Analysis 12, 359 – 376 (1992)

Analysis

ALGEBRAIC APPELL-LAURICELLA FUNCTIONS

Paula Beazley Cohen and Jürgen Wolfart

Received: February 24, 1992

Abstract: We completely classify the algebraic Appel-Lauricella functions F_1 in several variables, so extending the famous 1873 list of H.A. Schwarz of the algebraic Gauss hypergeometric functions. We indicate some consequences for the other Appell-Lauricella functions and extend Terada's list of functions F_1 with euclidean discontinuous monodromy group. One new aspect of our techniques is the intervention of certain families of abelian varieties with generalized complex multiplication type.

AMS - classification: 33 C 45 , 14 K 20

One of the most famous results about Gauss' hypergeometric functions in the 19th century is the complete determination of the algebraic functions among them by H.A. Schwarz in 1873 [Schw]. F. Beukers and G. Heckmann extended Schwarz' list to the one-variable generalizations ${}_nF_{n-1}$, and Takeshi Sasaki treated the algebraic Appell - Lauricella functions F_1 [Ssk]. Since Sasaki's paper, a lot of new techniques in this area have been introduced ([DM], [Ho], [M], [Sa], [Te 2], [Y] , to mention a few recent publications), so we present here new approaches to these finite monodromy groups of F_1 and new applications, in particular an extension of Terada's list [Te 3] of functions F_1 with euclidean discontinuous monodromy group.

The paper is organized as follows: § 1 introduces the necessary notations and presents as Theorem 1 the main result saying that *for the functions* F_1 *in two and three variables there is essentially just one case respectively with finite monodromy group* (thus slightly correcting Sasaki's list); some consequences for the other Appell - Lauricella functions are discussed. § 2 contains three finiteness proofs for the groups in question, based on a) Mostow - Deligne's results on the discontinuity of the monodromy groups ([DM], [M]), b) the interpretation of the monodromy groups as modular groups for some families of polarized abelian varieties [CW 1,2], and c) direct calculations using generators and relations as in [Ssk], [Te 2].

In § 3 the completeness of the list in Theorem 1 is shown by induction over the dimension. As an application we give in § 4, Theorem 2 a complete list of discontinuous monodromy groups for the case where one exponential parameter of F_1 is an integer; the method of proof is different from previous work on this subject by Picard [P], Deligne - Mostow [DM] and Terada [T3].

The material of the present article was announced by the second author at the Katata workshop on Special Differential Equations (August 1991). Some of its results are touched also in our forthcoming paper [CW 2].

1. Notations. The main result

With the usual notation $(a,0) := 1$ and $(a,n) := a(a+1) \cdot \ldots \cdot (a+n-1)$ for $a \in \mathbb{C}$, $n \in \mathbb{N}$, let us define the Appell - Lauricella function F_1 of the N complex variables $x_2, \ldots, x_{N+1}$ with parameters $a, b_2, \ldots, b_{N+1}$ and $c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, by the series - usually called F_D in the literature if $N > 2$ -

$$F_1(a, b_2, \ldots, b_{N+1}; c; x_2, \ldots, x_{N+1}) = \sum \frac{(a, \sum_j n_j) \prod_j (b_j, n_j)}{(c, \sum_j n_j) \prod_j (1, n_j)} \prod_j x_j^{n_j},$$

where j runs from 2 to $N+1$, each n_j runs from 0 to ∞, and the series converges if all $|x_j| < 1$. Almost everywhere we will use instead of this series its integral representation

$$\frac{1}{B(1-\mu_1, 1-\mu_{N+2})} \int u^{-\mu_o} (u-1)^{-\mu_1} \prod_{j=2}^{N+1} (u-x_j)^{-\mu_j} \, du ,$$

where the *exponential parameters* $\mu_o, \mu_1, \ldots, \mu_{N+2}$ are related to $a, b_1, \ldots, b_{N+1}, c$ by

$$\mu_j = b_j \quad \text{for} \quad j = 2, \ldots, N+1$$

$$\mu_o = c - \sum b_j , \qquad \mu_1 = 1 + a - c$$

$$\sum_{j=o}^{N+2} \mu_j = 2 , \qquad \text{i.e.} \quad \mu_{N+2} = 1 - a .$$

If μ_1 and μ_{N+2} are < 1, the integration path can be chosen between $u = 1$ and $u = \infty$ avoiding the other singularities and choosing an appropriate branch of the differential

$$(1) \qquad \omega := u^{-\mu_o} (u-1)^{-\mu_1} \prod (u-x_j)^{-\mu_j} \, du .$$

If μ_1 or μ_{N+2} are > 1, but not in $\mathbb{Z}$, the integration path has to be replaced by a Pochhammer cycle around 1 and ∞ and the Beta-value in front of the integral has to be multiplied by some factor $\neq 0$ (see [Y], pp. 9 to 11).

The function F_1 satisfies a system $L_1(\mu_o, \dots, \mu_{N+2})$ of linear partial differential equations with a vector space V of solutions of dimension $N+1$, a basis of which can always be given in the form of integrals $\int\omega$ where the integration has to be carried out either over loops on the Riemann surface of ω or between the singularities of ω, i.e. between points $u = 0, 1, \infty$ or x_j, $j = 2, \dots, N+1$. The case of integer μ_j will be treated separately in § 4, so we assume for the moment all μ_j to be $\notin \mathbb{Z}$, and all integration paths can be assumed to be Pochhammer cycles around the singularities. Outside the *characteristic surfaces* $x_j = 0$, 1 or ∞ or $x_i = x_j$ $(i \neq j)$ the solutions and the coefficients of $L_1(\mu_o, \dots, \mu_{N+2})$ are holomorphic. The *monodromy group* Δ of F_1 is now defined as a subgroup of GL(V) obtained by the analytic continuation of the elements of V along closed paths in the space Q of the variables outside the characteristic surfaces, starting from a fixed point $(x_2, \dots, x_{N+1})$. Any basis of V defines a locally biholomorphic and globally multivalent map of Q into $\mathbb{P}^N(\mathbb{C})$, the *"developing map"* Φ. The multivalence of Φ is described just by the natural action of Δ on its image.

We choose an unnatural numerotation of the variables to emphasize that every singularity $0, 1, x_2, \dots, x_{N+1}, \infty$ of ω has the same importance for the calculation of the monodromy group: if we suppose $y_o, y_1, \dots, y_{N+1}, y_{N+2}$ to be pairwise distinct points of $\mathbb{P}^1(\mathbb{C})$, the integrals over Pochhammer cycles around all pairs y_j, y_k

$$\int \prod_{i=o}^{N+2} (u - y_i)^{-\mu_i}\, du$$

(if some $y_i = \infty$, the factor $u - y_i$ has to be omitted) have the following invariance property: under common $PGL_2\,\mathbb{C}$ - transformations of the y_i they are invariant up to common simple factors $\neq 0$. So we can replace the components of Φ by integrals of this more general kind, and it is reasonable to define the set of variables as

the points of $\mathbb{P}^1(\mathbb{C})^{N+3}$ with at least three different coordinates
modulo the diagonal action of $PGL_2\,\mathbb{C}$.

In this formulation, the singularity set of L_1 and Φ is the union of the characteristic surfaces

$$S(i,j) := \left\{ (y_o, \dots, y_{N+2}) \bmod PGL_2\mathbb{C} \;\middle|\; y_i = y_j \right\}$$

(now including $S(0,1)$, $S(0, N+2)$, $S(1, N+2)$). One can of course normalize the coordinates such that three of them, say y_o, y_1, y_{N+2}, take the values $0, 1, \infty$ respectively. In this case we write as usual

$y_2 = x_2 = x$, $y_3 = x_3 = y$, $y_4 = x_4 = z$
$b_2 = b$, $b_3 = b'$, $b_4 = b''$.

Before formulating the main result we recall some well-known facts:

1) If the μ_j are not integers as we assume here with the exception of § 4, the group Δ is irreducible [Te 1] , so the finitely many branches of an algebraic F_1 generate the whole space V . Hence as in the classical case N = 1 we have the equivalence

$$\Delta \text{ finite} \iff F_1 \text{ algebraic.}$$

(If some $\mu_j \in \mathbb{Z}$ this equivalence does not hold since Δ can have invariant subspaces of algebraic functions without being finite, see § 4).

2) Up to isomorphism, Δ depends only on the residue classes μ_j mod $\mathbb{Z}$ ([Y] or the calculations in § 2; this remark also becomes false if some $\mu_j \in \mathbb{Z}$, see § 4 or [Kl] for N = 1). Permutations of the μ_j induce permutations of the generators of Δ only.

3) As in the classical case, algebraic functions F_1 can only have algebraic ramifications. Since the fundamental solutions of L_1 behave in S(i,j) (outside the other characteristic surfaces) either like holomorphic functions or like holomorphic functions times $(y_i - y_j)^{1-\mu_i-\mu_j}$, we may assume that all the μ_j are rational numbers.

THEOREM 1 : *Assume all* $\mu_o , \dots , \mu_{N+2} \in \mathbb{Q} - \mathbb{Z}$.

Then there are no algebraic F_1 *in more than three variables.*

The function F_1 (x,y) *is algebraic if and only if* $\mu_o , \dots , \mu_4$ *are up to permutations and a common change of the sign* $\equiv \frac{1}{3} , \frac{1}{6} , \frac{1}{6} , \frac{1}{6} , \frac{1}{6}$ mod $\mathbb{Z}$.

Then Δ *is the symmetry group of the extended Hesse polytope of order* 1296, *projectively isomorphic to a subgroup of order* 216 *of the group* $PSL_3\mathbb{F}_3$.

The function F_1 (x,y,z) *is algebraic if and only if* $\mu_o, \dots , \mu_5$ *are all* $\equiv \frac{1}{6}$ mod $\mathbb{Z}$ *or all* $\equiv -\frac{1}{6}$ mod $\mathbb{Z}$. *Then* Δ *is the symmetry group of the Witting polytope of order* 155 520 , *projectively isomorphic to the group* $PSp_4\, \mathbb{F}_3$ *of order* 25 920 .

In the case N = 2 this means that $F_1(a,b,b';c;x,y)$ is algebraic if and only if up to a common change of sign

$a \underset{\mathbb{Z}}{\equiv}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$
$b \equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$b' \equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$c \equiv$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$

We will prove Theorem 1 in §§ 2 and 3. For the moment we give some obvious consequences. The first is based on the following lemma obtained by straightforward calculations (the notations are the same as in [Y]) using

$$F_1(a,b,b';c;x,y) = \sum_{m=0} F(a+m,\ b,\ c+m;x)\,\frac{(b',m)}{(1,m)}\,y^m$$

and the analytic continuation of F around the point $x = 1$:

LEMMA: *In the common domain of convergence of the series involved, we have*

$$F_1(a,b,b';c;x,y) = \frac{B(c,c-a-b)}{B(c-a,c-b)}\,F_2(a,b,b';a+b+1-c,\ c-b;1-x,y)$$
$$+\ \frac{B(c,a+b-c)}{B(a,b)}\,(1-x)^{c-a-b}\,F_2(c-b,\ c-a,\ b';\ c+1-a-b,\ c-b;\ 1-x,y)\ .$$

In particular, $F_2(a,b,b';a+b+1-c,\ c-b;1-x,y)$ *is a solution of* $L_1(\mu_0, \dots, \mu_4)$.

(N. Takayama kindly informed the authors that this transformation formula was found previously by M. Saigo and N.T. Hai [SH]. For a very general theory of such transformations see Takayama's forthcoming paper [Ta].)
So if the function F_1 of the lemma is algebraic, both functions F_2 are also algebraic. For F_3 one can use in the same way [AK], p. 68 or the remark in [Y], p. 62, for F_4 see [B], p. 81 and 102.

COROLLARY: *The function* $F_2(a,b,b';c,c';x,y)$ *is an algebraic function if up to a common change of sign and the permutation* $(b,c) \longleftrightarrow (b',c')$

$a \underset{\mathbb{Z}}{\equiv}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$b \equiv$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$b' \equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$c \equiv$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$c' \equiv$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

The function $F_3(a,a',b,b';c;x,y)$ *is algebraic if up to a common change of sign and the permutation* $(a,b) \longleftrightarrow (a',b')$

a	$\overset{\mathbb{Z}}{\equiv}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
a'	$\equiv$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$
b	$\equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
b'	$\equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
c	$\equiv$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{3}$.

The function $F_4(a,b;c,c';x,y)$ *is algebraic if* $F(a,b,c;z)$ *is a Gauss hypergeometric function algebraic in* z *(see the table in §3) and* $c' \equiv a+b-c \bmod \mathbb{Z}$, *or if up to a common change of sign and the exchange of* c *and* c'

a	$\overset{\mathbb{Z}}{\equiv}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$
b	$\equiv$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
c	$\equiv$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$
c'	$\equiv$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$

It must be noted that for all these algebraic functions the branches only generate a 3-dimensional $\mathbb{C}$-vector space. Since F_2, F_3, F_4 satisfy a system of partial linear differential equations with a solution space of dimension 4, we have here always a proper subspace invariant under the respective monodromy group, hence we are in a reducible case of the respective system of partial differential equations. From [B, p.79, (3) and p.102, Ex.20 (iii) to (v)] one can deduce even more algebraic cases in which the branches only generate a one- or 2-dimensional space.

2. Finiteness

First proof of the finiteness of the groups Δ occuring in Theorem 1: According to Mostow [M], for $N = 4$ and

$$\mu_0 = \dots = \mu_4 = \frac{1}{6} \quad , \quad \mu_5 = \mu_6 = \frac{7}{12}$$

we have a monodromy group Δ acting discontinuously on a ball of dimension 4 .

The characteristic surface $S(5,6)$ is not semistable in the sense of Deligne - Mostow since $\mu_5 + \mu_6 > 1$, but nevertheless the developing map Φ extends continuously to

$$S'(5,6) := S(5,6) - \text{the points on the other characteristic surfaces,}$$

since there is always a basis of solutions of $L_1(\mu_0, \dots, \mu_6)$ holomorphic on $S'(5,6)$ with the exception of one holomorphic function multiplied by a factor $(y_5 - y_6)^{1-\mu_5-\mu_6}$. So this extension of Φ contracts $S'(5,6)$ to a point of the ball or rather all Δ - images of a point; in fact, a closer look at the components of Φ shows that these image points are the same as the Φ - images of the (stable) point

$$S(0,1,2,3,4), \quad \text{i.e.} \quad y_0 = y_1 = y_2 = y_3 = y_4 .$$

Now let Δ_S be a subgroup of Δ fixing such an image point, and let $\Delta_{5,6}$ be the monodromy group obtained from Δ by replacing the pair μ_5, μ_6 by the single exponent $\mu_5 + \mu_6 = \frac{7}{6}$, now for $N = 3$. Restricting the action of Δ_S to the components of Φ holomorphic on $S'(5,6)$ we obtain a homomorphism of Δ_S onto $\Delta_{5,6}$. Since Δ acts discontinuously, Δ_S and hence $\Delta_{5,6}$ must be finite - and $\Delta_{5,6}$ coincides just with the monodromy group of the algebraic function $F_1(x,y,z)$ of Theorem 1. Changing the signs of the μ_j mod $\mathbb{Z}$ induces an algebraic conjugation of the matrix entries which can be chosen in $\mathbb{Q}(\zeta)$, $\zeta = \exp \frac{2\pi i}{3}$.

The finiteness of the first monodromy group of Theorem 1 ($N = 2$) is proven either by setting $z = 0$ in the previous algebraic $F_1(x,y,z)$ or by applying the same arguments as above on the Mostow example

$$\mu_0 = \dots = \mu_3 = \frac{1}{6}, \quad \mu_4 = \mu_5 = \frac{2}{3} .$$

For the identification of these finite monodromy groups we look at their generators, obtained by analytic continuation of the fundamental solutions along small loops around $S(i,j)$: we obtain complex reflections with N eigenvalues 1 and one eigenvalue $\exp(2\pi i(1 - \mu_i - \mu_j))$. Our groups are therefore irreducible unitary reflection groups, and it is not hard to see - e.g. by restriction on the characteristic surfaces - that they are primitive groups. For $N = 3$ the generators are all of order 3, and for $N = 2$ there are generators of order 3 and of order 2, so the well-known classification of unitary reflection groups ([Cox], [ST], [Y]) leaves as the only possibilities the symmetry groups of the Witting and the extended Hesse polytope. For the identification of the projective groups see also [CC], [CM].

Second proof: Consider the family of nonsingular projective algebraic curves $X(x,y,z)$ given by affine models

$$w^6 = u(u-1)(u-x)(u-y)(u-z)$$

depending on three pairwise different complex parameters $x,y,z \neq 0,1$. These curves arise for $\mu_0 = \dots = \mu_4 = \frac{1}{6}$, $\mu_5 = \frac{7}{6}$ as the Riemann surfaces on which the differential (1) is defined as the differential $\omega = \frac{du}{w}$ of the second kind. Their

Jacobians admit morphisms m_3 and m_2 onto the Jacobians of the corresponding curves with w^6 replaced by w^3 and w^2 respectively. Let $T(x,y,z)$ be the connected component containing 0 of the common kernel of m_2 and m_3 . Then the methods of [CW 1,2] show that

a) $\dim T(x,y,z) = 4$,
b) the endomorphism algebra of $T(x,y,z)$ contains $\mathbb{Q}(\zeta)$ whose action on T is induced by the automorphism $(u,w) \mapsto (u, -\zeta w)$ of the curve,
c) the induced operation of $\mathbb{Q}(\zeta)$ on the space $H^0(T(x,y,z),\Omega)$ of differentials of the first kind splits this space into a "+" and a "-" eigenspace (in which $a \in \mathbb{Q}(\zeta)$ acts via multiplication by a or $\bar{a}$) with dimensions $r_n := -1 + \sum_{j=0}^{5} \langle n\mu_j \rangle$, where $\langle \alpha \rangle := \alpha - [\alpha]$ denotes the fractional part of $\alpha \in \mathbb{R}$, $n \in (\mathbb{Z}/6\mathbb{Z})^*$ and all $\mu_j = \frac{1}{6}$ in our case; so we obtain

$$r_1 = 0 \quad \text{and} \quad r_{-1} = 4 .$$

d) All polarized abelian varieties T with this endomorphism algebra $\mathbb{Q}(\zeta)$ and this splitting of $H^0(T,\Omega)$ form a family of polarized abelian varieties, with "generalized complex multiplication by $\mathbb{Q}(\zeta)$ of type (0,4)" in the sense of Shimura [S]. This family is parametrized by a complex bounded symmetric domain B . There is a natural injection of the monodromy group Δ of $L_1\left(\frac{1}{6},\dots;\frac{1}{6},\frac{Z}{6}\right)$ into the modular group of this family [CW 1,2].

On the other hand, $\dim B = r_1 \cdot r_{-1} = 0$, hence this modular group itself has to be finite. - Changing the signs of the μ_j mod $\mathbb{Z}$ changes only the differential on the curve. Moreover, the case N = 2 can be treated in the same way using the curve $\qquad w^6 = u(u-1)(u-x)(u-y)$

for $\omega = \frac{du}{w}$, $\mu_0 = \dots = \mu_3 = \frac{1}{6}$, $\mu_4 = \frac{4}{3}$. The identification of the groups can be done as in the first proof.

REMARK: [Wo, §6]: Even in the case N = 1 the finiteness of the monodromy groups in Schwarz' list (see §3 and let d be the least common denominator of $\mu_0, \mu_1, \mu_2, \mu_3$) can be deduced from the fact that the dimensions

$$r_n = -1 + \sum_{j=0}^{3} \langle n\mu_j \rangle$$

take only the extreme values 0 and 2 for all $n \in (\mathbb{Z}/d\mathbb{Z})^*$. This condition is in turn equivalent to the p - curvature condition [Ka] for the algebraicity of the function $F(a,b,c;x)$ which says that 1 and $\exp(2\pi inc)$ always seperate $\exp(2\pi ina)$ and $\exp(2\pi inb)$ on the unit circle: An easy reformulation shows that

$$r_n = \langle nb \rangle - \langle na \rangle + \langle nc - nb \rangle + \langle na - nc \rangle ,$$

so $\quad r_n = 0 \quad$ iff $\quad \langle nb \rangle < \langle nc \rangle < \langle na \rangle$,

and $\quad r_n = 2 \quad$ iff $\quad \langle na \rangle < \langle nc \rangle < \langle nb \rangle$.

The *third proof* on the one hand uses the explicit matrix representation of some generators of Δ ([Y], [Ssk]) and on the other hand the simple explicit presentations of the symmetry groups in question found by Coxeter [Cox]. To obtain the generators of Δ, fix a point $a = (a_0, \dots, a_5)$ in Q (case $N = 3$), fix a simply connected region $T(a) \subset \mathbb{P}^1$ bounded by a simple curve passing through $a_0, a_1, a_2, a_3, a_4, a_5$ in this order and leaving $T(a)$ at its left. Choose an appropriate set of fundamental solutions of L_1 from the integrals $I(i,j)$ of ω between a_i and a_j (or around a_i and a_j if the paths are Pochhammer cycles). The integration paths have to be chosen inside $T(a)$ up to small discs around the a_j if necessary. For $i < j$ let $T(i,j)$ be the generators of Δ obtained by analytic continuation of the above generators along the following loop in Q : a_i moves inside $T(a)$ up to a_j, turns counterclockwise around a_j in a small simple loop and moves inside $T(a)$ back to its initial position, all other a_k remaining fixed. The effect of $T(i,j)$ ($=: T(j,i)$ for any $i \neq j$) on the $I(k,l)$ is easily obtained graphically as in [Y], p. 149. If we identify $j = 6$ with $j = 0$, and always denote by k and l numbers $\neq j, j+1$, we find the following effect of $T(j, j+1)$ on the integrals:

$$I(k,l) \mapsto I(k,l)$$
$$I(k,j) \mapsto I(k,j) + [\exp(-2\pi i \mu_j) - \exp(-2\pi i(\mu_j + \mu_{j+1}))]\, I(j,j+1)$$
$$I(k,j+1) \mapsto I(k,j) + \exp(-2\pi i \mu_j)\, I(j,j+1)$$
$$I(j,j+1) \mapsto \exp(-2\pi i(\mu_j + \mu_{j+1}))\, I(j,j+1)$$

(be aware that our μ_j have a different meaning from those in [Y]!). Now take $I(j, j+1)$ and some other three $I(k,l)$ as a basis of L_1 (possible); then it is easily seen that $T(j,j+1)$ is a complex reflection whose order is the order of the root of unity $\exp(-2\pi i(\mu_j + \mu_{j+1}))$, i.e. always 3 in our case. The same argument shows that $T(j, j+1)$ commutes with any $T(k, k+1)$ if $k \neq j-1, j+1$. To produce relations between $T(j, j+1)$ and $T(j-1, j)$ take $I(j-1,j)$, $I(j,j+1)$ and some other $I(k,l)$ with k and $l \neq j-1, j, j+1$ as a basis of L_1 (possible), the latter being unchanged by both $T(j, j+1)$ and $T(j-1, j)$. A straightforward calculation shows that $T(j-1, j)\, T(j, j+1)\, T(j-1, j)$ is described by

$$I(j-1, j) \mapsto \zeta^2\, I(j, j+1)$$
$$I(j,j+1) \mapsto -\zeta\, I(j-1, j)\ ,$$

taking into account that all $\exp(-2\pi i \mu_j) = -\zeta$ where $\zeta = \exp\frac{2\pi i}{3}$. On the other hand we see in the same way that $T(j, j+1)\, T(j-1, j)\, T(j, j+1)$ has the same effect on these fundamental solutions. So we obtain in this special case

(2) $$T(j-1, j)\ T(j,j+1)\ T(j-1,j) = T(j,j+1)\ T(j-1, j)\ T(j,j+1)$$

for all $j = 1, 2, 3$ and

(3) $$T(j, j+1)^3 = 1$$

for all $j = 0, 1, 2, 3$. The same procedure shows

$$T(j-1, j+1) = T(j, j+1)^{-1} \; T(j-1, j) \; T(j, j+1)$$

and by induction on the difference $|k - l|$ that all $T(k, l)$ are generated by $T(j, j+1)$, $j = 0, 1, 2, 3$; note that we can always assume a_5 to be fixed and move only the other a_j .

Coxeter [Cox, p. 148] has given a presentation of the symmetry group of the Witting polytope by generators R_0, R_1, R_2, R_3 and relations

$$R_j^3 = 1 \qquad \text{for all } j = 0, 1, 2, 3$$

$$R_{j-1} R_j R_{j-1} = R_j R_{j-1} R_j \qquad \text{for all } j = 1, 2, 3$$

$$R_i R_j = R_j R_i \qquad \text{for all } i, j \text{ with } |i-j| > 1 .$$

Hence $R_j \to T(j, j+1)$ defines by (2) and (3) a homomorphism of the finite symmetry group onto our monodromy group Δ showing again the finiteness. This homomorphism is in fact an isomorphism by the same arguments as in the first proof.

Terada worked out in [Te 2] (see also [M']) the following connection between the monodromy groups Δ of the system L_1 and braid groups: Let B_{N+2} be the braid group on $N+2$ strings and H_{N+2} the kernel of its natural projection onto the symmetric group S_{N+2} . The group Δ provides a natural representation of H_{N+2} in the solution space of $L_1(\mu_0, \dots, \mu_{N+2})$. For our particular Δ above, with the $T(j,j+1)$, $j = 0, \dots, 3$, taken as the images of the standard generators of B_5, we have a representation of this group as well as of H_5 $(N = 3)$. However, as the $T(j,j+1)$, $j = 0, \dots, 3$, have order 3 , they are also the images of the inverse squares of the generators of B_5 . These squares are pure braids generating the whole of the image of H_5 in this case, as we remark above. (One may also argue as in [M', p. 229])

The other finiteness proof $(N = 2)$ can be given in the same way or by restriction of the case $N = 3$ on any characteristic surface, say $S(0,4)$.

3. Completeness of the list

Now suppose $\mu_0, \dots, \mu_4$ to be the exponential parameters of an algebraic Appell function $F_1(x,y)$. The specialization $x = 0$ gives an algebraic function in y , and the integral representation shows that this is a Gauss hypergeometric function with exponential parameters $\mu_0 + \mu_2, \mu_1, \mu_3, \mu_4$. The same argument applies to the other fundamental solutions of L_1 , specialized to other characteristic surfaces, so that any quadruple obtained from $\mu_0, \dots, \mu_4$ by replacing a pair (μ_i, μ_j) by the sum $\mu_i + \mu_j$ must give a quadruple of exponential parameters for a Gauss hypergeometric algebraic function $(N = 1)$.

We give therefore a table of these exponential parameters based on Schwarz' well-known table of the finite monodromy groups in the case N = 1 . As usual, $2\pi\lambda$, $2\pi\nu$ and $2\pi\rho$ denote the rotation angles of the generators of the monodromy group; they are related to the other parameters of the table by the equations

$$\begin{aligned}
\lambda &= 1 - c &&= 1 - \mu_o - \mu_2 &&= \mu_1 + \mu_3 - 1 \\
\nu &= b - a &&= \mu_2 + \mu_3 - 1 &&= 1 - \mu_o - \mu_1 \\
\rho &= c - a - b &&= 1 - \mu_1 - \mu_2 &&= \mu_o + \mu_3 - 1 .
\end{aligned}$$

Here the μ_0 , ... , μ_3 are the exponential parameters for the case N = 1 . Any hypergeometric function in one variable with finite monodromy group has a quadruple of exponential parameters obtained from one of the quadruples of Schwarz' table by the following operations:

a) permutation of the μ_j

b) replacement of all μ_j by $1 - \mu_j$

c) replacement of all μ_j by $\mu_j + k_j$, all $k_j \in \mathbb{Z}$ with $\sum k_j = 0$.

In other words: λ, ν, ρ can be permuted, can (individually) change their sign and can be replaced by $\lambda + m$, $\nu + n$, $\rho + r$ with m , n , $r \in \mathbb{Z}$ and $m + n + r$ even. The parameters a, b, c can be replaced by other representatives in their residue class mod $\mathbb{Z}$, can simultanuously change their sign and can be replaced by any of the 24 triples occuring in Kummer's transformation formulas, i.e. by

$$(c - a, c - b, c) , \; (a, c - b, c) , \; (c - a, b, c)$$

and so on [EMOT p. 105 f.] .

Now we return to the classification problem and assume as a first possibility that on some characteristic surface the restriction will give a finite Schwarz group of dihedral type; w.l.o.g. we may assume that on S(1,4) , say,

$$\mu_1 + \mu_4 \equiv \mu_o \equiv \frac{r}{2} \quad \text{and} \quad \mu_2 \equiv \mu_3 \equiv \frac{1}{2} - \frac{r}{2} \mod \mathbb{Z}$$

with some $r \in \mathbb{Q} - \mathbb{Z}$. This means that on another characteristic surface, say S(0,2) , we have another finite Schwarz group with exponential parameter $\mu_0 + \mu_2 \equiv \frac{1}{2} \mod \mathbb{Z}$. This exponential parameter occurs however only once in Schwarz' table and implies therefore

$$\mu_1 \equiv \mu_3 \equiv \mu_4 \equiv \pm \frac{1}{6} \mod \mathbb{Z} \quad ,$$

hence also $\mu_2 \equiv \pm \frac{1}{6}$ and $\mu_o \equiv \pm \frac{1}{3} \mod \mathbb{Z}$, giving just the first finite group of the theorem. In the same way one may see that if the third line of Schwarz' list occurs on a characteristic surface, Δ is forced to be of the same type.

A case by case analysis shows that only the first and the third line of Schwarz' table can occur as quadruples of exponential parameters restricted to characte-

λ	ν	ρ	a	b	c	μ_0	μ_1	μ_2	μ_3	
$\frac{1}{2}$	$\frac{1}{2}$	r	$-\frac{r}{2}$	$\frac{1}{2}-\frac{r}{2}$	$\frac{1}{2}$	$\frac{r}{2}$	$\frac{1}{2}-\frac{r}{2}$	$\frac{1}{2}-\frac{r}{2}$	$1+\frac{r}{2}$	dihedral case, $r \in \mathbb{Q} \cap \,]0, \frac{1}{2}]$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{13}{12}$	tetrahedral case
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{7}{6}$	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{24}$	$\frac{7}{24}$	$\frac{1}{2}$	$\frac{5}{24}$	$\frac{11}{24}$	$\frac{7}{24}$	$\frac{25}{24}$	octahedral case
$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{6}$	$\frac{13}{12}$	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$-\frac{1}{60}$	$\frac{19}{60}$	$\frac{1}{2}$	$\frac{11}{60}$	$\frac{29}{60}$	$\frac{19}{60}$	$\frac{61}{60}$	icosahedral case
$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{30}$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{11}{30}$	$\frac{3}{10}$	$\frac{31}{30}$	
$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{30}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{19}{30}$	$\frac{1}{6}$	$\frac{31}{30}$	
$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{1}{20}$	$\frac{7}{20}$	$\frac{1}{2}$	$\frac{3}{20}$	$\frac{9}{20}$	$\frac{7}{20}$	$\frac{21}{20}$	
$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$-\frac{1}{15}$	$\frac{4}{15}$	$\frac{2}{5}$	$\frac{2}{15}$	$\frac{8}{15}$	$\frac{4}{15}$	$\frac{16}{15}$	
$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$-\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{11}{10}$	
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$-\frac{1}{10}$	$\frac{7}{30}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{17}{30}$	$\frac{7}{30}$	$\frac{11}{10}$	
$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{7}{10}$	$\frac{1}{10}$	$\frac{11}{10}$	
$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$-\frac{7}{60}$	$\frac{17}{60}$	$\frac{1}{2}$	$\frac{13}{60}$	$\frac{23}{60}$	$\frac{17}{60}$	$\frac{67}{60}$	
$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{7}{30}$	$\frac{2}{5}$	$\frac{1}{6}$	$\frac{13}{30}$	$\frac{7}{30}$	$\frac{7}{6}$	

H. A. Schwarz' table

ristic surfaces, and so the uniqueness of the finite Δ is done in the case $N = 2$. This case by case analysis is supported by the following observation: If e.g. $(\mu_1 + \mu_4, \mu_o, \mu_2, \mu_3)$ is a quadruple occurring in Schwarz' table (by restriction on $S(1,4)$), then any sum of two components of it must (up to sign and mod $\mathbb{Z}$, of course) again occur as an exponential parameter in Schwarz' table, e.g. $\mu_o + \mu_2$ by restriction on $S(0,2)$ or $-(\mu_1 + \mu_4 + \mu_o) \equiv \mu_2 + \mu_3$ mod $\mathbb{Z}$ by restriction on $S(2,3)$. - So the quadruple of the second line is excluded because $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ only occurs in the third line just treated above, from which one cannot obtain the denominator 12 by summing. The octahedral and icosahedral cases may be similarly treated.

The uniqueness of Δ in the case $N = 3$ is proven in the same way but is in fact much easier since the 14 lines of Schwarz' table are replaced by one single line $\pm\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ mod $\mathbb{Z}$ of candidates for exponential parameters, restricted to the characteristic surfaces. By the same arguments, finite monodromy groups are impossible for $N > 3$.

4. Discontinuous euclidean monodromy groups

Felix Klein [Kl, § 49], pointing out an error in Riemann's papers, discovered that in the case of integral exponential parameters for $N = 1$, i. e. for the Gauss hypergeometric functions, the monodromy groups no longer depend only on the residue classes of the μ_j mod $\mathbb{Z}$. For $N > 1$ there are also at least two classes of monodromy groups to consider, and we will explain their difference by the distinction of the types of differentials and integration paths to be used. The monodromy groups which act as discontinuous groups are then almost immediately determined.

Since all relevant phenomena already arise if only one of the μ_j is integral, we will concentrate on the case $N = 2$, $\mu_o, \mu_1, \mu_2, \mu_4 \in \mathbb{Q} - \mathbb{Z}$ and $\mu_3 \in \mathbb{Z}$, where $\sum \mu_j = 2$ as before. Then

$$\omega = u^{-\mu_o} (u-1)^{-\mu_1} (u-x)^{-\mu_2} (u-y)^{-\mu_3} \, du$$

can be considered as a differential $\dfrac{du}{w \cdot (u-y)^{\mu_3}}$ on the nonsingular projective algebraic curve $X(x)$ with model

$$w^d = u^{d\mu_o} (u-1)^{d\mu_1} (u-x)^{d\mu_2} ,$$

where d denotes the least common denominator of $\mu_o, \mu_1, \mu_2, \mu_4$; again, $x, y, 0, 1, \infty$ are supposed pairwise distinct. As $\int\omega = 0$ over any Pochhammer cycle around y and any other of the points $u = 0, 1, x, \infty$, one has to choose the integration paths differently to obtain a fundamental system of solutions L_1. Two solutions S_1, S_2 can be chosen as before as Pochhammer integrals of ω around e.g. $1, \infty$ and $1, x$. But the third solution cannot be given in a uniform way; we have to distinguish two cases:

A) If $\mu_3 \leq 0$, ω is a differential of the second kind (i.e. meromorphic with vanishing residues) on $X(x)$ and the third solution S_0 can be chosen as an integral $\int_\gamma \omega$ whose path simply goes from y to 1 if $\mu_1 < 1$ or, if $\mu_1 > 1$, makes a small loop around $u = 1$ and goes back to $u = y$ avoiding all other $u = 0$, x, ∞ . Now analytic continuation in y along cycles in $\mathbb{P}_1 - \{0, 1, \infty, x\}$ does not change S_1 and S_2 , but transforms S_0 into

$$\rho S_0 + a S_1 + b S_2$$

where ρ runs through the d-th roots of unity according to the change of the sheet of $X(x)$ on which γ lies, and (a,b) runs through a sublattice of finite index in $\mathbb{Z}[\zeta_d] \times \mathbb{Z}[\zeta_d]$, with $\zeta_d := \exp \frac{2\pi i}{d}$. On the other hand, $(u - y)^{-\mu_3}$ is a polynomial in u and y , so S_1 and S_2 are polynomials in y also

$$S_\nu = \sum_{k=0}^{m} f_{\nu k}(x)\, y^k \quad ;$$

for every k , the functions f_{1k} and f_{2k} are solutions of the same hypergeometric differential equation with exponential parameters congruent to μ_0, μ_1, μ_2, μ_4 mod $\mathbb{Z}$. So analytic continuation in x along cycles in $\mathbb{P}_1 - \{0, 1, \infty, y\}$ transforms S_0 as above and S_1, S_2 by the matrices of the corresponding monodromy group Δ_x for $N = 1$ with exponential parameters μ_0, μ_1, μ_2, μ_4 mod $\mathbb{Z}$, i.e. the triangle group for the angles $\pi(1 - \mu_0 - \mu_1)$, $\pi(1 - \mu_0 - \mu_2)$, $\pi(1 - \mu_0 - \mu_4)$. In short, in this basis Δ looks like

$$\Delta = \left\{ \begin{pmatrix} \rho & a\ b \\ \begin{matrix} 0 \\ 0 \end{matrix} & A \end{pmatrix} \;\middle|\; \begin{matrix} \rho \text{ a } d^{th} \text{ root of unity, } (a,b) \in \text{ sublattice} \\ \text{in } \mathbb{Z}[\zeta_d] \times \mathbb{Z}[\zeta_d] \text{ of finite index, } A \in \Delta_x \end{matrix} \right\} .$$

Even with the subgroup $A = E$ (unit matrix $\longleftrightarrow$ analytic continuation in y) it is easily seen that Δ acts nowhere discontinuously in $\mathbb{P}^2$ with the exception $\mathbb{Z}[\zeta_d] = \mathbb{Z}$. Here $d = 2$ and $\mu_0, \mu_1, \mu_2, \mu_4 \equiv \frac{1}{2} \bmod \mathbb{Z}$, the triangle group Δ_x has signature $[\infty, \infty, \infty]$, and Δ becomes a subgroup of finite index in Jacobi's modular group acting discontinuously on $\mathbb{C} \times \mathfrak{H}$ (or $\mathbb{C} \times \mathfrak{H}^-$, where $\mathfrak{H}$ and $\mathfrak{H}^-$ denote the upper and the lower half planes respectively). In our case, S_0/S_2 plays the role of the coordinate in $\mathbb{C}$ and S_1/S_2 of the one in $\mathfrak{H}$.

B) Now suppose $\mu_3 > 0$. Then ω is no longer of second kind: it has vanishing residues in $u = 0, 1, x$ and ∞ and generically nonvanishing residues in the d points $u = y$ on $X(x)$. As a third solution S_0 we can choose the integral $\int_\gamma \omega$ over a small loop around y, i.e. essentially the residue of ω in $u = y$;

on the different sheets, these residues differ by factors of d^{th} roots of unity and this is the only effect on S_o of any analytic continuation along cycles in the space Q of regular points of L_1. For S_1 and S_2 the analytic continuation in Q can be calculated by moving the integration paths as in [Y, p.149] or in [Kl, p.102]. They are transformed again by the monodromy group Δ_x as in case A, but now every time a moving integration path passes a point u = y, we have to add a residue of ω at this point. Therefore now

$$\Delta = \left\{ \begin{pmatrix} \rho & 0 & 0 \\ a & & \\ b & A & \end{pmatrix} \;\middle|\; \begin{matrix} \rho \text{ a } d^{th} \text{ root of unity, a and } b \in \mathbb{Z}[\zeta_d] \\ \text{and} \quad A \in \Delta_x \end{matrix} \right\}$$

Hence Δ acts as a transformation group on the affine space $\cong \mathbb{C}^2 \subset \mathbb{P}^2$ given by $S_o \neq 0$ with a subgroup of translations $\cong \mathbb{Z}[\zeta_d] \times \mathbb{Z}[\zeta_d]$ and a subgroup Δ_x fixing (0,0) and contained in $GL_2\,\mathbb{Z}[\zeta_d]$. It is relatively easy to see that Δ acts discontinuously on $\mathbb{C}^2$ if and only if $\mathbb{Z}[\zeta_d] \times \mathbb{Z}[\zeta_d]$ is discrete in $\mathbb{C}^2$ and Δ_x is finite. A look at Schwarz' table shows that d = 4 (first line with $r = \frac{1}{2}$) and d = 6 (first line with $r = \frac{1}{3}$ or third line) are the only cases to consider.

Exactly the same arguments, together with the classification of the finite monodromy groups in higher dimensions given in Theorem 1, lead to

<u>THEOREM 2</u>: *Suppose $N \geq 2$, one of the $\mu_o, \ldots, \mu_{N+2}$ is in $\mathbb{Z}$, but the others not in $\mathbb{Z}$. Then the group Δ acts as a discontinuous group if and only if one of the following conditions hold :*

1) *The integral $\mu_j \leq 0$, all others are $\equiv \frac{1}{2} \bmod \mathbb{Z}$ and $N = 2$ (then Δ acts discontinuously on $\mathbb{C} \times \mathfrak{H}$ as a subgroup of finite index in Jacobi's modular group),*

2) *The integral $\mu_j > 0$, all others are up to permutations and a common change of sign, congruent $\bmod \mathbb{Z}$ to*

a)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$			N = 2
b)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$			N = 2
c)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$			N = 2
d)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$		N = 3
e)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	N = 4

(then Δ acts discontinuously on $\mathbb{C}^N$).

REMARKS: 1) Cases 1) and 2a) are contained in Le Vavasseur's list and are mentioned in [DM] and [T3] , 2a) is the subject of Picard's note [P] .

2) The other examples implicitly arise in Mostow's work [M] in the context of the semistable but nonstable characteristic surfaces, i.e., on S(i,j) one has $\mu_i + \mu_j = 1$; the developing map maps S(i,j) onto cusps on the border of the ball, and the restriction of the solutions of L_1 which remain finite on S(i,j) gives functions with euclidean monodromy group and exponential parameters with the pair (μ_i, μ_j) replaced by the sum $\mu_i + \mu_j$ (just as the algebraic F_1 arise as restrictions on non-semistable characteristic surfaces, see the first proof in § 2); one obtains e.g. 2e) of Theorem 2 on the characteristic surface S(6,7) of the Mostow example

$$\mu_j = \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6} .$$

3) In the same way, cases with more than one integral parameter can be handled. For N = 2 nothing new beyond Le Vavasseur's and Terada's list [Te3] is found.

4) In the euclidean cases 2) of Theorem 2 , the quotient $\mathbb{C}^N/G$ by the subgroup G of translations of Δ is an abelian variety splitting into elliptic curves with complex multiplication by either the Gauss or the Eisenstein integers.

5) As already remarked in § 1 , here we have infinite Δ but invariant subspaces of algebraic solutions of L_1, namely S_0 and its multiples if the integer μ_j is positive.

Literature

[AK] P.Appell, M.J.Kampé de Fériet: Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite. Gauthier-Villars 1926.

[B] W.N. Bailey: Generalized hypergeometric series. Stechert-Hafner Service Agency 1964.

[BH] F. Beukers, G. Heckmann: Monodromy for the hypergeometric function ${}_nF_{n-1}$. Inventiones **95** (1989) 325 - 354 .

[CW 1] P.Cohen, J.Wolfart: Monodromie des fonctions d'Appell, variétés abéliennes et plongement modulaire. Preprint MPI 1989 - 80 . To appear in the proceedings of the Journées Arithmétiques Luminy 1989.

[CW 2] P. Beazley Cohen, J. Wolfart: Fonctions hypergéométriques en plusieurs variables et espaces de modules de variétés abéliennes. Preprint MPI Bonn 8/92.

[CC] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson : Atlas of finite groups. Clarendon Press 1985 .

[Cox] H.S.M. Coxeter: Regular Complex Polytopes. Cambridge University Press 1974.

[CM] H.S.M. Coxeter, W.O.J. Moser : Generators and Relations for Discrete Groups. Springer 1965 .

[DM] P. Deligne, G.D. Mostow: Monodromy of Hypergeometric Functions and Non-Lattice Integral Monodromy. Publ. IHES **63** (1986) 5-89.

[EMOT] A. Erdelyi, W. Magnus, F. Oberhettinger, F. Tricomi: Higher transcendental functions. The Bateman manuscript project, Vol. 1, Mc Graw-Hill 1953.

[Ho] R.-P.Holzapfel: Geometry and Arithmetic around Euler Partial Differential Equations. Reidel 1986.

[Ka] N.M.Katz: Algebraic solutions of differential equations (p - curvature and the Hodge filtration), Inventiones **18** (1972) 1 - 118.

[Kl] F. Klein: Vorlesungen über die hypergeometrische Funktion. Springer Grundlehren **39** (1981).

[M] G.D. Mostow : Generalized Picard Lattices arising from Half - Integral Conditions. Publ. IHES **63** (1986) 91-106.

[M'] G.D.Mostow: Braids, Hypergeometric Functions and Lattices. Bull. AMS **16**, No.2 (1987) 225-246.

[P] E.Picard: Sur les séries hypergéométriques de deux variables. CRAS **104** (1887) 896-897.

[S] G. Shimura : On analytic families of polarized abelian varieties and automorphic functions. Ann. Math. **78** (1963) 149-192.

[Sa] J.K. Sauter, jr.: Isomorphisms among Monodromy Groups and Applications to Lattices in PU (1,2). Pacific J. of Math. **146** (1990) 331-384.

[Schw] H.A. Schwarz: Über diejenigen Fälle, in welchen die Gaußische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt. J.Reine Angew. Math. **75** (1873) 292-335.

[SH] H. Saigo, N.T. Hai: The asymptotic representations of Appell series F_1 and Horn series H_3 near boundaries of their convergence regions. Fukuoka University Science Reports **21** (1991) 17 - 23.

[Ssk] T. Sasaki: On the finiteness of the monodromy group of the system of hypergeometric differential equations (F_D) . J. Fac. Sci. Univ. of Tokyo **24** (1977) 565-573.

[ST] G.C. Shepard, J.A. Todd: Finite unitary reflection groups. Canad. J. Math. **6** (1954) 274-304.

[Ta] N. Takayama: Propagation of singularities of solutions of the Euler-Darboux equation and a global structure of the space of holonomic solutions. Preprint.

[Te 1] T.Terada: Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables. J. Math. Kyoto Univ. **13** (1973) 557-578.

[Te 2] T. Terada: Quelques Propriétés Géométriques du Domaine de F_1 et le Groupe de Tresses Colorées. Publ. RIMS **17** (1981) 95-111.

[Te 3] T. Terada: Hypergeometric function F_1 and automorphic functions III. Case with some integer parameters. M.P.I. Bonn 1987 - **50** .

[Wo] J.Wolfart: Fonctions hypergéométriques, arguments exceptionnels et groupes de monodromie. Publ. Math. de l'Univ. P. et M. Curie, Problèmes Diophantiens 1985-86, IX.1 - IX.24.

[Y] M.Yoshida: Fuchsian Differential Equations. Aspects of Mathematics E 11, Vieweg 1987.

Paula Beazley Cohen
URA 747 au CNRS
Collège de France
3 rue d'Ulm
F-75 005 Paris
France

Jürgen Wolfart
Mathematisches Seminar der
Universität Frankfurt
Robert-Mayer-Str. 10
D-6000 Frankfurt a.M. 1
FRG