image for smooth morphisms of smooth k-varieties and the direct image for étale morphisms.

The notions of direct image and higher direct image by a smooth morphism of higher relative dimension is more difficult and will be the object of study of Chapter VII.

5.1 Inverse image

Let $u : X \to Y$ be a morphism of smooth k-varieties. Then u induces a functor

$$\begin{array}{rccc} (5.1.1) \qquad & u^* : \mathbf{MIC}(Y) & \longrightarrow & \mathbf{MIC}(X) \\ & (\mathcal{M}, \nabla) & \longmapsto & (u^*\mathcal{M}, u^*\nabla)\ , \end{array}$$

where $u^*\nabla$ is defined as follows: by sheaf-theoretic inverse image, ∇ defines a k-linear mapping

$$(5.1.2) \qquad u^{-1}\mathcal{M} \longrightarrow u^{-1}\Omega^1_Y \otimes_{u^{-1}\mathcal{O}_Y} u^{-1}\mathcal{M}$$

sitting in a canonical diagram

$$(5.1.3) \qquad \begin{array}{ccc} u^{-1}\mathcal{M} & \longrightarrow & u^{-1}\Omega^1_Y \otimes_{u^{-1}\mathcal{O}_Y} u^{-1}\mathcal{M} \\ \big\downarrow & & \big\downarrow \\ u^*\mathcal{M} & & \Omega^1_X \otimes_{\mathcal{O}_X} u^*\mathcal{M} \end{array}$$

(recall that the canonical morphism $u^{-1}\Omega^1_Y \otimes_{u^{-1}\mathcal{O}_Y} u^{-1}\mathcal{M} \to u^{-1}(\Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{M})$ is an isomorphism by [50, 0.4.3]). There is a unique integrable connection $u^*\nabla$ on $u^*\mathcal{M}$ such that

$$(5.1.4) \qquad u^*\nabla : u^*\mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} u^*\mathcal{M}$$

completes (5.1.3) into a commutative square. Actually, it is not immediate that (5.1.2), which is not $u^{-1}\mathcal{O}_Y$-linear, extends to (5.1.4)[1], but the difficulty is easily overcome using Atiyah's linear description of connections: ∇ corresponds to an $\mathcal{O}_Y$-linear section J^1 of $p_Y \otimes 1_{\mathcal{M}}$, which defines (by composition with the canonical morphism $u^*\mathcal{P}^1_{Y/k} \to \mathcal{P}^1_{X/k}$) an $\mathcal{O}_X$-linear section u^*J^1 of $p_X \otimes 1_{u^*\mathcal{M}}$ [48, 16.7.9]. This defines a connection $u^*\nabla$. One checks that this connection is integrable.

The functor u^* is right-exact.

5.1.5. Here is the relation to differential modules. The function field $F' = \kappa(X)$ is an extension of $F = \kappa(Y)$, and any k-derivation ∂ of F extends to a k-derivation of F'. Let (M, ∇_∂) (resp. (M', ∇'_∂)) be the differential module over (F, ∂) (resp. (F', ∂)) attached to the generic fiber of $(\mathcal{M}, \nabla)$ (resp. $(u^*\mathcal{M}, u^*\nabla)$) and ∂. Then

$$(M', \nabla'_\partial) = (M, \nabla_\partial)_{F'}.$$

5.1.6. The inverse image of the trivial connection on $\mathcal{O}_Y$ is the trivial connection on $\mathcal{O}_X$.

[1] The same slight difficulty arises in defining the internal $\otimes$ and is solved in the same way.

5.2 Direct image by an étale morphism

As before, let $u : X \to Y$ be a morphism of smooth k-varieties. Let us assume in addition that the natural morphism $u^*\Omega^1_Y \to \Omega^1_X$ is an isomorphism (this occurs for instance if u is étale). Then one has a functor

$$\begin{array}{rccc} & u_* : \mathbf{MIC}(X) & \longrightarrow & \mathbf{MIC}(Y) \\ (5.2.1) & (\mathcal{M}', \nabla') & \longmapsto & (u_*\mathcal{M}', u_*\nabla') \ , \end{array}$$

where $u_*\nabla'$ is defined via the projection formula

$$u_*(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}') \cong u_*(u^*\Omega^1_Y \otimes_{\mathcal{O}_X} \mathcal{M}') \cong \Omega^1_Y \otimes_{\mathcal{O}_Y} u_*(\mathcal{M}').$$

The easiest description of the direct image connection is given in terms of differential modules: the action of a derivation in $\mathcal{D}er_Y$ on a section of $u_*\mathcal{M}'$ is exactly the action of the corresponding derivation in $\mathcal{D}er_X$ on that section. The integrability of $u_*\nabla'$ then follows immediately from that of ∇'. The functor u_* is left-exact.

5.2.2. If u is finite, the function field $F' = \kappa(X)$ is a finite extension of $F = \kappa(Y)$ and the extension of ∂ from F to F' is unique. Let (M', ∇'_∂) (resp. (M, ∇_∂)) be the differential module over (F', ∂) (resp. (F, ∂)) attached to the generic fiber of $(\mathcal{M}', \nabla')$ (resp. $(u_*\mathcal{M}', u_*\nabla')$) and ∂. Then

$$(M, \nabla_\partial) = {}_F(M', \nabla'_\partial).$$

5.2.3. If u is finite étale, then there is a strict relation between the direct image of the trivial connection on X and the local sections of the map u itself. In fact for any local section $s : Y \to X$ we have the morphism $s^\sharp : \mathcal{O}_X \to s_*\mathcal{O}_Y$. Taking the direct image by u we have a morphism $u_*s^\sharp : u_*\mathcal{O}_X \to u_*s_*\mathcal{O}_Y = \mathcal{O}_Y$ which is clearly a local solution of $(u_*\mathcal{O}_X, u_*d_X)$ because of the commutative diagram

$$\begin{tikzcd} u_*\mathcal{O}_X \arrow[r, "u_*d_X"] \arrow[d, "s^\sharp"'] & u_*\Omega^1_X = \Omega^1_Y \otimes u_*\mathcal{O}_X \arrow[d, "\mathrm{id}_{\Omega^1_Y} \otimes s^\sharp"] \\ \mathcal{O}_Y \arrow[r, "d_Y"] & \Omega^1_Y \end{tikzcd}$$

We have therefore a canonical inclusion of the sheaf (of finite sets) $\mathit{Sect}(u)$ of local sections of u in the sheaf (of k-vector spaces) $\mathit{Sol}(u_*\mathcal{O}_X, u_*d_X)$ of local solutions of the direct image of the trivial connection on X.

Moreover, possibly outside of the branch locus of u, the sheaf $\mathit{Sect}(u)$ generates (as k-vector space) the sheaf $\mathit{Sol}(u_*\mathcal{O}_X, u_*d_X)$. This may be verified after completion of the localization at any closed point y of Y, out of the branch locus. The completed local ring $\widehat{\mathcal{O}}_{Y,y}$ is then a ring of power series, and $(u_*\mathcal{O}_X)_y$ is an $\widehat{\mathcal{O}}_{Y,y}$-algebra generated (as module) by a finite set of idempotent orthogonal elements e_i. A local section of u gives a morphism of $\widehat{\mathcal{O}}_{Y,y}$-algebras $(u_*\mathcal{O}_X)_y \to \widehat{\mathcal{O}}_{Y,y}$, and this forces the morphism to send one of the e_i's to 1, the others to 0. These morphisms are horizontal and generate all morphisms of $\widehat{\mathcal{O}}_{Y,y}$-modules $(u_*\mathcal{O}_X)_y \to \widehat{\mathcal{O}}_{Y,y}$, from which the assertion follows.

Chapter III

Regularity: formal theory

Introduction

With the example of Gauss hypergeometric equation at hand, we first recall the classical Fuchs-Frobenius theory of meromorphic linear differential systems: Fuchs' numerical criterion of regularity and Frobenius' convergence of the asymptotic expansions of the solutions.

We then turn to the algebraic viewpoint which will be prevalent in the book, and present the classical theory of regularity of formal differential modules in one variable (after Yu. Manin, N. Katz, ...). We abstract the canonical decomposition according to exponents into a general formalism of Jordan decompositions which will be also used later in the theory of irregularity.

We then tackle the theory of regular formal integrable connections in several variables, and present a detailed account of the (little known) theory of Gérard-Levelt lattices.

6 Hypergeometric equations and local monodromy

6.1 Singular points of hypergeometric equations

Let us consider again, from the complex-analytic point of view, the hypergeometric differential operator $L_{a,b,c}$ of (1.1.1), with $a, b, c \in \mathbb{C}$, namely

(6.1.1) $$L_{a,b,c} = x(1-x)\,\partial_x^2 + \big(c - (a+b+1)x\big)\,\partial_x - ab,$$

or, equivalently, using the operator $\vartheta_x = x\partial_x$,

(6.1.2) $$xL_{a,b,c} = (1-x)\,\vartheta_x^2 + \big(c - 1 - (a+b)x\big)\,\vartheta_x - abx.$$

We already pointed out that, if $c \notin \mathbb{Z}$, a full set of solutions of $L_{a,b,c}$ at 0 is given by the two functions ${}_2F_1(a, b, c; x)$ and $x^{1-c}{}_2F_1(a+1-c, b+1-c, 2-c; x)$, where

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x^{1-c} is the multivalued function obtained by analytic continuation from one fixed determination at a fixed point $x_0 \neq 0$, for example

$$x^{1-c} = \Big(1 + \frac{x - x_0}{x_0}\Big)^{1-c} = \sum_n \binom{1-c}{n} \Big(\frac{x - x_0}{x_0}\Big)^n.$$

The power series part of the previous multivalued solutions has radius of convergence 1, as one can easily check from d'Alembert's ratio test, while any determination of x^{1-c} at any $x = x_0 \neq 0$, converges in the maximal open disk centered at x_0 and contained in $D^* = D \setminus \{0\}$. One could hardly expect a more complete description of the analytic solutions of the differential equation $L_{a,b,c}y = 0$ in the open unit disk D centered at $0 \in \mathbb{C}$. The point $x = 0$ is a *singular point* of the previous equation in D, while every other point $x_0 \in D^*$ is *non-singular*[1] for (6.1.1), in the sense that there exist two linearly independent converging power series solutions in the variable $x - x_0$. At 0 instead, the vector space of power series solutions is one-dimensional and spanned by ${}_2F_1(a, b, c; x)$. This is a general principle:

Theorem 6.1.3 (Cauchy). *Let $U \subseteq \mathbb{C}$ be an open connected domain and let*

$$L = \partial_x^{\mu} + a_{\mu-1}\, \partial_x^{\mu-1} + \cdots + a_1\, \partial_x + a_0, \tag{6.1.4}$$

be a differential operator with coefficients $a_i \in \mathcal{O}(U)$ that are holomorphic on U, $i = 0, \ldots, \mu - 1$. For any $x_0 \in U$, let $D_{x_0} \subseteq U$ be the maximal open disk in U centered at x_0. Then, for any $x_0 \in U$, the solutions of L in the differential ring $(\mathbb{C}[[x - x_0]], \partial_x)$ form a vector space of dimension μ. Moreover, they are Taylor expansions at x_0 of holomorphic functions on the full disk D_{x_0}.

In simple words, *the solutions of a differential equation at a non-singular point converge up to the nearest singularity.* An enhanced version of Cauchy's theorem states that *the power series part of solutions of a differential equation at a* regular *singular point converge up to the next nearest singularity.* A typical example of the latter situation is precisely (6.1.1). We refer to [87, Chap. V Sect. 17] for the proof, or the second appendix of [38] for more details. This chapter is dedicated to the explanation of the formal meaning of the term *regular* singular point appearing in the previous statement, and to its higher-dimensional generalizations.

Another generalization of Cauchy's theorem, in the complex-analytic context, is: every integrable connection is locally trivial (see 29.1.1).

Cauchy's theorem leads to the classical *theory of monodromy*, as we now explain.

[1] The classical name for *non-singular point* is *ordinary point*. However, we prefer to avoid this term, because the term *ordinary differential equation* is also used in the classical literature as an antonym to *partial differential equation*.

6.2 Local monodromy

6.2.1. Let D_ϵ^* be the punctured disk of radius ϵ centered at $0 \in \mathbb{C}$, and let $\mathbb{C}\{\{x\}\}$ be $\bigcup_\epsilon \mathcal{O}(D_\epsilon^*)$ viewed as a differential field. The field $\mathbb{C}(\{x\})$ of germs of meromorphic functions at 0 is a differential subfield of $\mathbb{C}\{\{x\}\}$. We consider a system of linear differential equations of the form

$$\frac{d}{dx}\,\vec{y} = G(x)\,\vec{y}, \tag{6.2.2}$$

where $G(x) \in M_\mu(\mathbb{C}\{\{x\}\})$. Here $\vec{y}$ is a column vector of unknown functions of size μ, but it also makes sense to replace $\vec{y}$ by a $\mu \times \mu$ matrix Y. We may assume that 0 is the only singularity of G in some disk D around 0.

6.2.3. Let $z_0 \in D^* := D \smallsetminus \{0\}$, and let $\mathcal{O}_{z_0}$ be the ring of germs of holomorphic functions at z_0. By Cauchy's theorem, the system has a solution matrix Y at z_0 in $\mathrm{GL}_\mu(\mathcal{O}_{z_0})$. This is called a *fundamental solution matrix at* z_0. Any other solution matrix Z of equation (6.2.2) in $M_\mu(\mathcal{O}_{z_0})$ is necessarily of the form $Z = YC$, for $C \in M_\mu(\mathbb{C})$, because $C := Y^{-1}Z \in M_\mu(\mathcal{O}_{z_0})$, and by a trivial manipulation, $\frac{dC}{dx} = 0$. There is a unique fundamental solution matrix at z_0 such that $Y(z_0) = I_\mu$. We shall denote it by $Y_{z_0}(x)$.

6.2.4. If $\gamma_0 : [0,1] \to D^*$, continuous with $\gamma_0(0) = \gamma_0(1) = z_0$, denotes a loop in D^*, starting at z_0 and turning once counterclockwise around 0, analytic continuation along γ_0 then defines an automorphism of $\mathbb{C}$-algebras

$$T_{\gamma_0} : \mathcal{O}_{z_0} \longrightarrow \mathcal{O}_{z_0}, \tag{6.2.5}$$

which is non-trivial in general. For example, for any determination of $\log x$ (resp. x^a) at z_0, $T_{\gamma_0} \log x = 2\pi i + \log x$ (resp. $T_{\gamma_0} x^a = e^{2\pi i a} x^a$).

6.2.6. Analytic continuation commutes with differentiation, and therefore transforms the solution matrix Y_{z_0} of equation (6.2.2) at z_0 into another fundamental solution matrix at z_0, $T_{\gamma_0} Y_{z_0} = Y_{z_0} C(\gamma_0)$, with $C(\gamma_0) \in \mathrm{GL}_\mu(\mathbb{C})$. Analytic continuation along a homotopic path $\gamma_0' \sim \gamma_0$, leads to the same result. The fundamental group of D^* with base point z_0 is the group of homotopy classes of continuous paths $\gamma : [0,1] \to D^*$, with $\gamma(0) = \gamma(1) = z_0$, where the product $[\gamma_0][\gamma_1]$ is represented by the path

$$\gamma(s) = \begin{cases} \gamma_0(2s), & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \gamma_1(2s-1), & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

It follows that, if we compose analytic continuation of $Y_{z_0} C(\gamma_0)$ along γ_1 with analytic continuation of Y_{z_0} along γ_0, we obtain the matrix $Y_{z_0} C(\gamma_1) C(\gamma_0)$, which represents analytic continuation of Y_{z_0} along the product path $\gamma_0\gamma_1$. The *monodromy representation* associated to (6.2.2) is the group anti-homomorphism

$$\begin{array}{rccc} \rho : & \pi_1(D^*, z_0) & \longrightarrow & \mathrm{GL}_\mu(\mathbb{C}) \\ & \gamma & \longmapsto & C(\gamma). \end{array} \tag{6.2.7}$$

For any $A \in M_\mu(\mathbb{C})$, let us define x^A as

$$x^A = \exp(A \log x) = \sum_{s=0}^{\infty} \frac{(A \log x)^s}{s!},$$

for some fixed determination of $\log x$ at z_0. Then

$$T_{\gamma_0}(x^A) = \exp(A \log x + 2\pi i A) = x^A \exp(2\pi i A).$$

So, if we take A such that $\exp(2\pi i A) = C(\gamma_0)$, then

$$T_{\gamma_0}(Y_{z_0} x^{-A}) = Y_{z_0} C(\gamma_0) C(\gamma_0)^{-1} x^{-A} = Y_{z_0} x^{-A},$$

so that

$$W = Y_{z_0} x^{-A} \tag{6.2.8}$$

is *uniform*, that is, analytic in D^*. This is the *complex monodromy theorem.*

6.2.9. Let us now assume that the coefficients of the system (6.2.2) lie in the differential subfield $\mathbb{C}(\{x\})$ of $\mathbb{C}\{\{x\}\}$. It then makes sense to distinguish between two cases:

(1) W has coefficients in $\mathbb{C}(\{x\})$, i.e., has at worst a pole at 0, or

(2) W has an essential singularity at 0.

In the first case the singularity 0 is called (meromorphically) *regular*; in the second *irregular*. This dichotomy depends only on the differential module M over $\mathbb{C}(\{x\})$ attached to the system (6.2.2). It turns out that it can be detected on the *formal* differential module $\widehat{M}$ over $\mathbb{C}((x))$ (the formal completion of M). As we will see, the notion of regularity makes sense for differential modules over $\mathbb{C}((x))$.

6.2.10. The convergence of formal solutions at a regular singularity will be at the root of the comparison theorem for algebraic and complex-analytic de Rham cohomology (cf. 31.3.3).

6.3 Fuchs-Frobenius theory

6.3.1. By the lemma of the cyclic vector (3.3.1), the system (6.2.2) is equivalent to a scalar equation $Ly = 0$, where

$$L = \partial_x^\mu + a_{\mu-1}\, \partial_x^{\mu-1} + \cdots + a_1\, \partial_x + a_0 \tag{6.3.2}$$

is a linear differential operator of order μ, with the coefficients a_i holomorphic in a possibly smaller punctured disk contained in D^*, and with at most a pole at 0.

In the next definitions we relax the assumptions on L, as in (6.3.2), and only assume that the coefficients $a_i \in K((x))$, for a field K of characteristic 0; ord_0 denotes again the x-adic valuation.

Definition 6.3.3 (**Fuchs number**). *We define the* Fuchs number (*or irregularity, following* [78, Def. 1.5]) *at* 0 *of a differential operator*

$$L = a_\mu \partial_x^\mu + a_{\mu-1}\,\partial_x^{\mu-1} + \cdots + a_1\,\partial_x + a_0$$

with $a_i \in K((x))$ *by*

$$i_0(L) = \max\{(i - \mathrm{ord}_0 a_i) - (\mu - \mathrm{ord}_0 a_\mu) : i = 0, \ldots, \mu\}.$$

It is clear from the definition that $i_0(L) \geqslant 0$. In the case of the monic operator L as in (6.3.2), we have

$$i_0(L) = \max\{0, \max\{(i - \mathrm{ord}_0 a_i) - \mu : i = 0, \ldots, \mu - 1\}\}.$$

Definition 6.3.4 (**Fuchs condition**). *We say that L satisfies the* Fuchs condition *at* 0*, or that* 0 *is a* regular singularity *for L, if $i_0(L) = 0$, namely if*

$$\mathrm{ord}_0(a_i) \geqslant i - \mu, \quad \textit{for } \ i = 0, \ldots, \mu - 1.$$

These definitions are motivated by the following classical theorem of Fuchs and Frobenius (see [42], [41]).

Theorem 6.3.5 (**Frobenius-Fuchs**). *Let $F = \mathbb{C}(\{x\})$ be the field of germs of meromorphic functions on $\mathbb{C}$ at $x = 0$. The differential system* (6.2.2)*, with coefficients in F, is regular at* 0 *if and only if one* (*and therefore any*) *scalar differential equation obtained from* (6.2.2) *by application of* (3.3.2) *over the differential field* (F, ∂_x) *satisfies the Fuchs condition at* 0*.*

Most of this chapter is dedicated to the purely formal theory of regular singularities in one and several variables, which greatly generalizes the formal questions raised by the theorem of Frobenius and Fuchs.

6.3.6. Thanks to this theorem, if the singularity is regular, the calculation of the monodromy matrix $C(\gamma_0)$ is easy. The logarithms of its eigenvalues are the zeros of the indicial polynomial of L at 0 (see Definition 7.3.1).

A similar discussion can be formally carried out in the irregular case, but does not lead to a calculation of $C(\gamma_0)$, because of the appearance of divergent series which only represent asymptotic expansions at 0 of holomorphic solutions of L in circular sectors of D^*.

7 The classical formal theory of regular singular points

The classical definition of regularity at 0 which we gave in the previous section has a meaning in pure differential algebra, as we now explain. The purpose of this section is to expose the formal aspects of the classical one-dimensional theory of regular singular points, on which the rest of the theory is ultimately based.

7.1 The exponential formalism x^A

7.1.1. Let K be a field of characteristic 0, and $\overline{K}$ be a fixed algebraic closure of K. We work with the differential field $F = K((x))$ of formal Laurent series equipped with derivation

$$\partial = \partial_x = \frac{d}{dx} \quad \text{or} \quad \vartheta_x = x\frac{d}{dx}.$$

First notice that there exists a differential field extension $(\mathcal{F}, \vartheta_x)$ of $(K((x)), \vartheta_x)$ such that:

(1) $\mathcal{F}$ contains a solution $\log x$ of the differential equation $\vartheta_x y = 1$;

(2) the natural homomorphism $(\mathbb{Z}, +) \to (\mathcal{F}^\times, \cdot)$, $m \mapsto x^m$, extends to an injective homomorphism $(\mathbb{Q}, +) \to (\mathcal{F}^\times, \cdot)$, $\frac{m}{n} \mapsto x^{\frac{m}{n}}$;

(3) the homomorphism $(\mathbb{Q}, +) \to (\mathcal{F}^\times, \cdot)$ extends to an injective map $(\overline{K}, +) \to (\mathcal{F}^\times, \cdot)$, $\alpha \mapsto x^\alpha$, such that

   (a) $x^{\alpha+m} = x^\alpha x^m$, for $m \in \mathbb{Z}$;

   (b) $x^{-\alpha} = (x^\alpha)^{-1}$;

   (c) x^α is a solution of the differential equation $\vartheta_x y = \alpha y$;

(4) $\mathcal{F}^{\vartheta_x} = \overline{K}$.

The map $\alpha \mapsto x^\alpha$ is not, in general, a group homomorphism, but for any $\alpha, \beta \in \overline{K}$, we have $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta}$, for some constant $c_{\alpha,\beta} \in \overline{K}$ satisfying suitable compatibility conditions (more precisely: $c_{\alpha,-\alpha} = 1 = c_{\alpha,m}$, $c_{\alpha,\beta} = c_{\beta,\alpha}$ and $c_{\alpha,\beta} c_{\alpha+\beta,\gamma} = c_{\alpha,\beta+\gamma} c_{\beta,\gamma}$ for any $\alpha, \beta, \gamma \in \overline{K}$ and $m \in \mathbb{Z}$).

If $\overline{K}$ embeds into $\mathbb{C}$, then upon choosing a branch of the logarithm we may define x^α to be $\exp(\alpha \log(x))$; with this choice we have $c_{\alpha,\beta} = 1$ for all $\alpha, \beta \in \overline{K}$.[2]

Lemma 7.1.2. *Let $q(t) \in \overline{K}[t]$, be a non-zero polynomial with constant coefficients, and let $\alpha \in \overline{K}$. Then $x^\alpha q(\log x) \in K((x))$ if and only if $\alpha \in \mathbb{Z}$, $\deg q = 0$, and $q \in K$.*

Proof. Let d be the degree of q. Let $w = x^\alpha q(\log x) = \sum_n b_n x^n \in K((x))$. We have $\vartheta_x q(\log x) = (\partial_t q)(\log x)$, hence $\vartheta_x^{d+1}(q(\log x)) = (\partial_t^{d+1} q)(\log x) = 0$. On the other hand, $\vartheta_x^{d+1}(q(\log x)) = \vartheta_x^{d+1}(x^{-\alpha} w) = x^{-\alpha}(\vartheta_x - \alpha)^{d+1} w$. Therefore $\sum_n b_n (n-\alpha)^{d+1} x^n = 0$. We deduce that either $w = 0$, or $\alpha \in \mathbb{Z}$ and $w = b_\alpha x^\alpha$. In either case $x^{-\alpha} w \in K$, so $q(\log x) \in K$. Since $\log x$ is not in $\overline{K}$, we conclude that $d = 0$ and $q \in K$. □

[2] An abstract construction of $\mathcal{F}$ is provided by Picard-Vessiot theory: let $A \sqcup 0$ be a set of representatives in $\overline{K}$ of elements of $\overline{K}$ modulo translation by integers and modulo sign. Let us endow the Laurent polynomial algebra $\overline{K}((x))[x_\alpha, x_\alpha^{-1}]_{\alpha \in A}$ with an action of ϑ_x given by $\vartheta_x(x_\alpha) = \alpha x_\alpha$. Its quotient by any maximal differential ideal is a simple differential algebra, hence a domain, and its fraction field $\mathcal{F}$, together with the images x^α of x_α, have the required properties (see e.g., [88, ch. 1])).

7.1.3. Let $A \in M_\mu(K)$, and let consider the differential system (7.2.1), with $G = A$ (that is, with constant coefficients). Then we may specify a solution matrix $x^A \in \mathrm{GL}_\mu(\mathcal{F})$ as follows:

(1) if $A = \Delta = \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_\mu \end{pmatrix}$ is diagonal, then $x^\Delta = \begin{pmatrix} x^{\Delta_1} & 0 & \cdots & 0 \\ 0 & x^{\Delta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{\Delta_\mu} \end{pmatrix}$;

(2) if $A = N$ is nilpotent, then $x^N = \exp(N \log x) = \sum_{j=0}^{\infty} \frac{N^j}{j!} (\log x)^j$;

(3) if $A = P(\Delta + N)P^{-1}$ gives the Jordan canonical form (with $P \in \mathrm{GL}_\mu(\overline{K})$), then $x^A = P x^\Delta x^N P^{-1}$.

Definition 7.1.4 (non-resonance). *We say that a matrix $A \in M_\mu(K)$ is* non-resonant *if the difference between any eigenvalues A in $\overline{K}$ is not a non-zero integer.*

This is a widespread, but not universal terminology: [38] says that A has *prepared eigenvalues.*

7.2 Non-resonance

We consider a differential system

$$\vartheta_x Y = GY, \quad \text{with } G \in M_\mu(K((x))). \tag{7.2.1}$$

Lemma 7.2.2. *Assume that the matrix G satisfies*

(1) *$G \in M_\mu(K[[x]])$;*

(2) *$G(0)$ is non-resonant.*

Then, if K' denotes any extension of K containing all eigenvalues of $G(0)$, the system has a fundamental solution matrix of the form $Y = W x^{G(0)}$, with $W \in \mathrm{GL}_\mu(K'[[x]])$ and $W(0) = I_\mu$.

Proof. Let $G(x) = \sum_{i=0}^{\infty} G_i x^i$ (with $G_i \in M_\mu(K)$; in particular, $G_0 = G(0)$ is non-resonant). We look for a matrix $W = \sum_{i=0}^{\infty} W_i x^i$ with $W \in M_\mu(K')$ and $W_0 = I_\mu$ such that $W x^{G_0}$ is a solution of (7.2.1). The condition gives

$$\vartheta_x W + W G_0 = GW,$$

that is,

$$\vartheta_x \big(\sum_{i=0}^{\infty} W_i x^i \big) + \big(\sum_{i=0}^{\infty} W_i x^i \big) G_0 = \big(\sum_{i=0}^{\infty} W_i x^i \big) \big(\sum_{i=0}^{\infty} G_i x^i \big)$$

and finally

$$\sum_{i=0}^{\infty}\big(iW_i+W_iG_0\big)x^i=\sum_{i=0}^{\infty}\big(\sum_{j=0}^{i}G_jW_{i-j}\big)x^i.$$

Equating the coefficient of x^i for any $i=0,1,\dots$ we have an infinite system of equations

$$iW_i+W_iG_0-G_0W_i=\sum_{j=1}^{i}G_jW_{i-j},$$

which allows us to use recursion on i. Starting with $W_0=I_\mu$, the i-th equation may be solved for W_i (the right-hand side depends only on $W_0,\dots,W_{i-1}$) if the linear transformation $\psi_{G_0}:M_\mu(K')\to M_\mu(K')$ given by $X\mapsto G_0X-XG_0$, has no non-zero integer eigenvalues. Since the eigenvalues of ψ_{G_0} are precisely the differences of eigenvalues of G_0, that condition is satisfied, because G_0 is non-resonant. $\square$

The method of *shearing transformations* allows to pass from a system (7.2.1) with $G\in M_\mu(K[[x]])$, to an equivalent system

$$\vartheta_xY=G_PY, \tag{7.2.3}$$

where $G_P\in M_\mu(K'[[x]])$ and $G_P(0)$ is non-resonant, where K' is the splitting field over K of the characteristic polynomial of $G(0)$ and $P\in \mathrm{GL}_\mu(K'[x,x^{-1}])$. Here is the precise statement.

Lemma 7.2.4 (**Shearing transformations**). *Let $G\in M_\mu(K[[x]])$, and let $\alpha\in K'$ be an eigenvalue of $G(0)$. There exists $P\in \mathrm{GL}_\mu(K'[x,x^{-1}])$ such that $G_P\in M_\mu(K'[[x]])$ and the eigenvalues of $G_P(0)$ and of $G(0)$ are the same and have the same multiplicities, except for α which is replaced by $\alpha+1$. Similarly for $\alpha-1$.*

Proof. We may assume that $G(0)=\begin{pmatrix}J&B\\0&D\end{pmatrix}$ with $J\in M_{\mu-s}(K')$, $D\in M_s(K')$ upper-triangular and $J=\begin{pmatrix}\alpha&*&\cdots&*\\0&\alpha&\ddots&\vdots\\\vdots&\ddots&\ddots&*\\0&\cdots&0&\alpha\end{pmatrix}$, $D=\begin{pmatrix}\alpha_1&*&\cdots&*\\0&\alpha_2&\ddots&\vdots\\\vdots&\ddots&\ddots&*\\0&\cdots&0&\alpha_s\end{pmatrix}$, $\alpha\neq\alpha_i$, for $i=1,\dots,s$. We then take $P=\begin{pmatrix}xI_{\mu-s}&0\\0&I_s\end{pmatrix}$ and check directly that $G_P(0)=\begin{pmatrix}J+I_{\mu-s}&0\\ *&D\end{pmatrix}$. To get the eigenvalue $\alpha-1$, one works in a similar way with lower triangular matrices. $\square$

7.3 Indicial polynomials

Definition 7.3.1 (**indicial polynomial**). *Let*

$$\Lambda=a_{\mu-1}\partial_x^{\mu}+a_{\mu-1}\partial_x^{\mu-1}+\cdots+a_j\partial_x^{j}+\cdots+a_0\in K((x))\langle\partial_x\rangle$$

be a differential polynomial. We define the indicial polynomial of Λ at $x = 0$ *as the polynomial* $\mathrm{ind}_{\Lambda,0}(t) \in K[t]$ *determined by the condition*

$$\Lambda(x^t) = (\mathrm{ind}_{\Lambda,0}(t) + o(x))x^{t+r}, \tag{7.3.2}$$

where $o(x)$ *denotes an element of* $xK[t][[x]]$. *The indicial polynomial* $\mathrm{ind}_{\Lambda,0}(t)$ *will be denoted also* $\mathrm{ind}_0(t)$ *if the there is no ambiguity on* Λ.

7.3.3. The dominant coefficient of $\mathrm{ind}_0(t)$ will be indicated by γ_0 or γ, and the roots by $\alpha_{0,i}$ or α_i, so that

$$\mathrm{ind}_0(t) = \gamma_0 \prod_i (t - \alpha_{0,i}).$$

7.3.4. If $\Lambda \in K(x)\langle\partial_x\rangle$, then for any $x_0 \in K \cup \{\infty\}$ we define the indicial polynomial $\mathrm{ind}_{\Lambda,x_0}(t)$ of Λ at $x = x_0$ in the same way (by the change of variable $x' = x - x_0$ if $x_0 \in K$, or $x' = 1/x$ for $x_0 = \infty$).

Example 7.3.5. Let $L_{n,m} = x^n \partial_x^m + 1$. Then we have

$$\mathrm{ind}_{L_{n,m},0}(t) = \begin{cases} t(t-1)\cdots(t-m+1), & \text{if } n < m, \\ t(t-1)\cdots(t-m+1)+1, & \text{if } n = m, \\ 1, & \text{if } n > m. \end{cases}$$

Remark 7.3.6. Let write the differential polynomial Λ of 7.3.1 in the form

$$\Lambda = b_\mu \vartheta_x^\mu + b_{\mu-1}\vartheta_x^{\mu-1} + \cdots + b_j \vartheta_x^j + \cdots + b_0 \in K((x))\langle\vartheta_x\rangle.$$

Set

$$r = \min_i \mathrm{ord}_0 b_i \quad \text{and} \quad \nu = \max\{i : \mathrm{ord}_0 b_i = r\}.$$

Then $\mathrm{ind}_{\Lambda,0}(t) = \sum_{i=0}^{\nu} (x^{-r} b_i)_{x=0} t^i$. In particular, the degree of the indicial polynomial $\mathrm{ind}_0(t)$ is always between 0 and μ. It is exactly μ if and only if $b_i \in \mathbb{C}[[x]]$, for all i.

Remark 7.3.7 (Indicial polynomial as a characteristic polynomial). Let us consider the system $\vartheta_x Y = GY$ constructed from a differential operator

$$\Lambda = \vartheta_x^\mu + b_{\mu-1}\vartheta_x^{\mu-1} + \cdots + b_j \vartheta_x^j + \cdots + b_0 \in K[[x]]\langle\vartheta_x\rangle.$$

Then the shape (2.3.5) of that system shows that the characteristic polynomial of $G(0)$ is

$$\det(I_\mu t - G(0)) = t^\mu + b_{\mu-1}(0)t^{\mu-1} + \cdots + b_j(0)t^j + \cdots + b_0(0), \tag{7.3.8}$$

and therefore coincides with the indicial polynomial $\mathrm{ind}_{\Lambda,0}(t)$ of Λ at 0.

7.4 Regularity of differential systems

The first half of the algebraic version of the theorem of Fuchs and Frobenius (6.3.5) is the following result about the system (7.2.1).

Proposition 7.4.1. *The following conditions are equivalent:*

(1) *there exists a finite Galois extension K' of K such that the system admits a solution of the form $Y = Wx^A$, with $A \in M_\mu(K')$ and $W \in \mathrm{GL}_\mu(K'((x)))$;*

(2) *there exists P in $\mathrm{GL}_\mu(K(x))$ (or, equivalently, in $\mathrm{GL}_\mu(K((x)))$), such that $G_P \in M_\mu(K[[x]])$;*

(3) *there exists a finite Galois extension K' of K and $P \in \mathrm{GL}_\mu(K'((x)))$ such that $G_P \in M_\mu(K')$ is in standard Jordan canonical form*

$$G_P = \begin{pmatrix} \alpha_1 + N_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 + N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_r + N_r \end{pmatrix}$$

where $\alpha_i - \alpha_j \notin \mathbb{Z}$, if $i \neq j$, and

$$N_i = \begin{pmatrix} 0 & \varepsilon_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_{n_1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a standard upper-triangular nilpotent matrix with $\varepsilon_i \in \{0, 1\}$ for all i.

Proof. (1) $\Rightarrow$ (2) By remark (2.6.10), it suffices to show that (1) implies (2) with K replaced by K'. This is clear if $W \in \mathrm{GL}_\mu(K'[[x]])$, since, by differentiation, we get $G = \vartheta_x(W)W^{-1} + WAW^{-1} \in M_\mu(K'[[x]])$. In general, $W \in \mathrm{GL}_\mu(K'((x)))$ and one can always find $H \in \mathrm{GL}_\mu(K'(x))$ such that $HW \in \mathrm{GL}_\mu(K'[[x]])$. But HWx^A is then a solution of the system $(\vartheta_x - G_H)Y = 0$, so that it follows that $G_H \in M_\mu(K'[[x]])$.

The implication (2) $\Rightarrow$ (1) follows directly from lemmas (7.2.2) and (7.2.4).

The equivalence (1) $\Leftrightarrow$ (3) is clear (for the "only if" part, use $P = W^{-1}$ to obtain $G_P = A$). □

Definition 7.4.2 (Regularity for differential systems). *One says that the differential system (7.2.1)*

$$\vartheta_x Y = GY \quad (\textit{or the equivalent system } \partial_x Y = \tfrac{1}{x} GY)$$

is regular *– or that* 0 is a regular singularity *– if the equivalent conditions of the previous proposition are satisfied.*

Remark 7.4.3. The second point of 7.4.1 suggests that regularity may be defined "formally", i.e., requiring the existence of a $K[[x]]$-lattice of the differential module over $K((x))$ stable under the action of ϑ_x, as we will do in the next section.

7.5 Regularity criterion for differential equations

We say that a differential operator L is equivalent to the differential system (7.2.1) if the associated differential system is equivalent in the sense of definition 2.3.9.

Proposition 7.5.1 (Fuchs regularity criterion). *The following conditions are equivalent:*

(1) *the system* (7.2.1) *is regular;*

(2) *for* some (*resp.* any) *scalar differential equation* $Ly = 0$ *equivalent to* (7.2.1), *where*

$$L = \partial_x^\mu + a_{\mu-1}\,\partial_x^{\mu-1} + \cdots + a_1\,\partial_x + a_0, \quad a_i \in K((x)),$$

we have $\mathrm{ord}_0(a_i) \geqslant i - \mu$, *for* $i = 0, \ldots, \mu - 1$*;*

(3) *for* some (*resp.* any) *scalar differential equation* $\Lambda y = 0$ *equivalent to* (7.2.1), *where*

$$\Lambda = \vartheta_x^\mu + b_{\mu-1}\,\vartheta_x^{\mu-1} + \cdots + b_1\,\vartheta_x + b_0, \quad b_i \in K((x)),$$

we have $b_i \in K[[x]]$, *for* $i = 0, \ldots, \mu - 1$.

Proof. The equivalence of (2) and (3) follows from 2.2.6.

The implication (3) $\Rightarrow$ (1) is also clear by the construction of the system (2.3.4) associated to the scalar differential equation (2.3.1).

We are only left to prove (1) $\Rightarrow$ (3); we will apply Proposition 7.4.1. Thus, the system (7.2.1) admits a fundamental solution matrix of the form Wx^A, with $W \in \mathrm{GL}_\mu(K((x)))$ and $A \in M_\mu(K)$. Any scalar differential operator of the form of Λ, which is equivalent to our system may be obtained using $Y = x^A$: we choose a row $(y_1, \ldots, y_\mu) = (u_1, \ldots, u_\mu)x^A$, with $u_i \in K((x))$, such that the wronskian

$$w(y_1, \ldots, y_\mu) := \det \begin{pmatrix} y_1 & \cdots & y_\mu \\ \vartheta_x y_1 & \cdots & \vartheta_x y_\mu \\ \vdots & \ddots & \vdots \\ \vartheta_x^{\mu-1} y_1 & \cdots & \vartheta_x^{\mu-1} y_\mu \end{pmatrix} \tag{7.5.2}$$

is non-zero, and construct the monic linear differential operator $\Lambda \in \mathcal{F}\langle\vartheta_x\rangle$ of minimal order that has the solutions $(y_1, \ldots, y_\mu)$. By the condition on the wronskian, $y_1, \ldots, y_\mu$ are linearly independent over the field of constants $\overline{K}$, and the order of Λ is μ. We have

$$\vartheta_x(y_1, \ldots, y_\mu) = (\vartheta_x(u_1, \ldots, u_\mu) + (u_1, \ldots, u_\mu)A)x^A,$$

and $(\vartheta_x(u_1,\dots,u_\mu)+(u_1,\dots,u_\mu)A)$ is a row of elements of $K((x))$. Iterating, we get

$$\text{(7.5.3)}\qquad \begin{pmatrix} y_1 & \cdots & y_\mu \\ \vartheta_x y_1 & \cdots & \vartheta_x y_\mu \\ \vdots & \ddots & \vdots \\ \vartheta_x^\mu y_1 & \cdots & \vartheta_x^\mu y_\mu \end{pmatrix} = \begin{pmatrix} v_{01} & \cdots & v_{0\mu} \\ v_{11} & \cdots & v_{1\mu} \\ \vdots & \ddots & \vdots \\ v_{\mu 1} & \cdots & v_{\mu\mu} \end{pmatrix} x^A = Vx^A,$$

with $v_{ij}\in K((x))$. Let V_i be the matrix obtained by removing from V the row of index i, e.g.,

$$\text{(7.5.4)}\qquad V_\mu = \begin{pmatrix} v_{01} & \cdots & v_{0\mu} \\ v_{11} & \cdots & v_{1\mu} \\ \vdots & \ddots & \vdots \\ v_{\mu-1,1} & \cdots & v_{\mu-1,\mu} \end{pmatrix}.$$

The wronskian matrix of $(y_1,\dots,y_\mu)$ is

$$\text{(7.5.5)}\qquad W(y_1,\dots,y_\mu)=V_\mu x^A,$$

so that $w(y_1,\dots,y_\mu)\neq 0$ is equivalent to $\det V_\mu \neq 0$. Explicitly,

$$\text{(7.5.6)}\qquad \Lambda(y)=(-1)^\mu(\det V_\mu x^A)^{-1}\det\begin{pmatrix} y & y_1 & \cdots & y_\mu \\ \vartheta_x y & \vartheta_x y_1 & \cdots & \vartheta_x y_\mu \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_x^{\mu-1}y & \vartheta_x^{\mu-1}y_1 & \cdots & \vartheta_x^{\mu-1}y_\mu \\ \vartheta_x^\mu y & \vartheta_x^\mu y_1 & \cdots & \vartheta_x^\mu y_\mu \end{pmatrix}$$

We get $\Lambda=\vartheta_x^\mu+b_{\mu-1}\vartheta_x^{\mu-1}+\cdots+b_j\vartheta_x^j+\cdots+b_0$, where

$$\text{(7.5.7)}\qquad b_j=(-1)^{\mu-j}\frac{\det(V_jx^A)}{\det(V_\mu x^A)}=(-1)^{\mu-j}\frac{\det V_j}{\det V_\mu}$$

showing that $\Lambda\in F\langle\vartheta_x\rangle$. Next, we have to prove that $b_j\in K[[x]]$, for all j. For this we may use the elementary proof of [38, III, Theorem 8.9], that we repeat for the reader convenience. By base change with a matrix $C\in \mathrm{GL}_\mu(K')$ we may assume that the matrix A is of the form $\left(\begin{smallmatrix}\alpha I+N & 0\\ 0 & A'\end{smallmatrix}\right)$, where N is a standard nilpotent matrix (all the entries are 0, outside the upper diagonal with entries all equal to 1). Changing Λ to $x^{-\alpha}\Lambda x^\alpha$ (which is in $K[[x]]\langle\vartheta_x\rangle$ if and only if Λ is) we may assume that $A=\left(\begin{smallmatrix}N & 0\\ 0 & A'\end{smallmatrix}\right)$. In particular, $y_1=u_1\neq 0$ and changing Λ to $u_1^{-1}\Lambda u_1$ (which is in $K[[x]]\langle\vartheta_x\rangle$ if and only if Λ is) we may assume that $y_1=u_1=1$ and the equality

$$(1,y_2,\dots,y_\mu)=(1,u_2,\dots,u_\mu)x^{\left(\begin{smallmatrix}N & 0\\ 0 & A'\end{smallmatrix}\right)}$$

holds, with $w(1, y_2, \ldots, y_\mu) = w(y_2, \ldots, y_\mu) \neq 0$. The differential system is then

$$\begin{aligned}\vartheta_x(1, y_2, \ldots, y_\mu) &= \vartheta_x\Big((1, u_2, \ldots, u_\mu)x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)}\Big) \\ &= ((0, \vartheta_x u_2, \ldots, \vartheta_x u_\mu) + (1, u_2, \ldots, u_\mu)\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right))x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)} \\ &= (0, v_2, \ldots, v_\mu)x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)},\end{aligned}$$

and erasing the first terms we have a system

$$\vartheta_x(y_2, \ldots, y_\mu) = (v_2, \ldots, v_\mu)x^{\left(\begin{smallmatrix} N' & 0 \\ 0 & A' \end{smallmatrix}\right)},$$

where N' is the standard nilpotent matrix obtained by removing first row and column of N. We can now conclude by induction on μ that the differential operator Λ' associated to this system has coefficients in $K[[x]]$. Therefore also $\Lambda = \Lambda'\vartheta_x$ has the same property. □

Remark 7.5.8. We shall give another proof of these criteria in 15.2.

7.6 Exponents

The remaining part of the classical theory of regular singular points deals with exponents.

Proposition-Definition 7.6.1 (Exponents). *Let G be the matrix of a regular system as in (7.2.1), and let G_P be the matrix of an equivalent system with entries in $K[[x]]$. Then the classes in $\overline{K}/\mathbb{Z}$ of the eigenvalues of $G_P(0)$ (counted with multiplicity), do not depend on G_P. They are called the* exponents *of the differential system (7.2.1).*

Here is a more precise form of this result.

Lemma 7.6.2. *Let $G, H \in M_\mu(K((x))$, and $P \in \mathrm{GL}_\mu(K((x)))$ be such that $H = G_P$. Assume that there exist matrix solutions of $(\vartheta_x - G)Y = 0$ (resp. of $(\vartheta_x - H)Y = 0$) of the form $Y_G x^A$ (resp. $Y_H x^B$) with $A, B \in M_\mu(K)$, $Y_G, Y_H \in \mathrm{GL}_\mu(K((x)))$. Let*

$$A = C_2^{-1}(\Delta + N)C_2, \quad B = C_1(\Delta' + N')C_1^{-1}, \quad C_1, C_2 \in \mathrm{GL}_\mu(K)$$

with Δ, Δ' diagonal, N, N' nilpotent, $\Delta N = N\Delta$, $\Delta' N' = N'\Delta'$. Then there exists $\overline{C} \in \mathrm{GL}_\mu(K)$ such that

$$x^{\Delta'}\overline{C}x^{-\Delta} \in \mathrm{GL}_\mu(K[x, x^{-1}]).$$

Proof. Let us write $\Delta = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_\mu \end{pmatrix}$ and $\Delta' = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_\mu \end{pmatrix}$. We have $PY_G x^A = Y_H x^B \widetilde{C}$, for some $\widetilde{C} \in \mathrm{GL}_\mu(\overline{K})$. Let $\overline{P} = Y_H^{-1} P Y_G \in \mathrm{GL}_\mu(K((x)))$, so $\overline{P} x^A = x^B \widetilde{C}$. But this is

$$\overline{P} C_2^{-1} x^{\Delta + N} C_2 = C_1 x^{\Delta' + N'} C_1^{-1} \widetilde{C},$$

so that

$$x^{\Delta'} x^{N'} C_1^{-1} \widetilde{C} C_2^{-1} x^{-N} x^{-\Delta} = C_1^{-1} \overline{P} C_2^{-1} \in \mathrm{GL}_\mu(K((x))).$$

Set $\overline{C} := C_1^{-1} \widetilde{C} C_2^{-1} = (c_{ij})_{ij} \in \mathrm{GL}_\mu(\overline{K})$; then

$$x^{\Delta'} x^{N'} \overline{C} x^{-N} x^{-\Delta} \in \mathrm{GL}_\mu(K((x))).$$

Now $x^{N'} \overline{C} x^{-N}$ is of the form $(P_{ij}(\log x))_{ij}$, for some $P_{ij}(T) \in \overline{K}[T]$ so that

$$x^{\Delta'} x^{N'} \overline{C} x^{-N} x^{-\Delta} = (x^{\beta_i - \alpha_j} P_{ij}(\log x))_{ij} \in \mathrm{GL}_\mu(K((x))).$$

By Lemma 7.1.2, we conclude that each $P_{ij}(T)$ is an element of K, so that $x^{N'} \overline{C} x^{-N} = \overline{C}$ and

$$x^{\Delta'} \overline{C} x^{-\Delta} = (c_{ij} x^{\beta_i - \alpha_j})_{ij} \in \mathrm{GL}_\mu(K((x))). \tag{7.6.3}$$

□

Definition 7.6.4 (Multiplicities). *Formula (7.6.3) shows that* $(\alpha_1 + \mathbb{Z}, \dots, \alpha_\mu + \mathbb{Z}) \in (\overline{K}/\mathbb{Z})^\mu$ *is a permutation of* $(\beta_1 + \mathbb{Z}, \dots, \beta_\mu + \mathbb{Z})$*. The number of times a given element* $\overline{\alpha}$ *of* $\overline{K}/\mathbb{Z}$ *appears in those rows is called the* multiplicity of the exponent $\overline{\alpha}$ *of the system* $(\vartheta_x - G)Y = 0$.

8 Jordan decomposition of differential modules

In this section we introduce some general tools which we will use in this chapter for the theory of regular differential modules, and in Chapter V for the formal theory of irregular differential modules. The main tool is a sort of generalization of the classical Jordan decomposition of linear operators to the case of differential modules, and its stability under the action of commuting derivations. The main results of this chapter are the existence of the Jordan decomposition in the regular case, and a variant with parameters.

8.1 Jordan theory for differential modules

8.1.1. Let (F, ∂) be a differential field and let $K = F^\partial$ be its field of constants (of characteristic 0, as always). For any differential module (M, ∇_∂) over (F, ∂), for any $\phi \in F$, and any $\nu = 0, 1, \dots$, we set (identifying ϕ with a homothety of M)

$$K_\phi^{(\nu)} = K_\phi^{(\nu)}(M) = \mathrm{Ker}_M (\nabla_\partial - \phi)^\nu,$$

and

$$M_\phi^{(\nu)} = \mathrm{Im}(F \otimes_K K_\phi^{(\nu)}(M) \longrightarrow M).$$

The $K_\phi^{(\nu)}$'s are K-vector subspaces of M (endowed with a nilpotent endomorphism $\nabla_\partial - \phi$), and the $M_\phi^{(\nu)}$'s are differential submodules of M. We have

$$(0) = K_\phi^{(0)} \subseteq K_\phi^{(1)} \subseteq \cdots$$

and

$$(0) = M_\phi^{(0)} \subseteq M_\phi^{(1)} \subseteq \cdots$$

For all $\phi, \psi \in F$ and for any $\lambda, \nu = 0, 1, \ldots$, we have

$$K_\phi^{(\lambda)}(M_\psi^{(\nu)}) = K_\phi^{(\lambda)}(M) \cap M_\psi^{(\nu)}$$

while, *a priori*, only

$$(M_\psi^{(\nu)})_\phi^{(\lambda)} \subseteq M_\phi^{(\lambda)} \cap M_\psi^{(\nu)}.$$

Lemma 8.1.2. *If $K_\phi^{(\nu)} = K_\phi^{(\nu+1)}$, then $K_\phi^{(\nu+1)} = K_\phi^{(\nu+2)} = \cdots$.*

Proof. Let $m \in K_\phi^{(\nu+2)}$, so that $(\nabla_\partial - \phi)m \in K_\phi^{(\nu+1)} = K_\phi^{(\nu)}$. Then $(\nabla_\partial - \phi)^{\nu+1} m = 0$, and $m \in K_\phi^{(\nu+1)}$. □

8.1.3 (Condition $(*)$). The following conditions on (F, ∂) are equivalent:

(1) $\mathrm{Ker}_F \partial^n \subseteq \mathrm{Ker}_F \partial$ $(= K)$ for all $n \geqslant 1$ (so the equalities hold);

(2) $\mathrm{Ker}_F \partial^2 \subseteq \mathrm{Ker}_F \partial$ $(= K)$ (so the equality holds);

(3) $\mathrm{Im}_F(\partial) \cap \mathrm{Ker}_F(\partial) = 0$, that is, for all $a \in F$ we have: $\partial(a) \in K$ (if and) only if $\partial(a) = 0$.

(4) for all $a \in F$ we have: $\partial^n(a) \in K$ for some $n \geqslant 1$ (if and) only if $\partial(a) = 0$.

The non-trivial implication "(2) implies (1)" is proved by induction; the equivalence of (2) and (3) follows from the trivial remark that $\partial^2(a) = 0$ if and only if $\partial(a) \in K (= \mathrm{Ker}_F \partial)$; the equivalence of (4) and (3) (or (4) and (1)) is straightforward.

Remark 8.1.4. The above conditions are satisfied by $F = K((x))$ and $\partial = x^r \partial_x$ if and only if $r = 1$. More generally, a derivation (of $F = K((x))$) of the form $g(x) x \partial_x$ satisfies the conditions if and only if $g(x)$ is a unit of $K[[x]]$.

Proposition 8.1.5. *Let (M, ∇_∂) be a differential module over a differential field (F, ∂) satisfying conditions $(*)$ of 8.1.3. Then, for any $\phi \in F$ and $\nu = 1, 2, \ldots$, the natural map*

$$F \otimes_K K_\phi^{(\nu)}(M) \longrightarrow M$$

is an injection.

Proof. We proceed by induction on ν. Let $m_1, \dots, m_s$ be elements of $K_\phi^{(1)} = \mathrm{Ker}_M(\nabla_\partial - \phi)$ with the following properties:

(1) $m_1, \dots, m_s$ are linearly independent over K;

(2) there exists a relation $m_s = \sum_{i=1}^{s-1} a_i m_i$, $a_i \in F$;

(3) s is minimal for the properties (1) and (2).

We apply $\nabla_\partial - \phi$ to m_s and obtain a relation $\displaystyle\sum_{i=1}^{s-1} (\partial a_i) m_i = 0$. By minimality, $\partial a_i = 0$, so that $a_i \in K$, for all $i = 1, \dots, s-1$, which contradicts property (1). This proves the case $\nu = 1$, for any (M, ∇_∂) and $\phi \in F$. Let us now assume that $F \otimes_K K_\phi^{(\nu)}(M)$ injects into M. Let N be the quotient differential module $M/M_\phi^{(1)}$. The kernel of the natural projection

$$\pi : K_\phi^{(\nu+1)}(M) \longrightarrow K_\phi^{(\nu)}(N)$$

is $\mathrm{Ker}(\pi) = K_\phi^{(\nu+1)}(M) \cap M_\phi^{(1)} = K_\phi^{(\nu+1)}(M) \cap (F \otimes_K K_\phi^{(1)}(M))$, by the case $\nu = 1$. Let $\displaystyle\sum_{i=1}^{s} a_i \otimes m_i \in \mathrm{Ker}\, \pi$, with $(m_1, \dots, m_s)$ a K-basis of $K_\phi^{(1)}$ and $a_i \in F$. We have

$$\begin{aligned} 0 = (\nabla_\partial - \phi)^{\nu+1} \sum_{i=1}^{s} a_i m_i &= (\nabla_\partial - \phi)^{\nu} \Big(\sum_{i=1}^{s} (\partial a_i) m_i + \sum_{i=1}^{s} a_i \underbrace{(\nabla_\partial - \phi) m_i}_{=0} \Big) \\ &= (\nabla_\partial - \phi)^{\nu} \sum_{i=1}^{s} (\partial a_i) m_i = \cdots = \sum_{i=1}^{s} (\partial^{\nu+1} a_i) m_i. \end{aligned}$$

Therefore, $\partial^{\nu+1} a_i = 0$, hence (by the condition $(*)$ of 8.1.3) $a_i \in K, \forall\, i = 1, \dots, s$, so that $\mathrm{Ker}\, \pi = K_\phi^{(1)}$. We then have an isomorphism

$$K_\phi^{(\nu+1)}(M)/K_\phi^{(1)}(M) \xrightarrow{\sim} K_\phi^{(\nu)}(N).$$

By the induction assumption (for the differential module $N = M/M_\phi^{(1)}$)

$$\frac{F \otimes_K K_\phi^{(\nu+1)}(M)}{F \otimes_K K_\phi^{(1)}(M)} = F \otimes_K \frac{K_\phi^{(\nu+1)}(M)}{K_\phi^{(1)}(M)} \cong F \otimes_K K_\phi^{(\nu)}(N) \hookrightarrow N = \frac{M}{F \otimes_K K_\phi^{(1)}(M)}.$$

Therefore, $F \otimes_K K_\phi^{(\nu+1)}(M)$ injects into M. □

Corollary 8.1.6. *For any $\lambda, \nu = 0, 1, \dots,$ and any $\phi \in F$, we have*

$$(M/M_\phi^{(\lambda)})_\phi^{(\nu)} = M_\phi^{(\lambda+\nu)}/M_\phi^{(\lambda)} \,. \tag{8.1.7}$$

Proof. We already showed, in the proof of the lemma, that the natural projection $\pi : M \to N = M/M_\phi^{(1)}$ identifies $M_\phi^{(\nu+1)}/M_\phi^{(1)}$ to $N_\phi^{(\nu)}$. This is the present statement for $\lambda = 1$, which we call the *basic case.* So, we now assume the statement to hold for some λ, for any M and any $\nu = 0, 1, \dots,$ and show that it holds for $\lambda + 1$ and any M and ν. We write

$$\begin{aligned}
&(M/M_\phi^{(\lambda+1)})_\phi^{(\nu)} \\
&= ((M/M_\phi^{(1)})/(M_\phi^{(\lambda+1)}/M_\phi^{(1)}))_\phi^{(\nu)} \quad \text{(by the basic case)} \\
&= ((M/M_\phi^{(1)})/(M/M_\phi^{(1)})_\phi^{(\lambda)})_\phi^{(\nu)} \quad \text{(by the inductive assumption)} \\
&= (M/M_\phi^{(1)})_\phi^{(\lambda+\nu)}/(M/M_\phi^{(1)})_\phi^{(\lambda)} \quad \text{(by two instances of the basic case)} \\
&= (M_\phi^{(\lambda+\nu+1)}/M_\phi^{(1)})/(M_\phi^{(\lambda+1)}/M_\phi^{(1)}) = M_\phi^{(\lambda+\nu+1)}/M_\phi^{(\lambda+1)}. \qquad \square
\end{aligned}$$

Corollary 8.1.8. *For any $\nu = 0, 1, \dots,$ and $\phi \in F$, the following conditions are equivalent:*

(1) $K_\phi^{(\nu)} = K_\phi^{(\nu+1)}$;

(2) $M_\phi^{(\nu)} = M_\phi^{(\nu+1)}$.

In particular, $\dim_K K_\phi^{(\nu)} \leqslant \mu := \dim_F M$, *and the sequence $K_\phi^{(\nu)}$ (resp. $M_\phi^{(\nu)}$) is stationary from $\nu = \mu$ on.*

Proposition 8.1.9. *Let (M, ∇_∂) be a differential module over a differential field (F, ∂) satisfying assumption $(*)$ of 8.1.3, and let $\phi, \psi \in F$. If* $\mathrm{Ker}_F(\partial - \phi + \psi) = (0)$, *then $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$ for any (λ, ν). Otherwise, $M_\phi^{(\nu)} = M_\psi^{(\nu)}$ for any ν.*

Proof. Let us assume that $\mathrm{Ker}_F(\partial - \phi + \psi) = (0)$. We first prove that $M_\phi^{(1)} \cap M_\psi^{(1)} = (0)$. Let $m = \sum_{i=1}^s a_i m_i = \sum_{j=1}^r b_j n_j \in M_\phi^{(1)} \cap M_\psi^{(1)}$, with $m_1, \dots, m_s$ linearly independent in $K_\phi^{(1)}$ (resp. $n_1, \dots, n_r$ linearly independent in $K_\psi^{(1)}$), and with s *minimal.* Then

$$\nabla_\partial m = \sum_{i=1}^s (\partial a_i) m_i + \phi m = \sum_{j=1}^r (\partial b_j) n_j + \psi m.$$

Combining the two equations

$$\sum_{i=1}^s (\partial a_i + \phi a_i) m_i = \sum_{j=1}^r (\partial b_j + \psi b_j) n_j \quad \text{and} \quad \sum_{i=1}^s a_i m_i = \sum_{j=1}^r b_j n_j$$

we get

$$\sum_{i=1}^{s-1} ((\partial a_s + \phi a_s) a_i - a_s(\partial a_i + \phi a_i)) m_i = \sum_{j=1}^r ((\partial a_s + \phi a_s) b_j - a_s(\partial b_j + \psi b_j)) n_j\,,$$

so that all coefficients of the m_i's and n_j's figuring here must vanish (by the minimality of s). In particular, $(\partial a_s + \phi a_s)b_j - a_s(\partial b_j + \psi b_j)) = 0\ \forall\, j$, may be read, if $b_j \neq 0$, as $\partial(a_s/b_j) = (\psi - \phi)a_s/b_j$, which implies (by hypothesis) $a_s = 0$, a contradiction to the minimality of s. So, $b_j = 0\ \forall\, j$, hence $m = 0$.

We now prove by induction on λ that $M_\phi^{(\lambda)} \cap M_\psi^{(1)} = (0)$, for all $\lambda = 1, 2, \dots$. We set again $N = M/M_\phi^{(1)}$, and denote by $\pi : M \to N$ the natural projection. Then π identifies $M_\phi^{(\lambda+1)}/M_\phi^{(1)}$ to $N_\phi^{(\lambda)}$, by Corollary 8.1.6, and $M_\psi^{(1)}$ to $N_\psi^{(1)}$, by the case $\lambda = 1$ treated above. By the induction assumption, $N_\phi^{(\lambda)} \cap N_\psi^{(1)} = (0)$. We get

$$M_\phi^{(\lambda+1)} \cap (M_\phi^{(1)} + M_\psi^{(1)}) = M_\phi^{(1)},$$

whence

$$M_\phi^{(\lambda+1)} \cap M_\psi^{(1)} = M_\phi^{(1)} \cap M_\psi^{(1)} = (0).$$

We now prove by induction on ν that $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$, for all $\nu = 1, 2, \dots$, the case $\nu = 1$ being already discussed. We set here $N = M/M_\psi^{(1)}$, and denote by $\pi : M \to N$ the natural projection. As before, π identifies $M_\psi^{(\nu+1)}/M_\psi^{(1)}$ to $N_\psi^{(\nu)}$, by Corollary 8.1.6, and $M_\phi^{(\lambda)}$ to $N_\phi^{(\lambda)}$, by the previous case. Induction on ν shows that $N_\phi^{(\lambda)} \cap N_\psi^{(\nu)} = (0)$, so that $(M_\phi^{(\lambda)} + M_\psi^{(1)}) \cap M_\psi^{(\nu+1)} = M_\psi^{(1)}$, and finally $M_\phi^{(\lambda)} \cap M_\psi^{(\nu+1)} = (0)$. Let us now prove the second assertion. We first remark that if $\partial f = cf$, for $c \in F$ and $f \in F^\times$, then $K_{\phi+c}^{(\nu)} = fK_\phi^{(\nu)}$ for every $\phi \in F$ and every ν. Therefore, if $\phi, \psi \in F$ are such that $\mathrm{Ker}_F(\partial - \phi + \psi) \neq (0)$, then $M_\phi^{(\nu)} = M_\psi^{(\nu)}$, for every ν. □

8.1.10 (Logarithmic derivatives). We denote by $\partial \log F^\times$ the additive subgroup of F consisting of *logarithmic derivatives*, i.e., elements of the form $f^{-1}\partial f$ for $f \in F^\times$.

An element u of F is in $\partial \log F^\times$ if and only if there exists $f \in F^\times$ such that $\partial f = uf$, that is, if and only if $\mathrm{Ker}(\partial - u) \neq \{0\}$.

As an example, take $F = K((x))$ and $\partial = \vartheta_x = x\partial_x$. Then $\partial \log F^\times$ is $\mathbb{Z} \oplus xK[[x]]$ (in particular, we have $\partial \log F^\times \cap K\left[\frac{1}{x}\right] = \mathbb{Z}$ and $\partial \log F^\times + K\left[\frac{1}{x}\right] = F$) and the quotient $F/\partial \log F^\times$ is isomorphic to $K\left[\frac{1}{x}\right]/\mathbb{Z} \cong K/\mathbb{Z} \oplus \frac{1}{x}K\left[\frac{1}{x}\right]$.

8.1.11. Notice that for each ϕ_i and for any logarithmic derivative $\psi = \partial \log(u)$ we have a canonical isomorphism of K-vector spaces $K_\phi^{(\nu)} \cong K_{\phi+\psi}^{(\nu)}$ (sending f to uf), and the equality $M_\phi^{(\nu)} = M_{\phi+\psi}^{(\nu)}$. In particular one may add a logarithmic derivative without changing the differential submodule $M_{\phi_i}^{(\mu)}$.

Definition 8.1.12 (Jordan differential modules, characters). *Let (F, ∂) be a differential field satisfying $(*)$ of 8.1.3. We say that the differential module (M, ∇_∂) of rank μ over (F, ∂)* admits a Jordan decomposition, *or* is a Jordan module, *if there exist $\phi_1, \ldots, \phi_r \in F$, such that*

$$M = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)},$$

where the ϕ_i's are pairwise distinct modulo $\partial \log F^\times$. Their classes $\overline{\phi}_i$ in $F/\partial \log F^\times$ are called the characters[3] *of the Jordan module. We define also the multiplicities of the character ϕ to be* $\dim_K(K_\phi^{(\mu)}) = \dim_F(M_\phi^{(\mu)})$.

Remark 8.1.13. The characters $\overline{\phi}_1, \ldots, \overline{\phi}_r \in F/\partial \log F^\times$ of (M, ∇_∂) and their multiplicities are obviously uniquely defined by our construction. In case $F = K((x))$ and $\partial' = u(x)\partial$ for a unit $u(x) \in K[[x]]^\times$, the F-vector space M which supports the differential module (M, ∇_∂) over (F, ∂) also supports the differential module $(M, \nabla_{\partial'} = u(x)\nabla_\partial)$ over (F, ∂'). The characters $\overline{\phi_1}, \ldots, \overline{\phi_r}$ of the former get respectively changed into $\overline{u(x)\phi_1}, \ldots, \overline{u(x)\phi_r}$, the modules $(M_{\phi_1}^{(\mu)}, \ldots, M_{\phi_r}^{(\mu)})$ into $(M_{u(x)\phi_1}^{(\mu)}, \ldots, M_{u(x)\phi_r}^{(\mu)})$, and the multiplicity of $\overline{u(x)\phi_i}$ for $(M, \nabla_{\partial'})$ over (F, ∂') coincides with the one of $\overline{\phi}_i$ for (M, ∇_∂) over (F, ∂).

8.1.14. The set of characters of M will be denoted by $\mathrm{Chr}(M)$. It is the set of (classes of) $\phi \in F$ for which

$$K_\phi^{(\mu)}(M) = \mathrm{Ker}_M(\nabla_\partial - \phi)^\mu \neq 0.$$

For example, consider again $F = K((x))$ and $\partial = \vartheta_x = x\partial_x$. Then the trivial differential module $M = F$ has $\mathrm{Chr}(M) = \{0\}$, while $M = F.x^\alpha$ for $\alpha \in K$ has $\mathrm{Chr}(M) = \{\alpha\}$. The differential module $M = F.\exp(\phi)$ for $\phi \in F$ has $\mathrm{Chr}(M) = \{\partial(\phi)\}$.

In general, differential modules of rank 1 are Jordan modules, and they are parametrized by characters as elements of $F/\partial \log F^\times$. By a recursive argument, any differential module is a direct sum of a Jordan module and a differential module without rank 1 submodules or quotients.

Proposition 8.1.15 (Structure of Jordan modules). *Every Jordan module over a differential field (F, ∂), satisfying $(*)$ of 8.1.3, is an iterated extension of differential modules of rank 1. Conversely, a differential module (M, ∇_∂) over a differential field (F, ∂) which is an iterated extension of differential modules of rank 1, is a Jordan module. Its characters are the characters of its components of rank 1 and*

[3] This terminology is not standard but will be convenient. It is justified by the tannakian viewpoint: the characters of a Jordan module M are the characters of the tannakian group attached to the tannakian category of differential modules generated by M.

the multiplicity of a character coincides with the multiplicity of the corresponding component in the sense of Jordan-Hölder.

Proof. This is achieved by writing $K_{\phi_i}^{(\mu)}$ as a successive extension of $K[\nabla_\partial - \phi_i]$-modules of K-rank 1 (note that $\nabla_\partial - \phi_i$ is a nilpotent endomorphism of $K_{\phi_i}^{(\mu)}$), tensoring by F and summing up over i. $\square$

8.1.16. The structure of Jordan modules up to isomorphism is completely determined by the characters ϕ_i and for any character the sequence of integer numbers

$$d_{\phi_i}^{(\nu)} = \dim_K(K_{\phi_i}^{(\nu)}) = \dim_F(M_{\phi_i}^{(\nu)})$$

for $\nu = 1, \ldots, \mu$. The following immediate consequence of Corollary 8.1.6 will be used in the next subsection.

Lemma 8.1.17. *Any element m of a Jordan module M such that $(\nabla_\partial - \phi)(m) \in M_\phi^{(\nu)}$ belongs to $M_\phi^{(\nu+1)}$.*

Lemma 8.1.18. *Subquotients, extensions, tensor products and duals of Jordan modules are Jordan modules.*

If M is a Jordan module over F, then $M_{F'}$ is a Jordan module over any differential extension F' of F with the same characters, taken modulo $\partial \log F'^\times$.

For characters, one has the formulas (comparing subsets of the abelian group $F/\partial \log F^\times$):

$$\mathrm{Chr}(M) = \mathrm{Chr}(M_1) \cup \mathrm{Chr}(M_2)$$

if M is an extension of M_1 by M_2, and

$$\begin{aligned} \mathrm{Chr}(M \otimes M') &= \mathrm{Chr}(M) + \mathrm{Chr}(M'), \\ \mathrm{Chr}(M^\vee) &= -\mathrm{Chr}(M), \\ \mathrm{Chr}(M_{F'}) &= \mathrm{Chr}(M). \end{aligned}$$

The proof is straightforward. $\square$

Proposition 8.1.19. *Let (M, ∇_∂) be a differential module over a differential field (F, ∂) satisfying assumption $(*)$ of 8.1.3. Suppose that there exists a differential extension (G, ∂) such that the module M_G admits a Jordan decomposition (over G). Then there exists a minimal extension F' of F under which $M_{F'}$ admits a Jordan decomposition. Moreover, F' is a Galois extension of F, it is generated over F by the characters of M (in G), and the degree is a divisor of $\mu!$.*

Proof. We may suppose that G is a Galois extension of F of Galois group Γ. Let $\phi_1, \ldots, \phi_r \in G$ the elements appearing in the Jordan decomposition of M_G (pairwise distinct modulo $\partial \log G^\times$). Then the elements of Γ permute the classes $\overline{\phi}_i$. Therefore, using the normal subgroup Γ' of Γ acting trivially on the characters of M_G, the projectors which define the Jordan decomposition of M_G descend to define a Jordan decomposition of $M_{F'}$ where F' is the Galois extension of F corresponding to Γ'. The degree of F' over F is $[\Gamma : \Gamma']$, which divides $\mu!$. $\square$

8.2 Action of commuting derivations

We keep the notation of the previous subsection: namely, (F, ∂) is a differential field of characteristic 0, and $K = F^{\partial}$ is its field of constants. Let Δ be a set of derivations δ of F which commute with ∂. One thus has $\Delta(K) \subseteq K$. Let $k = K^{\Delta} = F^{\partial,\Delta}$ be the field of simultaneous constants of ∂ and the δ's. Notice that k is algebraically closed in K.

Let us assume (M, ∇_∂) is endowed with a map

$$\Delta \longrightarrow \operatorname{End}_{k\langle\partial\rangle} M, \quad \delta \longmapsto \nabla_\delta$$

such that every ∇_δ satisfies the Leibniz rule (hence makes (M, ∇_δ) into a differential module over (F, δ)).

Proposition 8.2.1 (Stability of Jordan decomposition). *If (M, ∇_∂) has a Jordan decomposition*

$$M = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)},$$

this decomposition is stable under ∇_δ for any $\delta \in \Delta$.

Proof. We prove, by induction on ν, that $\nabla_\delta(M_\phi^{(\nu)}) \subseteq M_\phi^{(\nu+1)}$. This is trivial for $\nu = 0$, so let us deduce the case ν from the case $\nu - 1$. It is enough to show that for any $m \in K_\phi^{(\nu)}$, $\nabla_\delta(m)$ belongs to $M_\phi^{(\nu+1)}$. We know that $(\nabla_\partial - \phi)(m)$ belongs to $K_\phi^{(\nu-1)}$. By the induction hypothesis, $\nabla_\delta(\nabla_\partial - \phi)(m) \in M_\phi^{(\nu)}$. It follows that

$$(\nabla_\partial - \phi)\nabla_\delta(m) = \nabla_\delta(\nabla_\partial - \phi)(m) + \partial(\phi)m$$

belongs to $M_\phi^{(\nu)}$. Then, according to Lemma 8.1.17, $\nabla_\delta(m)$ belongs to $M_\phi^{(\nu+1)}$. $\square$

When the ϕ_i's belong to K, there is a more straightforward way to get this result. In fact, one has the following lemma, which holds without assuming that M is a Jordan module.

Lemma 8.2.2. *For any $\alpha \in K$, we have that $\nabla_\delta(K_\alpha^{(\nu)}) \subseteq K_\alpha^{(\nu+1)}$. A fortiori, $(K_\alpha^{(\mu)}, \nabla_\delta)$ is a differential module over (K, δ).*

Proof. The statement is trivial for $\nu = 0$. We proceed by induction: assume that the statement holds for $\nu < \nu_0$ ($\nu_0 \geqslant 1$) and let $m \in K_\alpha^{(\nu_0)}$. So, $(\nabla_\partial - \alpha)^{\nu_0} m = 0$ and $(\nabla_\partial - \alpha)m \in K_\alpha^{(\nu_0-1)}$. Then

$$(\nabla_\partial - \alpha)^{\nu_0+1}\nabla_\delta m = (\nabla_\partial - \alpha)^{\nu_0}\nabla_\delta(\nabla_\partial - \alpha)m + (\nabla_\partial - \alpha)^{\nu_0}(\delta\alpha)m = 0,$$

since $\delta\alpha \in K$ and by the induction assumption. $\square$

Theorem 8.2.3. *Assume that (M, ∇_∂) has a Jordan decomposition*

$$M = \bigoplus_{i=1}^{r} M^{(\mu)}_{\alpha_i}$$

with $\alpha_i \in K$. *Then:*

(1) *there exists a basis $\mathbf{m} = (m_1, \ldots, m_\mu)$ of M over F such that*

$$\nabla_\partial \,\mathbf{m} = \mathbf{m}\, H_\partial, \quad \nabla_\delta \,\mathbf{m} = \mathbf{m}\, H_\delta \ \ (\forall\, \delta \in \Delta),$$

with both H_∂ and H_δ in $M_\mu(K)$.

(2) *For any basis $\mathbf{m}$ of M as in* (1)*, the eigenvalues of H_∂ are in $k = K^\Delta$.*

Proof. (1): by Lemma 8.2.2, the $K^{(\mu)}_\alpha((M, \nabla_\partial))$ are stable under ∇_δ. This shows (1).

(2): from $\nabla_\partial \nabla_\delta = \nabla_\delta \nabla_\partial$ we get (cf. (4.4.4)) $[H_\partial, H_\delta] = \delta(H_\partial) - \partial(H_\delta) = \delta(H_\partial)$, and by induction, $[H^m_\partial, H_\delta] = \delta(H^m_\partial)$. To show that the case m implies the case $m+1$, we compute

$$\begin{aligned}\delta(H^{m+1}_\partial) &= \delta(H^m_\partial) H_\partial + H^m_\partial \delta(H_\partial)\\ &= (H^m_\partial H_\delta - H_\delta H^m_\partial) H_\partial + H^m_\partial (H_\partial H_\delta - H_\delta H_\partial)\\ &= -H_\delta H^{m+1}_\partial + H^{m+1}_\partial H_\delta\ .\end{aligned}$$

So, $\delta(\mathrm{Tr}(H^m_\partial)) = \mathrm{Tr}(\delta(H^m_\partial)) = 0$. Therefore, the coefficients of the characteristic polynomial of H_∂ are δ-constants, for any $\delta \in \Delta$, hence are in k. A fortiori, the eigenvalues of H_∂ are algebraic over k. Since they coincide with the $\alpha_i \in K$ modulo logarithmic derivatives, hence are in K, and since k is algebraically closed in K, we conclude that the eigenvalues of H_∂ are in k. □

8.3 The regular case

8.3.1. We come back to the case $(F, \partial) = (K((x)), \vartheta_x = x\partial_x)$. Condition $(*)$ of 8.1.3 is then satisfied, and the only logarithmic derivatives which belong to K are the integers ($n = \vartheta_x(x^n)/x^n$):

$$K/\mathbb{Z} \subseteq F/\partial \log F^\times .$$

For any $u \in K((x))^\times$, passing from $(M, \nabla_{u\partial_x})$ to $(M, \nabla_{\vartheta_x} = x\nabla_{\partial_x})$ identifies differential modules over $(K((x)), u\partial_x)$ and differential modules over $(K((x)), \vartheta_x)$. We shall just say that M is a differential module over $K((x))$.

The following definition formalizes the notion of regular differential module over $(F, \partial) = (K((x)), \vartheta_x = x\partial_x)$.

Definition 8.3.2 (Regularity for differential modules). *A differential module M over $K((x))$ is* regular *if it contains a $K[[x]]$-lattice (i.e., a free $K[[x]]$-module which spans M) stable under $x\nabla_{\partial_x} = \nabla_{\vartheta_x}$.*

The discussion of Section 7 on regular differential systems may be summarized in the following statements in the language of differential modules.

Theorem 8.3.3 (Regularity criteria for differential modules). *The following conditions on a differential module M of rank μ over $K((x))$ are equivalent:*

(1) *M is regular (that is, M admits a $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice);*

(2) *for every $m \in M$, the smallest $K[[x]]$-submodule of M containing m and stable under $x\nabla_{\partial_x}$ is finitely generated;*

(3) *the monic differential operator $\Lambda \in K((x))\langle x\partial_x\rangle$ attached to some (resp. any) cyclic vector m of M has coefficients in $K[[x]]$.*

Proof. The equivalence of the first two items is clear. The equivalence with the third item follows from the Fuchs criterion 7.5.1. □

Theorem 8.3.4 (Structure of regular differential modules). *Let M be a regular differential module M of rank μ over $K((x))$. Then there exists a finite extension $K' = K[\alpha_1, \dots, \alpha_r]$ of K such that $M_{K'((x))} = K' \otimes_K M$ admits a Jordan decomposition (with respect to $\vartheta_x = x\partial_x$)*

$$M_{K'((x))} = \bigoplus_{i=1}^{r} M_{\alpha_i}^{(\mu)}, \tag{8.3.5}$$

where $M_{\alpha_i}^{(\mu)} = K'((x)) \otimes_{K'} \mathrm{Ker}_{M_{K'((x))}}(x\nabla_{\partial_x} - \alpha_i)^{\mu}$ and the $\alpha_i \in K'$ are pairwise distinct modulo $\mathbb{Z}$. This decomposition is independent of the choice of the α_i's modulo $\mathbb{Z}$ and is canonical.

Proof. The theorem follows from 7.4.1 and the definition of a regular module. The sense in which uniqueness and canonicity hold was described in general in Remark 8.1.13. □

In fact the theorem asserts an equivalence, since the converse is obvious: a differential module over $K((x))$ is regular if and only if it admits a Jordan decomposition with characters in a finite extension of K.

Theorem 8.3.6 (Exponents). *Let M be a regular differential module M of rank μ over $K((x))$ as in the previous theorem. The characters $\overline{\alpha}_i = \alpha_i + \mathbb{Z} \in \overline{K}/\mathbb{Z}$ are called the* exponents of M, *and the set of exponents is denoted by* $\mathrm{Exp}\,(M)$, *instead of* $\mathrm{Chr}(M)$. *The following conditions on an element of $\overline{K}/\mathbb{Z}$ are equivalent:*

(1) *$\overline{\alpha} \in \mathrm{Exp}\,(M)$;*

(2) *M has a non-zero solution in $x^{\overline{\alpha}}\,\overline{K}((x))$, in the sense that for any lift α of $\overline{\alpha}$ in $\overline{K}$, there is a horizontal morphism from M to the $\overline{K}((x))$-differential module generated by an element x^α such that $\vartheta_x(x^\alpha) = \alpha x^\alpha$;*

(3) *$\overline{\alpha}$ is the class modulo $\mathbb{Z}$ of an eigenvalue of the value at $x = 0$ of the matrix of $x\nabla_{\partial_x}$ in some basis of some (or any) $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice of M (notice that the exponents of M are the exponents of any associated differential system (cf. 2.6), because the relation $G(0) = {}^tH(0)$ preserves eigenvalues);*

(4) *$\overline{\alpha}$ is the class modulo $\mathbb{Z}$ of a root of the indicial polynomial of Λ (more precisely, the dimension of $\mathrm{Ker}_{M_{K'((x))}}(x\nabla_{\partial_x} - \alpha_i)^\mu$ is the sum of the multiplicities of the roots of the indicial polynomial of Λ which are congruent to α_i modulo $\mathbb{Z}$).*

Proof. This follows from the definition 8.1.12 of characters and 7.6. □

Any one of the previous conditions allows one to associate to each exponent a well-defined multiplicity (coherenty with 7.6.4).

Theorem 8.3.7 (Descent of the decomposition). *Let $\widetilde{M}$ be a $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice such that the value at $x = 0$ of the matrix of $x\nabla_{\partial_x}$ in some basis of $\widetilde{M}$ is non-resonant (in the sequel, we simply say that $\widetilde{M}$ is* non-resonant*). Then, taking α_i to be the eigenvalues of that matrix, the decomposition (8.3.5) descends to a decomposition*

$$\widetilde{M}_{K'[[x]]} = \bigoplus_{i=1}^{r} K'[[x]] \otimes_{K'} \mathrm{Ker}_{\widetilde{M}_{K'[[x]]}}(x\nabla_{\partial_x} - \alpha_i)^\mu. \tag{8.3.8}$$

In the class $\overline{\alpha}_i$, viewed as a ordered set, α_i is characterized as the supremum among the lifts α for which $\widetilde{M}$ has a solution in $x^\alpha K'[[x]]$.

Proof. This is a translation of item (3) of Proposition 7.4.1 in terms of differential modules, using the correspondence established in 2.6. □

The proof of the following stability properties with respect to the usual operations (2.7, 2.8) is straightforward and left to the reader (see also [38, III,8]).

Proposition 8.3.9. *Regularity of differential modules over $F = K((x))$ is stable under taking subquotients, extensions, tensor products, and duals.*

If M is a regular differential module over F, then $M_{F'}$ is regular over any differential extension of the form $F' = K'((x^{1/e}))$ (equipped with the K'-linear extension of the derivation ∂_x, K' being some extension of K). If K'/K is finite, and if M' is a regular differential module over F', then ${}_FM'$ is regular over F; moreover, if M is a differential module over F and $M_{F'}$ is regular as a differential module over F', then M is a regular differential module over F.

For exponents, one has the formulas (comparing subsets of the divisible abelian group $\overline{K}/\mathbb{Z}$):

$$\mathrm{Exp}\,(M) = \mathrm{Exp}\,(M_1) \cup \mathrm{Exp}\,(M_2)$$

if M is an extension of M_1 by M_2, and

$$\begin{aligned} \mathrm{Exp}\,(M \otimes M') &= \mathrm{Exp}\,(M) + \mathrm{Exp}\,(M'), \\ \mathrm{Exp}\,(M^\vee) &= -\mathrm{Exp}\,(M), \\ \mathrm{Exp}\,(M_{F'}) = e \cdot \mathrm{Exp}\,(M), &\quad \mathrm{Exp}\,({}_F M') = \frac{1}{e} \cdot \mathrm{Exp}\,(M'). \end{aligned}$$

In the last two formulas, it is understood that F' is endowed with the derivation $x' \frac{d}{dx'} = \frac{x}{e} \frac{d}{dx}$. □

Finally, as in Subsection 8.2, we consider a set Δ of derivations of F commuting with $x\partial_x$ (for instance, derivations of K, which one extends to F by setting $\delta(x) = 0$), and $k = K^\Delta$.

Theorem 8.3.10. *Assume M is a regular differential module over $K((x))$, endowed, for every $\delta \in \Delta$, with an action ∇_δ commuting with ∇_{∂_x} and making (M, ∇_δ) into a differential module over (F, δ). Assume that the exponents of (M, ∇_{∂_x}) are in $K/\mathbb{Z}$ (rather than just in $\overline{K}/\mathbb{Z}$). Then:*

(1) *there exists a basis $\mathbf{m} = (m_1, \ldots, m_\mu)$ of M over $K((x))$ such that*

$$\nabla_{\partial_x} \mathbf{m} = \mathbf{m}\, H_{\partial_x}, \quad \nabla_\delta\, \mathbf{m} = \mathbf{m}\, H_\delta \ \ (\forall\, \delta \in \Delta),$$

with both H_{∂_x} and H_δ in $M_\mu(K)$.

(2) *For any basis $\mathbf{m}$ of M as in (1), the eigenvalues of H_{∂_x} are in k.*

Proof. This follows from point (3) of theorem 8.3.4, and theorem 8.2.3. □

8.4 Variant with parameters

8.4.1. We address now the situation where K is the fraction field of a noetherian, integrally closed k-algebra R. We consider a differential module (M, ∇_{∂_x}) over the differential ring $(R((x)), \partial_x)$, where $R((x)) = R[[x]][\frac{1}{x}]$, which is "generically regular", i.e., such that $M_{K((x))}$ is regular.

The theme of this section is: to which extent does the decomposition (8.3.5) descend to $R'((x))$, where R' denotes the integral closure of R in K'?

Theorem 8.4.2. *In the setting 8.4.1:*

(1) *The exponents $\overline{\alpha}_i$ belong to $R'/\mathbb{Z}$.*

(2) *If the differences between exponents are constant, i.e., belong to the algebraic closure k' of k in R' (modulo $\mathbb{Z}$), the decomposition (8.3.5) descends to a decomposition of $M_{R'((x))}$:*

$$M_{R'((x))} = \bigoplus_{i=1}^{r} R'((x)) \otimes_{R'} \mathrm{Ker}_{M_{R'((x))}} (x\nabla_{\partial_x} - \alpha_i)^\mu. \tag{8.4.3}$$

Remark 8.4.4. Some condition on the exponents is required to descend the decomposition: for instance, the differential module M attached to the differential operator $(x\partial_x)^2 - y^2$ with parameter y (for which one can take $\overline{\alpha}_1 = y+\mathbb{Z}$, $\overline{\alpha}_2 = -y+\mathbb{Z}$) decomposes over $K[y, \frac{1}{y}]((x))$, but not over $K[y]((x))$, since $M_{|y=0}$ is not semisimple.

Of course, such a phenomenon cannot occur in the integrable case, where the α's are indeed constant.

Proof. For simplicity, let us replace M by $M_{R'((x))}$ and drop all $'$ from the notation. Since R is normal, it suffices to show that for any divisorial valuation (that is, a valuation associated to prime ideal of height 1) v of R, the α's are v-integral. We may thus replace R by its v-adic completion $\widehat{R}$ (which is a complete discrete valuation ring, with fraction field $\widehat{K}$), and M by $\widehat{M} = M_{\widehat{R}((x))}$. We fix an isomorphism $\widehat{R} \cong \kappa[[y]]$, κ being the residue field (which we regard as a subfield of $\widehat{R}$). Weierstrass' division theorem implies that the ring $\widehat{R}((x))$ is principal. Therefore, the projective $\widehat{R}((x))$-module $\widehat{M}$ is free (of rank μ).

Up to replacing $\widehat{R}$ by a finite extension, we first construct a ϑ_x-stable $\widehat{R}[[x]]$-lattice in $\widehat{M}$. For this purpose, we use a cyclic vector, at the cost of introducing apparent singularities.

More precisely, we first choose a cyclic vector $\overline{m}$ of the $\kappa((x))$-differential module $\widehat{M}_{\kappa((x))} = \widehat{M}/y\widehat{M}$. Any lift $m \in \widehat{M}$ of $\overline{m}$ is a cyclic vector of the differential module $\widehat{M}_{\mathrm{Frac}(\widehat{R}((x)))}$. Indeed, it suffices to see that it becomes a cyclic vector over some extension of $\mathrm{Frac}(\widehat{R}((x))) = \mathrm{Frac}(\kappa[[x, y]])$, and the fraction field of the v-adic completion of $\widehat{R}((x)) = \kappa[[y]]((x))$ (which is nothing but $\kappa((x))((y))$) is such an extension: the smallest ϑ_x-stable $\kappa((x))[[y]]$-submodule of $\widehat{M}_{\kappa((x))[[y]]}$ containing m coincides with $\widehat{M}_{\kappa((x))[[y]]}$ (by Nakayama's lemma, since this holds modulo y by assumption).

Since the connection is generically regular, the matrix of ϑ_x in the cyclic basis

$$\mathbf{m} = (m, \nabla_{\vartheta_x} m, \ldots, \nabla_{\vartheta_x}^{\mu-1} m)$$

of $\widehat{M}_{\mathrm{Frac}(\widehat{R}((x)))}$ has no pole in the variable x. This means that

$$\nabla_{\vartheta_x}(\mathbf{m}) = \mathbf{m}\widetilde{H},$$

with $\widetilde{H} \in M_\mu((\widehat{R}[[x]])_{(x)}) \subseteq M_\mu(\widehat{K}[[x]])$.

In order to clear out the denominators, we use a classical technique (cf. e.g., [38, V.5.1]). We fix a basis $\mathbf{n}$ of $\widehat{M}$ over $\widehat{R}((x))$, and consider the matrix from $\mathbf{n}$ to $\mathbf{m}$:

$$\mathbf{m} = \mathbf{n}Q.$$

Since the elements of $\mathbf{m}$ are in $\widehat{M}$, we have

$$Q \in \mathrm{GL}_\mu(\mathrm{Frac}(\widehat{R}((x)))) \cap M_\mu(\widehat{R}((x))).$$

On the other hand, since $\mathbf{m}$ modulo y is a cyclic basis of $\widehat{M} \otimes \kappa((x))$, Q modulo y lies in $\mathrm{GL}_\mu(\kappa((x)))$. We use now the following assertion.

8.4.5. Claim. *Replacing $\widehat{R} \cong \kappa[[y]]$ by a finite extension if necessary, one can write Q as a product $Q'(Q'')^{-1}$, where*

$$Q' \in \mathrm{GL}_\mu(\widehat{R}((x))), \quad Q'' \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)}).$$

Proof of the claim. Since $\det Q$ modulo y is non-zero, $\det Q$ can be written, according to Weierstrass' preparation theorem ([25, VII, 3.8, prop. 6]), as the product of a monic polynomial $q \in \widehat{R}[x]$ and a unit $u \in \widehat{R}((x))^\times$. Replacing $\widehat{R}$ by a finite extension, we may and shall assume that the zeroes of q belong to $y\widehat{R}$.

Let ξ be one of these zeroes. By induction, it suffices to find a matrix

$$Q''_1 \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)})$$

such that $QQ''_1 \in M_\mu(\widehat{R}((x)))$, $\det(QQ''_1)$ divides $\det(Q)$, and $\mathrm{ord}_\xi \det(QQ''_1) < \mathrm{ord}_\xi \det(Q)$. Let $\lambda_1, \dots, \lambda_\mu \in \widehat{R}$ be the coefficients of a non-trivial dependence relation between the *columns* of $Q_{|x=\xi}$. We assume, as we may, that one of these numbers, say $\lambda_i = 1$. Then

$$Q''_1 = \left(\begin{array}{c|c|c} & \lambda_1/(x-\xi) & \\ \mathbf{I}_{i-1} & \vdots & \mathbf{O}_{i-1,\mu-i} \\ & \lambda_{i-1}/(x-\xi) & \\ \hline 0\cdots 0 & 1/(x-\xi) & 0\cdots 0 \\ \hline & \lambda_{i+1}/(x-\xi) & \\ \mathbf{O}_{\mu-i,i-1} & \vdots & \mathbf{I}_{\mu-i} \\ & \lambda_\mu/(x-\xi) & \end{array}\right),$$

whose inverse is

$$\left(\begin{array}{c|c|c} & -\lambda_1 & \\ \mathbf{I}_{i-1} & \vdots & \mathbf{O}_{i-1,\mu-i} \\ & -\lambda_{i-1} & \\ \hline 0\cdots 0 & x-\xi & 0\cdots 0 \\ \hline & -\lambda_{i+1} & \\ \mathbf{O}_{\mu-i,i-1} & \vdots & \mathbf{I}_{\mu-i} \\ & -\lambda_\mu & \end{array}\right),$$

is easily seen to satisfy the required property. □

We continue the proof of the theorem. The basis

$$\mathbf{n}' := \mathbf{n}Q' = \mathbf{m}Q''$$

generates a ϑ_x-stable $\widehat{R}[[x]]$-lattice in $\widehat{M}$. On the one hand, $\mathbf{n}'$ is a basis of $\widehat{M}$ over $\widehat{R}((x))$, and therefore the matrix $H = \widetilde{H}_{[Q'']}$ such that

$$\nabla_{\vartheta_x}\mathbf{n}' = \mathbf{n}'H$$

belongs to $M_\mu(\widehat{R}((x)))$. On the other hand,

$$\mathbf{n}' = \mathbf{m}Q'',$$

with $Q'' \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)}) \subseteq \mathrm{GL}_\mu(\widehat{K}[[x]])$, guarantees that $H \in M_\mu(\widehat{K}[[x]])$. But $\widehat{R}((x)) \cap \widehat{K}[[x]] = \widehat{R}[[x]]$, so that $H \in M_\mu(\widehat{R}[[x]])$. The eigenvalues of $H_{|x=0}$ then coincide, modulo $\mathbb{Z}$, with the α's. In particular, the α's are v-integral. This proves item (1) of the theorem.

Let us now prove item (2). We first notice that if M is projective of finite type over a ring R, K' an extension of $K = \mathrm{Frac}(R)$ and R' a subring of K' s.t. $R' \cap K = R$, then a decomposition of M_K descends to a decomposition of M if and only if the corresponding decomposition of $M_{K'}$ descends to a decomposition of $M_{R'}$. In fact, the decomposition is just defined by its projectors $n \in \mathrm{End}(M)$. Now $\mathrm{End}(M) = M^\vee \otimes M$ is projective of finite type, therefore there exists a set of generators $v_1, \dots, v_n$ of $\mathrm{End}(M)$ and a set $v_1^\vee, \dots, v_n^\vee$ of $\mathrm{End}(M)^\vee$ (R-dual of M) such that $n = \sum_\ell v_\ell^\vee(n)v_\ell$ (see for example [23, ch.II, par. 2, prop. 12]). In particular, the projectors have coordinates in R' if and only if they have coordinates in R.

Therefore, it is sufficient to show that the decomposition extended to $\widehat{K}((x))$ descends to $\widehat{R}((x))$ (for one completion). From the hypothesis we deduce that the exponents are constant, and by applying suitable shearing transformations in $\mathrm{GL}_\mu(\widehat{R}[x, x^{-1}])$ (cf. Lemma 7.2.4), we may assume that the eigenvalues of $H_{|x=0}$ are prepared.

It suffices now to find a basis $\mathbf{n}''$ of $\widehat{M}$ in which ϑ_x has matrix $H_0 := H_{|x=0}$. We know that there is a (unique) matrix $Y \in \mathrm{GL}_\mu(\widehat{K}[[x]])$ with $Y_{|x=0} = I$ and $\vartheta_x(Yx^{-H_0}) = -HYx^{-H_0}$. It is thus enough to show that, under the condition that the α's are constant and distinct modulo $\mathbb{Z}$, the entries of Y lie in $\widehat{R}[[x]]$. The condition on the exponents implies that, for every $n \in \mathbb{Z}_{>0}$, the endomorphism

$$\overline{Y} \longmapsto n\overline{Y} + H_0\overline{Y} - \overline{Y}H_0$$

of $M_\mu(\widehat{R}/y\widehat{R}) = M_\mu(\kappa)$ is injective, hence bijective by reason of dimension. Let H_n and Y_n denote the n^{th} coefficients of H and Y, respectively. The Y_n's satisfy the recursion ([38, III.8.5])

$$nY_n + H_0Y_n - Y_nH_0 = -(H_1Y_{n-1} + H_2Y_{n-2} + \cdots + H_n).$$

It order to conclude, it suffices to notice that the endomorphism $Y_n \mapsto nY_n + H_0Y_n - Y_nH_0$ of $M_\mu(\widehat{R})$ is surjective, since it is so modulo y. $\square$

9 Formal theory of integrable connections (several variables)

In this section we introduce the theory of formal connections in several variables, following in the exposition the fundamental paper [44] of Gérard and Levelt, with some complements. The main results are the decomposition of Gérard-Levelt structures (a generalization to several variables of the Jordan decomposition for regular modules), and the characterization of regularity for differential modules in several variables, in terms of regularity for each variable separately.

9.1 Outline of Gérard-Levelt theory

The basic object in the paper [44] is the following:

Definition 9.1.1 (**Gérard-Levelt structure**). *A d-dimensional Gérard-Levelt structure (a "GL-structure") over k is a set of data $(\Lambda, (\nabla_1, \dots, \nabla_d))$ where Λ is a torsion free $k[[x_1, \dots, x_d]]$-module of finite type, endowed with k-linear endomorphisms $\nabla_1, \dots, \nabla_d$ which commute and satisfy the Leibniz rule*

$$\nabla_j(fm) = x_j \frac{\partial f}{\partial x_j} m + f\nabla_j(m), \tag{9.1.2}$$

for any $f \in k[[x_1, \dots, x_d]]$ and $m \in \Lambda$.

9.1.3 (**Generic and special ranks**). We denote by $k((\mathbf{x}))$ the fraction field of $k[[\mathbf{x}]] = k[[x_1, \dots, x_d]]$ and regard Λ as a "lattice" in the $k((\mathbf{x}))$-vector space $V = \Lambda \otimes k((\mathbf{x}))$, i.e., as a finitely generated sub-$k[[\mathbf{x}]]$-module of V which spans V over $k((\mathbf{x}))$. The action of $\nabla_1, \dots, \nabla_d$ may be uniquely extended to V by imposing the rule (9.1.2) for any $f \in k((\mathbf{x}))$.

Let $p = \dim_{k((\mathbf{x}))} V$ (resp. $q = \dim_k \Lambda/(x_1, \dots, x_d)\Lambda$) denote the *generic* (resp. *special*) rank of Λ . Then $q \geqslant p$ and Λ is a free $k[[\mathbf{x}]]$-module if and only if $p = q$.

9.1.4. For any d-dimensional GL-structure $(\Lambda, (\nabla_1, \dots, \nabla_d))$ over k, the operators $\nabla_1, \dots, \nabla_d$ induce commuting k-linear operators $\delta_1, \dots, \delta_d$ on $\Lambda/(x_1, \dots, x_d)\Lambda$. A straightforward Jordan theory[4] attaches to the k-vector space $\Lambda/(x_1, \dots, x_d)\Lambda$ equipped with the commuting operators $(\delta_1, \dots, \delta_d)$ a notion of *multiplicity e_α of a vector $\alpha = (\alpha_1, \dots, \alpha_d) \in k^d$ as a multiexponent of* $(\delta_1, \dots, \delta_d)$, namely

$$e_\alpha = \dim_k \bigcap_{i=1}^{d} \mathrm{Ker}(\delta_i - \alpha_i)^M,$$

for any $M \gg 0$. Obviously, $e_\alpha = 0$ for almost all α ("α is not a multiexponent of $(\delta_1, \dots, \delta_d)$" in that case) and $\sum_\alpha e_\alpha = q$.

[4] developed in a more general context in Section 3 of [15].

Definition 9.1.5 (*GL*-**exponents, multiplicities).** *With the previous notation, we will say that e_α is the multiplicity of the GL-exponent $\alpha \in k^d$ of $(\Lambda, (\nabla_1, \dots, \nabla_d))$.*

Theorem 9.1.6 ([**44**, **Thm. 3.4**]). *Let $(\Lambda, (\nabla_1, \dots, \nabla_d))$ be a d-dimensional GL-structure over k, of generic rank p and special rank q. Let $\alpha_k = (\alpha_{1,k}, \dots, \alpha_{d,k})$, for $k = 1, \dots, q$ be the GL-exponents of $(\Lambda, (\nabla_1, \dots, \nabla_d))$, repeated according to their multiplicities.*

(i) *If*

$$\alpha_{i,j} - \alpha_{i,h} \notin \mathbb{Z} \smallsetminus \{0\} \quad \text{for all } i, j, h, \tag{9.1.7}$$

then $p = q$ (hence Λ is a free module), and there exists a $k[[\mathbf{x}]]$-basis $\mathbf{e} = (e_1, \dots, e_p)$ of Λ such that, for all $i = 1, \dots, d$,

$$\nabla_i \, \mathbf{e} = \mathbf{e} \, C_i, \tag{9.1.8}$$

for a $p \times p$ matrix C_i with coefficients in k.

(ii) *There exists a free lattice Λ' in V, stable under the action of $\nabla_1, \dots, \nabla_d$, satisfying assumption (9.1.7).*

Proof. (i) The main ingredient is the Jordan decomposition of a k-linear operator ϕ on a finite-dimensional k-vector space as the sum of its semisimple part ${}^{\mathrm{s}}\phi$ and of its nilpotent part ${}^{\mathrm{n}}\phi$, $\phi = {}^{\mathrm{s}}\phi + {}^{\mathrm{n}}\phi$, with $[{}^{\mathrm{s}}\phi, {}^{\mathrm{n}}\phi] = 0$. This elementary fact is applied to the filtered k-vector space $(\Lambda, \{(x_1, \dots, x_d)^M \Lambda\}_M)$, and to its endomorphisms $\nabla_1, \dots, \nabla_d$. One thus gets a Jordan decomposition $\nabla_i = {}^{\mathrm{s}}\nabla_i + {}^{\mathrm{n}}\nabla_i$, for $i = 1, \dots, d$, with the property that, for all $i = 1, \dots, d$ and $M = 1, 2, \dots$, the k-linear endomorphism induced by ${}^{\mathrm{s}}\nabla_i$ (resp. by ${}^{\mathrm{n}}\nabla_i$) on the finite-dimensional k-vector space $\Lambda/(x_1, \dots, x_d)^M \Lambda$ is semisimple (resp. nilpotent). The k-linear endomorphisms $\nabla_1, \dots, \nabla_d, {}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d, {}^{\mathrm{n}}\nabla_1, \dots, {}^{\mathrm{n}}\nabla_d$, commute. Moreover, ${}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d$ satisfy

$${}^{\mathrm{s}}\nabla_j(fm) = x_j \frac{\partial f}{\partial x_j} m + f \, {}^{\mathrm{s}}\nabla_j(m) \tag{9.1.9}$$

for any $f \in k[[\mathbf{x}]]$ and $m \in \Lambda$, while ${}^{\mathrm{n}}\nabla_1, \dots, {}^{\mathrm{n}}\nabla_d$ are $k[[\mathbf{x}]]$-linear and nilpotent. By semisimplicity, there exists a k-linear subspace W of Λ that complements $(x_1, \dots, x_d)\Lambda$, and is stable under ${}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d$. The projection

$$W \xrightarrow{\sim} \Lambda/(x_1, \dots, x_d)\Lambda$$

identifies the restriction $({}^{\mathrm{s}}\nabla_i)_{|W}$ with ${}^{\mathrm{s}}\delta_i$, for every i. Hence, there exists a k-basis $(e_1, \dots, e_q)$ of W, such that

$${}^{\mathrm{s}}\nabla_i(e_j) = \alpha_{i,j} e_j \ , \tag{9.1.10}$$

for all i, j. We note that in any case $(e_1, \dots, e_q)$ generate Λ over $k[[\mathbf{x}]]$. So, in order to show that Λ is free, it suffices to prove that $(e_1, \dots, e_q)$ are linearly independent over $k((\mathbf{x}))$. Let us recall the argument of [44]. If $p < q$, we can write $e_{r+1} = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with $\lambda_i \in k((\mathbf{x}))$ (not all in k) and r minimal. Applying ${}^s\nabla_j$, we get $\alpha_{j,r+1}e_{r+1} = \sum_{\ell \leqslant r}(x_j \frac{\partial}{\partial x_j}\lambda_\ell + \alpha_{j,\ell}\lambda_\ell)e_\ell$, and on combining these equalities,

$$x_j \frac{\partial}{\partial x_j}\lambda_1 + (\alpha_{j,1} - \alpha_{j,r+1})\lambda_1 = \cdots = x_j \frac{\partial}{\partial x_j}\lambda_r + (\alpha_{j,r} - \alpha_{j,r+1})\lambda_r = 0.$$

Let us remark that for any $\lambda \in k((\mathbf{x}))$, $\lambda \neq 0$, $\beta_j \in k$, the simultaneous equations $x_j \frac{\partial}{\partial x_j}\lambda = \beta_j \lambda$ imply that all β_j are integers. This shows that if some $\lambda_\ell \neq 0$, then $\alpha_{j,\ell} = \alpha_{j,r+1}$ for all j, so that $\lambda_\ell \in k$. This shows $\lambda_\ell \in k$ for *all* ℓ, a contradiction. One also needs to show that ${}^{\mathrm{n}}\nabla_i(W) \subseteq W$, for all i. This is a general fact, under the assumption (9.1.7): for any $k((\mathbf{x}))$-linear endomorphism T of V such that $[{}^{\mathrm{s}}\nabla_i, T] = 0$, for all i, a simple argument shows that $T(W) \subseteq W$. So, the isomorphism $W \xrightarrow{\sim} \Lambda/(x_1, \ldots, x_d)\Lambda$ also identifies the restriction $({}^{\mathrm{n}}\nabla_i)_{|W}$ with ${}^{\mathrm{n}}\delta_i$, hence $(\nabla_i)_{|W}$ with δ_i, for every i.

(ii) One uses the method of *shearing transformations* 7.2.4 Let us pick $(e_1, \ldots, e_q)$ as in (9.1.10). They generate Λ over $k[[\mathbf{x}]]$. If we multiply each e_ℓ by a suitable Laurent monomial

$$m_\ell = \prod_{i=1}^{d} x_i^{\gamma_{i,\ell}}$$

in the $(x_1, \ldots, x_d)$, with $\gamma_{i,\ell} \in \mathbb{Z}$, we can arrange that condition (9.1.7) of the theorem is satisfied for the lattice Λ' spanned in V by $(e'_1, \ldots, e'_q)$, where $e'_\ell = m_\ell e_\ell$, for each ℓ. Of course, $(e'_1, \ldots, e'_q)$ need not be linearly independent over k, but they satisfy

$$\text{(9.1.11)} \qquad {}^{\mathrm{s}}\nabla_i(e'_j) = \alpha'_{i,j}e'_j,$$

for all i, j, with $\alpha'_{i,j} = \alpha_{i,j} + \gamma_{i,j}$. The k-vector space W' generated by $(e'_1, \ldots, e'_q)$ in Λ' is a complement of $(x_1, \ldots, x_d)\Lambda'$, and is stable under ${}^{\mathrm{s}}\nabla_1, \ldots, {}^{\mathrm{s}}\nabla_d$. So, we may apply the discussion of part (i) to any maximal k-linearly independent subset of $(e'_1, \ldots, e'_q)$. Such a subset will be a free system of generators of Λ'. □

Remark 9.1.12. Notice that if in the previous discussion we replace $(e_1, \ldots, e_q)$ by a vector $(v_1, \ldots, v_q)$ of elements of Λ, with $v_j - e_j \in (x_1, \ldots, x_d)^M\Lambda$, for $M \gg 0$, the lattice spanned by $(v'_1, \ldots, v'_q)$, where $v'_\ell = m_\ell v_\ell$, for each ℓ, coincides with Λ'. This remark will be used in the sequel, when we have to replace the constructions of Gérard-Levelt in the formal setting by algebraic approximations.

Corollary 9.1.13. *Let $(\Lambda, (\nabla_1, \ldots, \nabla_d))$ be as in Theorem 9.1.6 and let $\tau = (\tau_1, \ldots, \tau_d)$ be any vector of set-theoretic sections τ_i of the projection $k \to k/\mathbb{Z}$.*

(i) *There is a unique free lattice Λ_τ in V, stable under the action of $\nabla_1, \ldots, \nabla_d$, and such that for every i the eigenvalues of the k-linear endomorphism induced by ∇_i on $\Lambda_\tau/(x_1, \ldots, x_d)\Lambda_\tau$ are in the image of τ_i.*

(ii) *There exists a free system of generators* $(e_1,\dots,e_p)$ *of* Λ_τ *over* $k[[x_1,\dots,x_d]]$*, such that* (9.1.10) *holds, with* $\alpha_{i,j}\in\operatorname{Im}\tau_i$.

(iii) *If* $\nabla_i\Lambda\subseteq x_i\Lambda$ *and* $\tau_i(\mathbb{Z})=0$ *for for* $i=s+1,\dots,d$*, the lattice* Λ_τ *satisfies* $\nabla_i\Lambda_\tau\subseteq x_i\Lambda_\tau$ *for those* i*'s.*

Proof. It is clear that in the discussion of (9.1.11) we can arrange that the $\alpha'_{i,j}$'s are in the image of τ_i, for all i,j. It is also clear from the previous discussion that the lattice Λ' generated by $(e'_1,\dots,e'_q)$ is free of rank p, and admits a basis $(e_1,\dots,e_p)$ as in the statement. Suppose now that we have two bases $\mathbf{e}=(e_1,\dots,e_p)$ and $\mathbf{e}'=(e'_1,\dots,e'_p)$ of V such that

$$^{\mathrm{s}}\nabla_i(e_j)=\alpha_{i,j}e_j,\quad {}^{\mathrm{s}}\nabla_i(e'_j)=\alpha'_{i,j}e'_j$$

with $\alpha_{i,j},\alpha'_{i,j}\in\operatorname{Im}\tau_i$ for all i,j. Suppose one of the $\{e'_1,\dots,e'_p\}$, say e'_1, were k-linearly independent of $(e_1,\dots,e_p)$. For all i,j, $\alpha_{i,j}-\alpha'_{i,1}\notin\mathbb{Z}\smallsetminus\{0\}$. So, the argument in the proof of 9.1.6 (i), shows that $(e'_1,e_1,\dots,e_p)$ is a set of $k((\mathbf{x}))$-linearly independent vectors in V. This is a contradiction; therefore $\mathbf{e}$ and $\mathbf{e}'$ span the same k-linear subspace of V[5]. □

Remark 9.1.14. The following special case of the previous theorem deserves attention. Assume the d-dimensional GL-structure $(\Lambda,(\nabla_1,\dots,\nabla_d))$ has the property that, for $i=s+1,\dots,d$, $\nabla_i\Lambda\subseteq x_i\Lambda$. Then $\delta_i=0$ for those i's. Let us pick a k-supplement W of $(x_1,\dots,x_d)\Lambda$, stable under all the ${}^{\mathrm{s}}\nabla_i$'s . Then $({}^{\mathrm{s}}\nabla_i)_{|W}=0$ for $i=s+1,\dots,d$. Assume that (9.1.7) is verified. Then Λ is free and W is also stable under the ${}^{\mathrm{n}}\nabla_i$'s and the ∇_i's. Since $({}^{\mathrm{n}}\nabla_i)_{|W}$ and $(\nabla_i)_{|W}$ identify with ${}^{\mathrm{n}}\delta_i$ and δ_i respectively, they are both 0 and we deduce that ${}^{\mathrm{n}}\nabla_i=0$ for $i=s+1,\dots,d$.

Notice however that, if we apply part (ii) to modify Λ so that (9.1.7) will be satisfied, but insist to preserve the property $\nabla_i\Lambda\subseteq x_i\Lambda$ for $i=s+1,\dots,d$, then we must take $\gamma_{i,j}=0$, for $i=s+1,\dots,d$ and all j's. In fact, $\alpha_{i,j}=0$ already for those i,j.

This is the reason why one assumes that the sections $\tau_1,\dots,\tau_N$ satisfy $\tau_i(\mathbb{Z})=0$, for all i. In down-to-earth terms, one wants to avoid introducing unnecessary "apparent singularities".

Theorem 9.1.15 (**Decomposition of** ***GL*****-structures).** *Let* $(\Lambda,(\nabla_1,\dots,\nabla_d))$ *be a* d*-dimensional GL-structure over* k*. Then there exist a finite extension* k' *of* k *and a canonical decomposition*

$$\Lambda_{k'((\mathbf{x}))}=\bigoplus_{\alpha\in k'^d}k'[[\mathbf{x}]]\Big[\frac{1}{x_1\cdots x_d}\Big]\otimes_k K_\alpha,$$

where the $\alpha\in k'^d$ *are pairwise distinct modulo* $\mathbb{Z}^d$ *and*

$$K_\alpha=\bigcap_{i=1}^{d}\operatorname{Ker}_{\Lambda_{k'[[\mathbf{x}]][(x_1\cdots x_d)^{-1}]}}(\nabla_{\vartheta_i}-\alpha_i)^{\mu}$$

[5] for more information see Section 3.4 of [15].

(for $\mu \gg 0$) is different from 0 if and only if $\alpha \in k'^d$ is (up to elements of $\mathbb{Z}^d$) a GL-exponent of the Gérard-Levelt structure. Moreover, if the GL-structure satisfies the condition (9.1.7), then the decomposition descends to $\Lambda \otimes_k k'$.

Proof. We proceed by induction on d, the case $d = 1$ being given by Theorem 8.3.4. The inductive step follows from Lemma 8.2.2, taking into account that in the proof of 9.1.6 part (ii) we need only for Laurent monomials (and not the whole fraction field $k'((\mathbf{x}))$). $\square$

9.1.16. Let $(\Lambda, (\nabla_1, \dots, \nabla_d))$ be a *GL*-structure of dimension d over k. For every i, let $\kappa_i := k((x_1, \dots, \widehat{x_i}, \dots, x_d))$ (where the single variable x_i is missing). The x_i-adic completion of the discrete valuation ring $k[[\mathbf{x}]]_{(x_i)}$ can be identified with $\kappa_i[[x_i]]$.

Similarly, the x_i-adic completion of the quotient field $k((\mathbf{x}))$ of $k[[\mathbf{x}]]_{(x_i)}$, can be identified with $\kappa_i((x_i))$. The standard κ_i-linear, x_i-adically continuous derivation $x_i \frac{\partial}{\partial x_i}$ of $\kappa_i((x_i))$ coincides with the extension by continuity of the derivation $x_i \frac{\partial}{\partial x_i}$ of $k((\mathbf{x}))$ considered in our previous discussion of Theorem 9.1.6. We consider the x_i-adic completion of Λ (resp. V) and call it $\Lambda_i = \kappa_i[[x_i]] \otimes \Lambda$ (resp. $V_i = \kappa_i((x_i)) \otimes V$). Since $\kappa_i[[x_i]]$ is a discrete valuation ring, the lattice Λ_i of V_i is a free $\kappa_i[[x_i]]$-module of rank p.

The extension $\nabla_i^{(1)}$ of the endomorphism ∇_i of Λ (resp. V) by x_i-adic continuity to Λ_i (resp. V_i) produces a 1-dimensional *GL*-structure over the field κ_i. The characteristic polynomial of the κ_i-linear endomorphism $\delta_i^{(1)}$ induced by $\nabla_i^{(1)}$ on $\Lambda_i/x_i\Lambda_i$ has coefficients in k and will be denoted by $P_i(t)$.

Corollary 9.1.17. *With the notation of 9.1.16, $P_i(t)$ divides the characteristic polynomial of the k-linear endomorphism δ_i of $\Lambda/(x_1, \dots, x_d)\Lambda$. If Λ is a free lattice, the two polynomials coincide.*

Proof. The natural map $\kappa_i \otimes_k \Lambda/(x_1, \dots, x_d)\Lambda \longrightarrow \Lambda_i/x_i\Lambda_i$ is a surjection and $\delta_i^{(1)}$ is induced by $1 \otimes \delta_i$. Therefore, ${}^{\mathrm{s}}\delta_i{}^{(1)}$ is induced by $1 \otimes {}^{\mathrm{s}}\delta_i$. If Λ is a free lattice, the last map is an isomorphism. $\square$

9.2 Regularity and logarithmic extensions

We keep the notation of 9.1.16.

Remark 9.2.1. Let (V, ∇) a differential module over $k((x_1, \dots, x_d))$. Notice that if we consider $V_i = V \otimes \kappa_i((x_i))$ as a differential module over $\kappa_i((x_i))$ with derivation $\vartheta_{x_i} = x_i \frac{\partial}{\partial x_i}$, then the regularity condition (along x_i) is equivalent to the existence of a free 1-dimensional *GL*-structure (Λ_i, ∇_i) over κ_i inducing (V_i, ∇_i) by base change.

Theorem 9.2.2. *Let (M, ∇) be a finite locally free differential module over the ring $R_{d,s} = k[[x_1, \dots, x_d]][\frac{1}{x_1 \cdots x_s}]$. Then the following conditions are equivalent:*

(1) *for all* $i = 1, \dots, s$, $M_i = M \otimes \kappa_i((x_i))$ *is regular* (*i.e., admits a free* 1-*dimensional GL-structure* (Λ_i, ∇_i) *over* κ_i);

(2) (M, ∇) *admits a* d-*dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over* k, *i.e., there exists a* d-*dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over* k *inducing* (M, ∇) *by base change;*

(3) (M, ∇) *admits a free* d-*dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over* k *and a basis of* Λ *such that the matrices of* ∇_i *are non-resonant with coefficients in* k (*and zero if* $i > s$).

Remark 9.2.3. Item (2) may be considered as a strong regularity condition for (M, ∇). The equivalence of (1) and (2) says that regularity in the strong sense is equivalent to regularity "in each variable" separately.

On the other hand, one can relax the assumption of local freeness by using the notion of stable freeness, cf. [79, 1.3].

Proof. The equivalence of (2) and (3) follows from Theorem 9.1.6 of the Gérard-Levelt theory, and Remark 9.1.14. Clearly (2) implies (1).

The crucial point is to show that (1) implies (2). By hypothesis, for all i we have free $\kappa_i[[x_i]]$-lattices Λ_i in M_i, stable under ∇_i. Let $c_i : M \to M_i$ be the canonical morphisms. We define

$$\Lambda = \bigcap_i c_i^{-1}(\Lambda_i) = \{v \in M \ : \ c_i(v) \in \Lambda_i \quad \forall i\},$$

which is clearly a $k[[x]]$-submodule of M (hence torsion-free), stable under $\nabla_i = \nabla_{\vartheta_{x_i}}$ and such that $\Lambda \otimes_{k[[x]]} R_{d,s} = M$. We have to show that Λ is finite as $k[[x]]$-module. Since M is finite projective, we may consider a set of generators $(v_1, \dots, v_n)$ of M and the set $(v_1^\vee, \dots, v_n^\vee)$ of $M^\vee$ ($R_{d,s}$-dual of M) with the property that for any $x \in M$ we have $x = \sum_\ell v_\ell^\vee(x) v_\ell$ (see, for example, [23, Ch.II, par. 2, Prop. 12]). Let us consider now the $\kappa_i[[x_i]]$-lattice $\Lambda_i^\vee$ inside $M_i^\vee \cong M^\vee \otimes_{R_{d,s}} \kappa_i((x_i))$. Then there exists an integer d_i such that $x_i^{d_i} v_\ell^\vee \in \Lambda_i^\vee$ for any $\ell = 1, \dots, n$. Therefore, $\left(\prod_i x_i^{d_i}\right)\Lambda$ is contained in the $k_i[[\mathbf{x}]]$-lattice Λ' generated by $(v_1, \dots, v_n)$. We deduce that $\Lambda \subseteq \left(\prod_i x_i^{-d_i}\right)\Lambda'$, which is of finite type over $k[[\mathbf{x}]]$, and so is also Λ (since $k[[\mathbf{x}]]$ is noetherian). □

Definition 9.2.4 (**logarithmic extensions**). *The data of a free GL-structure as in item* (3) *is called a logarithmic extension of* (M, ∇). *It is unique if we ask that the GL-exponents are in the images of sections* τ_i *of the projection* $k \to k/\mathbb{Z}$ (*see* 9.1.13).

Remark 9.2.5. Theorem 9.2.2 does not extend to differential modules over the field $F = k((x_1, \dots, x_d))$, as the example $M = F.\exp\left(\frac{1}{x_1 + x_2}\right)$ over $F = k((x_1, x_2))$ shows.

Chapter IV

Regularity: geometric theory

Introduction

The central topic of this chapter is the notion of regularity in several variables. For an algebraic integrable connection ∇ on the complement of a prime divisor D in an algebraic variety X, the notion of regularity along D may be defined, or characterized, in at least four different algebraic ways:

(a) in terms of the iterated action of any *single* vector field ∂ generically transversal to D: the order of the poles occurring in the action of ∂^n is at most $n + \text{constant}$;

(b) by the fact that the logarithmic differential operators of increasing order act with poles of bounded order at the generic point of D;

(c) via the classical notion of regularity in one variable, applied to the restriction of ∇ to sufficiently many smooth curves in X intersecting D transversally;

(d) by the existence of an extension with logarithmic poles along D.

Although the equivalence of these viewpoints is "well known", the proofs given in the literature are most often transcendental, and usually rely on Hironaka's resolution of singularities. J. Bernstein in his notes [16] requested an algebraic proof for these equivalences.

The aim of this chapter is to provide a purely algebraic, systematic treatment of these questions (avoiding resolution of singularities beyond the case of surfaces). Actually, the most advanced questions (such as refined versions of (c), and the absence of confluence for integrable regular connections) will be completely established only later in 21.1.2, using some irregularity theory. We first consider issues which are generic on the polar divisor D, then the case of a normal crossing polar divisor D. We study in detail the crucial issue of restriction to curves meeting the polar divisor at an arbitrary point, in several contexts of increasing generality,

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*,
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using increasingly sophisticated tools. On the way, for a regular connection $\mathcal{M}$ with a simple normal crossings polar divisor D, we give a purely algebraic construction of the (locally free) logarithmic extension of $\mathcal{M}$ across D with exponents in any given section τ of $k \to k/\mathbb{Z}$.

In the global setting, we start from a *birational* definition, and give the following equivalent characterizations of a *regular* connection ∇ on X (the complete proof is achieved in 21.1.2):

(e) the restriction of ∇ to every smooth curve in X is regular in the classical sense;

(f) ∇ is regular along the one-codimensional part of the boundary of X in some normal compactification.

Besides, we introduce the *exponents* of a regular connection along a divisor and show that they coincide with the classical exponents of the differential module induced on any curve meeting the divisor transversally. We also introduce and study a global, birational notion of exponents of a regular connection and give several characterizations.

10 Regularity and exponents along prime divisors

Let k be an *algebraically closed* field (of characteristic 0 as usual). We begin the study of regularity in several variables in the geometric framework of integrable connections.

Let X be a normal algebraic k-variety of dimension d, and let D be a *prime* divisor (i.e., a closed irreducible algebraic subvariety of codimension one) on X. Since X is normal, $\mathcal{O}_{X,D}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_{X,D}$, with fraction field $\kappa(X)$, and residue field $K := \kappa(D)$. Let $\mathcal{I}_D$ be the sheaf of ideals of D, and $\eta = \eta_D$ the generic point of D. We put $U = X \smallsetminus D$ and we call $j : U \to X$ the open immersion of U in X. Let $\widehat{\mathcal{O}}_{X,D}$ and $\widehat{\mathfrak{m}}_{X,D}$ denote the completion of $\mathcal{O}_{X,D}$ and $\mathfrak{m}_{X,D}$, respectively.

We denote by Ω^1_X the $\mathcal{O}_X$-module of differential forms on X and by $\mathcal{T}_X$ the $\mathcal{O}_X$-dual of Ω^1_X (see Section 4). Their stalks at η_D, namely $\Omega^1_{X,D}$ and $\mathcal{T}_{X,D}$, are free $\mathcal{O}_{X,D}$-modules of rank d, dual to each other.

We also introduce the sheaf $\Omega^1_X(\log D)$ of differential forms on X with logarithmic poles along D as the sub-$\mathcal{O}_X$-module of $j_*\Omega^1_U$ generated by Ω^1_X and the logarithmic differentials df/f for any local section f of $j_*\mathcal{O}_U^\times$. We let $\mathcal{T}_X(\log D)$ be the $\mathcal{O}_X$-dual of $\Omega^1_X(\log D)$. The associated $\mathcal{O}_{X,D}$-modules $\Omega^1_{X,D}(\log D)$ and $\mathcal{T}_{X,D}(\log D)$ are again free of rank d, and are dual to each other.

We observe that any element $\partial \in \mathcal{T}_{X,D}(\log D)$ naturally acts as a derivation of $\kappa(X)$ via $\partial(f) = f(\partial \circ \frac{df}{f})$ for any $f \in (j_*\mathcal{O}_U^\times)_{\eta_D} = \kappa(X)^\times$. Any germ $\partial \in \mathcal{T}_{X,D}$ (resp. $\partial \in \widehat{\mathcal{T}}_{X,D} = \mathcal{T}_{X,D} \otimes_{\mathcal{O}_{X,D}} \widehat{\mathcal{O}}_{X,D}$) of tangent vector field (resp. formal tangent vector field) at the generic point of D may be identified with a k-linear derivation of $\mathcal{O}_{X,D}$ (resp. $\widehat{\mathcal{O}}_{X,D}$). This produces a natural embedding

$$\mathcal{T}_{X,D}(\log D) \hookrightarrow \mathcal{T}_{X,D}.$$

Similarly, for the completions along D,

$$\widehat{\Omega}^1_{X,D} \subset \widehat{\Omega}^1_{X,D}(\log D), \quad \widehat{\mathcal{T}}_{X,D}(\log D) \subset \widehat{\mathcal{T}}_{X,D}.$$

Proposition 10.0.1. *$\mathcal{T}_{X,D}(\log D)$ is the $\mathcal{O}_{X,D}$-submodule of $\mathcal{T}_{X,D}$ consisting of derivations ∂ of $\mathcal{O}_{X,D}/k$ such that $\partial\mathfrak{m}_{X,D} \subseteq \mathfrak{m}_{X,D}$.*

Proof. If $x = x_1$ is a parameter of $\mathcal{O}_{X,D}$ and $x_2, \ldots, x_d$ is a transcendence basis of K/k, then

$$\frac{dx}{x}, dx_2, \ldots, dx_d$$

is a free system of $\mathcal{O}_{X,D}$-generators of $\Omega^1_X(\log D)$, with dual system of generators of $\mathcal{T}_{X,D}(\log D)$

$$x_1\frac{d}{dx_1}, \frac{d}{dx_2}, \ldots, \frac{d}{dx_d}.$$

Therefore, $\partial = a_1\frac{d}{dx_1} + a_2\frac{d}{dx_2} + \cdots + a_d\frac{d}{dx_d}$, for $a_i \in \mathcal{O}_{X,D}$, is the derivation of $\mathcal{O}_{X,D}$ associated to an element of $\mathcal{T}_{X,D}(\log D)$ if and only if $a_1 \in \mathfrak{m}_{X,D}$, hence if and only if $\partial\mathfrak{m}_{X,D} \subseteq \mathfrak{m}_{X,D}$. $\square$

10.1 Transversal derivations and integral curves

Definition 10.1.1 (Transversal and logarithmic derivations). *We say that $\partial \in \mathcal{T}_{X,D}$ (resp. $\partial \in \widehat{\mathcal{T}}_{X,D}$) is* transversal to D *if*

$$\partial\mathfrak{m}_{X,D} \not\subseteq \mathfrak{m}_{X,D} \quad (\textit{resp. } \partial\widehat{\mathfrak{m}}_{X,D} \not\subseteq \widehat{\mathfrak{m}}_{X,D}),$$

and that it is logarithmic with respect to D *if*

$$\partial\mathfrak{m}_{X,D} \subseteq \mathfrak{m}_{X,D} \text{ and } \partial\mathfrak{m}_{X,D} \not\subseteq \mathfrak{m}^2_{X,D} \quad (\textit{resp. } \partial\widehat{\mathfrak{m}}_{X,D} \subseteq \widehat{\mathfrak{m}}_{X,D} \text{ and } \partial\widehat{\mathfrak{m}}_{X,D} \not\subseteq \widehat{\mathfrak{m}}^2_{X,D}).$$

Notice that $\mathcal{T}_{X,D}(\log D)$ admits a free system of generators consisting of logarithmic derivations.

Example 10.1.2. Let us consider $X = \mathbb{A}^2_k$ (with coordinates x_1, x_2) with the divisor D defined by the ideal (x_1). Then $x_1\frac{d}{dx_1}$ and $\frac{d}{dx_2}$ are logarithmic derivations (for the first it is obvious, and for the second we note that $\frac{d}{dx_2}(x_1x_2) = x_1 \in \mathfrak{m}_{X,D} \smallsetminus \mathfrak{m}^2_{X,D}$), and they are a basis of $\mathcal{T}_{X,D}(\log D)$. A derivation of the form $f_1\frac{d}{dx_1} + f_2\frac{d}{dx_2}$ is transversal if and only if f_1 is not divisible by x_1, that is, it is invertible in the local ring $\mathcal{O}_{X,D}$.

10.1.3. There is a natural exact sequence of finite-dimensional K-vector spaces [48, 17.2.5.1]

$$0 \longrightarrow (\mathcal{I}_D/\mathcal{I}_D^2)_{\eta_D} \longrightarrow \Omega^1_{X,D} \otimes_{\mathcal{O}_{X,D}} K = \widehat{\Omega}^1_{X,D} \otimes_{\widehat{\mathcal{O}}_{X,D}} K \longrightarrow \Omega^1_{K/k} \longrightarrow 0\,. \tag{10.1.4}$$

As in loc.cit., the *conormal sheaf* $\mathcal{I}_D/\mathcal{I}_D^2$ of the embedding $D \hookrightarrow X$ is viewed as a $\mathcal{O}_D$-module. However, contrary to the convention of loc.cit., we reserve the notation $\mathcal{N}_{D|X}$ to the *normal sheaf* of $D \hookrightarrow X$, namely the $\mathcal{O}_D$-dual $\mathcal{O}_X(-D)$ of $\mathcal{I}_D/\mathcal{I}_D^2$. Notice that the K-dual of (10.1.4) is then the sequence

$$0 \longrightarrow \operatorname{Der}_k K \longrightarrow \mathcal{T}_{X,D} \otimes_{\mathcal{O}_{X,D}} K = \widehat{\mathcal{T}}_{X,D} \otimes_{\widehat{\mathcal{O}}_{X,D}} K \longrightarrow (\mathcal{N}_{D|X})_{\eta_D} \longrightarrow 0 \tag{10.1.5}$$

The stalk at η_D of $\mathcal{N}_{D|X}$ is the K-vector space $(\mathfrak{m}_{X,D}/\mathfrak{m}^2_{X,D})^\vee$ generated by the image of any transversal derivation ∂.

10.1.6. Let us fix a generator x of the ideal $\mathfrak{m}_{X,D}$ (so that $x = 0$ is a local equation for D around its generic point η_D). Note that for any transversal derivation $\partial \in \widehat{\mathcal{T}}_{X,D}$, $\partial(x)$ is invertible and the derivation $\partial_x = \frac{1}{\partial(x)}\partial$ satisfies $\partial_x(x) = 1$. Moreover, $x\partial_x$ is a logarithmic derivation with $x\partial_x(x) = x$. As the previous example shows, it is false in general that a logarithmic derivation is of the form $x\partial$ for some transversal derivation ∂.

Lemma 10.1.7. *The following data are equivalent:*

(a) *a transversal derivation $\partial_x \in \widehat{\mathcal{T}}_{X,D}$ satisfying $\partial_x(x) = 1$;*

(b) *a logarithmic derivation $\vartheta_x \in \widehat{\mathcal{T}}_{X,D}$ satisfying $\vartheta_x(x) = x$ and acting trivially on the residue field K of $\widehat{\mathcal{O}}_{X,D}$.*

Proof. In fact, given ∂_x as in (a), we can take $\vartheta_x = x\partial_x$. Conversely, for ϑ_x as in (b), $\vartheta_x\widehat{\mathcal{O}}_{X,D} \subset \widehat{\mathfrak{m}}_{X,D}$, so that we take $\partial_x := x^{-1}\vartheta_x$. □

Proposition 10.1.8 (**Formal tubular neighborhoods**)**.** *Let us assume that X is affine, D is defined by the equation $x = 0$, and $\partial_x \in \widehat{\mathcal{T}}_{X,D}$ satisfies $\partial_x(x) = 1$.*

(1) (a) *There is a unique isomorphism of differential rings over k*

$$(\widehat{\mathcal{O}}_{X,D}, \partial_x) \;\cong\; (K[[x]], \frac{d}{dx}), \tag{10.1.9}$$

which induces id_K modulo x.

(b) *This allows to lift any k-derivation δ of K to an element of $\widehat{\mathcal{T}}_{X,D}$ by setting $\delta(x) = 0$, and provides a lift of the exact sequence (10.1.5) to $\widehat{\mathcal{O}}_{X,D}$, which splits:*

$$\widehat{\mathcal{T}}_{X,D} \cong \widehat{\mathcal{O}}_{X,D}.\operatorname{Der}_k K \oplus \widehat{\mathcal{O}}_{X,D}.\partial_x \tag{10.1.10}$$

Moreover, ∂_x commutes with any element $\delta \in \operatorname{Der}_k K$.

(c) *The sub-$\widehat{\mathcal{O}}_{X,D}$-module $\widehat{\mathcal{T}}_{X,D}(\log D)$ of $\widehat{\mathcal{T}}_{X,D}$ dual to $\widehat{\Omega}^1_{X,D}(\log D)$ (the completion of the stalk at η_D of the sheaf of differential forms with logarithmic poles along D) is*

$$\widehat{\mathcal{T}}_{X,D}(\log D) \cong \widehat{\mathcal{O}}_{X,D}.\mathrm{Der}_k\, K \oplus \widehat{\mathcal{O}}_{X,D}.x\partial_x. \tag{10.1.11}$$

(2) *Assume that ∂_x extends to a derivation D_x of $\mathcal{O}(\widehat{X}_D)$. Then (10.1.9) extends to a unique isomorphism of differential rings over k*

$$(\mathcal{O}(\widehat{X}_D),\, D_x) \;\cong\; (\mathcal{O}(D)[[x]],\, \frac{d}{dx}), \tag{10.1.12}$$

which induces $id_{\mathcal{O}(D)}$ modulo x. In particular, one can attach to D_x a canonical decomposition of formal schemes over D

$$\widehat{X}_D \cong \widehat{\mathbb{A}}^1_k \times D \tag{10.1.13}$$

(*in this situation, $\widehat{X}_D$ is called a* formal tubular neighborhood of D).

Proof. (1)(a) In the style of 4.5.1, let us define

$$\Pi := \sum_n (-1)^n \frac{x^n}{n!}\,{\partial_x}^n.$$

One checks that this is a ring endomorphism of $\mathcal{O}_{X,D}$, that $\Pi^2 = \Pi$, $\Pi(\mathcal{O}_{X,D}) = (\mathcal{O}_{X,D})^{\partial_x}$, $\ker \Pi = x\mathcal{O}_{X,D}$, Π is the identity modulo x. One deduces that $\Pi(\mathcal{O}_{X,D}) = \mathcal{O}_{X,D}/(x) = K$ and that the natural morphism $(\Pi(\mathcal{O}_{X,D}))[[x]] = K[[x]] \to \mathcal{O}_{X,D}$ is an isomorphism (cf. [81, Thm. 30.1] for a generalization to several variables).

(b) For any $\delta \in \mathrm{Der}_k\, K$, $[\partial_x, \delta]$ is a derivation of $\widehat{\mathcal{O}}_{X,D}$ which kills both x and K, hence vanishes.

(c) is clear, since $\Omega^1_{X,D}(\log D)$ is free with basis $\frac{dx}{x}, dx_2, \dots dx_d$ for any system of local coordinates $(x_1, x_2, \dots, x_d)$ around the generic point of D with $x_1 = x$.

(2) Same proof as (1)(a), replacing ∂_X by D_x, $\mathcal{O}_{X,D}$ by $\mathcal{O}(\hat{X}_D)$ and K by $\mathcal{O}(D)$. □

Remark 10.1.14. In (2) the condition that ∂_x extends to a derivation of $\mathcal{O}(\hat{X}_D)$ is restrictive. However, if ∂_x comes from $\mathcal{T}_{X,D}$, it holds after replacing X by a suitable neighborhood of the generic point of D. But since elements of $\widehat{\mathcal{O}}_{X,D}$ are not necessarily defined in such a neighborhood, ∂_x does not extend in general to a derivation of $\mathcal{O}(\hat{X}_D)$ after shrinking X.

Proposition 10.1.15 (Integral curves). (1) *Let $\partial_x \in \widehat{\mathcal{T}}_{X,D}$ be transversal to D defined by $x = 0$. Then the following conditions are equivalent:*

(i) *∂_x is algebraic, i.e., belongs to $\mathcal{T}_{X,D}$, and* $\mathrm{tr.deg}_k\, \kappa(X)^{\partial_x} = \dim X - 1$;

(ii) *after replacing X and D by suitable open neighborhoods of η_D, the isomorphism (10.1.13) corresponding to ∂_x algebraizes, i.e., comes by completion from an étale morphism*

$$(x, f) : X \longrightarrow \mathbb{A}^1_k \times S \tag{10.1.16}$$

such that $f_{|D}$ is finite étale over S (the fibers of f are called integral curves *of ∂_x).*

(2) (a) *A derivation ∂_x satisfying the above conditions exists. One has $\langle \partial_x, f^*\omega \rangle = 0$ for any local section ω of Ω^1_S.*

(b) *This gives rise to a decomposition*

$$\mathcal{T}_{X,D} \cong \mathcal{O}_{X,D}.\mathrm{Der}_k\, \kappa(S) \oplus \mathcal{O}_{X,D}.\partial_x \tag{10.1.17}$$

(c) *The sub-$\mathcal{O}_{X,D}$-module $\mathcal{T}_{X,D}(\log D)$ of $\mathcal{T}_{X,D}$ dual to $\Omega^1_{X,D}(\log D)$ is*

$$\mathcal{T}_{X,D}(\log D) \cong \mathcal{O}_{X,D}.\mathrm{Der}_k\, \kappa(S) \oplus \mathcal{O}_{X,D}.x\partial_x. \tag{10.1.18}$$

Proof. (1) We first notice that $\kappa(X)^{\partial_x} \smallsetminus \{0\} \subseteq (\mathcal{O}_{X,D})^\times$: if y is a parameter of $\mathcal{O}_{X,D}$, and if $y^n u$, with $n > 0$ and $u \in (\mathcal{O}_{X,D})^\times$, were a non-zero element of $\kappa(X)^{\partial_x}$, the equation $0 = \partial_x(y^n u) = ny^{n-1}(\partial_x y)u + y^n \partial_x u$ would contradict the transversality of ∂_x.

So, by reduction modulo $\mathfrak{m}_{X,D}$, we get an injection $\pi : \kappa(X)^{\partial_x} \hookrightarrow K$, which is an algebraic extension of fields, since they have the same transcendence degree. A standard argument of spreading out then furnishes the étale morphism (x, f) (note that $\langle (\frac{d}{dx}, 0), \omega \rangle = 0$, whence $\langle \partial_x, f^*\omega \rangle = 0$).

The converse is immediate.

(2)(a) follows from the existence of f as in item (*b*) of (1), when D and X are smooth; this is standard.

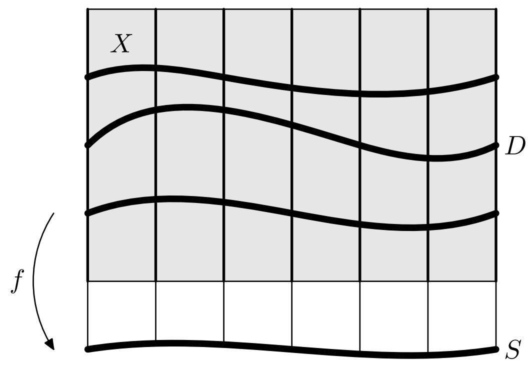
(2)(b) and (2)(c) follow as in the previous proposition. □

The data of the item (1)(b) of the previous proposition is called an algebraic tubular neighborhood for D. More precisely:

Definition 10.1.19 (Algebraic tubular neighborhoods). *Let X be a smooth algebraic k-variety, and let D be a* prime *divisor of X. Then X is said to be an* algebraic tubular neighborhood of *D if there exists an étale morphism*

$$(x, f) : X \longrightarrow \mathbb{A}^1_k \times S \tag{10.1.20}$$

such that the ideal sheaf $\mathcal{I}_D$ of D in X is freely generated by x and the composition $D \hookrightarrow X \to \mathbb{A}^1_k \times S \to S$ is finite étale.



The rational elementary fibrations which will occur in 25.1.4 are a special case of algebraic tubular neighborhood.

The following lemma shows how curves which are transversal to D can be incorporated in a family of integral curves.

Lemma 10.1.21. *Let X be a smooth affine variety and D be a prime divisor defined by a local equation $x = 0$. Let us consider curves C and C', closed points $P \in C$ and $P' \in C'$ and closed immersions $C \to X$ and $C' \to X$ such that C meets D transversally at P and C' meets D transversally at P'. Suppose $P \notin C'$ and $P' \notin C$. Then there exist an open common neighborhood U of P and P' in X and an algebraic tubular neighborhood*

$$(x, f) : U \longrightarrow \mathbb{A}^1_k \times S$$

of $D \cap U$, such that $C \cap U = f^{-1}(f(P))$ and $C' \cap U = f^{-1}(f(P'))$.

Proof. One can find elements $\lambda_1, \ldots, \lambda_{d-1}$ (resp. $\mu_1, \ldots, \mu_{d-1}$) of $\mathcal{J}_C$ (resp. $\mathcal{J}_{C'}$) which are local generators at P (resp. P') and such that the λ_i (resp. μ_i) do not vanish at P' (resp. P). Then $((\lambda_i, \mu_i)_{i=1,\ldots,d-1})$ defines a regular map $g : V \to S := (\mathbb{P}^1)^{d-1}$ on some open common neighborhood V of P and P' in X. Now, shrinking V we can obtain the asserted algebraic tubular neighborhood. More precisely, since C and D meet transversally at the closed point P, [48, 17.13.8.1] gives an isomorphism

$$\mathcal{J}_P/\mathcal{J}_P^2 \cong (\mathcal{J}_D/\mathcal{J}_D^2 \otimes_{\mathcal{O}_Z} \kappa(P)) \oplus (\mathcal{J}_C/\mathcal{J}_C^2 \otimes_{\mathcal{O}_C} \kappa(P)),$$

where by $\mathcal{J}_Y$ we denote the $\mathcal{O}_X$-ideal corresponding to a closed subscheme Y of X. Since $\mathcal{J}_P/\mathcal{J}_P^2 \cong \Omega^1_{X/K} \otimes_{\mathcal{O}_X} \kappa(P)$, for any system of sections x_1 (resp. $x_2, \ldots, x_d$) of $\mathcal{O}_X$ in a neighborhood U of P, representing a system of generators of $\mathcal{J}_D/\mathcal{J}_D^2 \otimes_{\mathcal{O}_Z} \kappa(P)$ (resp. of $\mathcal{J}_C/\mathcal{J}_C^2 \otimes_{\mathcal{O}_C} \kappa(P)$), we have that $(x_1, \ldots, x_d)$ are étale coordinates at P. We can shrink V around P, so that the morphism $(x, f) : U \to \mathbb{A}^1_k \times S$ (where f is the restriction of g to U) is an algebraic tubular neighborhood of $D \cap U$ and the fiber $f^{-1}(f(P))$ is connected. Certainly $C \cap U \subseteq f^{-1}(f(P))$, so by connectedness we have equality. The same argument works for C' and P'. $\square$

10.2 Regular connections along prime divisors

Let (X, D) be as 10: namely, X is a normal variety (over an algebraically closed field k), D is a prime divisor in X, and U is the open complement.

Let $\mathcal{M}$ be a locally free $\mathcal{O}_U$-module of finite rank μ endowed with an integrable connection

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_U \otimes_{\mathcal{O}_U} \mathcal{M}.$$

In this subsection and the next one, we shall introduce the notion of regularity and exponents of ∇ along D and establish basic properties. Our plan is to derive these notions and basic properties from the earlier ones for differential modules over the differential ring $(K((x)), \frac{d}{dx})$ (x being a local equation for D in X), using integrability.

Let $\widehat{\kappa(X)}$ denote the completion of the function field $\kappa(X)$ of X with respect to the valuation attached to D, and similarly let

$$M = \widehat{\mathcal{M}}_\eta$$

be the completion of the generic fibre of $\mathcal{M}$. Thus, M is a $\widehat{\kappa(X)}$-module and the connection ∇ gives for any $\partial \in \widehat{\mathcal{T}}_{X,D}$ an action ∇_∂, so that (M, ∇_∂) becomes a differential module over $(\widehat{\kappa(X)}, \partial)$. In analogy with 8.3.2 we give the following definition.

Definition 10.2.1 (Regularity along a prime divisor). *One says that* $(\mathcal{M}, \nabla)$ *is* regular along D *if there is a* $\widehat{\mathcal{O}}_{X,D}$*-lattice* $\widetilde{M}$ *in* M *stable under* $\widehat{\mathcal{T}}_{X,D}(\log D)$*.*

10.2.2. Any transversal derivation $\partial \in \widehat{\mathcal{T}}_{X,D}$ can be written in a unique way as $\partial = u\partial_x$, with $\partial_x(x) = 1$ and $u \in \widehat{\mathcal{O}}^\times_{X,D}$. By item (1) in Proposition 10.1.8, this gives rise to an identification of differential rings

$$(\widehat{\mathcal{O}}_{X,D}, \partial) \;\cong\; (K[[x]], u\frac{d}{dx}), \quad u \in K[[x]]^\times.$$

It thus makes sense to consider the regularity of M (endowed with ∇_{∂_x}), interpreted as a differential module over $K((x))$ (endowed with ∂_x), as in Subsection 8.3.

Recall that the sub-$\widehat{\mathcal{O}}_{X,D}$-module $\widehat{\mathcal{T}}_{X,D}(\log D)$ of $\widehat{\mathcal{T}}_{X,D}$ dual to $\widehat{\Omega}^1_{X,D}(\log D)$ decomposes as in (10.1.11). In particular, it contains $x\partial$ for *any* derivation $\partial \in \widehat{\mathcal{T}}_{X,D}$.

Proposition 10.2.3. *The following conditions are equivalent:*

(1) $(\mathcal{M}, \nabla)$ *is regular along* D*;*

(2) (M, ∇_∂) *is regular over* $(\widehat{\mathcal{O}}_{X,D}, \partial)$ *for* some (*resp.* any) *transversal derivation* $\partial \in \widehat{\mathcal{T}}_{X,D}$*;*

(3) (M, ∇_{∂_x}) *admits a Jordan decomposition:* $M \cong \bigoplus_i K((x)) \otimes_K K_{\alpha_i}^{(\mu)}$ *with* $\alpha_i \in k$;

(4) *there is an* $\mathcal{O}_{X,D}$*-lattice* $\widetilde{\mathcal{M}}_\eta$ *in* $\mathcal{M}_\eta$ *stable under* $\mathcal{T}_{X,D}(\log D)$.

Proof. Notice that $\mathrm{Der}_k K$ commutes with ∂_x, and that $k = K^{\mathrm{Der}_k K}$ (since k is algebraically closed). Then we can apply Theorem 8.3.10 (with $\Delta = \mathrm{Der}_k K$) and deduce that the exponents of (M, ∇_{∂_x}) belong to $k/\mathbb{Z}$. Thus, by Theorem 8.3.4, items (2) and (3) are equivalent.

Since it is clear that (1) implies (2), it is enough to show that (3) implies (1). Given the Jordan decomposition as in (3), the $K[[x]]$-lattice $\widetilde{M} := \bigoplus_i K[[x]] \otimes_K K_{\alpha_i}^{(\mu)}$ is stable under $x\partial_x$, and it is also stable under $\mathrm{Der}_k K$ in view of Lemma 8.2.2. Therefore, using (10.1.11), we conclude that $\widetilde{M}$, seen as an $\widehat{\mathcal{O}}_{X,D}$-lattice via the isomorphism 10.1.9, is stable under the whole $\widehat{\mathcal{T}}_{X,D}^{\log}$, which proves (1).

For (4), recall that $\mathcal{T}_{X,D}(\log D)$ contains $x\partial$ for *any* derivation $\partial \in \mathcal{T}_{X,D}$. It suffices to take the intersection $\mathcal{M}_\eta \cap \widetilde{M}$ in M. The converse is obvious. $\square$

Remarks 10.2.4. Let x be a local equation for D in X and let $\partial_x \in \widehat{\mathcal{T}}_{X,D}$ be a transversal derivation as in Lemma 10.1.7. By the use of the shearing transformation 7.2.4, we can always assume that the $\widehat{\mathcal{O}}_{X,D}$-lattice $\widetilde{M}$ in M is non-resonant with respect to $x\partial_x$.

The Jordan decomposition itself does not descend to $\mathcal{O}_{X,D}$ (the hypergeometric differential equation provides a counterexample, as soon as the parameters are not integral, since in that case it is irreducible).

Let $\tau : k/\mathbb{Z} \to k$ be any set-theoretic section of the projection (such that $\tau(\mathbb{Z}) = 0$ if not otherwise stated).

Proposition-Definition 10.2.5 (**τ-extensions**)**.** *There exists a unique* $\widehat{\mathcal{O}}_{X,D}$*-lattice* $\widetilde{M}$ *of* M *as in Definition* 10.2.1 (*resp.* $\mathcal{O}_{X,D}$*-lattice* $\widetilde{\mathcal{M}}_\eta$ *of* $\mathcal{M}_\eta$) *such that the elements* $\alpha_i \in k$ *occurring in the associated Jordan decomposition belong to the image of* τ*. It is called the* τ-extension *of* M (*resp. of* $\mathcal{M}_\eta$).

Proof. Starting from any lattice $\widetilde{M}$ as in definition 10.2.1 (and any basis of this lattice), the method of shearing transformations (cf. Lemma 7.2.4, with $K' = K$) allows to modify $\widetilde{M}$ into a τ-extension. Note that the matrix of the specialization at $x = 0$ of $\vartheta_x = x\nabla_{\partial_x}$ in any basis of $\widetilde{M}$ is then non-resonant (i.e., a τ-extension is automatically non-resonant), and 0 is the only eigenvalue in $\mathbb{Z}$.

Let us turn to uniqueness. Let $\widetilde{M}, \widetilde{M}'$ be two τ-extensions. Then $\widetilde{N} := \widetilde{M}^\vee \otimes \widetilde{M}'$ is a ϑ_x-stable lattice of the differential module $N := \mathrm{End}(M)$. By the last assertion of 8.3.7 we know that any solution of $\widetilde{N}$ in $K((x))$ is a solution in $K[[x]]$. It follows that $\mathrm{id}_M \in \widetilde{N}$, and, reversing the role of $\widetilde{M}$ and $\widetilde{M}'$, that $\widetilde{M} = \widetilde{M}'$. $\square$

10.3 Exponents along prime divisors

Let $\widetilde{M}$ be as in definition 10.2.1, and α_i be the elements of k occurring in the associated Jordan decomposition (Proposition 10.2.3). We know by Theorem 8.3.6 that these numbers are the eigenvalues of the value at $x = 0$ of the matrix (in some basis) of $\nabla_{x\partial_x}$ (this matrix is non-resonant by assumption).

Proposition-Definition 10.3.1 (Exponents along prime divisors). *The elements $\alpha_i \in k$ depend only on the choice of the lattice $\widetilde{M}$, and not on the choice of the local parameter x or of the transversal derivation ∂. Moreover, their classes $\overline{\alpha}_i$ modulo $\mathbb{Z}$ do not depend on $\widetilde{M}$. They are called the* exponents of $\mathcal{M}$ along D, *and the set of exponents is denoted by* $\mathrm{Exp}_D\mathcal{M}$ *(as subset of $k/\mathbb{Z}$).*

Proof. The first assertion comes from the easy observation that, given a lattice $\widetilde{M}$, the specialization at $x = 0$ of $\nabla_{x\partial_x}$ is an intrinsic k-linear endomorphism of the specialization at $x = 0$ of $\widetilde{M}$. The second assertion follows from Theorem 8.3.6. □

Proposition 10.3.2. *Regularity along D is stable under taking subquotients, extensions, tensor products and duals. Moreover, one has*

$$\mathrm{Exp}_D\mathcal{M} = \mathrm{Exp}_D\mathcal{M}_1 \cup \mathrm{Exp}_D\mathcal{M}_2$$

if $\mathcal{M}$ is an extension of $\mathcal{M}_1$ by $\mathcal{M}_2$, and

$$\mathrm{Exp}_D(\mathcal{M} \otimes \mathcal{M}') = \mathrm{Exp}_D\mathcal{M} + \mathrm{Exp}_D\mathcal{M}', \quad \mathrm{Exp}_D\mathcal{M}^\vee = -\mathrm{Exp}_D\mathcal{M}.$$

Proof. This is an immediate transposition of (part of) Proposition 8.3.9. □

Proposition 10.3.3. *Let $f : X' \to X$ be a dominant morphism of normal varieties, D', D prime divisors such that $U' = X' \setminus D'$ and $U = X \setminus D$ are smooth, and such that f induces a dominant map $D' \to D$. Let e_u be the ramification index of the extension $\kappa(X') = \kappa(U')$ (valuation induced by D') over $\kappa(X) = \kappa(U)$ (valuation induced by D).*

(1) *If $(\mathcal{N}, \nabla)$ is an integrable connection on U, regular along D, then $f^*(\mathcal{N}, \nabla)$ (with the inverse image connection, see 5.1) is regular along D', and*

$$\mathrm{Exp}_{D'}(f^*\mathcal{N}) = e_u \mathrm{Exp}_D(\mathcal{N}).$$

(2) *Suppose in addition that f induces a finite étale map $u : U' \to U$. Let $(\mathcal{M}', \nabla')$ be an integrable connection on U' regular along D'; then $u_*(\mathcal{M}', \nabla')$ (with the direct image connection, see 5.2) is regular along D, and*

$$\mathrm{Exp}_D(f_*\mathcal{M}') = \frac{1}{e_u}\mathrm{Exp}_{D'}(\mathcal{M}').$$

Proof. This is an immediate transposition of (part of) Proposition 8.3.9, and of the constructions 5.1 and 5.2. □

11 Regularity and exponents along a normal crossing divisor

We assume all over this Section 11 that the k-variety X is smooth.

11.1 Connections with logarithmic poles, and residues

11.1.1. We shall examine the case when the lattices $\widetilde{M}$ and $\widetilde{\mathcal{M}}_\eta$ considered above actually come from a global $\mathcal{O}_X$-module: this is the situation of integrable connections with logarithmic singularities along D.

It will be useful to drop the assumption that D is irreducible, and assume instead that *D is a normal crossing divisor* in the étale sense: there are local étale coordinates $x_1, \ldots, x_d$ on X such that the divisor D admits the equation $x_1 \cdots x_r = 0$. The divisor D is called a *strict normal divisor* if it is a normal divisor and its irreducible components are smooth. As usual, U is the complement of D and $j : U \to X$ the open inclusion.

We shall also consider, for $n \leqslant d$, the $\mathcal{O}_X$-modules of differential n-forms $\Omega^n_X(\log D) = \bigwedge^n \Omega^1_X(\log D)$ with logarithmic poles along D [35]. This is a locally free $\mathcal{O}_X$-module of rank $\binom{d}{n}$: in fact, $(x_1, \ldots, x_d)$ being a system of local coordinates at P in the étale neighborhood of a point P where D has the equation $x_1 \cdots x_r = 0$, a basis of $\Omega^1_X(\log D)$ is given by $\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_d$. A local section ω of $j_*\Omega^n_U$ is a section of $\Omega^n_X(\log D)$ if and only if ω and $d\omega$ have a simple pole at D, [35, II, 3.2 and 3.3.1] (the proof in given in the analytic context, but is easily translated into algebraic terms, working locally for the étale topology on X).

Definition 11.1.2 (**Connections with logarithmic poles along a divisors**). *Let $\widetilde{\mathcal{M}}$ be a coherent torsion-free $\mathcal{O}_X$-module. A* connection on $\widetilde{\mathcal{M}}$ with logarithmic poles along D *is a k-linear map*

$$\widetilde{\nabla} : \widetilde{\mathcal{M}} \longrightarrow \Omega^1_X(\log D) \otimes_k \widetilde{\mathcal{M}}$$

satisfying the Leibniz rule. It is said to be integrable *if the restriction of $(\widetilde{\mathcal{M}}, \widetilde{\nabla})$ to $U = X \smallsetminus D$ is an integrable connection. Such connections form a k-linear category denoted by* $\mathbf{MIC}(X, \log D)$.

For an integrable connection with logarithmic poles, one can construct the *logarithmic de Rham complex*

$$\mathrm{DR}_X(\widetilde{\mathcal{M}}, \widetilde{\nabla})) := [\widetilde{\mathcal{M}} \longrightarrow \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}} \longrightarrow \Omega^2_X(\log D) \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}} \longrightarrow \cdots]$$

in the usual way.

11.1.3 (**Residues**). Let us now assume that D is a *strict* normal crossing divisor: its irreducible components $D_i\,(i \in I)$ are smooth. We denote by ι_j the closed embedding $D_j \hookrightarrow X$ and by $\widetilde{\mathcal{M}}_{D_j}$ the $\mathcal{O}_{D_j}$-module ${\iota_j}^*\widetilde{\mathcal{M}}$.

In this situation, there is a well-defined $\mathcal{O}_{D_j}$-linear endomorphism

$$\operatorname{Res}_{D_j}\widetilde{\nabla} : \widetilde{\mathcal{M}}_{D_j} \longrightarrow \widetilde{\mathcal{M}}_{D_j}, \tag{11.1.4}$$

called the *residue* of $\widetilde{\nabla}$ at D_j. It is induced, locally, by the action of $\widetilde{\nabla}(x_j\frac{\partial}{\partial x_j})$, where $(x_1,\dots,x_d)$ denotes a system of local coordinates such that $D_i = V(x_i)$, for $i = 1,\dots,r$, $r \leqslant d$ [35, II.3.8]. In particular, $\operatorname{Res}_{D_j}\widetilde{\nabla} = 0$ for $r < j \leqslant d$. The coherent and locally free $\mathcal{O}_{D_j}$-module $\widetilde{\mathcal{M}}_{D_j}$ carries an integrable connection

$$\widetilde{\nabla}_{D_j} : \widetilde{\mathcal{M}}_{D_j} \longrightarrow \Omega^1_{D_j}(\log D \cap D_j) \otimes \widetilde{\mathcal{M}}_{D_j}, \tag{11.1.5}$$

with logarithmic poles along $D \cap D_j$, induced by the action of the $\widetilde{\nabla}(x_i\frac{\partial}{\partial x_i})$'s, for $i \neq j$ [35, II.3.9]. The endomorphism $\operatorname{Res}_{D_i}\widetilde{\nabla}$ of the $\mathcal{O}_{D_i}$-module $\widetilde{\mathcal{M}}_{D_i}$ is *horizontal* with respect to the connection $\widetilde{\nabla}_{D_i}$. Its characteristic polynomial $\operatorname{ind}_{D_i}(t) \in \mathcal{O}(D_i)[t]$ is therefore *constant along* D_i, i.e., $\operatorname{ind}_{D_i}(t) \in k[t]$ (cf. [35, II.3.10]).

The roots of $\operatorname{ind}_{D_i}(t)$ modulo $\mathbb{Z}$ are the *exponents* of $\mathcal{M} = \widetilde{\mathcal{M}}_{|U}$ along D_i. If these roots belong to the image of τ_i, then the localization of $\widetilde{\mathcal{M}}$ at the generic point of D_i is the τ_i-extension of $\mathcal{M}_{\eta,i}$.

Theorem 11.2.2 below states the converse, in some sense. It is an enhanced global form of Propositions 10.2.5 and 12.1.2.

11.2 Extensions with logarithmic poles

Definition 11.2.1. *Let $(\mathcal{M},\nabla)$ be an $\mathcal{O}_U$-module with integrable connection. We say that $(\mathcal{M},\nabla)$ is* regular along D *if it is regular along any component D_i of D.*

For each $i \in I$, we fix a set-theoretic section $\tau_i : k/\mathbb{Z} \to k$ of the projection satisfying $\tau_i(0) = 0$.

Theorem 11.2.2 (**τ-extension**). *A coherent $\mathcal{O}_U$-module with integrable connection $(\mathcal{M},\nabla)$ is regular along D if and only if there exists a coherent locally free sub-$\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$ of $j_*\mathcal{M}$ endowed with a connection $\widetilde{\nabla}$ with logarithmic pole along D and restricting to $\mathcal{M}$ on U.*

This extension $(\widetilde{\mathcal{M}},\widetilde{\nabla})$ is unique under the condition that the exponents along D_i belong to the image of τ_i, and in that case it is called the (logarithmic) τ-extension of $(\mathcal{M},\nabla)$ along D.

Remark 11.2.3. In the case where X is proper over $k = \mathbb{C}$, Deligne proved 11.2.2 using a transcendental monodromy argument, which is ascribed by Deligne [35, II.5.4] to Manin [80].

Remark 11.2.4. Since the sections τ_i are not additive, the formation of the τ-extension is neither compatible with tensor products, nor with taking inverse images in general.

11.3 On reflexivity

We first recall a few general facts about extensions of locally free $\mathcal{O}_X$-modules over a dense open subset of a normal variety X. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *reflexive* if the natural morphism from $\mathcal{F}$ to its bidual $\mathcal{F}^{\vee\vee}$ is an isomorphism. Any locally free module of finite rank is reflexive, but the converse is not true in general. However, this is true if X is smooth and $\operatorname{rk}\mathcal{F} = 1$ or $\dim X \leqslant 2$ [26, Ch. X, §4, n° 1, Cor. 3 of Prop. 3].

Let us only assume that X is *normal*, let $T \overset{\epsilon}{\hookrightarrow} X$ be a closed subvariety of codimension $\geqslant 2$, and let $\mathcal{F}$ be a locally free $\mathcal{O}_{X\smallsetminus T}$-module of finite rank. Then $\epsilon_*\mathcal{F}$ is a reflexive coherent $\mathcal{O}_X$-module, cf. [48, Cor. 5.11.4]. This is the unique reflexive extension of $\mathcal{F}$ to X (up to a unique isomorphism) [95, Prop. 7, and §3, Rem.2] (the point is that if $\widetilde{\mathcal{F}}$ is a reflexive coherent $\mathcal{O}_X$-module, then

$$\epsilon_*\epsilon^*\widetilde{\mathcal{F}} = \epsilon_*\mathcal{H}om(\epsilon^*\widetilde{\mathcal{F}}^\vee, \mathcal{O}_{X\smallsetminus T}) = \mathcal{H}om(\widetilde{\mathcal{F}}^\vee, \epsilon_*\mathcal{O}_{X\smallsetminus T}) = \mathcal{H}om(\widetilde{\mathcal{F}}^\vee, \mathcal{O}_X) = \widetilde{\mathcal{F}}\,;$$

applying this to $\widetilde{\mathcal{F}} = (\epsilon_*\mathcal{F})^{\vee\vee}$, one deduces that $(\epsilon_*\mathcal{F})^{\vee\vee} = \epsilon_*(\epsilon^*\widetilde{\mathcal{F}}) = \epsilon_*\mathcal{F}$, i.e., the reflexivity of $\epsilon_*\mathcal{F}$ stated above).

However, $\epsilon_*\mathcal{F}$ need not be locally free even if X is smooth. The following is a classical counterexample: if $X = \mathbb{A}^3$, $T = \{0\}$, $\pi : X \smallsetminus T \to \mathbb{P}^2$ is the natural projection, then $\mathcal{F} = \pi^*\Omega^1_{\mathbb{P}^2}$ does not extend to a locally free $\mathcal{O}_X$-module [95, 5.(a)].

11.4 Construction (and uniqueness) of $(\widetilde{\mathcal{M}}, \widetilde{\nabla})$

We come back to the proof of Theorem 11.2.2. In order to construct a coherent reflexive sub-$\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$ of $j_*\mathcal{M}$ such that the completion at the generic point of each D_i is the τ_i-extension, we may replace X by the complement of any closed subset T of codimension 2. We may thus assume that the branches D_i of D are smooth and disjoint. We may also assume that the $\mathcal{O}_{X,D_i}$-lattice $\widetilde{\mathcal{M}}_{\eta,i}$ provided by Proposition 10.2.5 extends to a locally free $\mathcal{O}_{U_i}$-module $\widetilde{\mathcal{M}}_i$ for some open neighborhood U_i of the generic point of D_i, and that U together with the U_i's form an open cover of X.

Notice that for $i \neq j$, one has $U_i \cap U_j \subseteq U$, so that $\mathcal{M}$ and the $\mathcal{M}_{\eta,i}$'s patch together to a locally free $\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$ which has the required properties.

Moreover, since each $\widetilde{\mathcal{M}}_{\eta,i}$ is unique (cf. Proposition 10.2.5), so is $\widetilde{\mathcal{M}}$. We now have to construct the logarithmic connection $\widetilde{\nabla}$.

Since for any i, $\widetilde{\mathcal{M}}_{\eta,i}$ is stable under $\mathcal{T}_{X,D_i}(\log D_i) = (\Omega^1_{X,D_i}(\log D_i))^\vee$, one can choose T (of codimension 2 in X) in such a way that ∇ extends to a (unique) integrable connection on the restriction of $\widetilde{\mathcal{M}}$ to $\widetilde{X} := X \smallsetminus T$ with logarithmic poles along $\widetilde{D} := D \smallsetminus D \cap T$. Let $\epsilon : \widetilde{X} \hookrightarrow X$, $j : U \hookrightarrow X$, $\widetilde{j} : U \hookrightarrow \widetilde{X}$ denote the inclusion morphisms (so that $\epsilon \circ \widetilde{j} = j$):

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "\epsilon"'] & \widetilde{D} \arrow[l, hook'] \arrow[d] \\
U \arrow[ur, "\widetilde{j}"] \arrow[r, hook, "j"] & X & D \arrow[l, hook']
\end{tikzcd}$$

Lemma 11.4.1. *For any $n \in \{1, \dots, d\}$, one has equality of sheaves $\epsilon^* j_* \Omega_U^n = \widetilde{j}_* \Omega_U^n$, and of their subsheaves*

$$\epsilon^* \Omega_X^n(\log D) = \Omega_{\widetilde{X}}^n(\log \widetilde{D}) \ .$$

Moreover, the canonical morphism $\Omega_X^n(\log D) \to \epsilon_* \Omega_{\widetilde{X}}^n(\log \widetilde{D})$ [50, 0_I, 4.4.3.2] is an isomorphism.

Proof. The equality $\epsilon^* j_* \Omega_U^n = \widetilde{j}_* \Omega_U^n$ follows from [48, Prop. 5.9.4] applied to $\mathcal{F} = \Omega_U^n$. The second equality is clear from the explicit local description of $\Omega_X^n(\log D)$ and $\Omega_{\widetilde{X}}^n(\log \widetilde{D})$, respectively. Taking into account the fact that $\Omega_X^n(\log D)$ is locally free, the third isomorphism follows according to [48, 5.10.2]. $\square$

Applying the functor ϵ_* allows to extend (uniquely) the connection on $\widetilde{\mathcal{M}}_{|\widetilde{X}}$ with logarithmic poles along $\widetilde{D}$ to a connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{M}}$ with logarithmic poles along D: the point is that, by the lemma and the projection formula,

$$\epsilon_*(\Omega_{\widetilde{X}}^1(\log \widetilde{D}) \otimes \widetilde{\mathcal{M}}_{|\widetilde{X}}) = \Omega_X^1(\log D) \otimes \epsilon_* \widetilde{\mathcal{M}}_{|\widetilde{X}} = \Omega_X^1(\log D) \otimes \widetilde{\mathcal{M}}.$$

11.5 Local freeness of $\widetilde{\mathcal{M}}$

In order to establish that $\widetilde{\mathcal{M}}$ is locally free, we shall use Gérard-Levelt's lattices as expounded in Section 9. We prove the following more precise lemma, which concludes the proof of Theorem 11.2.2.

Lemma 11.5.1. *Let $(\widetilde{\mathcal{M}}, \widetilde{\nabla})$ be a coherent $\mathcal{O}_X$-module with an integrable connection with logarithmic poles along D, such that the eigenvalues of the residues of $\widetilde{\nabla}$ along D_i lie in the image of τ_i. Then $\widetilde{\mathcal{M}}$ is locally free if and only if it is reflexive.*

Let T be a closed subset of X of codimension 2 as in 11.4 and set $\widetilde{X} = X \smallsetminus T$, $\widetilde{D} = D \smallsetminus (D \cap T)$. We denote by $\epsilon : \widetilde{X} \hookrightarrow X$ the inclusion morphism. We know that $\epsilon^* \widetilde{\mathcal{M}}$ is locally free. We have to prove that $\widetilde{\mathcal{M}} = \epsilon_* \epsilon^* \widetilde{\mathcal{M}}$ itself is locally free. It will be enough to show that for any closed point $P \in T = X \smallsetminus \widetilde{X}$, the completed stalk $\widehat{\widetilde{\mathcal{M}}}_P$ is a free module over the complete local ring $\widehat{\mathcal{O}}_{X,P}$.

Since the question is local for the étale topology on X, we may assume that X is affine and admits a system of global coordinates $(x_1, \dots, x_d)$ such that P corresponds to the origin and $D_i = V(x_i)$, for $i = 1, \dots, r$, $r \leqslant d$. If $r < d$, it will be convenient to choose set-theoretic sections τ_i with $\tau_i(0) = 0$ for $i = r+1, \dots, d$ as well.

We identify $\widehat{\mathcal{O}}_{X,P}$ with $k[[\mathbf{x}]]$ and consider the d-dimensional GL-structure

$$(\Lambda, (\nabla_1, \dots, \nabla_d)) = \Big(\widehat{\widetilde{\mathcal{M}}_P}, (\nabla_{\vartheta_{x_1}}, \dots, \nabla_{\vartheta_{x_d}})\Big).$$

We have to prove that Λ is free.

Let $q \geqslant p$ be the residual degree of $(\Lambda, (\nabla_1, \dots, \nabla_d))$ and let $(\alpha_1, \dots, \alpha_q)$ be the GL-multiexponents, iterated according to their multiplicities. Let $(e_1, \dots, e_q)$ be as in (9.1.10) for this GL-structure. By corollary 9.1.17, the polynomial $P_i(t)$ (recall the notations of 9.1.16) divides $\prod_{\ell=1}^{q}(t - \alpha_{i,\ell})$, and has its roots in the image of τ_i by assumption.

We then choose integers $\gamma_{i,j}$ such that $\alpha'_{i,j} := \alpha_{i,j} + \gamma_{i,j} \in \operatorname{Im}\tau_i$, for all i, j. Replacing e_ℓ by $e'_\ell = \prod_{i=1}^{d} x_i^{\gamma_{i,\ell}} e_\ell$, for each ℓ, then produces the *free* lattice Λ_τ considered in 9.1.13, which is generated by $(e'_1, \dots, e'_q)$ over $k[[\mathbf{x}]]$.

Let us approximate $(e_1, \dots, e_q)$ by a string $(v_1, \dots, v_q)$ of elements in $\widetilde{\mathcal{M}}_P$ up to the order $M \gg 0$, and use $(v'_1, \dots, v'_q) = (m_1 v_1, \dots, m_q v_q)$. The lattice generated by $(v'_1, \dots, v'_q)$ is the same Λ_τ by remark 9.1.12. As before, Λ_τ is freely generated by $(v'_1, \dots, v'_p)$, while $(v'_{p+1}, \dots, v'_q)$ are superfluous if $q > p$. Up to further shrinking X around P, we may even assume that the v_i's $(i = 1, \dots q)$ are global sections of $\widetilde{\mathcal{M}}$, and that the v'_i's (for $i = 1, \dots, p$) generate a free $\mathcal{O}_X$-module $\widetilde{\mathcal{M}}'$. By Corollary 9.1.17 again, the polynomial $P'_i(t)$ attached to it divides $\prod_{\ell=1}^{p}(t - \alpha'_{i,\ell})$, hence has its roots in the image of τ_i. By the uniqueness of the τ-extension, $\epsilon^*\widetilde{\mathcal{M}}' = \epsilon^*\widetilde{\mathcal{M}}$, hence, by applying ϵ_*, $\widetilde{\mathcal{M}}' = \widetilde{\mathcal{M}}$.

12 Base change

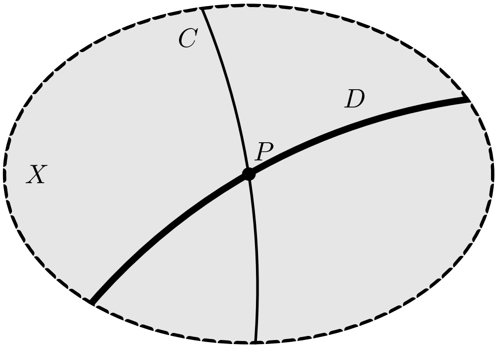
In this section X is a normal algebraic k-variety, D a closed subvariety of codimension 1 such that the complement $U = X \setminus D$ is smooth, $j : U \hookrightarrow X$ the inclusion morphism, D_i the irreducible components of D. We consider a locally free $\mathcal{O}_U$-module with connection $(\mathcal{M}, \nabla)$.

We consider pull-backs of a regular connection in increasing degree of generality, bringing into play increasingly sophisticated tools.

12.1 Restriction to curves I. The case when C meets D transversally at a smooth point

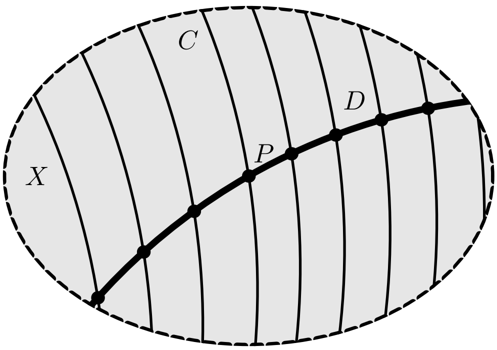
Proposition 12.1.1. *Assume that X is smooth and D is a smooth prime divisor, and let C be a curve in X which meets D transversally at a closed point P.*

If $(\mathcal{M}, \nabla)$ is regular at D, then $(\mathcal{M}, \nabla)_{|C}$ is regular at P. Moreover, $\operatorname{Exp}_P \mathcal{M}_{|C} = \operatorname{Exp}_D \mathcal{M}$.



Using Lemma 10.1.21, this follows from the more precise:

Proposition 12.1.2. $(\mathcal{M}, \nabla)$ *is regular along* D *if and only if for some* (*resp. any*) *family of integral curves w.r.t. a derivation transversal to* D (*see* 10.1.15), *with* X *and* D *replaced by suitable neighborhoods of* η_D, *the inverse image of* $(\mathcal{M}, \nabla)$ *to any curve* C (*defined over* k) *in the family is regular at any point* P *of* $C \cap D$. *Moreover,* $\mathrm{Exp}_P \mathcal{M}_{|C} = \mathrm{Exp}_D \mathcal{M}$.



Proof. Since the question is local around η_D, we may assume that D is given by a global equation $x = 0$ and that X is affine. Let us choose a cyclic vector (cf. 3.3.1) of $\mathcal{M}_\eta$ with respect to $\vartheta_x = x\partial_x$ for some transversal derivation ∂_x satisfying $\partial_x(x) = 1$. If one shrinks X appropriately around η_D, one may assume that this cyclic vector comes from a section m of $\mathcal{M}$ over U such $(m, \vartheta_x m, \dots, \vartheta_x^{\mu-1} m)$ is a basis of $\mathcal{M}$ over U. The corresponding monic differential operator $\Lambda = \sum b_i \vartheta_x^i$ has coefficients in $\mathcal{O}(U)$, and $\mathcal{M}$ is regular if and only if there is no pole at $x = 0$ (i.e., $b_i \in \mathcal{O}(X)$) by the Fuchs criterion 8.3.4 (item 4). It is clear that this is equivalent to the condition that the pull-back of Λ to any curve C has no pole at $x = 0$, i.e., that the inverse image of $\mathcal{M}$ to C is regular along $C \cap D$.

The second assertion comes from the identification, modulo $\mathbb{Z}$, of the exponents with the roots of the characteristic polynomial of Λ (7.3.7). □

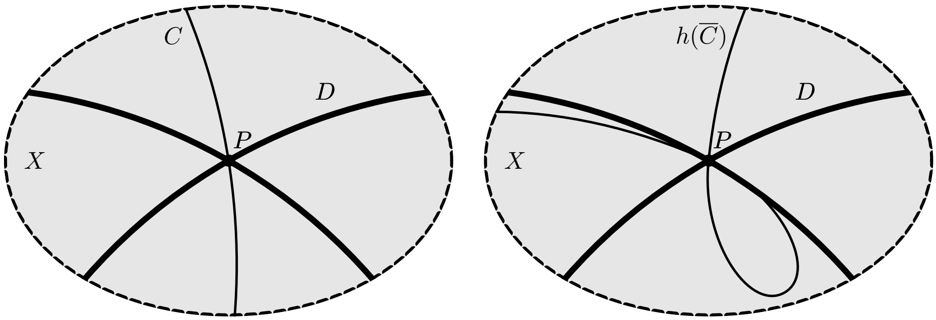
Corollary 12.1.3. *Suppose X is an algebraic tubular neighborhood of D as in 10.1.19, and let $(\mathcal{M}, \nabla)$ be a connection on $X \smallsetminus D$ regular along D. Let ∂ a derivation transversal to D, and suppose that there exists a cyclic vector for $(\mathcal{M}, \nabla)$ with respect to ∂ with associated monic differential operator Λ. Then the dominant term of the indicial polynomial of Λ (see 7.3.1) is invertible in $\mathcal{O}(S)$.*

Proof. Let X_s be the fiber of f at the point $s \in S$. By Proposition 12.1.2, the restriction of $(\mathcal{M}, \nabla)$ to X_s is regular for any $s \in S$, so that the degree of the indicial polynomial is the rank of $\mathcal{M}$, which does not depend on the point s. This proves the assertion. □

12.2 Restriction to curves II. The case when D is a strict normal crossing divisor

Proposition 12.2.1. *Assume that X is smooth and D is a strict normal crossing divisor, and let $h : \overline{C} \to X$ be a morphism from a smooth curve such that $h(\overline{C}) \not\subseteq D$. Let $C = h^{-1}(U)$.*

If $(\mathcal{M}, \nabla)$ is regular at D, then $(\mathcal{M}, \nabla)_{|C}$ is regular at any point $Q \in \overline{C}$ such that $P = h(Q) \in D$. Moreover, $\mathrm{Exp}_P \mathcal{M}_{|C} \subseteq \sum \mathrm{Exp}_{D_i} \mathcal{M}$, where the sum runs over the components D_i of D through P.



Proof. We denote by j_C the open immersion of C into $\overline{C}$. We use the existence of an extension $(\widetilde{\mathcal{M}} \subseteq j_* \mathcal{M}, \widetilde{\nabla})$ of $(\mathcal{M}, \nabla)$ on X with logarithmic poles along D (see 11.2.2). The functoriality map $h^*(j_* \Omega^1_X) \to j_{C*} \Omega^1_C$ induces a map

$$h^* \Omega^1_X(\log D) \longrightarrow \Omega^1_{\overline{C}}(\log Q),$$

so the map $\widetilde{\nabla}$ gives rise to a composite mapping

$$h^* \widetilde{\nabla} : h^* \widetilde{\mathcal{M}} \longrightarrow h^* \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\overline{C}}} h^* \Omega^1_X(\log D) \longrightarrow h^* \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\overline{C}}} \Omega^1_{\overline{C}}(\log Q)$$

which is an extension of $\nabla_{|C}$ with logarithmic poles at Q. Hence $\nabla_{|C}$ is regular at Q.

Let us address the assertion about exponents. We may assume that C is the complement of $\{Q\}$ in $\overline{C}$. We have the τ-extension $(\widetilde{\mathcal{M}}, \widetilde{\nabla})$ on X at our disposal (see 11.2.2). We may assume that $\widetilde{\mathcal{M}}$ is free, that each irreducible component D_i of D is globally described by an equation $x_i = 0$, and that $\Omega^1_{\overline{C}/k}(\log Q)$ is free with basis $\frac{dx}{x}$ for some global coordinate x on $\overline{C}$. The functoriality map $h^*\Omega^1_{X/k}(\log D) \longrightarrow \Omega^1_{\overline{C}/k}(\log Q)$ induces a linear map

$$T_h^* : \frac{h^*\Omega^1_{X/k}(\log D)}{h^*\Omega^1_{X/k}} \longrightarrow \frac{\Omega^1_{\overline{C}/k}(\log Q)}{\Omega^1_{\overline{C}/k}}$$

of skyscraper sheaves of k-vector spaces (concentrated at Q).

A basis of $h^*\Omega^1_{X/k}(\log D)/h^*\Omega^1_{X/k}$ (resp. $\Omega^1_{\overline{C}/k}(\log Q)/\Omega^1_{\overline{C}/k}$) at Q is given by the classes of the $h^*\big(\frac{dx_i}{x_i}\big)$ (resp. by the class of $\frac{dx}{x}$), and one has $T_h^*\big(h^*\big(\frac{dx_i}{x_i}\big)\big) = e_i\frac{dx}{x}$ (mod. $\Omega^1_{\overline{C}/k}$) for some $e_i \in \mathbb{Z}_{\geqslant 0}$ (e_i is the valuation of the image in $\mathcal{O}_{\overline{C},Q}$ of a local equation of D_i in $\mathcal{O}_{X,P}$).

Now $\widetilde{\nabla}$ induces a composite map

$$\begin{tikzcd}
(h^*\widetilde{\mathcal{M}})_Q \otimes \kappa(Q) \arrow[rr, "u=(u_i=(\mathrm{Res}_{D_i}\nabla)_P)"] \arrow[dr, "\mathrm{Res}_Q h^*\nabla"'] & & (h^*\widetilde{\mathcal{M}})_Q \otimes \frac{h^*\Omega^1_{X'/k}(\log D)}{h^*\Omega^1_{X'/k}} \arrow[d, "v=1\otimes T_h^*"] \\
& (h^*\widetilde{\mathcal{M}})_Q \otimes \kappa(Q) \cong & (h^*\widetilde{\mathcal{M}})_Q \otimes \frac{\Omega^1_{\overline{C}/k}(\log Q)}{\Omega^1_{\overline{C}/k}}
\end{tikzcd}$$

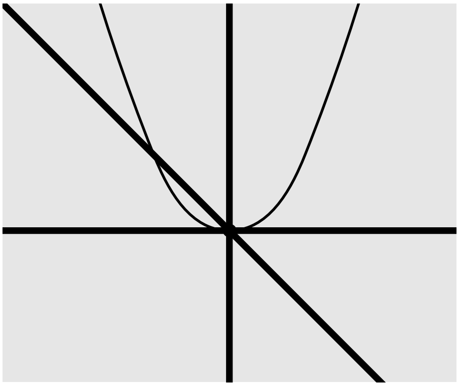
The map u_i is nothing but the specialization of $\mathrm{Res}_{D_i}\nabla$ at P, and $v \circ u$ is nothing but $\mathrm{Res}_Q h^*\nabla$. Since the $\mathrm{Res}_{D_i}\nabla$'s commute, the endomorphisms $v \circ u_i = e_i u_i$ of the finite-dimensional $k = \kappa(Q)$-vector space $(h^*\widetilde{\mathcal{M}})_Q \otimes \kappa(Q)$ also commute and the eigenvalues of $\sum_i v \circ u_i$ are of the form $\sum_i e_i\lambda_i$, for eigenvalues λ_i of u_i. We conclude that $\mathrm{Exp}_Q(h^*\nabla) \subseteq \sum_i e_i \mathrm{Exp}_{Z_i}(\nabla)$. $\square$

One cannot expect that these logarithmic techniques will apply beyond the case of a normal crossing polar divisor without running into serious problems, as the following example shows.

Bernstein's example[1]. In this example, $X = \mathbb{A}^2_k$, $D_1 = V(x_1), D_2 = V(x_2), D_3 = V(x_1+x_2)$, D is the union of the three lines, P is the origin, $\epsilon : X^* = X \smallsetminus P \hookrightarrow X$,

[1]This example due to J. Bernstein shows that the "proof" of 13.1.5 in the general case which was proposed in the first edition of this book was not correct (the problem was in Lemma I.5.5). See [14] for a more detailed discussion of this counterexample.

$D^* = D \cap X^*$.



On $U := X \setminus D = X^* \setminus D^*$, the differential form

$$\omega := \frac{1}{x_2 - x_1}\Big(\frac{dx_2}{x_2} + \frac{dx_1}{x_1} - 2\frac{d(x_1+x_2)}{x_1+x_2}\Big) = \frac{x_2 dx_1 - x_1 dx_2}{x_1 x_2 (x_1 + x_2)}$$

extends to a differential form on X^* with logarithmic poles along D^*; in fact, one has

$$d\omega = \frac{dx_1 \wedge dx_2}{x_1 x_2 (x_1 + x_2)}.$$

The claim is that the connection ∇ on $\mathcal{O}^2_{X^*}$ given in the standard basis by $d + G$, where

$$G = \begin{pmatrix} 0 & 0 \\ \omega & \frac{dx_1}{2x_1} + \frac{dx_2}{2x_2} \end{pmatrix},$$

is in $\mathbf{MIC}^\tau(X^*(\log D^*))$ for a suitable section τ, and the extension $\epsilon_*(\mathcal{O}^2_{X^*}, \nabla) = (\mathcal{O}^2_X, \epsilon_*\nabla)$ is regular along the closed subset $D \subseteq X$, but its pull-back to the parabola $C = V(x_1 - x_2^2)$ has a double pole at the origin.

The integrability of ∇ is proved by the explicit computation that $dG + G \wedge G = 0$. If one chooses τ_i in such a way that $\tau_i(\frac{1}{2}) = \frac{1}{2}$ for $i = 1, 2$, then $(\mathcal{O}^2_{X^*}, \nabla)$ belongs to $\mathbf{MIC}^\tau(X^*(\log D^*))$, hence is the τ-extension of $(\mathcal{O}^2_U, \nabla)$ to X^*. To verify the regularity of $\epsilon_*\nabla$ it suffices to show that, for a smooth modification $\pi : Y \to X$ such that $\pi^{-1}(D)$ is a divisor with normal crossings in Y, $\pi^*(\mathcal{O}^2_X, \epsilon_*\nabla)$ is regular along the components of $\pi^{-1}(D)$. This is immediately seen by taking for π the blow-up of $P \in X$. One gets, in coordinates (x_1, x_2, s, t), with $x_2 = tx_1$, $x_1 = sx_2$,

$$\omega = -\frac{1}{x_1}\frac{dt}{t(t+1)} = \frac{1}{x_2}\frac{ds}{s(s+1)},$$

$$G = \begin{pmatrix} 0 & 0 \\ -\frac{1}{x_1}\frac{dt}{t(t+1)} & \frac{dx_1}{x_1} + \frac{dt}{2t} \end{pmatrix}.$$

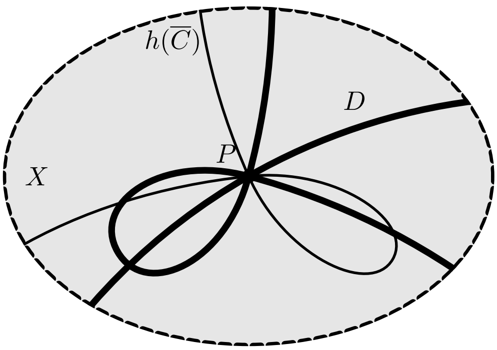
Direct substitution of $x_1 = u$ and $x_2 = u^2$ in the form ω immediately shows that there is a pole of order 2 in u, so the restriction of ω to the curve C does not have simple poles at P.

12.3 Restriction to curves III. The general case

Theorem 12.3.1. *Let $h : \overline{C} \to X$ be a morphism from a smooth curve such that $h(\overline{C}) \nsubseteq D$, and $C = h^{-1}(U)$. Let $Q \in \overline{C}$ map to a point $P \in D$.*

If $(\mathcal{M}, \nabla)$ is regular at every component D_i of D which contains P, then $(\mathcal{M}, \nabla)_{|C}$ is regular at Q. Moreover, $\mathbb{Q}\mathrm{Exp}_Q \mathcal{M}_{|C} = \mathbb{Q}\mathrm{Exp}_D \mathcal{M}$.

Remark 12.3.2. The assertion concerning regularity is due to Deligne [35], who proved it using Hartogs' theorem and resolution of singularities. The assertion concerning exponents is due to Kashiwara [58].



Note that the theorem has an easy converse:

Proposition 12.3.3. *If the restriction of $(\mathcal{M}, \nabla)$ to every locally closed smooth curve in X is regular at ∞ (i.e., at the points of a smooth compactification of C) with exponents contained in a subset $\Sigma \subseteq k/\mathbb{Z}$, then $(\mathcal{M}, \nabla)$ is regular at every component of D with exponents contained in Σ.*

Proof. It suffices to consider curves which meet the components D_i transversally, and to apply Proposition 12.1.2. $\square$

The proof of Theorem 12.3.1 will only be completed in Chapter 6 (21.1.2), using some irregularity theory. Here, we content ourselves to reduce the question to the case where X is a smooth surface (following [5]), and to prove the second assertion about exponents, assuming regularity.

12.3.4 (Reduction to the case of a normal surface). This is essentially a "Bertini argument":

Lemma 12.3.5. *For any $P \in D$, there is a locally closed surface $X' \subseteq X$ containing P such that $U' := X' \cap U$ is smooth and the components D'_i of $D' := X' \cap D$ are the components of D containing P.*

Proof. Replacing X by a quasi-projective neighborhood of P, and taking the normalization of its Zariski closure in some projective embedding, we may assume that X is projective. The lemma follows from the following

Claim. *Let X be a closed normal projective subvariety of dimension $d \geqslant 2$ in $\mathbb{P}^N$, and let U, $D = \bigcup_i D_i$, $h : C \to X$ and $h(Q) = P \in D$ as before. Then for any sufficiently large δ there exists an irreducible complete intersection $Y \subseteq \mathbb{P}^N$ of dimension $N - d + 2$ and multidegree $(\delta, \dots, \delta)$ such that:*

(i) *Y contains $h(C)$ and cuts U transversally at the generic point of $h(C)$;*

(ii) *$Y \cap X$ is a normal connected surface and Y cuts $U \smallsetminus (h(C) \smallsetminus P)$ transversally;*

(iii) *in a neighborhood of P, Y cuts each D_j transversally out of P and does not cut any irreducible component of $X \smallsetminus U$ of codimension > 1 in X, nor the singular locus of D, except in P.*

For lack of an adequate reference, let us give some detail. For $N = 2$ or $d = 2$ there is nothing to prove, so we may suppose $d > 2$. For short, we change a little notation and now write C for the closure of $h(C)$ in X, with reduced structure (an irreducible, possibly singular, curve). Let $\pi : \widetilde{\mathbb{P}} \to \mathbb{P}^N$ be the blow-up centered at C, and let $E \subseteq \widetilde{\mathbb{P}}$ be the exceptional divisor.

We set $\mathcal{I}_C = \mathrm{Ker}(\mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_C)$. Then $\mathrm{Im}(\pi^*\mathcal{I}_C \to \mathcal{O}_{\widetilde{\mathbb{P}}}) = \mathcal{O}(-E)$ and, for $\delta \gg 0$, $\pi^*(\mathcal{O}_{\mathbb{P}^N}(\delta)) \otimes \mathcal{O}(-E)$ is very ample: a basis of global sections defines an embedding of $\widetilde{\mathbb{P}}$ into $\mathbb{P}^M$. On the other hand, since π is birational and $\mathbb{P}^N$ is normal, one has $\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}} = \mathcal{O}_{\mathbb{P}^N}$. It follows that the two natural arrows $\mathcal{I}_C \to \pi_*(\mathrm{Im}(\pi^*\mathcal{I}_C \to \mathcal{O}_{\widetilde{\mathbb{P}}})) = \pi_*\mathcal{O}(-E) \to \pi_*\mathcal{O}_{\widetilde{\mathbb{P}}} = \mathcal{O}_{\mathbb{P}^N}$ are inclusions of ideals of $\mathcal{O}_{\mathbb{P}^N}$. Let $Z \subseteq C$ be the closed subscheme corresponding to $\pi_*\mathcal{O}(-E)$. If $Z \neq C$, then Z would be punctual and $1 \in \mathcal{O}_{\mathbb{P}^N}(\mathbb{P}^N \smallsetminus Z^{\mathrm{red}})$ would correspond to a function on $\widetilde{\mathbb{P}} \smallsetminus (\pi^{-1}(Z))^{\mathrm{red}}$ vanishing on $E \smallsetminus (\pi^{-1}(Z))^{\mathrm{red}}$, a contradiction. Hence $Z = C$ and therefore $\pi_*\mathcal{O}(-E) = \mathcal{I}_C$. From the projection formula, one deduces that

$$\pi_*(\pi^*(\mathcal{O}_{\mathbb{P}^N}(\delta)) \otimes \mathcal{O}(-E)) \cong \mathcal{O}_{\mathbb{P}^N}(\delta) \otimes \mathcal{I}_C \cong \mathcal{I}_C(\delta),$$

whence

$$\begin{aligned} \mathrm{Ker}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta)) \longrightarrow H^0(C, \mathcal{O}_C(\delta))) &= H^0(\mathbb{P}^N, \mathcal{I}_C(\delta)) \\ &= H^0(\widetilde{\mathbb{P}}, \pi^*(\mathcal{O}_{\mathbb{P}^N}(\delta)) \otimes \mathcal{O}(-E)), \end{aligned}$$

and the linear system of hypersurfaces of degree δ in $\mathbb{P}^N$ containing C gives rise to a locally closed embedding $\mathbb{P}^N \smallsetminus C \hookrightarrow \mathbb{P}^M = \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_C(\delta)))$, with Zariski closure $\widetilde{\mathbb{P}}$. The canonical bijection between hyperplanes $\mathcal{H}$ of $\mathbb{P}^M$ and hypersurfaces H of degree δ in $\mathbb{P}^N$ containing C, is such that the intersection $\mathcal{H} \cap (\mathbb{P}^N \smallsetminus C)$ (in $\mathbb{P}^M$) equals $H \smallsetminus C$. So, the intersection of $X \smallsetminus C$ with a general complete intersection Y of multidegree $(\delta, \dots, \delta)$ ($1 \leqslant s \leqslant d-1$ entries) in $\mathbb{P}^N$ containing η_C, is the intersection of $X \smallsetminus C$ with a general linear subvariety $\mathcal{Y}$ of codimension

s in $\mathbb{P}^M$. By [2, Exp. XI, Thm. 2.1. (i)], Y cuts $X \smallsetminus C$ (resp. the smooth part of $D \smallsetminus (C \cap D)$) transversally and intersects properly any irreducible component of $\partial X \smallsetminus (C \cap \partial X)$. Since $s < d$, Bertini's theorem ([54, II,8.18] or [54, III,10.8]) shows that the intersection of $\mathcal{Y}$ with the strict transform of X in $\mathbb{P}^M$ is normal and connected. On the other hand, since η_C is a simple point of X, it is well known that a general complete intersection of s hypersurfaces of degree δ in $\mathbb{P}^N$ containing η_C intersects X transversally at this point. Applying these considerations for $s = d-2$, one obtains (i), (ii) and (iii), which proves the lemma. $\square$

If $(\mathcal{M}, \nabla)$ is regular along each $D_1, \ldots, D_I$, so is its pullback $(\mathcal{M}, \nabla)_{|U'}$ along $D_1 \cap X', \ldots, D_t \cap X'$; this reduces 12.3.1 to the case where X is a (projective) normal surface.

12.3.6 (**Reduction to the case of a smooth surface).** One cannot simply blow up P since it is not clear a priori that the pull-back of $(\mathcal{M}, \nabla)$ will be regular along the exceptional divisors. Instead, we project to a plane:

Lemma 12.3.7. *We now assume that X is a projective normal surface. There exists a morphism $g : X \to \mathbb{P}^2_k$ finite in a neighborhood V of P, such that $g(h(C) \cap V)$ is not contained in the branch locus, and such that $g(P) \notin g(T)$ for any irreducible component T of $X \smallsetminus U$ if $P \notin T$.*

Proof. We may suppose that X is a normal projective surface in $\mathbb{P}^N$. Let $\mathbb{G}(N-3, \mathbb{P}^N)$ be the Grassmannian of linear subvarieties of $\mathbb{P}^N$ of codimension 3, and let G be its dense open subset consisting of linear subvarieties which do not intersect X.

For any $\gamma \in G$, X may be considered as a closed subvariety of the blow-up $\widetilde{\mathbb{P}}_\gamma$ of $\mathbb{P}^N$ at γ, and the projection $p_{//\gamma} : \widetilde{\mathbb{P}}_\gamma \to \mathbb{P}^2$ with center γ induces a morphism $g_\gamma : X \to \mathbb{P}^2$.

Let $\mathbf{\Lambda} \subseteq \mathbb{G}(N-2, \mathbb{P}^N) \times \mathbb{G}(N-3, \mathbb{P}^N)$ be the incidence subvariety (locus of (λ, α) such that λ contains α), and let p_2, p_3 be the natural projections. Notice that p_3 is a fibration with fiber $\mathbb{P}^2$ and admits a section above G: for any $\gamma \in G$ there is a unique $\lambda_\gamma \in \mathbb{G}(N-2, \mathbb{P}^N)$ of $\mathbb{P}^N$ of codimension 2 passing through P and containing γ. When γ varies, the λ_γ form (via p_2) a dense open subset of the hyperplane section of $\mathbb{G}(N-2, \mathbb{P}^N)$ of linear subvarieties containing P.

On the other hand, λ_γ may be identified with the fiber of $p_{//\gamma}$ above $g_\gamma(P) \in \mathbb{P}^2$, and $\lambda_\gamma \cap X$ with $g_\gamma^{-1}(g_\gamma(P))$. By Bertini, one deduces that there is an open dense subset $G' \subseteq G$ such that for any $\gamma \in G'$, g_γ is finite in a neighborhood of P.

Moreover, if there exists $\lambda \in p_3^{-1}(\gamma)$ which intersects X transversally and cuts $h(C) \cap V$ (resp. which passes through P and avoids the T's), then $g_\gamma(h(C) \cap V)$ is not contained in the branch locus of g_γ (resp. $g_\gamma(P) \notin g_\gamma(T)$). One deduces that there is an open dense subset $G'' \subseteq G'$ such that for any $\gamma \in G''$, $g = g_\gamma$ satisfies the conditions in the lemma. $\square$

The map g induces an étale morphism

$$X' = V \smallsetminus (g_{|V}^{-1}(B) \cup g_{|V}^{-1}(g(D))) \longrightarrow \mathbb{P}^2_{\mathbb{C}} \smallsetminus (B \cup g(D))$$

which is finite over its image. Up to replacing C by a suitable neighborhood of P, we have $h(C \smallsetminus P) \subseteq U \cap X'$. Using 13.1.6 and 12.3.7, one sees that the push-forward $g_*((\mathcal{M}, \nabla)_{|U\cap X'})$ on $g(U \cap X')$ is regular along each component of $\mathbb{P}^2_{\mathbb{C}} \smallsetminus g(U \cap X')$ passing through $g(Q)$ (which are either the $g(D_j)$'s or else are contained in B).

On the other hand, if the connection $(g \circ h)^* g_*((\mathcal{M}, \nabla)_{|U\cap X'})$ on $C \smallsetminus Q$ is regular at Q, the same is true for its subconnection $h^*((\mathcal{M}, \nabla)_{|U\cap X'})$. Therefore, in order to prove the theorem, one may substitute to X any Zariski neighborhood of $g(P)$ in the projective plane. $\square$

12.3.8 (Exponents and restriction to curves). Let us prove that $\mathbb{Q}\mathrm{Exp}_Q \mathcal{M}_{|C} = \mathbb{Q}\mathrm{Exp}_D \mathcal{M}$ assuming that $\mathcal{M}_{|C}$ is regular at the points Q above P. Unfortunately, we don't know a purely algebraic proof (beyond the case where D is a normal crossing divisor). Here is an argument due to O. Gabber.

As before, one reduces to the case where X is a smooth surface (or even an affine neighborhood of the origin in the complex plane). Let X' be an iterated blow-up with center P such that the exceptional divisors $E_j, j \leqslant J$, together with the strict transforms D'_i of the $D_i, i \leqslant I$, form a divisor with strict normal crossings, whose complement we denote by U'. Let ∇' be the pull-back of ∇ on X'. By 21.1.2, we know that ∇' is *regular*.

It suffices to show that $\sum_{i\leqslant I} \mathbb{Q}\mathrm{Exp}_{D'_i}\nabla'$ contains $\sum_{j\leqslant J} \mathbb{Q}\mathrm{Exp}_{E_j}\nabla'$. Equivalently: any $\mathbb{Q}$-linear form $\phi : \big(\sum_{j\leqslant J} \mathrm{Exp}_{E_j}\nabla' + \sum_{i\leqslant I} \mathrm{Exp}_{D'_i}\nabla'\big) / \sum_{i\leqslant I} \mathrm{Exp}_{D'_i}\nabla' \to \mathbb{Q}$ is 0.

Let U_k be a tubular neighborhood of E_k in $X'(\mathbb{C})$ and set $U_k^0 = U_k \cap U'(\mathbb{C})$. According to [84], $\pi_1(U_k^0)$ is generated by elements α_j (turning around E_j, $j \leqslant J$) and elements β_i (turning around D'_i, $i \leqslant I$), such that α_k is central and

$$\gamma_k := \alpha_1^{(E_1.E_k)} \cdots \alpha_J^{(E_J.E_k)} \beta_1^{(D'_1.E_k)} \cdots \beta_I^{(D'_I.E_k)}$$

is a commutator (see [74] for detail about base points). Since α_k is central, the monodromy representation of $\pi_1(U_k^0)$ attached to ∇' admits subrepresentations $V_{k\ell}$ where α_k has a unique eigenvalue, and since γ_k is a commutator, one has $\det(\gamma_{k|V_{k\ell}}) = 1$, whence

$$\begin{aligned} &\det(\alpha_{1|V_{k\ell}})^{(E_1.E_k)} \cdots \det(\alpha_{J|V_{k\ell}})^{(E_J.E_k)} \\ &= \det(\beta_{1|V_{k\ell}})^{-(D'_1.E_k)} \cdots \det(\beta_{I|V_{k\ell}})^{-(D'_I.E_k)}. \end{aligned} \tag{12.3.9}$$

On the other hand, $V_{k\ell}$ corresponds to a subconnection $\nabla_{k\ell}$ of $\nabla'_{|U_k^0}$ which has only one exponent e_k (up to an integer) along E_k, and taking logarithms, the last equation translates into a linear equation between exponents of ∇' along the E_j's and the D'_i's. We choose $e_k = e_k^{\pm}$ such that $\phi(e_k^{\pm})$ is the maximal (resp. minimal) value among the images by ϕ of all exponents along all E_k. Combining (12.3.9) and the fact that ϕ kills the exponents along D'_j, and the fact that $(E_j.E_k) \geqslant 0$ if $j \neq k$, we get the inequalities $\pm\sum_j (E_j.E_k)\phi(e_j^{\pm}) \geqslant 0$ for any k, thus $\sum_{jk}(E_j.E_k)(\phi(e_j^+) - \phi(e_j^-))(\phi(e_k^+) - \phi(e_k^-)) \geqslant 0$. Using the fact that the matrix $(E_j.E_k)$ is negative definite, one concludes that $\phi(e_k^+) = \phi(e_k^-)$, and further, that they are all 0. $\square$

12.4 Pull-back of a regular connection along D

Theorem 12.4.1. *Let X' be a normal algebraic k-variety, D' a closed subvariety of codimension 1 such that the complement $U' = X' \setminus D'$ is smooth, and let $f : X' \to X$ be a morphism which sends D' to D.*

If $(\mathcal{M}, \nabla)$ is regular at every component of D, then $f_{U'}^*(\mathcal{M}, \nabla)$ is regular at every component of D'. Moreover, $\mathbb{Q}\mathrm{Exp}_Q f_{U'}^*\mathcal{M} = \mathbb{Q}\mathrm{Exp}_D\mathcal{M}$.

Proof. We may replace X' and D' by smooth dense open subsets. Proposition 12.1.2 then reduces the question to the case where X' is a curve, which is treated in the previous theorem. $\square$

13 Global regularity and exponents

In this final section, we now change our viewpoint: instead of looking at a connection on the complement U of a fixed (polar) divisor in an ambient variety X and at its behaviour along D, we fix U and look at the behaviour of the connection along the boundary of arbitrary boundary components in partial compactifications of U.

With this change of perspective in mind, we also change notation. Let X be a smooth connected variety over k (algebraically closed field of characteristic zero) and $(\mathcal{M}, \nabla)$ is a locally free $\mathcal{O}_X$-module of finite rank with integrable connection ∇.

13.1 Global regularity

Definition 13.1.1 (Divisorial valuations). *A divisorial valuation v of X (or of $\kappa(X)$) is a discrete valuation of $\kappa(X)$ trivial over k, normalized by the condition that its value group is exactly $\mathbb{Z}$, and such that $\mathrm{tr.deg}_k k(v) = \mathrm{tr.deg}_k \kappa(X) - 1$, where $k(v) \supseteq k$ is the residue field of the valuation. We will use R_v and $\mathfrak{m}_v$ for the valuation ring and its maximal ideal.*

13.1.2 (Models of divisorial valuations). For any divisorial valuation v of X there exists a birational map $u : X \to \widetilde{X}$, with X' a normal k-variety, and a prime divisor D of $\widetilde{X}$ such that the valuation v is induced by D, i.e., in the isomorphism $\kappa(\widetilde{X}) \to \kappa(X)$ the local ring $\mathcal{O}_{\widetilde{X},D}$ corresponds to the valuation ring of v (so that $\kappa(D)$ corresponds to $k(v)$). In this situation, the pair $(\widetilde{X}, D)$ is said to be a model of v.

13.1.3 (**Regularity along a divisorial valuation**). Let $(\mathcal{M}, \nabla)$ be an object of $\mathbf{MIC}(X)$. One says that $(\mathcal{M}, \nabla)$ is regular along a divisorial valuation v of X if it is regular along D (see 10.2.1) for any model $(\widetilde{X}, D)$ of v.

The regularity condition along a divisorial valuation v of X is then equivalent to the regularity the generic fiber of $\mathcal{M}$ w.r.t. the valuation v (and a transversal derivation of $\kappa(X)$ w.r.t. v), that is to say, to the existence of an R_v-lattice in $\mathcal{M}_\eta$ stable for the corresponding logarithmic derivation, see 8.3.2).

Definition 13.1.4 (**Global regularity, fuchsian connections**). *The differential module* $(\mathcal{M}, \nabla)$ *is said to be* (globally) regular *or* fuchsian *if it is regular at every divisorial valuation of* X.

This is a birational notion, in the sense that if $f : X' \to X$ is a birational morphism (and X' is smooth), then $(\mathcal{M}, \nabla)$ is regular if and only if $f^*(\mathcal{M}, \nabla)$ is regular.

In the following assertion, we use the existence of normal compactification $\overline{X}$ of X; such a compactification is obtained by normalizing any (Nagata) compactification of X.

Theorem 13.1.5. *The following conditions are equivalent:*

(1) $(\mathcal{M}, \nabla)$ *is* (*globally*) *regular;*

(2) *for every* normal *compactification* $\overline{X}$, ∇ *is regular along every component* D_i *of* $\overline{X} \setminus X$ *of codimension* 1 *in* $\overline{X}$*;*

(3) *for some* normal *compactification* $\overline{X}$, ∇ *is regular along every component* D_i *of* $\overline{X} \setminus X$ *of codimension* 1 *in* $\overline{X}$.

Proof. The equivalence of (1) and (2) is elementary: it follows from the fact that given a compactification $\overline{X}$ as in (2), for any divisorial valuation v of $\kappa(X)$ there exists a closed irreducible subscheme W of $\overline{X}$ (whose points $x \in \overline{X}$ have the property that the local ring $\mathcal{O}_{\overline{X},x}$ is contained in the valuation ring R_v), and the regularity at v corresponds to the regularity along the divisor obtained by blowing-up W (in $\overline{X}$).

For the equivalence of (2) and (3), Theorem 12.3.1 and Proposition 12.3.3 allow to pass from a normal compactification to another using specialization along enough curves. □

Theorem 13.1.6 (**Stability properties**). *Global regularity is stable under taking subquotients, extensions, tensor products and duals.*

It is invariant under direct image by finite étale morphisms.

It is invariant under inverse image by any morphism; moreover, if the morphism is dominant, the opposite implication is true: a connection is globally regular if and only if its inverse image is globally regular for a dominant morphism.

Proof. Everything follows from the elementary equivalence (1) $\Leftrightarrow$ (2) together 10.3.2 and 10.3.3, except for base change, which follows from Theorem 12.4.1. □

Remark 13.1.7. In the language of $\mathcal{D}$-modules, the regularity of holonomic $\mathcal{D}$-modules is defined alternatively in terms of local conditions (see [59], or [16]) close to the definition in the language of integrable connections, or (see [60]) in terms of the radicality of the ideal of definition of the characteristic variety (for suitable good filtration). The equivalence of the two definitions is considered well known, and a proof, in the case of normal crossing divisors, is given in [28].

13.2 Global exponents

13.2.1 (**Exponents along a divisorial valuation).** Let X be a smooth k-variety and let $(\mathcal{M}, \nabla)$ be a differential module on X regular along the divisorial valuation v of X. The exponents $\mathrm{Exp}_v(\mathcal{M}, \nabla) = \mathrm{Exp}_v(\mathcal{M}) = \mathrm{Exp}_v(\nabla)$ of $(\mathcal{M}, \nabla)$ along v are the exponents of an extension of $(\mathcal{M}, \nabla)$ to a model of v, as elements of $k/\mathbb{Z}$. The definition is well posed (see Section 10.3) and to each exponent we may associate a multiplicity (see 8.1.12).

Definition 13.2.2 (**Global exponents).** *Let X be a smooth k-variety and let $(\mathcal{M}, \nabla)$ be a regular differential module on X. We define the (finitely generated) abelian group of* global exponents $\mathbb{Z}\mathrm{Exp}(\mathcal{M}, \nabla) = \mathbb{Z}\mathrm{Exp}(\mathcal{M}) = \mathbb{Z}\mathrm{Exp}(\nabla)$ *as the subgroup of $k/\mathbb{Z}$ generated by* $\mathrm{Exp}_v(\mathcal{M})$ *for all divisorial valuations v of X. We say that* $(\mathcal{M}, \nabla)$ *has* rational exponents *if* $\mathbb{Z}\mathrm{Exp}(\nabla)$ *is finite.*

Let define also $\mathbb{Q}\mathrm{Exp}(\mathcal{M})$ *as the sub-$\mathbb{Q}$-vector space of $k/\mathbb{Q}$ generated by (the image of)* $\mathrm{Exp}(\mathcal{M})$.

These definitions are birational.

Theorem 13.2.3. *One has* $\mathbb{Q}\mathrm{Exp}(\mathcal{M}) = \sum \mathbb{Q}\mathrm{Exp}_{D_i}(\mathcal{M})$ *for any* normal *compactification $\overline{X}$, where the D_i's are the components $\overline{X} \setminus X$ of codimension 1 in $\overline{X}$.*

Proof. As in 13.1.5, Theorem 12.3.1 and Proposition 12.3.3 allow us to pass from a normal compactification to another using specialization along enough curves. $\square$

Chapter V

Irregularity: formal theory

Introduction

With the example of Kummer hypergeometric equation at hand, we first recall some typical issues due to an irregular singularity: presence of divergent series, tension between the topological viewpoint (monodromy) and the formal algebraic viewpoint and its resolution in terms of asymptotic expansions in sectors and the Stokes phenomenon.

We then turn and settle to the algebraic viewpoint which is prevalent in the book. The central theme is the structure of formal differential modules in one variable: the slope filtration (and its associated Newton polygon) and the more refined Turrittin-Levelt decomposition.

We start by defining the maximal slope (the Poincaré rank) in terms of spectral norms of derivations, and study in detail the case of a cyclic differential module. We establish the Turrittin-Levelt decomposition and study a number of variants: notion of turning point in the presence of parameters, existence of a similar decomposition at a crossing point of the polar divisor.

14 Confluent hypergeometric equations and phenomena related to irregularity

14.1 Solutions of the confluent hypergeometric equation

Let us consider the confluent hypergeometric differential equation (1.2.1), for $c \notin \mathbb{Z}$, namely

(14.1.1) $$L_{a,c}y = x\,\partial_x^2\, y + (c - x)\,\partial_x\, y - a\,y = 0.$$

By an easy computation, the Fuchs number at ∞ (see 6.3.3) is $i_\infty(L_{a,c}) = 1$, so that the differential equation is irregular at ∞.

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*, Progress in Mathematics 189, https://doi.org/10.1007/978-3-030-39719-7_5

It is easily seen that a basis of solutions at any point of the complex plane is represented by the two converging expressions

$$ {}_1F_1(a,c;x) = \sum_0^\infty \frac{(a)_n}{(c)_n n!} x^n \quad \text{and} \quad x^{1-c}{}_1F_1(a+1-c, 2-c; x). $$

It will be useful in the sequel to observe that *Kummer's transformation* $y(x) \mapsto e^x y(-x)$ transforms $L_{a,c}$ in $L_{c-a,c}$, so that it provides alternative expressions of the same solutions, namely

$$ \begin{aligned} {}_1F_1(a,c;x) &= e^x {}_1F_1(c-a, c; -x), \\ x^{1-c}{}_1F_1(a+1-c, 2-c; x) &= x^{1-c} e^x {}_1F_1(1-a, 2-c; -x). \end{aligned} $$

At ∞, a formal solution is given by

$$ x^{-a}{}_2F_0(a, a-c+1; x^{-1}), \tag{14.1.2} $$

where ${}_2F_0(a, a-c+1; x^{-1})$ is the divergent series

$$ \sum_{n=0}^\infty \frac{(a)_n (a-c+1)_n}{n!} \frac{(-1)^n}{x^n}. $$

To determine a second (formal) solution, we twist $xL_{a,c} = \vartheta_x^2 + (c-1-x)\vartheta_x - ax$, first by x^{-a} and then by e^x. We use the fact that $x^a \circ \vartheta_x \circ x^{-a} = \vartheta_x - a$ and $e^{-x} \circ \vartheta_x \circ e^x = \vartheta_x - x$ to find $x^a e^{-x} \circ xL_{a,c} \circ x^{-a} e^x = \vartheta_x^2 + (c-1-2a+x)\vartheta_x + (c-2a)x + a(a-c+1)$. We then determine the unique solution of this equation in $\mathbb{C}[[\frac{1}{x}]]$ and find a second solution at infinity of (14.1.1):

$$ x^{-a} e^x E(a, a-c+1; x^{-1}), \tag{14.1.3} $$

where $E(a, a-c+1; x^{-1})$ is the divergent series

$$ {}_3F_1(a, a-c+1, 1; 2a-c+1; x^{-1}) = \sum_{n=0}^\infty \frac{(a)_n (a-c+1)_n}{(2a-c+1)_n} x^{-n}. $$

This provides a full set of *formal solutions* of (14.1.1) (in a differential field extension of $\mathbb{C}((x^{-1}))$ containing x^{-a} and e^x), namely

$$ \begin{aligned} \hat{u}(a,c;x^{-1}) &:= x^{-a}{}_2F_0(a, a-c+1; -x^{-1}), \\ \hat{v}(a,c;x^{-1}) &:= x^{-a} e^x E(a, a-c+1; x^{-1}). \end{aligned} \tag{14.1.4} $$

Since this is a purely formal decomposition, it is not a priori clear that this might be of any use in the understanding of the complex-analytic theory of the equation (14.1.1).

14.2 Meromorphic coefficients and Stokes multipliers

Given a differential module M over $\mathbb{C}(\{x\})$, we may view it as a differential module with analytic coefficients over a small disk punctured at 0; the classification is then given by the monodromy (cf. 6.2.9).

On the other hand, we may look at its formal completion $\widehat{M}$ over $\mathbb{C}((x))$. In this chapter, we will study the classification of irregular differential modules over $\mathbb{C}((x))$.

In the case where 0 is a regular singularity, we have seen the formal classification in Chapter III and how it fits with the analytic viewpoint. In general, as we saw in the confluent hypergeometric case, divergent series occur and the relation to the analytic theory is much more delicate; it involves asymptotic expansions in suitable sectors, Gevrey series, and the Stokes phenomenon. This goes beyond the scope of this book (cf. [88]).

15 Poincaré rank

15.1 Spectral norms

15.1.1. Let $(K, |\text{-}|)$ be a valued field of characteristic 0, complete with respect to a *non archimedean* absolute value $|\text{-}| : K \to \mathbb{R}_{\geqslant 0}$. This means that the ultrametric inequality

$$|x + y| \leqslant \max\{|x|, |y|\} \ \forall x, y \in K,$$

holds. If the image of $|\text{-}|$ is $\{0, 1\}$, we say that $|\text{-}|$ is the *trivial absolute value*. We do not exclude this case. We define $v(\text{-}) := -\log|\text{-}| : K \to \mathbb{R} \cup \{\infty\}$ as the corresponding valuation. In this situation K is called a *non-archimedean field*.

We will set $R_v := \{f \in K \mid |f| \leqslant 1\}$ = the local ring of v-integers of K, $\mathfrak{m}_v := \{f \in K \mid |f| < 1\}$ = the maximal ideal of R_v, $\kappa(v) := R_v/\mathfrak{m}_v$ = the residue field of K at v.

Example 15.1.2 (**x-adic valuation).** Let K be a field of characteristic 0 and $F = K((x))$, endowed with the x-adic valuation $v(\text{-}) = \mathrm{ord}_x(\text{-})$. So, $|x| = a^{-1} < 1$, and for any $f \in F$, we have $|f(x)| = a^{-\mathrm{ord}_x f}$, where $a > 1$ is chosen arbitrarily. Then F is a non-archimedean field and $R_v = K[[x]]$, $\kappa(v) = K$. The closed subfield K of F is trivially valued. The field $(K((x)), |\text{-}|)$ is the basic object of this book.

15.1.3. Let K be a non-archimedean field. A non-archimedean K-Banach space is a K-vector space M endowed with a *Banach norm* (or K-*Banach norm*, for precision), that is, a map $|\text{-}|_M : M \to \mathbb{R}_{\geqslant 0}$ satisfying

(1) $|m|_M = 0$ if and only if $m = 0$;

(2) $|am|_M = |a||m|_M$, for $a \in K$, $m \in M$;

(3) $|m + n|_M \leqslant \max\{|m|_M, |n|_M\}$;

(4) M equipped with the distance $d_M(m,n) = |m-n|_M$ is a complete metric space.

15.1.4. A subset S of a non-archimedean K-Banach space $(M, |\text{-}|_M)$ is *orthonormal* if for any finite subset $\{v_1, \dots, v_n\} \subset S$, and coefficients $c_1, \dots, c_n \in K$,

$$\Big|\sum\nolimits_i c_i v_i\Big|_M = \max_i |c_i|.$$

Two K-Banach norms $|\text{-}|_1$ and $|\text{-}|_2$ on a K-vector space M are *equivalent* if there exist constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$|\text{-}|_1 \leqslant c_2|\text{-}|_2 \leqslant c_1|\text{-}|_1.$$

Any two K-Banach norms on a finite-dimensional K-vector space M are equivalent. In particular, on any finite-dimensional K-Banach space $(M, |\text{-}|_M)$ there exists an equivalent norm which admits an orthonormal basis.

Remark 15.1.5. It will be sometimes more convenient to use the *Banach valuation*

$$v_M(\text{-}) := -\log|\text{-}|_M$$

on M.

15.1.6 (Operator norm). Let K be a non-archimedean field. For two K-Banach spaces $(M, |\text{-}|_M)$, $(N, |\text{-}|_N)$ we define $\mathcal{L}_K(M,N)$ as the K-Banach space of bounded K-linear maps, equipped with the *operator norm* $|\text{-}|_{M,N}$: for $\varphi \in \mathcal{L}_K(M,N)$,

$$\begin{aligned} |\varphi|_{\mathcal{L}_K(M,N)} &= \inf\{C > 0 \mid |\varphi m|_N \leqslant C|m|_M \ \forall m \in M\} \\ &= \sup\Big\{\frac{|\varphi m|_N}{|m|_M} \mid m \in M,\ m \neq 0\Big\}. \end{aligned} \tag{15.1.7}$$

For a further K-Banach space $(P, |\text{-}|_P)$ and $\varphi \in \mathcal{L}_K(M,N)$ and $\psi \in \mathcal{L}_K(N,P)$, we have

$$|\psi \circ \varphi|_{\mathcal{L}_K(M,P)} \leqslant |\psi|_{\mathcal{L}_K(N,P)}|\varphi|_{\mathcal{L}_K(M,N)}. \tag{15.1.8}$$

For $M = N$, we simply write $(\mathcal{L}_K(M), |\text{-}|_M)$ for this K-Banach space. It is in fact a *K-Banach algebra*, since for $\varphi, \psi \in \mathcal{L}_K(M)$, $|\psi \circ \varphi|_M \leqslant |\psi|_M|\varphi|_M$.

15.1.9 (Tensor product). For two K-Banach spaces $(M, |\text{-}|_M)$, $(N, |\text{-}|_N)$ we define the topological tensor product $(M\widehat{\otimes}_K N, |\text{-}|_{M\widehat{\otimes}_K N})$ of $(M, |\text{-}|_M)$ and $(N, |\text{-}|_N)$ as the completion of $(M \otimes_K N, |\text{-}|_{M\otimes_K N})$, where

$$|\ell|_{M\otimes_K N} = \inf\{\max_i |m_i|_M |n_i|_N\},$$

over all representations of $\ell \in M \otimes_K N$ as $\ell = \sum_i m_i \otimes n_i$.

15.1.10. Let F/K be an extension of valued fields with F non-trivially valued, and let $\mathcal{L}_K(F)$ be the K-Banach algebra of bounded K-linear endomorphisms of F. If M and N are finite-dimensional F-vector spaces endowed with F-Banach ultranorms $|\text{-}|_M$ and $|\text{-}|_N$, then $(M, |\text{-}|_M)$ and $(N, |\text{-}|_N)$ are at the same time K-Banach spaces. We have a canonical continuous injection

$$\mathrm{Hom}_F(M,N) = \mathcal{L}_F(M,N) \hookrightarrow \mathcal{L}_K(M,N)$$

and, for any $L \in \mathcal{L}_F(M,N)$, $|L|_{\mathcal{L}_F(M,N)} = |L|_{\mathcal{L}_K(M,N)}$. Let $m_1, \dots, m_r$ (resp. $n_1, \dots, n_s$) be an orthonormal F-basis of M (resp. N). Then any $\varphi \in \mathcal{L}_K(M,N)$ can be uniquely expressed as

$$\varphi = \sum_{i,j} \varphi_{i,j}(m_i^\vee \widehat{\otimes}_K n_j)$$

with $\varphi_{i,j} \in \mathcal{L}_K(F)$. We have $|\varphi|_{\mathcal{L}_K(M,N)} = \max_{i,j} |\varphi_{i,j}|_{\mathcal{L}_K(F)}$.

Proposition 15.1.11. *Let $(M, |\text{-}|_M)$, $(N, |\text{-}|_N)$ be finite-dimensional F-Banach spaces which both admit F-orthonormal bases. Then, for any $m \in M$ and $n \in N$,*

$$|m \otimes_F n|_{M \otimes_F N} = |m \widehat{\otimes}_K n|_{M \widehat{\otimes}_K N} = |m|_M |n|_N.$$

Proof. Notation as above. Then there exist unique coefficients $a_1, \dots, a_r$ and $b_1, \dots, b_s$ in F such that $m = \sum_i a_i m_i$, $n = \sum_j b_j n_j$. Then $\{m_i \otimes_F n_j\}_{i,j}$ is an F-orthonormal basis of $M \otimes_F N$, and $|m \otimes_F n|_{M \otimes_F N} = \max_{i,j} |a_i||b_j| = \max_i |a_i| \max_j |b_j| = |m|_M |n|_N$. □

Remark 15.1.12. The second equality in the previous statement holds for any K-Banach spaces if K is non-trivially valued, but its proof is somewhat subtle. We refer the interested reader to [94, Prop. 17.4].

If M and N are finite-dimensional F-vector spaces endowed with F-Banach norms $|\text{-}|_M$ and $|\text{-}|_N$, the F-vector space $M \otimes_F N$ is already complete for the F-norm $|\text{-}|_{M \otimes_F N}$, and we have a canonical bounded projection

$$p_{M,N} : M \widehat{\otimes}_K N \longrightarrow M \otimes_F N\ .$$

In the previous situation, we have

Proposition 15.1.13. *Assumptions as in Proposition 15.1.11. Then, for any $\ell \in M \otimes_F N$,*

$$|\ell|_{M \otimes_F N} = \inf_{\ell' \mapsto \ell} |\ell'|_{M \widehat{\otimes}_K N} = \min_{\ell' \mapsto \ell} |\ell'|_{M \widehat{\otimes}_K N}.$$

Proof. Notation as in the proof of Proposition 15.1.11. We have uniquely $\ell = \sum_{i,j} a_{i,j} m_i \otimes_F n_j$ with $a_{i,j} \in F$, and

$$|\ell|_{M \otimes_F N} = \max_{i,j} |a_{i,j}|.$$

Suppose now $\ell' = \sum_\alpha x_\alpha \widehat{\otimes}_K y_\alpha \mapsto \ell$ with $x_\alpha \in M$ and $y_\alpha \in N$. We find uniquely determined coefficients $a_{\alpha,i}, b_{\alpha,j} \in F$ such that

$$x_\alpha = \sum_{i=1}^{r} a_{\alpha,i} m_i, \quad y_\alpha = \sum_{j=1}^{s} b_{\alpha,j} n_j.$$

Then

$$\ell' = \sum_{i,j} \Big(\sum_\alpha a_{\alpha,i} \widehat{\otimes}_K b_{\alpha,j} \Big) (m_i \widehat{\otimes}_K n_j) \longmapsto \ell = \sum_{i,j} a_{i,j} m_i \otimes_F n_j,$$

so that for the product map $\mu_F : F \widehat{\otimes}_K F \to F$,

$$\mu_F \Big(\sum_\alpha a_{\alpha,i} \widehat{\otimes}_K b_{\alpha,j} \Big) = \sum_\alpha a_{\alpha,i} b_{\alpha,j} = a_{i,j}.$$

The statement is then reduced to the case of $M = N = F$. But then for any convergent expression in F

$$a = \sum_\alpha a_\alpha b_\alpha$$

we have

$$|a| = \Big| \sum_\alpha a_\alpha b_\alpha \Big| \leqslant \sup_\alpha |a_\alpha| |b_\alpha|.$$

The result follows. □

Definition 15.1.14 (Spectral norms and valuations). *For any K-Banach algebra $(\mathcal{A}, ||\text{-}||)$ one defines the* spectral norm *of $f \in \mathcal{A}$ as the number*

$$|f|_{\mathrm{sp}} = \lim_{n \to \infty} ||f^n||^{1/n} = \inf_{n \geqslant 1} ||f^n||^{1/n}. \tag{15.1.15}$$

We define also the spectral valuation *of $f \in \mathcal{A}$ as*

$$v_{\mathrm{sp}}(f) = -\log |f|_{\mathrm{sp}} = \lim_{n \to \infty} \frac{1}{n} v(f^n) = \sup_{n \geqslant 1} \frac{1}{n} v(f^n),$$

for $v(\text{-}) = -\log ||\text{-}||$.

The existence of the limit and its coincidence with the inf are standard consequences of Fekete's lemma: for any integers $n \geqslant 0$ and $m > 0$ we can take the euclidean division $n = q(n)m + r(n)$ with $0 \leqslant r(n) < m$ so that $c||f^n||^{1/n} \leqslant ||f^m||^{q(n)/n} ||f^{r(n)}||^{1/n} = (||f^m||^{q(n)m/n})^{1/m} ||f^{r(n)}||^{1/n}$. Letting n go to ∞, we have $\limsup_n ||f^n||^{1/n} \leqslant ||f^m||^{1/m}$. Therefore,

$$\limsup_{n \to \infty} ||f^n||^{1/n} \leqslant \inf_m ||f^m||^{1/m} \leqslant \liminf_{n \to \infty} ||f^n||^{1/n},$$

from which the existence of the limit and the equality with the inf follow at once.

The choice of an equivalent norm on $\mathcal{A}$ does not affect $||\text{-}||_{\mathrm{sp}}$.

Remark 15.1.16. For any $f \in \mathcal{A}$ we have $|f|_{\text{sp}} \leqslant ||f||$ or, equivalently, $v_{\text{sp}}(f) \geqslant v(f)$.

15.1.17. Notice that in general the "spectral norm" is not an algebra norm, nor even a semi-norm on the Banach space $\mathcal{A}$. However, it defines an algebra semi-norm on any commutative sub-algebra of $\mathcal{A}$, in the following sense.

15.1.18. For any non-archimedean K-Banach algebra $\mathcal{A}$, and any commuting elements $f, g \in \mathcal{A}$, we have the following properties:

(1) $|f+g|_{\text{sp}} \leqslant \max\{|f|_{\text{sp}}, |g|_{\text{sp}}\}$, with $=$ holding if $|f|_{\text{sp}} \neq |g|_{\text{sp}}$,

(2) $|fg|_{\text{sp}} \leqslant |f|_{\text{sp}}|g|_{\text{sp}}$,

(3) $|f^n|_{\text{sp}} = |f|^n_{\text{sp}}$.

See for instance [90, pp. 222–223].

Lemma 15.1.19. *Let $\mathcal{J}$ be a closed square-zero bilateral ideal of the non-archimedean K-Banach algebra $\mathcal{A}$. Let $(\overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}, |\text{-}|_{\overline{\mathcal{A}}})$ be the quotient K-Banach algebra, and let $f \mapsto \overline{f}$ be the canonical projection. For any $f \in \mathcal{A}$ we have*

$$|f|_{A,\text{sp}} = |\overline{f}|_{\overline{\mathcal{A}},\text{sp}}.$$

Proof. Let us first show that $|f|_{A,\text{sp}}$ depends only on $\overline{f}$. By symmetry, it suffices to show that for any $j \in \mathcal{J}$, $|f+j|_{A,\text{sp}} \leqslant |f|_{A,\text{sp}}$. Because $\mathcal{J}^2 = 0$, one has $(f+j)^n = f^n + \sum_{k=0}^{n-1} f^k j f^{n-1-k}$. In particular

$$|(f+j)^n|_A \leqslant \max\big\{|f^n|_A, \max_{k<n} |f^k|_A |f^{n-1-k}|_A |j|_A\big\},$$

which gives

$$\begin{aligned}|f+j|_{A,\text{sp}} = \lim |(f+j)^n|_A^{1/n} &\leqslant \max\Big\{\lim_n |f^n|_A^{1/n},\ \lim_n \max_{k<n} (|f^k|_A |f^{n-1-k}|_A)^{1/n}\Big\} \\ &= \max\Big\{|f|_{A,\text{sp}},\ \lim_m \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}}\Big\}.\end{aligned}$$

But

$$\begin{aligned}&\lim_m \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}} = \sup_\ell \lim_{m\geqslant \ell} \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}} \\ &= \max\Big\{\sup_\ell \lim_{m\geqslant\ell} \max_{\ell\leqslant k\leqslant m/2} \big(|f^k|_A^{\frac{1}{k}}\big)^{\frac{k}{m}} \cdot \big(|f^{m-k}|_A^{\frac{1}{m-k}}\big)^{1-\frac{k}{m}}, \\ &\qquad \sup_\ell \lim_{m\geqslant\ell} \max_{k<\ell} \big(|f^k|_A^{\frac{1}{k}}\big)^{\frac{k}{m}} \cdot \big(|f^{m-k}|_A^{\frac{1}{m-k}}\big)^{1-\frac{k}{m}}\Big\},\end{aligned}$$

and both terms are bounded by $|f|_{A,\text{sp}}$, as wanted. Now

$$|\overline{f}|_{\overline{\mathcal{A}},\text{sp}} = \inf_n \inf_{j\in\mathcal{J}} |(f+j)^n|_A^{1/n} = \inf_{j\in\mathcal{J}} \inf_n |(f+j)^n|_A^{1/n} = \inf_{j\in\mathcal{J}} |f+j|_{A,\text{sp}} = |f|_{A,\text{sp}}.$$

$\square$

Lemma 15.1.20. *Let M, N be finite-dimensional F-vector spaces and $|\text{-}|_M$, $|\text{-}|_N$ be F-Banach norms on them. For any $D \in \mathcal{L}_K(M)$ and $D' \in \mathcal{L}_K(N)$, we have*

$$|D \widehat{\otimes}_K D'|_{M\widehat{\otimes}_K N,\mathrm{sp}} = |D|_{M,\mathrm{sp}} |D'|_{N,\mathrm{sp}}$$

and

$$|D \widehat{\otimes}_K 1_N + 1_M \widehat{\otimes}_K D'|_{M\widehat{\otimes}_K N,\mathrm{sp}} \leqslant \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

Moreover, let $\mathcal{L}_K(D, D') \in \mathcal{L}_K(\mathcal{L}_K(M,N))$ be $D' \circ - \circ D$; then one has

$$|\mathcal{L}_K(D, D')|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D|_{M,\mathrm{sp}} |D'|_{N,\mathrm{sp}}$$

and

$$|\mathcal{L}_K(1_M, D') - \mathcal{L}_K(D, 1_N)|_{\mathcal{L}_K(M,N),\mathrm{sp}} \leqslant \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

If $|D|_{M,\mathrm{sp}} \neq |D'|_{N,\mathrm{sp}}$, equality holds in the previous formulas.

Proof. Under the assumptions of Proposition 15.1.11 the stronger formula

$$|D \widehat{\otimes}_K D'|_{M\widehat{\otimes}_K N} = |D|_M |D'|_N$$

holds. Indeed, by 15.1.10 and 15.1.11, the tensor product

$$(\mathcal{L}_K(M), |\text{-}|_M) \widehat{\otimes}_K (\mathcal{L}_K(N), |\text{-}|_N)$$

embeds isometrically as a closed subspace of $(\mathcal{L}_K(M \widehat{\otimes}_K N, |\text{-}|_{M\widehat{\otimes}_K N})$, so that the formula follows from Proposition 15.1.11 applied to the former tensor product. This proves the first part of the statement.

We next note that the operators $D \widehat{\otimes}_K 1_N$ and $1_M \widehat{\otimes}_K D'$ on $M \widehat{\otimes}_K N$ such that $D \widehat{\otimes}_K 1_N \circ 1_M \widehat{\otimes}_K D' = D \widehat{\otimes}_K D'$ (resp. $\mathcal{L}_K(1_M, D')$ and $\mathcal{L}_K(D, 1_N)$ on $\mathcal{L}_K(M, N)$, such that $\mathcal{L}_K(1_M, D') \circ \mathcal{L}_K(D, 1_N) = \mathcal{L}_K(D, D')$) commute. On the other hand, $|D \widehat{\otimes}_K 1_N|_{M\widehat{\otimes}_K N,\mathrm{sp}} = |D|_{M,\mathrm{sp}}$, and $|1_M \widehat{\otimes}_K D'|_{M\widehat{\otimes}_K N,\mathrm{sp}} = |D'|_{N,\mathrm{sp}}$ (resp. $|\mathcal{L}_K(D, 1_N)|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D|_{M,\mathrm{sp}}$, and $|\mathcal{L}_K(1_M, D')|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D'|_{N,\mathrm{sp}}$). So, the second and fourth assertions hold.

It remains to prove the third formula. Again, under the assumptions of Proposition 15.1.11 the stronger formula

$$|\mathcal{L}_K(D, D')|_{\mathcal{L}_K(M,N)} = |D|_M |D'|_N$$

holds. In fact, let us write

$$D = \sum_{i,j=1}^{r} D_{i,j}(m_i^\vee \widehat{\otimes}_K m_j) \quad \text{and} \quad D' = \sum_{h,k=1}^{s} D'_{h,k}(n_h^\vee \widehat{\otimes}_K n_k), \ \ D_{i,j}, D'_{h,k} \in \mathcal{L}_K(F),$$

so that $|D|_M = \max_{i,j} |D_{i,j}|_{\mathcal{L}_K(F)}$ and $|D'|_N = \max_{h,k} |D'_{h,k}|_{\mathcal{L}_K(F)}$. On the other hand, a simple computation shows that

$$\mathcal{L}_K(D, D')(m_u^\vee \widehat{\otimes}_K n_v) = \sum_{u,v} D_{i,u} D'_{v,k} m_i^\vee \widehat{\otimes}_K n_k,$$

so that the formula follows.

The last assertion in the statement follows from the properties of the spectral norm recalled above. □

Lemma 15.1.21. *Let*

$$(P, |\text{-}|_P) = (M, |\text{-}|_M) \oplus (N, |\text{-}|_N) \tag{15.1.22}$$

be an orthogonal direct sum of K-Banach spaces and let $F \in \mathcal{L}_K(P)$ be an operator such that $F(M) \subset M$. Let $D = F_{|M} \in \mathcal{L}_K(M)$ and $D' \in \mathcal{L}_K(N)$ be the operators induced by F. Then

$$|F|_{P,\mathrm{sp}} = \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

Proof. Consider the operator $L' = D \oplus D' \in \mathcal{L}_K(P)$. Obviously, $|L'|_{P,\mathrm{sp}} = \max\{|D'|_{N,\mathrm{sp}}, |D|_{M,\mathrm{sp}}\}$. On the other hand, $L = L' + H$, where H is an operator that kills M and sends P to M. Such operators form a bilateral ideal $\mathcal{J}$, with $\mathcal{J}^2 = 0$, in the sub-K-Banach algebra $\mathcal{A}$ of $\mathcal{L}_K(P)$ of operators preserving M. We may then apply Lemma 15.1.19 to $(\mathcal{A}, \mathcal{J})$ and to $L, L' \in \mathcal{A}$, which reduce modulo $\mathcal{J}$ to the same element $\overline{L} = \overline{L'} \in \overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}$. We deduce that $|L|_{P,\mathrm{sp}} = |L'|_{P,\mathrm{sp}}$. □

15.1.23. Let E be a subfield of F: it inherits an absolute value induced by $|\text{-}|_F$. If the ramification index $e = (|F^*| : |E^*|)$ is finite (which in particular is the case if F is a finite extension of E), it will often be convenient to renormalize the absolute value on E by setting $|\text{-}|_E = |\text{-}|_F^{1/e}$. We write w for the associated valuation on E, and write $e = e(v/w)$.

Lemma 15.1.24. *Assume that $|\text{-}|$ is trivial on K. For any $L \in \mathcal{L}_K(M) = \mathcal{L}_K({}_EM)$, $|L|_{M,\mathrm{sp}} = |L|^{e(v/w)}_{{}_EM,\mathrm{sp}}$.*

Proof. Indeed, ${}_EM$ is the same K-space as M, endowed with the norm $|\text{-}|^{1/e}$. □

15.2 Christol-Dwork-Katz theorem

15.2.1. Let F/K be as in 15.1.10 with K a field of characteristic 0; in particular, the absolute value of F is non-trivial. Let ∂ be a bounded K-linear derivation of F and let (M, ∇_∂) be a differential module over (F, ∂) in the sense of 2.4. So, ∇_∂ is a K-linear endomorphism of M which satisfies the Leibniz rule w.r.t. the elements of F and *is bounded with respect to any F-Banach norm on M*. Since M is of finite dimension over F, all these norms are equivalent, so that the condition does not depend on the norm.

The operator norm of ∇_∂ depends on the norm $|\text{-}|_M$ we choose on M, but the spectral norm of ∇_∂ defined by (15.1.15) does not.

We now present the computation of the spectral norm of ∇_∂ in terms of its action on a cyclic vector of (M, ∇_∂).

Theorem 15.2.2 (Christol-Dwork-Katz). *Let (M, ∇_∂) be a differential module over the differential field (F, ∂), as described above, and let $m \in M$ be a cyclic vector. Let us write*

$$\nabla_\partial^\mu(m) = \sum_{i=0}^{\mu-1} a_i \nabla_\partial^i(m), \quad a_i \in F,$$

so that, in terms of the basis $\mathbf{m} = (m, \nabla_\partial(m), \ldots, \nabla_\partial^{\mu-1}(m))$,

$$\nabla_\partial \mathbf{m} = \mathbf{m} \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{\mu-2} \\ 0 & 0 & \cdots & 1 & a_{\mu-1} \end{pmatrix}.$$

For $n = 0, 1, 2, \ldots$, let us write $\nabla_\partial^n \mathbf{m} = \mathbf{m} H_n$, with $H_n \in M_\mu(F)$.

(1) *For any $\sigma \geqslant |\partial|$, the following conditions are equivalent:*

  (i) $|\nabla_\partial|_{\mathrm{sp}} \leqslant \sigma$*;*

  (ii) $|a_j| \leqslant \sigma^{\mu-j}$ *for any $j = 0, 1, \ldots, \mu-1$,*

  (iii) $|H_n| \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$ *for any $n \in \mathbb{N}$.*

(2) *Suppose that there exists ξ in F with $|\xi| = \min\{1, |a_j|^{-\frac{1}{\mu-j}}\}$, and let L the matrix of $\xi\nabla_\partial$ in the basis $\mathbf{n} = \mathbf{m}\Xi$, where Ξ is the diagonal matrix with diagonal entries $1, \xi, \xi^2, \ldots, \xi^{\mu-1}$. Then $|L| \leqslant 1$, and if $|\nabla_\partial|_{\mathrm{sp}} > 1$, the reduction $L(0)$ is not nilpotent.*

Proof. We follow the proof of [30, 1.5].
(1) (iii)⇒(i) is clear: $|\nabla_\partial^n| \leqslant \max_{i+j=n}(|\partial^i|\,|H_j|) \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$.

(ii)⇒(iii) is seen by induction on n: define $a_{j,n} \in F$, $n \in \mathbb{N}$ by $\nabla_\partial^n(m) = \sum_{j=0}^{\mu-1} a_{j,n} \nabla_\partial^j(m)$. one has the recursion $a_{j,n} = \partial(a_{j,n-1}) + a_{j-1,n-1} + a_{\mu-1,n-1} a_j$. Since $|\partial(a)| \leqslant \sigma|a|$, using *(ii)* we get by induction on n that $|a_{j,n}| \leqslant \sigma^{n-j}$, so that $|H_n| \leqslant \max_{i,j \leqslant \mu-1} \sigma^{n+i-j} \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$.

(i)⇒(ii) (by contraposition): suppose (ii) does not hold, so that there exists ξ in a finite separable extension of F with $|\xi|^{-1} = \max_j |a_j|^{1/\mu-j} > \sigma$. This implies

$$|a_j \xi^{\mu-j}| \leqslant 1, \quad |\xi^{-1}\partial\xi| < 1.$$

It suffices to show that $|\nabla_\partial|_{\mathrm{sp}} \geqslant |\xi|^{-1}$.

Consider the base change $\mathbf{n} = \mathbf{m}\Xi$ where Ξ is the diagonal matrix with entries $1, \ldots, \xi^{\mu-1}$. Then the matrix H of ∇_∂ is changed to $H' = \Xi^{-1} H \Xi + \Xi^{-1}\partial(\Xi)$, explicitly $H' = \xi^{-1}(A+B)$, where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0\xi^{\mu} \\ 1 & 0 & \cdots & 0 & a_1\xi^{\mu-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & a_{\mu-2}\xi^2 \\ 0 & 0 & \cdots & 1 & a_{\mu-1}\xi \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & & & & \\ & \partial(\xi) & & & \\ & & 2\partial(\xi) & & \\ & & & \ddots & \\ & & & & (\mu-1)\partial(\xi) \end{pmatrix}.$$

We define H'_n by $\nabla_\partial^n \mathbf{n} = \mathbf{n} H'_n$. Let us show that

$$|H'^n| = |H'|^n = |\xi|^{-n} = |H'_n|.$$

In fact, the first two equalities follow since the characteristic polynomial $t^\mu - \sum_i a_i t^i$ of ξA (equal to that of H) has the property that $|a_j| \leqslant |\xi|^{j-n}$ (with at least one equality), and the characteristic polynomial $t^\mu - \sum_i b_i t^i$ of H' has the same property. In particular, its Newton polygon has a side of slope $v(\xi)$, and H' has an eigenvalue (in some algebraic extension of F) of valuation $|\xi|^{-1}$. We conclude that $|\xi|^{-n} \leqslant |H'^n| \leqslant |H'|^n = |\xi|^{-n}$. The last equality follows by induction on n using the relations

$$H_n - H^n = \partial(H_{n-1}) + H(H_{n-1} - H^{n-1}).$$

From these equalities, using the inequalities $|H'_n| \leqslant |\nabla_\partial^n| \leqslant \max\{|\partial|, |H'|\}^n$, we deduce that $|\nabla_\partial^n| = |\xi|^{-n}$, and then $|\nabla_\partial|_{\text{sp}} = |\xi|^{-1} > \sigma$, contradicting (i). This concludes the proof of the first part of the theorem.

(2) With the above notation, $L = \xi H' = A + B$, with $|A| = 1$ and $|\nabla_\partial|_{\text{sp}} = |\xi|^{-1}$. Therefore $|\nabla_\partial|_{\text{sp}} > 1$ implies $|\xi| < 1$ and $|B| < 1$. The reduction modulo x of the characteristic polynomial of L coincides with the reduction modulo x of the characteristic polynomial of A, which is different from t^μ because one of the coefficients has norm 1. □

Remark 15.2.3. Notice that this gives a new proof of some of results seen in Chapter III, in the regular case (cf. 7.5.1 and 8.3.3). The theorem (with the same proof) also holds for any valued field of characteristic 0 as in 15.1, see [6].

15.3 Poincaré rank

We now specialize the definitions and results presented in this section to the case of a trivially valued field K of characteristic 0, and of $F = K((x))$, as in example 15.1.2. We still allow some freedom in the choice of the F/K-derivation ∂.

Definition 15.3.1 (Poincaré rank). *Given a differential module* (M, ∇_∂) *over* $(K((x)), \partial)$ *as above* (15.2.1)*, we define the* Poincaré rank (*a.k.a.* Poincaré-Katz rank*, a.k.a.* rank of irregularity) *of* (M, ∇_∂) *as*

$$\rho_v(M, \nabla_\partial) = \max\{0, \log|\nabla_\partial|_{\text{sp}} - \log|\partial|_{\text{sp}}\}.$$

Remark 15.3.2. Lemma 6.2.4 of [67] asserts that

$$|\nabla_\partial|_{\mathrm{sp}} \geqslant |\partial|_{\mathrm{sp}},$$

so that in fact

$$\rho_v(M, \nabla_\partial) = \log|\nabla_\partial|_{\mathrm{sp}} - \log|\partial|_{\mathrm{sp}} \geqslant 0.$$

Lemma 15.3.3. *Let M, M_1, M_2 be F/K-differential modules. Then*

(i) $\rho_v(M^\vee) = \rho_v(M)$*;*

(ii) $\rho_v(M_1 \otimes_F M_2) \leqslant \max\{\rho_v(M_1), \rho_v(M_2)\}$*;*

(iii) $\rho_v(\mathrm{Hom}_F(M_1, M_2)) \leqslant \max(\rho_v(M_1), \rho_v(M_2))$*;*

(iv) *if* $\rho_v(M_1) \neq \rho_v(M_2)$ *we have equality in* (ii) *and* (iii)*;*

(v) *if* $0 \to M_1 \to M \to M_2 \to 0$ *is an exact sequence of differential modules, then* $\rho_v(M) = \max\{\rho_v(M_1), \rho_v(M_2)\}$.

Proof. Parts (i) to (iv) follow from 15.1.20; (v) follows from Lemma 15.1.21. □

Lemma 15.3.4. *Let F' be a finite field extension of F (endowed with the unique extension v' of the valuation of F) with ramification index e. Assume that $|\cdot|$ is trivial on K. Then, using the normalization of* 15.1.23*, we have the following properties:*

(i) *if M is a F/K-differential module, then* $\rho_{v'}(M_{F'}) = e\rho_v(M)$*;*

(ii) *if M' is a F'/K-differential module, then* $\rho_v({}_F M') = e^{-1}\rho_{v'}(M')$.

Proof. (ii) follows from 15.1.24. For (i), note that $\rho_v(M) = \rho_v({}_F(M_{F'}))$ by 15.3.3 (iii) (with $M_2 = (F', d_{F'/K})$), so that (i) follows from (ii). □

Example 15.3.5 (Regularity). In case $\partial = \vartheta_x$ we obviously have $|\partial|_{\mathrm{sp}} = |\partial|_F = 1$. So, $\rho_v(M, \nabla_{\vartheta_x}) = \log|\nabla_{\vartheta_x}|_{\mathrm{sp}}$. Let us show that $(M, \nabla_{\vartheta_x})$ is regular if and only if $|\nabla_{\vartheta_x}|_{\mathrm{sp}} = 1$, i.e., if and only if $\rho_v(M, \nabla_{\vartheta_x}) = 0$.

In fact, from item (2) of 8.3.3 we see that the regularity condition for a differential module over $K((x))$ is equivalent to the condition that for any (or some) $K[[x]]$-lattice Λ of M the action of the operators $\nabla^n_{\vartheta_x}$ (for $n \in \mathbb{N}$) is bounded for the x-adic valuation of M defined by the lattice. More precisely, $(M, \nabla_{\vartheta_x})$ is regular if and only if for any (or some) $K[[x]]$-lattice Λ of M we have

$$\sup_n -v(\nabla^n_{\vartheta_x}(\Lambda)) \leqslant c$$

for a constant c, where v is the valuation on M defined by $v(m) = i$ if $m \in x^i\Lambda$ and $m \notin x^{i+1}\Lambda$ (two such valuations induced by different lattices differ by a constant). □

We list now some useful consequence of the Christol-Dwork-Katz theorem.

Corollary 15.3.6. *If* $|\partial|_{\mathrm{sp}} = |\partial|$, *then* $\rho_v(\nabla_\partial) = \max\left\{\log|\partial|_{\mathrm{sp}},\ \max_j \frac{-v(a_j)}{\mu-j}\right\}$. *If moreover* $|\partial| = 1$, *then* $\rho_v(\nabla_\partial) = \max\left\{0,\ \max_j \frac{-v(a_j)}{\mu-j}\right\}$ *is a rational number with denominator bounded by* μ*; in addition, the function*

$$n \longmapsto \log|\nabla_\partial^n| - n\rho_v(\nabla_\partial)$$

for $n \in \mathbb{N}$ *is non-negative and bounded.*

Proof. The first two assertions follow immediately from item (1)(i) of Theorem 15.2.2; the last one from item (1)(iii). □

Corollary 15.3.7. *We have* $\rho_v(\nabla_\partial) \leqslant \rho$ *if and only if there exists a basis of* M *such that the matrix of* ∇_∂ *in that basis has valuation* $\geqslant -\rho$.

Proof. It suffices to take the basis $\mathbf{n}$ as in Theorem 15.2.2. □

15.3.8. Let us choose $\partial = \frac{d}{dx}$ or $x\frac{d}{dx}$. In both cases $|\partial|_{\mathrm{sp}} = |\partial|$, and $|x\frac{d}{dx}| = 1$. The Poincaré rank was introduced by Poincaré himself [86, p. 305] for differential operators in $\mathbb{C}[z, \frac{d}{dz}]$, with respect to the singularity at ∞: his definition coincides with the formula in 15.3.7 for $x = 1/z$, the x-adic valuation of a polynomial in $\mathbb{C}[z]$ being identified with its degree.

16 Turrittin-Levelt decomposition and variants

In this section, we give the structure theorem for formal differential modules in one variable. Herein $F = K((x))$, with its x-adic valuation[1].

Any finite extension $K((x))$ is a complete valued field of ramification index e of the form $K'((x'))$, with $(x')^e = x$ and K' a finite extension field of K. The extension $K'((x'))/K((x))$ is Galois if and only if K'/K is a Galois extension containing the e-th roots of unity.

16.1 The Turrittin-Levelt decomposition

16.1.1. Consider now a finite extension $F' = K'((x^{1/e}))$ of F and the operator $\partial = \vartheta_x = x\partial_x$. We have that $\partial\log(F'^\times) = \frac{1}{e}\mathbb{Z} \oplus x^{1/e}K'[[x^{1/e}]]$. Notice that $\partial\log(F'^\times) \cap K'[x^{-1/e}] = \frac{1}{e}\mathbb{Z}$, and that $\partial\log(F'^\times) + K'[x^{-1/e}] = F'$, so that

$$F'/\partial\log(F'^\times) \;\cong\; K'[x^{-1/e}]/\tfrac{1}{e}\mathbb{Z} \;\cong\; (K'/\tfrac{1}{e}\mathbb{Z}) \oplus x^{-1/e}K'[x^{-1/e}].$$

As a consequence, if we choose a section τ of the canonical projection $K' \to K'/\frac{1}{e}\mathbb{Z}$, we may extend it canonically to a section τ of $F' \to F'/\partial\log(F'^\times)$.

[1] For statements over more general valued differential fields, with the same proof, see [5, 2.3].

Theorem 16.1.2 (Turrittin-Levelt-Jordan decompositions). *Let $(M, x\nabla_{\partial_x})$ be a rank μ differential module over F/K. There is a finite extension $F' = F(\phi_1, \dots, \phi_r)$ of F over which $(M_{F'}, x\nabla_{\partial_x})$ admits a Jordan decomposition of F'/K differential modules*

$$M_{F'} = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)}$$

with characters $\overline{\phi}_i \in K'[x^{-1/e}]/\frac{1}{e}\mathbb{Z}$, where $M_{\phi_i}^{(\mu)} = F' \otimes_{K'} \mathrm{Ker}_{M_{F'}}(x\nabla_{\partial_x} - \phi_i)^{\mu}$.

Bringing together the summands $M_{\phi}^{(\mu)}$ for which the characters ϕ_i's differ only by the constant terms, we get the Turrittin-Levelt decomposition *of F'/K differential modules*

$$M_{F'} = \bigoplus_{j} L_{\psi_j} \otimes_{F'} R_j,$$

where L_{ψ_j} is of F'-dimension one, $\psi_j \in x^{-1/e}K'[x^{-1/e}]$, and R_j is regular.

Remark 16.1.3. In the Turrittin-Levelt-Jordan decomposition the characters are parametrized by $\phi \in F'/\partial \log(F'^{\times})$, while the Turrittin-Levelt decomposition is indexed by $\psi \in F'/R_v$. There is a canonical projection $\pi : F'/\partial \log(F'^{\times}) \to F'/R_v$ induced by the inclusion of $\partial \log(F'^{\times})$ in R_v. Then the sum of the terms $M_{\phi}^{(\mu)}$ of the Turrittin-Levelt-Jordan decomposition of M with $\pi(\phi) = \psi$ gives the term $L_{\psi} \otimes R$ of the Turrittin-Levelt decomposition. Viceversa, a term $L_{\psi} \otimes R$ of the Turrittin-Levelt decomposition can be written according the Jordan decomposition of the regular part R (see 8.3.4), to obtain the Turrittin-Levelt-Jordan decomposition.

Proposition-Definition 16.1.4 (Turrittin index). *There is a unique minimal extension field $K'((x'))$ of $K((x))$ on which the Turrittin-Levelt-Jordan decomposition holds. It is a Galois extension and it is generated by $\phi_1, \dots, \phi_r$ over $K((x))$. The ramification index e of $K'((x'))/K((x))$ (a divisor of* l.c.m.$(2, \dots, \mu)$*) is called the* Turrittin index *of M.*

The extension K'/K is generated by the primitive e-th roots of unity and the coefficients of the terms of the ϕ_j's of degree $\leqslant 0$ in x'.

Proof. The first part of the proposition follows from the general Jordan theory (see 8.1.19). For the ramification index, notice that in the proof of the decomposition the only ramification that needs to be introduced at each step arises from the denominator of the Poincaré rank of M. □

Definition 16.1.5 (Turrittin exponents). *The constant terms (in $K'/(\frac{1}{e}\mathbb{Z})$) of the $\overline{\phi}_j$'s (viewed as polynomials in $x^{-1/e}$) are called the* Turrittin exponents *of M.*

Proposition 16.1.6. *In the situation of Theorem 16.1.2, let δ be any x-adically continuous derivation of F and ∇_{δ} an action on M commuting with $x\nabla_{\partial_x}$ and making (M, ∇_{δ}) a differential module over (F, δ). Then the Turrittin-Level-Jordan decomposition is stable under ∇_{δ}.*

In particular, the submodules of the decomposition are defined over (F, δ), and δ kills the Turrittin exponents of M.

Proof. From the general theory of Section 8.2, it follows that the Turrittin-Levelt-Jordan decomposition is stable under any ∇_δ for any derivation δ commuting with ϑ_x. $\square$

16.2 Proof of the decomposition

Theorem 16.1.2 can be proved using the nilpotent orbit method of Babbitt and Varadarajan (see [10]), and also using Hensel's lemma for differential operators (see [89]). We don't follow these paths, but derive the theorem from the Dwork-Katz-Turrittin theorem using a decomposition lemma of van den Essen and Levelt (see [40]).

Proposition 16.2.1 (Splitting lemma). *Let R be a complete noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field k (of any characteristic). Let δ be a derivation of R such that $\delta(R) \subseteq \mathfrak{m}$ and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Let E be a free R-module of finite type, endowed with an additive action ∇_δ of δ satisfying the Leibniz rule (w.r.t. δ), and let $\overline{\nabla}_\delta$ denote the induced k-linear action of δ on $\overline{E} := E \otimes_R k$. Let*

$$\overline{E} = \bigoplus \overline{E}_j$$

be a decomposition into k-spaces such that the sets of eigenvalues of $\overline{\nabla}_\delta$ (in any extension of k) on the $\overline{E}_j$'s are pairwise disjoint. Then this decomposition lifts uniquely to a decomposition

$$E = \bigoplus E_j$$

into ∇_δ-stable R-submodules. Moreover, if δ' is another derivation of R, and $\nabla_{\delta'}$ is an additive action of δ' on E satisfying the Leibniz rule (w.r.t. δ') and commuting with ∇_δ, then the decomposition is stable under $\nabla_{\delta'}$.

Proof. It suffices to treat the case of two factors.
Existence. Let $\mathbf{e} = (e_1, \ldots, e_\mu)$ be a basis of E such that the image of $(e_1, \ldots, e_\nu)$ (resp. $(e_{\nu+1}, \ldots, e_\mu)$) is a basis of $\overline{E}_1$ (resp. $\overline{E}_2$), and let us write the matrix of ∇_δ in this basis in block form $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Its reduction modulo $\mathfrak{m}$ is $\begin{pmatrix} \overline{P} & 0 \\ 0 & \overline{S} \end{pmatrix}$, where $\overline{P}$ and $\overline{S}$ have no common eigenvalues. This assumption will be used in the guise that the endomorphism

$$\overline{H} \longmapsto \overline{PH} - \overline{HS} \quad \text{of} \quad M_{\nu,\mu-\nu}(k)$$

is injective, hence surjective, which implies that, for any n, the endomorphism

$$H \longmapsto PH - HS \quad \text{of} \quad M_{\nu,\mu-\nu}(\mathfrak{m}^n)$$

is also surjective. We look for a matrix $T = \begin{pmatrix} I & X \\ Y & I \end{pmatrix}$, with $\overline{X} = 0$, $\overline{Y} = 0$, such that in the basis $\mathbf{e}T$, the matrix of ∇_δ takes the block form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. This amounts to solving the two equations

$$\begin{aligned} \delta(X) + PX - XS - XRX + Q &= 0, \\ \delta(Y) + SY - YP - YQY + R &= 0. \end{aligned}$$

Let us show that the first equation admits a solution $X \in M_{\nu,\mu-\nu}(\mathfrak{m})$ (the second equation is similar, after transposition). Since R is $\mathfrak{m}$-adically complete, one can proceed by successive approximation. Let $X \in M_{\nu,\mu-\nu}(\mathfrak{m})$ be a solution of the congruence

$$Z := \delta(X) + PX - XS - XRX + Q \equiv 0 \text{ modulo } \mathfrak{m}^n,$$

and let us look for a solution $X + H$ of the congruence modulo $\mathfrak{m}^{n+1}$, with $H \in M_{\nu,\mu-\nu}(\mathfrak{m}^n)$. Since $\delta(\mathfrak{m}^n) \subseteq \mathfrak{m}^{n+1}$, the latter congruence amounts to

$$PH - HS + Z \equiv 0 \text{ modulo } \mathfrak{m}^{n+1},$$

which is indeed solvable in H by the remark at the beginning of the proof.

Uniqueness. Let $E'_1 \oplus E'_2$ be another decomposition as in the proposition. The natural homomorphism $\phi : E_1 \to E'_2$ then commutes with ∇_δ. To show that $\phi = 0$, we proceed by induction, assuming that $\phi(E_1) \subseteq \mathfrak{m}^n E'_2$. Let

$$\overline{\phi} : \overline{E}_1 \longrightarrow \mathfrak{m}^n E'_2 \otimes_R k \cong \overline{E}_2 \otimes_R (\mathfrak{m}^n/\mathfrak{m}^{n+1})$$

be the induced homomorphism. Then $(\overline{\nabla}_{\delta|\overline{E}_2} \otimes 1) \circ \overline{\phi} = \overline{\phi} \circ \overline{\nabla}_{\delta|\overline{E}_1}$, and since $\overline{\nabla}_{\delta|\overline{E}_1}$ and $\overline{\nabla}_{\delta|\overline{E}_2}$ have no common eigenvalues, $\overline{\phi} = 0$, that is, $\phi(E_1) \subseteq \mathfrak{m}^{n+1} E'_2$. So we deduce that $\phi = 0$. Similarly, the canonical morphism $E_2 \to E'_1$ is zero, and the decompositions coincide.

Stability. Consider the morphism $\phi : E_1 \to E_2$ given by the restriction to E_1 of the composition of $\nabla_{\delta'}$ with the projection onto E_2. It is easy to see that ϕ is an R-linear morphism and commutes with ∇_δ. Then the previous argument show that $\phi = 0$, so that $\nabla_{\delta'}$ is stable on E_1, and similarly for E_2. $\square$

Notation. We let L_ϕ denote the differential module over the differential field $(K((x)), \partial)$ generated by one element ℓ subject to the action $\nabla_\partial(\ell) = \phi\ell$. For example, if $\partial = \frac{d}{dx}$ (resp. $\partial = \vartheta_x = x\frac{d}{dx}$), then $\ell = \exp(\psi)$ where ψ is a primitive of ϕ (resp. ψ is a primitive of ϕ/x). The Poincaré rank of L_ϕ is $\max\{0, -\mathrm{ord}_x(\phi)\}$ (resp. $\max\{0, 1 - \mathrm{ord}_x(\phi)\}$).

Proof of Theorem 16.1.2. The proof proceeds by induction on pairs ($\mu \in \mathbb{N}$, $\rho \in \frac{1}{\mu!}\mathbb{N}$), using the lexicographic order: $(\mu, \rho) < (\mu', \rho')$ if either $\mu < \mu'$, or $\mu = \mu'$

and $\rho < \rho'$. In this discussion $\mu = \dim_{K((x))} M$, $\rho = \rho(M)$, applied to any M as before.

The induction starts at $(\mu, 0)$, for any μ (it is the regular case), and at $(1, \rho)$, for any $\rho \in \mathbb{N}$ (it is the rank-one case).

Write ρ as an irreducible fraction l/m, $m \leqslant \mu$. Choose a cyclic basis $\mathbf{m} = (m, \dots, \vartheta_x^{\mu-1} m)$, and write

$$\vartheta_x \mathbf{m} = \mathbf{m} \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \cdots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{\mu-1} \end{pmatrix}.$$

According to Corollary 15.3.6, $\rho = \max\left\{0, \max_{j=0,\dots,\mu-1}\left(-\frac{ord_x(a_j)}{\mu-j}\right)\right\}$. As in the proof of Theorem 15.2.2 (using $\partial = \vartheta_x$ and $\xi = x^\rho$), we then modify $\mathbf{m}$, viewed as a basis of $M_{K((x'))}$, with $x' = x^{1/m}$, to $\mathbf{n} = \mathbf{m}\Xi$, where Ξ is the diagonal matrix with entries $1, x^\rho, \dots, x^{(\mu-1)\rho}$. The matrix of $\delta := x^{\rho+1}\frac{d}{dx} = x^\rho \vartheta_x = m^{-1}(x')^{m\rho+1}\frac{d}{dx'} = m^{-1}(x')^{m\rho}\vartheta_{x'}$ in this new basis is

$$B_{-\rho} = \begin{pmatrix} 0 & \cdots & 0 & x^{\mu\rho} a_0 \\ 1 & \cdots & 0 & x^{(\mu-1)\rho} a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & x^{rho} a_{\mu-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \rho x^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mu-1)\rho x^\rho \end{pmatrix} \in M_\mu(K[[x^{1/m}]]).$$

If $\rho = 0$, we are in the regular case, which is the starting point of the induction.

If $\rho > 0$, we may apply the splitting lemma (with $F' = \overline{K}[[x^{1/m}]]$).

If $B_{-\rho}(0)$ has at least two distinct eigenvalues in $\overline{K}$, this reduces the situation to the case of rank $< \mu$, and the induction assumption applies.

If $B_{-\rho}(0)$ has only one eigenvalue $\zeta \in \overline{K}$, this eigenvalue cannot be 0 (since $\rho > 0$, cf. (2) of 15.2.2), and in fact one has

$$x^{(\mu-j)\rho} a_j + \binom{\mu}{j}(-\zeta)^{\mu-j} \quad \in x^{1/m} K[\zeta][[x^{1/m}]] \quad \text{for all } j = 0, \dots, \mu-1.$$

Therefore,

$$\rho = -\frac{\mathrm{ord}_x(a_j)}{\mu - j}, \quad \text{for all } j = 0, \dots, \mu-1,$$

whence $m = 1$, and $\rho = l$ is a positive integer in this case.

Tensoring M with the rank-one $K((x))/K$-differential module $L_{-\zeta x^{-\rho}}$ (with generator $\ell = \exp(\frac{\zeta}{\rho} x^{-\rho})$ and action $\vartheta_x \ell = -\zeta x^{-\rho}\ell$), one checks that $\mathbf{m}' = (m' := m \otimes \ell, \dots, (x\frac{d}{dx})^{\mu-1} m')$ is a cyclic basis, since it may be written using a triangular invertible matrix in terms of the basis $\mathbf{m} \otimes \ell$, and write

$$\vartheta_x \mathbf{m}' = \mathbf{m}' \begin{pmatrix} 0 & \cdots & 0 & a'_0 \\ 1 & \cdots & 0 & a'_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a'_{\mu-1} \end{pmatrix}.$$

Using the basis $\mathbf{n}' = \mathbf{m}'\Xi$ with Ξ as before, we have that the matrix $B'_{-\rho}$ of the operator $\delta = x^{\rho+1}\frac{d}{dx} = x^\rho \vartheta_x$ has the property that $B'_{-\rho}(0)$ is nilponent, whence $\mathrm{ord}_x(x^{(\mu-j)\rho} a'_j) > 0$ for all $j = 0, \ldots, \mu-1$. We deduce that

$$-\frac{\mathrm{ord}_x(a'_j)}{\mu - j} < \rho \quad \text{for all } j = 0, \ldots, \mu - 1\ ,$$

hence $\rho(M \otimes L) < \rho$ and the induction assumption applies. □

17 Slopes and Newton polygons

17.1 Slope decomposition

Definition 17.1.1 (Slopes). *For any element $\overline{\phi}$ of $F/\partial \log F^\times$, we define the* slope *of $\overline{\phi}$ to be* 0 *if $\overline{\phi} = 0$, and to be $-v(\overline{\phi})$ if $\overline{\phi} \neq 0$.*

The definition is well-posed, since $v(\partial \log F^\times) = 0$.

17.1.2. In the case of $F = K((x))$, we have that the slope of ϕ is $-\mathrm{ord}_x(\phi)$. If $F' = K'((x')$ with $x'^e = x$, then $\mathrm{ord}_{x'} = e\,\mathrm{ord}_x$, that is $v' = ev$.

Proposition-Definition 17.1.3. *Using the notation of 16.1.2, for $\lambda \in \mathbb{Q}_{\geqslant 0}$, let us set*

$$M'_{(\lambda)} = \begin{cases} \displaystyle\bigoplus_{\mathrm{ord}_{x'}(\phi_j) \geqslant 0} M^{(\mu)}_{\phi_j}, & \text{if } \lambda = 0, \\ \displaystyle\bigoplus_{\mathrm{ord}_{x'}(\phi_j) = -e\lambda} M^{(\mu)}_{\phi_j}, & \text{if } \lambda > 0. \end{cases}$$

Then the decomposition $M_{F'} = \bigoplus_\lambda M'_{(\lambda)}$ descends to a decomposition (the slope decomposition) *over $K((x))[x\frac{\partial}{\partial x}]$,*

$$M = \bigoplus_\lambda M_{(\lambda)},$$

where $M'_{(\lambda)} = (M_{(\lambda)})_{K'((x'))}$. The dimension μ_λ of $M_{(\lambda)}$ as a $K((x))$-vector space is called the multiplicity of the slope λ of M*. The Poincaré rank of $M_{(\lambda)}$ is λ.*

Proof. This follows from Galois descent, cf. 16.1.4. □

Remarks 17.1.4. (1) For any $\lambda \in \mathbb{Q}_{\geqslant 0}$, $F_\lambda(M) := \bigoplus_{\lambda' \leqslant \lambda} M_{(\lambda)}$ is the maximal differential $K((x))/K$-submodule of M of Poincaré rank $\leqslant \lambda$; its elements are those $m \in M$ for which the monic operator of minimal order

$$\Gamma_m = \vartheta_x^n - \sum_{j=0}^{n-1} a_j \vartheta_x^j, \qquad a_j \in K((x)),$$

such that $\Gamma_m(m) = 0$, satisfies, for any j, $\frac{\mathrm{ord}_x(a_j)}{n-j} \geqslant -\lambda$.

Starting from the slope filtration, there is an abstract procedure, using duality, to recover the slope decomposition , see [7, §10].

(2) As a consequence of 16.1.4 and 16.1.5, the Turrittin index e_λ of $M_{(\lambda)}$ is a divisor of l.c.m.$(2, \ldots, \mu_\lambda)$, and the Turrittin index of M is a divisor of l.c.m.$(2, \ldots, \max_\lambda \mu_\lambda)$.

(3) Even if the slopes of M are all integral, one may need some ramification to obtain the Turrittin-Levelt decomposition. An example is obtained by starting with an M having Turrittin index > 1, and twisting it by any $L = L_{x^{-\lambda}}$, of positive integral slope λ strictly bigger than the slopes of M. The resulting $K((x))/K$-differential module $L \otimes M$ has the single slope λ, but its characters are given by $\phi_i + x^{-\lambda}$ (the ϕ_i being the characters of M), and in particular its Turrittin index e equals the Turrittin index of M.

Lemma 17.1.5. (1) *For every λ, $M^\vee_{(\lambda)}$ is dual to $M_{(\lambda)}$.*

(2) *Let M_1 and M_2 be $K((x))/K$-differential modules purely of slopes λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$; then* $\mathrm{Hom}_{K((x))\langle\vartheta_x\rangle}(M_1, M_2) = 0$.

(3) *If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of differential modules, then for every λ, there is an exact sequence* $0 \to M_{1(\lambda)} \to M_{(\lambda)} \to M_{2(\lambda)} \to 0$.

Proof. This follows from the Turrittin-Levelt-Jordan decomposition and 8.1.18. □

Definition 17.1.6. *The submodule $M_{(0)} = F_0(M)$ is called the* regular part *of M. The Turrittin exponents of the regular part of M are also called the* Fuchs exponents *of M.*

Proposition 17.1.7. (1) *Let M be the cyclic module attached to the differential operator $\Lambda \in K((x))\langle \frac{d}{dx} \rangle$ $(F = K((x)))$. The roots of* $\mathrm{ind}_{\Lambda,0}(t)$ *are the Fuchs exponents.*

(2) *Let us assume that for some set Δ of derivations of K, there is an action of $\delta \in \Delta$ on M commuting with $\frac{d}{dx}$ and which makes M a differential module over $(K((x)), \delta)$. Then the Fuchs exponents of M are in K^Δ.*

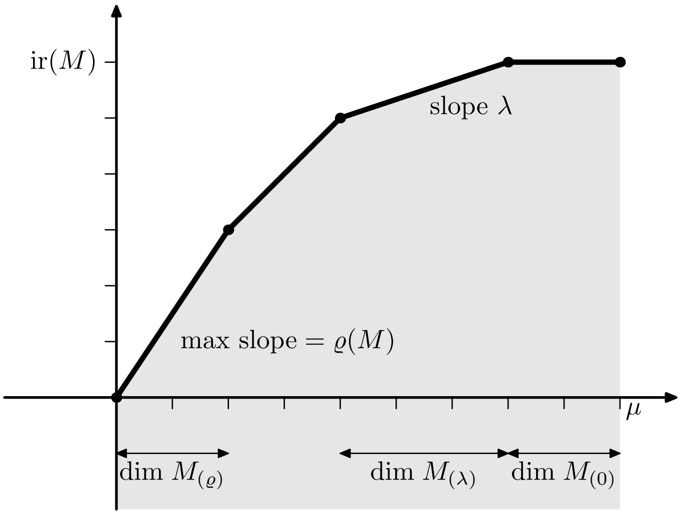
Proof. (1) One can write $\Lambda = \Lambda_{>0}\Lambda_0$, with $M_{>0} \cong F[\partial]/F[\partial]\Lambda_{>0}$, $M_{(0)} \cong F[\partial]/F[\partial]\Lambda_0$ (see 3.2.10). It is clear that $\mathrm{ind}_{\Lambda,0}(t) = \mathrm{ind}_{\Lambda_0,0}(t)$ up to a multiplicative constant.

(2) follows from (1) and Proposition 16.1.6. □

17.2 Newton polygons

Definition 17.2.1 (**Newton polygon and irregularity**). *Let (M, ∇) be a F/K-differential module of rank μ, and let $M = \bigoplus_\lambda M_{(\lambda)}$ be its slope decomposition. The* Newton polygon *of (M, ∇) is the convex hull $\mathrm{NP}(M, \nabla) \subseteq [0, \mu] \times \mathbb{R}$ of the vertical half-line $\{ x_1 = 0\,,\ x_2 \leqslant 0\}$, and the concave piecewise affine curve such that the edge of slope λ has horizontal length μ_λ.*

The irregularity *of M, denoted by $\mathrm{ir}(M, \nabla)$, is the maximum of the ordinates of the vertices of $\mathrm{NP}(M, \nabla)$ (alternatively, it is the ordinate of the extreme right vertex).*



Lemma 17.2.2. *Let M, M_1, M_2 be F/K-differential modules. Then*

(i) $\mathrm{NP}(M^\vee) = \mathrm{NP}(M)$;

(ii) *if $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of differential modules, then $\mathrm{NP}(M) = \mathrm{NP}(M_1) + \mathrm{NP}(M_2)$ (algebraic sum of convex plane sets). In particular,*

$$\mathrm{ir}(M) = \mathrm{ir}(M_1) + \mathrm{ir}(M_2).$$

Proof. Lemma 17.1.5 allows one to reduce to the case of a single slope, where this follows from the analogous properties for the Poincaré rank 15.3.3. □

Remark 17.2.3. This additivity property accounts in part for the fact that, as a measure of irregularity, $\mathrm{ir}(M)$ is more commonly used in the literature than the Poincaré rank $\rho(M)$, which is the maximal slope of $\mathrm{NP}(M)$. Another reason is its occurrence in index formulas, both in the local and the global cases [78], [35]. A third reason is the analogy between irregularity and Swan conductor in arithmetics.

For any $r, s \in \mathbb{Q}_{>0}$, let us denote by $\varphi_{r,s}$ the automorphism of $\mathbb{R}^2$ given by $\varphi_{r,s}(x, y) = (rx, sy)$.

Lemma 17.2.4. *Let F' be a finite field extension of F (endowed with the unique extension v' of the valuation of F) with ramification index e. Assume that $|\text{-}|$ is trivial on K. Then, using the notation of 15.1.23, we have the following properties:*

(i) *if M is a F/K-differential module, then $\mathrm{NP}(M_{F'}) = \varphi_{1,e}\mathrm{NP}(M)$;*

(ii) *if M' is a F'/K-differential module, then $\mathrm{NP}({}_FM') = \varphi_{d,d/e}\mathrm{NP}(M')$.*

Proof. Item (ii) of the preceding lemma allows one to reduce to the case of a single slope, where this follows from the analogous properties for the Poincaré rank 15.3.4. □

Remark 17.2.5 (Action of commuting derivations). Let $(M, \nabla_{x\partial_x})$ be a rank-μ differential module over F/K. Let δ be a derivation of $F = K((x))$ of norm 1 which commutes with $x\partial_x$, and let ∇_δ be an action on M commuting with $\nabla_{x\partial_x}$ such that (M, ∇_δ) is a differential module. Then we have

$$\rho_v(\nabla_\delta) \leqslant \rho_v(\nabla_{x\partial_x}) \quad \text{and} \quad \mathrm{NP}(M, \nabla_\delta) \subseteq \mathrm{NP}(M, \nabla_{x\partial_x}).$$

In particular, if $(M, \nabla_{x\partial_x})$ is regular, then so is (M, ∇_δ).

Using the stability of the Turrittin-Levelt decomposition for ∇_δ, the proof is reduced to the case of an irreducible module, that is, of a Newton polygon with a single slope: then it is enough to verify the first assertion. Then by the μ-exterior product the proof is reduced to the rank-one case. In that case, using a generator m, we have $\nabla_{x\partial_x}(m) = \vartheta m$, $\nabla_\delta(m) = \eta m$ and $\delta(\vartheta) = d\partial_x(\eta)$. Then we have

$$\begin{aligned}\rho_v(\nabla_\delta) &= \max(0, -v(\eta)) = \max(0, -v(\partial_x\eta)) \\ &= \max(0, -v(\delta\vartheta)) \leqslant \max(0, -v(\vartheta)) = \rho_v(\nabla_{x\partial_x}).\end{aligned}$$

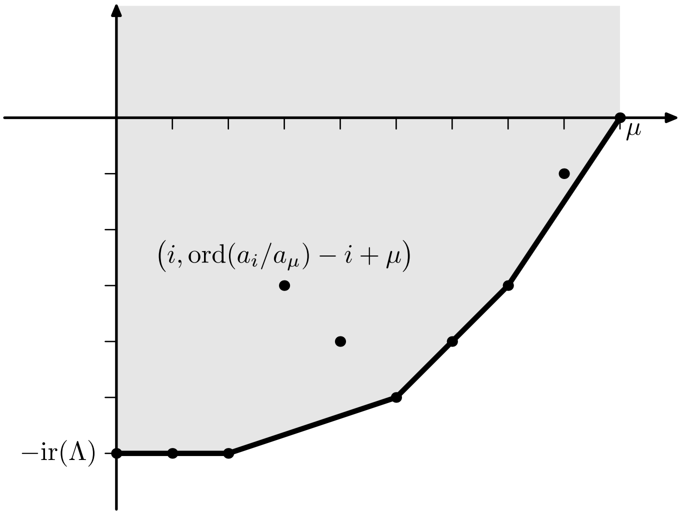
17.3 Newton polygons of cyclic modules

Definition 17.3.1 (Newton polygon and Fuchs number of differential operators). *Let $\Lambda = \sum_0^\mu a_i \frac{d^i}{dx^i}$, with $\alpha_\mu \neq 0$, be a differential operator with coefficients in $F = K((x))$. The* Newton polygon *of Λ is the convex hull $\mathrm{NP}(\Lambda)$ of the union of the vertical strips $S_0, \ldots, S_\mu$, where*

$$S_i = \{\, 0 \leqslant x_1 \leqslant i\,,\ x_2 \geqslant \mathrm{ord}_x(a_i/a_\mu) - i + \mu \,\} \subseteq \mathbb{R}^2, \quad \textit{for } i = 0, \ldots, \mu.$$

The height of $\mathrm{NP}(\Lambda)$ (i.e., the maximal distance between the ordinates of the vertices) is the Fuchs number *of Λ (cf. 6.3.3).*

Notice that the points $(0, -ir(\Lambda))$ and $(\mu, 0)$ are the extreme vertices of $\mathrm{NP}(\Lambda)$.



Remarks 17.3.2. (1) If we write

$$\frac{x^\mu}{a_\mu}\Lambda = \vartheta_x^\mu - \sum_{i=0}^{\mu-1} b_i \vartheta_x^i$$

and set $b_\mu = -1$, $\mathrm{NP}(\Lambda)$ can also be described as the convex hull of the union of the strips

$$S_i' = \{x_1 = i\,,\ x_2 \geqslant \mathrm{ord}_x(b_i)\,\} \subseteq \mathbb{R}^2, \quad \text{for } i = 0, \ldots, \mu.$$

(2) The Newton polygon (resp. the irregularity) of a product of differential operators is the (algebraic) sum of the Newton polygons (resp. the irregularities) of the factors.

(3) Substituting the variable $x^{1/e}$ to x in Λ has the effect of applying $\varphi_{1,e}$ to $\mathrm{NP}(\Lambda)$ and multiplying the irregularity by e.

Theorem 17.3.3. *If $M = F\langle\partial\rangle/F\langle\partial\rangle\Lambda$ is the cyclic module attached to a differential operator Λ with respect to $\partial = x\frac{\partial}{\partial x}$, then $\mathrm{NP}(M)$ coincides with the reflection around the point $(\mu/2, 0)$ (sending (x, y) to $(\mu - x, -y)$) of the Newton polygon of the associated differential operator.*

In particular, $\mathrm{ir}(M)$ coincides with the Fuchs number of Λ and $\rho(M)$ with the maximal slope of $\mathrm{NP}(\Lambda)$.

With the above notation:

$$\begin{aligned}
\mathrm{ir}(M) &= \max\Big\{0, \max_{i=0,\dots,\mu-1}(-\mathrm{ord}_x(a_i/a_\mu)+i-\mu)\Big\}\\
&= \max\Big\{0, \max_{i=0,\dots,\mu-1}(-\mathrm{ord}_x(b_i))\Big\};\\
\rho(M) &:= \max\Big\{0, \max_{i=0,\dots,\mu-1}\big(-\frac{\mathrm{ord}_x(a_i/a_\mu)}{\mu-i}-1\big)\Big\}\\
&= \max\Big\{0, \max_{i=0,\dots,\mu-1}\big(-\frac{\mathrm{ord}_x(b_i)}{\mu-i}\big)\Big\}.
\end{aligned}$$

The reason for the reflection is a matter of conventions: for Λ, we have followed the traditional normalization of Newton polygon for (differential) polynomial, while for M we have followed the convention from the general theory of slope filtrations (see [7]). For more on the computation of these invariants, see [55].

Proof. We first recall that if $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of F/K-differential modules, one can find Λ_1, Λ_2 and $\Lambda = \Lambda_1\Lambda_2$ in $F\langle\partial\rangle$, all monic polynomials in ∂, such that $M_i = F\langle\partial\rangle/F\langle\partial\rangle\Lambda_i$ and $M = F\langle\partial\rangle/F\langle\partial\rangle\Lambda$ (see 3.2.10). Using remarks 17.2.4 and 17.3.2, the Turrittin-Levelt-Jordan decomposition reduces the question to the rank-one case (after a ramification and taking successive extensions), which is immediate. □

Corollary 17.3.4. *The vertices of* $\mathrm{NP}(M)$ *have integral coordinates.*

Remark 17.3.5. Let M be a differential module over $K((X))$ with respect to $\partial = x\frac{\partial}{\partial x}$, and let L be the monic differential operator associated to a cyclic vector of M. Then, compatible with the slope decomposition

$$M = M_{(\lambda_1)} \oplus \cdots \oplus M_{(\lambda_r)}$$

with $\lambda_1 < \cdots < \lambda_r$, we have a factorization

$$L = L_{(\lambda_1)} \cdots L_{(\lambda_r)}$$

with $\mathrm{NP}(L_{(\lambda_i)}) = \mathrm{NP}(M_{(\lambda_i)})$ for $i = 1, \dots r$ (Newton polygons with only one slope). This follows from 3.2.10 and Theorem 17.3.3. For more on the formal factorization of differential operators, see [56].

17.4 Index of operators and Malgrange's definition of irregularity

Definition 17.4.1. *For any field k and any morphism of k-vector spaces $\phi : E \to F$, we say that ϕ* has an index *if both* $\mathrm{Ker}\,\phi$ *and* $\mathrm{Coker}\,\phi$ *are finite-dimensional, and we define the* index of ϕ *as the integer*

$$\chi(\phi) = \dim_k \mathrm{Ker}\,\phi - \dim_k \mathrm{Coker}\,\phi.$$

The emphasis on index theorems for differential operators is due to Malgrange, cf. [78].

Remark 17.4.2. Let (M, ∇) be a $k((x))/k$-differential module. Let $M_1 \subset M_2$ be two $k[[x]]$-lattices in M such that $\nabla(M_1) \subseteq \frac{dx}{x} \otimes M_2$. Malgrange in [78, Prop. 5.1] shows that $\nabla : M_1 \to \frac{dx}{x} \otimes M_2$ has an index and that the integer

$$i(M, \nabla) := \chi \left(\nabla : M_1 \to \frac{dx}{x} \otimes M_2 \right) + \dim_k M_2/M_1 \tag{17.4.3}$$

is independent of the choice of the pair (M_1, M_2) (cf. [78, (5.3)]). He also shows that $i(M, \nabla)$ is non-negative, by reduction to the cyclic case, where an explicit formula holds [78, (5.4)]. This formula coincides with the formula for irregularity deduced by Theorem 17.3.3, and shows that the Malgrange definition of irregularity $i(M, \nabla)$ coincides with the irregularity $\mathrm{ir}(M, \nabla)$ defined in (17.2.1).

Lemma 17.4.4. *Let (M, ∇) be a $k((x))/k$-differential module. Then, there exists a free $k[x^{-1}]$-submodule $M_0 \subset M$ such that $M = M_0 \otimes_{k[x^{-1}]} k((x))$ and such that*

$$\nabla(M_0) \subseteq \frac{d\,x}{x} \otimes M_0.$$

Moreover, $\nabla : M \to \Omega^1_{k((x))/k} \otimes M$ has an index and that index is 0.

Proof. The first part follows from the Turrittin-Levelt formal decomposition by Galois descent to $k((x))$. The second part then follows from [78, Thm. 2.1(b)]. One can also proceed directly as follows. Clearly, for $F' = k((x'))$ and $(M' = M_{F'}, \nabla')$ as in the Turrittin-Levelt decomposition 16.1.2, we have:

$$\chi\big(\nabla : M \longrightarrow \Omega^1_{k((x))/k} \otimes M\big) = 0 \text{ if and only if } \chi\big(\nabla' : M' \longrightarrow \Omega^1_{k((x'))/k} \otimes M'\big) = 0.$$

So, it remains to prove the statement for a $k((x))/k$-differential module of the form $(k((x)), \nabla_\phi)$, with $\phi \in k[1/x]$ and $\nabla_\phi(1) = \phi \frac{dx}{x}$. Then, ∇_ϕ is bijective unless ϕ is an integer $n \in \mathbb{Z}$, in which case $\nabla_n x^i = (n+i)\, x^i \frac{d\,x}{x}$, so that $\ker \nabla_n = k\, x^{-n}$ and $\operatorname{coker} \nabla_n$ is generated by the class of x^{-n}. □

17.5 Variant with parameters. Turning points

17.5.1. We consider the situation where K is the fraction field of a noetherian, integrally closed k-algebra R, and $(M, \nabla_{\vartheta_x})$ a differential module over $(R((x)) = R[[x]][\frac{1}{x}], \vartheta_x)$. Let P be a point of $\operatorname{Spec} R$, and let $M_{(P)}$ be the specialization of M over the differential field $(\kappa(P)((x)), x\partial_x)$.

We explore here the following question: to which extent does the Turrittin-Levelt-Jordan decomposition 16.1.2 descend to $R'((x^{1/e}))$, where R' denotes the integral closure of R in K'?

Clearly, in the rank-one case the decomposition descends without further hypothesis. However, in the general case, suitable hypothesis are needed, as the following example shows. Let consider the rank-two differential module M over $R((x))$ with basis m_1, m_2 and

$$\nabla_{\vartheta_x} m_1 = \frac{y}{x} m_1, \quad \nabla_{\vartheta_x} m_2 = -m_1.$$

Then m_2 is a cyclic vector of $(M, \nabla_{\vartheta_x})$ over $(R((x)), \vartheta_x)$ with associated differential operator $\Lambda = x\vartheta_x^2 - y\vartheta_x$. Theorem 17.3.3 and and the calculation of Remark 17.3.2 (1) show that the slopes of M are 1 and 0, so that M decomposes over $(F = K((x)), \vartheta_x)$ into a direct sum of a summand of slope 1 and one of slope 0.

For $y = 0$ we obtain an indecomposable regular differential module, in particular the decomposition does not descend from F to $k[[y]]((x))$.

17.5.2. Let $(M, \nabla_{\vartheta_x})$ be as before. We assume that, when extended to $K((x))$, $(M, \nabla_{\vartheta_x})$ has Poincaré rank ρ and Turrittin index e. It follows from Theorem 16.1.2 that there is a finite extension K'/K such that $(M, \nabla_{\vartheta_x})$, extended over $F' = K'((x^{1/e}))$, is a Jordan module over (F', ϑ_x) in the sense of Definition 8.1.12. We let $\phi_1, \ldots, \phi_r$ be representatives in $K'[x^{-1/e}]$ of the distinct characters $\overline{\phi}_1, \ldots, \overline{\phi}_r$ of $(M, \nabla_{\vartheta_x})$ in $F'/\vartheta_x \log(F')^\times$. We recall that the constant terms (in K', well-defined modulo $\frac{1}{e}\mathbb{Z}$) of the polynomials $\phi_1, \ldots, \phi_r$ are called the Turrittin exponents of M (cf. Definition 16.1.5)

Theorem 17.5.3. *In the previous situation*

(1) *The coefficient of $x^{-\rho}$ in each of $\phi_1, \ldots, \phi_r$ belongs to R'.*

(2) *Assume*

1. *the differences between Turrittin exponents are constant, i.e., belong to the algebraic closure k' of k in R' (modulo $\frac{1}{e}\mathbb{Z}$),*
2. *condition*

$$\text{the polynomials } \phi_i \in K'[x^{-1/e}] \text{ and their differences are invertible elements of } R'((x^{1/e})) \text{ unless they are } 0, \tag{17.5.4}$$

is satisfied,

then the Turrittin-Levelt-Jordan decomposition 16.1.2 descends to a decomposition of $M_{R'((x^{1/e}))}$:

$$M_{R'((x^{1/e}))} = \bigoplus_{i=1}^{r} R'((x^{1/e})) \otimes_{R'} \mathrm{Ker}_{M_{R'((x^{1/e}))}} (x\nabla_{\partial_x} - \phi_i)^\mu. \tag{17.5.5}$$

Proof. The proof is similar to that of 8.4.2. After completion w.r.t. a divisorial valuation we can suppose $\widehat{R} \cong \kappa[[y]]$. As in the proof of the Turrittin-Levelt decomposition theorem, we choose an element $m \in \widehat{M}$ which generates a cyclic basis $\mathbf{m}$ of $\widehat{M} \otimes_{\kappa[[y]]((x))} \kappa((y))((x))$, and modify it into a basis $\mathbf{n} = \mathbf{m}\Xi$ with

$\Xi = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ 0 & x^\rho & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & x^{(\mu-1)\rho} \end{pmatrix}$, in which the matrix $B_{-\rho}$ of $\delta := x^{\rho+1}\frac{\partial}{\partial x}$ has entries in $\kappa((y))[[x]]$. The eigenvalues of $B_{-\rho\,|x=0}$ are the coefficients of $x^{-\rho}$ in the ϕ_j's.

Actually, we choose m as in the proof of 8.4.2 to be a lifting of a cyclic vector $\overline{m}$ of $\widehat{M} \otimes_{\kappa[[y]]((x))} \kappa((x))$, and we eliminate the apparent singularities as in loc.cit. (via a matrix $Q'' \in \mathrm{GL}_\mu(\kappa((y))[x]_{(x)})$). We get in this way a basis $\mathbf{n}'$ of $\widehat{M}$ which generates a δ-stable $\kappa[[x,y]]$-lattice in $\widehat{M}$. The matrix of δ in this new basis is $B'_{-\rho} = (Q'')^{-1}B_{-\rho}Q'' + (Q'')^{-1}\delta Q''$. The eigenvalues of $B'_{-\rho\,|x=0} = (Q'')^{-1}_{|x=0}B_{-\rho\,|x=0}Q''_{|x=0}$ are again the coefficients $\phi_{j,-\rho}$ of $x^{-\rho}$ in the ϕ_j's (with some multiplicity). In particular $\phi_{j,-\rho} \in \kappa[[y]]$. This proves (1).

For item (2), if there are at least two slopes (i.e., if not all ϕ_j's are of degree $\rho > 0$ in $1/x$), condition (17.5.4) implies that $\phi_{j,-\rho}$ is a unit in $\kappa[[y]]$ if and only if it is non-zero. The Splitting Lemma 16.2.1 thus applies (with the two-dimensional local ring $\widehat{R} = \kappa[[x,y]]$) and separates out the slope-ρ part of $\widehat{M}$. We may thus assume that there is a unique slope $\rho > 0$.

If there are at least two ϕ_j's (of the same slope ρ), then (17.5.4) implies that the $\phi_{j,-\rho}$'s are not all equal modulo $\mathfrak{m}$ and the Splitting Lemma applies. So, we reduce to the case of a single ϕ_i which is obvious. $\square$

Remark 17.5.6. By induction on ρ after twisting by $\kappa[[y]]((x)) \cdot \exp\big(-\frac{\phi_{i,-\rho}}{\rho}x^{-\rho}\big)$, we can also prove that the characters ϕ_j's belongs to $R'[\frac{1}{x}]$.

Condition (1) could actually be omitted, but it is traditionally part of the following definition.

Definition 17.5.7 (Turning points). *A point P of $\operatorname{Spec} R$ for which the stability condition (17.5.4) does not hold is called a turning point.*

From Krull's Hauptidealsatz, one gets:

Lemma 17.5.8. *The set of turning points is a closed subset of $\operatorname{Spec} R$ which is either empty, or of codimension one.*

17.6 Variation of the Newton polygon

We consider again the situation where K is the fraction field of a noetherian, integrally closed k-algebra R. We consider a differential module $(M, x\nabla_{\partial_x})$ over $(R((x)) = R[[x]][\frac{1}{x}], x\partial_x)$. For any point P of $\operatorname{Spec} R$ we let $M_{(P)}$ denote the specialization of M over the differential field $(\kappa(P)((x)), x\partial_x)$.

Proposition 17.6.1 (Turning points in terms of NP). *In the above situation, the following conditions are equivalent:*

(1) *P is not a turning point for $(M, x\nabla_{\partial_x})$ (see 17.5.7: the characters and their differences are invertible in $R'((x^{1/e}))$ in a Zariski neighborhood of P);*

(1′) *the coefficients of the terms of minimal order of the characters ϕ_i and their differences $\phi_i - \phi_j$ are invertible in R'_P (if not zero);*

(2) *we have equalities of Newton polygons*

$$\mathrm{NP}(M_{(P)}) = \mathrm{NP}(M) \quad \textit{and} \quad \mathrm{NP}(\mathrm{End}(M_{(P)})) = \mathrm{NP}(\mathrm{End}(M));$$

(3) *we have equalities of Newton polygons*

$$\mathrm{NP}(M_C) = \mathrm{NP}(M) \quad \textit{and} \quad \mathrm{NP}(\mathrm{End}(M_C)) = \mathrm{NP}(\mathrm{End}(M))$$

for any formal curve C in $\mathrm{Spf}R[[x]]$ *cutting* $\mathrm{Spec}(R)$ *transversally in P.*

Proof. The equivalence of (1) and (1′) is standard. The condition (1′) clearly implies (2) and (3) since the slopes of the Newton polygons $\mathrm{NP}(M)$ (resp. $\mathrm{NP}(M_{(P)})$, $\mathrm{NP}(\mathrm{End}(M))$, $\mathrm{NP}(\mathrm{End}(M_{(P)}))$) are given by the order in $1/x$ of the terms ϕ_i (resp. $\phi_i(P)$, $\phi_i - \phi_j$, $(\phi_i - \phi_j)(P)$). Condition (3) clearly implies (2), so it remains to prove that (2) implies (1).

Replacing R with the completion of R_P, we may suppose that R is a complete local ring, and P the closed point of $\mathrm{Spec}(R)$.

We proceed by induction on the rank μ of M (notice that the condition (2) is stable under taking direct summand). For $\mu = 1$ or $\rho = \rho(M) = 0$ the result is obvious, so we may suppose $\rho > 0$. If there is only one character ϕ, by 17.5.3 (part (1) and the remark following the theorem) the character has coefficients in $R' \cap K = R$ and $\phi_{j,-\rho}$ is a unit in R_P. If there are at least two different characters, we choose ϕ_j one of the characters and put

$$\rho' = \max_{i \neq j} -v_x(\phi_i - \phi_j) \in (0, \rho].$$

Let $\phi_j^{\leqslant \rho'}$ denote the component of ϕ_j of degree $\leqslant \rho'$; again by 17.5.3, $\phi_j^{\leqslant \rho'}$ has coefficients in R'. Now, the differential module (over $R'((x))$)

$$M' = M \otimes L_{-\phi_j^{\leqslant \rho'}}$$

has Poincaré rank equal to ρ', equal to the Poincaré rank of $\mathrm{End}(M')$ (since M' has at least two slopes). Using the hypothesis (2), we have that $\mathrm{NP}(\mathrm{End}(M'_{(P)})) = \mathrm{NP}(\mathrm{End}(M'))$, so that the coefficients $\phi_{i,-\rho'} - \phi_{j,-\rho'}$ are zero or invertible in R'. Then we can apply the Splitting Lemma 16.2.1 using the derivation $\delta = x^{\rho'+1}\partial_x$ of $R'[[x^{1/e}]]$ to the lattice of M' generated over $R'[[x^{1/e}]]$ by a basis of M' in which the matrix of $\nabla_{x\partial_x}$ is of the form $x^{-\rho'}G(x)$ with $G(x) \in M_\mu(R'[[x^{1/e}]])$. In this way we obtain a decomposition of $M'_{R'[[x^{1/e}]]}$, hence of $M_{R'[[x^{1/e}]]}$, while respecting the hypothesis on Newton polygons, which permits to reduce μ. $\square$

It turns out that at turning points, the Newton polygon can only drop. We start with an easier observation:

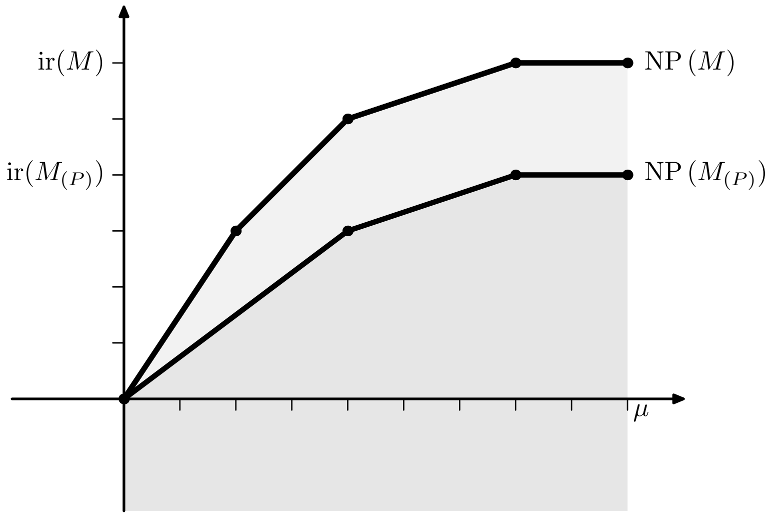
Lemma 17.6.2. *$\rho(M_{(P)}) \leqslant \rho(M)$ (in particular, if M is regular, so is $M_{(P)}$).*

Proof. Let us consider an x-adic norm for M, and the quotient norm for $M_{(P)}$. Then for every n we have the inequality

$$|\nabla^n_{(P),\vartheta_x}|_{M_{(P)}} \leqslant |\nabla^n_{\vartheta_x}|_M$$

and passing to the limit, we have the result. □

Proposition 17.6.3. *In the above situation, we have* $\mathrm{NP}(M_{(P)}) \leqslant \mathrm{NP}(M)$*, that is,* $\mathrm{NP}(M_{(P)})$ *lies below* $\mathrm{NP}(M)$*, that is,* $\mathrm{NP}(M_{(P)}) \subseteq \mathrm{NP}(M)$.



Proof. [2] By successive specializations, we may assume that R has dimension one, in fact a discrete valuation ring with fraction field K and residue field k, and that P is the closed point of $\operatorname{Spec} R$. Then M is free of rank μ over $R((x))$, and we denote by M_K and $M_k = M_{(P)}$ the corresponding differential modules over $K((x))$ and $k((x))$, respectively. After ramification of x, we may assume that their slopes are integers, and we set

$$\begin{aligned} \mu_{(\lambda)} &= \text{ rank of the } \lambda\text{-slope part of } M, \\ \overline{\mu}_{(\lambda)} &= \text{ rank of the } \lambda\text{-slope part of } M_k. \end{aligned}$$

We may also suppose, eventually after ramification, that the slopes λ are integers. Following the strategy of [43], we choose an $R[[x]]$-lattice Λ of M, and define by

[2] Taken from [6]. The “proof” proposed in the first edition of this book (II.4.2.3) was not correct, because of the use of an incorrect cyclic vector lemma, see note at the proof of 3.3.2.

induction a double sequence of lattices:

$$\Lambda_{m,0} = \Lambda \text{ for all } m \in \mathbb{N}\ ,$$
$$\Lambda_{m,n+1} = \Lambda_{m,n} + \nabla_{x^{m+1}\partial_x}(\Lambda_{m,n}).$$

We notice that the quotients $\Lambda_{m,n}/\Lambda$ are R-modules of finite type, and define

$$\lambda_m = \lim_n \frac{1}{n} \dim_K(\Lambda_{m,n})_K/\Lambda_K,$$
$$\overline{\lambda}_m = \lim_n \frac{1}{n} \dim_k \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k).$$

Using the Turrittin-Levelt decomposition 16.1.2 and 15.3.6 we get the equalities

$$\lambda_m = \sum_{\lambda > m} (\lambda - m)\mu_{(\lambda)}, \quad \overline{\lambda}_m = \sum_{\lambda > m} (\lambda - m)\overline{\mu}_{(\lambda)}.$$

Moreover,

$$\lambda_m \geqslant \overline{\lambda}_m \geqslant 0.$$

Indeed[3], the $R[[x]]$-bidual $\Lambda^{**}_{m,n}$ of $\Lambda_{m,n}$ is free as an $R[[x]]$-module (since $R[[x]]$ is regular of dimension 2), and the quotient $\Lambda^{**}_{m,n}/\Lambda$ is flat as an R-module (since it is a submodule of M/Λ, which has no torsion over the discrete valuation ring R). From the exact sequence $\Lambda_{m,n}/\Lambda \to \Lambda^{**}_{m,n}/\Lambda \to \Lambda^{**}_{m,n}/\Lambda_{m,n} \to 0$ tensored with k, we get $\operatorname{Ker}((\Lambda^{**}_{m,n})_k/\Lambda_k \to (\Lambda^{**}_{m,n}/\Lambda_{m,n})_k) = \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k)$. Therefore,

$$\begin{aligned} \dim_K((\Lambda_{m,n})_K/\Lambda_K) &= \dim_k((\Lambda^{**}_{m,n})_k/\Lambda_k) \\ &= \dim_k \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k) + \dim_k(\Lambda^{**}_{m,n}/\Lambda_{m,n})_k, \end{aligned}$$

from which the inequality follows.

Consider now the functions $f(x)$ (resp. $\overline{f}(x)$) defined by the convex Newton polygon associated to $\mathrm{NP}(M)$ (resp. $\mathrm{NP}(M_k)$), that is, the Newton polygon associated to the corresponding differential operator. Their Legendre transforms

$$f^*(\xi) = \sup_t (t\xi - f(t)), \quad \overline{f}^*(\xi) = \sup_t (t\xi - \overline{f}(t))$$

(for $\xi \in [0, \infty[$), satisfy the Young inequalities

$$t\xi \leqslant f(t) + f^*(\xi), \quad t\xi \leqslant \overline{f}(t) + \overline{f}^*(\xi).$$

Using the previous results we have that

$$f^*(m) = \mu m + \lambda_m, \quad \overline{f}^*(m) = \mu m + \overline{\lambda}_m$$

and the Young inequality gives

$$\lambda_m \geqslant -f(t) - (\mu - t)m, \quad \overline{\lambda}_m \geqslant -\overline{f}(t) - (\mu - t)m,$$

[3] This argument is the Deligne lemma in [36], lettre à N. Katz (1/12/1976)

for all $t \in [0,\mu]$. For $t = t_m < \mu$ a vertex of $f(t)$, and $\xi = m$ the right-slope at that vertex, the left inequality is an equality and since $\lambda_m \geqslant \overline{\lambda}_m$ we deduce that

$$-f(t_m) - (\mu - t_m)m \geqslant -\overline{f}(t_m) - (\mu - t_m)m.$$

In particular, we obtain that $f(t_m) \leqslant \overline{f}(t_m)$, from which the result follows. □

18 Varia

18.1 Cyclic vectors in the neighborhood of a non-turning singular point

Theorem 18.1.1. *Let M be a $R((x))/R$-differential module. Assume that the Turrittin index is 1, and that there is no turning point. Then M has a cyclic vector.*

Remark 18.1.2. Remark 3.3.3 shows that the assumption that there is no turning point cannot be dropped.

Proof. Let μ be the rank of M. We apply Theorem 17.5.3 to obtain a decomposition (taking into account that the Turrittin index is 1)

$$M = \bigoplus_{i=1}^{r} R((x)) \otimes_R \operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_i)^{\mu},$$

where $\nabla_{\vartheta_x} = x\nabla_{\partial_x}$, the characters ϕ_i belong to $R\big[\frac{1}{x}\big]$ and the differences $\phi_i - \phi_j$ for $i \neq j$ are invertible in $R((x))$ and not integers.

For any $\nu = 1, \dots, \mu$, there exist elements $e_{\nu,j,1}, \dots, e_{\nu,j,s_{\nu,j}}$ in $\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu}$ whose classes form an R-basis in the quotient $\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu}/\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu-1}$. Altogether, the elements $e_{\nu,j,k}$ form an $R((x))$-basis of M. We reorder them in lexicographical order:

$$(\nu, j, k) < (\nu', j', k') \Leftrightarrow ((\nu > \nu') \text{ or } (\nu = \nu'\ , j > j') \text{ or } (\nu = \nu'\ , j = j', k > k')),$$

and rename them $(m_0, \dots, m_{\mu-1})$, so that $\nabla_{\vartheta_x} m_j = \psi_j m_j + \sum_{k>j} a_{kj} m_k$, where ψ_j is one of the ϕ_l, and $a_{kj} \in R$.

We now choose an integer $n > \mu \cdot \rho(M)$, and set

$$m := m_0 + x^{-n} m_1 + \cdots + x^{-nj} m_j + \cdots + x^{-n(\mu-1)} m_{\mu-1} \in M.$$

We shall prove that m is a cyclic vector, with the help of an auxiliary sequence $m^{(0)} := m, m^{(1)}, \dots$. Let us set $m_j^{(0)} = x^{-nj} m_j$, and rewrite $m^{(0)} = m_0 + \sum_{j=1}^{\mu-1} m_j^{(0)}$, $\psi_j^{(0)} := \psi_j - nj$, so that

$$\nabla_{\vartheta_x} m_j^{(0)} = \psi_j^{(0)} m_j^{(0)} + \sum_{k>j} a_{kj}^{(0)} m_k^{(0)},$$

with $\operatorname{ord}_x a_{kj}^{(0)} > 0$ and, for $j \neq k$, $\psi_j^{(0)} - \psi_k^{(0)}$ is a unit in $R((x))$, with $\operatorname{ord}_x (\psi_j^{(0)} - \psi_k^{(0)}) \leqslant 0$.

Next, we define

$$m^{(1)} := \left(\psi_1^{(0)} - \psi_0^{(0)} + a_{1,0}^{(0)}\right)^{-1}\left(\nabla_{\vartheta_x} - \psi_0^{(0)}\right)m = m_1^{(0)} + \sum_{k>1} u_{k,1} m_k^{(0)},$$

where $u_{k,1} = \left(\psi_1^{(0)} - \psi_0^{(0)} + a_{1,0}^{(0)}\right)^{-1}\left(\psi_k^{(0)} - \psi_0^{(0)} + \sum_{k>j} a_{kj}^{(0)}\right)$ is a unit in $R((x))$, of x-order between $-\rho(M)$ and $+\rho(M)$. It is clear that $(m^{(0)}, m^{(1)})$, and hence also $(m, \nabla_{\vartheta_x} m)$, generate $M/\sum_{k>1} R((x))m_k$.

If we set $m_j^{(1)} = u_{j,1} m_j^{(0)}$, so that $m^{(1)} = m_1^{(0)} + \sum_{j>1} m_j^{(1)}$, $\psi_j^{(1)} = \psi_j^{(0)} + \frac{\nabla_{\vartheta_x}(u_{j,1})}{u_{j,1}}$ for $j > 1$, we get

$$\nabla_{\vartheta_x} m_j^{(1)} = \psi_j^{(1)} m_j^{(1)} + \sum_{k>j} a_{kj}^{(1)} m_k^{(1)},$$

with $a_{kj}^{(1)} = u_{k,1} a_{kj}^{(0)}$. Note that $\operatorname{ord}_x a_{kj}^{(1)} > 0$ (using the fact that $n > \rho(M)$) and, for $j \neq k$, $\psi_j^{(1)} - \psi_k^{(1)}$ is a unit in $R((x))$, with $\operatorname{ord}_x\left(\psi_j^{(1)} - \psi_k^{(1)}\right) \leqslant 0$.

We then construct

$$m^{(2)} := \left(\psi_2^{(1)} - \psi_1^{(1)} + a_{2,1}^{(1)}\right)^{-1}\left(\nabla_{\vartheta_x} - \psi_1^{(1)}\right)m^{(1)} = m_2^{(1)} + \sum_{k>2} u_{k,2} m_k^{(1)}.$$

It is clear that $(m^{(0)}, m^{(1)}, m^{(2)})$, hence also $(m, \nabla_{\vartheta_x} m, \nabla_{\vartheta_x}^2 m)$, generate $M/\sum_{k>2} R((x))m_k$.

Then we construct $m_j^{(2)}$, $\psi_j^{(2)}$, $a_{jk}^{(2)}$ (of positive x-order since $n > 2\rho(M)$), as before, and iterate $\mu - 1$ times. Iteration step ν shows that $(m^{(0)}, \ldots, m^{(\nu)})$, hence also $(m, \ldots, \nabla_{\vartheta_x}^{\nu} m)$, generate $M/\sum_{k>\nu} R((x))m_k$. For $\nu = \mu$, we conclude that m is cyclic. $\square$

18.2 Turrittin decomposition around crossing points of the polar divisor

18.2.1. We finish this chapter with an exploration of the Turrittin decomposition around a crossing point of the polar divisor of an integrable connection in two variables. We will need these considerations in the next chapter.

For deeper work in this direction (notably Sabbah's conjecture and its proof), we refer to [91], [83] and [66]).

We set $Y = \operatorname{Spf} k[[y_1, y_2]]$, $D =$ the divisor $xy = 0$, $P = Y_{\mathrm{red}}$ the crossing point $(y_1 = y_2 = 0)$, and a finitely generated $k[[y_1, y_2]][\frac{1}{y_1 y_2}]$-module M with integrable connection ∇ (w.r.t. continuous derivations of $k[[y_1, y_2]]$). Any such module is projective, and even free since $k[[y_1, y_2]][\frac{1}{y_1 y_2}]$ is a principal domain.

In the rank-one case, such a module is determined by $\omega := \nabla(1) \in \Omega^1_Y(*D)$, a closed form. If we set $\vartheta_i = y_i \partial_{y_i}$, $\psi_i = \nabla_{\vartheta_i}(\omega) = \langle \vartheta_i, \omega \rangle \in \mathcal{O}_Y(*D)$ for $i = 1, 2$, the integrability condition translates into the equation $\vartheta_1(\psi_2) = \vartheta_2(\psi_1)$. We denote this module by L_ω.

18.2.2 (Nice formal structure). Any M as above gives rise to a differential module M_{F_1} over $F_1 = k((y_2))((y_1))$ (resp. M_{F_2} over $F_2 = k((y_1))((y_2))$). After ramification $(y_1, y_2) \mapsto (y_1^{\frac{1}{e}}, y_2^{\frac{1}{e}})$ (with an index e dividing $\mu!$), one has the Turrittin decomposition $M_{F_1} = \bigoplus_j M^{(\mu)}_{\phi_{1,j}}$, with $\phi_{1,j} \in \frac{1}{y_1} k((y_2))[\frac{1}{y_1}]$ of order at least $-\rho_1$ w.r.t. y_1 and equal to $-\rho_1$ for at least one index; similarly for M_{F_2}.

We say that M has *a nice formal structure at* P[4] if there exists a decomposition

$$M = \bigoplus_h M_{\overline{\omega}_h} \quad \text{with} \quad M_{\overline{\omega}_h} \cong L_{\omega_h} \otimes R_h,$$

where $\omega_h \in \Omega^1_Y(*D)$ are closed and have distinct classes $\overline{\omega}_h$ modulo $\Omega^1_Y(\log D)$, L_{ω_h} is the rank-one integrable connection associated to ω_h, and R_h is a free module of finite rank over $\mathcal{O}_Y(*D)$ with a regular integrable connection (w.r.t. ϑ_1, ϑ_2).

In this situation, the decomposition is unique and induces, after extension to F_1 and F_2, a refinement of the Turrittin-Levelt-Jordan decompositions:

$$M^{(\mu)}_{\phi_{1,j}} = \bigoplus_{h:\pi_1(\psi_{1,h})=\phi_{1,j}} M_{\overline{\omega}_h} \otimes F_1 \quad \text{and} \quad M^{(\mu)}_{\phi_{2,j}} = \bigoplus_{h:\pi_2(\psi_{2,h})=\phi_{2,j}} M_{\overline{\omega}_h} \otimes F_2. \tag{18.2.3}$$

Here π_1 and π_2 refer to the canonical projections

$$\pi_1 : \frac{k[[y_1, y_2]]\big[\frac{1}{y_1 y_2}\big]}{k[[y_1, y_2]]} \longrightarrow \frac{1}{y_1} k((y_2))\big[\frac{1}{y_1}\big] \quad \text{and}$$
$$\pi_2 : \frac{k[[y_1, y_2]]\big[\frac{1}{y_1 y_2}\big]}{k[[y_1, y_2]]} \longrightarrow \frac{1}{y_2} k((y_1))\big[\frac{1}{y_2}\big].$$

18.2.4. Let us consider a sequence of formal blow-ups

$$\pi : Y' \longrightarrow Y$$

of P, then of crossing points of successive exceptional divisors. One knows that such a sequence of *toric* blow-ups corresponds to a regular fan of the first quadrant of $\mathbb{R}^2$. Toric charts are isomorphic to $\mathbb{A}^2$ (completed). The trace of the pull-back of D in such a chart is the union of the coordinate axes. In the one associated

[4] cf. [5]; this is equivalent to saying that P is semi-stable for M in the sense of [6], but weaker than saying that M has a good formal structure in the sense of [91].

to the cone bounded by edges passing through $(a,b)\in\mathbb{N}^2$ and $(c,d)\in\mathbb{N}^2$, with $ad-bc=1$, adapted coordinates (y_1',y_2') are given by

$$y_1=(y_1')^a(y_2')^b,\quad y_2=(y_1')^c(y_2')^d. \tag{18.2.5}$$

If $\omega=\phi_1\frac{dy_1}{y_1}+\phi_2\frac{dy_2}{y_2}$, then $\pi^*\omega=\phi_1'\frac{dy_1'}{y_1'}+\phi_2'\frac{dy_2'}{y_2'}$, where

$$\phi_1'=a.\phi_1+c.\phi_2\ ,\quad \phi_2'=b.\phi_1+d.\phi_2. \tag{18.2.6}$$

Dually,

$$y_1\frac{\partial}{\partial y_1}=\pi_*\Big(d\,y_1'\frac{\partial}{\partial y_1'}-c\,y_2'\frac{\partial}{\partial y_2'}\Big),\quad y_2\frac{\partial}{\partial y_2}=\pi_*\Big(-b\,y_1'\frac{\partial}{\partial y_1'}+a\,y_2'\frac{\partial}{\partial y_2'}\Big). \tag{18.2.7}$$

Note also that the ramification $(y_1,y_2)\mapsto(y_1^{\frac{1}{e}},y_2^{\frac{1}{e}})$ commutes with toric blow-ups.

Proposition 18.2.8 (Nice formal structure by toric blow-ups). *There exists a finite sequence of formal toric blow-ups $\pi:Y'\to Y$ (starting with a blow-up of P) such that π^*M has a nice formal structure at any crossing point P' of $\pi^{-1}(P)$ (after ramification in y_1 and y_2).*

This result was first proved by Sabbah (in [91, III,4.3.1]) using a generalization in dimension 2 of the nilpotent orbit method of Babbitt and Varadarajan. We sketch here the proof given in [6, 5.4], which only uses elementary calculations on toric blow-ups.

Proof (Sketch). We proceed by induction on the rank μ of M (there is nothing to prove for $\mu=1$). For any $\Lambda\subseteq k$, we denote by $k[\lambda]_\Lambda$ the localization $k[\lambda,\frac{1}{\lambda-\lambda_0}]_{\lambda_0\in\Lambda}$.

18.2.9. *Claim. After ramification, there exist a sequence π' of toric blow-ups, a finite set $\Lambda\subseteq k$ and, in each toric chart, a basis $\mathbf{n}(\lambda)$ of the pull-back of*

$$M^\Lambda:=M\otimes_{k[[y_1,y_2]][\frac{1}{y_1y_2}]}k[\lambda]_\Lambda[[y_1,y_2]]\big[\frac{1}{y_1y_2}\big]$$

in which the matrix of

$$\vartheta(\lambda):=\vartheta_1+\lambda\vartheta_2$$

has no pole or can be written, in adapted coordinates (y_1',y_2'), in the form

$$(y_1')^{-r_1}(y_2')^{-r_2}G(\lambda,y_1',y_2'),$$

where $r_1,r_2\geqslant 0$ (not both 0), $G(\lambda,y_1',y_2')\in M_\mu(k[\lambda]_\Lambda[[y_1',y_2']])$, and where for any $\lambda_0\notin\Lambda$, $G(\lambda_0,0,0)\in M_\mu(k)$ is not nilpotent.

Proof of the claim. Let $m \in M$ be a cyclic vector of $M \otimes \mathrm{Frac}(k[[y_1, y_2]])$ w.r.t. $\vartheta(0) = \vartheta_1$. In some basis of $\bigwedge^\mu M \cong k[\lambda][[y_1, y_2]]\big[\frac{1}{y_1 y_2}\big]$, one can write

$$m \wedge \vartheta(0)m \wedge \cdots \wedge \vartheta(0)^{\mu-1}m = g(y_1, y_2) \in k[[y_1, y_2]],$$

where g is not divisible by y_1 or y_2. On the other hand, we may write $m \wedge \vartheta(\lambda)m \wedge \cdots \wedge \vartheta(\lambda)^{\mu-1}m = y_1^{-s_1} y_2^{-s_2}.g(\lambda, y_1, y_2) \in k[\lambda][[y_1, y_2]]$, where $g(\lambda, y_1, y_2)$ is not divisible by y_1 or y_2 and $g(0, y_1, y_2) = y_1^{s_1} y_2^{s_2} g(y_1, y_2) \neq 0$.

Then for every λ_0 outside a finite set $\Lambda' \subseteq k$,

$$\mathrm{ord}_{y_1} g(\lambda_0, y_1, 0) = \mathrm{ord}_{y_1} g(\lambda, y_1, 0), \quad \mathrm{ord}_{y_2} g(\lambda_0, 0, y_2) = \mathrm{ord}_{y_2} g(\lambda, 0, y_2),$$

and there exists a sequence π'' of toric blow-ups such that in each chart, and for every $\lambda_0 \notin \Lambda'$, the strict transform of $g(\lambda_0, y_1, y_2) = 0$ does not meet any crossing of $(\pi'')^{-1}(P)$. It follows that

$$\mathbf{m}(\lambda) = (m, \vartheta(\lambda)(m), \ldots, \vartheta(\lambda)^{\mu-1}(m))$$

is a cyclic basis of the pull-back of $M^{\Lambda'}$ w.r.t. $\vartheta(\lambda)$, and induces a cyclic basis after specialization at every $\lambda_0 \notin \Lambda'$. The same remains true if one performs a ramification $(y_1, y_2) \mapsto (y_1^{\frac{1}{e}}, y_2^{\frac{1}{e}})$ at the beginning ($\vartheta(\lambda)$ becomes $\vartheta(\lambda)/e$), which allows us to assume that the Poincaré ranks ρ_1, ρ_2 along ${y'}_1 = 0, {y'}_2 = 0$ (in each toric chart) are integers.

Let us then modify $\mathbf{m}(\lambda)$ into a basis

$$\mathbf{m}(\lambda) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & (y'_1)^{\rho_1}(y'_2)^{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (y'_1)^{\rho_1(\mu-1)}(y'_2)^{\rho_2(\mu-1)} \end{pmatrix}.$$

Let us write the matrix of $\vartheta(\lambda)$ in this new basis in the form

$${y'}_1^{-\rho_1} {y'}_2^{-\rho_2} H(\lambda, {y'}_1, {y'}_2)$$

with $H(\lambda, {y'}_1, {y'}_2) \in M_\mu(k[\lambda]_{\Lambda'}[[{y'}_1, {y'}_2]][\frac{1}{y'_1 y'_2}])$. The theorem of Dwork, Katz and Turrittin 15.2.2 applies (over the complete field $k(\lambda)(({y'}_2))(({y'}_1))$ endowed with the y'_1-adic valuation), and shows that the entries of $H(\lambda, {y'}_1, {y'}_2)$ lie in $k(\lambda)(({y'}_2))[[{y'}_1]]$ and in case $\rho_1 > 0$, $H(\lambda, 0, {y'}_2)$ is not nilpotent (item 2.). Idem by exchanging y'_2 and y'_1.

Moreover, if $\rho_1 = 0$ (and similarly if $\rho_2 = 0$), one can still reduce to the case where $H(\lambda, 0, {y'}_2)$ is not nilpotent on multiplying the basis by y'_1 (which adds to H the matrix $(y'_2)^{\rho_2} \frac{\vartheta(\lambda)(y'_1)}{y'_1} I_\mu$).

One thus finds a basis $\mathbf{n}(\lambda)$ in which the matrix of $\vartheta(\lambda)$ has no pole if $\rho_1 = \rho_2 = 0$, and otherwise has the form ${y'}_1^{-\rho_1} {y'}_2^{-\rho_2} H(\lambda, {y'}_1, {y'}_2)$, with $H(\lambda, {y'}_1, {y'}_2) \in M_\mu(k[\lambda]_{\Lambda'}[[{y'}_1, {y'}_2]])$, $H(\lambda, 0, {y'}_2)$ being $H(\lambda, {y'}_1, 0)$ non-nilpotent. In particular,

$$h(\lambda_0, {y'}_1, {y'}_2, t) := \det(t.I_\mu - H(\lambda_0, {y'}_1, {y'}_2)) - t^\mu \in k[[{y'}_1, {y'}_2]][t]$$

is not divisible by ${y'}_1$ or ${y'}_2$, and the same holds if one specializes t at some $t_0 \in k$. Therefore, for every λ_0 outside a finite set $\Lambda \supseteq \Lambda'$, one has

$$\begin{aligned}\operatorname{ord}_{y'_1} h(\lambda_0, y'_1, 0, t_0) &= \operatorname{ord}_{y'_1} h(\lambda, y'_1, 0, t_0),\\ \operatorname{ord}_{y'_2} h(\lambda_0, 0, y'_2, t_0) &= \operatorname{ord}_{y'_2} h(\lambda, 0, y'_2, t_0).\end{aligned}$$

Thanks to these uniform bounds, there exists a sequence of toric blow-ups such that in each chart and for every $\lambda_0 \notin \Lambda$, the strict transform of $h(\lambda_0, {y'}_1, {y'}_2, t_0) = 0$ does not meet any crossing of the divisor above P. Each of these charts admits adapted coordinates y''_1, y''_2 with $y'_1 = (y''_1)^a (y''_2)^b$, $y'_2 = (y''_1)^c (y''_2)^d$, $a, b, c, d \geqslant 0$, $ad - bc = 1$. It follows that in the basis $\mathbf{n}(\lambda)$, the matrix of $\vartheta(\lambda)$ is of the form $(y''_1)^{-r_1}(y''_2)^{-r_2} G(\lambda, y''_1, y''_2)$ with $G(\lambda, y''_1, y''_2) = H(\lambda, {y'}_1, {y'}_2) \in M_\mu(k[\lambda]_\Lambda[[y''_1, y''_2]])$, and

$$\det(t.I_\mu - G(\lambda_0, y''_1, y''_2)) - t^\mu = h(\lambda_0, {y'}_1, {y'}_2, t) \in k[[y''_1, y''_2]][t]$$

does not vanish at $y''_1 = y''_2 = 0$. Therefore, for every $\lambda_0 \notin \Lambda$, $G(\lambda_0, 0, 0) \in M_\mu(k)$ is not nilpotent, which proves the claim. □

From here, the proof of the proposition follows [91, III.4.3.1]: twisting M by a suitable L_ω, we may assume that $\bigwedge^\mu M$ is regular. One completes the sequence of toric blow-ups by refining the corresponding fan in such a way that each cone has an edge with primitive vector (a, c), the image of a in k being non-zero and the image of c/a being outside Λ, and such that the associated divisor is a component of the polar divisor of $\vartheta(c/a)$ (if the latter is non-empty).

Let us fix a toric chart, with adapted coordinates y'_1, y'_2. Assume that y'_1 corresponds to an edge whose slope $\lambda_0 = c/a$ lies outside Λ. In the basis $\mathbf{n}(c/a)$ of the inverse of M, the matrix of $y'_1 \frac{\partial}{\partial y'_1} = a.\vartheta(c/a)$ is of the form $(y'_1)^{-r_1}(y'_2)^{-r_2} G(y'_1, y'_2)$ with $G(y'_1, y'_2) \in M_\mu(k[[y'_1, y'_2]])$, and $G(0,0)$ is not nilpotent if r_1 and r_2 are not both 0. Since $\bigwedge^\mu M$ is regular, $G(0,0)$ has trace 0, hence has two distinct eigenvalues. If r_1 and r_2 are not both 0, the Splitting Lemma 16.2.1 applies to the derivations $\delta = (y'_1)^{r_1+1}(y'_2)^{r_2} \frac{\partial}{\partial y'_1}$ and $\delta' = (y'_1)^{s_1}(y'_2)^{s_2+1} \frac{\partial}{\partial y'_2}$ of $k[[y'_1, y'_2]]$ (for suitable $s_1, s_2 \geqslant 0$) and to the $k[[y'_1, y'_2]]$-lattice spanned by $\mathbf{n}(c/a)$. This yields a decomposition of M, which allows us to reduce to smaller rank μ. If, on the contrary, $r_1 = r_2 = 0$, the pull-back of M is still regular along $y'_1 = 0$ and $y'_2 = 0$. □

Chapter VI

Irregularity: geometric theory

Introduction

This chapter completes our study of irregularity in several variables, in the geometric framework of integrable connections. For an algebraic integrable connection ∇ on the complement of a prime divisor D in an algebraic variety X, we define its Newton polygon and its Poincaré rank (maximal slope). We study the behaviour of the Newton polygon by restriction to curves, and introduce a stratification of the polar divisor D indexed by "secondary" Newton polygons.

We also present an algebraic construction of the analog of the logarithmic extension in the irregular case (Malgrange lattice).

The main result of this chapter is Theorem 21.1.1 on the semi-continuity of the Poincaré rank (in an appropriate sense): if X is fibered as a family of curves over a base S, the *sum* of the Poincaré ranks of ∇_s at the points of D above s defines a lower semi-continuous function on S. In particular, for regular integrable connections, there is no confluence at crossing points of the polar divisor.

19 Poincaré rank and Newton polygon along a prime divisor

19.1 Poincaré rank along a prime divisor

As in Section 10, we consider a smooth k-variety X, a prime divisor D, and $U = X \smallsetminus D$, the open complement.

Let $\mathcal{M}$ be a locally free $\mathcal{O}_U$-module of finite rank μ endowed with integrable connection

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

In this subsection we shall introduce the notion of Poincaré rank of ∇ along D

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*,
Progress in Mathematics 189, https://doi.org/10.1007/978-3-030-39719-7_6

and establish basic properties. Let M denote the completion of the generic fibre of $\mathcal{M}$ (with respect to the valuation attached to D). As in 10.2 we consider M as a differential module over $K((x))$, and we endow $K((x))$ with the x-adic valuation v.

Definition 19.1.1 (**Poincaré rank along a divisor**). *We define the Poincaré rank of $\mathcal{M}$ to be*

$$\sup_{\partial} \rho_v(M, \nabla_\partial),$$

where ∂ runs over all continuous derivations in $\widehat{\mathcal{T}}_{X,D}$ (see 10.1 and 15.3). We denote it by $\rho_D(\mathcal{M}, \nabla)$, $\rho_D(\mathcal{M})$, or $\rho_D(\nabla)$.

Proposition 19.1.2. (1) *$\rho_D(\mathcal{M}) = 0$ if and only if ∇ is regular along D.*

(2) *For any transversal derivation ∂, one has $\rho_D(\mathcal{M}) = \rho_v(M, x\nabla_\partial)$.*

Proof. (1): see Remark 15.3.5.
(2) follows from Remark 17.2.5 together with Lemma 2.2.7. □

Lemma 19.1.3. *Let $(\mathcal{M}, \nabla)$,$(\mathcal{M}_1, \nabla_1)$,$(\mathcal{M}_2, \nabla_2)$ be as before.*

(1) *If $\mathcal{M}$ sits in a horizontal exact sequence*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_2 \longrightarrow 0,$$

then $\rho_D(\mathcal{M}) = \max\{\rho_D(\mathcal{M}_1), \rho_D(\mathcal{M}_2)\}$;

(2) *we have*

$$\rho_D(\mathcal{M}_1 \otimes \mathcal{M}_2) \leqslant \max\{\rho_D(\mathcal{M}_1), \rho_D(\mathcal{M}_2)\}$$

and

$$\rho_D(\mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2)) \leqslant \max\{\rho_D(\mathcal{M}_1), \rho_D(\mathcal{M}_2)\}$$

with equality if $\rho_D(\mathcal{M}_1) \neq \rho_D(\mathcal{M}_2)$.

Proof. This follows from Lemma 15.3.3. □

Lemma 19.1.4. *Let $f : X \to Y$ be a finite morphism of smooth k-varieties, and let E be a smooth connected divisor of Y. Let us assume that $f^{-1}E = \sum e_i D_i$, where D_i are smooth connected divisors of X, and that f is étale on $X \setminus \bigcup D_i$. Let $(\mathcal{M}, \nabla)$ be a coherent sheaf with integrable connection on $X \setminus \bigcup D_i$. Then $\rho_E(f_* \mathcal{M}) \leqslant \max\{e_i^{-1} \rho_{D_i}(\mathcal{M})\}$.*

Proof. Let w be the valuation of $\kappa(Y)$ attached to E. Since f is finite, each valuation v of $\kappa(X)$ above w is attached to some D_i. The result then follows from Lemma 15.3.4. □

19.2 Newton polygon along a prime divisor

In the same setting, it follows from Remark 17.2.5 together with Lemma 2.2.7 that for any transversal derivation ∂ and any derivation δ with $|\delta|_v \leqslant 1$, $\mathrm{NP}_v(M, \delta) \subseteq \mathrm{NP}_v(M, x\partial)$. It follows that $\mathrm{NP}_v(M, x\partial)$ is independent of ∂ and is the supremum of all $\mathrm{NP}_v(M, \delta)$. This is the *Newton polygon of $\mathcal{M}$ along D*, denoted by $\mathrm{NP}_D(\mathcal{M}, \nabla)$, $\mathrm{NP}_D(\mathcal{M})$ or $\mathrm{NP}_D(\nabla)$.

Since the definition is around the generic point of D, we may extend it to a not necessarily smooth (but irreducible) divisor D. The Poincaré rank $\rho_D(\mathcal{M})$ is nothing but the *maximal slope* of $\mathrm{NP}_D(\mathcal{M})$.

If D is not irreducible, and has components D_i, we shall later use the notation $\mathrm{NP}_D(\mathcal{M})$ to mean the collection of the $\mathrm{NP}_{D_i}(\mathcal{M})$.

Lemma 19.2.1. $\mathrm{NP}_D(\mathcal{M}^\vee) = \mathrm{NP}_D(\mathcal{M})$*, and* $\mathrm{NP}_D(\mathcal{M}) = \mathrm{NP}_D(\mathcal{M}_1) + \mathrm{NP}_D(\mathcal{M}_2)$ *for an exact sequence* $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$.

Proof. This follows from Lemma 17.2.2. □

Let $\phi_{r,s}$ denote transformation $(x, y) \mapsto (rx, sy)$.

Lemma 19.2.2. *Let $f : X \to Y$ be a finite morphism of smooth k-varieties, and let E be a smooth connected divisor of Y. Let us assume that $f^{-1}E = \sum e_i D_i$, where D_i are smooth connected divisors of X, and that f is étale of degree d on $X \smallsetminus \bigcup D_i$; let d_i be the local degree at D_i, so that $d = \sum d_i e_i$. Then* $\mathrm{NP}_E(f_*\mathcal{M}) = \sum \phi_{d_i, d_i/e_i} \mathrm{NP}_{D_i}(\mathcal{M})$.

Proof. This follows from Lemma 17.2.4. □

Here is a variant of Corollary 12.1.3 in the irregular case.

Lemma 19.2.3. *Suppose X is an algebraic tubular neighborhood of D as in 10.1.19, with smooth affine basis S, and let $(\mathcal{M}, \nabla)$ a connection on $X \smallsetminus D$. Let ∂ a derivation transversal to D, and suppose that there exists a cyclic vector for $(\mathcal{M}, \nabla)$ with respect to ∂. Consider the differential operator Λ associated to the cyclic vector. If the length of the horizontal edge of* $\mathrm{NP}_Q(\mathcal{M}, \nabla)_{f^{-1}f(Q)}$ *is independent of $Q \in D$, then the dominant term of the indicial polynomial of Λ is invertible in $\mathcal{O}(S)$.*

Proof. Indeed, this length is nothing but the degree $\mathrm{ind}_{Q, \Lambda_{f^{-1}f(Q)}}$. □

Here is an algebraic version of Theorem 18.1.1.

Theorem 19.2.4. *Suppose X is an algebraic tubular neighborhood of D as in 10.1.19, with smooth affine basis S and D connected, and let $(\mathcal{M}, \nabla)$ a connection on $X \smallsetminus D$. Let ∂ a derivation transversal to D. Assume that the Turrittin index is* 1*, and there is no turning point on D. Then after restricting S to a neighborhood of any given point P, there is a divisor D' in X disjoint from D such that* $(\mathcal{M}, \nabla)_{|X \setminus (D \cup D')}$ *is cyclic with respect to ∂.*

Proof. We may replace S by a finite étale covering and assume that D is a union of images D_i of sections of f given by equations $x = \theta_i$, so that the completion of $\mathcal{O}(X)$ at D_i is $\mathcal{O}(S)[[x - \theta_i]]$. We may then replace S by the local affine scheme $\operatorname{Spec} \mathcal{O}_{S,P}$, and X by the semi-local affine scheme $\operatorname{Spec} \mathcal{O}(X)_D$. The completion of $\mathcal{O}(X)_D$ along D is $\Pi \mathcal{O}(S)[[x - \theta_i]]$.

Note that $\mathcal{M}$ extends to a free $\mathcal{O}(X)_D$-module $\overline{M}$, and fix a basis $e_1, \dots e_\mu$. According to 18.1.1, there is a cyclic vector $m_i \in \overline{M} \otimes_{\mathcal{O}(X)_D} \mathcal{O}(S)((x - \theta_i))$, which we may choose in $\overline{M} \otimes_{\mathcal{O}(X)_D} \mathcal{O}(S)[[x - \theta_i]]$.

We then approximate simultaneously the m_i's at $(x - \theta_i)$-order μ by some vector $m \in \overline{M}$. Then $m_i \wedge \partial m_i \wedge \cdots \wedge \partial^{\mu-1} m_i = (x - \theta_i)^{n_i} u_i e_1 \wedge \cdots \wedge e_\mu$ for some unit u_i in $\mathcal{O}(S)[[x - \theta_i]]$ and $n_i \in \mathbb{N}$. Then $m \wedge \partial m \wedge \cdots \wedge \partial^{\mu-1} m = u e_1 \wedge \cdots \wedge e_\mu$ for some element $u \in \mathcal{O}(X)_D$ which approximates $(x - \theta_i)^{n_i} u_i$ at order μ. It follows that u is a unit in each $\mathcal{O}(S)((x - \theta_i))$, hence also in the semi-local ring $\mathcal{O}(X)_D\left[\frac{1}{\prod_i (x - \theta_i)}\right]$. □

19.3 Stratification of the polar divisor by Newton polygons

Proposition 19.3.1. *Let $D \subseteq X$ be a (reduced) divisor in a smooth k-variety, D_i its irreducible components, and D^{sm} the smooth part of D. Let $(\mathcal{M}, \nabla)$ be a coherent $\mathcal{O}_{X \smallsetminus D}$-module with integrable connection on $X \smallsetminus D$.*

(1) *There is an open dense subset $D^0 \subseteq D^{\mathrm{sm}}$ such that for any locally closed curve C which meets D transversally at some point Q of D_i, $\mathrm{NP}_Q(\nabla_{|(C \smallsetminus Q)}) = \mathrm{NP}_{D_i}(\nabla)$ (in fact, it suffices to take for D^0 the set of non-turning points in D^{sm} (cf. Definition 17.5.7)).*

(2) *Moreover, for any locally closed curve C which meets D_i transversally at $Q \in D^{\mathrm{sm}}$, one has $\mathrm{NP}_Q(\nabla_{|(C \smallsetminus Q)})$ is contained in (i.e., lies below) $\mathrm{NP}_{D_i}(\nabla)$.*

Proof. (1) It is clear, by the definition of a non-turning point P, the formal Turrittin decomposition of ∇ around P induces the formal Turrittin decomposition of $\nabla_{|(C \smallsetminus Q)}$ by restriction to the curve. By the characterization in terms of Newton polygons (17.6.1), we get $\mathrm{NP}_Q(\nabla_{|(C \smallsetminus Q)}) = \mathrm{NP}_{D_i}(\nabla)$.
(2) follows from Proposition 17.6.3. □

Corollary 19.3.2. *Let X be a smooth k-variety, D be a reduced divisor, and $(\mathcal{M}, \nabla)$ be a coherent $\mathcal{O}_{X \smallsetminus D}$-module with integrable connection on $X \smallsetminus D$. Then ∇ is regular along D if and only if for any $P \in D^{\mathrm{sm}}$ (resp. any P in a dense subset of D^{sm}), and for all (resp. there exists some) locally closed smooth curve C in X which meets D transversally at P such that $\nabla_{|(C \smallsetminus C \cap D)}$ is regular at P.*

Proof. The regularity clearly implies the condition (13.1.5). Viceversa, from the condition of the corollary and the proposition we deduce that the Newton polygon of $(\mathcal{M}, \nabla)$ is horizontal, therefore the regularity. □

Remark 19.3.3. Item (2) of Proposition 19.3.1 no longer holds if C does not meet D transversally, as is shown for instance by the case of the rank-one connection with formal solution $e^{\frac{1}{y-x^2}}$ and $C =$ the x-axis (which is tangent to D).

Item (2) no longer holds if Q is not a smooth point of D, as is shown for instance by the case of the rank-one connection with formal solution $e^{\frac{1}{xy}}$. We shall deal with this problem in 21. In particular, we will see that it disappears in the regular case.

Let $\mathrm{Inf}^1 X$ be the scheme of infinitely near points of first order on X, i.e., the projective bundle of tangent directions $\mathbb{P}(\Omega^1_{X/k})$. Let Q be a point of X. We say that a property $\mathcal{P}$ is shared by *almost every* smooth locally closed curve containing Q if there is a dense open subset U of $\mathrm{Inf}^1 X_{|Q}$ such that every such curve with tangent direction in U staisfies $\mathcal{P}$.

Theorem 19.3.4. *Let D be a divisor in X, D_i its irreducible components, and let $(\mathcal{M}, \nabla)$ be a coherent $\mathcal{O}_{X \smallsetminus D}$-module with integrable connection on $X \smallsetminus D$.*

(1) *There exist a finite set $\mathfrak{N}$ of polygons with integral vertices, and a unique finite partition of D into non-empty constructible subsets Z_N indexed by the elements of $\mathfrak{N}$, each of which is a finite disjoint union of smooth subvarieties $Z_{N,\alpha}$, with the following property:*

   ($*$) *for every closed point $Q \in Z_{N,\alpha}$ and almost every smooth locally closed curve C which meets $Z_{N,\alpha}$ transversally at Q, $\mathrm{NP}_Q(\nabla_{|(C \smallsetminus Q)}) = N$.*

(2) *Moreover, if D is smooth irreducible, $Z_{N'} \subseteq \overline{Z}_N$ implies $N' \subseteq N$; in particular, $\bigcup_{N' \subseteq N} Z_{N'}$ is closed in D.*

Proof. (1) It is clear that the partition $D = \coprod Z_N$ is unique if it exists (a constructible subset is determined by its subset of closed points). We can show its existence locally on D. By applying Proposition 19.3.1 to $X \setminus \bigcup_{j \neq i} D_j$, we obtain dense open subsets $D_i^0 \subseteq D_i$ such that $\mathrm{NP}_Q(\nabla_{|(C \setminus Q)}) = \mathrm{NP}_{D_i^0}(\nabla)$ for any smooth locally closed curve C which meets D_i^0 transversally at some point Q. We then blow up $D \setminus \bigcup D_i^0$ (which is of codimension $\geqslant 2$ in X). By applying Proposition 19.3.1 to the smooth part of the components of the exceptional divisor, we obtain dense open subsets $D_i^1 \subseteq (D \setminus \cup D_i^0)$ such that $\mathrm{NP}_Q(\nabla_{|(C \setminus Q)})$ is constant for every point $Q \in D_i^1$ and almost every smooth locally closed curve C which meets each relevant component of D transversally at Q. We then blow up $(D \setminus \bigcup D_i^0) \setminus \bigcup D_i^1$ (which is of codimension $\geqslant 3$ in X), and iterate the process. The construction of the partition is complete in finitely many steps: at the end, we glue together the strata corresponding to the same Newton polygon, taking into account the fact that a finite union of constructible subsets is constructible.

(2) We have to show that if $P' \in Z_{N',\alpha}$ lies in the Zariski closure of Z_N, then $N' \subseteq N$. If $P' \in \overline{Z}_N^{\mathrm{sm}}$, the assertion follows from Proposition 17.6.3 after blowing up $\overline{Z}_N$ and looking at the smooth part of the exceptional divisor.

If $P' \notin \overline{Z}_N^{\mathrm{sm}}$, we make a detour. We replace X by an affine neighborhood of P'. Let C' be a smooth locally closed curve in X which meets D transversally at P', such that $\mathrm{NP}_{P'}(\nabla_{|(C'\setminus P')}) = N'$, and let C be a smooth locally closed curve in X which meets D transversally at some point $P \in Z_N$, such that $\mathrm{NP}_P(\nabla_{|(C\setminus P)}) = N$. We may and shall assume that $P' \notin C$, and $P \notin C'$.

According to Lemma 10.1.21, there exist an open common neighborhood U of P and P' in X and a tubular neighborhood $(U \xrightarrow{f} S, t)$ of $D \cap U$, such that $C \cap U = f^{-1}(f(P))$, $C' \cap U = f^{-1}(f(P'))$.

Let T be the normalization of some curve joining $f(P)$ and $f(P')$ in $f(Z_N^0)$. Let X_T (resp. D_T) be the pull-back of U (resp. $D \cap U$) on T, and let ∇_{X_T} be the pull-back of ∇ on $X_T \setminus D_T$. We have $\mathrm{NP}_{D_T}(\nabla_{X_T}) = N$. The result then follows from Proposition 17.6.3 applied to the closed immersion of C' in X_T. $\square$

Remark 19.3.5. Even if D is smooth irreducible, one cannot remove the restriction "for almost every" in $(*)$, as is shown by the case of the rank-one connection ∇ with formal solution $e^{\frac{y}{x^2}}$: the curve $y = 0$ is exceptional for the stratum $\{0\}$, since the restriction of ∇ to it is regular, while the restriction of ∇ to almost every curve passing through the origin has Poincaré rank 1.

Remark 19.3.6. Even if D is smooth irreducible, one cannot expect that $\bigcup_{N'\subseteq N} Z_{N'} = \overline{Z}_N$ (in item (2)), as is shown by the case of the rank-one connection with formal solution $e^{\frac{y(x+y-1)}{x^2}}$: in the polar divisor $x = 0$, the big stratum (slope 2) is the complement of the two points $y = 0$ and $y = 1$, which are strata with slope 0 and 1, respectively.

In the same vein, one cannot expect that $\overline{Z}_N$ is a union of strata, as is shown by the case of the rank-one connection with formal solution $e^{\frac{y(x+y-1)z}{x^2}}$.

Remark 19.3.7. A different study of irregularity, in the context of $\mathcal{D}$-modules, has been developed by Y. Laurent and Z. Mebkhout [72], [82], [73]. Their main theme is the geometric construction, based upon the idea of the V-filtration, of some algebraic cycles which turn out to be positive and to correspond, through the "comparison theorem", to the perverse sheaf of irregularity and its Gevrey filtration introduced in [Me2]; in the process, they attach a Newton polygon to any holonomic $\mathcal{D}$-module along a hypersurface, and even a collection of Newton polygons indexed by the components of the characteristic variety of the sheaf of irregularity. It would be interesting to understand the relationship to the Newton polygons appearing above.

20 Turrittin-Levelt decomposition and τ-extensions

20.1 Formal Turrittin decomposition along a divisor

20.1.1. Let X be a normal variety over the algebraically closed field k. Let $D = \bigcup D_i \subseteq X$ be a reduced divisor with irreducible components D_i, and let us assume that $U := X \smallsetminus D$ is smooth.

As in 11.2, we fix a set-theoretic section τ of the canonical projection $k \to k/\mathbb{Z}$ such that $\tau(\mathbb{Z}) = 0$.

Consider on U a coherent (hence automatically locally free) $\mathcal{O}_U$-module $\mathcal{M}$ of rank μ equipped with an integrable connection

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_U \otimes \mathcal{M}.$$

20.1.2. According to 17.5, for each i, we can find an affine smooth dense open neighborhood U_i in X of the generic point η_{D_i} of D_i, such that

(1) *$D_i \cap U_i$ is smooth, defined by an equation $x_i = 0$;*

(2) *$\mathcal{M}_{|U_i \smallsetminus (D\cap U_i)}$ is free; we denote by M_i the corresponding free $\mathcal{O}(U_i)\big[\frac{1}{x_i}\big]$-module;*

(3) *there is a finite étale covering $D'_i \to D_i \cap U_i$ such that, if we set $x'_i = x_i^{1/\mu!}$, we have a canonical decomposition of $\mathcal{O}(D'_i)((x'_i))$-modules with integrable formal connections* (*i.e., integrable connections with respect to x_i-adically continuous derivations*)

$$M_i \otimes_{\mathcal{O}(U_i)[\frac{1}{x_i}]} \mathcal{O}(D'_i)((x'_i)) = \bigoplus_j L_{i,j} \otimes R_{i,j}, \tag{20.1.3}$$

where $L_{i,j}$ is a rank-one (formal) connection of type

$$\mathcal{O}(D'_i)((x'_i)) \cdot \exp\big(-\int \phi_{i,j} dx'_i/x'_i\big)$$

with $\phi_{i,j} \in \frac{1}{x'_i}\mathcal{O}(D'_i)\big[\frac{1}{x'_i}\big]$, and $R_{i,j}$ is a regular *(formal) connection.*

20.2 τ-extensions of irregular connections

20.2.1. In this subsection (not used in the rest of this book), we present a partial generalization of the theory of τ-extensions of a regular integrable connection (developed in 11.2.2), allowing some irregular singularities. This is an algebraic version of the Deligne-Malgrange construction of canonical lattices of meromorphic integrable connections (cf. also [79]).

In the $\mathcal{O}(D'_i)((x'_i))$-modules $L_{i,j}$ (resp. $R_{i,j}$), we consider the $\mathcal{O}(D'_i)[[x'_i]]$-lattice

$$L_{0,i,j} = \mathcal{O}(D'_i)[[x'_i]] \cdot \exp\big(-\textstyle\int \phi_{i,j} dx'_i/x'_i\big),$$

(resp. $R_{0,i,j}$ = the τ_i-*extension* of $R_{i,j}$). We may and shall assume that $R_{0,i,j}$ is free.

Definition 20.2.2. *A coherent $\mathcal{O}_X$-module $\overline{\mathcal{M}}$ extending the $\mathcal{O}_U$-module $\mathcal{M}$ is called a τ-extension of $\mathcal{M}$ if and only if*

(i) $\overline{\mathcal{M}}$ *is reflexive* (*i.e., isomorphic to its bidual*)*;*

(ii) *for each* i*, set* $\overline{M}_i = \Gamma(U_i, \overline{\mathcal{M}})$*; then one has a decomposition of free* $\mathcal{O}(D'_i)[[x'_i]]$*-modules extending the decomposition* (20.1.3)*:*

$$\overline{M}_i \otimes_{\mathcal{O}(U_i)} \mathcal{O}(D'_i)[[x'_i]] = \bigoplus_j L_{0,i,j} \otimes R_{0,i,j}.$$

Remark 20.2.3. When X is smooth, D is a divisor with normal crossings, and ∇ is regular along D, a τ-extension $\mathcal{E}$ is nothing but the underlying $\mathcal{O}_X$-module of the τ-extension with logarithmic poles constructed in 11.2. In that situation, $\overline{\mathcal{M}}$ is automatically locally free. In the general case, however, there is no reason to expect $\overline{\mathcal{M}}$ to be locally free.

Theorem 20.2.4. *Any coherent* $\mathcal{O}_U$*-module with integrable connection* $(\mathcal{M}, \nabla)$ *admits a* τ*-extension, which is unique up to a unique isomorphism.*

Proof (sketch). Since X is normal, the reflexivity of $\overline{\mathcal{M}}$ implies that $\overline{\mathcal{M}} = j_* j^* \overline{\mathcal{M}}$ for any open immersion $j : X' \hookrightarrow X$ of the complement of a closed subset of codimension at least 2. Therefore (as in the regular case), it is enough to replace X by a neighborhood of the generic point of any of the D_i's. We may thus assume that $X = U_i$, $D = D_i$, and drop the subscript i from the notation.
Uniqueness: let $\overline{\mathcal{M}}'$ and $\overline{\mathcal{M}}''$ be two τ-extensions of $(\mathcal{M}, \nabla)$, and set

$$\overline{\mathcal{N}} = \mathcal{H}om(\overline{\mathcal{M}}', \overline{\mathcal{M}}'')$$

(a reflexive coherent $\mathcal{O}(X)$-module). We have a canonical element $h \in \mathcal{N} := \overline{\mathcal{N}} \otimes \mathcal{O}(U)$ corresponding to the identity map of $\mathcal{M}$, and we have to show that $h \in \overline{\mathcal{N}}$ (h will then automatically come from an *isomorphism* between $\overline{\mathcal{M}}'$ and $\overline{\mathcal{M}}''$). We note that $h \in \mathcal{N}^{\nabla}$. Then

$$\mathcal{N} \otimes \mathcal{O}(D')((x')) = \bigoplus_{j,k} (L_{0,j}^{\vee} \otimes L_{0,k}) \otimes (R_{0,j}^{\vee} \otimes R_{0,k}).$$

Since the rank-one differential module $(L_{0,j}^{\vee} \otimes L_{0,k})$ is regular if and only if $j = k$, we have

$$(\mathcal{N} \otimes \mathcal{O}(D')((x')))^{\nabla} = \bigoplus\nolimits_j \operatorname{End}^{\nabla} R_{0,j},$$

which coincides with $\bigoplus_j \operatorname{End}^{\nabla} R_j$, since the differences of the exponents of R_j are never non-zero integers. Therefore, $h \in \mathcal{N} \cap (\overline{\mathcal{N}} \otimes \mathcal{O}(D')[[x']]) = \mathcal{N}$.
Existence: one proceeds in two steps: algebraization and descent.
• algebraization: recall that M is free, and choose a basis $\mathbf{e}$. Recall that R_j is also assumed to be free on $\mathcal{O}(D')[[x']]$, and choose a basis $\mathbf{f}_j$. The elements $\widehat{\mathbf{e}}_j = \exp(-\int \phi_j dx'/x') \otimes \mathbf{f}_j$ together form a basis $\widehat{\mathbf{e}}$ of $M \otimes \mathcal{O}(D')((x'))$. Write $\mathbf{e} = \widehat{\mathbf{e}}P$, $P \in \mathrm{GL}_\mu(\mathcal{O}(D')((x')))$. By truncating the series occuring in P, one constructs

a new basis $\widetilde{\mathbf{e}}$ of $M \otimes_{\mathcal{O}(D)[x]} \mathcal{O}(D')[x']$ such that $\widetilde{\mathbf{e}}$ and $\widehat{\mathbf{e}}$ span the same $\mathcal{O}(D')[[x']]$-lattice in $M \otimes \mathcal{O}(D')((x'))$.

• descent: let M be the intersection of M and the $\mathcal{O}(D')[x']$-module spanned by $\widetilde{\mathbf{e}}$ in $M \otimes_{\mathcal{O}(D)[x]} \mathcal{O}(D')[x']$. Since $\mathcal{O}(D')[x']$ is a (finite) faithfully flat extension of $\mathcal{O}(D)[x]$, M is a projective $\mathcal{O}(X)$-lattice in M. $\square$

21 Main theorem on the Poincaré rank

21.1 Statement of the main theorem

The setting is now the following: X is a normal k-variety, U a dense open subset, $j : U \to X$ the inclusion, $D = \bigcup_i D_i$ is the part of $X \smallsetminus U$ purely of codimension one in X (the D_i's being the irreducible components), and $(\mathcal{M}, \nabla)$ is a locally free $\mathcal{O}_U$-module of finite rank with integrable connection.

Theorem 21.1.1. *Let $h : \overline{C} \to X$ be a morphism from a smooth curve such that $h(\overline{C}) \not\subseteq D$, and $C = h^{-1}(U) \cap \overline{C}$. Let $Q \in \overline{C}$ map to a point $P \in D$ (if any). Then*

$$\rho_Q(\nabla_C) \leqslant \sum (h(\overline{C}).D_i)_P \; \rho_{D_i}(\nabla).$$

In the case where $\rho_{D_i}(\nabla) = 0$, we get the following corollary, which completes the proof of Theorem 12.3.1 in the general case:

Corollary 21.1.2. *In the same situation, if ∇ is regular along every D_i, then ∇_C is regular at Q.*

Corollary 21.1.3 (**Semi-continuity of the Poincaré rank**)**.** *Let $f : X \to S$ be a smooth morphism of smooth k-varieties of relative dimension 1, and D a (reduced) divisor in X. Let $(\mathcal{M}, \nabla)$ be a locally free $\mathcal{O}_U$-module of finite rank with integrable connection ∇. The function*

$$S \ni P \longmapsto \sum_{Q \in D, f(Q) = P} \rho_Q(\nabla)$$

is lower semi-continuous.

In case D is étale over S, this already readily follows from Proposition 17.6.3 (or 19.3.1). The corollary shows that *summing* the Poincaré ranks over all preimages Q of P compensates the possible upper jumps which may arise, as in the case of the rank-one connection with formal solution $e^{\frac{1}{xy}}$.

Let us explain how to derive the corollary from the theorem. Constructibility of the function follows from item (1) of Theorem 19.3.4. It remains to show that the function drops by specialization $P \rightsquigarrow P'$. By base-change, one may assume that P is the generic point of S. The assertion then comes from the inequality of the theorem applied to the curve $C = f^{-1}(P)$. $\square$

Remark 21.1.4. The corollary also holds for the *irregularity* in place of the Poincaré rank. This was *Malgrange's conjecture*, and we refer to [6] for the proof, which is slightly more involved than the proof of 21.1.3.

Malgrange's conjecture was motivated by its ℓ-adic analog (with the heuristic dictionary: irregularity $\leftrightarrow$ Swan conductor), which occurred in an influential work by Deligne and Laumon. However, it seems that the ℓ-adic analogs of Corollary 21.1.3 and of its variant with Poincaré rank replaced by irregularity are still incompletely understood, cf. [57].

21.2 Proof of the main theorem

21.2.1. The strategy of the proof of the main Theorem 21.1.1 follows the papers [6][5]. We have already reduced the problem to the case of smooth surfaces, then to the case of the plane (see 12.3.6). Changing notation, we shall denote by C the (possibly singular) curve which was denoted above by $h(\overline{C})$. This reduces 21.1.1 to the following

Lemma 21.2.2. *Let $D = D_1 \cup \cdots \cup D_r$ be a union of germs of curves in $Y = \operatorname{Spf} k[[y_1, y_2]]$, C another germ of curve, $\widetilde{C} \to C$ the normalization of C, and $Q = \widetilde{C}_{\mathrm{red}}$ (with image the origin $P = Y_{\mathrm{red}}$). Then for any formal connection M on Y with poles along D, i.e., any $\mathcal{O}_Y(*D)$-module projective of finite type endowed with an integrable connection over k, we have that*

$$\rho_Q(M|_{\widetilde{C}}) \leqslant \sum_{i=1}^{r} (C, D_i)_P \; \rho_{D_i}(M)$$

where $(C, D_i)_P = \dim_k k[[y_1, y_2]]/(\mathcal{I}_C + \mathcal{I}_{D_i})$ is the intersection multiplicity of C and D_i (at the origin).

We start with a finite sequence of formal blow-ups $\pi : Y' \to Y$ (first blowing-up P in Y) satisfying the following conditions:

(1) $\pi^{-1}(D \cup C)$ has strict normal crossing;

(2) any irreducible component of the exceptional divisor $E = \pi^{-1}(P)$ cuts at most one of the strict transforms D_i' of the components D_i of D;

(3) the pull-back M' of M has a nice formal structure at crossing points of E (the Turrittin-Levelt decomposition "extends" at crossing points; this may be achieved by 18.2.8).

In that situation, for each component of the exceptional divisor, we shall define suitable divisors associated to the slopes of the pull-back of the connection. Estimates for the Poincaré ranks will then follow from the study of the intersection properties of that divisor with the exceptional divisors of the sequence of blow-ups.

21.2.3. In the previous situation, let Z be an irreducible component of $E = \pi^{-1}(P)$, let $X = \operatorname{Spf} R[[x]]$ be a formal open affine subscheme of Y' such that $X_{\text{red}} = Z \cap X (=: Z')$. We will associate to the restriction of (M, ∇) to X a well-defined divisor $\Phi_{Z,\lambda}(M)$, depending on a non-zero slope λ of (M, ∇), of the normal bundle $N_Z(X)$.

Consider the slope decomposition 17.1.3 of (M, ∇) over F' (suitable extension of $F = K((x))$, where K is the fraction field of R), and recall that the Galois group of F'/F permutes the terms $M^{(\mu)}_{\phi_j}$ since it permutes the characters ϕ_j (see the proof of 8.1.19).

Let λ be a non-zero slope of M_F. Define

$$J_{(\lambda)} = \{j : \phi_j \in F' \text{ s.t. } M^{(\mu)}_{\phi_j} \subseteq M_{(\lambda)}\}$$

(set of indices appearing in the λ-component of the slope decomposition), and for $j \in J_{(\lambda)}$ define μ_j to be the dimension of the regular part relative to the character ϕ_j in the Turrittin-Levelt-Jordan decomposition, so that $\mu_{(\lambda)} = \sum_{j \in J_{(\lambda)}} \mu_j$ is the dimension of $M_{(\lambda)}$. Notice that $\lambda\mu_{(\lambda)}$ is the irregularity of $M_{(\lambda)}$.

Now set

$$\varphi_\lambda(x) = \prod_{j \in J_{(\lambda)}} (x^\lambda - \phi_{j,-\lambda})^{\mu_j}$$

where $\phi_{j,-\lambda}$ ($\in R'$ by 17.5.3) is the coefficient of $x^{-\lambda}$ in ϕ_j. By a standard Galois argument, we have that $\varphi_\lambda(x) \in K[x]$ (of degree $\lambda\mu_{(\lambda)}$).

Since M' has a nice formal structure at crossing points of $Z' = \operatorname{Spec}(R)$, we have $\varphi_\lambda(x) \in R[x]$ and we can define

$$\Phi_{Z,\lambda}(M) = (\varphi_\lambda(x))$$

(a positive Weil divisor of $\operatorname{Spec} R[x]$). Every component of $\Phi_{Z,\lambda}(M)$ is finite over Z, and étale over stable points.

If $Z' = \operatorname{Spec}(R)$ is regularly embedded in $X = \operatorname{Spec}(R[x])$, we can identify $\operatorname{Spec}(R[x])$ with the normal bundle $N_{Z'}(X) = \mathbb{V}(\mathcal{I}_{Z'}/\mathcal{I}^2_{Z'})$ of Z' in X (where $\mathcal{I}_{Z'}$ is the ideal of Z' in X) using the decomposition in 10.1.18 by the choice of a transversal derivation. Then the Weil divisor $\Phi_{Z,\lambda}(M)$ is well defined as a divisor of the normal bundle $N_{Z'}(X)$, that is, it does not depend on the choice of the transversal derivation ∂. In fact, any other derivation whose restriction to Z' is ∂ is given, using the decomposition 10.1.15, by $u\partial + xv\partial'$, where u is 1 modulo $\mathcal{I}_{Z'}$, v is a x-adic integer and ∂' commutes with ∂. Therefore the term $xv\partial'$ does not change the terms of $\varphi_\lambda(x))$ of smaller degree in x, so that its inverse $x^\lambda\phi^{-1}_{j,-\lambda}$ is well definied in $\Gamma(\mathcal{I}^\lambda_{Z'}/\mathcal{I}^{\lambda+1}_{Z'})$.

Consider the Zariski closure $\overline{\Phi}_{Z,\lambda}(M)$ of $\Phi_{Z,\lambda}(M)$ in the projective bundle $\mathbb{P}(N_Z(X))$ of $N_Z(X)$. The following lemma determines the intersection multiplicities of the divisor $\overline{\Phi}_{Z,\lambda}(M)$ with the divisor (∞) of the section at infinity of $\mathbb{P}(N_Z(X))$ over Z.

21.2.4. *Claim. One has the following properties:*

(1) $\overline{\Phi}_{Z,\lambda}(M)$ *cuts the divisor* (∞) *of* $\mathbb{P}(N_Z(X))$ *only over* $Z \smallsetminus Z'$*;*

(2) *if* $\lambda = \rho_Z(M)$*, then* $\overline{\Phi}_{Z,\lambda}(M)$ *cuts the divisor* (∞) *of* $\mathbb{P}(N_Z(X))$ *only over points of intersection of* Z *and* E *or* D*;*

(3) *if the connection has a nice formal structure at a crossing point* Q *of* Z *and some irreducible component* W *of* E *or of* D*, then*

$$(\overline{\Phi}_{Z,\lambda}(M), (\infty))_Q \leqslant \mu_{(\lambda)}\ \rho_W(M),$$

where $(\overline{\Phi}_{Z,\lambda}(M), (\infty))_Q$ *is the intersection multiplicity of* $\overline{\Phi}_{Z,\lambda}(M)$ *and* (∞) *at* Q*.*

The first item follows directly from the equations of the divisor $\Phi_{Z,\lambda}(M)$. For the second point, let Q be a point of Z which is not a crossing point with E or D, and consider a local coordinate x' on Z having a zero of order one on Q (then the formal completion of Y' at Q is isomorphic to $\operatorname{Spf} R[[x, x']]$). From the equation of $\Phi_{Z,\lambda}(M)$ (up to ramification in order to have integral orders) we deduce that

$$(\overline{\Phi}_{Z,\lambda}(M), (\infty))_Q = \max\Big(0, -\sum_{j \in J_{(\lambda)}} \operatorname{ord}_{x'} \phi_{j,-\lambda}\Big)$$

and, for $\lambda = \rho_Z(M)$, we have $\operatorname{ord}_{x'} \phi_{j,-\lambda} \geqslant 0$ (by 17.5.3), therefore the intersection multiplicity is 0.

For the last item, we use the notation of 18.2.2 with $y_1 = x$ and y_2 a local equation for W. Let ν_h denote the rank of $M_{\overline{\omega}_h}$, by $H(\lambda)$ the set of indices h such that $\max(0, -\operatorname{ord}_{y_1} \pi_1(\psi_{1,h})) = \lambda$ (in particular, $\mu_{(\lambda)}$ is the sum of ν_h for $h \in H(\lambda)$). For $h \in H(\lambda)$, let $\psi_{1,h,-\lambda}$ be the coefficient of minimal degree in y_1 of the term $\psi_{1,h}$ (or the term $\pi_1(\psi_{1,h})$, since $\lambda > 0$).

Using the second decomposition of 18.2.3 we have

$$\max(0, -\operatorname{ord}_{y_2} \psi_{2,h}) = \max(0, -\operatorname{ord}_{y_2} \pi_2(\psi_{2,h})) \leqslant \rho_Z(M),$$

while using the first decomposition of 18.2.3 we see that the equation of $\overline{\Phi}_{Z,\lambda}(M)$ is given by

$$\varphi_\lambda(y_1) = \prod_{h \in H(\lambda)} (y_1^\lambda - \psi_{1,h,-\lambda})^{\nu_h},$$

so that $(\overline{\Phi}_{Z,\lambda}(M), (\infty))_Q = \max\big(0, -\sum_{h \in H(\lambda)} \nu_h \operatorname{ord}_{y_2} \psi_{1,h,-\lambda}\big)$.

Therefore, we have to prove that

$$\max(0, -\operatorname{ord}_{y_2} \psi_{1,h,-\lambda}) \leqslant \max(0, -\operatorname{ord}_{y_2} \psi_{2,h}).$$

This follows from the fact that $\operatorname{ord}_{y_2} \psi_{1,h,-\lambda} \geqslant \operatorname{ord}_{y_2} \psi_{1,h}$ and the following relations where the integrability condition is used:

$$\begin{aligned}\max(0, -\operatorname{ord}_{y_2} \psi_{1,h}) = \max(0, -\operatorname{ord}_{y_2} \vartheta_2 \psi_{1,h}) &= \max(0, -\operatorname{ord}_{y_2} \vartheta_1 \psi_{2,h}) \\ &\leqslant \max(0, -\operatorname{ord}_{y_2} \psi_{2,h}).\end{aligned}$$

$\square$

21.2.5. We finally prove Lemma 21.2.2. Let π a finite sequence of formal blow-ups as above. The exceptional divisor E is a tree, and we consider the dual tree $\mathbb{T}$ whose vertices v are the components E_v of $E = \pi^{-1}P$ (there is an edge between v and w if and only if E_v cuts E_w). Any component D_i of the divisor D can be indexed using the vertex v of the component E_v of the exceptional divisor with $D'_i \cap E_v \neq \emptyset$. Let V be the finite set of vertices involved in that indexing, and v_0 the vertex with $E_{v_0} \cap C' \neq \emptyset$. Let A be the opposite of the symmetric matrix of intersection for the E_v's with $v \in \mathbb{T}$: the diagonal terms are $A_{vv} = -\deg(N_{E_v}Y')$ and the other terms A_{vw} are -1 if $E_v \cap E_w \neq \emptyset$ and 0 otherwise. The inverse matrix of A has the following property: for each $v \in \mathbb{T}$, let C_v be a germ of curve which cuts transversally E_v outside the crossing points. Then $(A^{-1})_{u,v} = (\pi C_u, \pi C_v)_P$, and in particular it is non-negative (the proof of this fact, by induction on the number of blow-ups in π, is reported for example in [91, I.3.2.8]).

Now, let Z be as in 21.2.3 an irreducible component of E. For any closed curve $T \subseteq \mathbb{P}(N_Z(X))$ finite of degree δ over Z,

$$-\deg(N_Z(X))\,\delta = T\cdot(\infty) - T\cdot(0) \leqslant T\cdot(\infty).$$

Using as T the components of $\overline{\Phi}_{Z,\lambda}(M)$, which is finite over Z of degree $\lambda\mu_{(\lambda)}$, for $\lambda = \rho_Z(M')$ and using 21.2.4 we have

$$-\deg(N_Z(X))\;\rho_Z(M') \leqslant \sum_{W:W\cap Z\neq\emptyset} \rho_W(M').$$

Hence, for any irreducible component E_v of E we have

$$\sum_{w\in\mathbb{T}} A_{vw}\;\rho_{E_w}(M') \leqslant \begin{cases} \rho_{D'_v}(M'), & \text{if } v\in V,\\ 0, & \text{if } v\notin V.\end{cases}$$

Using $C_{v_0} = C'$ and $C_v = D'_v$ (for $v \in V$), we deduce that

$$\rho_{E_{v_0}}(M') \leqslant \sum_{v\in V}(A^{-1})_{v_0 v}\;\rho_{D'_v}(M') = \sum_{v\in V}(C, D_v)_P\;\rho_{D_v}(M).$$

On the other hand, by 17.6.2,

$$\rho_Q(M|_{\widetilde{C}}) = \rho_{P'}(M'|_{C'}) \leqslant \rho_{E_{v_0}}(M'),$$

which proves the lemma, and hence the theorem. $\square$

Chapter VII

de Rham cohomology and Gauss-Manin connection

Introduction

This chapter opens the cohomological part of the book. It focuses on the (higher) direct images of a locally free $\mathcal{O}_X$-module with integrable connection $(\mathcal{M}, \nabla)$ with respect to a smooth morphism $f : X \to S$. These are $\mathcal{O}_S$-modules with integrable connection (the Gauss-Manin connection), defined as derived functors of the functor $\mathcal{M} \mapsto f_*\mathcal{M}^{\nabla_{X/S}}$ of the $\mathcal{O}_S$-module of relative horizontal solutions.

After briefly reviewing the Gauss hypergeometric equation as an $R^1 f_*(\mathcal{M}, \nabla)$, we propose a number of descriptions (or equivalent definitions) of $R^q f_*(\mathcal{M}, \nabla)$, notably in terms of the relative de Rham complex, and indicate general properties: flat base change, vanishing, computation for an affine f of relative dimension 1 in terms of solutions and cosolutions.

22 Hypergeometric equation and Euler representation

We outline here Euler's integral representation of the hypergeometric function. This is a special case and the main motivation for the study of the higher direct images of connections under a smooth morphism.

Let a, b, c be positive rational numbers with $c > a$, and let

$$u = t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}$$

be the algebraic function of t which occurs as the integrand of Euler's integral representation of the hypergeometric function

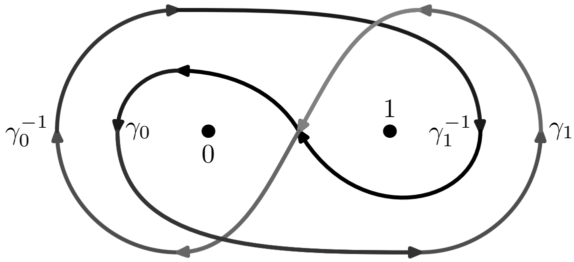
$${}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u\, dt.$$

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*, Progress in Mathematics 189, https://doi.org/10.1007/978-3-030-39719-7_7

Up to a multiplicative constant, this coincides with the (Pochhammer) loop integral

$$y = \int_\gamma u\,dt,$$

where γ is a double contour around 0 and 1.



The derivatives of y then have similar integral representations

$$\partial_x^i y = \int_\gamma u\,v_i\,dt, \quad v_0 = 1,\ v_1 = \frac{bt}{1-tx},\ v_2 = \frac{b(b+1)t^2}{(1-tx)^2}, \dots,\ v_i \in \mathbb{Q}\big[x, t, \frac{1}{1-tx}\big].$$

Now let S be $\operatorname{Spec}\mathbb{Q}[x, \frac{1}{x}, \frac{1}{1-x}]$, let X be the complement in the affine line over S of the étale divisor D given by $t(1-t)(1-tx) = 0$, and let $f : X \to S$ be the natural projection.

The connection

$$\nabla = u^{-1} \circ d \circ u = d + \frac{a-1}{t} - \frac{c-a-1}{1-t} + \frac{bx}{1-xt} : \mathcal{O}_X \longrightarrow \Omega^1_X$$

induces a relative connection

$$\nabla_{X/S} := u^{-1} \circ d_{X/S} \circ u : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$$

by considering only derivatives with respect to the "vertical" variable t. Then the linear map

$$\omega \in f_*\Omega^1_{X/S} \longmapsto \int_\gamma u\,\omega$$

factors through the cokernel of $f_*\nabla_{X/S}$ by Stokes' theorem:

$$\int_\gamma u\nabla_{X/S}(v) = \int_\gamma d_{X/S}(uv) = 0.$$

It turns out that $H^1_{\mathrm{DR}}(X/S, \nabla_X) := u \cdot \operatorname{Coker} f_*\nabla_{X/S}$ is free of rank 2 on S, generated by udt and $uv_1\,dt$.

On the other hand, as a consequence of the fact that $\nabla_{X/S}$ comes from an "absolute" connection ∇, one gets a connection on $H^1_{\mathrm{DR}}(X/S, \nabla_X)$. In fact, the class of udt is a cyclic vector and the connection is expressed by the hypergeometric operator

$$\Lambda_{a,b,c} := x(1-x)\,\partial_x^2 + (c-(a+b+1)x)\,\partial_x - ab = (a+x\partial_x)(b+x\partial_x) - (c+x\partial_x)(1+x\partial_x)\frac{1}{x}.$$

Indeed, one has

$$\Lambda_{a,b,c}(u)\,dt = u\nabla_{X/S}\left(\frac{bt(1-t)}{1-tx}\right),$$

which shows that $\Lambda_{a,b,c}(y)=0$, hence $\Lambda_{a,b,c}({}_2F_1(a,b,c;x))=0$.

23 de Rham cohomology and the Gauss-Manin connection

23.1 Direct image and higher direct images

23.1.1. In 5.2, we have defined the direct image of an integrable connection under an étale morphism. We now extend the construction to an arbitrary *smooth* morphism $f : X \to S$. We recall the (locally split) exact sequence of differentials

$$0 \longrightarrow f^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

and the dual exact sequence of tangent sheaves (sheaves of derivations)

$$0 \longrightarrow \mathcal{T}_{X/S} \longrightarrow \mathcal{T}_X \longrightarrow f^*\mathcal{T}_S \longrightarrow 0.$$

For any $(\mathcal{E},\nabla)$ in $\mathbf{MIC}(X)$ we may restrict the action of $\nabla : \mathcal{T}_X \to \mathcal{E}nd(\mathcal{E})$ to $\mathcal{T}_{X/S}$, yielding $\nabla_{X/S} : \mathcal{T}_{X/S} \to \mathcal{E}nd(\mathcal{E})$, so that $\mathcal{E}^{\nabla_{X/S}}$ admits an action of $f^*\mathcal{T}_S$, and finally $f_*\mathcal{E}^{\nabla_{X/S}}$ is an element of $\mathbf{MIC}(S)$.

23.1.2. This give rise to a functor

$$\begin{aligned} R^0_{\mathrm{DR}}f_* : \mathbf{MIC}(X) &\longrightarrow \mathbf{MIC}(S)\\ (\mathcal{E},\nabla) &\longmapsto (f_*\mathcal{E}^{\nabla_{X/S}}, \text{induced connection}) \end{aligned}$$

which is left-exact. If f is étale, we simply write f_* instead of $R^0_{\mathrm{DR}}f_*$, and we recover the definition in 5.2.

Proposition 23.1.3. *Let $f : X \to S$ be a smooth morphism of smooth k-varieties with connected geometric fibers. Then for any coherent $(\mathcal{E},\nabla) \in \mathbf{MIC}(X)$, the natural morphism $f^*R^0_{\mathrm{DR}}f_*(\mathcal{E},\nabla) \to (\mathcal{E},\nabla)$ is a monomorphism.*

Proof. We shall write ∇ instead of $\nabla_{X/S}$. Let x be any closed point of X. We first show that $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,f(x)}} \mathcal{E}_x^\nabla \to \mathcal{E}_x$ is injective. Since the geometric fibers of f are connected, $\mathrm{Frac}(\mathcal{O}_{X,x})$ is a regular extension of $\mathrm{Frac}(\mathcal{O}_{S,f(x)})$ and

$$\mathrm{Frac}(\mathcal{O}_{X,x}) \otimes_{\mathrm{Frac}(\mathcal{O}_{S,f(x)})} \mathrm{Frac}(\widehat{\mathcal{O}}_{S,f(x)}) \longrightarrow \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})$$

is injective; hence,

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,f(x)}} \widehat{\mathcal{O}}_{S,f(x)} \longrightarrow \widehat{\mathcal{O}}_{X,x}$$

is itself injective. Since $\mathcal{E}_x$ is finitely generated, it embeds into its $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{E}}_x$, and $\mathcal{E}_x^\nabla \subseteq \widehat{\mathcal{E}}_x^\nabla$. We are thus reduced to showing that $\widehat{\mathcal{O}}_{X,x} \otimes_{\widehat{\mathcal{O}}_{S,f(x)}} \widehat{\mathcal{E}}_x^\nabla \to \widehat{\mathcal{E}}_x$ is injective. This is in fact an isomorphism by the argument of Taylor series in [61, 8.9]. We have shown that $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{E}^\nabla \to \mathcal{E}$ is injective, and since $f^* R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} f^{-1} f_* \mathcal{E}^\nabla$, we conclude by noticing that f is flat and $f^{-1} f_* \mathcal{E}^\nabla \to \mathcal{E}^\nabla$ is injective (because $f^{-1} f_* \mathcal{E} \to \mathcal{E}$ is, $\mathcal{E}$ being locally free and X connected). □

Definition 23.1.4. *The q-th derived functor of $R^0_{\mathrm{DR}} f_*$ is denoted by*

$$R^q_{\mathrm{DR}} f_* : \mathbf{MIC}(X) \longrightarrow \mathbf{MIC}(S),$$

i.e., we define $R^q_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) := R^q(R^0_{\mathrm{DR}} f_*)(\mathcal{E}, \nabla)$. We call it the q-th higher (de Rham) direct image functor.

If S is a point, we also write $H^q_{\mathrm{DR}}(X, (\mathcal{E}, \nabla))$ for the k-vector space $R^q_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$.

Remark 23.1.5. For a morphism $f : X \to S$ of smooth algebraic varieties, a direct image functor f_+ is defined from the derived category of left $\mathcal{D}_X$-modules to the derived category of left $\mathcal{D}_Y$-modules by means of a transfer module, as in [19] or [16]. In the equivalence of categories between left $\mathcal{D}$-modules and integrable connections, the construction of f_+ corresponds to the notion of higher (de Rham) direct image up to a shift. This has been for long considered a well-known fact. The first explicit proof of this assertion appears in the paper [37] and uses as main technical tool M. Saito's equivalence between the derived category of $\mathcal{D}$-modules and a localized category of differential complexes. A more elementary proof was then given in [27].

23.2 de Rham and Spencer complexes

In preparation of more concrete descriptions of the Gauss-Manin connection, we record the notion of de Rham and Spencer complexes in the context of $\mathcal{D}_X$-modules: namely

$$\begin{aligned}&\mathrm{DR}^\bullet_{X/S}(\mathcal{D}_X)\\ &:= [\mathcal{D}_X \longrightarrow \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \cdots \longrightarrow \Omega^d_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}_X]\end{aligned}$$

(the maps being defined as in the de Rham complex of $\mathcal{D}_X$ as a left $\mathcal{D}_X$-module). This is a locally free left resolution of $\omega_{X/Y} = \Omega^d_{X/S}$ in the category of right $\mathcal{D}_X$-modules, and

$$\begin{aligned}&\mathrm{Sp}^\bullet_{X/S}(\mathcal{D}_X)\\ &:= [\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^d \mathcal{T}_{X/S} \longrightarrow \cdots \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^2 \mathcal{T}_{X/S} \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{X/S} \longrightarrow \mathcal{D}_X]\end{aligned}$$

where the differentials are defined by

$$\begin{aligned}\partial\otimes\xi_1\wedge\cdots\wedge\xi_k\longmapsto &\sum_i(-)^{i-1}\partial\xi_i\otimes\xi_1\wedge\cdots\widehat{\xi_i}\cdots\wedge\xi_k\\ &+\sum_{i<j}(-)^{i+j}\partial[\xi_i,\xi_j]\otimes\xi_1\wedge\cdots\widehat{\xi_i}\cdots\widehat{\xi_j}\cdots\wedge\xi_k\end{aligned}$$

(for local sections ∂ of $\mathcal{D}_X$ and ξ_i of $\mathcal{T}_{X/S}$) which is a locally free left resolution (locally a Koszul complex) of $f^*\mathcal{D}_S\cong\mathcal{D}_X/(\mathcal{D}_X\mathcal{T}_{X/S})$ in the category of left $\mathcal{D}_X$-modules.

23.2.1. For any left $\mathcal{D}_X$-module $\mathcal{M}$ we have

$$\mathrm{DR}^\bullet_{X/S}(\mathcal{M})\cong\mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^\bullet_{X/S}(\mathcal{D}_X),\mathcal{M}),$$

so that this complex represents $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S,\mathcal{M})$ (right derived functor of the solution functor $\mathcal{M}^{\nabla_{X/S}}=\mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S,\mathcal{M})$). This shows that, for any complex of $\mathcal{O}_S$-modules representing $\mathbf{R}f_*\mathrm{DR}^\bullet_{X/S}(\mathcal{M})$, the cohomology sheaves carry an integrable S/k connection. This is the Gauss-Manin connection.

We now recall the construction of $R^j_{\mathrm{DR}}f_*(\mathcal{E},\nabla)$ given in [65] and [61] using the higher direct images $\mathbf{R}^qf_*$ (for complexes of abelian sheaves or, what amounts to the same, of $f^{-1}\mathcal{O}_S$-modules on X) of the relative de Rham complex $\mathrm{DR}_{X/S}(\mathcal{E},\nabla)$) of $(\mathcal{E},\nabla)$, endowed with the Gauss-Manin connection.

23.2.2. Let us filter the absolute de Rham complex $\mathrm{DR}_{X/k}(\mathcal{E},\nabla)$ by the subcomplexes

$$F^p=\mathrm{Im}\left(f^*(\Omega^p_S)\otimes_{\mathcal{O}_X}\Omega^{\bullet-p}_X\otimes_{\mathcal{O}_X}\mathcal{E}\longrightarrow\Omega^\bullet_X\otimes_{\mathcal{O}_X}\mathcal{E}\right)$$

so that

$$gr^p=F^p/F^{p+1}\cong f^*(\Omega^p_S)\otimes_{\mathcal{O}_X}\mathrm{DR}^{\bullet-p}_{X/S}(\mathcal{E},\nabla).$$

One now considers the spectral sequence of the direct image functor f_* w.r.t. the finitely filtred complex $\mathrm{DR}_{X/k}(\mathcal{E},\nabla)$:

$$\begin{aligned}E_1^{p,q}&=\mathbf{H}^{p+q}(f_*gr^p)=\mathbf{R}^{p+q}f_*(f^*(\Omega^p_S)\otimes_{\mathcal{O}_X}\mathrm{DR}^{\bullet-p}_{X/S}(\mathcal{E},\nabla))\\&=\Omega^p_S\otimes_{\mathcal{O}_S}\mathbf{R}^qf_*\mathrm{DR}^\bullet_{X/S}(\mathcal{E},\nabla).\end{aligned}$$

Definition 23.2.3. *The differentials $E_1^{p,q}\to E_1^{p+1,q}$ for $p=0$ gives the Gauss-Manin connection*

$$\nabla^q_S:\mathbf{R}^qf_*\mathrm{DR}^\bullet_{X/S}(\mathcal{E},\nabla)\longrightarrow\Omega^1_S\otimes_{\mathcal{O}_S}\mathbf{R}^qf_*\mathrm{DR}^\bullet_{X/S}(\mathcal{E},\nabla)$$

whose de Rham complex is exactly the complex $E_1^{\bullet,q}$ (in particular, it is an integrable connection).

By construction of the spectral sequence, we have

Proposition 23.2.4. *The Gauss-Manin connection is the coboundary map in the long cohomology sequence of the* $\mathbf{R}^q f_*$ *arising from the short exact sequence*

$$0 \longrightarrow gr^1 \longrightarrow F^0/F^2 \longrightarrow gr^0 \longrightarrow 0.$$

23.2.5. We show now that the higher direct image $R^q_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is, as an object in $\mathbf{MIC}(S)$, canonically isomorphic to the hypercohomology $\mathbf{R}^q f_* \mathrm{DR}^\bullet_{X/S}(\mathcal{E}, \nabla)$ endowed with the Gauss-Manin connection, that is, the Gauss-Manin connection on $\mathbf{R}^q f_* \mathrm{DR}^\bullet_{X/S}(\mathcal{E}, \nabla)$ gives the q-derived functor of $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$.

Claim. Let $(\mathcal{I}, \nabla)$ *be an injective object in* $\mathbf{MIC}(X)$*. Then* $\mathrm{DR}^\bullet_{X/S}(\mathcal{I}, \nabla)$ *is an injective resolution of* $\mathcal{I}^{\nabla_{X/S}}$*, and* $\mathcal{I}^{\nabla_{X/S}}$ *is a flabby* $\mathcal{O}_X$*-module.*

Proof of the claim. Indeed, since $(\mathcal{I}, \nabla)$ is injective, the functor $\mathrm{Hom}_{\mathcal{D}_X}(-, \mathcal{I})$ is exact, and therefore $\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{I})$ is exact. Applying this functor to the Spencer resolution of $f^*\mathcal{D}_S$

$$Sp_{X/S}(\mathcal{D}_X) \longrightarrow f^*\mathcal{D}_S \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{I}) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(Sp_{X/S}(\mathcal{D}_X), \mathcal{I})$$

and $\mathrm{DR}^\bullet_{X/S}(\mathcal{I}, \nabla) \cong \mathcal{H}om_{\mathcal{D}_X}(Sp_{X/S}(\mathcal{D}_X), \mathcal{I})$ is a resolution of $\mathcal{I}^{\nabla_{X/S}} \cong \mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{I})$.

For the injectivity, notice that

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathrm{DR}^p_{X/S}(\mathcal{I}, \nabla)) = \mathrm{Hom}_{\mathcal{D}_X}(- \otimes_{\mathcal{O}_X} Sp^p_{X/S}(\mathcal{D}_X), (\mathcal{I}, \nabla))$$

is an exact functor, since $Sp^p_{X/S}(\mathcal{D}_X)$ is locally free over $\mathcal{D}_X$ and $\mathcal{I}$ is injective.

Finally, since $\mathcal{I}$ is injective, for any left $\mathcal{D}_X$-module $\mathcal{G}$ and any open inclusion $j : U \to X$, the canonical inclusion $\mathcal{G}_U \to \mathcal{G}$ gives a surjective morphism $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G}_U, \mathcal{I})$, which shows that the sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})$ is flabby. In particular, using $\mathcal{G} = f^*\mathcal{D}_S$, we have that $\mathcal{I}^{\nabla_{X/S}} = \mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{I})$ is flabby. □

To prove our assertion it is now sufficient to remark that the functors $\mathbf{R}^q f_* \mathrm{DR}^\bullet_{X/S}(\mathcal{E}, \nabla)$ form an effaceable δ-functor ($\mathcal{I}^{\nabla_{X/S}}$ is f_*-acyclic, being flabby) and $\mathbf{R}^0 f_* \mathrm{DR}^\bullet_{X/S}(\mathcal{E}, \nabla)$ coincides with $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$. Then one applies [54, III, 1.3A and 1.4] or [51, II]. □

Remark 23.2.6. Essentially the same argument was used in [53] in terms of derived functors: since $\mathcal{E}^{\nabla_{X/S}} = \mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{E})$, the derived functor is represented by $R\mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{E}) = \mathrm{DR}_{X/S}(\mathcal{E})$. Then the derived functor of the composite $f_*\mathcal{E}^{\nabla_{X/S}}$ is represented by the composite $Rf_* R\mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{E}) = Rf_* \mathrm{DR}_{X/S}(\mathcal{E})$ (since $\mathcal{I}^{\nabla_{X/S}}$ is acyclic for f_* for any injective $\mathcal{I}$).

Remark 23.2.7. By [47] [0_{III}, 12.4.3] applied to the morphism of ringed spaces

$$f : (X, f^{-1}\mathcal{O}_S) \longrightarrow (S, \mathcal{O}_S)$$

and $\mathcal{K}^\bullet = \mathrm{DR}(X/S, (\mathcal{E}, \nabla))$, the sheaf underlying $R^q_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is the sheaf associated to the presheaf

$$\mathcal{H}^q(f, \mathcal{K}^\bullet) : U \longmapsto \mathbf{H}^q(f^{-1}(U), \mathcal{K}^\bullet_{|f^{-1}(U)})$$

on S. We see in particular that the formation of $R^q_{\mathrm{DR}} f_*$ is compatible with Zariski localization on S.

23.3 Some spectral sequences

23.3.1 (Leray spectral sequences). Since $R^q_{\mathrm{DR}} f_*$ is the q-th derived functor of $R^0_{\mathrm{DR}} f_*$, we can write the spectral sequence corresponding to the composition of two maps $g : Y \to X$ and $f : X \to S$. In fact, we have $R^0_{\mathrm{DR}}(f \circ g)_* = R^0_{\mathrm{DR}} f_* \circ R^0_{\mathrm{DR}} g_*$ (and $R^0_{\mathrm{DR}} g_*$ sends injective sheaves into flabby sheaves, acyclic for $R^0_{\mathrm{DR}} f_*$), so we have the composition of derived functors

$$RR^0_{\mathrm{DR}}(f \circ g)_* = RR^0_{\mathrm{DR}} f_* \circ RR^0_{\mathrm{DR}} g_*$$

and the corresponding spectral sequence

$$R^p_{\mathrm{DR}} f_* \circ R^q_{\mathrm{DR}} g_* \Longrightarrow R^{p+q}_{\mathrm{DR}}(f \circ g)_*$$

or, more explicitly,

$$R^p_{\mathrm{DR}} f_*(R^q_{\mathrm{DR}} g_*(\mathcal{G}, \nabla_Y), \nabla^q_X) \Longrightarrow R^{p+q}_{\mathrm{DR}}(f \circ g)_*(\mathcal{G}, \nabla_Y), \tag{23.3.2}$$

where $(\mathcal{G}, \nabla_Y)$ is an object of $\mathbf{MIC}(Y/k)$ and ∇^q_X indicates the Gauss-Manin connection on $R^q_{\mathrm{DR}} g_*(\mathcal{G}, \nabla_Y)$.

23.3.3 (Čech spectral sequences). Let $\{U_\alpha\}$ be an open cover of X (not necessarily finite). In the following lemma, which is a variation of [EGA 0_{III}, 12.4.6], $\mathbf{h}^q(f, \mathcal{K}^\bullet)$ is the presheaf $U \mapsto \mathbf{R}^q f_{|U*}(\mathcal{K}^\bullet_{|U})$, while $h^q(f, \mathcal{K}^j)$ is $U \mapsto R^q f_{|U*}(\mathcal{K}^j_{|U})$, both presheaves on X taking values in the category of *sheaves* of $\mathcal{O}_S$-modules, and $h^q(f, \mathcal{K}^\bullet)$ is the complex of presheaves obtained from a term by term application of the latter. For a presheaf $\mathcal{F}$ (resp. a complex of presheaves $\mathcal{F}^\bullet$) on X, with values in any abelian category, the notation $C^\bullet(\{U_\alpha\}, \mathcal{F})$ (resp. $C^\bullet(\{U_\alpha\}, \mathcal{F}^\bullet)$) refers to the usual Čech complex (resp. bicomplex). If $\mathcal{F}$ is a presheaf of abelian groups on X, we also use the notation $\mathcal{C}^i_f(\{U_\alpha\}, \mathcal{F}) = \bigoplus_{\alpha_0 < \alpha_1 < \cdots < \alpha_i} f_{\underline{\alpha}*}(\mathcal{F}_{|U_{\underline{\alpha}}})$ for the abelian presheaf of (alternating) Čech cochains.

Lemma 23.3.4. *Let $\mathcal{K}^\bullet$ be a complex of $f^{-1}\mathcal{O}_S$-modules on X. There exist two regular spectral functors ${}_IE^{\bullet,\bullet}_\bullet$ and ${}_{II}E^{\bullet,\bullet}_\bullet$ in $\mathcal{K}^\bullet$ taking values in the category of*

$\mathcal{O}_S$-modules and converging to the relative hypercohomology $\mathbf{R}^\bullet f_*(\mathcal{K}^\bullet)$. The E_1 terms are given by

$$
\begin{aligned}
{}_IE_1^{p,q} &= (C^\bullet(\{U_\alpha\}, h^q(f,\mathcal{K}^\bullet)))^p_{\mathrm{tot}} = \bigoplus_{i+j=p} C^i(\{U_\alpha\}, h^q(f,\mathcal{K}^j)) = \bigoplus_{\substack{i+j=p \\ \alpha_0<\alpha_1<\cdots<\alpha_i}} R^q f_{\underline{\alpha}*}(\mathcal{K}^j_{|U_{\underline{\alpha}}}),\\
{}_{II}E_1^{p,q} &= C^p(\{U_\alpha\}, \mathbf{h}^q(f,\mathcal{K}^\bullet)) = \bigoplus_{\alpha_0<\alpha_1<\cdots<\alpha_p} \mathbf{R}^q f_{\underline{\alpha}*}(\mathcal{K}^\bullet_{|U_{\underline{\alpha}}}),
\end{aligned}
$$

respectively. If $\mathcal{K}^\bullet$ is bounded from below, the above spectral sequences are biregular.

Proof. Following [47, 0_{III}, 12.4.6], we consider an injective Cartan-Eilenberg resolution $\mathcal{L}^{\bullet,\bullet}$ of $\mathcal{K}^\bullet$ and the tricomplex of $\mathcal{O}_S$-modules

$$\mathcal{C}_f^\bullet(\{U_\alpha\}, \mathcal{L}^{\bullet,\cdot}) = \bigoplus_{i,j,k} \mathcal{C}_f^i(\{U_\alpha\}, \mathcal{L}^{j,k}) \ .$$

We first regard $\mathcal{C}_f^\bullet(\{U_\alpha\}, \mathcal{L}^{\bullet,\cdot})$ as a bicomplex for the degrees $(i, j+k)$. The second spectral sequence of this bicomplex is regular and degenerate, since H_I^q is the associated sheaf to the presheaf $V \mapsto H^q(\{U_\alpha \cap f^{-1}(V)\}, \mathcal{L}^{j,k})$ on S, which is 0 for $q>0$, $\mathcal{L}^{j,k}_{|f^{-1}(V)}$ being flasque. So, $H^n(\mathcal{C}_f^\bullet(\{U_\alpha\}, \mathcal{L}^{\bullet,\cdot})_{\mathrm{tot}}) \xrightarrow{\sim} \mathbf{R}^n f_*\mathcal{K}^\bullet$. We then define ${}_IE_\bullet^{\bullet,\bullet}$ as the first spectral sequence of $\mathcal{C}_f^\bullet(\{U_\alpha\}, \mathcal{L}^{\bullet,\bullet})$, viewed as a bicomplex for the degrees $(i+j, k)$ (this is the spectral sequence in the statement of *loc.cit.*). Here $H_{II}^q = \bigoplus_{i,j} H^q(\mathcal{C}_f^i(\{U_\alpha\}, \mathcal{L}^{j,\cdot})) = \bigoplus_{i,j} C^i(\{U_\alpha\}, h^q(f,\mathcal{K}^j))$, since $\mathcal{L}^{j,\cdot}_{|U}$ is an injective resolution of $\mathcal{K}^j_{|U}$. As ${}_{II}E_\bullet^{\bullet,\bullet}$ we take instead the first spectral sequence of $\mathcal{C}_f^\bullet(\{U_\alpha\}, \mathcal{L}^{\bullet,\bullet})$, viewed as a bicomplex for the degrees $(i, j+k)$. $\square$

Remark 23.3.5 (Zariski spectral sequences). In the special case of $\mathcal{K}^\bullet = \mathrm{DR}_{X/S}(\mathcal{E},\nabla)$, the spectral sequence ${}_{II}E_\bullet^{\bullet,\bullet}$ of the lemma is the *Zariski spectral sequence* of $\mathcal{O}_S$-modules as considered in [61, 3.5.1.0]:

$$E_1^{p,q} = \bigoplus_{\alpha_0<\alpha_1<\cdots<\alpha_p} R^q_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}} \Longrightarrow R^{p+q}_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \tag{23.3.6}$$

If we replace $\mathrm{DR}_{X/S}(\mathcal{E},\nabla)$ by the short exact sequence in 23.2.4, and use the fact that hypercohomology of a direct image is a cohomological functor, we obtain a coboundary morphism of spectral sequences

$$
\begin{tikzcd}
E_1^{p,q} \cong E_1^{p,q}(gr^0) \arrow[r, Rightarrow] \arrow[d] & R^{p+q}_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \arrow[d] \\
E_1^{p,q+1}(gr^1) \cong \Omega^1_{S/k} \otimes_{\mathcal{O}_S} E_1^{p,q}(gr^0) \arrow[r, Rightarrow] & \Omega^1_{S/k} \otimes_{\mathcal{O}_S} R^{p+q}_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \cong \mathbf{R}^{p+q+1} f_*(gr^1)
\end{tikzcd}
$$

which expresses the fact that 23.3.6 "is" a spectral sequence in $\mathbf{MIC}(S)$.

23.4 Local construction of the Gauss-Manin connection

This subsection, which will not be used in the sequel, is devoted to a local description of the Gauss-Manin connection. The E_2^{pq} term of the spectral sequence ${}_IE_\bullet^{\bullet,\bullet}$ of Lemma 23.3.4 is

$${}_IE_2^{p,q} = \text{total cohomology of degree } p \text{ of the bicomplex } C^\bullet(\{U_\alpha\}, h^q(f, \mathcal{K}^\bullet)) \ .$$

This implies, in particular, that if each U_α is affine, and if $\mathcal{E}$ is quasi-coherent, then $R^j_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ *may be calculated "à la Čech"*, as the j-th (total) homology $\mathcal{O}_S$-module of the bicomplex of $\mathcal{O}_S$-modules $\mathcal{C}_f^\bullet(\{U_\alpha\}, \Omega^\bullet_{X/S} \otimes \mathcal{E})$, of alternating Čech cochains on the nerve of the cover $\{U_\alpha\}$.

Proposition 23.4.1 (§3 of [65])**.** *The Gauss-Manin connection comes, upon passage to homology, from a* (*not integrable*) *connection on the bicomplex*

$$C^\bullet(\{U_\alpha\}, h^q(f, \mathcal{K}^\bullet)) \quad \textit{for} \quad \mathcal{K}^\bullet = \mathrm{DR}_{X/S}(\mathcal{E}, \nabla).$$

Namely, if ∂ is a vector field on S, and if ∂_α is a lifting of ∂ on U_α, the value of this connection at ∂ can be expressed as the sum of the Lie derivative $\mathrm{L}(\partial_{\alpha_0})$ *and the* interior product $I(\partial_{\alpha_0} - \partial_{\alpha_1})$ (*up to sign*)*.*

Proof. The proof in [65] depends on a local calculation. We present here an alternative argument, based on functoriality.

Denote by d_{DR} the differential of the de Rham complex $\Omega^\bullet_{X/S} \otimes \mathcal{E}$ or of $C_f^p(\{U_\alpha\}, \Omega^\bullet_{X/S} \otimes \mathcal{E})$, and by $\check{d}$ the Čech differential on $C_f^p(\{U_\alpha\}, \Omega^\bullet_{X/S} \otimes \mathcal{E})$. We follow the convention of [65] (not the one in [47]) by putting the total differential $\delta = d_{\mathrm{DR}} + (-)^q \check{d}$ on the term

$$C_f^{p,q} = C_f^p(\{U_\alpha\}, \Omega^q_{X/S} \otimes \mathcal{E})$$

of the bicomplex. Denote by $C_f^\bullet = C_f^\bullet(\mathcal{E}, \nabla)$ the associated simple complex, so that

$$H^i(C_f^\bullet(\mathcal{E}, \nabla)) \cong \mathbf{R}^i f_*(\Omega^\bullet_{X/S} \otimes \mathcal{E}),$$

functorially in $(\mathcal{E}, \nabla)$. For each vector field ∂ on S, we define an additive endomorphism $\widetilde{\aleph}(\partial)$ of C_f^i by

$$\widetilde{\aleph}(\partial) = \mathrm{L}(\partial_{\alpha_0}) + (-)^q I(\partial_{\alpha_0} - \partial_{\alpha_1})$$

on $C_f^{i-q,q}$. This endomorphism satisfies the Leibniz rule, because the Lie derivative does; hence the collection of $\widetilde{\aleph}(\partial)$'s defines a connection $\widetilde{\aleph}$ on $C_f^i(\mathcal{E}, \nabla)$, which is functorial in $(\mathcal{E}, \nabla)$ (because the Lie derivative $\mathrm{L}(\partial_{\alpha_0}) : \Omega^q_{U_{\alpha_0}/S} \otimes \mathcal{E}_{|U_{\alpha_0}} \to \Omega^q_{U_{\alpha_0}/S} \otimes \mathcal{E}_{|U_{\alpha_0}}$ is functorial).

On the other hand, we have the formulas

(i) $[\mathrm{L}(\partial_{\alpha_0}), d_{\mathrm{DR}}] = 0$,

(ii) $\mathrm{L}(\partial_{\alpha_0}) - \mathrm{L}(\partial_{\alpha_1}) = d_{\mathrm{DR}} \circ I(\partial_{\alpha_0} - \partial_{\alpha_1}) + I(\partial_{\alpha_0} - \partial_{\alpha_1}) \circ d_{\mathrm{DR}}$,

(iii) $I(\partial_{\alpha_0} - \partial_{\alpha_1}) + I(\partial_{\alpha_1} - \partial_{\alpha_2}) + I(\partial_{\alpha_2} - \partial_{\alpha_0}) = 0$.

From (ii) we deduce that

(iv) $[\mathrm{L}(\partial_{\alpha_0}), (-)^q \check{d}] + [(-)^q I(\partial_{\alpha_0} - \partial_{\alpha_1}), d_{\mathrm{DR}}] = 0$,

and (iii) implies

(v) $[(-)^q I(\partial_{\alpha_0} - \partial_{\alpha_1}), (-)^q \check{d}] = 0$.

Formulas (i), (iv), (v) put together show that $[\widetilde{\aleph}(\partial), \delta] = 0$, i.e., $\widetilde{\aleph}(\partial)$ is an (additive) endomorphism of the complex $C_f^\bullet(\mathcal{E}, \nabla)$. We thus obtain functors

$$\begin{aligned} \aleph^i : \mathbf{MIC}(X) &\longrightarrow \mathbf{MC}(S) \\ (\mathcal{E}, \nabla) &\longmapsto (H^i(C_f^\bullet(\mathcal{E}, \nabla)) \cong \mathbf{R}^i f_*(\Omega^\bullet_{X/S} \otimes \mathcal{E}), H^i(\widetilde{\aleph})), \end{aligned}$$

where $\mathbf{MC}(S)$ stands for the abelian category of $\mathcal{O}_S$-modules with (not necessarily integrable) connection.

By functoriality of $\widetilde{\aleph}$ with respect to $(\mathcal{E}, \nabla)$, it is clear that the collection of the $\aleph^i$ forms a δ-functor. This δ-functor is effaceable (because any $(\mathcal{E}, \nabla)$ imbeds into an injective $(\mathcal{I}, \nabla)$ of $\mathbf{MIC}(X)$ and $\mathbf{R}^i f_*(\Omega^\bullet_{X/S} \otimes \mathcal{I}) = 0$, for $i > 0$), hence $\aleph^i = R^i \aleph^0$. Let $\omega : \mathbf{MIC}(S) \to \mathbf{MC}(S)$ stand for the (exact) imbedding. We have $\aleph^0 = \omega \circ R^0_{\mathrm{DR}} f_*$, because $\widetilde{\aleph}(\partial)$ is just the Gauss-Manin connection on $H^0(C_f^\bullet) \cong f_* \mathcal{E}^{\nabla_{|\mathcal{D}er_{X/S}}}$, being induced by the Lie derivative in the direction ∂ acting on $\mathcal{E}$. We conclude that $\aleph^i = R^i(\omega \circ R^0_{\mathrm{DR}} f_*) = \omega \circ R^i_{\mathrm{DR}} f_*$, which takes values in $\mathbf{MIC}(S)$. $\square$

23.5 Flat base change

Let

$$\begin{array}{ccc} X^\sharp & \xrightarrow{u^\sharp} & X \\ {\scriptstyle f^\sharp}\downarrow & & \downarrow{\scriptstyle f} \\ S^\sharp & \xrightarrow{u} & S \end{array}$$

be a fibered square, u being a morphism of smooth k-varieties. We have $u^{\sharp *} \Omega^\bullet_{X/S} \simeq \Omega^\bullet_{X^\sharp/S^\sharp}$.

When we try to define $u^{\sharp *} \Omega^\bullet_{X/S}$, a difficulty occurs because the differentials of the de Rham complex are not $\mathcal{O}_X$-linear. One can overcome it in exactly the same way we did in order to define inverse images of connections (see 5.1): the differentials d in $\mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$ are differential operators of order one (see 4); they factor as

$$\mathcal{E} \otimes \Omega^n_{X/S} \xrightarrow{d_{\mathrm{univ}}} \mathcal{P}^1_{X/S}(\mathcal{E} \otimes \Omega^n_{X/S}) \xrightarrow{\bar{d}} \mathcal{E} \otimes \Omega^{n+1}_{X/S}$$

where the first map is the universal differential operator of order one, and the second map $\bar{d}$ is $\mathcal{O}_X$-linear. One then defines $u^{\sharp *}\Omega^\bullet_{X/S}$ using $u^{\sharp *}(\bar{d})$.

One has a canonical isomorphism $u^{\sharp *}\mathrm{DR}_{X/S}(\mathcal{E},\nabla) \simeq \mathrm{DR}_{X^\sharp/S^\sharp}(u^{\sharp *}(\mathcal{E},\nabla))$.

Moreover, if $\mathcal{E}$ is quasi-coherent, and if u is *flat* (e.g., an étale localization), we have an isomorphism

$$u^* R^q_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \xrightarrow{\sim} R^q_{\mathrm{DR}} f^\sharp_*(u^{\sharp *}(\mathcal{E},\nabla)),$$

by flat base change for hypercohomology ([47]). One can also choose an affine open cover of X, use the first spectral sequence of Lemma 23.3.4 (where ${}_I E_1^{p,q} = 0$ for $q > 0$), and flat base change for $f_{\underline{\alpha}*}(\Omega^j_{U_{\underline{\alpha}}/S})$.

23.6 Vanishing and computation

Assume $\mathcal{E}$ is quasi-coherent and let d be the maximum among the dimensions of the fibers of f. Then

$$R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla) = 0 \quad \text{for} \quad j \notin [0, d + \dim X],$$

as can be seen from the Hodge-de Rham spectral sequence

$$E_1^{p,q} = R^q f_*(\Omega^p_{X/S} \otimes \mathcal{E}) \Longrightarrow R^{p+q}_{\mathrm{DR}} f_*(\mathcal{E},\nabla).$$

Assume moreover that f is *affine*. Then

$$R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla) = 0 \quad \text{for} \quad j \notin [0, d].$$

To see this, let us look at the first spectral sequence of hypercohomology [47, 0_{III}, 12.4.1.1] for $\mathcal{K}^\bullet = \mathrm{DR}_{X/S}(\mathcal{E},\nabla)$

$$\mathcal{E}_2^{p,q} = \mathcal{H}^p(R^q f_* \mathrm{DR}_{X/S}(\mathcal{E},\nabla)) \Longrightarrow \mathbf{R}^{p+q} f_* \mathrm{DR}_{X/S}(\mathcal{E},\nabla).$$

Since f is affine, $R^q f_* \mathrm{DR}_{X/S}(\mathcal{E},\nabla) = 0$ for $q > 0$, hence the spectral sequence degenerates at $\mathcal{E}_2$, so that

$$R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \cong \mathcal{H}^j(f_* \mathrm{DR}_{X/S}(\mathcal{E},\nabla)).$$

In particular $R^1_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \cong \mathrm{Coker} f_*(\nabla_{X/S})$, if $d = 1$[1].

Suppose moreover S is *affine* (hence so is X); then $R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla)$ may thus be computed as the $\mathcal{O}_S$-module attached to the j-th cohomology $\Gamma(S,\mathcal{O}_S)$-module of the global relative de Rham complex

$$\mathcal{E}(X) \longrightarrow \mathcal{E}(X) \otimes \Omega^1_{X/S}(X) \longrightarrow \cdots \ .$$

[1] This should not be confused with $f_*(\mathrm{Coker}\nabla_{X/S})$, which arises in the *second* spectral sequence of hypercohomology

$$\mathcal{E}_2^{p,q} = R^p f_* \mathcal{H}^q(\mathrm{DR}_{X/S}(\mathcal{E},\nabla)) \Rightarrow \mathbf{R}^{p+q} f_* \mathrm{DR}_{X/S}(\mathcal{E},\nabla).$$

We shall apply this computation in the case where f is an elementary fibration.

More generally, if S is affine and $\{U_\alpha\}$ is an affine open cover of X, then $R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla)$ is the $\mathcal{O}_S$-module attached to the j-th total cohomology group of the Čech bicomplex $C^\bullet(\{U_\alpha\},\Omega^\bullet_{X/S}\otimes\mathcal{E})$.

For algorithmic aspects of the computation of de Rham cohomology and the Gauss-Manin connection, see [96][92][93][71].

24 Index formula

24.1 Deligne's global index formula on algebraic curves

24.1.1. Let X be a smooth projective complex algebraic curve of genus g_X, and let $Z \subset X(k)$ be a finite subset of closed points of X; we set $U := X\backslash Z$ and $j : U \hookrightarrow X$ for the open embedding. Let $\mathcal{E}$ be a coherent and locally free $\mathcal{O}_U$-module and let $\nabla : \mathcal{E} \to \Omega^1_{U/k} \otimes \mathcal{E}$, be a connection on $\mathcal{E}$.

24.1.2. We recall that the de Rham cohomology of U with coefficients in $(\mathcal{E},\nabla)$ is defined as the hypercohomology of the de Rham complex

$$\begin{aligned} H^\bullet_{\mathrm{DR}}(U;(\mathcal{E},\nabla)) &:= \mathbf{H}^\bullet\big(U, 0 \to \mathcal{E} \xrightarrow{\nabla} \Omega^1_{U/k}\otimes\mathcal{E} \to 0\big) \\ &= H^\bullet_{\mathrm{DR}}(X; j_*(\mathcal{E},\nabla)) = \mathbf{H}^\bullet\big(X, 0 \to j_*\mathcal{E} \xrightarrow{j_*\nabla} \Omega^1_{X/k}\otimes j_*\mathcal{E} \to 0\big). \end{aligned}$$

In particular, for the trivial connection, we have that

$$0 \longrightarrow \mathcal{O}_X(*Z) \xrightarrow{d_{X/k}} \Omega^1_{X/k}(*Z) \longrightarrow 0$$

is, locally at each $z \in Z$, a direct sum of the global subcomplex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d_{X/k}} \Omega^1_{X/k}(\log Z) \longrightarrow 0$$

and of the exact subcomplexes

$$0 \longrightarrow k\,x^j \xrightarrow{d_{X/k}} k\,x^j\,\frac{d\,x}{x} \longrightarrow 0,$$

for $z \in Z$ and $j \leqslant -1$, where x denotes a local parameter at z. Therefore,

$$\begin{aligned} H^\bullet_{\mathrm{DR}}(U/k) &:= \mathbf{H}^\bullet\big(U, 0 \longrightarrow \mathcal{O}_U \xrightarrow{d_{U/k}} \Omega^1_{U/k} \longrightarrow 0\big) \\ &= H^\bullet_{\mathrm{DR}}(X; j_*(\mathcal{O}_U, d_{U/k})) = \mathbf{H}^\bullet\big(X, 0 \longrightarrow \mathcal{O}_X(*Z) \xrightarrow{d_{X/k}} \Omega^1_{X/k}(*Z) \longrightarrow 0\big) \end{aligned}$$

equals, as a graded k-vector space,

$$\mathbf{H}^\bullet(X, 0 \longrightarrow \mathcal{O}_X \xrightarrow{d_{X/k}} \Omega^1_{X/k}(\log Z) \longrightarrow 0) = H^\bullet_{\mathrm{DR}}(X/k) \oplus \Gamma(X, \mathcal{J}_Z^{-1}/\mathcal{O}_X)[-1],$$

where $\mathcal{J}_Z$ denotes the sheaf of ideals of Z in X.

We are interested in a formula relating the Euler-Poincaré characteristic of the de Rham cohomology

$$\chi_{\mathrm{DR}}(U;(\mathcal{E},\nabla)) = \sum_{i=0,1,2} (-1)^i \dim_k H^i_{\mathrm{DR}}(U,(\mathcal{E},\nabla)),$$

to the Euler-Poincaré characteristic of U

$$\chi_{\mathrm{DR}}(U) = \sum_{i=0,1,2} (-1)^i \dim_k H^i_{\mathrm{DR}}(U/k) = \chi_{\mathrm{DR}}(X) - \mathrm{card}(Z),$$

cf. [35, p.111].

Theorem 24.1.3 (global index formula). *In the context of 24.1.1, for any $z \in Z$ we define the* irregularity *of $(\mathcal{E},\nabla)$ at z as $\mathrm{ir}_z(\mathcal{E},\nabla) := \mathrm{ir}(\widehat{j_*\mathcal{E}_z}, \widehat{\nabla}_z)$ (as $\widehat{\mathcal{O}_{U,z}}/k$ differential module, see 17.2.1). In the notation in 24.1.2, we have*

$$(\mathrm{rk}\,\mathcal{E})\chi_{\mathrm{DR}}(U) - \chi_{\mathrm{DR}}(U;(\mathcal{E},\nabla)) = \sum_{z\in Z} \mathrm{ir}_z(\mathcal{E},\nabla). \tag{24.1.4}$$

A consequence of the theorem and of the definition of irregularity is

Corollary 24.1.5. *In the above notation, for any dense open subset $W \subset U$,*

$$(\mathrm{rk}\,\mathcal{E})\chi_{\mathrm{DR}}(W) - \chi_{\mathrm{DR}}(W;(\mathcal{E},\nabla)_{|W}) = (\mathrm{rk}\,\mathcal{E})\chi_{\mathrm{DR}}(U) - \chi_{\mathrm{DR}}(U;(\mathcal{E},\nabla))\,.$$

24.2 Proof of the global index formula

Here is an elementary purely algebraic proof of (24.1.4). Let $\mathcal{E}_1 \subset \mathcal{E}_2 \subset j_*\mathcal{E}$ be any coherent extensions of $\mathcal{E}$ to X such that $(j_*\nabla)\mathcal{E}_1 \subset \Omega_{X/k}(\log Z)\otimes\mathcal{E}_2$. We have a short exact sequence of complexes of abelian sheaves on X

$$\begin{array}{c}
0 \\
\downarrow \\
[0 \longrightarrow \mathcal{E}_1 \xrightarrow{j_*\nabla} \Omega_{X/k}(\log Z)\otimes\mathcal{E}_2 \longrightarrow 0] \\
\downarrow \\
[0 \longrightarrow j_*\mathcal{E} \xrightarrow{j_*\nabla} \Omega^1_{X/k}\otimes j_*\mathcal{E} \longrightarrow 0] \\
\downarrow \\
\bigoplus_{z\in Z}[0 \longrightarrow j_*\mathcal{E}_z/\mathcal{E}_{1,z} \xrightarrow{\overline{\nabla}_z} \Omega_{X/k}(\log Z)_z \otimes j_*\mathcal{E}_z/\mathcal{E}_{2,z} \longrightarrow 0] \\
\downarrow \\
0
\end{array}$$

so that

$$\chi\Big(\mathcal{E}_1 \xrightarrow{j_*\nabla} \Omega_{X/k}(\log Z)\otimes\mathcal{E}_2\Big) - \chi_{\mathrm{DR}}(U,(\mathcal{E},\nabla))$$
$$+\sum_{z\in Z}\chi\Big(j_*\mathcal{E}_z/\mathcal{E}_{1,z} \xrightarrow{\overline{\nabla}_z} \Omega_{X/k}(\log Z)_z\otimes j_*\mathcal{E}_z/\mathcal{E}_{2,z}\Big) = 0.$$

For any $z\in Z$, and $i=1,2$, we have $j_*\mathcal{E}_z/\mathcal{E}_{i,z}\cong \widehat{j_*\mathcal{E}}_z/\widehat{\mathcal{E}}_{i,z}$, and by Lemma 17.4.4 we have $\chi\Big(\widehat{\nabla}_z : \widehat{j_*\mathcal{E}}_z \to \Omega^1_{\widehat{X/Z}/k,z}\otimes\widehat{j_*\mathcal{E}}_z\Big)=0$. Therefore,

$$\begin{aligned}\mathrm{ir}_z(\mathcal{E},\nabla) &= \chi\big(\widehat{\nabla}_z : \widehat{\mathcal{E}}_{1,z} \longrightarrow \frac{dx}{x}\otimes\widehat{\mathcal{E}}_{2,z}\big) + \dim_k \mathcal{E}_{2,z}/\mathcal{E}_{1,z}\\ &= -\chi\big(\overline{\nabla}_z : \widehat{j_*\mathcal{E}}_z/\widehat{\mathcal{E}}_{1,z} \longrightarrow \frac{dx}{x}\otimes\widehat{j_*\mathcal{E}}_z/\widehat{\mathcal{E}}_{2,z}\big) + \dim_k \mathcal{E}_{2,z}/\mathcal{E}_{1,z}\end{aligned}$$

and

$$\begin{aligned}\chi\Big(\mathcal{E}_1 \xrightarrow{j_*\nabla} \Omega_{X/k}(\log Z)\otimes\mathcal{E}_2\Big) &= \chi(X,\mathcal{E}_1) - \chi(X,\Omega_{X/k}(\log Z)\otimes\mathcal{E}_2)\\ &= \chi(X,\mathcal{E}_1) - \chi(X,\Omega_{X/k}(\log Z)\otimes\mathcal{E}_1)\\ &\quad - \chi(X,\Omega_{X/k}(\log Z)\otimes\mathcal{E}_2/\mathcal{E}_1)\\ &= \mathrm{rk}(\mathcal{E})\chi_{\mathrm{DR}}(U) - \sum_{z\in Z}\dim_k \mathcal{E}_{2,z}/\mathcal{E}_{1,z}\ .\end{aligned}$$

Summing up, we have

$$\mathrm{rk}(\mathcal{E})\chi_{\mathrm{DR}}(U) - \sum_{z\in Z}\dim_k \mathcal{E}_{2,z}/\mathcal{E}_{1,z} - \chi_{\mathrm{DR}}(U,(\mathcal{E},\nabla)) - \mathrm{ir}_z(\mathcal{E},\nabla)$$
$$+\sum_{z\in Z}\dim_k \mathcal{E}_{2,z}/\mathcal{E}_{1,z} = 0,$$

which yields the global index formula. $\square$

Chapter VIII

Elementary fibrations and applications to Gauss-Manin

Introduction

This chapter develops a new approach to the study of direct images of connections (i.e., de Rham cohomology with coefficients, endowed with the Gauss-Manin connection) with respect to a smooth, not necessarily proper, morphism of smooth algebraic varieties in characteristic 0.

We present elementary and purely algebraic proofs of the *generic finiteness and base change* theorems, as well as of the fundamental *finiteness, regularity, monodromy and base change theorems for direct images of regular algebraic connections.*

Our proofs use neither resolution of singularities (beyond the classical case of embedded resolution of curves in surfaces), nor the theory of holonomy. In fact, in contrast to the now standard methods which consist of trying to extend all objects to a good compactification and studying ramification at infinity, our strategy relies upon a dévissage inspired by Artin's theory of elementary fibrations. This approach allows us to reduce problems to the simple case of an ordinary differential operator in one variable (but in a relative situation), which we handle directly.

We consider properties $\mathcal{P}$ of connections which are étale-local and strongly exact. Our main lemma of *dévissage*[1] asserts that *in order to establish that for any smooth morphism of smooth K-varieties $X \xrightarrow{f} S$, for any (quasi-coherent) connection $(\mathcal{E}, \nabla)$ on X satisfying $\mathcal{P}$, and for any $i \geqslant 0$, the restriction of $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ to the Artin set $A(f) \subseteq S$ satisfies $\mathcal{P}$, it suffices to consider rational elementary*

[1] The idea of this dévissage has already appeared in our work on G-functions [9]. It is interesting to note that Artin's technique of elementary fibrations was also used, independently, as a basic tool by A. de Jong in his now famous alteration theorem in [34], as we learned shortly after beginning this work.

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*, Progress in Mathematics 189, https://doi.org/10.1007/978-3-030-39719-7_8

fibrations f *(in which case* $A(f) = S$*), and* $i = 0, 1$. This essentially reduces the problem to the study of the kernel and cokernel of a connection in one variable.

25 Elementary fibrations and dévissage

25.1 Elementary fibrations

Let k be an algebraically closed field of characteristic 0. A *k-variety* will always be meant to be a reduced separated k-scheme of finite type, not necessarily irreducible.

Definition 25.1.1 (**Elementary fibrations**). *A morphism* $f : X \to S$ *of smooth algebraic k-varieties is called an* elementary fibration *if it can be embedded in a commutative diagram*

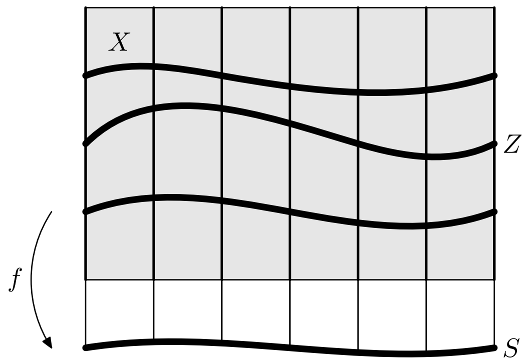
(25.1.2)
$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, hook'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

where

(i) Z *is a closed reduced subscheme of* $\overline{X}$ *and* j *is the complementary open immersion;*

(ii) $\overline{f}$ *is projective and smooth with geometric fibers irreducible of dimension* 1*;*

(iii) g *is an étale covering.*



Lemma 25.1.3. *An elementary fibration is a surjective affine morphism.*

Proof. The question is local for the étale topology on S. Therefore, We may assume that Z is a disjoint union of divisors isomorphic to S via g. For each point $s \in S$, $Z_s = g^{-1}(s)$ is then a positive effective divisor on the projective curve $\overline{X}_s/\kappa(s)$, hence it is ample [54, Chap. IV, Cor. 3.3]. By [47, 4.7.1], Z is then a relatively ample divisor in the relative projective curve $\overline{X}/S$. We conclude by [46, 5.5.7] that f is affine. □

Definition 25.1.4 (**Rational elementary fibrations**). *An elementary fibration f as in 25.1.1 is* rational *if $\overline{X} = \mathbb{P}^1_S$, $\overline{f}$ is the projection on S and Z is a disjoint union of images of sections σ_i, $i = 1, \dots, r, \infty$, of $\overline{f}$, one of which is the section $\sigma_\infty : S \to Z_\infty = \{\infty\} \times S \subseteq \mathbb{P}^1_S$. We thus have the diagram*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \mathbb{A}^1_S \arrow[d, "\mathrm{pr}_S"] & \coprod_{i=1}^r \sigma_i(S) \arrow[l, hook'] \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{25.1.5}$$

Remark 25.1.6. If $k = \mathbb{C}$, any rational elementary fibration is thus topologically locally trivial for the classical complex topology.

Definition 25.1.7 (**Coordinatized elementary fibrations**). *An elementary fibration f as in 25.1.1 is* coordinatized *if it can be embedded in a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] & Z \arrow[l, hook'] \arrow[d, "\pi'"] \\ Y \arrow[r, hook, "j'"] \arrow[dr, "f'"'] & \mathbb{P}^1_S \arrow[d, "\mathrm{pr}_S"] & Z' \arrow[l, hook'] \arrow[dl, "g'"] \\ & S & \end{tikzcd} \tag{25.1.8}$$

where

(i) *$\overline{\pi}$ is finite and π is an étale covering;*

(ii) *$f = f' \circ \pi$, $\overline{f} = \mathrm{pr}_S \circ \overline{\pi}$, $g = g' \circ \pi'$;*

(iii) *the lower part of the diagram is an elementary fibration with $j'(Y) \subseteq \mathbb{A}^1_S$.*

Remark 25.1.9. We notice that the two squares in (25.1.8) are necessarily cartesian and that π' is an étale covering. A rational elementary fibration is *ipso facto* coordinatized. Conversely, *given a coordinatized elementary fibration as in* (25.1.8), *there exists an étale covering $S' \to S$ such that the lower part of the pull-back of* (25.1.8) *over S' is a rational elementary fibration.*

Remark 25.1.10. Let $f : X \to S$ be a smooth morphism of smooth k-varieties, of relative dimension d. Assume S is connected. It follows from M. Artin's theorem [2, Exp. XI, prop. 3.3] on "good neighborhoods" that there exist an étale dominant morphism $\epsilon : S' \to S$ and a finite open (affine) cover $\{U_\alpha\}$ of $X' = X \times_S S'$, such that each $U_\alpha \to S'$ is a tower of elementary fibrations

$$U_\alpha = V_{\alpha,d} \longrightarrow V_{\alpha,d-1} \longrightarrow \cdots \longrightarrow V_{\alpha,0} = S' \quad . \tag{25.1.11}$$

The integer $d \geqslant 0$ is called the *height* of the tower (a tower of height 0 is then an isomorphism $U_\alpha \xrightarrow{\cong} S'$). This result will play a fundamental role in the present chapter, in the following more precise form.

Proposition 25.1.12. *Let $f : X \to S$ be a smooth morphism of smooth k-varieties, with S connected. Then there exist an étale dominant morphism $\epsilon : S' \to S$ and a finite open (affine) cover $\{U_\alpha\}$ of $X' = X \times_S S'$ such that each $U_\alpha \to S'$ is a tower of coordinatized elementary fibrations.*

Proof. We may assume that f is of pure relative dimension $d \geqslant 0$. Standard thickening arguments allow us to replace S by its geometric generic point $\operatorname{Spec} k(S)^{alg}$, and then replace the ground field k by $k(S)^{alg}$. The cases $d = 0, 1$ are then trivial.

By induction on d ($d \geqslant 2$), it suffices to find, for an arbitrary closed point $\xi \in X$, an open neighborhood U of ξ in X and a coordinatized elementary fibration $U \to T$.

Let us first recall, for convenience, Artin's construction of an elementary fibration $U \to T$, let alone the condition of being "coordinatized". We may assume X affine and consider a normal projective closure $\overline{X}$. Let $\mathcal{L}$ be a very ample line bundle on $\overline{X}$ and let us embed $\overline{X}$ in a projective space $\mathbb{P}^N$ via $\mathcal{L}^{\otimes 2}$. Then a "general" linear space $L \subseteq \mathbb{P}^N$ of codimension $d-1$ passing through ξ cuts $\overline{X}$ and $\overline{X} \smallsetminus X$ transversally and avoids the singular locus of $\overline{X}$. Let $H \subseteq \mathbb{P}^N$ be a "general" hyperplane (not passing through ξ and avoiding $L \cap (\overline{X} \smallsetminus X)$), and let us consider the blow-up $\widetilde{\mathbb{P}^N}$ of $\mathbb{P}^N$ along $L \cap H$, and the morphism $p : \widetilde{\mathbb{P}^N} \to \mathbb{P}^{d-1}$ induced by the linear projection with center $L \cap H$. Let $\widetilde{\overline{X}}$ be the strict transform of $\overline{X}$ in $\widetilde{\mathbb{P}^N}$, and let $Z \subseteq \widetilde{\overline{X}}$ be the disjoint union of $\overline{X} \smallsetminus X$ (considered as a subscheme of $\widetilde{\overline{X}}$ of codimension 1) and of the exceptional divisor $\simeq \mathbb{P}^{d-1} \times (X \cap L \cap H)$. Then $\widetilde{\overline{X}} \smallsetminus Z$ identifies with an open subscheme of X containing ξ, and the pull-back of the diagram

$$\begin{tikzcd} \widetilde{\overline{X}} \smallsetminus Z \arrow[r, hook] \arrow[dr] & \widetilde{\overline{X}} \arrow[d, "p_{|\widetilde{\overline{X}}}"'] & Z \arrow[l, hook'] \arrow[dl] \\ & \mathbb{P}^{d-1} & \end{tikzcd} \tag{25.1.13}$$

over a suitable (affine) neighborhood T of $p(\xi)$ in $\mathbb{P}^{d-1}$ defines an elementary fibration. Moreover, the fiber of $p_{|\widetilde{\overline{X}}}$ above $p(\xi)$ identifies with $\overline{X} \cap L$. The proposition is then a consequence of the following lemma.

Lemma 25.1.14. *Given an elementary fibration as in* (25.1.2)

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, hook'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

and a closed point $\xi \in X$, with image $\zeta = f(\xi)$, there is an open neighborhood U of ξ (resp. T of ζ), such that f induces a coordinatized elementary fibration $U \to T$.

Proof. We denote by $\{\xi_1, \dots, \xi_r\}$ the finite set $\{\xi\} \cup Z_\zeta$ of closed points of $\overline{X}_\zeta$. Let us choose a closed S-immersion $\overline{X} \to \mathbb{P}^M_S$. The fiber $\overline{X}_\zeta$ is a closed smooth curve in $\mathbb{P}^M$ to which the theory of Lefschetz pencils applies [3, exp. XVII]. There exists an open dense subset $\mathcal{U}$ of the Grassmannian of lines in the dual projective space $\check{\mathbb{P}}^M$ such that for any $D_\zeta \in \mathcal{U}$:

(a)$_\zeta$ D_ζ is a Lefschetz pencil; in particular, the axis of D_ζ (which has codimension 2 in $\mathbb{P}^M$) does not cut $\overline{X}_\zeta$, so that D_ζ gives rise to a finite morphism

$$\overline{\pi}_\zeta : \overline{X}_\zeta \longrightarrow \mathbb{P}^1 \simeq D_\zeta;$$

(b)$_\zeta$ if $i \neq j$, $\overline{\pi}_\zeta(\xi_i) \neq \overline{\pi}_\zeta(\xi_j)$;

(c)$_\zeta$ for each i, $\overline{\pi}_\zeta$ is étale at each point of $\overline{\pi}_\zeta^{-1}\overline{\pi}_\zeta(\xi_i)$.

In order to justify (b)$_\zeta$ and (c)$_\zeta$, it suffices to show that the generic pencil satisfies these conditions; but this follows from the fact [3, exp. XVII, 3.2.8] that for any hyperplane $H_i \subseteq \mathbb{P}^M$ containing ξ_i, but not ξ_j for $j \neq i$, and cutting $\overline{X}_\zeta$ transversally, there exists a Lefschetz pencil of hyperplanes containing H_i.

Let us fix $D_\zeta \in \mathcal{U}$. Up to changing the coordinate in $\mathbb{P}^1 \simeq D_\zeta$, we may assume in addition that

(d)$_\zeta$ the hyperplane H_∞ corresponding to $\infty \in D_\zeta$ cuts $\overline{X}_\zeta$ transversally, and does not meet the points ξ_i.

Let us now extend D_ζ to a relative line D/S in the dual projective space $\check{\mathbb{P}}^M_S$ (D is defined by an S-point in a suitable Grassmannian). After replacing S by a suitable Zariski neighborhood T of ζ, we may assume that the axis of D does not cut $\overline{X}$, so that D gives rise to a finite T-morphism

$$\overline{\pi} : \overline{X} \longrightarrow \mathbb{P}^1_T.$$

We may also assume that:

(b) $\overline{\pi}(Z)$ is étale over T, and does not contain $\overline{\pi}(\xi)$;

(c) $\overline{\pi}$ is étale at any point of $\overline{\pi}^{-1}\overline{\pi}(Z)$;

(d) $\overline{\pi}^{-1}(\infty \times T)$ does not meet $\overline{\pi}^{-1}\overline{\pi}(Z)$ nor $\{\xi\}$, and the restriction of $\overline{\pi}$ to $\overline{\pi}^{-1}(\infty \times T)$ is étale.

According to [3, exp. XVII, 6.3], we may assume furthermore that:

(a) for any closed point $\zeta' \in T$, $D_{\zeta'}$ is a Lefschetz pencil, and the branch locus $B \subseteq \mathbb{P}^1_T$ of $\overline{\pi}$ is étale over T.

Then there is at most one ramification point in each fiber of $\overline{\pi}^{-1}(B)$, so that $\overline{\pi}^{-1}(B)$ is étale over T, and does not meet $\overline{\pi}^{-1}\overline{\pi}(Z) \sqcup \overline{\pi}^{-1}(\infty \times T)$ nor $\{\xi\}$. It is now clear that

$$U = \overline{X} \smallsetminus \big(\overline{\pi}^{-1}(B) \sqcup \overline{\pi}^{-1}\overline{\pi}(Z) \sqcup \overline{\pi}^{-1}(\infty \times T)\big)$$

is a coordinatized elementary fibration (via $\overline{\pi}$), and $\xi \in U \subseteq X$.

This concludes the proof of lemma, hence of the proposition. □

In the next chapter, we shall need the following slight generalization:

Lemma 25.1.15. *Given an elementary fibration* (25.1.2)

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, hook'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

and a finite set of closed points $F \subseteq X$, there is an open neighborhood U of F (resp. T of $f(F)$), such that f induces a coordinatized elementary fibration $U \to T$.

Proof. Let us choose a closed S-embedding $\overline{X} \to \mathbb{P}^M_S$. For each $\zeta \in f(F)$, we denote by $\{\xi_1^\zeta, \dots, \xi_r^\zeta\}$ the finite set $F \cup Z_\zeta$ of closed points of $\overline{X}_\zeta$. There exists an open dense subset $\mathcal{U}^\zeta$ of the Grassmannian of lines in the dual projective space $\check{\mathbb{P}}^M$ such that for any $D_\zeta \in \mathcal{U}^\zeta$:

(a)$_\zeta$ D_ζ is a Lefschetz pencil; in particular, the axis of D_ζ (which has codimension 2 in $\mathbb{P}^M$) does not cut $\overline{X}_\zeta$, so that D_ζ gives rise to a finite morphism $\overline{\pi}_\zeta : \overline{X}_\zeta \longrightarrow \mathbb{P}^1 \simeq D_\zeta$;

(b)$_\zeta$ if $i \neq j$, then $\overline{\pi}_\zeta(\xi_i^\zeta) \neq \overline{\pi}_\zeta(\xi_j^\zeta)$;

(c)$_\zeta$ for each i, $\overline{\pi}_\zeta$ is étale at each point of $\overline{\pi}_\zeta^{-1}\overline{\pi}_\zeta(\xi_i^\zeta)$.

There exists a relative line D/S in the dual projective space $\check{\mathbb{P}}^M_S$ such that $D_\zeta \in \mathcal{U}^\zeta$ for each ζ. Up to changing the coordinate in $\mathbb{P}^1 \simeq D_\zeta$, we may moreover assume that

(d)$_\zeta$ the hyperplane H_∞ corresponding to $\infty \in D_\zeta$ cuts $\overline{X}_\zeta$ transversally, and does not meet the points ξ_i^ζ.

We then finish the proof as in Lemma 25.1.14. □

Remark 25.1.16. In fact, Proposition 25.1.12 holds for a separably closed ground field k of any characteristic. The proof is the same, except that one must replace $k(S)^{\mathrm{alg}}$ by the separable closure of $k(S)$, and that in Lemma 25.1.14 it may be necessary to replace the given embedding $\overline{X} \hookrightarrow \mathbb{P}_S^N$ by a Veronese multiple (in characteristic 2, one should also invoke [3, exp. XVII, 6.4], and replace $\mathcal{L}^{\otimes 2}$ by $\mathcal{L}^{\otimes 3}$ in (the proof of) 25.1.12).

25.2 Artin sets

Let $f : X \to S$ be a smooth morphism of k-varieties. We denote by d the maximum of the dimensions of the fibers. Let $\{U_\alpha\}_{\alpha=0,\dots,r}$ be a finite open cover of X. For $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$, $0 \leqslant \alpha_0 < \alpha_1 < \cdots < \alpha_p \leqslant r$, $|\underline{\alpha}| = p+1$, we denote by $U_{\underline{\alpha}}$ the open subset $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ of X, and by $f_{\underline{\alpha}}$ the restriction of f to $U_{\underline{\alpha}}$.

We introduce a decreasing chain of subsets of S, the Artin sets of f, which will be well-suited for the dévissage in Section 25.3 below.

Definition 25.2.1 (Artin sets). *The* Artin set of f of level $i \geqslant 0$, *denoted by* $A_i(f)$, *is the union of the images* $\epsilon(S') \subseteq S$, *for all étale morphisms* $\epsilon : S' \to S$, *finite over their image, such that:*

(1) *$X_{S'}$ admits a finite open cover $\{U_\alpha\}$ such that the restriction of $f_{S'}$ to each U_α is a tower of coordinatized elementary fibrations;*

(2) *each $U_{\underline{\alpha}}$, $|\underline{\alpha}| > 1$, admits a finite open cover $\{U_{\underline{\alpha},\beta}\}$ such that the restriction of $f_{S'}$ to each $U_{\underline{\alpha},\beta}$ is a tower of coordinatized elementary fibrations;*

(3) *each $U_{\underline{\alpha},\underline{\beta}}$, $|\underline{\beta}| > 1$, admits a finite open cover $\{U_{\underline{\alpha},\underline{\beta},\gamma}\}$ such that the restriction of $f_{S'}$ to each $U_{\underline{\alpha},\underline{\beta},\gamma}$ is a tower of coordinatized elementary fibrations;*

… *and so on, for all $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \dots)$ such that*

$$(|\underline{\alpha}| - 1) + (|\underline{\beta}| - 1) + (|\underline{\gamma}| - 1) + \cdots \leqslant i.$$

Obviously, $A_i(f) \supseteq A_{i+1}(f)$ for all i, and, for any $U_{\underline{\alpha}}$ appearing in the definition, $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) \supseteq A_i(f)$.

We set

$$A(f) = A_{d+\dim X}(f)$$

(*this definition is the most important and it will be justified later*).

From Proposition 25.1.12, we deduce:

Corollary 25.2.2. *For any $i \geqslant 0$, $A_i(f)$ is a dense open subset of the image of the* (*open*) *morphism f in S.*

Remarks 25.2.3. (i) All Artin sets of a tower $X \to S$ of coordinatized elementary fibrations coincide with S itself.

(ii) If f is étale, they coincide with the maximal open subset of S on which f is an étale covering.

25.3 Dévissage

In the sequel, we consider properties of $\mathcal{O}_X$-modules with integrable connection on arbitrary smooth k-varieties X. We write $\mathcal{P}((\mathcal{E},\nabla))$ to indicate that $(\mathcal{E},\nabla)$ satisfies property $\mathcal{P}$.

Definition 25.3.1. *We say that $\mathcal{P}$ is*

(1) local for the étale topology *if for any X, any $(\mathcal{E},\nabla) \in$ $\mathbf{MIC}(X)$ and any étale cover $\{V_i\}$ of X, one has*

$$\mathcal{P}((\mathcal{E},\nabla)) \Longleftrightarrow (\ \forall\, i,\ \ \mathcal{P}((\mathcal{E},\nabla)_{|V_i})\ );$$

(2) stable under finite étale direct image *if for any X, any $(\mathcal{E},\nabla) \in$ $\mathbf{MIC}(X)$ and any finite étale $\pi : X \to X'$, one has*

$$\mathcal{P}((\mathcal{E},\nabla)) \Longrightarrow \mathcal{P}(\pi_*(\mathcal{E},\nabla)).$$

Here we write simply $\pi_*(\mathcal{E},\nabla)$ instead of $R^0_{\mathrm{DR}}\pi_*(\mathcal{E},\nabla)$, the underlying $\mathcal{O}_{X'}$-module being $\pi_*\mathcal{E}$.

Definition 25.3.2. *We say in general that a property $\mathcal{P}$ of objects of an abelian category $\mathcal{A}$ is* strongly exact *if $\mathcal{P}(0)$ holds and for any exact sequence*

$$E_1 \longrightarrow E \longrightarrow E_2$$

in $\mathcal{A}$, one has

$$(\mathcal{P}(E_1)\ \text{ and }\ \mathcal{P}(E_2)\ ) \Longrightarrow \mathcal{P}(E).$$

Strong exactness is equivalent to the conjunction of exactness in the sense of [47, 3.1.1] and stability under taking subobjects.

Lemma 25.3.3. *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k-varieties which is strongly exact and local for the étale topology. Then $\mathcal{P}$ is stable under finite étale direct images.*

Proof. Let $(\mathcal{E},\nabla) \in$ $\mathbf{MIC}(X)$ and $\pi : X \to Z$ be an étale covering. We may and shall assume that X is connected. Let $\tau : Y \to X$ be the Galois closure of X relative to π [1, Exp. V, §4 and §7], and let G be the associated Galois group of $\pi\tau$, so that $Z \cong Y/G$. Let $\sigma : Z' \to Z$ be an étale covering such that $Y_{Z'} \to Z'$ [1, Exp. V, Prop. 2.6, (ii bis)] is the trivial Galois covering $\mathrm{pr}_1 : Z' \times G \longrightarrow Z'$. We have the commutative diagram with fibered squares

$$\begin{array}{ccccc} Y & \xrightarrow{\ \tau\ } & X & \xrightarrow{\ \pi\ } & Z \\ \uparrow{\scriptstyle \sigma''} & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \sigma} \\ Z' \times G = Y_{Z'} & \xrightarrow{\ \tau'\ } & X_{Z'} & \xrightarrow{\ \pi'\ } & Z' \end{array}$$

where all morphisms are finite étale and $\pi' \tau' = \mathrm{pr}_1$.

Since $Z' \times G = \coprod_{\gamma \in G} Z' \times \{\gamma\}$, an object $(\mathcal{M}, \nabla)$ of $\mathbf{MIC}(Z' \times G)$, consists of local components $(\mathcal{M}, \nabla)_\gamma$ in $\mathbf{MIC}(Z' \times \{\gamma\}) = \mathbf{MIC}(Z')$, and $\mathcal{P}((\mathcal{M}, \nabla)) \Leftrightarrow (\mathcal{P}((\mathcal{M}, \nabla)_\gamma)\ ,\ \forall\, \gamma \in G)$. Since G operates on $Z' \times G$ by Z'-automorphisms, for any $\gamma \in G$

$$(\mathrm{pr}_1^* \ \mathrm{pr}_{1*}(\mathcal{M}, \nabla))_\gamma \cong \bigoplus_{\delta \in G} (\mathcal{M}, \nabla)_\delta$$

and

$$\mathcal{P}((\mathcal{M}, \nabla)) \Longleftrightarrow \mathcal{P}(\mathrm{pr}_1^* \ \mathrm{pr}_{1*}(\mathcal{M}, \nabla)).$$

Applying this to $(\mathcal{M}, \nabla) = \tau'^* \sigma'^* (\mathcal{E}, \nabla)$, we derive $\mathcal{P}(\mathrm{pr}_{1*} \tau'^* \sigma'^* (\mathcal{E}, \nabla))$. On the other hand,

$$\mathrm{pr}_{1*} \tau'^* \sigma'^* = \sigma^* \pi_* \tau_* \tau^*,$$

whence $\mathcal{P}(\pi_* \tau_* \tau^* (\mathcal{E}, \nabla))$. Finally, since $(\mathcal{E}, \nabla)$ is a direct summand of $\tau_* \tau^* (\mathcal{E}, \nabla)$, we conclude $\mathcal{P}(\pi_*(\mathcal{E}, \nabla))$. $\square$

Lemma 25.3.4 (**Lemma on dévissage, first form).** *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k-varieties, which is strongly exact and local for the étale topology. Assume that*

(∗) *for any rational elementary fibration $f : X \to S$ with S affine, étale over some affine k-space, and for any quasi-coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$,*

$$\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(R^j_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)), \quad j = 0, 1.$$

Then for any $i \geqslant 0$, any smooth morphism $f : X \to S$ of smooth k-varieties, and any quasi-coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$,

$$\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))_{|A_i(f)}).$$

Proof. We proceed in three steps.

25.3.5. The case when f has relative dimension 0 is a simple consequence of the stability of $\mathcal{P}$ under finite étale direct image; we discard this trivial case in the sequel. Because $\mathcal{P}$ is local, we may assume that S is connected. On the other hand, if $f = f_1 \sqcup f_2 : X = X_1 \sqcup X_2 \to S$, then $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) = R^i_{\mathrm{DR}} f_{1*}((\mathcal{E}, \nabla)_{|X_1}) \oplus R^i_{\mathrm{DR}} f_{2*}((\mathcal{E}, \nabla)_{|X_2})$; because $\mathcal{P}$ is strongly exact, we may thus assume that X is connected so that f has pure relative dimension d. We may also assume that

$$0 \leqslant i \leqslant d + \dim X,$$

otherwise $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) = 0$, in which case the result is trivial.

25.3.6. We may replace at once S by $A_i(f)$, and then (because $\mathcal{P}$ is local for the étale topology and $R^j_{\mathrm{DR}} f_*$ is compatible with étale localization for quasi-coherent $\mathcal{E}$, see 23.1) by an affine connected étale neighborhood S' (étale over some affine space if we wish) such that $X_{S'}$ admits a finite open cover $\{U_\alpha\}$ with the properties listed in definition 25.2.1 Let us consider the Zariski spectral sequence in $\mathbf{MIC}(S)$ 23.3.5:

$$E_1^{p,i-p} = \bigoplus_{\alpha_0<\alpha_1<\cdots<\alpha_p} R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}} \Longrightarrow R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla).$$

By induction on i, since $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) = A_i(f) = S$, $\mathcal{P}(R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}})$ for all $\underline{\alpha}$ such that $|\underline{\alpha}| = p+1 > 1$. Thus, by strong exactness of $\mathcal{P}$, this spectral sequence reduces us to proving $\mathcal{P}(R^i_{\mathrm{DR}} f_{\alpha*}(\mathcal{E},\nabla)_{|U_\alpha})$; hence, we may assume that f itself is a tower

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \cdots \longrightarrow X_0 = S$$

of coordinatized elementary fibrations over an affine connected base S (étale over some affine space if we wish). Such a tower gives rise to Leray spectral sequences in **MIC** (see 23.3) and by strong exactness of $\mathcal{P}$ we are reduced to proving $\mathcal{P}$ for each level of the tower, i.e., we may assume that f itself is a coordinatized elementary fibration.

25.3.7. In case f is a coordinatized elementary fibration, 25.1.8

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] & Z \arrow[l, hook'] \arrow[d] \\
Y \arrow[r, hook, "j'"] \arrow[dr, "f'"'] & \mathbb{P}^1_S \arrow[d] & Z' \arrow[l, hook'] \arrow[dl] \\
& S &
\end{tikzcd}$$

($f = f' \circ \pi$), we may assume that $i = 0, 1$: other cohomology sheaves are 0, since f is affine of relative dimension 1 (cf. 23.6). Because $\mathcal{P}$ is étale-local, we may replace S by an étale covering so that f' becomes a rational elementary fibration. Since π is an étale covering, the Leray spectral sequence for $f = f' \circ \pi$ degenerates: $R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla) = R^i_{\mathrm{DR}} f'_*(\pi_*(\mathcal{E},\nabla))$. Then $\mathcal{P}((\mathcal{E},\nabla)) \Rightarrow \mathcal{P}(\pi_*(\mathcal{E},\nabla)) \Rightarrow \mathcal{P}(R^i_{\mathrm{DR}} f'_*(\pi_*(\mathcal{E},\nabla)))$, by $(*)$. This proves the lemma. $\square$

Remarks 25.3.8. (i) In lemma 25.3.4, we may replace $A_i(f)$ by $A(f) = A_{d+\dim X}(f)$ since

$$R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla) = 0 \ \text{ for } \ j > d + \dim X.$$

(ii) If in lemma 25.3.4, assumption $(*)$ holds only for $j = 0$, the conclusion keeps holding for $i = 0$.

Lemma 25.3.9 (Lemma on dévissage, second form). *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k-varieties that implies coherence, is strongly exact and local for the étale topology. Assume that*

(∗) *for any rational elementary fibration $f : X \to S$ with S affine, étale over some affine k-space, and for any simple $(\mathcal{E}, \nabla) \in$ $\mathbf{MIC}(X)$, which is cyclic with respect to a generator of $T_{X/S}$, there exists an open dense subset $U \subseteq S$, such that*

$$\mathcal{P}((\mathcal{E}, \nabla)) \Longrightarrow \mathcal{P}((R^j_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))_{|U}), \quad j = 0, 1.$$

Then for any smooth morphism $f : X \to S$ of smooth k-varieties, and any $(\mathcal{E}, \nabla) \in$ $\mathbf{MIC}(X)$, there exists a non-empty open subset $U \subseteq S$ (depending upon $(\mathcal{E}, \nabla)$) such that, for any $i \geqslant 0$,

$$\mathcal{P}((\mathcal{E}, \nabla)) \Longrightarrow \mathcal{P}((R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))_{|U}).$$

Proof. We proceed in four steps.

25.3.10. We first show that (∗) implies the apparently more stringent condition

(∗)′ *The conclusion of* (∗) *holds true for every elementary fibration $f : X \to S$ and every $(\mathcal{E}, \nabla) \in$ $\mathbf{MIC}(X)$.*

We show that (∗) ⇒ (∗)′. We first observe that, for any elementary fibration $f : X \to S$, if the conclusion of (∗) holds true when $(\mathcal{E}, \nabla)$ is simple, then it holds true for any $(\mathcal{E}, \nabla)$. In fact, let $(\mathcal{E}, \nabla)$ be any object of $\mathbf{MIC}(X)$ satisfying $\mathcal{P}$. Since $\mathcal{P}$ implies coherence, $(\mathcal{E}, \nabla)$ has finite length. Applying the strong exactness of $\mathcal{P}$ to long exact cohomology sequences, we see that it is enough to establish (∗) for the simple subquotients of $(\mathcal{E}, \nabla)$. Hence, we may assume that $(\mathcal{E}, \nabla)$ is simple.

We remark that we are free to replace S by some open dense subset; because $\mathcal{P}$ is étale-local, we may even replace S by an affine étale neighborhood. Hence we may even assume that the restriction of $(\mathcal{E}, \nabla)$ to the geometric generic fiber of f is simple.

It is clear that any elementary fibration may be coordinatized after replacing S by an open dense subset and then the argument in 25.3.7 further reduces the case of a coordinatized elementary fibration to the case of a rational one. So we may assume that $f : X \to S$ is indeed a rational elementary fibration.

On the other hand, according to Proposition 4.6.1, there exists a finite open affine cover $\{U_\alpha\}_{\alpha=0,\dots,r}$ of X, such that, for every α, $(\mathcal{E}, \nabla)_{|U_\alpha}$ admits a cyclic vector v_α with respect to any derivation $\partial_\alpha \in \mathcal{D}er_{X/k}(U_\alpha)$ which has no zero on U_α. Then after replacing S by some affine étale neighborhood, we may assume that for any α (resp. α, β) $f_{U_\alpha} : U_\alpha \to S$ (resp. $f_{U_\alpha \cap U_\beta} : U_\alpha \cap U_\beta \to S$) is a rational elementary fibration and $(\mathcal{E}, \nabla)_{|U_\alpha}$ (resp. $(\mathcal{E}, \nabla)_{|U_\alpha \cap U_\beta}$) admits a cyclic vector with respect to a generator of the relative tangent space $T_{X/S}$. The Čech spectral sequence for $\{U_\alpha\}$ then reduces us to the cyclic case, so that (∗)′ follows from (∗).

We are left to prove that the conclusion of Lemma 25.3.9 holds if (∗)′ is verified.

25.3.11. As in 25.3.5, we reduce to the case when S is connected, f is equidimensional of relative dimension $d \geqslant 1$, and $0 \leqslant i \leqslant d + \dim X$. By Artin's theorem (i.e., Proposition 25.1.12, without insisting on "coordinatized" elementary fibrations), there exists an étale map $\epsilon : S' \to S$ such that $X_{S'}$ admits a finite open affine cover $\{U_\alpha\}$ such that the restriction of $f_{S'}$ to each U_α is a tower of elementary fibrations. We may then replace S by S', and assume from the beginning that X admits a finite open affine cover $\{U_\alpha\}$ such that the restriction of f to each U_α is a tower of elementary fibrations.

Granting the previous reduction on $f : X \to S$, we further reduce the statement, by induction on i, to the case when f itself is a tower of elementary fibrations over a dense open subset of S. In fact, let us assume $\mathcal{P}((\mathcal{E}, \nabla))$, and let us consider the Zariski spectral sequence in $\mathbf{MIC}(S)$:

$$E_1^{p,i-p} = \bigoplus_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}} \Longrightarrow R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla).$$

If $i = 0$, we are reduced to the case when f itself is a tower of d elementary fibrations and the result follows trivially from $(*)'$. By induction on i, we know that after further replacing S by a dense affine open subset, we may suppose that $R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}((\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}})$ satisfies $\mathcal{P}$, for all $\underline{\alpha}$ such that $|\underline{\alpha}| = p + 1 > 1$. Thus, by the strong exactness of $\mathcal{P}$, this spectral sequence reduces us to proving $\mathcal{P}(R^i_{\mathrm{DR}} f_{\alpha*}(\mathcal{E}, \nabla)_{|U_\alpha})$, after replacing S by a dense affine open subset. Hence we may assume that f is a tower of elementary fibrations over an affine connected base S.

25.3.12. We argue by induction on the height d of the tower. The induction hypothesis will be:

$(*)_d$ *For any tower of elementary fibrations of height d,*

$$X_d \xrightarrow{f_{d-1}} X_{d-1} \longrightarrow \cdots \xrightarrow{f_0} X_0,$$

with X_0 affine connected, and any $(\mathcal{E}, \nabla) \in$ $\mathbf{MIC}(X_d)$, there exists a dense affine open subset X_0' of X_0 such that for any $(i_0, \ldots, i_{d-1}) \in \{0,1\}^d$, we have $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R^{i_0}_{\mathrm{DR}} f_{0*} \ldots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_0'})$.

Notice that $(*)_1 = (*)'$. We point out that via the Leray spectral sequences for $f = f_0 \circ \cdots \circ f_{d-1}$, and taking into account the strong exactness of $\mathcal{P}$, it follows from $(*)_d$ that for any $i \geqslant 0$, $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))_{|X_0'})$.

25.3.13. We now consider a tower of height $d \geqslant 2$ as in $(*)_d$, and $(\mathcal{E}, \nabla) \in$ $\mathbf{MIC}(X_d)$ satisfying $\mathcal{P}$ (in particular, $\mathcal{E}$ is torsion-free over $\mathcal{O}_{X_d}$). We apply the induction hypothesis $(*)_{d-1}$ to $X_d \longrightarrow \cdots \longrightarrow X_1$, to the effect that there exists a dense affine open subset $X_1' \subseteq X_1$ such that the $\mathcal{O}_{X_1'}$-module with integrable connection $(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'}$ satisfies $\mathcal{P}$.

By shrinking X_0, we may assume that the restriction of f_0, say $f_0' : X_1' \to X_0$, is an elementary fibration and that X_1' is the complement of a smooth divisor in X_1. We then apply $(*)_1$ to

$$(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'},$$

with respect to f_0'. Upon replacing X_0 again by a dense affine open subset, we obtain

$$\mathcal{P}(R^{i_0}_{\mathrm{DR}} f'_{0*}(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'}), \quad \text{for any } (i_0, \ldots, i_{d-1}) \in \{0,1\}^d.$$

We summarize the geometric situation, obtained at the price of shrinking X_0, in the diagram

$$\begin{tikzcd}
X_d' \arrow[r, "f_{d-1}'"] \arrow[d, "j_d"'] & X_{d-1}' \arrow[r] \arrow[d, "j_{d-1}"'] & \cdots \arrow[r] & X_1' \arrow[r, "f_0'"] \arrow[d, "j_1"'] & X_0 \arrow[d, equal] \\
X_d \arrow[r, "f_{d-1}"] & X_{d-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "f_0"] & X_0
\end{tikzcd}$$

in which the horizontal arrows are elementary fibrations, the vertical arrows are affine open embeddings, and all squares not involving the 0-subscript are fibered. Moreover, we may and will assume that, for any $k = 1, \ldots, d$, $X_k \smallsetminus X_k'$ is a smooth divisor in X_k.

From these properties it follows that, for any $k \in \{1, \ldots, d-1\}$ and any quasi-coherent $(\mathcal{N}, \nabla) \in \mathbf{MIC}(X_{k+1})$,

$$\begin{aligned}
j_{k*} j_k^* R^{i_k}_{\mathrm{DR}} f_{k*}(\mathcal{N}, \nabla) &= j_{k*} R^{i_k}_{\mathrm{DR}} f'_{k*} j_{k+1}^*(\mathcal{N}, \nabla) \\
&= R^{i_k}_{\mathrm{DR}} (j_k \circ f_k')_* j_{k+1}^*(\mathcal{N}, \nabla) \\
&= R^{i_k}_{\mathrm{DR}} f_{k*}(j_{k+1*} j_{k+1}^*(\mathcal{N}, \nabla)).
\end{aligned}$$

By iteration, we obtain

$(**)_k$

$$j_{k*} j_k^*(R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla)) = R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla)).$$

On the other hand, since $\mathcal{E}$ is torsion-free, there is an exact sequence

$$0 \longrightarrow (\mathcal{E}, \nabla) \longrightarrow j_{d*} j_d^*(\mathcal{E}, \nabla) \longrightarrow \mathrm{Coker} \longrightarrow 0.$$

By descending induction on $k \in \{0, \ldots, d-1\}$, we shall show that the sequence of natural morphisms

$$(***)_k \qquad \begin{array}{c} 0 \\ \downarrow \\ R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \\ \downarrow \\ R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla)) \\ \downarrow \\ R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}\mathrm{Coker} \\ \downarrow \\ 0 \end{array}$$

is a short *exact sequence* ($i_k \in \{0, 1\}$).

By descending induction on k, and via the long exact sequence attached to $R^{\bullet}_{\mathrm{DR}} f_{k*}$, it is enough to show that

$$(***)'_k \qquad R^0_{\mathrm{DR}} f_{k*}(R^{i_{k+1}}_{\mathrm{DR}} f_{k+1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}\mathrm{Coker}) = 0.$$

By $(**)_{k+1}$ and $(***)_{>k}$,

$$\begin{aligned} & R^{i_{k+1}}_{\mathrm{DR}} f_{k+1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}\mathrm{Coker} \\ & = \mathrm{Coker} \begin{pmatrix} R^{i_{k+1}}_{\mathrm{DR}} f_{k+1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \\ \downarrow \\ j_{k+1*} j^*_{k+1} R^{i_{k+1}}_{\mathrm{DR}} f_{k+1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \end{pmatrix} \end{aligned}$$

which is torsion, supported by $X_{k+1} \smallsetminus X'_{k+1}$. This implies $(***)'_k$. Indeed, let $m \neq 0$, if any, be a section of $R^{i_{k+1}}_{\mathrm{DR}} f_{k+1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}\mathrm{Coker}$ supported on the component of $X_{k+1} \smallsetminus X'_{k+1}$ of local equation t. Then $t^h m = 0$, for some $h \in \mathbb{Z}_{>0}$, which we assume to be minimal. So, for the $\mathcal{O}_{X_k}$-linear derivation $\partial_t = \frac{d}{dt}$,

$$0 = \nabla_{\partial_t}(t^h m) = h t^{h-1} m + t^h \nabla_{\partial_t}(m).$$

This shows that $\nabla_{\partial_t}(m) \neq 0$, hence the assertion $(***)'_k$. We have now proved $(***)'_0$, which shows that $R^{i_0}_{\mathrm{DR}} f_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla)$ is a subobject of

$$\begin{aligned} & R^{i_0}_{\mathrm{DR}} f_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla)) \\ & = R^{i_0}_{\mathrm{DR}} f_{0*} j_{1*} j_1^* R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \\ & = R^{i_0}_{\mathrm{DR}} f'_{0*}((R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X'_1}). \end{aligned}$$

Since the latter satisfies $\mathcal{P}$, this completes the proof of $(*)_d$ and of the lemma. $\square$

Remark 25.3.14. The argument also shows that, for any $k = d-1, \dots, 1, 0$, the natural map

(25.3.15)
$$R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \longrightarrow R^{i_k}_{\mathrm{DR}} f_{k*} j_{k+1*} j^*_{k+1} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla)$$

or, equivalently, the map

$$R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \longrightarrow R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(j_{d*} j^*_d(\mathcal{E}, \nabla)) \tag{25.3.16}$$

is an isomorphism if $(i_k, i_{k+1} \dots, i_{d-1}) \neq (1, 1, \dots, 1)$. Indeed, assume h, $k \leqslant h \leqslant d-1$, is maximal with the condition that $i_h = 0$. Then (25.3.15) is an isomorphism for $k = h$. But then, it is also an isomorphism for $k = h-1, \dots, 1, 0$, since it is obtained by applying the functor $R^{i_k}_{\mathrm{DR}} f_{k*} \cdots R^{i_{h-1}}_{\mathrm{DR}} f_{h-1*}$ to the isomorphism (25.3.16) for $k = h$.

Remark 25.3.17. One can give an *alternative proof* of Lemma 25.3.9, under the extra assumption that for any closed immersion $i : Y \to X$ of smooth k-varieties, $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(i^*(\mathcal{E}, \nabla))$. This second proof avoids the use of Proposition 25.1.12 (see [9] in a special case, using the language and results of [19]).

Remark 25.3.18. One can avoid spectral sequences and give shorter proofs of these devissage lemmas using derived categories of $\mathcal{D}$-modules, see [9].

26 Main theorems on the Gauss-Manin connection

26.1 Generic finiteness of direct images

Theorem 26.1.1. *For any smooth morphism $f : X \to S$ of smooth k-varieties and any coherent $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla)$, there exists a dense open subset $U \subseteq S$ such that for every $j \geqslant 0$, $R^j_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)_{|U}$ is locally free of finite rank.*

For the proof, we may and shall assume that f is dominant.

26.1.2. We treat separately the case $j = 0$, for which one can in fact take $U = A_0(f)$, the Artin set of level 0 (see 25.2.2), independently of $(\mathcal{E}, \nabla)$.

Lemma 26.1.3. *Let $f : X \to S$ be a smooth morphism with $S = A_0(f)$. Then for any coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$, $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is locally free of finite rank as an $\mathcal{O}_S$-module.*

Proof. (of the lemma) It suffices to check the coherence of $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$, and this is an étale-local condition on S. By the definition of $A_0(f)$, we thus may assume that f is Zariski-locally on X a tower of elementary fibrations, hence (separating the connected components of X) that f has non-empty connected geometric fibers. The coherence of $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ then follows from Proposition 23.1.3 and faithfully flat descent [48, 2.5.2]. □

26.1.4. For $j > 0$, we use the lemma on dévissage in its second form 25.3.9, with $\mathcal{P}$ = *coherence* (i.e., $(\mathcal{E}, \nabla)$ satisfies $\mathcal{P}$ if and only if $\mathcal{E}$ is $\mathcal{O}_X$-coherent). Strong exactness and étale-localness are fulfilled. Therefore, we are left to show that for any *rational elementary fibration* $f : X \to S$ with S smooth affine, and for any $(\mathcal{E}, \nabla)$ *cyclic* with respect to a generator of $T_{X/S}$, there is a dense open subset $U \subseteq S$ such that $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)_{|U}$ is $\mathcal{O}_U$-coherent; in other words, such that the cokernel of the relative de Rham complex $\Gamma(U, \mathcal{E}) \to \Gamma(U, \mathcal{E})dx$ is finitely generated.

26.1.5. Let us write the diagram pertaining to a rational elementary fibration over a smooth affine k-variety S:

(26.1.6)
$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \mathbb{P}^1_S \arrow[d, "\mathrm{pr}_S"] & \sigma_\infty(S) \sqcup \coprod_{i=1}^r \sigma_i(S) \arrow[l, hook'] \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

where $\sigma_\infty(S) = \infty \times S$, $\sigma_i : \zeta \mapsto (\theta_i(\zeta), \zeta)$, $\theta_i \in \mathcal{O}(S)$, $\theta_i - \theta_j \in \mathcal{O}(S)^\times$. As usual, x denotes the affine coordinate on $\mathbb{P}^1$ and ∂_x the corresponding generator of $T_{\mathbb{A}^1_S/S}$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module endowed with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/k}$. Assume that $\mathcal{E}$ is cyclic with respect to ∂_x with cyclic vector $v \in E := \Gamma(X, \mathcal{E})$, and let $(\mathcal{E}, \nabla_{\partial_x})$ be the sheaf of $(\mathcal{O}_X, \partial_x)$-differential modules naturally obtained from $(\mathcal{E}, \nabla)$. Then $(\mathcal{E}, \nabla_{\partial_x})$ is also cyclic with cyclic vector v. The dual connection $(\mathcal{E}^\vee, \nabla^\vee)$ is also cyclic with respect to ∂_x and the construction $(\mathcal{E}^\vee, \nabla^\vee_{\partial_x})$ gives the sheaf of $(\mathcal{O}_X, \partial_x)$-differential modules dual to $(\mathcal{E}, \nabla_{\partial_x})$. We assume $v^\vee \in E^\vee := \Gamma(X, \mathcal{E}^\vee)$ is a ∂_x-cyclic vector, and let

$$\Lambda = \partial_x^\mu + \sum_{k=0}^{\mu-1} a_k \partial_x^k \in \mathcal{O}(X)\langle \partial_x \rangle$$

the associated differential operator (such that $\nabla^\vee_\Lambda v^\vee = 0$).

We recall (see 7.3.1) that the *indicial polynomial* of Λ at θ_i, denoted by $\mathrm{ind}_{\theta_i}(t)$ (resp. $\mathrm{ind}_\infty(t)$), is the element of $\mathcal{O}(S)[t]$ defined by the condition that, for any $m \in \mathbb{Z}$,

$$\Lambda \frac{1}{(x-\theta_i)^m} = \mathrm{ind}_{\theta_i}(-m) \frac{1}{(x-\theta_i)^{m+r_i}} + \text{ lower order terms in } \frac{1}{x-\theta_i}, \tag{26.1.7}$$

(resp.

$$\Lambda x^m = \mathrm{ind}_\infty(-m) x^{m+r_\infty} + \text{ l.o.t. in } x\ ), \tag{26.1.8}$$

where $r_i = \max(k - \mathrm{ord}_{\theta_i} a_k) \geqslant 0$ (resp. $r_\infty = \max(\deg_x a_k - k) \geqslant 0$).

Let $\gamma_i \in \mathcal{O}(S) \smallsetminus \{0\}$ (resp. $\gamma_\infty \in \mathcal{O}(S) \smallsetminus \{0\}$) be the *leading coefficient* of the indicial polynomial $\operatorname{ind}_{\theta_i}(t)$ (resp. $\operatorname{ind}_{\theta_\infty}(t)$), and let us set $\gamma = \gamma_\infty \prod_i \gamma_i$. Since ∇ is integrable, it follows from 8.2.3 that

$$\frac{\operatorname{ind}_{\theta_i}(t)}{\gamma_i}, \ \frac{\operatorname{ind}_\infty(t)}{\gamma_\infty} \in k[t]. \tag{26.1.9}$$

Lemma 26.1.10. *In the above geometric situation and with the above notation, assume moreover that:*

- $(\mathcal{E}, \nabla_{\partial_x})$ *is a cyclic* $(\mathcal{O}_X, \partial_x)$*-differential module;*
- γ *is unit in* $\mathcal{O}(S)$*.*

Then $\operatorname{Coker}(\nabla_{\partial_x})$ *is a locally free* $\mathcal{O}_S$*-module of finite rank. In general, if* $U \subseteq S$ *is the open subset on which* γ *is invertible, and we restrict* $(\mathcal{E}, \nabla)$ *on* $X_U = f^{-1}(U)$ *and* f *to* $X_U \to U$*, then* $\operatorname{Coker}(\nabla_{\partial_x})$ *is* $\mathcal{O}_U$*-coherent.*

Proof of the lemma. We use the dual situation of Lemma 3.2.14. Let

$$(v^\vee, \nabla^\vee_{\partial_x} v^\vee, \dots, (\nabla^\vee_{\partial_x})^{\mu-1} v^\vee)$$

be the cyclic basis of $E^\vee = \Gamma(X, \mathcal{E}^\vee)$ corresponding to $v^\vee$ and let

$$(v_0, v_1, \dots, v_{\mu-1})$$

be the $\mathcal{O}(X)$-basis of E, *dual* to it. The diagram (3.2.15) produces here a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\quad\Lambda\quad} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ \Gamma(X, \mathcal{E}) = \bigoplus_{k=0}^{\mu-1} \mathcal{O}(X) v_k & \xrightarrow[\nabla_{\partial_x}]{} & \bigoplus_{k=0}^{\mu-1} \mathcal{O}(X) v_k = \Gamma(X, \mathcal{E}) \end{array} \tag{26.1.11}$$

where the vertical arrows are $\eta \mapsto (\eta, \dots, \partial_x^{\mu-1}\eta)$ and $\eta \mapsto (0, \dots, 0, \eta)$, respectively. Diagram (26.1.11) induces an isomorphism between the cokernel (resp. kernel) of Λ and the cokernel (resp. kernel) of ∇_{∂_x}. Therefore,

$$\operatorname{Coker}\left(\Gamma(X, \mathcal{E}) \xrightarrow{\nabla_{\partial_x}} \Gamma(X, \mathcal{E})\right) \simeq \mathcal{O}(X)/\Lambda\mathcal{O}(X).$$

Let

$$P(x) = \prod_{i=1}^r (x - \theta_i)^{s_i} \in \mathcal{O}(S)[x]$$

be such that

$$\Lambda' = P(x)\Lambda = \sum_{k=0}^{\mu} a'_k \partial_x^k \in \mathcal{O}(S)\big[x\big]\langle \partial_x \rangle \tag{26.1.12}$$

with $s_i \geqslant 0$ minimal. We also have

$$\mathrm{Coker}(\Gamma(X,\mathcal{E}) \xrightarrow{\nabla_{\partial_x}} \Gamma(X,\mathcal{E})) \simeq \mathcal{O}(X)/\Lambda'\mathcal{O}(X).$$

26.1.13. The indicial polynomials of Λ and Λ' (denoted by ind'_{θ_i}) at θ_i differ by multiplication by a unit in $\mathcal{O}(S)$. Therefore, in the statement of the lemma we can replace γ by the corresponding element γ' pertaining to Λ'. We have

$$\Lambda' \frac{1}{(x-\theta_i)^m} = \mathrm{ind}'_{\theta_i}(-m)\frac{1}{(x-\theta_i)^{m+r'_i}} + \text{ l.o.t. in } \frac{1}{x-\theta_i} \ ,$$

(resp.

$$\Lambda' x^m = \mathrm{ind}'_\infty(-m)x^{m+r'_\infty} + \text{ l.o.t. in } x\ ),$$

where $r'_i = \max(k - \mathrm{ord}_{\theta_i} a'_k) \geqslant 0$ (resp. $r'_\infty = \max(\deg_x a'_k - k) \geqslant 0$), and $\frac{\mathrm{ind}'_{\theta_i}(t)}{\gamma'_i}$, $\frac{\mathrm{ind}'_\infty(t)}{\gamma'_\infty} \in k[t]$.

Let us now finish the proof of the lemma: we use the Mittag-Leffler decomposition

$$\mathcal{O}(X) = \mathcal{O}(S)[x] \oplus \bigoplus_{i=1}^{r} \frac{1}{x-\theta_i}\mathcal{O}(S)\Big[\frac{1}{x-\theta_i}\Big]. \tag{26.1.14}$$

Let $M \geqslant \max\ r_i$ be such that $\mathrm{ind}'_\infty(-n+r'_\infty)\prod_{j=1,\dots,r}\mathrm{ind}'_{\theta_j}(-n+r'_j) \neq 0$ for $n > M$. Then

$$\frac{\gamma'_i}{(x-\theta_i)^n} \in \Lambda' \frac{\sigma_{i,n}}{(x-\theta_i)^{n-r'_i}} + \frac{1}{(x-\theta_i)^{n-1}}\mathcal{O}(S)[x], \quad \sigma_{i,n} \in \mathcal{O}(S)^\times,$$

and

$$\gamma'_\infty x^n \in \Lambda'(\sigma_{\infty,n}x^{n-r'_\infty}) + \bigoplus_{k=0}^{n-1} x^k\mathcal{O}(S), \quad \sigma_{\infty,n} \in \mathcal{O}(S)^\times.$$

Let F be the $\mathcal{O}(S)$-submodule of $\mathcal{O}(X)/\Lambda'\mathcal{O}(X)$ generated by the images of

$$1, x, \dots, x^M, \frac{1}{(x-\theta_1)}, \dots, \frac{1}{(x-\theta_1)^M}, \dots, \frac{1}{(x-\theta_r)}, \dots, \frac{1}{(x-\theta_r)^M}.$$

By the Mittag-Leffler decomposition, for any element $[f] = f + \Lambda'\mathcal{O}(X)$ of $\mathcal{O}(X)/\Lambda'\mathcal{O}(X)$, there is a power $(\gamma')^m$, for some $m = 0, 1, \dots,$ such $(\gamma')^m[f] \in F$. Since $\gamma' \in \mathcal{O}(S)^\times$, we conclude that $F = \mathcal{O}(X)/\Lambda'\mathcal{O}(X)$ so that the latter is finitely generated. On the other hand, $\mathcal{O}(X)/\Lambda'\mathcal{O}(X)$ carries an integrable connection, so that it is projective of finite rank. This completes the proof of the lemma. $\square$

From the preceding lemma, one gets the following purity statement:

Corollary 26.1.15. *Let $f : X \to S$ be a rational elementary fibration, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module with integrable connection. Assume that there is a closed subset $T \subseteq S$ of codimension $\geqslant 2$ such that $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is locally free of finite rank over $S \smallsetminus T$. Then $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is locally free of finite rank over S.*

Remark 26.1.16. The proof does not use the full hypothesis that v is a cyclic vector on the whole of X (i.e., that one may take $P(x) = \prod_{i=1}^r (x-\theta_i)^{s_i}$). It would suffice to assume that v is a cyclic vector outside some divisor D whose Zariski closure in $\mathbb{P}^1_S$ is contained in X.

(Counter)example 26.1.17. Unlike the case of $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$, it is not possible to choose an open subset $U \subseteq S$ of coherence for $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$, which is independent of $(\mathcal{E}, \nabla)$. Indeed, let $X = \mathbb{A}^2_k = \mathrm{Spec}\,(k[x,y]) \to S = \mathbb{A}^1_k = \mathrm{Spec}\,(k[y])$ be the second projection, and consider, for $y_0 \in k$, the integrable $\mathcal{O}_X$-connection of rank 1 $(\mathcal{E} = \mathcal{O}_X e, \nabla_{y_0})$, defined by

$$\begin{aligned}(\nabla_{y_0})_{\partial_x} e &= (y - y_0)e,\\ (\nabla_{y_0})_{\partial_y} e &= xe.\end{aligned}$$

The calculation of $\mathrm{Coker}((\nabla_{y_0})_{\partial_x})$ involves the ∂_x-adjoint of the operator $\partial_x - (y - y_0)$, that is, $\partial_x + (y - y_0)$. More precisely, the commutative square (26.1.11) takes the form

$$\begin{CD} k[x,y] @>{\partial_x + y - y_0}>> k[x,y] \\ @VVV @VVV \\ k[x,y]e @>>{\nabla_{\partial_x}}> k[x,y]e \end{CD} \tag{26.1.18}$$

where both vertical maps are $f(x,y) \mapsto f(x,y)e$, and induces an isomorphism of $k[y]$-modules

$$\mathrm{Coker}((\nabla_{y_0})_{\partial_x}) \cong k[x,y]/(\partial_x + (y - y_0))k[x,y]. \tag{26.1.19}$$

The indicial polynomial $\mathrm{ind}_\infty(t)$ for the derivation ∂_x is the "constant" $y - y_0$. From

$$(\partial_x + (y - y_0))(x^j) = jx^{j-1} + (y - y_0))x^j, \quad \forall j = 0, 1, 2 \ldots.$$

we obtain in $k[x,y]/(\partial_x + (y - y_0))k[x,y]$ the congruences

$$x^j \equiv \frac{j!}{(y_0 - y)^j}, \qquad \mathrm{mod}\ (\partial_x + y - y_0)k[x,y].$$

We then have an isomorphism of $k[y]$-modules

$$\begin{aligned} k[x,y]/(\partial_x + (y - y_0))k[x,y] &\longrightarrow k[y, \frac{1}{y - y_0}] \\ x^j &\longmapsto \frac{j!}{(y_0 - y)^j}, \end{aligned}$$

so that the maximal open subset of coherence of $\mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E}$ is $S \smallsetminus \{y_0\}$, which depends upon y_0.

In order to determine the structure of left $k[y]\langle\partial_y\rangle$-module of $\mathrm{Coker}((\nabla_{y_0})_{\partial_x})$, that is, the action $\Box_{\partial_y}$ of ∂_y on it, we need to fully replace $(\mathcal{E} = \mathcal{O}_X e, \nabla_{y_0})$ by the dual connection $(\mathcal{E}^\vee = \mathcal{O}_X e^\vee, \nabla_{y_0}^\vee)$ defined by

$$\begin{aligned}(\nabla_{y_0}^\vee)_{\partial_x} e^\vee &= (y_0 - y)e,\\ (\nabla_{y_0}^\vee)_{\partial_y} e^\vee &= -xe^\vee.\end{aligned}$$

Then, we must equip $k[x,y]$ in the upper line of the diagram (26.1.18) with the left $k[x,y]\langle\partial_y\rangle$-module in which ∂_y acts as $\partial_y + x$. (Notice that $\partial_y + x$ and $\partial_x + y - y_0$ commute.) Finally, the action $\Box_{\partial_y}$ of ∂_y on $\Gamma(S, \mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E}))$ is induced by $\partial_y + x$ the isomorphism (26.1.19). In the end, since the operators $\partial_y + x$ and $\partial_y - \frac{1}{y-y_0}$ coincide in $k[x,y]/(\partial_x + (y - y_0))k[x,y]$, the isomorphism of $k[y]$-modules

$$k[y, \frac{1}{y-y_0}] \longrightarrow \mathrm{Coker}((\nabla_{y_0})_{\partial_x}) \;\;,\;\; f(y) \longmapsto f(y)[e]$$

extends to an isomorphism of differential modules over $(k[y], \partial_y)$

$$(k[y, \frac{1}{y-y_0}], \partial_y - \frac{1}{y-y_0}) \xrightarrow{\;\sim\;} (\Gamma(S, \mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E})), \Box_{\partial_y}).$$

26.2 Generic base change for direct images

Theorem 26.2.1. *Let $f : X \to S$ be a smooth morphism of smooth k-varieties, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module with integrable connection. There is a dense open subset $U \subseteq S$ with the following property. For any smooth k-variety $S^\sharp$ and any morphism $u : S^\sharp \to S$, construct the fibered diagram*

$$\begin{tikzcd} X^\sharp \arrow[r, "f^\sharp"] \arrow[d, "u^\sharp"'] & S^\sharp \arrow[d, "u"] & U^\sharp \arrow[l, hook'] \arrow[d] \\ X \arrow[r, "f"] & S & U \arrow[l, hook'] \end{tikzcd}$$

Then, for any $i \geqslant 0$, the restriction to $U^\sharp$ of the base change morphism

$$\varphi^i : u^* R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \longrightarrow R^i_{\mathrm{DR}} f^\sharp_* u^{\sharp *}(\mathcal{E}, \nabla)$$

is an isomorphism in $\mathbf{MIC}(U^\sharp)$.

26.2.2 (Notation). In the sequel, we set $(\mathcal{E}^\sharp, \nabla^\sharp) = u^{\sharp *}(\mathcal{E}, \nabla)$.

Remarks 26.2.3. (i) $U^\sharp$ is an open subset of $S^\sharp$, possibly empty (see 26.2.9 below).

The following assertions are straightforward consequences of the compatibility of the formation of $R^i_{\mathrm{DR}} f_*$ with flat base change (23.5).

(ii) It would suffice to consider the case of a closed immersion u (write u as a closed immersion given by its graph, followed by a projection).

(iii) The statement is equivalent to the following. For any $u : S^\sharp \to S$ such that $\mathrm{Im}(u) \subseteq U$, φ^i is an isomorphism.

(iv) In order to prove Theorem 26.2.1, one may replace S by any S' étale and dominant over S.

26.2.4. Replacing S by affine étale neighborhoods of its connected components, we may assume that X admits a finite open affine cover $\{U_\alpha\}$ such that the restriction f_α of f to U_α is a tower of elementary fibrations. Considering an affine cover of $S^\sharp$, or using (ii) of the preceding remark, we may assume that $u^{\sharp-1}(U_\alpha)$ are affine. With the notation of 25.2, there is a natural morphism of Čech spectral sequences in $\mathbf{MIC}(S^\sharp)$

$$\begin{tikzcd}
\bigoplus\limits_{\alpha_0<\cdots<\alpha_p} u^* R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}} \arrow[r, Rightarrow] \arrow[d, "\varphi_{i-p}"'] & u^* R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla) \arrow[d, "\varphi_i"] \\
\bigoplus\limits_{\alpha_0<\cdots<\alpha_p} R^{i-p}_{\mathrm{DR}} f^\sharp_{\underline{\alpha}*}(\mathcal{E}^\sharp,\nabla^\sharp)_{|u^{\sharp-1}U_{\underline{\alpha}}} \arrow[r, Rightarrow] & R^i_{\mathrm{DR}} f^\sharp_*(\mathcal{E}^\sharp,\nabla^\sharp)
\end{tikzcd}$$

Arguing by induction on i, we may assume that *f is a tower of elementary fibrations.*

26.2.5. We argue by induction on the height d of the tower, as in the proof of 25.3.9. The induction hypothesis will be:

$(*)_d$ *For any tower of elementary fibrations of height d*

$$X_d \xrightarrow{f_{d-1}} X_{d-1} \longrightarrow \cdots \xrightarrow{f_0} X_0,$$

with X_0 affine connected, and any coherent $(\mathcal{E},\nabla) \in \mathbf{MIC}(X_d)$, there exists a dense affine open subset X_0' of X_0 with the following property.

For any smooth k-variety $X_0^\sharp$ and any morphism $u_0 : X_0^\sharp \to X_0$, construct the fibered diagram

$$\begin{tikzcd}
X_d^\sharp \arrow[r, "f_{d-1}^\sharp"] \arrow[d, "u_d"'] & X_{d-1}^\sharp \arrow[r] \arrow[d, "u_{d-1}"'] & \cdots \arrow[r, "f_0^\sharp"] & X_0^\sharp \arrow[d, "u_0"] \\
X_d \arrow[r, "f_{d-1}"] & X_{d-1} \arrow[r] & \cdots \arrow[r, "f_0"] & X_0
\end{tikzcd}$$

Then for any $(i_0, \dots, i_{d-1}) \in \{0,1\}^d$,

$$(R^{i_0}_{\mathrm{DR}} f_{0*} R^{i_1}_{\mathrm{DR}} f_{1*} \dots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X'_0}$$

is coherent, and the restriction to $u_0^{-1}(X'_0)$ *of the base change morphism*

$$\begin{aligned} &u_0^*(R^{i_0}_{\mathrm{DR}} f_{0*} R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}\mathcal{E}, \nabla)) \\ &\xrightarrow{\psi} R^{i_0}_{\mathrm{DR}} f^\sharp_{0*} R^{i_1}_{\mathrm{DR}} f^\sharp_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^\sharp_{d-1*} u_d^*(\mathcal{E}, \nabla) \end{aligned}$$

is an isomorphism in $\mathbf{MIC}(u_0^{-1}(X'_0))$.

We notice that via base change for the Leray spectral sequences of the composed morphism $f = f_0 \circ \cdots \circ f_{d-1}$, the statement 26.2.1 in the case of towers of elementary fibrations of height $\leqslant d$ follows from $(*)_{\leqslant d}$.

Remark 26.2.6. If $i_0 = i_1 = \cdots = i_{d-1} = 1$, then the base change morphism ψ is an isomorphism; this follows from the fact that for f affine smooth of relative dimension one the formation of $R^1_{\mathrm{DR}} f_*$ commutes with arbitrary base change, being a cokernel.

26.2.7. First consider $(*)_d$ in the case $d = 1$, i.e., f is an elementary fibration $X_1 = X \to S = X_0$. In this case, it turns out that *one can choose for* $X'_0 = U$ *any dense open subset such that* $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)_{|U}$ *is coherent.*

For simplicity of notation, assume that $S = U$. We note that only $i = 0$ or 1 occur. Since f is affine and flat, we have the exact sequence (cf. 23.6)

$$\begin{tikzcd} 0 \arrow[r] & R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \arrow[r] \arrow[d, equal] & f_*(\nabla) \arrow[r, "f_*(\nabla)"] & f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \arrow[r] & R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \arrow[r] \arrow[d, equal] & 0 \\ & \mathrm{Ker} f_*(\nabla) & & & \mathrm{Coker} f_*(\nabla) & \end{tikzcd}$$

in which all terms are flat over S, which we split into two short exact sequences of *flat* $\mathcal{O}_S$-modules:

$$0 \longrightarrow \mathrm{Im} f_*(\nabla) \longrightarrow f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \longrightarrow \mathrm{Coker} f_*(\nabla) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ker} f_*(\nabla) \longrightarrow f_*\mathcal{E} \longrightarrow \mathrm{Im} f_*(\nabla) \longrightarrow 0.$$

Let us compare the exact sequences obtained by taking u^* with the analogous exact sequences built in terms of $f^\sharp$ and $\mathcal{E}^\sharp$: since $u^* f_* \mathcal{E} = f^\sharp_* \mathcal{E}^\sharp$, and $u^* f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) = f^\sharp_*(\mathcal{E}^\sharp \otimes_{\mathcal{O}_{X^\sharp}} \Omega^1_{X^\sharp/S^\sharp})$ [50, 9.3.3], we see that these respective exact sequences coincide, so that

$$\begin{aligned} u^* \mathrm{Im} f_*(\nabla) &= \mathrm{Im} f^\sharp_*(\nabla^\sharp), \\ u^* \mathrm{Coker} f_*(\nabla) &= \mathrm{Coker} f^\sharp_*(\nabla^\sharp) \cong R^1_{\mathrm{DR}} f^\sharp_*(\mathcal{E}^\sharp, \nabla^\sharp), \\ u^* \mathrm{Ker} f_*(\nabla) &= \mathrm{Ker} f^\sharp_*(\nabla^\sharp) \cong R^0_{\mathrm{DR}} f^\sharp_*(\mathcal{E}^\sharp, \nabla^\sharp), \end{aligned}$$

hence φ^0 and φ^1 are isomorphisms in $\mathbf{MIC}(S^\sharp)$, which establishes $(*)_1$.

26.2.8. We now consider a tower of height $d \geqslant 2$ as in $(*)_d$. We apply the induction hypothesis $(*)_{d-1}$ to $X_d \longrightarrow \cdots \longrightarrow X_1$, to the effect that there exists a dense affine open subset $j_1 : X_1' \hookrightarrow X_1$ such that $(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'}$ is coherent and the natural morphism

$$\begin{aligned}&(u_1^*(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla)))_{|u_1^{-1}(X_1')} \\ &\xrightarrow{\varphi_{(1)}} (R^{i_1}_{\mathrm{DR}} f^{\sharp}_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^{\sharp}_{d-1*} u_d^*(\mathcal{E}, \nabla))_{|u_1^{-1}(X_1')}\end{aligned}$$

is an isomorphism in $\mathbf{MIC}(u_1^{-1}(X_1'))$.

Let us consider the fibered prism

$$\begin{tikzcd}
X_1'^{\sharp} \arrow[rr, "j_1^{\sharp}"] \arrow[dr, "f_0'^{\sharp}"] \arrow[dd, "u_1'"'] & & X_1^{\sharp} \arrow[dl, "f_0^{\sharp}"'] \arrow[dd, "u_1"] \\
& X_0^{\sharp} \arrow[dd, "u_0"] & \\
X_1' \arrow[rr, "j_1"] \arrow[dr, "f_0'"'] & & X_1 \arrow[dl, "f_0"] \\
& X_0 &
\end{tikzcd}$$

By shrinking X_0, we may assume that f_0' is an elementary fibration, and (by applying $(*)_1$ to f_0') that

(1) $R^{i_0}_{\mathrm{DR}} f'_{0*}((R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'})$ is coherent over X_0.

(2) The natural morphism

$$\begin{array}{c} u_0^* R^{i_0}_{\mathrm{DR}} f'_{0*}(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'} \\ \big\downarrow \\ R^{i_0}_{\mathrm{DR}} f'^{\sharp}_{0*} u_1'^{*}(R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla))_{|X_1'} \\ \| \wr \\ R^{i_0}_{\mathrm{DR}} f'^{\sharp}_{0*} j_1^{\sharp *} u_1^* R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \end{array}$$

is an isomorphism in $\mathbf{MIC}(X_0^{\sharp})$.

On combining with $\varphi_{(1)}$, we get a natural isomorphism φ in $\mathbf{MIC}(X_0^{\sharp})$

$$\begin{array}{c} u_0^* R^{i_0}_{\mathrm{DR}} f_{0*} j_{1*} j_1^* R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E}, \nabla) \\ \varphi \big\downarrow \cong \\ R^{i_0}_{\mathrm{DR}} f^{\sharp}_{0*} j^{\sharp}_{1*} j_1^{\sharp *} R^{i_1}_{\mathrm{DR}} f^{\sharp}_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^{\sharp}_{d-1*} u_d^*(\mathcal{E}, \nabla)\ . \end{array} \tag{3}$$

By Remark 26.2.6, we may assume that one of the i_j is 0; by Remark 25.3.14, one then has that the natural maps

$$R^{i_0}_{\mathrm{DR}} f_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E},\nabla) \longrightarrow R^{i_0}_{\mathrm{DR}} f_{0*} j_{1*} j_1^* R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E},\nabla),$$

$$R^{i_0}_{\mathrm{DR}} f^\sharp_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^\sharp_{d-1*} u_d^*(\mathcal{E},\nabla) \longrightarrow R^{i_0}_{\mathrm{DR}} f^\sharp_{0*} j^\sharp_{1*} j_1^{\sharp *} R^{i_1}_{\mathrm{DR}} f^\sharp_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^\sharp_{d-1*} u_d^*(\mathcal{E},\nabla)$$

are isomorphisms. In particular, $R^{i_0}_{\mathrm{DR}} f_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E},\nabla)$ is coherent by (1). Moreover, these isomorphisms fit in a commutative square

$$\begin{tikzcd}
u_0^* R^{i_0}_{\mathrm{DR}} f_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E},\nabla) \arrow[r, "\sim"] \arrow[d, "\psi"'] & u_0^* R^{i_0}_{\mathrm{DR}} f_{0*} j_{1*} j_1^* R^{i_1}_{\mathrm{DR}} f_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f_{d-1*}(\mathcal{E},\nabla) \arrow[d, "\varphi"', "\cong (3)"] \\
R^{i_0}_{\mathrm{DR}} f^\sharp_{0*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^\sharp_{d-1*} u_d^*(\mathcal{E},\nabla) \arrow[r, "\sim"] & R^{i_0}_{\mathrm{DR}} f^\sharp_{0*} j^\sharp_{1*} j_1^{\sharp *} R^{i_1}_{\mathrm{DR}} f^\sharp_{1*} \cdots R^{i_{d-1}}_{\mathrm{DR}} f^\sharp_{d-1*} u_d^*(\mathcal{E},\nabla)
\end{tikzcd}$$

We conclude that ψ (the base change map appearing in $(*)_d$) is an isomorphism, and this achieves the inductive proof of 26.2.1. $\square$

(Counter)example 26.2.9. Consider the fibered square

$$\begin{tikzcd}
\operatorname{Spec} k[x] \arrow[r, "u^\sharp"] \arrow[d, "f^\sharp"'] & \operatorname{Spec} k[x,y] = X \arrow[d, "f"'] \\
\{y_0\} = \operatorname{Spec} k \arrow[r, "u"] & \operatorname{Spec} k[y] = S
\end{tikzcd}$$

in which f is the projection and u is the embedding of the point defined by $y = y_0$. Next, consider the integrable connection on X of rank 1 $(\mathcal{E},\nabla_{y_0})$ defined in 26.1.17. It is clear that $R^0_{\mathrm{DR}} f_*(\mathcal{E},\nabla_{y_0}) = 0$, while $R^0_{\mathrm{DR}} f^\sharp_* u^{\sharp *}(\mathcal{E},\nabla_{y_0}) \cong k$, because $u^{\sharp *}(\mathcal{E},\nabla_{y_0}) \cong (k[x], d)$.

This shows the necessity of localizing in 26.2.1, when u is not flat. This example is also interesting in connection with the base change theorem of [19, VI, 8.4].

27 Gauss-Manin connection in the regular case

27.1 Main theorems (in the regular case)

Let i be a non-negative integer. Let $f : X \to S$ be a *smooth morphism of smooth k-varieties, such that the Artin subset $A_i(f)$ is S itself* (cf. 25.2.1). Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module endowed with an integrable *regular* connection ∇.

Theorem 27.1.1 (Finiteness theorem). *$R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla)$ is a coherent $\mathcal{O}_S$-module.*

Theorem 27.1.2 (Regularity theorem). *The Gauss-Manin connection $\aleph$ on $R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla)$ is regular.*

Theorem 27.1.3 (**Monodromy theorem).** *If the exponents of* ∇ *all belong to some* $\mathbb{Q}$*-subspace* Δ *of* k *(* $\Delta \supseteq \mathbb{Q}$*), so do the exponents of* $\aleph$*.*

Theorem 27.1.4 (**Base change theorem).** *For any smooth* k*-variety* $S^\sharp$ *and any morphism* $u : S^\sharp \to S$*, the base change morphism*

$$u^* R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \longrightarrow R^i_{\mathrm{DR}} f^\sharp_* u^{\sharp *}(\mathcal{E}, \nabla)$$

attached to the fibered square

$$\begin{tikzcd} X^\sharp \arrow[r, "u^\sharp"] \arrow[d, "f^\sharp"'] & X \arrow[d, "f"] \\ S^\sharp \arrow[r, "u"] & S \end{tikzcd}$$

is an isomorphism.

Proofs. Let us say that $(\mathcal{E}, \nabla)$ satisfies property $\mathcal{P}_\Delta$ if and only if $\mathcal{E}$ is coherent, ∇ is regular, and all exponents of ∇ belong to Δ. Then $\mathcal{P}_\Delta$ is a strongly exact property (cf. Proposition 10.3.2 and Section 13), local for the étale topology (cf. Section 13). By the lemma on dévissage 25.3.4, in order to prove Theorems 27.1.1, 27.1.2, 27.1.3 it suffices to consider the case when f is a rational elementary fibration and $i = 0$ or 1, which will be settled in Sections 27.2 and 27.3.

As for the base change theorem, one reduces, as in Section 26.2 to the case when f is a tower of coordinatized elementary fibrations. The associated Leray spectral sequences allow us to further reduce to the case when f is a single coordinatized elementary fibration, which will be settled in 27.2.19. $\square$

27.2 Coherence of the cokernel of a regular connection

27.2.1. In this section, we establish that *for any rational elementary fibration* $f : X \to S$ *as in* (26.1.6) *and any locally free* $\mathcal{O}_X$*-module* $(\mathcal{E}, \nabla)$ *of finite rank* μ *with regular integrable connection,* $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ *is locally free of finite rank* (*over the whole of* S)*.* This is the key point in the proof of 27.1.1 (and 27.1.4).

We may assume that S is affine and connected and of dimension $\geqslant 1$. The statement amounts to the finite generation of $\Gamma(X, \mathcal{E})/\nabla_{\partial_x}\Gamma(X, \mathcal{E})$ over $\mathcal{O}(S)$.

27.2.2. If $(\mathcal{E}, \nabla)$ is cyclic and regular, the leading coefficients of the indicial polynomials are units (cf. Proposition 12.1.3), hence the result follows from 26.1.10.

However, unlike the situation of 25.3.10, the reduction to the cyclic case is problematic: it is not clear that one can *cover* S by étale neighborhoods S' with the property that $X_{S'}$ admits a Zariski open cover $\{U_\alpha\}$ such that $U_\alpha \to S'$ and $U_\alpha \cap U_\beta \to S'$ are rational elementary fibrations and such that $(\mathcal{E}, \nabla)_{|U_\alpha}$ is cyclic outside some divisor whose Zariski closure in $\mathbb{P}^1_{S'}$ is contained in U_α, see Section 18.1. We shall take another approach, via τ-extensions.

27.2.3. For notational convenience, we shall assume, by translating the "vertical" variable x, that one of the $\theta_1, \dots, \theta_r$, say θ_1, is 0 (if $r \neq 0$).

Let us fix a set-theoretic section τ of the projection $k \to k/\mathbb{Z}$ with $\tau(0) = 0$.

Let us consider the τ-extension $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ of $(\mathcal{E}, \nabla)$ on $\mathbb{P}^1_S$, as constructed in 10.2.5, 11.2.2: $\widetilde{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}^1_S}$-module, endowed with an integrable connection $\widetilde{\nabla} : \mathcal{D}er_Z\mathbb{P}^1_S \to \mathcal{E}nd_{\mathcal{O}_S}\widetilde{\mathcal{E}}$ with logarithmic poles along $\mathbb{P}^1_S \setminus X = Z = \coprod_{\nu=1,\dots r,\infty} Z_\nu$, such that the eigenvalues of the residue endomorphisms $\mathrm{Res}_{\theta_i}\widetilde{\nabla}$ belong to the image of τ.

Lemma 27.2.4. *Let $\mathcal{F}$ be a locally free $\mathcal{O}_{\mathbb{P}^1_S}$-module of finite rank. Then there exists a locally free $\mathcal{O}_S$-module $\mathcal{M}$ such that $\mathcal{F}_{|\mathbb{A}^1_S} \cong \mathrm{pr}^*\mathcal{M}$, where* pr *stands for the projection $\mathbb{A}^1_S \to S$.*

Proof. (of the lemma) The graded $\mathcal{O}(S)[x_0, x_1]$-module F attached to $\mathcal{F}$ is projective of finite rank (since so are its localizations with respect to the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}^1_S$ with $U_0 := \mathbb{A}^1_S$, $U_1 := \mathbb{P}^1_S \setminus (\{0\} \times S)$) and

$$F_{(x_i)} \cong \Gamma(U_i, \mathcal{F}),$$

for $i = 0, 1$. According to [26, X, p.144] there is a projective $\mathcal{O}(S)$-module M such that $F \cong M \otimes_{\mathcal{O}(S)} \mathcal{O}(S)[x_0, x_1]$. One may then take for $\mathcal{M}$ the locally free $\mathcal{O}_S$-module attached to M. □

Remark 27.2.5. Lemma 27.2.4 is a special, elementary case of a result of H. Lindel [75] extending the work of Quillen and Suslin on the Serre conjecture, and which says that in fact, for affine smooth S/k, any locally free $\mathcal{O}_{\mathbb{A}^1_S}$-module of finite rank has the form $\mathrm{pr}^*\mathcal{M}$, for some locally free $\mathcal{O}_S$-module $\mathcal{M}$.

27.2.6. We apply 27.2.4 to $\mathcal{F} = \widetilde{\mathcal{E}}$, to the effect that there are projective $\mathcal{O}(S)$-modules M and M' of rank μ, such that

$$\Gamma(\mathbb{A}^1_S, \widetilde{\mathcal{E}}) = M[x], \quad \Gamma(\mathbb{P}^1_S \setminus (\{0\} \times S), \widetilde{\mathcal{E}}) = M'[\frac{1}{x}],$$

and $M[x, \frac{1}{x}] = M'[x, \frac{1}{x}]$. Since M and M' are finitely generated, there is a positive integer N such that

$$x^N M'[x] \subseteq M[x] \subseteq x^{-N} M'[x].$$

27.2.7. Set

$$\Theta(x) = \prod_{i=1}^{r} (x - \theta_i). \tag{27.2.8}$$

We have $\widetilde{\nabla}_{\Theta(x)\partial_x} M[x] \subseteq M[x]$, and, for any $i = 1, \dots, r$, we are given an endomorphism $\mathrm{Res}_{\theta_i}\widetilde{\nabla}$ of M which enjoys the following properties (cf. Section 11.1):

(1) the characteristic polynomial of $\mathrm{Res}_{\theta_i}\widetilde{\nabla}$ has coefficients in k and roots in the image of τ (in particular, none of the roots is a non-zero integer);

(2) for any $m \in M$, identified with $m(x-\theta_i)^0 \in M[x-\theta_i] = M[x]$, we have

$$\widetilde{\nabla}_{(x-\theta_i)\partial_x} m - (\mathrm{Res}_{\theta_i}\widetilde{\nabla})m \ \in (x-\theta_i)M\Big[x, \frac{1}{\prod_{k\neq i}(x-\theta_k)}\Big].$$

Set $\gamma_i = \prod_{k\neq i}(\theta_i - \theta_k)$, which is a unit in $\mathcal{O}(S)$. It follows from (2) that for any $m \in M = M \cdot 1 \subseteq M[x, \frac{1}{x-\theta_i}]$,

(3) $\widetilde{\nabla}_{\Theta(x)\partial_x}(m(x-\theta_i)^{-n}) - \gamma_i(\mathrm{Res}_{\theta_i}\widetilde{\nabla} - n1_M)(m)(x-\theta_i)^{-n} \in (x-\theta_i)^{-n+1}M[x].$

On the other hand, (1) implies that $\mathrm{Res}_{\theta_i}\widetilde{\nabla} - n1_M$ is an invertible endomorphism of M for any $n > 0$ (by Cayley-Hamilton, the endomorphism $(\mathrm{Res}_{\theta_i}\widetilde{\nabla} - n1_M)^{-1}$ of $M \otimes_{\mathcal{O}(S)} \kappa(S)$ is a polynomial in $\mathrm{Res}_{\theta_i}\widetilde{\nabla} - n1_M$ with coefficients in k, hence induces an endomorphism of M). This shows that

$$m(x-\theta_i)^{-n} \in \widetilde{\nabla}_{\Theta(x)\partial_x}(M(x-\theta_i)^{-n}) + (x-\theta_i)^{-n+1}M[x].$$

By virtue of the Mittag-Leffler decomposition

$$M\Big[x, \frac{1}{\Theta(x)}\Big] = M \otimes \mathcal{O}(S)\Big[x, \frac{1}{\Theta(x)}\Big] = M[x] \oplus \bigoplus_{i=1}^{r} \frac{1}{x-\theta_i} M\big[\frac{1}{x-\theta_i}\big],$$

one concludes that

$$M\Big[x, \frac{1}{\Theta(x)}\Big] = \widetilde{\nabla}_{\Theta(x)\partial_x} M\Big[x, \frac{1}{\Theta(x)}\Big] + M[x]. \tag{4}$$

At this point the proof splits into two alternative arguments (27.2.9 and 27.2.13).

27.2.9 (First conclusion of the proof). From

$$\widetilde{\nabla}_{\frac{1}{x}\left(\prod_{i=2}^{r}(\frac{1}{x}-\frac{1}{\theta_i})\right)\partial_{\frac{1}{x}}} M'\big[\frac{1}{x}\big] \subseteq M'\big[\frac{1}{x}\big],$$

we deduce that

$$\widetilde{\nabla}_{x^{1-r}\Theta(x)\partial_x} M' \subseteq \bigoplus_{j=0}^{l} \frac{1}{x^j} M',$$

for sufficiently large $l \geqslant 1$. We consider the endomorphism $\mathrm{Res}_\infty \widetilde{\nabla}$ of M', which satisfies 27.2.7(1) above, and

$$\widetilde{\nabla}_{x\partial_x} m + (\mathrm{Res}_\infty\widetilde{\nabla})m \in \frac{1}{x}M'\big[\big[\frac{1}{x}\big]\big], \tag{27.2.10}$$

or, equivalently,

$$\widetilde{\nabla}_{x^{1-r}\Theta(x)\partial_x} m + (\mathrm{Res}_\infty\widetilde{\nabla})m \in \bigoplus_{j=1}^{l} \frac{1}{x^j} M', \tag{27.2.11}$$

for any $m \in M'$. So, for any $n \geqslant 0$ and $m \in M'$,

$$\widetilde{\nabla}_{x^{1-r}\Theta(x)\partial_x}(x^{n+l}m) + x^{n+l}(\mathrm{Res}_\infty \widetilde{\nabla} - (n+l)1_{M'})m \in \bigoplus_{j=0}^{l-1} x^j M'. \tag{27.2.12}$$

The invertibility of $\mathrm{Res}_\infty \widetilde{\nabla} - (n+l)1_{M'}$ implies that

$$M'[x]/\widetilde{\nabla}_{x^{1-r}\Theta(x)\partial_x} x^l M'[x]$$

is generated by $\bigoplus_{j=0}^{l-1} x^j M'$. So, for any $h \geqslant 0$,

$$x^{r-h-1}M'[x]/\widetilde{\nabla}_{\Theta(x)\partial_x} x^l M'[x]$$

is generated by $\bigoplus_{j=r-h-1}^{r+l-2} x^j M'$. We now pick l, h sufficiently large so that

$$x^l M'[x] \subseteq M[x] \subseteq x^{r-h-1}M'[x].$$

This proves that $M[x]/\widetilde{\nabla}_{\Theta(x)\partial_x} M[x]$ is finitely generated over $\mathcal{O}(S)$. A fortiori, the same holds for $\Gamma(X, \mathcal{E})/\nabla_{\partial_x}\Gamma(X, \mathcal{E})$. $\square_1$

27.2.13 (Alternative conclusion of proof). We come back to 27.2.7, and derive the coherence of $\mathrm{Coker}_{\Gamma(X,\mathcal{E})}\nabla_{\partial_x}$ using another argument which will be of importance in the next section.

Let us first remark that

$$M[x] \cap \widetilde{\nabla}_{\Theta(x)\partial_x} M[x, \frac{1}{\Theta(x)}] = \widetilde{\nabla}_{\Theta(x)\partial_x} M[x]. \tag{27.2.14}$$

Indeed, if an element of $M\big[x, \frac{1}{\Theta(x)}\big]$ has a pole of order $n \geqslant 0$ at θ_i, with Mittag-Leffler component $m(x-\theta_i)^{-n}$, and is sent to $M[x]$ by $\widetilde{\nabla}_{\Theta(x)\partial_x}$, then $m \in \mathrm{Ker}(\mathrm{Res}_{\theta_i}\widetilde{\nabla} - n1_M)$, in view of 27.2.7(3); hence, $n \in \mathrm{Im}\tau \cap \mathbb{Z} = \{0\}$. This argument also shows that

$$\mathrm{Ker}_{M[x]}\widetilde{\nabla}_{\Theta(x)\partial_x} = \mathrm{Ker}_{M\left[x, \frac{1}{\Theta(x)}\right]}\nabla_{\partial_x}. \tag{27.2.15}$$

Putting 27.2.7(4) and 27.2.14 together, we see that multiplication by $\Theta(x)^{-1}$ induces an isomorphism

$$M[x]/\widetilde{\nabla}_{\Theta(x)\partial_x} M[x] \cong M\Big[x, \frac{1}{\Theta(x)}\Big]/\nabla_{\partial_x} M\Big[x, \frac{1}{\Theta(x)}\Big]. \tag{27.2.16}$$

By the symmetry $x \mapsto \frac{1}{x}$, we also have

$$\mathrm{Ker}_{M'[\frac{1}{x}]}\widetilde{\nabla}_{\frac{1}{x}\left(\prod_{i=2}^r(\frac{1}{x}-\frac{1}{\theta_i})\right)\partial_{\frac{1}{x}}} = \mathrm{Ker}_{M'\left[x, \frac{1}{\Theta(x)}\right]}\nabla_{\partial_x} \tag{27.2.15$'$}$$

and

$$(27.2.16)' \quad M'[\frac{1}{x}]/\widetilde{\nabla}_{\frac{1}{x}\left(\prod_{i=2}^{r}(\frac{1}{x}-\frac{1}{\theta_i})\right)\partial_{\frac{1}{x}}} M'[\frac{1}{x}] \cong M\Big[x, \frac{1}{\Theta(x)}\Big]/\nabla_{\partial_x} M\Big[x, \frac{1}{\Theta(x)}\Big].$$

This can be summarized in the following diagram of isomorphisms:

$$(27.2.17) \quad \begin{array}{ccc} (\mathrm{Co})\mathrm{Ker}_{M[x]}\widetilde{\nabla}_{\Theta(x)\partial_x} & & \\ \downarrow & & \\ (\mathrm{Co})\mathrm{Ker}_{M[x,\frac{1}{x}]=M'[x,\frac{1}{x}]}\widetilde{\nabla}_{\Theta(x)\partial_x} & \xrightarrow{\cdot\Theta(x)^{-1}} & (\mathrm{Co})\mathrm{Ker}_{M\left[x,\frac{1}{\Theta(x)}\right]}\nabla_{\partial_x} \\ \Big\uparrow{\scriptstyle \cdot(-1)^r x^{r-1}\prod_{i=2}^r \theta_i} & & \\ (\mathrm{Co})\mathrm{Ker}_{M'[\frac{1}{x}]}\widetilde{\nabla}_{\frac{1}{x}\left(\prod_{i=2}^{r}(\frac{1}{x}-\frac{1}{\theta_i})\right)\partial_{\frac{1}{x}}} & & \end{array}$$

In terms of the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}^1_S$ with $U_0 := \mathbb{A}^1_S$, $U_1 := \mathbb{P}^1_S \setminus (\{0\} \times S)$, ($X \subseteq U_{01} := \mathbb{A}^1_S \setminus (\{0\} \times S)$), this diagram may be rewritten as follows. For $i = 0, 1$, the maps

$$\begin{array}{ccccc} \underline{h}^i\overline{f}_{|U_0*}(\Omega^\bullet_{U_0/S}(\log(U_0 \setminus X)) \otimes \widetilde{\mathcal{E}}_{|U_0}) & & & & \\ \downarrow & & & & \\ \underline{h}^i\overline{f}_{|U_{01}*}(\Omega^\bullet_{U_{01}/S}(\log(U_{01} \setminus X)) \otimes \widetilde{\mathcal{E}}_{|U_{01}}) & \longrightarrow & \underline{h}^i f_*(\Omega^\bullet_{X/S} \otimes \mathcal{E}) & \xrightarrow{\cong} & R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \\ \uparrow & & & & \\ \underline{h}^i\overline{f}_{|U_1*}(\Omega^\bullet_{U_1/S}(\log(U_1 \setminus X)) \otimes \widetilde{\mathcal{E}}_{|U_1}) & & & & \end{array}$$

are isomorphisms of $\mathcal{O}_S$-modules (in fact, isomorphisms in $\mathbf{MIC}(S)$).

Let us now compute $\mathbf{R}^i\overline{f}_*(\Omega^\bullet_{\mathbb{P}^1_S/S}(\log Z) \otimes \widetilde{\mathcal{E}})$, using the (2 by 2) Čech bicomplex of $\mathcal{O}_S$-modules $C^\bullet(\mathcal{U}, \Omega^\bullet_{\mathbb{P}^1_S/S}(\log Z) \otimes \widetilde{\mathcal{E}})$. The differentials $d_2^{0,0}$ and $d_2^{1,0}$ are given by $\widetilde{\nabla}$, while $d_1^{0,0}$ (resp. $d_1^{0,1}$) is induced by $(j_0^*, -j_1^*)$ (resp. $(-j_0^*, j_1^*)$), where j_k stands for the inclusion $U_{01} \subseteq U_k$. For $k = 0, 1$, (27.2.17) identifies $\mathrm{Ker}\, d_2^{k,0}$ (resp. $\mathrm{Coker}\, d_2^{k,0}$) with $2-k$ copies of $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ (resp. $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$), and for the first spectral sequence of the bicomplex, we find $\mathbf{I}_2^{0,0} \cong R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$, $\mathbf{I}_2^{0,1} \cong R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$, $\mathbf{I}_2^{1,0} = \mathbf{I}_2^{1,1} = 0$. Applying [29, chap. XV, sect. 6, case 4] we obtain:

Proposition. *The morphism of $\mathcal{O}_S$-modules*

$$(27.2.18) \quad \mathbf{R}^i\overline{f}_*(\Omega^\bullet_{\mathbb{P}^1_S/S}(\log Z) \otimes \widetilde{\mathcal{E}}) \longrightarrow \mathbf{R}^i\overline{f}_*(j_*(\Omega^\bullet_{X/S} \otimes \mathcal{E})) \cong R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla),$$

induced by the open S-embedding $j : X \hookrightarrow \mathbb{P}^1_S$, is an isomorphism.

Since $\mathbf{R}^i\overline{f}_*(\Omega^\bullet_{\mathbb{P}^1_S/S}(\log Z)\otimes\widetilde{\mathcal{E}})$ is the abutment of a spectral sequence starting with $E_1^{pq} = R^q\overline{f}_*(\Omega^p_{\mathbb{P}^1_S/S}(\log Z)\otimes\widetilde{\mathcal{E}})$, the coherence of $R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla)$ now follows from the stability of coherence under direct image by the projective morphism $\overline{f}$. $\square_2$

27.2.19 (**Base change**). Combining 27.2.1 and 26.2.7, we conclude that *for any fibered square of smooth k-varieties*

$$\begin{tikzcd} X^\sharp \arrow[r, "u^\sharp"] \arrow[d, "f^\sharp"'] & X \arrow[d, "f"] \\ S^\sharp \arrow[r, "u"] & S \end{tikzcd}$$

with f a rational elementary fibration, and for any regular object $(\mathcal{E},\nabla)$ of $\mathbf{MIC}(X)$*, the base change morphism*

$$u^*R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla) \longrightarrow R^i_{\mathrm{DR}}f^\sharp_*u^{\sharp *}(\mathcal{E},\nabla)$$

is an isomorphism for $i = 0, 1$.

In fact it is immediate to generalize 27.2.1 (coherence) and this base change property to the case where f is a coordinatized elementary fibration.

27.3 Regularity and exponents of the cokernel of a regular connection

27.3.1. In this section, we establish that *for any rational elementary fibration*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \\ & S & \end{tikzcd} \tag{27.3.2}$$

and for any locally free finite-rank $\mathcal{O}_X$-module $(\mathcal{E},\nabla)$ with regular integrable connection, the Gauss-Manin connection on $R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla)$ is regular for $i = 0, 1$; moreover, if the exponents of $(\mathcal{E},\nabla)$ belong to a certain $\mathbb{Q}$-space Δ ($\mathbb{Q} \subseteq \Delta \subseteq k$), so do the exponents of $R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla)$. This is the key point in the proof of 27.1.2 (and 27.1.3).

We notice that the case $i = 0$ is an easy consequence of 23.1.3.

27.3.3. By Theorem 13.1.5 and by base change 27.2.19, we may assume that S is the complement of a finite set of points Σ in a smooth projective *curve* $\overline{S}$. By the classical theory of embedded resolution of curves (Zariski), there exist a smooth projective surface $\overline{\overline{X}}$ containing $\overline{X}$, such that $\overline{Z} := \overline{\overline{X}} \smallsetminus X$ is a union $\bigcup_{i=1}^{s} \overline{Z}_j$ of smooth connected curves with normal crossings, and a projective morphism $\overline{\overline{f}} : \overline{\overline{X}} \to \overline{S}$ extending $\overline{f}$. By rearranging the $\overline{Z}_j$, we may assume that $\overline{Z}_j \cap \overline{X} \cong Z_j$, for $i \leqslant r$, while $\overline{\overline{f}}^{-1}(\Sigma) = \sum_{j=r+1}^{s} e_j \overline{Z}_j$, for some positive integers e_j. Following Katz [62], we shall construct a locally free extension of $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ on $\overline{S}$ and an extension of the Gauss-Manin connection with logarithmic poles along Σ.

27.3.4. Let us set

$$\Omega^p_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) = \bigwedge^p_{\mathcal{O}_{\overline{\overline{X}}}} (\Omega^1_{\overline{\overline{X}}}(\log \overline{Z}))/\overline{\overline{f}}^* \Omega^1_{\overline{S}}(\log \Sigma)).$$

Let $(\widetilde{\widetilde{\mathcal{E}}}, \widetilde{\widetilde{\nabla}})$ be a locally free extension of $(\mathcal{E}, \nabla)$ on $\overline{\overline{X}}$ with logarithmic poles along $\overline{Z}$ (see Section 11.1). Let us filter the logarithmic de Rham complex $\Omega^\bullet_{\overline{\overline{X}}}(\log \overline{Z}) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}}$ by

$$F^p = \text{image of } \overline{\overline{f}}^* \Omega^p_{\overline{S}}(\log \Sigma) \otimes \Omega^{\bullet - p}_{\overline{\overline{X}}}(\log \overline{Z}) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}},$$

so that

$$gr^p = F^p/F^{p+1} = \overline{\overline{f}}^* \Omega^p_{\overline{S}}(\log \Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \Omega^{\bullet - p}_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}}.$$

We define

$$R^i_{\mathrm{DR}} \overline{\overline{f}}_*(\widetilde{\widetilde{\mathcal{E}}}, \widetilde{\widetilde{\nabla}}) = \mathbf{R}^i \overline{\overline{f}}_*(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}})/\{\mathcal{O}_{\overline{S}} - \text{torsion}\},$$

which is a locally free $\mathcal{O}_{\overline{S}}$-module of finite rank because $\overline{\overline{f}}$ is projective and $\overline{S}$ is a curve. Moreover,

$$\mathbf{R}^i \overline{\overline{f}}_*(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}})_{|S} \cong \mathbf{R}^i \overline{f}_*(\Omega^\bullet_{\overline{X}/\overline{S}}(\log Z) \otimes_{\mathcal{O}_{\overline{X}}} \widetilde{\widetilde{\mathcal{E}}}_{|\overline{X}}) \cong R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla),$$

according to (27.2.18), so that $R^i_{\mathrm{DR}} \overline{\overline{f}}_*(\widetilde{\widetilde{\mathcal{E}}}, \widetilde{\widetilde{\nabla}})$ is a locally free extension of $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$. On the other hand, an extension of the Gauss-Manin connection

$$\mathbf{R}^i \overline{\overline{f}}_*(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}}) \longrightarrow \Omega^1_{\overline{S}}(\log \Sigma) \otimes_{\mathcal{O}_{\overline{S}}} \mathbf{R}^i \overline{\overline{f}}_*(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}})$$

is provided by the coboundary map of the long exact sequence

$$0 \longrightarrow gr^1 \longrightarrow F^0/F^2 \longrightarrow gr^0 \longrightarrow 0$$

(since $\Omega^1_{\overline{S}}(\log \Sigma) \otimes_{\mathcal{O}_{\overline{S}}} \mathbf{R}^i \overline{\overline{f}}_*(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{\overline{X}}}} \widetilde{\widetilde{\mathcal{E}}})$ may be identified with $\mathbf{R}^{i+1} \overline{\overline{f}}_*(gr^1)$). That map factors through torsion, and induces an extension of the Gauss-Manin connection on $R^i_{\mathrm{DR}} \overline{\overline{f}}_*(\widetilde{\widetilde{\mathcal{E}}}, \widetilde{\widetilde{\nabla}})$, with logarithmic poles along Σ.

27.3.5. Katz's argument [62, 7] applies without change to our situation and shows that the indicial polynomial of $R^i_{\mathrm{DR}}\overline{\overline{f}}_*(\overset{\approx}{\mathcal{E}}, \overset{\approx}{\nabla})$ at $s \in \Sigma$ divides

$$\prod_{j \in J} \prod_{f_j=0}^{e_j-1} \mathrm{ind}_j(e_j X - f_j), \tag{27.3.6}$$

where ind_j is the indicial polynomial of $(\overset{\approx}{\mathcal{E}}, \overset{\approx}{\nabla})$ at $\overline{Z}_j$, $\overline{\overline{f}}^{-1}(s) = \sum_{j \in J} e_j \overline{Z}_j$ (so, $J \subseteq \{r+1, \dots, s\}$). This completes the proof of the statement in 27.3.

Remarks 27.3.7. (i) A closer examination of Katz's argument allows to replace (27.3.6) by the least common multiple of the indicial polynomials $\mathrm{ind}_j(e_j X - f_j)$, for $j \in J$ and $f_j \in \{0, 1, \dots, e_j - 1\}$.

(ii) If $k \subseteq \mathbb{C}$ and τ is the canonical section $k/\mathbb{Z} \to k$, defined by $\mathrm{Re}(\tau(z)) \in [0, 1[$, and if $(\overset{\approx}{\mathcal{E}}, \overset{\approx}{\nabla})$ is the τ-extension of $(\mathcal{E}, \nabla)$ on $\overline{\overline{X}}$, then 27.3.5 shows that $R^i_{\mathrm{DR}}\overline{\overline{f}}_*(\overset{\approx}{\mathcal{E}}, \overset{\approx}{\nabla})$ is the τ-extension of $R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ on $\overline{S}$; indeed $\mathrm{Re}(z) \in [0, 1[$ implies that $\mathrm{Re}(\frac{z+f_j}{e_j}) \in [0, 1[$.

(iii) The arguments 27.3.3, 27.3.4, 27.3.5 work as well for any elementary fibration, not necessarily rational, over a curve.

Chapter IX

Complex and p-adic comparison theorems

Introduction

In this chapter, we show how to adapt Artin's strategy of proof of his comparison theorem for étale cohomology [2] in the de Rham context. The result is a particularly simple proof of the Grothendieck-Deligne comparison theorem (*algebraic versus complex-analytic de Rham cohomology with regular coefficients* [49], [35]). As a corollary, we obtain an elementary proof of Riemann's existence theorem for coverings, in higher dimensions (Appendix).

Not only this proof does not rely on resolution of singularities, it also does not make use of moderate growth conditions, nor even of the notion of monodromy. This enables us to transfer almost literally our argument to the p-adic setting, which yields a simple proof of the Kiehl-Baldassarri theorem (*algebraic versus rigid-analytic de Rham cohomology with regular coefficients* [68], [12]).

Furthermore, we prove that *in the p-adic setting, the comparison theorem extends to irregular connections*, as was conjectured by the second author (with the usual proviso on the field of definition).

28 The hypergeometric situation

Let us consider again the Gauss hypergeometric differential operator

$$L_{a,b,c} = x(1-x)\,\partial_x^2 + (c-(a+b+1)x)\,\partial_x - ab$$

with positive rational parameters. Since

$$L_{a,b,c}x^n = (a+n)(b+n)x^n - (c+n)(1+n)x^{n-1},$$
$$L_{a,b,c}(1-x)^n = (a+n)(b+n)(1-x)^n - (a+b+1-c+n)(1+n)(1-x)^{n-1},$$

Y. André et al., *De Rham Cohomology of Differential Modules on Algebraic Varieties*,
Progress in Mathematics 189, https://doi.org/10.1007/978-3-030-39719-7_9

it is easy to see that the cokernel of $L_{a,b,c}$ in $\mathcal{O}(S) = k[x, \frac{1}{x}, \frac{1}{1-x}]$ is a k-vector space of dimension 2 (in conformity with the global index formula 24.1.3), generated by $x^{-2}, (1-x)^{-2}$. It turns out that the natural map to the cokernel of $L_{a,b,c}$ in $\mathcal{O}(S^{\mathrm{an}})$ is an isomorphism (for $k = \mathbb{C}$).

The p-adic context exhibits the same behaviour. However, if we drop our usual assumption that the parameters are rational, and consider instead p-adic integers a, b, c which are very well approximated by infinitely many negative integers, it may happen that a series $\sum a_n x^n$ has finite radius of convergence, while $L_{a,b,c}(\sum a_n x^n) = \sum((a+n)(b+n)a_n - (c+1+n)(2+n)a_{n+1})x^n$ is an entire function; it fact, the cokernel of $L_{a,b,c}$ in $\mathcal{O}(S^{\mathrm{rig}})$ may be infinite-dimensional.

Let us now look at the Kummer hypergeometric differential operator

$$L_{a,c} = x\,\partial_x^2 + (c-x)\,\partial_x - a$$

with positive rational parameters a, c. It has an irregular singularity at ∞ with irregularity 1. Since

$$L_{a,c}x^n = (a+n)x^n - (c+n-1)nx^{n-1},$$

it is readily seen that the cokernel of $L_{a,c}$ in $\mathcal{O}(S) = k[x, \frac{1}{x}]$ is a k-vector space of dimension 1 (in conformity with the global index formula 24.1.3), generated by x^{-1}. In contrast to the Gauss case, the natural map to the cokernel of $L_{a,c}$ in $\mathcal{O}(S^{\mathrm{an}})$ sends x^{-1} to 0 (so that this cokernel is zero).

Indeed, let us consider two basic solutions $y_1, x^{1-c}y_2$ of $L_{a,c}$, with $y_1 = {}_1F_1(a, c; x)$, $y_2 = {}_1F_1(a+1, 2-c; x)$ (cf. 14.1.1). Up to a constant factor, their wronskian is $x^{-c}e^x$. Let z_2 be a primitive of $e^{-x}y_2$, and z_1 be x^{1-c} times a primitive of $x^{c-1}e^{-x}y_1$. The method of variation of the constant then shows that the entire function $y = z_1y_2 - z_2y_1$ is a solution of $L_{a,c}y = x^{-1}$.

In the p-adic situation, on the other hand (say when $k = \mathbb{C}_p$), the factorials appearing in the denominators "reduce" the radii of convergence. In particular, y_1, y_2 and y have finite radius of convergence, and the natural map to the cokernel of $L_{a,c}$ in $\mathcal{O}(S^{\mathrm{rig}})$ is an isomorphism.

29 Analytic contexts

29.1 Complex-analytic connections

29.1.1. Let $\mathcal{X}$ be a smooth complex analytic variety. The notion of an integrable connection on an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$ is defined as in the algebraic situation (Chapter II). It is equivalent to the notion of $\mathcal{D}_{\mathcal{X}}$-module. We denote by $\mathbf{MIC}(\mathcal{X})$ the category of $\mathcal{O}_{\mathcal{X}}$-modules with integrable connection. If $\mathcal{E}$ is coherent, it is then automatically locally free (same argument as in 4.5).

In the complex-analytic setting, every integrable connection is locally trivial by Cauchy's theorem in several variables. More precisely, for any $\mathcal{M}$ in $\mathbf{MIC}(\mathcal{X})$,

$\mathcal{M}^\nabla$ is a local system of complex vector spaces, and if one endows $\mathcal{M}^\nabla \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}}$ with the connection which is trivial on each open subset where $\mathcal{M}^\nabla$ is constant, then the natural morphism $\mathcal{M}^\nabla \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}} \to \mathcal{M}$ is an isomorphism in $\mathbf{MIC}(\mathcal{X})$. Moreover, $H^0_{\mathrm{DR}}(\mathcal{M}, \nabla) = \Gamma(\mathcal{X}, \mathcal{M}^\nabla)$.

29.1.2. Inverse images are constructed as in the algebraic case (5.1). For a smooth morphism of smooth complex analytic varieties $f : \mathcal{X} \to \mathcal{S}$, one constructs the direct image and the higher direct images $R^q_{\mathrm{DR}} f_*(\mathcal{E}, \nabla) \in Ob\ \mathbf{MIC}(\mathcal{S})$ of any $(\mathcal{E}, \nabla) \in Ob\ \mathbf{MIC}(\mathcal{X})$, as in the algebraic situation (Chapter VII). The formation of $R^q_{\mathrm{DR}} f_*$ is compatible with localization on $\mathcal{S}$.

29.1.3. One associates functorially to any (smooth) algebraic $\mathbb{C}$-variety X a (smooth) complex analytic variety X^{an}. The GAGA functor

$$\begin{aligned} \mathbf{MIC}(X) &\longrightarrow \mathbf{MIC}(X^{\mathrm{an}}) \\ (\mathcal{E}, \nabla) &\longmapsto (\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}}) \end{aligned}$$

commutes with inverse images (essentially because $\mathcal{O}_{X^{\mathrm{an}}}$ is flat over $\mathcal{O}_X$), but not with direct images in general. In the sequel, we limit ourselves to the case of analytic connections on algebraic complex analytic varieties (i.e., objects of $\mathbf{MIC}(X^{\mathrm{an}})$ for smooth algebraic varieties X over $\mathbb{C}$).

29.1.4. The discussion of Čech spectral sequences in 23.3 carries over for analytic connections, except that "quasi-coherent" should be replaced by "coherent" (and using the fact that affine varieties are Stein, and Cartan's theorem B).

29.1.5. Flat base change 23.5 is more delicate in the analytic context: the interpretation of $(u^{\sharp\,\mathrm{an}})^*$ as $(f^{\sharp\,\mathrm{an}})^{-1}\mathcal{O}_{S^{\sharp\,\mathrm{an}}} \otimes_{(u^{\sharp\,\mathrm{an}})^{-1}(f^{\mathrm{an}})^{-1}\mathcal{O}_{S^{\mathrm{an}}}} (u^{\sharp\,\mathrm{an}})^{-1}$ can only hold for a finite base change morphism u. For finite flat u, the argument of 23.5 (flat base change for ${}_I E_1^{p,q}$, using Čech complexes for affine covers as usual) carries over, and shows in particular that the formation of $R^q_{\mathrm{DR}} f^{\mathrm{an}}_*$ is compatible with étale localization (by factorizing an étale map as a composition of open immersions and finite flat morphisms[1]).

29.1.6. The computation $R^j_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}, \nabla) \cong H^j(f^{\mathrm{an}}_* \mathrm{DR}_{X^{\mathrm{an}}/S^{\mathrm{an}}}(\mathcal{E}, \nabla))$ of 23.6 for affine f (and vanishing for $j > d$), also holds.

However, for affine (or Stein) S, the computation of that cohomology sheaf in terms of the j-th cohomology module of the global relative de Rham complex seems to require the *a priori* knowledge of its coherence; using Cartan's theorem B, one then shows that the presheaves of cohomology of $f^{\mathrm{an}}_* \mathrm{DR}_{X^{\mathrm{an}}/S^{\mathrm{an}}}(\mathcal{E}, \nabla)$ are sheaves.

29.1.7. The first lemma of dévissage (25.3.4) also holds for properties $\mathcal{P}$ of modules with integrable connection on algebraic complex-analytic (with the same proof),

[1] A singular variety may appear in such a factorization. This does not make trouble since the $\mathcal{O}_S$-module $\mathbf{R}^q f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$ is well defined even if S is singular, provided f is smooth.

if one replaces "quasi-coherent" by "coherent". For instance, let $\mathcal{P}$ = (analytic) coherence, i.e., $(\mathcal{E},\nabla)$ on X^{an} satisfies $\mathcal{P}$ if and only if $\mathcal{E}$ is $\mathcal{O}_{X^{\mathrm{an}}}$-coherent. This property is clearly strongly exact and local for the étale topology. Moreover, since any rational elementary fibration $f' : X' \to S'$ is (locally on S') topologically trivial for the classical topology (Remark 25.1.6), and since for any $\zeta \in S'^{\mathrm{an}}$ and any coherent $(\mathcal{E}',\nabla') \in Ob\ \mathbf{MIC}(X'^{\mathrm{an}})$, $H^j\big(X'_\zeta, \mathcal{E}'^{\nabla'}_{|X'_\zeta}\big)$ is finite-dimensional (for $j = 0, 1$), it follows that $R^j_{\mathrm{DR}} f'^{\mathrm{an}}_*(\mathcal{E}',\nabla') \simeq R^j f'^{\mathrm{an}}_*(\mathcal{E}'^{\nabla'_{|\mathcal{D}er_{\mathrm{cont}} X^{\mathrm{an}}/S^{\mathrm{an}}}}) \otimes_{\mathbb{C}} \mathcal{O}_{S'^{\mathrm{an}}}$ is $\mathcal{O}_{S'^{\mathrm{an}}}$-coherent by [35, I, 2.28].

One concludes that *for any smooth morphism $f : X \to S$ of smooth complex algebraic varieties of pure relative dimension d, for any $j \geqslant 0$, and any coherent $(\mathcal{E},\nabla) \in Ob\ \mathbf{MIC}(X^{\mathrm{an}})$, $R^j_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E},\nabla)_{|U^{\mathrm{an}}}$ is coherent, with $U = A_j(f)$ if $j \leqslant d + \dim X$, and $U = S$ otherwise.*

29.1.8. However, we do not know whether the second lemma of dévissage (25.3.9) holds in the analytic category. While trying to adapt the given algebraic proof (or its mentioned alternative), one is confronted with the delicate question of comparing a de Rham complex with essential singularities along a divisor, with a meromorphic one.

29.1.9. Just as for the finiteness question (cf. 29.1.6), the analytic situation shows a simpler behaviour than the algebraic situation with respect to base change, namely:

Let $f : X \to S$ be a smooth morphism of smooth complex algebraic varieties, and let $(\mathcal{E},\nabla)$ be a coherent object of $\mathbf{MIC}(X^{\mathrm{an}})$. Then, for any smooth complex variety $S^\sharp$, any morphism $u : S^\sharp \to S$, and any $i \geqslant 0$, the restriction to $(u^{-1}A_i(f))^{\mathrm{an}}$ of the base change morphism

$$u^{\mathrm{an}*} R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E},\nabla) \longrightarrow R^i_{\mathrm{DR}} f^{\sharp\mathrm{an}}_*(u^{\sharp\mathrm{an}*}(\mathcal{E},\nabla))$$

is an isomorphism in $\mathbf{MIC}((u^{-1}A_i(f))^{\mathrm{an}})$.

Indeed, replacing S by affine étale neighborhoods of the connected components of $A_i(f)$, we may assume that X admits a finite open affine cover $\{U_\alpha\}$ with the properties listed in Definition 25.2.1. There is a morphism of Čech spectral sequences in $\mathbf{MIC}(S^{\sharp\mathrm{an}})$ analogous to the one considered in 26.2.4, which allows to reduce the argument to the case where f is a tower of (coordinatized) elementary fibrations. The result is then an easy consequence of the topological local triviality of the fibration f, which implies that

$$u^{\mathrm{an}*} R^i f^{\mathrm{an}}_* \mathcal{E}^{\nabla_{|\mathcal{D}er_{\mathrm{cont}} X^{\mathrm{an}}/S^{\mathrm{an}}}} \xrightarrow{\sim} R^i f^{\sharp\mathrm{an}}_* \mathcal{E}^{\sharp\nabla^\sharp_{|\mathcal{D}er_{\mathrm{cont}} X^{\sharp\mathrm{an}}/S^{\sharp\mathrm{an}}}},$$

is an isomorphism.

29.2 Rigid analytic connections

29.2.1. Similar comments apply to the rigid-analytic situation, over an algebraically closed field K of characteristic 0, complete with respect to a non-archimedean absolute value $|\text{-}|$ (e.g., $K = \mathbb{C}_p$) ([17], [20]). Notions and comments 29.1.1 to 29.1.5 may be repeated in this context. One uses the fact that affine covers of X give rise to admissible covers of X^{an}, that affine varieties are quasi-Stein, and Kiehl's analogue of theorem B, cf. [69].

However, Cauchy's theorem in this context is weaker (solutions do not converge up to the next singularity), and as a consequence, integrable connections are only *locally* trivial for the "wobbly topology" (i.e., in small balls), not in the sense of rigid geometry.

29.2.2. The discussion 23.3.4 also carries over if $\{U_\alpha\}$ is chosen to be an admissible affinoid cover of X^{an}. This remark allows us to prove finite flat base change (using the first spectral sequence of 23.3.4 for such a cover, and flat base change for $f_{\underline{\alpha}*}(\Omega^j_{U_{\underline{\alpha}}}/S)$ with affinoid $f_{\underline{\alpha}}$). Therefore, the formation of $R^i_{\mathrm{DR}} f^{\mathrm{an}}_*$ is again compatible with étale localization. The previous discussion of $R^i_{\mathrm{DR}} f^{\mathrm{an}}_*$ for affine f carries over in the rigid-analytic context (on replacing "Stein" by "quasi-Stein", and Stein covers by admissible affinoid covers), but the question of coherence of $R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}, \nabla)$ is more delicate.

In fact, since Dwork's early studies, it is well-known that one cannot expect generic finiteness of $R^1_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ in general, due to "Liouville phenomena". As we shall see later, in the case of an elementary fibration, the problem disappears when *the roots* α_{ij} *of the indicial polynomials* ind_i *are non-Liouville* (in the sense that the series $\sum\limits_{n \geqslant 0, n \neq -\alpha_{ij}} \frac{x^n}{\alpha_{ij}+n}$ converge).

29.2.3. As is well known, there are quite a few different available frameworks for non-archimedean analytic geometry: Tate's original rigid-analytic geometry, Raynaud's algebro-geometric approach using admissible blow-ups, valuative approaches à la Zariski-Riemann, Berkovich's geometry, Huber's geometry... To fix ideas, we state our results in rigid-analytic terms, but it is clear that they could be transposed without change into any other non-archimedean analytic setting. In fact, our "axiomatic" techniques are designed to use as little archimedean or non-archimedean analysis as possible.

30 Abstract comparison criteria

We present abstract comparison criteria for the kernel and cokernel of differential modules in various differential extensions.

30.1 First criterion

Let (C, ∂) be a differential $\mathbb{Q}$-algebra, and let A, B be differential sub-algebras, such that $A^\partial = C^\partial$ (equality of rings of constants), A is faithfully flat over A^∂, and $\partial_{|A}$ is surjective onto A.

Proposition 30.1.1 (First comparison criterion). *Let E be an $(A \cap B)\langle\partial\rangle$-module, projective of finite rank μ as a module over $A \cap B$. Assume that E is solvable in A, i.e., the canonical map*

$$(E \otimes A)^\partial \otimes_{A^\partial} A \longrightarrow E \otimes A$$

is an isomorphism. Then $\varphi_0 : \mathrm{Ker}_E \partial \to \mathrm{Ker}_{E\otimes B} \partial$ is an isomorphism and $\varphi_1 : \mathrm{Coker}_E \partial \to \mathrm{Coker}_{E\otimes B} \partial$ is injective.

Proof. For short, we write E_A, etc. instead of $E \otimes A$...

Injectivity of φ_0: this follows from the fact that, since E is flat over $A \cap B$, E injects into E_B.

Surjectivity of φ_0 and injectivity of φ_1: since E is solvable in A, which is faithfully flat over A^∂, the A^∂-module $(E_A)^\partial$ is projective (of rank μ), and, locally with respect to the Zariski topology on $\mathrm{Spec}\, A^\partial$, the $A[\partial]$-module E_A is a sum of copies of A. Since $A^\partial = C^\partial$, this in turn implies

$$(E_A)^\partial = (E_C)^\partial. \tag{30.1.2}$$

On the other hand, since $\partial : A \to A$ is surjective, and Zariski-locally on $\mathrm{Spec}\, A^\partial$, $E_A \cong A^\mu$ as $A\langle\partial\rangle$-modules, it follows that $\partial : E_A \to E_A$ is Zariski-locally (on $\mathrm{Spec}\, A^\partial$) surjective, hence

$$\partial : E_A \longrightarrow E_A \quad \text{is surjective.} \tag{30.1.3}$$

One thus has a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (E_A)^\partial \arrow[r] \arrow[d, "\cong"] & E_A \arrow[r] \arrow[d] & E_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (E_C)^\partial \arrow[r] & E_C \arrow[r] & E_C &
\end{tikzcd}$$

which implies that the right face of the cube

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dd] & & E_A \arrow[dd] \\
E \arrow[ur, "\partial"] \arrow[rr] \arrow[dd] & & E_A \arrow[ur, "\partial"] \arrow[dd] & \\
& E_B \arrow[rr] & & E_C \\
E_B \arrow[ur, "\partial"] \arrow[rr] & & E_C \arrow[ur, "\partial"] &
\end{tikzcd}$$

is cartesian; since the front and rear faces are also cartesian (since E is a locally free $(A\cap B)$-module, $E_A \cap E_B = E$), we conclude that the left face is cartesian as well. As a consequence, φ_0 is surjective and φ_1 is injective. □

30.2 Second criterion

Proposition 30.2.1 (**Second comparison criterion).** *In the setting of the preceding proposition* (*with the same assumptions*), *we assume moreover that C* (*resp. A*) *is a product of differential $\mathbb{Q}$-algebras C_i* (*resp. $A_i \subseteq C_i$*), *$i = 0, 1, \ldots, r$, containing a differential subalgebra C_i'* (*resp. $A_i' := A_i \cap C_i'$*).

We assume furthermore, for each i, the existence of

(i) *a differential A_i'-subalgebra T_i of C_i,*

(ii) *a multiple $\partial_i = u_i\partial$ by a unit $u_i \in A_i'$,*

(iii) *a sequence of elements of elements of A_i such that $\partial_i(\partial_i^{-j}1) = \partial_i^{-j+1}1$,*

subject to the conditions:

(a) *$\partial : C_i \to C_i$ is surjective and C_i is faithfully flat over C_i^∂;*

(b) (*b*) *the composed homomorphism of differential algebras $B \to C \xrightarrow{\mathrm{pr}_i} C_i$ factors through an injection $B \hookrightarrow C_i'$;*

(c) *the natural morphism $C_i^{\log} := \bigoplus_{j\geqslant 0} C_i'\partial_i^{-j}1 \to C_i$ is injective and identifies $C_i^{\log}$ with a differential subalgebra of C_i, and the natural morphism $T_i \otimes_{A_i'} C_i^{\log} \to C_i$ is an isomorphism of differential algebras;*

(d) *$C_0' \subseteq A_0' + \mathrm{pr}_0 B$, and, for $i > 0$, $C_i' \subseteq A_i' + \bigcap_{j<i} \mathrm{pr}_i(B \cap A_j)$* (*using* (b) *to make sense of this intersection*).

Then $\varphi_1 : \mathrm{Coker}_E\partial \to \mathrm{Coker}_{E\otimes B}\partial$ is surjective.

Proof. We remark that (via pr_i) E is solvable in A_i, which is faithfully flat over A_i^∂, and $A_i^\partial = C_i^\partial$. A fortiori, E is solvable in C_i. On the other hand, (b) implies that $\mathrm{pr}_{i|B}$ induces an injection $A\cap B \hookrightarrow A_i'$; via pr_i, $B \cap A_i = B \cap A_i'$, which we shall view either as a subring of B or of C_i', and $\bigcap_{j=0}^r B\cap A_j = A \cap B$. We have a cartesian diagram of inclusions

$$\begin{array}{ccc} E_{A_i'} & \longrightarrow & E_{A_i} \\ \downarrow & & \downarrow \\ E_{C_i'} & \longrightarrow & E_{C_i} \end{array}$$

The assumptions of Proposition 30.1.1 are satisfied on replacing E by $E_{(A_i\cap T_i)}$ and the differential algebras C, A, B by C_i, A_i and T_i, respectively. In particular (30.1.2), (30.1.3) apply in this setting, and give:

(30.2.2) $$(E_{A_i})^\partial = (E_{C_i})^\partial, \quad \partial : E_{A_i} \longrightarrow E_{A_i} \quad \text{is surjective.}$$

Let us fix an element $e = e_{-1}$ of E_B. We are looking for an element $f \in E$ and an element $e' \in E_B$ such that $e = f + \partial e'$.

To this aim, we shall construct recursively, for $i = 0, \dots, r$, an element $e'_i \in E \otimes (\bigcap_{j<i} B \cap A_j)$, such that $e_i := e_{i-1} - \partial e'_i$ belongs to $E \otimes (\bigcap_{j \leqslant i} B \cap A_j)$. It will then suffice to take $e' := \sum_{i=0}^r e'_i$ and $f := e_r$ (which belongs to $E \otimes (\bigcap_{j=0}^r B \cap A_j) = E$).

Let us start with $e_{i-1} \in E \otimes (\bigcap_{j<i} B \cap A_j) \subseteq E_{C'_i}$ (if $i = 0$, $e_{-1} \in E_B \subseteq E_{C'_i}$). There exists an element $f_i \in E_{C_i} = E_{T_i} \otimes_{A'_i} C_i^{\log}$ such that $e_{i-1} = \partial f_i$. By condition (c), we may write $f_i = \sum_{j=0}^N f_{i,j} \partial_i^{-j} 1$, with $f_{i,j} \in E_{T_i} \otimes_{A'_i} C'_i$. Equation $e_{i-1} = \partial f_i$ decomposes into the following set of equations

$$\begin{aligned} \partial f_{i,N} &= 0, \\ \partial f_{i,N-1} &= -u_i^{-1} f_{i,N}, \\ &\vdots \\ \partial f_{i,1} &= -u_i^{-1} f_{i,2}, \\ \partial f_{i,0} &= -u_i^{-1} f_{i,1} + e_{i-1}, \end{aligned}$$

from which we deduce, step by step, using (30.2.2), that $f_{i,N} \in (E_{A_i} \otimes_{A'_i} C'_i)^\partial = (E_{A_i})^\partial$, $f_{i,N-1}, \dots, f_{i,1} \in E_{A_i}$.

By condition (d), $f_{i,0} = e'_i + f'_i$, with $e'_i \in E \otimes (\bigcap_{j<i} B \cap A_j)$, $f'_i \in E_{A'_i}$. One then observes that the element $e_i := e_{i-1} - \partial e'_i$ of $E \otimes (\bigcap_{j<i} B \cap A_j)$ may also be written $u_i^{-1} f_{i,1} + \partial f'_i$, an element of E_{A_i}. Therefore, $e_i \in E \otimes (\bigcap_{j \leqslant i} B \cap A_j)$. □

31 Comparison theorem for algebraic vs. complex-analytic cohomology

31.1 Statement of the comparison

Theorem 31.1.1 ([35, II, 6.13]). *Let i be a non-negative integer, and let $f : X \to S$ be a smooth morphism of smooth complex algebraic varieties, with $A_i(f) = S$ (cf. Definition 25.2.1). Let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module endowed with an integrable* regular *connection. Then the canonical morphism in* $\mathbf{MIC}(S^{\mathrm{an}})$*:*

$$\varphi_i : (R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))^{\mathrm{an}} \longrightarrow (R^i_{\mathrm{DR}} f_*^{\mathrm{an}}(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}}))$$

is an isomorphism.

In the special case $S = \operatorname{Spec} \mathbb{C}$ and $(\mathcal{E}, \nabla) = (\mathcal{O}_X, d_{X/S})$, this gives (taking into account the analytic Poincaré lemma):

Corollary 31.1.2 ([49]). *For any smooth complex algebraic variety X and any $i \geqslant 0$, there are canonical isomorphisms $H^i_{\mathrm{DR}}(X) \cong H^i_{\mathrm{DR}}(X^{\mathrm{an}}) \cong H^i(X(\mathbb{C}), \mathbb{C})$.*

31.2 Reduction to the case of a rational elementary fibration

31.2.1. If f is of relative dimension 0, the statement amounts to the fact that for étale finite f, the canonical morphism $(f_*\mathcal{E})^{\mathrm{an}} \to f_*^{\mathrm{an}}\mathcal{E}^{\mathrm{an}}$ is an isomorphism, which is an elementary result of GAGA type (one may reduce, as in 25.3.3, to the case when f is a Galois covering, and then remark that $(f^*f_*\mathcal{E})^{\mathrm{an}} \xrightarrow{\sim} f^{\mathrm{an}*}f_*^{\mathrm{an}}\mathcal{E}^{\mathrm{an}}$).

31.2.2. As in 25.3.6, one then reduces to the case when f has pure relative dimension $d \geqslant 1$, $S = A_i(f)$ is affine connected, and $0 \leqslant i \leqslant d + \dim X$. Moreover, since the statement is local on S^{an}, and since the formation of algebraic or analytic de Rham cohomology commutes with étale localization on S (23.5, 29.1.5), we may replace S, as in 25.3.7, by an affine connected étale neighborhood S' such that $X_{S'}$ admits a finite open cover $\{U_\alpha\}$ with the properties listed in definition 25.1.4.

There is a natural morphism of Čech spectral sequences in $\mathbf{MIC}(S^{\mathrm{an}})$,

$$\begin{array}{ccc}
\bigoplus\limits_{\alpha_0<\cdots<\alpha_p}(R^{i-p}_{\mathrm{DR}}f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}})^{\mathrm{an}} & \Longrightarrow & (R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla))^{\mathrm{an}} \\
\Big\downarrow{\scriptstyle \varphi_{i-p}} & & \Big\downarrow{\scriptstyle \varphi_i} \\
\bigoplus\limits_{\alpha_0<\cdots<\alpha_p} R^{i-p}_{\mathrm{DR}}f^{\mathrm{an}}_{\underline{\alpha}*}(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|U^{\mathrm{an}}_{\underline{\alpha}}} & \Longrightarrow & R^i_{\mathrm{DR}}f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})
\end{array}$$

We argue by induction on i: since $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) = A_i(f) = S$, φ_{i-p} is an isomorphism for $p > 0$; hence we may assume that f itself is a tower of coordinatized elementary fibrations.

31.2.3. For f a tower of coordinatized elementary fibrations, we have a natural morphism of Leray spectral sequences, by which we are reduced to proving the comparison theorem for each level of the tower. Therefore, we may assume that f itself is a coordinatized elementary fibration, taking into account the fact that regularity is preserved under direct images by coordinatized elementary fibrations (27.3). In fact, since we are free to replace S by an étale covering, we may even assume that $f = f' \circ \pi$, where f' is a rational elementary fibration and π is an étale covering. We then have $R^i_{\mathrm{DR}}f_*(\mathcal{E},\nabla) = R^i_{\mathrm{DR}}f'_* \circ \pi_*(\mathcal{E},\nabla)$ (resp. $R^i_{\mathrm{DR}}f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}) = R^i_{\mathrm{DR}}f'^{an}_* \circ \pi^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})$), and we are reduced to proving the comparison theorem for a rational elementary fibration, and $i = 0, 1$.

At this point the proof splits into two alternative ways.

31.3 First way: reduction to an ordinary linear differential system

We use the fact that both the source and the target of φ_i are coherent $\mathcal{O}_{S^{\mathrm{an}}}$-modules (27.2 and 29.1.7, resp.). Therefore, it suffices to check that the fibre $(\varphi_i)_s$ is an isomorphism for any $s \in S(\mathbb{C})$. By base change (26.2, 29.1.9), we are finally reduced to the case of an ordinary linear (regular) differential system, i.e., $S = \operatorname{Spec}\mathbb{C}$, $X = \mathbb{A}^1 \setminus \{\theta_1,\dots,\theta_r\}$, and it is enough to deal with global sections. Hence, we have reduced Theorem 31.1.1 to the following special case.

Proposition 31.3.1. *Let $\theta_1, \dots, \theta_r$ be distinct complex numbers and let*

$$\Theta(x) = \prod_{i=1}^{r} (x - \theta_i) \in \mathbb{C}[x].$$

Consider the affine rational curve $X = \operatorname{Spec} \mathbb{C}[x, \frac{1}{\Theta(x)}]$ and let (E, ∇) be a projective $\mathcal{O}(X)$-module of finite rank μ endowed with a regular connection. Then the natural morphisms

$$\varphi_0 : \operatorname{Ker}_E \nabla_{\partial_x} \longrightarrow \operatorname{Ker}_{E \otimes \mathcal{O}(X^{\mathrm{an}})} \nabla_{\partial_x}$$

and

$$\varphi_1 : \operatorname{Coker}_E \nabla_{\partial_x} \longrightarrow \operatorname{Coker}_{E \otimes \mathcal{O}(X^{\mathrm{an}})} \nabla_{\partial_x}$$

are isomorphisms.

We shall deduce proposition 31.3.1 from the abstract comparison criteria of Section 30.

31.3.2. Let notation be as in Proposition 31.3.1. We further set $B := \mathcal{O}(X^{\mathrm{an}})$, and, for $i = 1, \dots, r$,

$C'_i := \bigcup_{\epsilon > 0} \mathcal{O}(D(\theta_i, \epsilon) \smallsetminus \{\theta_i\})$,
$\mathcal{M}_{\theta_i}$:= the field of germs of meromorphic functions at θ_i,
$A_i := \mathcal{M}_{\theta_i}[\log(x - \theta_i), (x - \theta_i)^{\alpha}]_{\alpha \in \mathbb{C}}$, with $(x - \theta_i)^{\alpha}(x - \theta_i)^{\beta} = (x - \theta_i)^{\alpha+\beta}$,
$C_i = A_i C'_i$.

We adopt similar definitions for $i = 0$, upon replacing θ_i by ∞.

We set

$$C := \prod_{i=0}^{r} C_i, \quad A := \prod_{i=0}^{r} A_i, \quad \text{and} \quad \partial = \frac{d}{dx}.$$

We have

$$A \cap B = \Big(\prod_{i=0}^{r} \mathcal{M}_{\theta_i} \Big) \cap B = \mathcal{O}(X) = \mathbb{C}\Big[x, \frac{1}{\Theta(x)}\Big],$$

diagonally embedded in A,

$$A_i^{\partial} = C_i^{\partial} = \mathbb{C}, \quad A'_i := A_i \cap C'_i = \mathcal{M}_{\theta_i}.$$

31.3.3. We now exploit the *regularity* of the $(A \cap B)[\partial]$-module E in the guise that E is *solvable in* A, or, what amounts to the same, in each A_i (Fuchs-Frobenius, the point being that formal solution series of a differential operator at a regular singular point are convergent). It remains to check that all assumptions in Section 30 are met. Both A and C are clearly faithfully flat over $A^{\partial} = C^{\partial} \cong \mathbb{C}^{r+1}$. Using the formula

$$x^{\alpha-1} \frac{\log^k x}{k!} = \partial\Big(\frac{x^{\alpha}}{\alpha} \sum_{i+j=k} \Big(-\frac{1}{\alpha} \Big)^i \frac{\log^j x}{j!} \Big),$$

for $\alpha \neq 0$, and

$$x^{-1}\frac{\log^k x}{k!} = \partial \frac{\log^{k+1} x}{(k+1)!},$$

one sees that ∂ acts surjectively both on A and on C. This proves the requirements for 30.2.1, and condition (a) of 30.2.1 in addition. Condition (b) is clear (by analytic continuation). Condition (c) holds for $i \geqslant 1$ if we set

$$u_i = x - \theta_i, \quad \partial_i^{-j} 1 = \frac{\log^j (x-\theta_i)}{j!}, \quad T_i = \mathcal{M}_{\theta_i}[(x-\theta_i)^\alpha]_{\alpha \in \mathbb{C}}.$$

For $i = 0$, we adopt similar definitions, replacing θ_i by ∞.

Finally, (d) holds in a stronger form. For any $i \in \{0, \dots, r\}$, $C'_i = A'_i + \bigcap_{j \neq i}(B \cap A_j)$, which reflects the decomposition of the Laurent expansion of an element of C'_i, into its positive and negative parts, respectively.

This completes the proof of 31.3.1 and of the comparison theorem. $\square_1$

Remark 31.3.4. A concrete realization of the elements $\log(x - \theta_i)$ as germs of complex-analytic functions at θ_i can be obtained as follows. For any $i = 1, \dots, r$, fix $\eta_i \in \mathbb{C} \smallsetminus \{\theta_i\}$ and define L_i to be the analytic solution of the differential equation $\partial Y = \frac{\eta_i - \theta_i}{x - \theta_i}$ in a neighborhood of the open real half-line $\sigma_i := \theta_i + \mathbb{R}_{>0}(\eta_i - \theta_i)$ which takes real values on σ_i and vanishes at η_i, namely

$$L_i = -\sum_{n=1}^{\infty} n^{-1}\Big(\frac{x-\eta_i}{\theta_i - \eta_i}\Big)^n.$$

We define the *germ of* L_i *at* θ_i *along* σ_i to be the inductive system of restrictions to L_i to the elements U of the filter $\mathcal{F}_i$ of open neighborhoods in $\mathbb{C} \smallsetminus \{\theta_i\}$ of some open interval $(\theta_i, \varepsilon(\eta_i - \theta_i))$, for $\varepsilon > 0$. Similarly for $i = 0$ we may take as $\log(x - \theta_0)$ the germ of

$$L_0 = -\sum_{n=1}^{\infty} n^{-1}\Big(\frac{x-1}{x}\Big)^n$$

on open neighborhoods in $\mathbb{C}$ of the real half-line $(N, +\infty)$, as $N \to +\infty$. Then, for any $\alpha \in \mathbb{C}$, we pick as $(x - \theta_i)^\alpha$ the germ at θ_i along σ_i of the analytic function $\exp(\alpha L_i)$. The rings A_i and C_i obtained as in section 31.3.2 from these definitions of $\log(x - \theta_i)$ and $(x - \theta_i)^\alpha$, for $i = 0, 1, \dots, r$ and $\alpha \in \mathbb{C}$, satisfy the requirements in Proposition 30.1.1 and in Proposition 30.2.1. The proof of Proposition 31.3.1 and of the comparison theorem is independent of the choices made.

31.4 Second way: dealing with the relative situation

The proofs 29.1.7, 29.1.9 of coherence and base change for $R^1_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$, which are used in 31.3, rely on topological arguments (monodromy). We can avoid such arguments by handling directly the case of a rational elementary fibration in the following way.

31.4.1. For $\mathcal{S}$ compact in $S(\mathbb{C})$ and $\epsilon > 0$, we introduce the compact tube in $S^{\mathrm{an}} \times (\mathbb{A}^1)^{\mathrm{an}}$

$$T_{\mathcal{S},i,\epsilon} = \{(s,x),\ s \in \mathcal{S},\ |x - \theta_i(s)| \leqslant \epsilon\},$$

and

$$T^*_{\mathcal{S},i,\epsilon} = \{(s,x),\ s \in \mathcal{S},\ 0 < |x - \theta_i(s)| \leqslant \epsilon\}.$$

Let Δ be a $\mathbb{Q}$-subspace of $\mathbb{C}$ containing $\mathbb{Q}$ (in the sequel, Δ will be the $\mathbb{Q}$-space generated by 1 and the exponents of the canonical extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$, taking into account Theorem 27.1.3 in the course of dévissage).

We define
$C'_{\mathcal{S},i} = \bigcup_{\epsilon>0} \mathcal{O}(T^*_{\mathcal{S},i,\epsilon})$,
$A_{\mathcal{S},i} = \bigcup_{\epsilon>0} \mathcal{O}(T_{\mathcal{S},i,\epsilon})[(x-\theta_i)^\alpha, \log(x-\theta_i)]_{\alpha\in\Delta}$.

Remark 31.4.2. As in Remark 31.3.4, we can here give an explicit analytic form to the elements $\log(x-\theta_i)$ and $(x-\theta_i)^\alpha$. We may, for example, pick a decreasing family of compact connected neighborhoods $\{\mathcal{S}_\varepsilon\}_\varepsilon$ of $\mathcal{S}$ in S^{an}, so that, for any $\varepsilon' < \varepsilon$, $\mathcal{S}_\varepsilon$ is a neighborhood of $\mathcal{S}_{\varepsilon'}$, and choose a permutation $i \mapsto j(i)$ of $\{0, 1, \ldots, r\}$ with no fixed point. Then, for any sufficiently small $\varepsilon > 0$, we may consider the analytic function

$$L_i = -\sum_{n=1}^{\infty} n^{-1} \Big(\frac{x - \theta_{j(i)}}{\theta_i - \theta_{j(i)}}\Big)^n$$

on an open connected neighborhood $U_{i,\varepsilon}$ in $T^*_{\mathcal{S}_\varepsilon,i,\epsilon}$ of the real hypersurface

$$x = \theta_i(s) + \rho\theta_{j(i)}(s),$$

for $\rho \in (0, \varepsilon]$ and $s \in \mathcal{S}_\varepsilon$. Then L_i defines a germ of analytic function on (small open neighborhoods in X^{an} of) the subsets in the filtering family $\{U_{i,\varepsilon}\}_{\varepsilon>0}$. We may take $\log(x-\theta_i)$ as that germ, and then coherently define $(x-\theta_i)^\alpha = \exp(\alpha \log(x-\theta_i))$.

We also set

$$C_{\mathcal{S},i} = C'_{\mathcal{S},i} A_{\mathcal{S},i}, \quad A'_{\mathcal{S},i} := C'_{\mathcal{S},i} \cap A_{\mathcal{S},i} = \bigcup_{\epsilon>0} \mathcal{O}(T_{\mathcal{S},i,\epsilon}) \Big[\frac{1}{x-\theta_i}\Big].$$

We adopt similar definitions for $i = 0$, upon replacing θ_i by ∞, and set

$$C_{\mathcal{S}} = \prod_{i=0}^{r} C_{\mathcal{S},i}, \quad A_{\mathcal{S}} = \prod_{i=0}^{r} A_{\mathcal{S},i}, \quad \partial = \frac{d}{dx}.$$

Finally, we introduce the inverse image $\mathcal{X}$ of $\mathcal{S}$ in X^{an}, and set $B_{\mathcal{S}} = \mathcal{O}(\mathcal{X})$, diagonally embedded into $C_{\mathcal{S}}$.

One then checks as in 31.3.3 that $A_{\mathcal{S}}$ is faithfully flat over $A^\partial_{\mathcal{S}} = C^\partial_{\mathcal{S}} = \mathcal{O}(\mathcal{S})^{r+1}$, that $\partial_{|A_{\mathcal{S}}}$ is surjective onto $A_{\mathcal{S}}$, and that conditions (a),…,(d) of 30.2.1

are fulfilled, with

$$\begin{aligned} u_i &= x - \theta_i, \\ \partial_i^{-j} 1 &= \frac{\log^j(x-\theta_i)}{j!}, \\ T_i &= \bigoplus_{\alpha \in \Delta \cap (\operatorname{Im}\tau \smallsetminus \{0\})} C'_{\mathcal{S},i}(x-\theta_i)^\alpha. \end{aligned}$$

Notice that $A_{\mathcal{S}} \cap B_{\mathcal{S}} = \mathcal{O}(\mathcal{S})\big[x, \frac{1}{\Theta}\big]$.

31.4.3. Because $\widetilde{\mathcal{E}}$ is locally free, there exists a Zariski covering of S by open subsets U such that $\widetilde{\mathcal{E}}_{|Z_i \cap f^{-1}(U)}$ is free over $\mathcal{O}_{Z_i \cap f^{-1}(U)}$ for every i, where $Z_i = V(x-\theta_i) \subseteq \mathbb{P}^1_S$ (Z_0 is the divisor at infinity). Then for every compact $\mathcal{S} \subseteq U(\mathbb{C})$, there exists $\epsilon_{\mathcal{S}} > 0$ such that $\widetilde{\mathcal{E}} \otimes \mathcal{O}(T_{\mathcal{S},i,\epsilon})$ is free over $\mathcal{O}(T_{\mathcal{S},i,\epsilon})$ if $\epsilon \leqslant \epsilon_{\mathcal{S}}$. We denote by $E_{\mathcal{S}}$ the $\mathcal{O}(\mathcal{S})\big[x, \frac{1}{\Theta(x)}\big]\langle \frac{d}{dx}\rangle$-module $\Gamma(\mathcal{E}) \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S})$.

Proposition 31.4.4. *$E_{\mathcal{S}}$ is solvable in $A_{\mathcal{S}}$.*

Proof. We have to show that $E_{\mathcal{S}}$ is solvable in $A_{\mathcal{S},i}$, for every $i = 0, \dots, r$. Let $\underline{e}$ be a basis of $\widetilde{\mathcal{E}} \otimes \mathcal{O}(T_{\mathcal{S},i,\epsilon})$ in which the connection may be written $\nabla\big((x-\theta_i)\frac{d}{dx}\big)\underline{e} = \underline{e}G$, where G has entries analytic in $T_{\mathcal{S},i,\epsilon}$, and $G_{|Z_i}$ is a constant matrix with eigenvalues in Δ: indeed, since $T_{\mathcal{S},i,\epsilon}$ is a compact tube, there exists a compact neighborhood $\mathcal{S}'$ of $\mathcal{S}$ such that $G \in M_\mu(\mathcal{O}(T_{\mathcal{S}',i,\epsilon}))$.

It suffices to show that the differential system $\big((x-\theta_i)\frac{d}{dx} + G\big)Y = 0$ has a solution of the form $Y = W \cdot (x-\theta_i)^{G_{|Z_i}}$, where $W \in \mathrm{GL}_\mu(\mathcal{O}(T_{\mathcal{S},i,\epsilon'}))$, for some $0 < \epsilon' < \epsilon$. We follow the classical method used in the case $\mathcal{S} =$ one point, cf. e.g., [38, III, 8.5 and App. II] a formal computation expresses the coefficients of the Taylor expansion of W at $x = \theta_i$ by a recursive formula, under the extra condition: $W_{|Z_i} = I$; one then estimates the growth of these coefficients (with respect to the sup-norm on $\mathcal{S}'$) as in [38, App. II] and one concludes that $W \in M_\mu(\mathcal{O}(T_{\mathcal{S},i,\epsilon''}))$, $0 < \epsilon'' < \epsilon$. Since $W_{|Z_i} = I$, it is clear that there exists ϵ', $0 < \epsilon' \leqslant \epsilon''$, such that $W \in \mathrm{GL}_\mu(\mathcal{O}(T_{\mathcal{S},i,\epsilon'}))$, as wanted. This concludes the proof of 31.4.4.

31.4.5. We can apply the criteria 30.1.1 and 30.2.1, and conclude that

$$(\mathrm{Co})\mathrm{Ker}_{E_{\mathcal{S}}} \nabla_{\partial_x} \cong (\mathrm{Co})\mathrm{Ker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})} \nabla_{\partial_x}.$$

Since $\mathcal{O}(\mathcal{S})$ is flat over $\mathcal{O}(S)$, this may be rewritten

$$(\mathrm{Co})\mathrm{Ker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x} \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \cong (\mathrm{Co})\mathrm{Ker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})} \nabla_{\partial_x},$$

which holds for any small enough compact neighborhood $\mathcal{S}$ of any point of $S(\mathbb{C})$. Therefore, the natural morphism

$$(R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))^{\mathrm{an}} \longrightarrow R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$$

is an isomorphism for $i = 0, 1$, and this completes the alternative proof of 31.1.1. $\square_2$

Corollary 31.4.6. *Let X be a smooth complex algebraic variety. The functor*

$$\begin{aligned}\{\text{regular connections on } X\} &\longrightarrow \{\text{local systems on } X(\mathbb{C})\} \\ (\mathcal{E}, \nabla) &\longmapsto (\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}\end{aligned}$$

is fully faithful.

Proof. Let $(\mathcal{E}, \nabla)$, $(\mathcal{E}', \nabla')$ be two regular objects in $\mathbf{MIC}(X)$. Then

$$\mathcal{H}om((\mathcal{E}, \nabla), (\mathcal{E}', \nabla'))$$

is a regular object in $\mathbf{MIC}(X)$ (see 13). By 31.1.1, with $i = 0$, we have

$$\begin{aligned}\mathrm{Hom}((\mathcal{E}, \nabla), (\mathcal{E}', \nabla')) &= H^0_{\mathrm{DR}}(X, \mathcal{H}om((\mathcal{E}, \nabla), (\mathcal{E}', \nabla'))) \\ &\cong H^0_{\mathrm{DR}}(X^{\mathrm{an}}, \mathcal{H}om((\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}}), (\mathcal{E}'^{an}, \nabla'^{an}))) \\ &\cong H^0(X(\mathbb{C}), \mathcal{H}om((\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}, (\mathcal{E}'^{an})^{\nabla'^{an}})) \\ &= \mathrm{Hom}((\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}, (\mathcal{E}'^{an})^{\nabla'^{an}}).\end{aligned}$$

Corollary 31.4.7. *Let X and $(\mathcal{E}, \nabla)$, $(\mathcal{E}', \nabla')$ be as before. Then the natural map*

$$\mathrm{Ext}^1_{\mathbf{MIC}(X)}((\mathcal{E}, \nabla), (\mathcal{E}', \nabla')) \longrightarrow \mathrm{Ext}^1_{\text{loc. syst.}}((\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}, (\mathcal{E}'^{an})^{\nabla'^{an}})$$

is a bijection.

Proof. Same argument as for 31.4.6, but with $i = 1$.

31.5 Deligne's GAGA version of the index formula

Many formulas in algebraic geometry, which hold over any field k, of characteristic 0 or even of positive characteristic p, were first proven by complex-analytic methods. A typical example is the Riemann-Roch formula for divisors Z on a smooth projective curve X over k,

$$\ell(Z) - \ell(K - Z) = \deg Z + 1 - g, \tag{31.5.1}$$

where $g := \dim_k H^0(X, \Omega^1_{X/k})$ is the genus of X, $\ell(Z) := \dim_k H^0(X, \mathcal{O}_X(Z))$, $\ell(K - Z) := \dim_k H^0(X, \Omega^1_{X/k}(-Z))$. Formula (31.5.1) admits a purely algebraic proof, but, in the case of $k = \mathbb{C}$, its meaning is strongly improved by a number of comparison theorems which relate different types of cohomology groups. The complete understanding, thanks to Grothendieck, of derived functor cohomology together with the sheaf-theoretic interpretation of classical invariants, show, for example, that the invariant g appearing in (31.5.1) coincides with the topological genus of the compact orientable surface underlying the compact Riemann surface X^{an} associated to X, and add an essential extra flavor to (31.5.1). Conversely, the discovery that a classical analytic procedure leads to an algebraic result is surprising and of fundamental importance.

In this vein, we recall that a formula related to (24.1.4) appears in Deligne's book [35, II.6] in a complex-analytic vs. complex-algebraic setting. Namely, formula 6.21.1 of loc.cit. says

$$\chi_{\mathrm{DR}}(U^{\mathrm{an}};(\mathcal{E},\nabla)^{\mathrm{an}})-\chi_{\mathrm{DR}}(U;(\mathcal{E},\nabla))=\sum_{z\in Z}\mathrm{ir}_z(\mathcal{E},\nabla). \tag{31.5.2}$$

The classical equivalence of categories $(\mathcal{E},\nabla)^{\mathrm{an}}\longleftrightarrow E$ between flat analytic vector bundles and complex local systems on the complex manifold U^{an}, comprises the statement

$$H^{\bullet}_{\mathrm{DR}}(U^{\mathrm{an}};(\mathcal{E},\nabla)^{\mathrm{an}})\cong H^{\bullet}(U^{\mathrm{an}};E), \tag{31.5.3}$$

which implies the equality

$$\chi_{\mathrm{DR}}(U^{\mathrm{an}};(\mathcal{E},\nabla)^{\mathrm{an}})=\chi_{\mathrm{top}}(U^{\mathrm{an}};E)=(\dim_{\mathbb{C}}E)\chi_{\mathrm{top}}(U^{\mathrm{an}})=(\mathrm{rk}\,\mathcal{E})\chi_{DR}(U^{\mathrm{an}})\ ,$$

while (6.20) (c) of loc.cit. gives the equality $\chi_{\mathrm{DR}}(U^{\mathrm{an}})=\chi_{\mathrm{DR}}(U)$. In the end we obtain the purely algebraic formula 24.1.4.

32 Comparison theorem for algebraic vs. rigid-analytic cohomology (regular coefficients)

32.1 Liouville numbers

In this section, k is a valued field extension of $\mathbb{Q}_p$, assumed to be complete and algebraically closed. We recall that the (p-adic) *Liouville type* of $a\in k$ is the radius of convergence of the series

$$\sum_{j=0,\,j\neq -a}^{\infty}\frac{x^j}{a+j}. \tag{32.1.1}$$

The formula [39, 21.2.4]

$$\frac{s!}{(a)_s}=\sum_{i+j=s}(-1)^j\binom{s}{j}\frac{1}{a+j-1}$$

shows that if the p-adic Liouville type of a is $\tau\in(0,1]$ then we have the estimate

$$|a(a+1)\cdots(a+s-1)|\leqslant\kappa\,p^{\frac{s}{p-1}}\,\tau^{-s},\quad\forall\,s=1,2,\dots, \tag{32.1.2}$$

for some $\kappa>0$. We also recall that algebraic numbers have Liouville type 1. See [67, §13.1] for proofs.

Definition 32.1.3. *We will say that $a \in k$ is a (p-adically)* Liouville number *if its Liouville type is* 0*. We will say that a is* non-Liouville*, otherwise. A polynomial $P(t) \in k[t]$ is (p-adically)* non-Liouville *if all of its roots are non-Liouville.*

A Liouville number $a \in k$ necessarily belongs to $\mathbb{Z}_p \subseteq k$. For a p-adically non-Liouville polynomial $P(t) \in k[t]$ of degree d and for any $n \in \mathbb{Z}$, we have an estimate of the form

(32.1.4)

$$|P(-n)P(-n-1)\cdots P(-n-s+1)| \leqslant \kappa\, ||P||^s (p^{\frac{1}{p-1}}\, \tau^{-1})^{sd}, \quad \forall\, s = 1, 2, \dots,$$

for some $\kappa > 0$, where $||P||$ denotes the Gauss norm of $P(t)$ (maximum of the absolute values of the coefficients).

32.2 Comparison

Theorem 32.2.1. *Let i be a non-negative integer, and let $f : X \to S$ be a smooth morphism of smooth k-varieties, with $A_i(f) = S$. Let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module endowed with an integrable regular connection, such that the additive subgroup of k generated by* 1 *and the exponents of ∇ contains no Liouville number. Then the canonical morphism*

$$\varphi_i : (R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))^{\mathrm{an}} \longrightarrow R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$$

is an isomorphism.

In the special case $S = \operatorname{Spec} k$ and $(\mathcal{E}, \nabla) = (\mathcal{O}_X, d_{X/S})$, this gives:

Corollary 32.2.2. *For any smooth algebraic k-variety X and any $i \geqslant 0$, $H^i_{\mathrm{DR}}(X) \cong H^i_{\mathrm{DR}}(X^{\mathrm{an}})$.*

In the case $S = \operatorname{Spec} k$, theorem 32.2.1 was already proven in [12] using resolution of singularities.

32.2.3. Our proof of 32.2.1 is very close to the proof in the complex case (in its second form, cf. 31.4). We reduce to the case of a rational elementary fibration, $i = 0, 1$, and apply the abstract criteria of Section 30. We introduce $\mathcal{S}$, $\mathcal{X}$, $T_{\mathcal{S},i,\epsilon}$, $A_{\mathcal{S},i}$, $C'_{\mathcal{S},i}$, $A_{\mathcal{S}}$, $C_{\mathcal{S}}$, $E_{\mathcal{S}}$, as in 31.4 (replacing the word “compact” by “affinoid”). The only difference is that for the solvability of $E_{\mathcal{S}}$ in $A_{\mathcal{S}}$ – more specifically, in the non-archimedean estimates replacing [38, Appendix II] –, as well as for the stability of $A_{\mathcal{S},i}$ and $C'_{\mathcal{S},i}$ by integration, we have to *use the fact that* Δ (the $\mathbb{Q}$-vector space generated by 1 and the exponents) *does not contain Liouville numbers*, which follows from the assumption in 32.2.1.

We conclude again that the natural morphism

$$(\mathrm{Co})\mathrm{Ker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x} \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \longrightarrow (\mathrm{Co})\mathrm{Ker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})} \nabla_{\partial_x}$$

is an isomorphism for any affinoid subspace $\mathcal{S}$ of U^{an}, where $\{U\}$ is a finite Zariski open covering of S as in 31.4.3.

32.2.4. Since (Co)Ker$_{\Gamma(X,\mathcal{E})}\nabla_{\partial_x}$ is finitely generated over $\mathcal{O}(S)$ (see 27.2), we see that

$$\mathcal{S} \longmapsto \mathrm{Ker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})}\nabla_{\partial_x} \quad (\text{resp.} \quad \mathcal{S} \longmapsto \mathrm{Coker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})}\nabla_{\partial_x})$$

is a coherent sheaf on U^{an}, which coincides with

$$R^0_{\mathrm{DR}}(f^{\mathrm{an}}_{|f^{\mathrm{an}-1}U^{\mathrm{an}}})_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|(f^{\mathrm{an}})^{-1}U^{\mathrm{an}}} \cong (R^0_{\mathrm{DR}}f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}))_{|U^{\mathrm{an}}}$$

(resp. $(R^1_{\mathrm{DR}}f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}))_{|U^{\mathrm{an}}}$).

By pasting, we conclude that φ_i is an isomorphism for $i = 0, 1$ in our special situation. This completes the proof of 32.2.1. $\square$

33 Rigid-analytic comparison theorem in relative dimension one

From here on, we drop any assumption of regularity on our connections.

33.1 On the coherence of the cokernel of a connection in the rigid analytic situation

We place ourselves in the situation of 26.1.5. We have the rational elementary fibration (26.1.6) where we assume that the base space S is affine and smooth. We let $(\mathcal{E},\nabla)$ be an object of $\mathbf{MIC}(X)$ which satisfies the assumptions of Lemma 26.1.10 with associated monic differential polynomial $\Lambda \in \mathcal{O}(X)\langle\partial_x\rangle$, as in 26.1.11; in particular, we assume that S is localized in such a way that the leading coefficients of the indicial polynomials (7.3.2)

$$\text{(33.1.1)} \qquad \mathrm{ind}_{\theta_i}(t) = \mathrm{ind}_{\Lambda,\theta_i}(t), \quad i = 1,\dots,r \ (\text{resp. } \mathrm{ind}_\infty(t) = \mathrm{ind}_{\Lambda,\infty}(t)),$$

of Λ at $x = \theta_i$ (resp. at $x = \infty$) are units in $\mathcal{O}(S)$.

We denote by $f^{\mathrm{an}} : X^{\mathrm{an}} \to S^{\mathrm{an}}$ and $(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}) \in \mathbf{MIC}(X^{\mathrm{an}})$ the corresponding analytic objects. Let also $\mathcal{S} \subseteq S^{\mathrm{an}}$ be an affinoid subspace, and let $\mathcal{X} \to \mathcal{S}$ denote the (quasi-Stein) morphism obtained from f^{an} by base change. The following proposition is a rigid analytic version of Lemma 3.2.15.

Proposition 33.1.2. *Assumptions and notation as in Lemma 3.2.15. We further assume:*

- *the polynomials* $\frac{\mathrm{ind}_i(t)}{\gamma_i} \in k[t]$ *are non-Liouville.*

Then the natural map

$$\text{(33.1.3)}\quad \mathrm{Coker}\Big(\Gamma(X,\mathcal{E}) \xrightarrow{\nabla_{\partial_x}} \Gamma(X,\mathcal{E})dx\Big)\otimes\mathcal{O}(\mathcal{S}) \longrightarrow \mathrm{Coker}\Big(\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}}) \xrightarrow{\nabla^{\mathrm{an}}_{\partial_x}} \Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})dx\Big)$$

is surjective.

The rest of this (sub)section is devoted to the proof of this assertion.

33.1.4. There is a commutative diagram (induced by (26.1.11))

$$
\begin{tikzcd}
\mathcal{O}(\mathcal{X}) \arrow[r, "\Lambda"] \arrow[d] & \mathcal{O}(\mathcal{X}) \arrow[d] \\
\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}}) = \bigoplus_{k=0}^{\mu-1}\mathcal{O}(\mathcal{X})v_k \arrow[r, "\nabla_{\partial_x}"'] & \bigoplus_{k=0}^{\mu-1}\mathcal{O}(\mathcal{X})v_k = \Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})
\end{tikzcd}
\tag{33.1.5}
$$

which gives rise to isomorphisms:

$$
(\mathrm{Co})\mathrm{Ker}\Big(\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}}) \xrightarrow{\nabla_{\partial_x}} \Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})dx\Big) \cong (\mathrm{Co})\mathrm{Ker}_{\mathcal{O}(\mathcal{X})}\Lambda \cong (\mathrm{Co})\mathrm{Ker}_{\mathcal{O}(\mathcal{X})}\Lambda'
$$

where $\Lambda' = P(x)\Lambda \in \mathcal{O}(\mathcal{S})[x]\langle\partial_x\rangle$ with $P(x)$ chosen as in (26.1.12).

33.1.6. We have a Mittag-Leffler decomposition (induced by (26.1.14))

$$
\mathcal{O}(\mathcal{X}) = \mathcal{O}(\mathbb{A}^{1\,\mathrm{an}}_{\mathcal{S}}) \oplus \bigoplus_{i=1}^{r} \frac{1}{x-\theta_i}\mathcal{O}(\mathbb{P}^{1\,\mathrm{an}}_{\mathcal{S}} \setminus \sigma_i(\mathcal{S})),
\tag{33.1.7}
$$

where

$$
\mathcal{O}(\mathbb{A}^{1\,\mathrm{an}}_{\mathcal{S}}) = \Big\{ \sum_{n\geqslant 0} a_{n,\infty}x^n \in \mathcal{O}(\mathcal{S})[[x]]\ ,\ \lim_{n\to\infty} ||a_{n,\infty}||^{1/n} = 0 \Big\},
$$

$$
\mathcal{O}(\mathbb{P}^{1\,an}_{\mathcal{S}} \setminus \sigma_i(\mathcal{S})) = \Big\{ \sum_{n\geqslant 0} \frac{a_{n,i}}{(x-\theta_i)^n} \in \mathcal{O}(\mathcal{S})[[\frac{1}{x-\theta_i}]]\ ,\ \lim_{n\to\infty} ||a_{n,i}||^{1/n} = 0 \Big\},
$$

for some (any) Banach norm $||\text{-}||$ on the Tate k-algebra $\mathcal{O}(\mathcal{S})$.

33.1.8. Let us restate more precisely the induction process in 26.1.5. We recall from (26.1.12) that

$$
\Lambda' = \sum_{k=0}^{\mu} a'_k\partial_x^k,\ \text{ with } a'_k \in \mathcal{O}(X),\ \forall\, k = 0,1,\dots,\mu.
$$

For ease of notation, we will denote by γ_∞ and γ_i (rather than γ'_∞ and γ'_i, as in 26.1.5) the leading coefficients of the indicial polynomials ind'_∞ and ind'_{θ_i} of Λ'. We will assume that they are invertible in $\mathcal{O}(S)$. In view of our explicit computation (and taking into account 26.1.13) we set, for $k = 0,1,\dots,\mu$,

$$
a'_k = \sum_{l=k-r'_i}^{k+r'_\infty} b_{k,l,i}(x-\theta_i)^l = \sum_{l=0}^{k+r'_\infty} b_{k,l}x^l,
$$

with $b_{k,l,i}$, $b_{k,l} \in \mathcal{O}(S)$, where r'_∞ and all r'_i's are $\geqslant 0$. For $i = 1, \dots, r$ we also set

$$\kappa_i = \max\{1, \max_{k,l} ||k! \gamma_i^{-1} b_{k,l,i}||\}, \text{ for } i = 1, \dots, r, \quad \text{and}$$
$$\kappa_\infty = \max\{1, \max_{k,l} ||k! \gamma_\infty^{-1} b_{k,l}||\}.$$

We claim that for $n > 0$ (resp. $n \geqslant 0$) we can solve for $\gamma_{n,l,i}, \delta_{n,l,i} \in \mathcal{O}(S)$

$$(*)_{n,i} \qquad \frac{1}{(x-\theta_i)^n} = \Lambda'\Big(\sum_{l=1}^{n-r'_i} \gamma_{n,l,i} \frac{1}{(x-\theta_i)^l}\Big) + \sum_{l=1}^{M} \delta_{n,l,i} \frac{1}{(x-\theta_i)^l},$$

with

$$||\gamma_{n,l,i}||\ ,\ ||\delta_{n,l,i}|| \leqslant \kappa_i^{\max(0,n-M)} \prod_{m=M+1}^{n} \max\Big\{1, \Big|\frac{\gamma_i}{\mathrm{ind}'_{\theta_i}(-m+r'_i)}\Big|\Big\}$$

and for $\gamma_{n,l,\infty}, \delta_{n,l,\infty} \in \mathcal{O}(S)$

$$(*)_{n,\infty} \qquad x^n = \Lambda'\Big(\sum_{l=0}^{n-r'_\infty} \gamma_{n,l,\infty} x^l\Big) + \sum_{l=0}^{M} \delta_{n,l,\infty} x^l,$$

with

$$||\gamma_{n,l,\infty}||\ ,\ ||\delta_{n,l,\infty}|| \leqslant \kappa_\infty^{\max(0,n-M)} \prod_{m=M+1}^{n} \max\Big\{1, \Big|\frac{\gamma_\infty}{\mathrm{ind}'_\infty(-m+r'_\infty)}\Big|\Big\}.$$

Indeed this is trivial for $n \leqslant M$. For $n > M$, we write

$$\frac{1}{(x-\theta_i)^n} = \Lambda'\Big(\frac{1}{\gamma_i} \cdot \frac{\gamma_i}{\mathrm{ind}'_{\theta_i}(-n+r'_i)} \cdot \frac{1}{(x-\theta_i)^{n-r'_i}}\Big)$$
$$+ \sum_{k=n-r'_i-r'_\infty}^{\max\{M, n-r'_i-r'_\infty\}} \beta_{n,k,i}(x-\theta_i)^{-k} + \sum_{k=\max\{M,n-r'_i-r'_\infty\}+1}^{n-1} \beta_{n,k,i}(x-\theta_i)^{-k},$$

with $|\beta_{n,k,i}| \leqslant \kappa_i \max\big\{1, \big|\frac{\gamma_i}{\mathrm{ind}'_{\theta_i}(-n+r'_i)}\big|\big\}$, and $(*)_{n,i}$ follows by induction, on applying $(*)_{<n,i}$ to the terms of the last sum (resp.

$$x^n = \Lambda'\Big(\frac{1}{\gamma_\infty} \cdot \frac{\gamma_\infty}{\mathrm{ind}'_\infty(-n+r'_\infty)} \cdot x^{n-r'_\infty}\Big) + \sum_{k=\max\{0,n-r'_\infty-\mu\}}^{\max\{M,n-r'_\infty-\mu\}} \beta_{n,k,\infty} x^k + \sum_{k=\max\{M,n-r'_\infty-\mu\}+1}^{n-1} \beta_{n,k,\infty} x^k,$$

with $|\beta_{n,k,\infty}| \leqslant \kappa_\infty \max\{1, |\frac{\gamma_\infty}{\mathrm{ind}'_\infty(-n+r'_\infty)}|\}$). On the other hand,

$$\prod_{m=M+1}^{n} \max\Big\{1, \Big|\frac{\gamma_i}{\mathrm{ind}'_{\theta_i}(-m+r'_i)}\Big|\Big\}$$

(resp.

$$\prod_{m=M+1}^{n} \max\Big\{1, \Big|\frac{\gamma_\infty}{\mathrm{ind}'_\infty(-m+r'_\infty)}\Big|\Big\}\ )$$

grows at most exponentially with n, because the polynomials $\mathrm{ind}'_{\theta_i}(t)$, for $i = 1,\dots,r$ (resp. $\mathrm{ind}'_\infty(t)$) are assumed to be non-Liouville. Hence $||\gamma_{n,l,i}||, ||\delta_{n,l,i}|| \leqslant \kappa'^n$, for some $\kappa' \geqslant 1$.

Therefore, any $\sum_{n\geqslant 0} a_{n,\infty}x^n + \sum_i \sum_{n\geqslant 1} \frac{a_{n,i}}{(x-\theta_i)^n} \in \mathcal{O}(\mathcal{X})$ may be written as $\Lambda' u + v$, where

$$u = \sum_{l\geqslant 0}\Big(\sum_{n\geqslant l+r'_\infty} a_{n,\infty}\gamma_{n,l,\infty}\Big)x^l + \sum_i \sum_{l\geqslant 1}\Big(\sum_{n\geqslant l+r'_i} a_{n,i}\gamma_{n,l,i}\Big)\frac{1}{(x-\theta_i)^l},$$

$$v = \sum_{k=0}^{M}\Big(\sum_{n\geqslant 0} a_{n,\infty}\delta_{n,k,\infty}\Big)x^k + \sum_i \sum_{k=1}^{M}\Big(\sum_{n\geqslant 1} a_{n,i}\delta_{n,k,i}\Big)\frac{1}{(x-\theta_i)^k},$$

and the previous estimates show that $u \in \mathcal{O}(\mathcal{X})$, $v \in \mathcal{O}(\mathcal{S})\big[x, \frac{1}{\Theta(x)}\big]$.

This shows that the natural map

$$\big(\mathrm{Coker}_{\mathcal{O}(X)}\Lambda'\big) \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \longrightarrow \mathrm{Coker}_{\mathcal{O}(\mathcal{X})}\Lambda'$$

is surjective, hence so is the map 33.1.3. □

Remark 33.1.9. As was already pointed out in the algebraic situation 26.1.16, the proof does not use the full hypothesis that v is a cyclic vector on the whole of X (i.e., that one may take $P(x) = \prod_{i=1}^{r}(x-\theta_i)^{s_i}$). It would suffice to assume that v is a cyclic vector outside some divisor D whose Zariski closure in $\mathbb{P}^1_S$ is contained in X.

33.2 Rigid analytic comparison theorem in relative dimension one

We now give a relative version of the main theorem of [11]. This result, bound to rigid-analytic comparison for morphisms of relative dimension one, will be generalized to arbitrary dimensions in the next section.

Proposition 33.2.1. *Let $f : X \to S$ be a smooth morphism of smooth algebraic k-varieties of* relative dimension one*, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module endowed with an integrable connection. We assume that f and $(\mathcal{E}, \nabla)$ may be defined over some algebraically closed subfield $k_0 \subseteq k$ which contains no Liouville number (e.g., $k_0 = \overline{\mathbb{Q}}$). Then there is a dense open subset $U \subseteq S$ such that for any $i \geqslant 0$, the restriction to U^{an} of the canonical morphism*

$$\varphi_i : (R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))^{\mathrm{an}} \longrightarrow R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$$

is an isomorphism in $\mathbf{MIC}(U^{\mathrm{an}})$.

The rest of the (sub)section is devoted to the proof of 33.2.1

33.2.2. We may replace S by any dense open subset, and then by any finite flat S-scheme S' (indeed, the formation of algebraic and analytic de Rham cohomologies commutes with finite flat base change 23.5, 29.2.2, $\mathcal{O}_{S'^{\mathrm{an}}}$ is faithfully flat over $\mathcal{O}_{S^{\mathrm{an}}}$, and the functor "inverse image of $\mathcal{O}_{S^{\mathrm{an}}}$-modules" on S^{an} is nothing but $- \otimes_{\mathcal{O}_{S^{\mathrm{an}}}} \mathcal{O}_{S'^{\mathrm{an}}}$). As we saw in 25.3.12, this allows us to assume that X admits a finite open cover $\{U_\alpha\}$ such that all $f_{\underline{\alpha}} := f_{|U_{\underline{\alpha}}=U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}}$ are coordinatized elementary fibrations for $p \leqslant 1 + \dim X$.

Thanks to the morphism of Čech spectral sequences in $\mathbf{MIC}(S^{\mathrm{an}})$,

$$\begin{array}{ccc} \bigoplus_{\alpha_0<\cdots<\alpha_p} (R^{i-p}_{\mathrm{DR}} f_{\underline{\alpha}*}(\mathcal{E},\nabla)_{|U_{\underline{\alpha}}})^{\mathrm{an}} & \Longrightarrow & (R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla))^{\mathrm{an}} \\ \Big\downarrow{\scriptstyle \varphi_{i-p}} & & \Big\downarrow{\scriptstyle \varphi_i} \\ \bigoplus_{\alpha_0<\cdots<\alpha_p} R^{i-p}_{\mathrm{DR}} f^{\mathrm{an}}_{\underline{\alpha}*}(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|U^{\mathrm{an}}_{\underline{\alpha}}} & \Longrightarrow & R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}) \end{array}$$

we may assume that f itself is a coordinatized elementary fibration.

After replacing S by an étale covering and arguing as in 25.3.10, we reduce at last to the case when f is a rational elementary fibration, and $i = 0, 1$.

33.2.3. If $S = \operatorname{Spec} k$ (as in the situation of [11]), we are left with an ordinary differential system in one variable. Hence the comparison theorem

$$H^i_{\mathrm{DR}}(X, (\mathcal{E},\nabla)) \cong H^i_{\mathrm{DR}}(X^{\mathrm{an}}, (\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}}))$$

for a curve is equivalent to the following statement (whose proof only uses the non-archimedean Turrittin theorem of [13]).

Proposition 33.2.4. *Let $\Theta(x) = \prod_{i=1}^r (x-\theta_i)$, with $\theta_1,\ldots,\theta_r$ distinct elements of k, let $X = \operatorname{Spec} k\big[x, \frac{1}{\Theta(x)}\big]$, and let (E,∇) be a projective $\mathcal{O}(X)$-module of finite rank μ with a connection, such that the $\mathbb{Q}$-vector subspace $\Delta \subseteq k$ generated by all Turrittin exponents (at the θ_i and at ∞) contains no Liouville numbers. Let X^{an} denote the rigid analytic space associated to the k-scheme X. Then*

$$\varphi_0 : \mathrm{Ker}_E \nabla_{\partial_x} \longrightarrow \mathrm{Ker}_{E\otimes\mathcal{O}(X^{\mathrm{an}})} \nabla_{\partial_x}$$

and

$$\varphi_1 : \mathrm{Coker}_E \nabla_{\partial_x} \longrightarrow \mathrm{Coker}_{E\otimes\mathcal{O}(X^{\mathrm{an}})} \nabla_{\partial_x}$$

are isomorphisms.

Proof. This is obtained by choosing B, C'_i as in 31.3.2,

$$T_i = \mathcal{M}_{\theta_i}\Big[(x-\theta_i)^\alpha, \exp\big(a(x-\theta_i)^{-\frac{k}{\mu!}}\big)\Big]_{\alpha\in\Delta, k\in\mathbb{Z}_{>0}, a\in k},$$

with the usual identification

$$\begin{aligned}&(x-\theta_i)^{\alpha}\exp\left(a(x-\theta_i)^{-\frac{k}{\mu!}}\right)\cdot(x-\theta_i)^{\beta}\exp\left(b(x-\theta_i)^{-\frac{k}{\mu!}}\right)\\&=(x-\theta_i)^{\alpha+\beta}\exp\left((a+b)(x-\theta_i)^{-\frac{k}{\mu!}}\right).\end{aligned}$$

Then

$$A_i = T_i[\log(x-\theta_i)],$$

and $C_i = A_i C_i'$ (for $i=0$ we are tacitly replacing θ_i by ∞, as in 31.3.2).

We also set $C=\prod_{i=0}^r C_i$, $A=\prod_{i=0}^r A_i$, $\partial=\frac{d}{dx}$.

Lemma 33.2.5. *$\partial : A \to A$ is surjective.*

Proof of the lemma. We tacitly replace A by A_i, and assume $\theta_i=0$. We also set $z=x^{-\frac{1}{\mu^2!}}$, an element of A. We may work with d/dz instead of ∂. Any element of A is a finite sum of terms of the form

$$u\, z^{\alpha}\,\frac{\log^m z}{m!}\, e^{a\, z^{-k}},$$

where $u\in\mathcal{M}_0[z]$ is a p-adic meromorphic function of z. We have to show that $\int u\, z^{\alpha}\,\frac{\log^m z}{m!}\, e^{a\, z^{-k}} dz$ is of the same form. We first note that if $k=0$ or $a=0$, the result is an easy consequence of the following formulas

$$z^{\alpha-1}\frac{\log^m z}{m!}=\partial\Big(\frac{z^{\alpha}}{\alpha}\sum_{i+j=m}\big(-\frac{1}{\alpha}\big)^i\frac{\log^j z}{j!}\Big),$$

for $\alpha\neq 0$, and

$$z^{-1}\frac{\log^m z}{m!}=\partial\frac{\log^{m+1} z}{(m+1)!}.$$

Let us then assume that $k>0$, $a\neq 0$. By integration by parts

$$\begin{aligned}&\int u\, z^{\alpha}\,\frac{\log^m z}{m!}\, e^{az^{-k}} dz\\&=\frac{\log^m z}{m!}\int u\, z^{\alpha}\, e^{az^{-k}}dz-\int z^{-1}\,\frac{\log^{m-1} z}{(m-1)!}\Big(\int u\, z^{\alpha}\, e^{az^{-k}}dz\Big)\, dz\end{aligned}$$

and descending induction on m, we reduce to the case where $m=0$. We then have to show that, for $\alpha\in\Delta$ and $a\in K$, there is a formal integral

$$y=u^{-1}z^{-\alpha}e^{-az^{-k}}\int uz^{\alpha}e^{az^{-k}}dz$$

that is a p-adic meromorphic function of z. We notice that y satisfies the inhomogenous differential equation

$$\frac{dy}{dz}=-\Big(\frac{1}{u}\frac{du}{dz}+\frac{\alpha}{z}-kaz^{-k-1}\Big)y+1$$

hence the homogenous differential equation of second order

$$\frac{d^2y}{dz^2} + \Big(\frac{1}{u}\frac{du}{dz} + \frac{\alpha}{z} - kaz^{-k-1}\Big)\frac{dy}{dz} + \Big(\frac{d}{dz}\Big(\frac{1}{u}\frac{du}{dz}\Big) - \frac{\alpha}{z^2} + k(k+1)az^{-k-2}\Big)y = 0.$$

Since u is meromorphic, $d/dz(u^{-1}du/dz)$ has at worst a double pole at 0, and since $k > 0$, we see that the indicial polynomial of this differential equation at 0 is

$$\mathrm{ind}_0(t) = ak(t-k-1).$$

As a simple instance of Clark's theorem [32], we conclude that the differential equation possesses a meromorphic solution y. This completes the proof of Lemma 33.2.5. $\square$

To apply proposition 25.3.9, we need to show that

$$A_i' := A_i \cap C_i' = \mathcal{M}_{\theta_i}. \tag{33.2.6}$$

We first show that

$$A_i \cap C_i' \subseteq \mathcal{M}_{\theta_i}\big[(x-\theta_i)^{\alpha}, \log(x-\theta_i)\big]_{\alpha\in\Delta}. \tag{33.2.7}$$

Indeed, every element $y \in A_i$ can be expanded in a convergent power series in $x - \zeta_i$, for $\zeta_i \neq \theta_i$ sufficiently close to θ_i:

$$y = y_0(x-\zeta_i) + \sum_{k=1}^{r} y_k(x-\zeta_i)\exp\Big(a\big((x-\zeta_i)^{-\frac{k}{\mu!}} - (\zeta_i-\theta_i)^{-\frac{k}{\mu!}}\big)\Big),$$

where $y_k(x-\zeta_i)$ is the expansion at ζ_i of an element y_k of $\mathcal{M}_{\theta_i}\big[(x-\theta_i)^{\alpha}, \log(x-\theta_i)\big]_{\alpha\in\Delta}$. So, the radius of convergence of $y_k(x-\zeta_i)$ is of the form $C|\zeta_i-\theta_i|$, for a positive constant C depending on y, but not on $|\zeta_i-\theta_i|$ [13, Lemma 6]. On the other hand, the radius of convergence of $\exp\Big(a\big((x-\zeta_i)^{-\frac{k}{\mu!}} - (\zeta_i-\theta_i)^{-\frac{k}{\mu!}}\big)\Big)$ is of the form $C|\zeta_i-\theta_i|^{1+k}$. So, for sufficiently small values of $|\zeta_i-\theta_i|$, these radii are all distinct, and the radius of convergence of y is the infimum of them; 33.2.7 follows.

On the other hand, by Dwork's lemma [4, Lemma 1], one has

$$\mathcal{M}_{\theta_i}\big[(x-\theta_i)^{\alpha}, \log(x-\theta_i)\big]_{\alpha\in\Delta} \cap C_i' \subseteq \mathcal{M}_{\theta_i}. \tag{33.2.8}$$

From 33.2.6 we deduce $A \cap B = (\prod_{i=0}^{r}\mathcal{M}_{\theta_i}) \cap B = \mathcal{O}(X)$ and $A_i^{\partial} = C_i^{\partial} = \mathbb{C}$. As for the solvability of (E,∇) in A, it follows from the non-archimedean Turrittin theorem of [13]. This concludes the proof of Proposition 33.2.4. $\square$

33.2.9. However, we cannot reduce the case of a rational elementary fibration to the case of "$\mathbb{P}^1$ minus a few points" as in 31.3, for lack of having yet proven coherence and base change for $R^i_{\mathrm{DR}}f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})$. Nevertheless, since we are free to replace

S by a dense affine open subset, the argument of 25.3.13 together with the use of Čech spectral sequences as above allows us to assume that $(\mathcal{E}, \nabla)$ is cyclic of rank μ, with associated differential operator $\Lambda \in \mathcal{O}(X)\langle \frac{d}{dx} \rangle$, and that the leading coefficients γ_i of the indicial polynomials of Λ are invertible in S. In particular, $\mathrm{Ker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x}$ and $\mathrm{Coker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x}$ are projective $\mathcal{O}(S)$-modules of finite rank.

We notice that all previous reductions could be carried over k_0, and in particular the roots of the indicial polynomials belong to k_0. By 17.1.7, all Turrittin exponents (at θ_i and at ∞) belong to k_0; we denote by Δ the $\mathbb{Q}$-subspace of k_0 that they generate (together with 1).

33.2.10. Let $\mathcal{S} \subseteq S^{\mathrm{an}}$ be an affinoid subspace with good reduction, and let $\mathcal{X} \to \mathcal{S}$ be the morphism obtained from f^{an} by base change. Since Δ does not contain any Liouville number, we are in the situation of 33.1.2.

On the other hand, by the non-archimedean Turrittin theorem [13], there is a Banach field $\mathcal{K}$ which is a finite extension of the completion of the fraction field of $\mathcal{O}(\mathcal{S})$ with respect to the spectral seminorm (which is a multiplicative norm because $\mathcal{S}$ has good reduction), and which enjoys the following property. If we set

$$E = \Gamma(X, \mathcal{E}) \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}), \quad B = \mathcal{O}(\mathcal{X}),$$

and choose $A \subseteq C$ as in 33.2.5, with k replaced by $\mathcal{K}$, then

$$A \cap B \cong \mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \subseteq \mathcal{O}(\mathcal{X}),$$

as follows by combining the Mittag-Leffler decomposition 33.1.7 and

$$A \cap \mathcal{O}\Big(\Big(\mathrm{Spec}\, \mathcal{K}\Big[x, \frac{1}{\Theta(x)}\Big]\Big)^{\mathrm{an}}\Big) = \mathcal{K}\Big[x, \frac{1}{\Theta(x)}\Big],$$

and $A^\partial = C^\partial = \mathcal{K}$, $B^\partial = (A \cap B)^\partial = \mathcal{O}(\mathcal{S})$. In this situation, E is solvable in A.

Proposition 30.1.1 (and flatness of $\mathcal{O}(\mathcal{S})$ over $\mathcal{O}(S)$) shows that the natural morphism

$$\mathrm{Ker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x} \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \longrightarrow \mathrm{Ker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})} \nabla_{\partial_x}$$

(resp.

$$\mathrm{Coker}_{\Gamma(X,\mathcal{E})} \nabla_{\partial_x} \otimes_{\mathcal{O}(S)} \mathcal{O}(\mathcal{S}) \longrightarrow \mathrm{Coker}_{\Gamma(\mathcal{X},\mathcal{E}^{\mathrm{an}})} \nabla_{\partial_x}\ )$$

is an isomorphism (resp. is injective).

On the other hand, we know by 33.1.2 that the latter morphism is surjective. As in 31.4.5 , we conclude that φ_i is an isomorphism for $i = 0, 1$ in this situation, and this completes the proof of 33.2.1. □

Proposition 33.2.1 yields the following purity statement:

Corollary 33.2.11. *Let $f : X \to S$ be a rational elementary fibration, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module endowed with an integrable connection. Assume* *that there is an open subset $U \subseteq S$ such that $S \smallsetminus U$ is of codimension $\geqslant 2$ in S and for any $i \geqslant 0$, the restriction to U^{an} of the canonical morphism*

$$\varphi_i : (R^i_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))^{\mathrm{an}} \longrightarrow R^i_{\mathrm{DR}} f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$$

is an isomorphism of coherent objects in $\mathbf{MIC}(U^{\mathrm{an}})$*.*

Then φ_i is in fact an isomorphism of coherent objects in $\mathbf{MIC}(S^{\mathrm{an}})$*.*

Proof. By localization, one may assume that there is a cyclic vector. The case $i = 0$ is easy. The coherence of $R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla))$ over U implies that the τ_i are units on U, hence on X, and the previous discussion applies. □

Remarks 33.2.12. (i) In proving the surjectivity of φ_1, we cannot apply 30.2.1, because the last condition (d) is not fulfilled.

(ii) In the situation of 33.2.1, assume moreover that S is affine, f is a rational elementary fibration, $(\mathcal{E}, \nabla)$ is cyclic outside some divisor which does not meet Z, and the leading coefficients of the indicial polynomials at the branches of Z are units in $\mathcal{O}(S)$. Then our argument shows that the conclusion of 33.2.1 holds with $U = S$ (cf. 33.1.9).

34 Comparison theorem for algebraic vs. rigid-analytic cohomology (irregular coefficients)

34.1 Statement

In this section, we tackle the problem of comparing algebraic and rigid-analytic de Rham cohomologies with possibly irregular coefficients, i.e., of extending Proposition 33.2.1 to the case of higher dimensional morphisms. We prove the following statement, which was conjectured in [11].

Theorem 34.1.1. *Let X be a smooth k_0-variety and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module with integrable connection* (*as before, k_0 denotes an algebraically closed subfield of the non-archimedean complete algebraically closed field k such that k_0 does not contain Liouville numbers, e.g., $k_0 = \overline{\mathbb{Q}}$*)*. Then for any $i \geqslant 0$,*

$$H^i_{\mathrm{DR}}(X, (\mathcal{E}, \nabla)) \otimes_{k_0} k \cong H^i_{\mathrm{DR}}(X^{\mathrm{an}}_k, (\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})).$$

Our proof extends the strategy which we have used before in the case of regular coefficients: combining 33.2.1 and dévissage 25.

34.2 Key propositions

Let $f : X \to S$ be a coordinatized elementary fibration, and let $(\mathcal{E}, \nabla)$ be a coherent object of $\mathbf{MIC}(X)$. We saw in 26.1 that $R^0_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)$ is locally free of finite rank, and that there exists a dense open subset $U \subseteq S$, depending on

$(\mathcal{E},\nabla)$, such that $R^1_{\mathrm{DR}} f_*(\mathcal{E},\nabla)_{|U}$ is locally free of finite rank (the other cohomology modules $R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla)_{|U}$ vanish). Since $U \neq S$ in general for irregular connections, this prevents us from using the first form of dévissage 25.3.4 in the study of $R^i_{\mathrm{DR}} f_*(\mathcal{E},\nabla)_{|U}$ for higher-dimensional f.

On the other hand, as was mentioned in 29.1.8, the analytic version of the second form of dévissage 25.3.9 is problematic. In order to tackle the problem of comparing algebraic and rigid-analytic de Rham cohomologies with irregular coefficients by an extension of our strategy, we thus need suitable refinements of the coherence results of 25.3.9.

More precisely, the proof of 34.1.1 relies upon the following two *key propositions*. The first one is purely algebraic, and makes essential use of the results of Chapter VI about irregularity in several variables:

Proposition 34.2.1. *Let X be a smooth k_0-variety and let $(\mathcal{E},\nabla)$ be a coherent $\mathcal{O}_X$-module with integrable connection. There exists a finite open affine cover (U_α) of X and, for each α, a coordinatized elementary fibration $f_\alpha : U_\alpha \to S_\alpha$ such that $R^0_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})$ and $R^1_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})$ are coherent $\mathcal{O}_{S_\alpha}$-modules.*

The coherence of $R^0_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})$ is in fact automatic (see 26.1.3), and is put here only for reference. In the irregular case, however, example 26.1.17 shows that the coherence of $R^1_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})$ is not automatic (when the dominant term of the indicial polynomial is not a unit).

The second proposition is a variant of 33.2.1, 33.2.12(ii):

Proposition 34.2.2. *In the situation of 34.2.1, one can moreover choose f_α in such a way that for $j = 0, 1$, the canonical morphism*

$$R^j_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})^{\mathrm{an}} \longrightarrow R^j_{\mathrm{DR}} f^{\mathrm{an}}_{\alpha*}((\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|U^{\mathrm{an}}_\alpha})$$

is an isomorphism in $\mathbf{MIC}(S^{\mathrm{an}}_\alpha)$.

34.3 Proof

Let us deduce 34.1.1 from 34.2.1 and 34.2.2. We consider a cover (U_α) as in 34.2.1 and 34.2.2. For $\underline{\alpha} = (\alpha_0,\dots,\alpha_p)$, $\alpha_0 < \cdots < \alpha_p$, we set $U_{\underline{\alpha}} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. There is a natural morphism of Čech spectral sequences,

$$\begin{array}{ccc} \bigoplus_{\alpha_0<\cdots<\alpha_p} H^{i-p}_{\mathrm{DR}}(U_{\underline{\alpha}},(\mathcal{E},\nabla))\otimes k & \Longrightarrow & H^i_{\mathrm{DR}}(X,(\mathcal{E},\nabla))\otimes k \\ \Big\downarrow{\scriptstyle \varphi_{i-p}} & & \Big\downarrow{\scriptstyle \varphi_i} \\ \bigoplus_{\alpha_0<\cdots<\alpha_p} H^{i-p}_{\mathrm{DR}}(U^{\mathrm{an}}_{\underline{\alpha}k},(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})) & \Longrightarrow & H^i_{\mathrm{DR}}(X^{\mathrm{an}}_k,(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})). \end{array}$$

By induction on i, we are left with proving the theorem for $X = U_\alpha$.

On the other hand, there is a natural morphism of Leray spectral sequences

$$\begin{array}{ccc}
\bigoplus_{0\leqslant j\leqslant i} H^{i-j}_{\mathrm{DR}}(S_\alpha, R^j_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha}))\otimes k & \Longrightarrow & H^i_{\mathrm{DR}}(X,(\mathcal{E},\nabla))\otimes k \\
\Big\downarrow{\scriptstyle \varphi'_{i-j}} & & \\
\bigoplus_{0\leqslant j\leqslant i} H^{i-j}_{\mathrm{DR}}(S^{\mathrm{an}}_{\alpha k}, (R^j_{\mathrm{DR}} f_{\alpha*}((\mathcal{E},\nabla)_{|U_\alpha})^{\mathrm{an}}) & & \Big\downarrow{\scriptstyle \varphi_i} \\
\Big\downarrow{\scriptstyle \varphi''_j} & & \\
\bigoplus_{0\leqslant j\leqslant i} H^{i-j}_{\mathrm{DR}}(S^{\mathrm{an}}_{\alpha k}, R^j_{\mathrm{DR}} f^{\mathrm{an}}_{\alpha*}((\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|U^{\mathrm{an}}_\alpha}) & \Longrightarrow & H^i_{\mathrm{DR}}(X^{\mathrm{an}}_k,(\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})).
\end{array}$$

By 34.2.2, we know that φ''_j is an isomorphism. By 34.2.1 and induction on the dimension, we may assume that φ'_{i-j} is an isomorphism. Hence the right vertical arrow φ_i is an isomorphism. □

34.4 Proof of 34.2.1

34.4.1 (Strategy). As we stated before, it suffices to consider the case of $R^1_{\mathrm{DR}} f_*$. By quasi-compactness, it is enough, given an arbitrary closed point $\xi \in X$, to find an open affine neighborhood V of ξ and a coordinatized elementary fibration $f : V \to S$ such that $R^1_{\mathrm{DR}} f_*((\mathcal{E},\nabla)_{|V})$ is a coherent $\mathcal{O}_S$-module. We may and shall also assume that X is affine of pure dimension $d \geqslant 2$ (the cases $d = 0, 1$ are trivial).

If we take *any* coordinatized elementary fibration as constructed in 25.1.12, and try to apply the method of 26.1, we encounter two serious obstacles:

(i) in general, there is no cyclic vector in the neighborhood of the singular divisors, as was pointed out in 17.5;

(ii) even if it happens that one can find such a cyclic vector, the leading coefficients of the indicial polynomials will not be units in $\mathcal{O}(S)$ in general.

We shall overcome these difficulties by

(1) taking $f : V \to S$ *sufficiently general* with respect to the parameters of the Artin construction of good neighborhoods, and

(2) applying the method of 25.1.12 to a suitable modification of f.

To reach this objective, we shall make full use of our study of irregularity in several variables (specifically, and in order of appearance: 19.3.1, 18.1.1, 19.2.3). We shall proceed in five steps.

34.4.2. Let us again consider in detail the construction of a coordinatized elementary fibration as in 25.1.12, 25.1.14, 25.1.15. It depends on the choice of

(i) a projective embedding $\overline{X} \to \mathbb{P}^N$ of a projective normal closure of X,

(ii) a general linear space $L \subseteq \mathbb{P}^N$ of codimension $d-1$ passing through ξ,

(iii) a general hyperplane $H \subseteq \mathbb{P}^N$,

and is obtained as a suitable restriction of the linear projection

$$p : \mathbb{P}^N \dashrightarrow \mathbb{P}^{d-1}$$

with center $L \cap H$. More precisely, let $\epsilon : \widetilde{\overline{X}} \to \overline{X}$ denote the blow-up with center $L \cap H \cap \overline{X}$, with exceptional divisor E, and let D denote the divisor $\overline{X} \smallsetminus X$. Let $T \subseteq D$ be any fixed closed subset of codimension at least 1 in D with the following properties:

(i) $D \smallsetminus T$ is a disjoint union of smooth divisors contained in the smooth part of $\overline{X}$;

(ii) for any component C of $D \smallsetminus T$, the coherent and locally free $\mathcal{O}_X$-module $\mathcal{E}$ extends to a coherent and locally free $\mathcal{O}_X$-module on $X \cup C$;

(iii) for any component C of $D \smallsetminus T$, the big cell $C^0 \subseteq C$ of Proposition 19.3.1 for both the connections $(\mathcal{E}, \nabla)$ and $\mathcal{E}nd(\mathcal{E}, \nabla)$, coincides with C.

A sufficiently general choice of L through ξ implies that D intersects neither $L \cap H \cap \overline{X}$ nor T, and that the curve $X \cap L$ does not intersect the singular locus of D.

According to the proof of Lemma 25.1.14, there is an affine open neighborhood S of $\zeta := p(\xi)$ in $\mathbb{P}^{d-1}$, a divisor F in $\widetilde{\overline{X}}_{|S}$ and a finite morphism $\overline{\pi} : \widetilde{\overline{X}}_{|S} \to \mathbb{P}^1_S$ such that the induced diagram

$$\begin{tikzcd}
\widetilde{\overline{X}}_{|S} \smallsetminus Z = U \arrow[r, hook, "j"] \arrow[d, "\pi"'] & \widetilde{\overline{X}}_{|S} \arrow[d, "\overline{\pi}"] & Z = (D \sqcup E)_{|S} \sqcup F \arrow[l, hook'] \arrow[d, "\pi'"] \\
U' \arrow[r, hook, "j'"] \arrow[dr, "f'"'] & \mathbb{P}^1_S \arrow[d, "\overline{f}'"] & Z' \arrow[l, hook'] \arrow[dl, "g'"] \\
& S &
\end{tikzcd} \tag{34.4.3}$$

pertains to a coordinatized elementary fibration 25.1.7, with

$$f := p_{|U} = f' \circ \pi, \quad \overline{f} := \overline{f}' \circ \overline{\pi}, \quad g := g' \circ \pi', \quad \overline{f}' = \mathrm{pr}_S,$$

and S affine and smooth. Moreover the fiber of $p_{|\widetilde{\overline{X}}}$ above ζ is $\overline{X} \cap L$ and

$$\widetilde{\overline{X}}_{|S} \cap T = \emptyset\ .$$

As in the proof of Lemma 25.1.14, in our diagram $F = (\pi')^{-1}(B)$, where $B \subseteq \mathbb{P}^1_S$ is the branching locus of the finite morphism $\overline{\pi}$ and is étale finite over S. In particular, $\overline{\pi}$ is finite étale above $\pi'(D \sqcup E)_{|S}$, F is finite étale above B (and S) and

$$(\overline{\pi}^{-1}\pi' D_{|S}) \cap Z = D_{|S}, \quad (\overline{\pi}^{-1}\pi' E_{|S}) \cap Z = E_{|S},$$

according to 25.1.14 (c),(d). Note that $D_{|S} \subseteq \overline{X}^{\mathrm{sm}}$ and that the map $C \mapsto \overline{C}$ which sends a component of $D_{|S}$ into its closure in $\overline{X}$ is a bijection onto the set of 1-codimensional components of $\overline{X} \smallsetminus X$.

34.4.4 (**Conclusions for a sufficiently general projection).** We conclude that

(i) $D_{|S}$ is a disjoint union of smooth divisors $\{D_i\}_i$, contained in the smooth part of $\overline{X}$, each of which is finite and étale above S;

(i)′ $\pi'(D_{|S})$ is a disjoint union of the smooth divisors $\{\pi'(D_i)\}_i$, contained in the smooth part of $\overline{X}$, each of which is finite and étale above S;

(ii) the $\mathcal{O}_U$-module $\mathcal{E}_{|U}$ extends to a locally free module over $\widetilde{\overline{X}}_{|S}$;

(ii)′ the $\mathcal{O}_{U'}$-module $\pi_*(\mathcal{E}_{|U})$ extends to a locally free module over $\mathbb{P}^1_{|S}$;

(iii) for any locally closed curve C which meets $D_{|S}$ transversally at some point Q of the component D_i, there are equalities of Newton polygons:

$$\mathrm{NP}_Q((\mathcal{E}, \nabla)_{|C \smallsetminus Q}) = \mathrm{NP}_{D_i}((\mathcal{E}, \nabla)_{|U}),$$

$$\mathrm{NP}_Q(\mathcal{E}nd(\mathcal{E}, \nabla)_{|C \smallsetminus Q}) = \mathrm{NP}_{D_i}(\mathcal{E}nd(\mathcal{E}, \nabla)_{|U});$$

(iii)′ for any locally closed curve C which meets $\pi'(D_{|S})$ transversally at some point Q of the component $\pi'(D_i)$, there are equalities of Newton polygons:

$$\mathrm{NP}_Q(\pi_*((\mathcal{E}, \nabla)_{|U})_{|C \smallsetminus Q}) = \mathrm{NP}_{\pi'(D_i)}(\pi_*((\mathcal{E}, \nabla)_{|U})),$$

$$\mathrm{NP}_Q(\mathcal{E}nd(\pi_*((\mathcal{E}, \nabla)_{|U}))_{|C \smallsetminus Q}) = \mathrm{NP}_{\pi'(D_i)}(\mathcal{E}nd(\pi_*((\mathcal{E}, \nabla)_{|U}))).$$

After replacing S by an étale neighborhood of $\zeta \in S$ we may assume that the lower part of diagram (34.4.3) is a rational elementary fibration as in Definition 25.1.4, in which

$$Z' = \left(\coprod_{i=1}^{r} \sigma_i(S) \right) \sqcup \sigma_\infty(S),$$

with

$$\sigma_i(S) = V(x - \theta_i), \quad \sigma_\infty(S) = \{\infty\} \times S,$$

for $\theta_i \in \mathcal{O}(S)$.

34.4.5 (A Kummer covering). *In order to be able to apply our result of 18.1.1 on cyclic vectors, which requires a differential module with Turrittin exponents equal to 1, we shall pull back $\pi_*(\mathcal{E},\nabla)$ to a suitable Kummer étale covering U_1 of U'.*

For $i = 1,\dots,r$, we denote by $L_i = \mathbb{P}^1_S \xrightarrow{\lambda_i} \mathbb{P}^1_S$ the ramified covering of degree $\mu!$ given by $x_i \mapsto x = x_i^{\mu!} + \theta_i$. It is totally ramified at Z'_i and at Z'_∞ and unramified elsewhere.

We blow up

$$L_1 \times_S L_2 \times_S \cdots \times_S L_r = (\mathbb{P}^1_S)^r = (\mathbb{P}^1)^r_S$$

along the closed subvariety $(\infty,\dots,\infty)_S$, and consider the strict transform $\overline{U}_1$ of

$$L_1 \times_{\mathbb{P}^1_S} L_2 \times_{\mathbb{P}^1_S} \cdots \times_{\mathbb{P}^1_S} L_r \subseteq (\mathbb{P}^1)^r_S.$$

We denote by

$$\overline{\epsilon} : \overline{U}_1 \longrightarrow \mathbb{P}^1_S$$

the canonical finite morphism.

We claim that:

(1) $\overline{U}_1$ is smooth over S.

(2) The map $\overline{\epsilon}$ is finite and locally free of degree $(\mu!)^r$. For any $i = 1,\dots,r,\infty$, $\overline{\epsilon}^{-1}(\sigma_i(S))$ is the disjoint union of a family of $(\mu!)^{r-1}$ smooth divisors B_{ik}. For any i,k, the map $\overline{\epsilon}$ has multiplicity $\mu!$ along B_{ik}, in the sense that, at any point $z \in B_{ik}$, a local equation of $\sigma_i(S)$ at $\overline{\epsilon}(z)$ is the $\mu!$-th power of a local equation for B_{ik} at z.

(3) The restriction of $\overline{\epsilon}$,

$$\epsilon :\, U_1 := \overline{U}_1 \smallsetminus \coprod_{i,k} B_{ik} \to U',$$

is a Galois étale covering with group

$$G = \mathrm{Gal}\Big(\, \kappa(S)(x,(x-\theta_1)^{\frac{1}{\mu!}},\dots,(x-\theta_r)^{\frac{1}{\mu!}})/\kappa(S)(x)\Big) \cong (\mathbb{Z}/\mu!\mathbb{Z})^r.$$

(4) The natural morphism

$$f_1 : U_1 \longrightarrow S$$

is an elementary fibration, sitting in the commutative diagram

(34.4.6)
$$\begin{tikzcd}
U_1 \arrow[r, hook, "j"] \arrow[d, "\epsilon"'] & \overline{U}_1 \arrow[d, "\overline{\epsilon}"] & \coprod_{i,k} B_{ik} \arrow[l, hook'] \arrow[d, "\epsilon'"] \\
U' \arrow[r, hook, "j'"] \arrow[dr, "f'"'] & \mathbb{P}^1_S \arrow[d, "\overline{f}'"] & Z' \arrow[l, hook'] \arrow[dl, "g'"] \\
& S &
\end{tikzcd}$$

whose lower part coincides with the lower part of diagram (34.4.3). In particular,

$$f_1 = f' \circ \epsilon, \quad \overline{f}' = \mathrm{pr}_S.$$

Indeed, since the branch loci of the λ_j do not intersect each other at finite distance, (2) and (3) are clear, except for $i = \infty$, and $\overline{U}_1$ is smooth over S outside $\overline{\epsilon}^{-1}(\sigma_\infty(S)) = \overline{\epsilon}^{-1}(\{\infty\} \times S)$. At ∞, we use the coordinates $y_j = 1/x_j$ on L_j and $y = 1/x$ on $\mathbb{P}^1_S$, so that $y_j^{\mu!} = \frac{y}{1-\theta_j y}$.

There is an affine cover $(\mathcal{V}_i)_{i=1,\dots,r}$ of a neighborhood of $\overline{\epsilon}^{-1}(y = 0)$ in $\overline{U}_1$ such that

$$\{y_i, t_{ij} := y_j/y_i \ \mid j = 1, \dots, r, \ j \neq i \}$$

are local coordinates on $\mathcal{V}_i$. We have

$$y_i^{\mu!} = \frac{y}{1-\theta_i y}, \quad t_{ij}^{\mu!} = \frac{1-\theta_i y}{1-\theta_j y}.$$

In particular, the t_{ij} are unramified at $y = 0$, and properties (1), (2), (3) follow.

Finally, (1) implies that $\overline{U}_1$ is normal. It is just the normalization of $\mathbb{P}^1_S$ in the Kummer extension $\kappa(S)(x, (x-\theta_1)^{\frac{1}{\mu!}}, \dots, (x-\theta_r)^{\frac{1}{\mu!}})/\kappa(S)(x)$, which implies (4). Finally, (5) follows from our construction.

34.4.7 (**Up and down).** The elementary fibration $f_1 : U_1 \to S$ of 34.4.5(4) has the advantage that the connections $(\mathcal{E}_1, \nabla_1) := \epsilon^*(\pi_*((\mathcal{E}, \nabla)_U)$ and $\mathcal{E}nd(\mathcal{E}_1, \nabla_1)$ in $\mathbf{MIC}(U_1/K)$, have Turrittin index 1 along any polar divisor. Moreover, if B_{ik} is a polar divisor for $(\mathcal{E}_1, \nabla_1)$ or $\mathcal{E}nd(\mathcal{E}_1, \nabla_1)$, the big cell B^0_{ik} for these connections in the sense of Proposition 19.3.1, coincides with the whole divisor B_{ik}. However, in order to reduce the proof of Proposition 34.2.1 to our result in Section 33, we need to re-project down $(\mathcal{E}_1, \nabla_1)$ and $\mathcal{E}nd(\mathcal{E}_1, \nabla_1)$ to $\mathbb{P}^1_S$, but this time via a finite morphism which is étale on the polar locus of those connections. This is our present intent.

Since we may replace S by an étale neighborhood of $\zeta = f(\xi)$, we may proceed as in Lemma 25.1.15 to complete the elementary fibration $f_1 : U_1 \to S$ of 34.4.5(4), as follows. We find a divisor F_1 in U_1, which is a split covering of S and does not intersect $\epsilon^{-1}\{\pi(\xi)\}$, such that the restriction of f_1 to $V_1 := U_1 \setminus F_1 \to S$ is a *coordinatized* elementary fibration:

(34.4.8)
$$\begin{tikzcd}
V_1 \arrow[r, hook, "j_1"] \arrow[d, "\pi_1"'] & \overline{V}_1 = \overline{U}_1 \arrow[d, "\overline{\pi}_1 = \overline{\epsilon}"] & (\overline{U}_1 \setminus U_1) \sqcup F_1 = (\coprod_{i,k} B_{ik}) \sqcup F_1 \arrow[l, hook'] \arrow[d, "\pi'_1"] \\
V'_1 \arrow[r, hook, "j'_1"] \arrow[dr, "f'_1"'] & \mathbb{P}^1_S \arrow[d, "\mathrm{pr}_S"] & Z'_1 \arrow[l, hook'] \arrow[dl, "g'_1"] \\
& S &
\end{tikzcd}$$

where, moreover,

$$\overline{\pi}_1 \text{ is étale above } \pi_1'(\overline{U}_1 \smallsetminus U_1). \tag{34.4.9}$$

We may also replace F_1 by the union of its conjugates under the Galois group G, hence assume that

$$V_1 \text{ is a } G\text{-covering of } V' := U' \smallsetminus \epsilon(F_1). \tag{34.4.10}$$

The conditions that $\pi(\xi) \notin \overline{\epsilon}(F_1)$ and that F_1 is disjoint from $\overline{U}_1 \smallsetminus U_1$ imply that $\widetilde{F} := \overline{\pi}^{-1}\overline{\epsilon}F_1 \subseteq \widetilde{\overline{X}}_{|S}$ is a divisor in U which does not contain ξ, and is étale finite over S; in particular,

$$V := U \smallsetminus \widetilde{F} \xrightarrow{f} S$$

is an elementary fibration.

The commutative diagram

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[dr, "f"'] & U' \arrow[d, "f'"] & U_1 \arrow[l, "\epsilon"'] \arrow[dl, "f_1"] \\ & S & \end{tikzcd} \tag{34.4.11}$$

(where the horizontal maps are étale coverings and the vertical maps are elementary fibrations) induces by restriction a commutative diagram

$$\begin{tikzcd} U \smallsetminus \widetilde{F} = V \arrow[r, "\pi"] \arrow[dr, "f"'] & V' = U' \smallsetminus \epsilon(F_1) \arrow[d, "f'"] & V_1 = U_1 \smallsetminus F_1 \arrow[l, "\epsilon"'] \arrow[dl, "f_1"] \\ & S & \end{tikzcd}$$

(with the same properties), which extends to a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "f"'] & V' \arrow[d, "f'"] & V_1 \arrow[l, "\epsilon"'] \arrow[r, "\pi_1"] \arrow[d, "f_1"] & V_1' \arrow[dl, "f_1'"] \\ & S \arrow[r, equal] & S & \end{tikzcd} \tag{34.4.12}$$

(since there is more danger of inflation than confusion with notation at this point, we use abusively the same letter for various restrictions of the same morphism).

Corollary 34.4.13. *We have:*

$$R^1_{\mathrm{DR}} f_*((\mathcal{E},\nabla)_{|V}) \cong R^1_{\mathrm{DR}} f'_*(\pi_*((\mathcal{E},\nabla)_{|V})) \cong R^1_{\mathrm{DR}} f'_*(\epsilon_*(\mathcal{E}_1,\nabla_1))^G, \tag{34.4.14}$$

$$R^1_{\mathrm{DR}} f'_*(\epsilon_*(\mathcal{E}_1,\nabla_1)) \cong R^1_{\mathrm{DR}} f'_{1*}(\pi_{1*}(\mathcal{E}_1,\nabla_1)). \tag{34.4.15}$$

34.4.16 (**Finiteness).** We claim that for this choice U of a neighborhood of ξ, $R^1_{DR} f_*((\mathcal{E},\nabla)_{|U})$ is a coherent $\mathcal{O}_S$-module.

In view of (34.4.14) and (34.4.15), it suffices to show that $R^1_{\mathrm{DR}} f'_{1*}(\pi_{1*}(\mathcal{E}_1,\nabla_1))$ is a coherent $\mathcal{O}_S$-module. Let us list a few properties of the latter.

(1) $\pi_{1*}\mathcal{E}_1$ extends to a locally free $\mathcal{O}_{\mathbb{P}^1_S}$-module.

This follows from the fact that $\mathcal{E}_{|U}$ extends to a locally free $\mathcal{O}_{\widetilde{\overline{X}}|S}$-module 34.4.4 and from the fact that all maps in the upper row of (34.4.12) extend to finite flat morphisms of relative projective completions.

(2) The Turrittin ramification indices of $\pi_{1*}\nabla_1$ are all 1.

Indeed, by the ramification property 34.4.5(4), the Turrittin indices of $(\mathcal{E}_1,\nabla_1) = \epsilon^*\pi_*((\mathcal{E},\nabla))$ at every branch B_{ik} of $\overline{U}_1 \smallsetminus U_1$ are 1, and this connection has no singularity at the branches of F_2; (2) then follows from (34.4.9).

Let us write $\mathbb{P}^1_S \smallsetminus V'_1 = \coprod_i W_i$, where the W_i are images of sections ${\sigma_i}'$ of $f'_1 = \mathrm{pr}_S$.

(3) For any component W_i and any point $Q'_1 \in W_i$, we get equalities of Newton polygons (using Lemma 19.2.2):

$$\mathrm{NP}_{Q'_1}(\pi_{1*}(\mathcal{E}_1,\nabla_1)_{|f_1'^{-1}(g'_1(Q'_1))}) = \mathrm{NP}_{W_i}(\pi_{1*}(\mathcal{E}_1,\nabla_1)),$$
$$\mathrm{NP}_{Q'_1}(\mathcal{E}nd(\pi_{1*}(\mathcal{E}_1,\nabla_1))_{|f_1'^{-1}(g'_1(Q'_1))}) = \mathrm{NP}_{W_i}(\mathcal{E}nd(\pi_{1*}(\mathcal{E}_1,\nabla_1))).$$

It is enough to verify the previous formulas at a point $Q'_1 \in \overline{\pi}_1(\overline{U}_1 \smallsetminus U_1)$, since other points are regular and easier to handle. But we recall that, by (34.4.9), $\overline{\pi}_1$ is étale above $\pi'_1(\overline{U}_1 \smallsetminus U_1)$, so that it suffices to prove the same statements for $(\mathcal{E}_1,\nabla_1)$ and $\mathcal{E}nd(\mathcal{E}_1,\nabla_1)$ along the components of $\overline{U}_1 \smallsetminus U_1$, that is along the various B_{ik}. In this case, the statement is obtained from the statement (iii)$'$ of Section 34.4.4 and the behavior of Newton polygons under pull-back along a ramification Lemma 19.2.2.

Properties (1), (2), (3) now allow us to apply Theorem 19.2.4 on the existence of cyclic vectors in neighborhoods of singularities: we conclude that, up to restricting S to a neighborhood of ζ, and outside a divisor D' disjoint from Z'_1, $\pi_{1*}\mathcal{E}_1$ admits a cyclic vector v. Let us denote by Λ the associated differential polynomial.

Property (3) also allows us to apply 19.2.3, and obtain that the leading coefficient of the indicial polynomial of Λ at W_i is a unit in $\mathcal{O}(W_i)$.

Finally, the method of 26.1 (see especially Remark 26.1.16) shows that $R^1_{\mathrm{DR}} f'_{1*}(\pi_{1*}(\mathcal{E}_1,\nabla_1))$ is coherent over $\mathcal{O}_S$, and so we conclude that $R^1_{\mathrm{DR}} f_*((\mathcal{E},\nabla)_{|V})$ is a coherent $\mathcal{O}_S$-module. This completes the proof of 34.2.1. □

34.5 Proof of 34.2.2

We claim that, with the same $V \xrightarrow{f} S$ as in 34.4, the canonical morphism

$$(R^j_{\mathrm{DR}} f_*(\mathcal{E},\nabla)_{|V})^{\mathrm{an}} \longrightarrow R^j_{\mathrm{DR}} f^{\mathrm{an}}_*((\mathcal{E}^{\mathrm{an}},\nabla^{\mathrm{an}})_{|V^{\mathrm{an}}})$$

is an isomorphism in $\mathbf{MIC}(S^{\mathrm{an}})$.

This is trivial for $j \neq 0, 1$, and the case $j = 0$ follows from the argument in 33.2.10. Let us now consider the case $j = 1$. In view of (34.4.14), (34.4.15) and by finite flat descent, it suffices to show that the map

$$(R^1_{\mathrm{DR}} f'_{1*}(\pi_{2*}(\mathcal{E}_1, \nabla_1)))^{\mathrm{an}} \longrightarrow R^1_{\mathrm{DR}} f'^{\mathrm{an}}_{1*}(\pi^{\mathrm{an}}_{1*}(\mathcal{E}^{\mathrm{an}}_1, \nabla^{\mathrm{an}}_1))$$

is an isomorphism in $\mathbf{MIC}(S^{\mathrm{an}})$. The argument in 33.2.10 also shows that this map is injective. In order to prove its surjectivity, let us consider again the flat morphism h of 34.4.16, and perform the cyclic reduction as above.

We can now conclude, by the method of 33.1, 33.2 (see especially the Remark 33.2.12(ii)) that

$$h^{\mathrm{an}*}(R^1_{\mathrm{DR}} f'_{1*}(\pi_{1*}(\mathcal{E}_1, \nabla_1)))^{\mathrm{an}} \longrightarrow h^{\mathrm{an}*} R^1_{\mathrm{DR}} f'^{\mathrm{an}}_{1*}(\pi^{\mathrm{an}}_{1*}(\mathcal{E}^{\mathrm{an}}_1, \nabla^{\mathrm{an}}_1))$$

is surjective, hence in fact an isomorphism of coherent objects in $\mathbf{MIC}(S'^{\mathrm{an}})$. Taking into account the flatness of h and 34.4.16(5), the purity statement 33.2.11 now shows that

$$(R^1_{\mathrm{DR}} f'_{1*}(\pi_{1*}(\mathcal{E}_1, \nabla_1)))^{\mathrm{an}} \longrightarrow R^1_{\mathrm{DR}} f'^{\mathrm{an}}_{1*}(\pi^{\mathrm{an}}_{1*}(\mathcal{E}^{\mathrm{an}}_1, \nabla^{\mathrm{an}}_1))$$

is an isomorphism, and we conclude finally that

$$(R^1_{\mathrm{DR}} f_*(\mathcal{E}, \nabla)_{|V})^{\mathrm{an}} \longrightarrow R^1_{\mathrm{DR}} f^{\mathrm{an}}_*((\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})_{|V^{\mathrm{an}}})$$

is an isomorphism in $\mathbf{MIC}(S^{\mathrm{an}})$.

This completes the proof of 34.2.2 and of the main Theorem 34.1.1. □

34.6 Properties of the GAGA functor

In cohomological degree 0, the statement of 34.1.1 reduces to the following:

Corollary 34.6.1. *Let $f : X \to S$ be a smooth morphism of smooth algebraic k_0-varieties, with $S = A_0(f)$. Then for any coherent $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla)$, the natural morphism*

$$(f_* \mathcal{E}^{\nabla_{|\mathcal{D}er(X/S)}}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\mathrm{an}}} \longrightarrow f^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}_{|\mathcal{D}er_{\mathrm{cont}}(X^{\mathrm{an}}/S^{\mathrm{an}})}}$$

is an isomorphism in $\mathbf{MIC}(S^{\mathrm{an}})$.

(*Simplified*) *proof.* Indeed, in this special case, one can provide a much shorter argument. With the help of Lemma 26.1.3 and Remark 25.3.8(ii), we may reduce by dévissage to the case when f is a coordinatized elementary fibration, or even a rational elementary fibration (arguing as in 31.2 or 32.2.3). The result then follows by application of 30.1.1 exactly as in 33.2.10. □

Corollary 34.6.2. *Let X be a smooth algebraic k-variety defined over an algebraically closed subfield $k_0 \subseteq k$ which does not contain any Liouville number. The functor*

$$\left\{\begin{matrix}\text{Coherent } \mathcal{O}_X\text{-modules} \\ \text{with integrable connection} \\ \text{defined over } k_0\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Coherent } \mathcal{O}_{X^{\mathrm{an}}}\text{-modules} \\ \text{with integrable connection}\end{matrix}\right\}$$
$$(\mathcal{E}, \nabla) \longmapsto (\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$$

is fully faithful.

Proof. Apply 34.1.1 to internal $\mathcal{H}om$ as in 31.4.6. □

Remark 34.6.3. This functor is however not essentially surjective, even for $X = \mathbb{A}^1$. Indeed, let h be any non-polynomial entire function. Then there is no entire function g such that $\frac{g'}{g} + h$ is a polynomial. This means that the rank-one connection defined by $\nabla_{\partial_x} e = he$ is not algebraizable.

This is in contrast with the complex-analytic situation, where the analogous functor is in fact essentially surjective, but not fully faithful.

Corollary 34.6.4. *Let $(\mathcal{E}, \nabla)$, $(\mathcal{E}', \nabla')$ be coherent $\mathcal{O}_X$-modules with integrable connections over X. Then the natural map*

$$\mathrm{Ext}^1_{\mathbf{MIC}(X_k)}((\mathcal{E}_k, \nabla_k), (\mathcal{E}'_k, \nabla'_k)) \longrightarrow \mathrm{Ext}^1_{\mathbf{MIC}(X_k^{\mathrm{an}})}((\mathcal{E}_k^{\mathrm{an}}, \nabla_k^{\mathrm{an}}), ({\mathcal{E}'_k}^{\mathrm{an}}, {\nabla'_k}^{\mathrm{an}}))$$

is a bijection.

Proof. Apply 34.1.1 with $j = 1$ to internal $\mathcal{H}om$ as in 31.4.7. □

Appendix A

Riemann's "existence theorem" in higher dimension, an elementary approach

Theorem A.1. *Let X be a smooth complex algebraic variety. Then the natural functor*

$$\left\{\begin{matrix}\text{algebraic étale}\\ \text{coverings of } X\end{matrix}\right\} \xrightarrow{(*)} \left\{\begin{matrix}\text{topological unramified}\\ \text{finite coverings of } X(\mathbb{C})\end{matrix}\right\}$$

is an equivalence of categories.

The problem of essential surjectivity of this functor, sometimes called Riemann's existence problem, was settled by Grauert and Remmert [45] and subsequently by Grothendieck [1, XII, thm. 5.1] using Hironaka's resolution of singularities.

In this appendix, we present an "elementary proof" by reduction to the one-dimensional case.

A.2. Recall that a topological unramified covering of $X(\mathbb{C})$ admits a canonical structure of an analytic variety $\mathcal{Y}$, endowed with a holomorphic étale morphism $\pi : \mathcal{Y} \to X^{\mathrm{an}}$. In the finite case, we have equivalences of categories

$$\begin{array}{ccc}\left\{\begin{matrix}\text{finite étale coverings}\\ \pi : \mathcal{Y} \longrightarrow X^{\mathrm{an}}\end{matrix}\right\} & \longrightarrow & \left\{\begin{matrix}\text{coherent } \mathcal{O}_{X^{\mathrm{an}}}\text{-modules}\\ \text{with integrable connection } (\mathcal{E}, \nabla)\\ \text{and a horizontal (commutative)}\\ \mathcal{O}_{X^{\mathrm{an}}}\text{-algebra structure } \times : \mathcal{E}^{\otimes 2} \to \mathcal{E}\end{matrix}\right\} \\ & & \downarrow \\ & & \left\{\begin{matrix}\text{étale } \mathcal{O}_{X^{\mathrm{an}}}\text{-algebras } (\mathcal{E}, \times)\\ \text{coherent as } \mathcal{O}_{X^{\mathrm{an}}}\text{-modules}\end{matrix}\right\}\end{array}$$

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given by

$$\pi \longmapsto (R^0_{\mathrm{DR}}\pi_*(\mathcal{O}_{\mathcal{Y}}, d), \times) = (\pi_*\mathcal{O}_{\mathcal{Y}}, \nabla, \times) \longmapsto (\pi_*\mathcal{O}_{\mathcal{Y}}, \times),$$

where $\times$ is deduced from the $\pi^{-1}\mathcal{O}_{X^{\mathrm{an}}}$-algebra structure of $\mathcal{O}_{\mathcal{Y}}$. These equivalences also take place in the algebraic categories:

$$\left\{\begin{matrix}\text{finite étale coverings}\\ \pi : Y \longrightarrow X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{coherent } \mathcal{O}_X\text{-modules}\\ \text{with integrable } \textit{regular} \text{ connection } (\mathcal{E}, \nabla)\\ \text{and a horizontal (commutative)}\\ \mathcal{O}_X\text{-algebra structure } \times : \mathcal{E}^{\otimes 2} \to \mathcal{E}\end{matrix}\right\}$$

$$\Big\downarrow$$

$$\left\{\begin{matrix}\text{finite étale}\\ \mathcal{O}_X\text{-algebras } \mathcal{E}\end{matrix}\right\}.$$

They permit to deduce that the functor $(*)$ is fully faithful, from the comparison theorem 31.4.6. (But this fact is of course much more elementary, cf. [2, Exp. XI, Prop. 4.3. *ii*]).

A.3. We will need an "extension lemma" for étale coverings, which, thanks to the dictionary A.2, will be obtained as a special case of a general extension lemma for regular connections.

Lemma A.4 (**Extension for regular connections).** *Let X be a smooth complex algebraic variety, and let $\mathcal{L}$ be a local system of finite-dimensional complex vector spaces on $X(\mathbb{C})$. Assume that there exist a dense open subset $U \subseteq X$ and a regular connection $(\mathcal{E}_U, \nabla_U)$ on U such that $(\mathcal{E}_U^{\mathrm{an}})^{\nabla_U^{\mathrm{an}}} \cong \mathcal{L}_{|U(\mathbb{C})}$. Then there exists a regular connection $(\mathcal{E}, \nabla)$ on X such that $(\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}} \cong \mathcal{L}$ and $(\mathcal{E}, \nabla)_{|U} \cong (\mathcal{E}_U, \nabla_U)$.*

Proof. Let Z be the smooth, purely 1-codimensional part of $X \smallsetminus U$. We denote by $j : U' := U \sqcup Z \hookrightarrow X$ the canonical immersion, and by $(\widetilde{\mathcal{E}}_U, \widetilde{\nabla}_U)$ the τ-extension of $(\mathcal{E}_U, \nabla_U)$ to U' (see 10). Because $(\mathcal{E}_U^{\mathrm{an}})^{\nabla_U^{\mathrm{an}}}$ extends to a local system on $U'(\mathbb{C})$ (even on $X(\mathbb{C})$), the local monodromy around Z is trivial. With the help of [35, II, 3.11], and taking into account the fact that $\tau(0) = 0$, we deduce that $\mathrm{Res}_Z \widetilde{\nabla}_U = 0$, hence $\widetilde{\nabla}_U$ is a genuine connection on U' (without pole at Z). We claim that $(\mathcal{E}, \nabla) := j_*(\widetilde{\mathcal{E}}_U, \widetilde{\nabla}_U)$ satisfies the requirements. Indeed, since $X \smallsetminus U'$ has codimension $\leqslant 2$, $\mathcal{E}$ is coherent, according to [48, 5.11.4] and the natural map $j^*(\mathcal{E}, \nabla) \to (\widetilde{\mathcal{E}}_U, \widetilde{\nabla}_U)$ is an isomorphism (which restricts to an isomorphism $(\mathcal{E}, \nabla)_{|U} \to (\mathcal{E}_U, \nabla_U)$ over U). On the other hand, $(\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}_{|U'(\mathbb{C})} \cong (\widetilde{\mathcal{E}}_U^{\mathrm{an}})^{\widetilde{\nabla}_U^{\mathrm{an}}} \cong \mathcal{L}_{|U'(\mathbb{C})}$. Since $(X \smallsetminus U')(\mathbb{C})$ has topological codimension $\leqslant 4$ in $X(\mathbb{C})$, $\pi_1((X \smallsetminus U')(\mathbb{C}), x) \to \pi_1(X(\mathbb{C}), x)$ is an isomorphism, and we conclude that $(\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}} \cong \mathcal{L}$. □

Lemma A.5 (**Extension for étale coverings).** *Assume that there exist an étale dominant morphism $X' \xrightarrow{\varphi} X$ finite over its image, and an étale covering $Y' \to X'$, such* *that Y'^{an} and $\mathcal{Y} \times_{X^{\mathrm{an}}} X'^{\mathrm{an}}$ are isomorphic coverings of X'^{an}. Then there exists an étale covering $Y \to X$ such that Y^{an} and $\mathcal{Y}$ are isomorphic coverings of X^{an}.*

Proof. Let us first assume that φ is surjective, hence an étale covering. Let Y''/Y' be an étale covering such that the induced covering Y''/X is Galois with group G. Then by the usual Galois theory of topological coverings, there is a subgroup $H \subseteq G$ such that $\mathcal{Y} \cong Y''^{\mathrm{an}}/H$. It then suffices to take $Y := Y''^{\mathrm{an}}/H$.

This first step allows us to assume that φ is an open dominant immersion $X' \hookrightarrow X$. Let $(R^0_{\mathrm{DR}}\pi_*(\mathcal{O}_{\mathcal{Y}}, d), \times)$ be the triple associated to π as in A.3. We know from the assumption that the restriction to X' of this triple is algebraizable; in particular, $R^0_{\mathrm{DR}}\pi_*(\mathcal{O}_{\mathcal{Y}}, d)$ comes from a regular connection on X'. By virtue of the extension lemma A.4, $R^0_{\mathrm{DR}}\pi_*(\mathcal{O}_{\mathcal{Y}}, d)$ itself comes from a regular connection $(\mathcal{E}, \nabla)$ on X. By the comparison theorem 31.4.6, the horizontal multiplication $\times : \mathcal{E}^{\mathrm{an}\otimes 2} \to \mathcal{E}^{\mathrm{an}}$ comes from a horizontal $\mathcal{O}_X$-linear map $\times : \mathcal{E}^{\otimes 2} \to \mathcal{E}$, which endows $\mathcal{E}$ with the structure of a finite $\mathcal{O}_X$-algebra. It then suffices to take $Y := \operatorname{Spec} \mathcal{E}$. $\square$

A.6. Let us prove the essential surjectivity of $(*)$ by induction on $\dim X$, assuming the (elementary) case $\dim X = 1$. Suppose given a finite étale covering $\mathcal{Y}$ of X^{an}. By A.5, we may replace X by any open dense subset. Hence we may assume that X is the total space of an elementary fibration

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

Let K be an algebraically closed subfield of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$, such that this elementary fibration comes from an elementary fibration

$$\begin{tikzcd} X_K \arrow[r, hook, "j_K"] \arrow[dr, "f_K"'] & \overline{X}_K \arrow[d, "\overline{f}_K"] & Z_K \arrow[l, hook'] \arrow[dl] \\ & S_K & \end{tikzcd}$$

defined over K. Let $s \in S(\mathbb{C})$ induce a geometric *generic* point of S_K, and let K' denote the algebraic closure of the image of $\kappa(S_K)$ in $\mathbb{C}$ (via the embedding given by s). Note that the smooth projective curve $\overline{X}_s$ comes from $\overline{X}_K \times_{S_K} \operatorname{Spec} K'$ by extension of scalars $K' \hookrightarrow \mathbb{C}$.

A.7. By virtue of Riemann's existence theorem in dimension 1, we know that the finite étale covering of X_s^{an} induced by $\mathcal{Y}$ is algebraizable, and in fact comes from a finite covering $\overline{T} \to \overline{X}_s$ unramified outside Z_s (inducing a finite étale covering $T \to X_s$). By a standard argument of (Weil) descent, such a covering is already defined over K', i.e., comes from a finite covering $\overline{T}_{K'} \to \overline{X}_K \times_{S_K} \operatorname{Spec} K'$, unramified outside $Z_K \times_{S_K} \operatorname{Spec} K'$ (thus inducing an étale covering $T_{K'} \to X_K \times_{S_K} \operatorname{Spec} K'$).

Moreover, by "spreading-out", one can find an étale morphism $S'_K \to S_K$, finite over its image, an embedding $\kappa(S'_K) \hookrightarrow K'$, a finite étale morphism $T'_K \to X_K \times_{S_K} S'_K$ and an isomorphism of coverings $T_{K'} \xrightarrow{\sim} T'_K \times_{S'_K} \operatorname{Spec} K'$. Let $s' \in S'_K(\mathbb{C})$ be the point above s defined by the embedding $\kappa(S'_K) \hookrightarrow K' \hookrightarrow \mathbb{C}$. Then the fiber at s' of the morphism $T'_K \to S'_K$ satisfies

$$T'^{\mathrm{an}}_{Ks'} \cong T^{\mathrm{an}} \cong \mathcal{Y}_s \cong (\mathcal{Y} \times_{S^{\mathrm{an}}} S'^{\mathrm{an}})_{s'}.$$

A.8. According to A.5, we can again replace X by an étale dominant neighborhood finite over its image in X, for instance $(X_K \times_{S_K} S'_K) \otimes_K \mathbb{C}$ or a Galois covering of this space lying above $T'_K \otimes_K \mathbb{C}$. This allows to assume that $\mathcal{Y}_s$ is a trivial covering of X_s. Since every elementary fibration is locally topologically trivial, we have a homotopy long exact sequence, a fragment of which being

$$\pi_1(X_s^{\mathrm{an}}, x) \longrightarrow \pi_1(X^{\mathrm{an}}, x) \longrightarrow \pi_1(S^{\mathrm{an}}, s) \longrightarrow 1$$

for any $x \in X(\mathbb{C})$ lying above s. We see that $\mathcal{Y}$ comes from an analytic étale covering of S^{an}, which is algebraizable by the induction hypothesis. Therefore $\mathcal{Y}$ is algebraizable. $\square$

It would be interesting to give a similar proof of the p-adic avatar, namely Lütkebohmert's theorem [76].

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