FINITENESS OF DE RHAM COHOMOLOGY.

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0. Introduction. Let k be a field of characteristic 0 and A be a finitely generated smooth k algebra. Let $\Omega^{\cdot}_{A/k}$ be the complex of algebraic differential forms on A. $\Omega^{\cdot}_{A/k}$ admits a degree 1 exterior differentiation, d. The homology groups of the complex $(\Omega^{\cdot}_{A/k}, d)$ will be denoted by $H_{DR}{}^{i}(A)$; they are the algebraic De Rham cohomology groups of A.

When $k = \boldsymbol{C}$ these groups have topological significance. Grothendieck proved in [1] that $H_{DR}{}^{i}(A) \approx H^{i}(V;\boldsymbol{C})$ where V is the complex manifold attached to A. From this one can see that $H_{DR}{}^{i}(A)$ is finite dimensional when $k = \boldsymbol{C}$, and the result extends easily to all k of characteristic zero.

But it is desirable (especially to a p-adic cohomologist) to give a purely algebraic proof of finite dimensionality. Such a proof has been given by Hartshorne (unpublished); it is global in nature and like Grothendieck's proof makes essential use of resolution of singularities. The object of this note is to give a purely local proof.

In Hartshorne's proof, the Gysin sequence is used to reduce a question concerning affine varieties to one on projective varieties. We also will use the Gysin sequence to reduce to the case when A is a localization of a polynomial ring. In this case $H_{DR}(A)$ turns out to be the homology of a Koszul complex of first order differential operators on a polynomial ring. Deformation techniques from [2] may then be used to handle this complex. Indeed our paper is nothing more than a simplification and reinterpretation of some results from [2] in terms of De Rham cohomology. That such translations may be made has been shown by Katz, and the map of Lemma 2.1 relating the De Rham complex to a Koszul complex on a polynomial ring comes from his thesis ([3]).

1. $H_{DR}(A)$ behaves nicely under ground field extension, so we may for simplicity suppose k algebraically closed.

LEMMA 1. *Let A be a smooth finitely generated domain over k and z a non-zero element of A. Then there exists an $m > 0$ and non-zero elements $t_1 \cdots t_m$ of A such that*

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(1) *$A/(t_1)$ and $A_{t_1\cdots t_{i-1}}/(t_i)$ are smooth over k.*

(2) $z^{-1} \in A_{t_1 t_2 \cdots t_m}$.

In other words we can excise a non-singular hypersurface from $\operatorname{Spec} A$, a second non-singular hypersurface from what's left over, and after a finite number of steps land inside $\operatorname{Spec} A_z$. The proof is by induction on the dimension of the singular set of the closed subscheme of $\operatorname{Spec} A$ defined by $z=0$. The essential step is finding a t_m in A vanishing on the singular set of $z=0$, and such that the singular set of $t_m=0$ is contained in but contains no component of the singular set of $z=0$. The argument is basically Bertini's theorem; for details see [4], Prop. 4, p. 134.

LEMMA 2. *Let S be a collection of smooth finitely generated k-algebras satisfying:*

(1) *Anything isomorphic to an element of S is in S.*

(2) *$k[X_1\cdots X_n]_h$ is in S for all n and all $h \in k[X_1\cdots X_n]$.*

(3) *S is closed under finite direct sums.*

(4) *Suppose A is a smooth finitely generated domain over k, $t \in A$, and A/t is smooth. Then if two of A, A/t, A_t are in S, so is the third.*

Then every finitely generated smooth A is in S.

Proof. We may assume A is a domain by (3); argue by induction on $n=\dim A$. For $n=0$, use (2). Suppose $n>0$. Every variety is birational to a hypersurface so we can find $A'=k[X_0\cdots X_n]/g$ with the same quotient field as A. Also we can find $h \notin (g)$ such that $A^*=A'_h$ is smooth over k. By (2) and (4), $A^* \in S$.

Since A and A^* have the same quotient field we can find $x \in A$ and $z \in A^*$ such that $A_x = A^*_z$. Choose $t_1\cdots t_m$ as in Lemma 1 (for A^* and z). Then $A^* \subset A^*_{t_1} \subset \cdots \subset A^*_{t_1\cdots t_m}$. Repeated applications of (4) together with the induction assumption on n show that $A^*_{t_1\cdots t_m} = A^*_{zt_1\cdots t_m}$ is in S. Repeating the argument but this time "going up" we see that $A^*_z \in S$. So A_x is in S, and arguing in the same way with A and x we find that $A \in S$.

The above result is particularly useful when combined with the Gysin sequence of the following lemma.

LEMMA 3. *Let A be a smooth finitely generated domain over k and $t \neq 0$ an element of A such that A/t is smooth. Then there is an exact sequence:*

$$\to H_{DR}^{i-2}(A/t) \to H_{DR}^{i}(A) \to H_{DR}^{i}(A_t) \to H_{DR}^{i-1}(A/t) \to.$$

A proof of Lemma 3 is indicated in the first few pages of [6]. We now use Lemmas 2 and 3 to prove:

THEOREM 1.1. *Suppose $H_{DR}(A)$ is finite dimensional whenever $A = k[X_1 \cdots X_n]_h$. Then $H_{DR}(A)$ is finite dimensional for any finitely generated smooth A.*

For let $S = \{A$ such that $H_{DR}(A)$ is finite dimensional$\}$. We only need check that S satisfies (4) of Lemma 2; this follows from Lemma 3.

Following Hartshorne, we now indicate how the above ideas combined with resolution of singularities may be used to give an algebraic proof of finite dimensionality. If V is a smooth k-variety let $H_{DR}(V)$ be the hypercohomology of the complex of sheaves $\Omega^{\cdot}_{V/k}$. The spectral sequence of hypercohomology shows that $H_{DR}(V) = H_{DR}(\Gamma(V))$ when V is affine, and that $H_{DR}(V)$ is finite dimensional when V is projective. We have the following global variations on Lemmas 2 and 3.

LEMMA 2′. *Let S be a collection of smooth quasi-projective k-varieties satisfying:*

(1) *Anything isomorphic to an element of S is in S.*

(2) *S contains all smooth projectives.*

(3) *S is closed under finite disjoint unions.*

(4) *Suppose V is irreducible smooth quasi-projective and W is a smooth divisor on V. Then if two of V, W, $V - W$ are in S, so is the third.*

Then every quasi-projective smooth V is in S.

LEMMA 3′. *Let V be an irreducible smooth k-variety and W be a smooth subvariety of V of pure codimension s. Then there is an exact sequence:*

$$\to H_{DR}^{i-2s}(W) \to H_{DR}^{i}(V) \to H_{DR}^{i}(V - W) \to H_{DR}^{i-2s+1}(W) \to.$$

To prove Lemma 2′ we argue by induction on dimension V. By resolution of singularities, V is birational to a projective smooth variety V^*. Then V^* is in S and we use the down and up, up and down argument of Lemma 2 to show that V is in S. Lemma 3′ is due to Hartshorne; he proves it by combining Lemma 3 with general arguments from local cohomology.

Now let $S = \{$quasi projective smooth V such that $H_{DR}(V)$ is finite dimensional$\}$. Lemma 3′ with $s = 1$ shows that S satisfies (4) of Lemma 2′. Thus $H_{DR}(V)$ is finite dimensional for all quasi-projective smooth V; in particular $H_{DR}(A)$ is finite dimensional for all finitely generated smooth A.

When $k=\boldsymbol{C}$ the above technique is helpful in proving Grothendieck's theorem that $H_{DR}(A)$ is the cohomology of the complex manifold attached to A. For following Grothendieck, it suffices to show the mapping of hypercohomology: $H^{\cdot}(V_{zar},\Omega^{\cdot})\to H^{\cdot}(V_{cl},\Omega^{\cdot}_{hol})$ is bijective for a smooth affine variety V. When V is projective this is a consequence of *GAGA*; the affine case may be reduced to the projective case by Lemmas 2′ and 3′, an analytic form of Lemma 3′, and the five lemma.

2. Let A be a vector space over k and $D_i\colon A\to A$ be commuting k linear maps $(1\leqq i\leqq n)$. We define the Koszul complex $K.(A;D_1\cdots D_n)$ as follows. As graded group, $K.=A\underset{k}{\otimes}\wedge(k^n)$. The degree -1 differentiation on $K.$ is given by:

$$d\,a(e_{i_1}\wedge\cdots\wedge e_{i_s})=\sum_{j=1}^{s}(-1)^{j+1}D_{i_j}(a)\,(e_{i_1}\wedge\cdots\hat{e}_{i_j}\cdots\wedge e_{i_s}).$$

The homology groups of $K.$ will be denoted by $H.(A;D_1\cdots D_n)$. We will be particularly concerned with complexes of the following type. Let A be a k-algebra, $E_1\cdots E_n$ be commuting k-linear derivations of A and f be an element of A. Set $f_i=E_i(f)$ and let $D_i\colon A\to A$ be the first order differential operator $g\to E_i(g)+f_ig$. The D_i commute, and we may form the complex $K.(A;D_1\cdots D_n)$. We shall refer to this complex as $K.(A;f;E_1\cdots E_n)$ and denote its homology by $H.(A;f;E_1\cdots E_n)$.

LEMMA 2.1. *Suppose A, E_i and f are as above with f a non-zero divisor. E_i prolongs to a derivation of A_f; denote the induced operator on A_i/A by E_i. Let T be an indeterminant over A. Then:*

$$H_i(A_f/A;E_1\cdots E_n)\approx H_i(A[T];Tf;E_1\cdots E_n,\frac{\partial}{\partial T}).$$

Proof. Let $\gamma\colon A[T]\to A_f/A$ be the A linear map such that $\gamma(T^i)=(-1)^i i!/f^{i+1}$. γ is onto and the kernel is generated as A module by f and $iT^{i-1}+fT^i$ $(i>0)$. So we have an exact sequence

$$0\to A[T]\xrightarrow{\frac{\partial}{\partial T}+L_f}A[T]\xrightarrow{\gamma}A_f/A\to 0.$$

We may identify the cokernel of $\partial/\partial T+L_f$ with A_f/A; under this identification the operator $E_i+L_{Tf_i}$ on $A[T]$ is easily seen to induce E_i on A_f/A. Now,

$$\begin{aligned}&H_i(A[T];Tf;E_1\cdots E_n,\frac{\partial}{\partial T})\\&\qquad=H_i(A[T];E_1+L_{Tf_1}\cdots E_n+L_{Tf_n},\frac{\partial}{\partial T}+L_f);\end{aligned}$$

the lemma follows.

THEOREM 2.1. *Suppose* $H_i(k[X_1\cdots X_n];h;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ *are finite dimensional for all* n *and all* $h\in k[X_1\cdots X_n]$. *Then* $H_{DR}(B)$ *is finite dimensional for all finitely generated smooth* B.

Proof. Let $A=k[X_1\cdots X_n]$ and $f\in A$, $f\neq 0$. By Lemma 2.1 and the hypotheses of Theorem 2.1, $H_i(A;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ and $H_i(A_f/A;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ are finite dimensional. It follows that $H_i(A_f;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ are finite dimensional. But $H_i(A_f;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ is easily seen to be isomorphic with $H_{DR}{}^{n-i}(A_f)$; now apply Theorem 1.1.

3. To conclude the proof of finiteness it suffices to show that the $H_i(k[X_1\cdots X_n];f;\frac{\partial}{\partial X_1}\cdots\frac{\partial}{\partial X_n})$ are finite dimensional. We shall first prove a relative finiteness theorem for a special type of f and then use deformation theory to handle arbitrary f.

LEMMA 1. *Let* R *be a ring,* ϵ *be a unit in* R *and* m *be an integer* >0. *Suppose* $\phi_i\colon R[X_1\cdots X_n]\to R[X_1,\cdots X_n]$ *are commuting* R *linear maps,* $(1\leqq i\leqq n)$, *such that* $\phi_i(g)=\epsilon X_i{}^m\cdot g$ *plus terms of lower degree. Then:*

(a) $H_i(R[X_1\cdots X_n];\phi_1\cdots\phi_n)=0\quad(i>0)$.

(b) $H_0(R[X_1\cdots X_n];\phi_1\cdots\phi_n)$ *is a free R-module on the monomials having degree* $<m$ *in each* X_i.

Proof. To prove (a) it is enough to show that

$$H_1(R[X_1\cdots X_n];\phi_1\cdots\phi_k)=0$$

for $k\leqq n$. In other words given g_i such that $\sum_1^k\phi_i(g_i)=0$ we must find a skew-symmetric set $\{c_{ij}\}$ such that $g_i=\sum_{j=1}^k\phi_j(c_{ij})$. We argue by induction on $d=\max(\deg g_i)$. Let $g_i{}^*$ be the degree d part of g_i. Then $\sum X_i{}^m g_i{}^*=0$. Consequently $g_i{}^*=\sum_j X_j{}^m a_{ij}$ where $\{a_{ij}\}$ is skew-symmetric and a_{ij} is homogeneous of degree $d-m$. Now let $h_i=g_i-\epsilon^{-1}\sum_j\phi_j(a_{ij})$. Then $\sum_1^k\phi_i(h_i)=0$. Since $\max(\deg h_i)<d$ we may argue by induction. A similar induction shows that the monomials of degree $<m$ in each X_i represent linearly independent elements of H_0; that they span H_0 is obvious.

From now on A_n will denote $k[X_1 \cdots X_n]$. Let Γ be an indeterminate over A_n, $A_{n+1} = A_n[\Gamma]$ and $A_{n+1}' = A_n[\Gamma, \Gamma^{-1}]$.

LEMMA 2. *Suppose $f \in A_n$. Let m be an integer $> \deg f$ and set $f^* \in A_{n+1} = f + \Gamma \sum_1^n X_i^m$. Then:*

(a) $V_i = H_i(A_{n+1}'; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}) = 0 \quad (i > 0)$.

(b) $V_0 = H_0(A_{n+1}'; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n})$ *is a finite free $k[\Gamma, \Gamma^{-1}]$ module.*

Proof. This is immediate from Lemma 1 with $R = k[\Gamma, \Gamma^{-1}]$ and $\phi_i = \partial/\partial X_i + L_{f^*_i}$.

Observe now that when Γ specializes to 0, f^* specializes to f. The next two lemmas are the heart of the deformation argument.

LEMMA 3.

(a) $H_i(A_n; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}) \approx H_i(A_{n+1}'/A_{n+1}; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma})$

(b) $H_i(A_{n+1}'/A_{n+1}; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma})$

$$\approx H_i(A_{n+1}'/A_{n+1}; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma}).$$

Remark. By abuse of language, $K.(A_{n+1}'/A_{n+1}; f; \cdots)$ denotes

$$K.(A_{n+1}'; f; \cdots)/K.(A_{n+1}; f; \cdots),$$

and $H.(A_{n+1}'/A_{n+1}; f; \cdots)$ is the homology of this quotient complex.

Proof. A_{n+1}'/A_{n+1} consists of sums $\sum_{-1}^{-r} a_i \Gamma^i$ with $a_i \in A_n$. Thus

$$\frac{\partial}{\partial \Gamma} : A_{n+1}'/A_{n+1} \rightarrow A_{n+1}'/A_{n+1}$$

is injective and the cokernel may be identified with A_n. This gives (a). To prove (b) note that $A_{n+1}'/A_{n+1} = A_n[\Gamma, \Gamma^{-1}]/A_n[\Gamma]$ and is isomorphic with $A_n[[\Gamma]][\Gamma^{-1}]/A_n[[\Gamma]]$. Now multiplication by $\exp(f^* - f)$ transforms this last space bijectively into itself and induces an isomorphism between the two Koszul complexes occurring in (b).

LEMMA 4. *The spaces $W_i = H_i(A_{n+1}'; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma})$ are finite dimensional over k.*

Proof. Let V_i be the $k[\Gamma, \Gamma^{-1}]$ modules occurring in Lemma 2. There is a standard Koszul exact sequence:

$$\to W_{i+1} \to V_i \xrightarrow{\Delta_i} V_i \to W_i \to$$

where Δ_i is the map induced by the operator $g \to \frac{\partial g}{\partial \Gamma} + (\frac{\partial f^*}{\partial \Gamma}) g$. By Lemma 2, $W_i = 0$ for $i > 1$ while W_1 and W_0 are isomorphic with the kernel and cokernel of Δ_0. So we must study the action of Δ_0 on V_0. Letting $0 = k[\Gamma, \Gamma^{-1}]$ and choosing an isomorphism of V_0 with 0^r we see that Δ_0 takes the form:

$$(u_1 \cdots u_r) \to (v_1 \cdots v_r) \text{ where } v_\iota = \frac{du_i}{d\Gamma} + \sum_j c_{ij} u_j.$$

(Here the c_{ij} are fixed elements of 0). So we must prove:

LEMMA 5. *Let* $0 = k[\Gamma, \Gamma^{-1}]$ *and* C *be an* $r \times r$ *matrix with entries in* 0. *Then the operator* $\frac{d}{d\Gamma} = C\colon 0^r$ *has finite dimensional kernel and cokernel over* k.

Proof. The kernel is easily handled. For example, we may imbed 0 in $k[[T]]$ by mapping Γ on $1 + T$. C becomes a matrix over $k[[T]]$ and we only need show that $d/dT + C\colon k[[T]]^r \to k[[T]]^r$ has finite dimensional kernel. But the kernel has dimension $\leqq r$, for a solution vector $u = (u_1 \cdots u_r)$ of $(d/d\Gamma + C)(u) = 0$ is determined by the contant terms of the u_i.

To handle the cokernel we use a suggestion of G. R. Allen, communicated to us by J. C. Robson. Let W_j be the subspace of 0^r consisting of vectors all of whose entries are k-linear combinations of Γ^i with $-j \leqq i \leqq j$. Then for c large, $\Delta = \frac{d}{d\Gamma} + C$ maps W_j into W_{j+c}. Let K_j and C_j be the kernel and cokernel of $\Delta\colon W_j \to W_{j+c}$. Then,

$$\dim C = (\dim W_{j+c} - \dim W_j) + \dim K_j \leqq 2rc + r.$$

As this bound is independent of j, the cokernel of $\Delta\colon 0^r \to 0^r$ also has dimension $\leqq 2rc + r$, completing the proof of Lemma 5.

We can now prove our main result.

THEOREM 3.1. *Suppose* $f \in A_n$. *Then* $H_i(A_n; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n})$ *are finite dimensional. Consequently* $H_{DR}(A)$ *is finite dimensional for any finitely generated smooth* k*-algebra.*

Proof. The exact sequence of groups $0 \to A_{n+1} \to A_{n+1}' \to A_{n+1}'/A_{n+1} \to 0$ gives rise to an exact sequence of Koszul complexes and a long exact sequence of homology:

$$W_i \to H_i(A_{n+1}'/A_{n+1}; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma})$$
$$\to H_{i-1}(A_{n+1}; f^*; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial \Gamma}).$$

By Lemma 3 the middle space in the above sequence may be identified with $H_i(A_n; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n})$. Now argue by induction on i. When $i = 0$, we find that $H_0(A_n; f; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}) \approx W_0$ and is finite dimensional by Lemma 4. For $i > 0$ we again use Lemma 4, the above exact sequence, and induction. The final assertion of Theorem 3.1 comes from Theorem 2.1.

We conclude with some remarks and open questions. The first part of Theorem 3.1 suggests generalizations. For we may interpret it as saying the following. If M is a rank 1 free module over a polynomial ring A and $M \to M \otimes \Omega^1{}_{A/k}$ is an integrable connection on M then the cohomology groups of the complex attached to the connection are finite dimensional. What happens when one considers more generally an integrable connection on a locally free sheaf over a smooth variety?

Many of the arguments of this paper go through in the study of p-adic cohomology (both in the sense of [2] and of [5]). The sticking point to proving finite dimensionality seems to be the p-adic analogue of Lemma 5, and in particular the question of the finiteness of the cokernel for certain ordinary differential operators on rings of p-adic analytic functions. A general theory of p-adic differential equations would be very desirable.

For $g \in A_n$, the $H_i(A_{n+1}; gT; \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}, \frac{\partial}{\partial T})$ are essentially the De Rham cohomology groups of $(A_n)_g$. In the course of the proof of Theorem 3.1 we deform gT into an element of $A_{n+1}[\Gamma]$ not divisible by T. Consequently the spaces V_0 and W_0 and the operator Δ_0 occurring in Lemma 4 have no obvious geometric interpretation. It seems possible however to give a more geometric proof in which the V_i are replaced by the relative De Rham cohomology groups of a family of varieties over a pnctured line. The W_i are then the cohomology groups of the total space of the family and the Δ_i are the operators induced by the derivation, $\frac{\partial}{\partial \Gamma}$, of the co-ordinate ring of the base.

In this setting, the exact sequence of Lemma 4 is an exact sequence attached to the Gauss-Manin connection, and the Δ_i are the classical Picard-Fuchs operators. Again see [3] for related questions.

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