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ON A CONJECTURE BY PIERRE CARTIER ABOUT A GROUP OF ASSOCIATORS

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Dedicated to Professor Gérard Jacob.

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Abstract In Cartier (Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents. Sém. BOURBAKI, 53ème 2000–2001, no. 885), Pierre Cartier conjectured that for any non-commutative formal power series Φ on $X = \{x_0, x_1\}$ with coefficients in a $\mathbb{Q}$-extension, A, subjected to some suitable conditions, there exists a unique algebra homomorphism φ from the $\mathbb{Q}$-algebra generated by the convergent polyzetas to A such that Φ is computed from the Φ_{KZ} Drinfel'd associator by applying φ to each coefficient. We prove that φ exists and that it is a free Lie exponential map over X. Moreover, we give a complete description of the kernel of ζ and draw some consequences about the arithmetical nature of the Euler constant and about an algebraic structure of the polyzetas.

Keywords Algebraic computation · Combinatorial Hopf algebra · Drinfel'd associators · Free Lie algebra · Noncommutative symbolic computation · Nonlinear dynamical systems · Polylogarithm · Polyzêta · Renormalization · Regularization · Special functions · Transcendence basis

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1 Introduction

1.1 Drinfel'd associator and polyzetas

In 1986, in order to study the linear representation of the braid group B_n coming from the monodromy of the Knizhnik–Zamolodchikov differential equations over $\mathbb{C}_*^n = \{\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$ [12]:

$$dF(\underline{z}) = \Omega_n(\underline{z})F(\underline{z}) \quad \text{with } \Omega_n(\underline{z}) = \frac{1}{2\mathrm{i}\pi}\sum_{1\le i<j\le n} t_{i,j}\frac{d(z_i - z_j)}{z_i - z_j}, \tag{1.1}$$

and $\{t_{i,j}\}_{i,j\ge 1}$ are noncommutative variables, Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series Φ is called an *associator*.

Since the system (1.1) is completely integrable then [7, 12].

$$d\Omega_n - \Omega_n \wedge \Omega_n = 0. \tag{1.2}$$

This is equivalent to the fact that the $\{t_{i,j}\}_{i,j\ge 1}$ satisfy the infinitesimal braid relations:

$$t_{i,j} = 0 \quad \text{for } i = j, \tag{1.3}$$

$$t_{i,j} = t_{j,i} \quad \text{for } i \neq j, \tag{1.4}$$

$$[t_{i,j}, t_{i,k} + t_{j,k}] = 0 \quad \text{for distinct } i, j, k, \tag{1.5}$$

$$[t_{i,j}, t_{k,l}] = 0 \quad \text{for distinct } i, j, k, l. \tag{1.6}$$

Example 1

- $\mathcal{T}_2 = \{t_{1,2}\}$.

$$\Omega_2(z_1, z_2) = \frac{t_{1,2}}{2\mathrm{i}\pi} \frac{d(z_1 - z_2)}{z_1 - z_2} \quad \text{with } F(z_1, z_2) = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}.$$

- $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, $[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$.

$$\Omega_3(z_1, z_2, z_3) = \frac{1}{2\mathrm{i}\pi}\left[t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right].$$

$$F(z_1, z_2, z_3) = G\left(\frac{z_1 - z_2}{z_1 - z_3}\right)(z_1 - z_3)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi},$$

where G satisfies the following Fuchsian differential equation with three regular singularities at 0, 1, and ∞:

$$\text{(DE)} \quad dG(z) = \bigl[x_0\, \omega_0(z) + x_1\, \omega_1(z)\bigr]G(z),$$

with

$$x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi} \quad \text{and} \quad \omega_0(z) := \frac{dz}{z},$$

$$x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1 - z}.$$

As already shown by Drinfel'd, the equation (DE) admits, on the simply connected domain $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$, two specific solutions:

$$G_0(z) \underset{z \rightsquigarrow 0}{\widetilde{}} \exp\bigl[x_0 \log(z)\bigr] \quad \text{and} \quad G_1(z) \underset{z \rightsquigarrow 1}{\widetilde{}} \exp\bigl[-x_1 \log(1 - z)\bigr]. \tag{1.7}$$

Drinfel'd also proved there exists the associator Φ_{KZ} such that $G_1^{-1}(z)G_0(z) = \Phi_{\mathrm{KZ}}$.

After that, Lê and Murakami expressed the coefficients of the Drinfel'd associator Φ_{KZ} in terms of *convergent* polyzetas [41], i.e. for $r_1 > 1$,

$$\zeta(r_1, \ldots, r_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{r_1} \cdots n_k^{r_k}}. \tag{1.8}$$

In [41], the authors also expressed the *divergent* coefficients as *linear* combinations of convergent polyzetas via a *regularization process* (see also [28]). This process is one of many ways to regularize the divergent terms.

1.2 Group of associators and regularized Chen generating series

The algebraic aspects of our regularization process based essentially on various products[1] among polyzetas (see [36]) and its analytical aspects will be described, in Sect. 3.1, as the *finite part*, of the asymptotic expansions in different scales of comparison[2] [5]. It will be seen also, in Sect. 3.2, as the action of the differential Galois group of the polylogarithms[3] (recalled in Sect. 2.1.2)

$$\mathrm{Li}_{r_1,\ldots,r_k}(z) = \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{r_1}\cdots n_k^{r_k}} \tag{1.9}$$

on the asymptotic expansion of polylogarithms, at $z=1$ and in the comparison scale $\{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, and the same action on the asymptotic expansions, at $+\infty$ and in the comparison scales $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and

$$\left\{n^a \mathrm{H}_1^b(n)\right\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$

of the harmonic sums (recalled in Sect. 2.1.1)

$$\mathrm{H}_{r_1,\ldots,r_k}(N) = \sum_{n_1>\cdots>n_k>0}^{N} \frac{1}{n_1^{r_1}\cdots n_k^{r_k}}. \tag{1.10}$$

This action leads then to a conjecture by Pierre Cartier ([8], conjecture C3) and to the description of the group of associators yielding the ideal of polynomial relations among coefficients of associators (Theorems 3.6 and 3.7). This group is in fact, closely linked to the group of the Chen generating series studied by K.T. Chen to describe the solutions of differential equations [10] and it turns out that each associator regularizes a Chen generating series of the differential forms ω_0 and ω_1 along the integration path on the simply connected domain $\mathbb{C}-(]-\infty,0]\cup[1,+\infty[)$.

1.3 Global renormalization and global regularization

In fact, our regularization process based essentially on two noncommutative generating series over the infinite alphabet $Y=\{y_i\}_{i\ge 1}$, which encodes the multi-indices $(r_1,\ldots,r_k)$ by the words $y_{r_1}\cdots y_{r_k}$ over the monoid generated by Y, denoted by Y^*, of polylogarithms and of harmonic sums (recalled in Sect. 2.2.1)

$$\Lambda(z) = \sum_{w\in Y^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) = \sum_{w\in Y^*} \mathrm{H}_w(N)\, w. \tag{1.11}$$

[1] First source of ambiguity leading to the problem of rewriting expressions of polyzetas in a canonical form using irreducible Lyndon words (see [31, 35]).

[2] Second source of ambiguity leading to the problem to determine the value of regularized polyzetas and its analytical meaning (see [27, 36]).

[3] Third source of ambiguity leading to the problem of fixing the integration path to solve (DE) and its monodromy group (see [33]) or its differential Galois group (see [28]).

Through the algebraic combinatorial aspects[4] [44] and the topological aspects [2] of formal power series in noncommutative variables, we have already showed the existence of noncommutative formal series over Y, Z_1 and Z_2 with constant terms, such that [29]

$$\lim_{z\to 1}\exp\left(y_1\log\frac{1}{1-z}\right)\Lambda(z)=Z_1, \tag{1.12}$$

$$\lim_{N\to\infty}\exp\left(\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N)=Z_2. \tag{1.13}$$

Moreover, Z_1 and Z_2 are equal and stand for the noncommutative generating series of all convergent polyzetas $\{\zeta(w)\}_{w\in Y^*-y_1Y^*}$ as shown by the factorized form indexed by Lyndon words (recalled in Sect. 2.2). This theorem enables, in particular, to explicit the counter-terms eliminating the divergence of the polylogarithms $\{\mathrm{Li}_w(z)\}_{w\in y_1Y^*}$, for $z\to 1$, and of the harmonic sums $\{\mathrm{H}_w(N)\}_{w\in y_1Y^*}$, for $N\to\infty$, and to calculate the Euler–Mac Laurin constants associated to the divergent polyzetas $\{\zeta(w)\}_{w\in y_1Y^*}$ (see Corollary 3.3). It allows also to give, in Sect. 3.3 and via identification of local coordinates in infinite dimension, a *complete* description of the kernel by its generators, of the following algebra homomorphism:[5]

$$\zeta:\big(A1_{Y^*}\oplus(Y-y_1)A\langle Y\rangle,\,⧢\big)\longrightarrow(\mathbb{R},.) \tag{1.14}$$

$$y_{r_1}\cdots y_{r_k}\longmapsto\sum_{n_1>\cdots>n_k>0}\frac{1}{n_1^{r_1}\cdots n_k^{r_k}}, \tag{1.15}$$

and the set of A-*irreducible* polyzetas forming a transcendence basis of the image of ζ, with $A=\mathbb{Q}[\mathrm{i}\pi]$ (see Corollary 3.9).

Finally, via the *indiscernibility* (recalled in Sect. 2.3) over the group of associators, this study makes precise the structure of the A-algebra generated by the convergent polyzetas (see Theorem 4.1) and concludes the main challenge of the *polynomial* relations among polyzetas indexed by convergent Lyndon words which are algebraically independent of the Euler constant and motivated [3, 31, 35, 48]. In particular, the A-algebra generated by the convergent polyzetas was conjectured to be *free* [31, 35] and the conjecture will be proved, thanks to Propositions 3.8, 3.9, and 3.10. Moreover, this free A-algebra is *graded by weight* meaning there is no *linear* relation among convergent polyzetas of different weight (see Theorem 4.1).

[4]See [44] to get an idea of these aspects of combinatorial Hopf algebra of the shuffle product, denoted by ⧢, and its co-product, denoted by $\Delta_{⧢}$. For the quasi-shuffle product, denoted by $\uplus$, and its co-product, denoted by $\Delta_{\uplus}$, see Appendix 1.

In our works, recalled in Appendix 2, these algebraic combinatorial aspects were explored systematically to expand the outputs of nonlinear controlled dynamical system with singular inputs (Corollary 6.1) on polylogarithmic functional basis [23, 30, 32]. In this way [29], polyzetas do appear then as fundamental arithmetical constant for the asymptotic analysis and for the renormalization of the outputs and their successive derivations (Corollary 6.2) via the extended Fliess fundamental formula (Theorem 6.2).

[5]Here, 1_{Y^*} stands for the empty word over Y.

2 Background: algebraic structures and analytical studies of harmonic sums and of polylogarithms

2.1 Structures of harmonic sums and of polylogarithms

2.1.1 Quasi-symmetric functions and harmonic sums

Let $\{t_i\}_{i\in\mathbb{N}_+}$ be an infinite set of variables. The elementary symmetric functions η_k and the power sums ψ_k are defined by (see [44])

$$\eta_k(\underline{t}) = \sum_{n_1>\cdots>n_k>0} t_{n_1}\cdots t_{n_k} \quad \text{and} \quad \psi_k(\underline{t}) = \sum_{n>0} t_n^k. \tag{2.1}$$

They are, respectively, coefficients of the following generating functions:

$$\eta(\underline{t}\mid z) = \prod_{i\geq 1}(1+t_i z) \quad \text{and} \quad \psi(\underline{t}\mid z) = \sum_{i\geq 1}\frac{t_i z}{1-t_i z}. \tag{2.2}$$

These generating functions satisfy a Newton identity:

$$z\frac{d}{dz}\log\eta(\underline{t}\mid z) = \psi(\underline{t}\mid -z). \tag{2.3}$$

The fundamental theorem from symmetric functions theory asserts that $\{\eta_k\}_{k\geq 0}$ are linearly independent, and provides remarkable identities like (with $\eta_0 = 1$):

$$\eta_k = \frac{(-1)^k}{k!}\sum_{\substack{s_1,\dots,s_k\geq 0\\ s_1+\cdots+ks_k=k+1}}\binom{k}{s_1,\dots,s_k}\left(-\frac{\psi_1}{1}\right)^{s_1}\cdots\left(-\frac{\psi_k}{k}\right)^{s_k}. \tag{2.4}$$

Let Y be the infinite alphabet $\{y_i\}_{i\geq 1}$ equipped with the order $y_1 > y_2 > y_3 > \cdots$ and let $\mathcal{L}ynY$ be the set of Lyndon words over Y. The length of $w = y_{s_1}\cdots y_{s_r} \in Y^*$ is denoted by $|w|$ and its degree equals $s_1+\cdots+s_r$.

The quasi-symmetric function F_w, of depth $r = |w|$ and of degree (or weight) $s_1+\cdots+s_r$, is defined by

$$\mathrm{F}_w(\underline{t}) = \sum_{n_1>\cdots>n_r>0} t_{n_1}^{s_1}\cdots t_{n_r}^{s_r}. \tag{2.5}$$

In particular, $\mathrm{F}_{y_1^k} = \eta_k$ and $\mathrm{F}_{y_k} = \psi_k$. The functions $\{\mathrm{F}_{y_1^k}\}_{k\geq 0}$ are linearly independent and the integrating differential equation (2.3) shows that functions $\mathrm{F}_{y_1^k}$ and F_{y_k} are linked by the formula

$$\sum_{k\geq 0}\mathrm{F}_{y_1^k} z^k = \exp\left(-\sum_{k\geq 1}\mathrm{F}_{y_k}\frac{(-z)^k}{k}\right). \tag{2.6}$$

Every $\mathrm{H}_w(N)$ can be obtained by specializing, in the quasi-symmetric function F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [39]:

$$\forall N\geq i\geq 1,\quad t_i = 1/i \quad \text{and} \quad \forall i > N,\quad t_i = 0. \tag{2.7}$$

In the same way, for $w \in Y^* - y_1Y^*$, the convergent polyzeta $\zeta(w)$ can be obtained by specializing, in F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [39]:

$$\forall N \geq i \geq 1, \quad t_i = 1/i. \tag{2.8}$$

The notation F_w is extended by linearity to all polynomials over Y.

If u (resp. v) is a word in Y^*, of length r and of weight[6] p (resp. of length s and of weight q), $\mathrm{F}_{u \uplus v}$ is a quasi-symmetric function of depth $r+s$ and of weight $p+q$, and $\mathrm{F}_{u \uplus v} = \mathrm{F}_u\, \mathrm{F}_v$, where $\uplus$ is the quasi-shuffle product[7] [39]. Hence,

$$\forall u, v \in Y^*, \quad \mathrm{H}_{u \uplus v} = \mathrm{H}_u\, \mathrm{H}_v \tag{2.9}$$

and then

$$\forall u, v \in Y^* - y_1Y^*, \quad \zeta(u \uplus v) = \zeta(u)\, \zeta(v). \tag{2.10}$$

The remarkable identity (2.4) can then be seen as

$$y_1^k = \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\cdots+ks_k=k+1}} \binom{k}{s_1,\ldots,s_k} \frac{(-y_1)^{\uplus s_1}}{1^{s_1}} \uplus \cdots \uplus \frac{(-y_k)^{\uplus s_k}}{k^{s_k}}. \tag{2.11}$$

2.1.2 Iterated integrals and polylogarithms

In all the sequel, we follow the notations of [2, 44].

Let X be the finite alphabet $\{x_0, x_1\}$ equipped with the order $x_0 < x_1$ and let

$$\mathcal{C} := \mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}\right] \quad \text{and} \quad \mathcal{G} := \left\{z, \frac{1}{z}, \frac{z-1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, 1-z\right\}. \tag{2.12}$$

This ring $\mathcal{C}$ is invariant under differentiation and under the homographic transformations belonging to the group $\mathcal{G}$ whose elements commute the singularities $\{0, 1, +\infty\}$.

The iterated integral over ω_0, ω_1 associated to the word $w = x_{i_1}\cdots x_{i_k}$ over X^* (the monoid generated by X) and along the integration path $z_0 \rightsquigarrow z$ is the multiple integral defined by

$$\int_{z_0\rightsquigarrow z} \omega_{i_1}\cdots\omega_{i_k} = \int_{z_0}^{z} \omega_{i_1}(t_1) \int_{z_0}^{t_1} \omega_{i_2}(t_2) \cdots \int_{z_0}^{t_{r-2}} \omega_{i_r}(t_{r-1}) \int_{z_0}^{t_{r-1}} \omega_{i_r}(t_r), \tag{2.13}$$

where $t_1\cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In a shortened notation, we denote this integral by $\alpha_{z_0}^z(w)$ and[8] $\alpha_{z_0}^z(1_{X^*}) = 1$. One can check that the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$ [25, 30]:

$$\mathrm{Li}_w(z) = \alpha_0^z\left(x_0^{s_1-1}x_1\cdots x_0^{s_r-1}x_1\right). \tag{2.14}$$

[6] The weight is as in (2.5).

[7] See Appendix 1, for the systematical study of the Hopf algebra of this quasi-shuffle product which is not included in [44].

[8] Here, 1_{X^*} stands for the empty word over X.

The definition of polylogarithms is extended over the words $w \in X^*$ by putting

$$\mathrm{Li}_{x_0}(z) := \log z. \tag{2.15}$$

The $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathcal{C}$-linearly independent [33, 35]. Thus, the following functions:

$$\forall w \in X^*, \quad \mathrm{P}_w(z) := (1-z)^{-1}\,\mathrm{Li}_w(z), \tag{2.16}$$

are also $\mathbb{C}$-linearly independent. Since, for any $w \in Y^*$, P_w is the ordinary generating function of the sequence $\{\mathrm{H}_w(N)\}_{N\geq 0}$ [27]:

$$\mathrm{P}_w(z) = \sum_{N\geq 0} \mathrm{H}_w(N)\, z^N \tag{2.17}$$

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ are also $\mathbb{C}$-linearly independent. Firstly, $\ker \mathrm{P} = \{0\}$ and $\ker \mathrm{H} = \{0\}$, and secondly, P is a morphism for the Hadamard product:

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \sum_{N\geq 0} \mathrm{H}_u(N)\mathrm{H}_v(N) z^N = \sum_{N\geq 0} \mathrm{H}_{u \,⧢\, v}(N) z^N = \mathrm{P}_{u \,⧢\, v}(z). \tag{2.18}$$

Proposition 2.1 [27] *Extended by linearity, the map*

$$\mathrm{P} : (\mathbb{C}\langle Y\rangle, ⧢) \longrightarrow \big(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot\big),$$
$$u \longmapsto \mathrm{P}_u$$

is an isomorphism of algebras. Moreover, the map

$$\mathrm{H} : (\mathbb{C}\langle Y\rangle, ⧢) \longrightarrow \big(\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .\big),$$
$$u \longmapsto \mathrm{H}_u = \big\{\mathrm{H}_u(N)\big\}_{N\geq 0}$$

is an isomorphism of algebras.

Studying the equivalence between action of $\{(1-z)^l\}_{l\in\mathbb{Z}}$ over $\{\mathrm{P}_w(z)\}_{w\in Y^*}$ and that of $\{N^k\}_{k\in\mathbb{Z}}$ over $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ (see [11]), we have

Theorem 2.1 [29] *The Hadamard $\mathcal{C}$-algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$ can be identified with that of $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$. In the same way, the algebra of harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ with polynomial coefficients can be identified with that of $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$.*

By Identity (2.11) and by applying the isomorphism H on the set of Lyndon words $\{y_r\}_{1\leq r\leq k}$, we obtain $\mathrm{H}_{y_1^k}$ as polynomials in $\{\mathrm{H}_{y_r}\}_{1\leq r\leq k}$ (which are algebraically independent), and

$$\mathrm{H}_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\cdots+ks_k=k+1}} \frac{(-1)^k}{s_1!\cdots s_k!}\left(-\frac{\mathrm{H}_{y_1}}{1}\right)^{s_1}\cdots\left(-\frac{\mathrm{H}_{y_k}}{k}\right)^{s_k}. \tag{2.19}$$

2.2 Results à la Abel for generating series of harmonic sums and of polylogarithms

2.2.1 *Generating series of harmonic sums and of polylogarithms*

Let $\mathrm{H}(N)$ be the noncommutative generating series of $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ [27]:

$$\mathrm{H}(N) := \sum_{w\in Y^*} \mathrm{H}_w(N)\, w. \tag{2.20}$$

Let $\{\Sigma_w\}_{w\in Y^*}$ and $\{\check{\Sigma}_w\}_{w\in Y^*}$ be, respectively, a PBW basis of the enveloping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$ and the quasi-shuffle algebra $(\mathbb{Q}\langle Y\rangle, ⧢\!\!\!\!+)$ (viewed as a $\mathbb{Q}$-module) on duality such that $\{\Sigma_l\}_{l\in\mathcal{L}ynX}$ and $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynX}$ are, respectively, a basis of the Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$ and a transcendence basis of the quasi-shuffle algebra (see Appendix 1).

Theorem 2.2 (Factorization of H) *Let*

$$\mathrm{H}_{\mathrm{reg}}(N) := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\check{\Sigma}_l}(N)\,\Sigma_l}.$$

Then $\mathrm{H}(N) = e^{\mathrm{H}_{y_1}(N)\,y_1}\mathrm{H}_{\mathrm{reg}}(N)$.

Proof See Appendix 1. □

For $l\in\mathcal{L}ynY-\{y_1\}$, the polynomial Σ_l is a finite combination of words in $Y^*-y_1Y^*$. Then we can state the following.

Definition 2.1 We set $Z_{⧢\!\!\!\!+} := \mathrm{H}_{\mathrm{reg}}(\infty)$.

The noncommutative generating series of polylogarithms [33, 35]

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\ w \tag{2.21}$$

satisfies Drinfel'd's differential equation (DE) of Example 1

$$d\mathrm{L} = (x_0\omega_0 + x_1\omega_1)\mathrm{L} \tag{2.22}$$

with boundary condition [13, 14]

$$\mathrm{L}(\varepsilon) \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_0\log\varepsilon}. \tag{2.23}$$

This enables us to prove that L is the exponential of a Lie series[9] [33, 35]. Hence,

Proposition 2.2 (Logarithm of L [28])

$$\log\mathrm{L}(z) = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^+}\mathrm{Li}_{u_1⧢\cdots⧢u_k}(z)\,u_1\cdots u_k$$

[9] *That is*, L is group-like for the co-product $\Delta_{⧢}$: $\Delta_{⧢}(\mathrm{L}) = \mathrm{L}\otimes\mathrm{L}$.

$$= \sum_{w \in X^*} \mathrm{Li}_w(z)\, \pi_1(w),$$

where $\pi_1(w)$ is the following Lie series:

$$\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k \in X^+} \langle w \mid u_1 ш \cdots ш u_k \rangle\, u_1 \cdots u_k.$$

Applying a theorem of Ree [43, 44], L satisfies the Friedrichs criterion [33, 35]:

$$\forall u, v \in X^*, \quad \mathrm{Li}_{u ш v} = \mathrm{Li}_u\ \mathrm{Li}_v\,. \tag{2.24}$$

Hence,

$$\forall u, v \in x_0 X^* x_1, \quad \zeta(u ш v) = \zeta(u)\, \zeta(v). \tag{2.25}$$

Proposition 2.3 (Successive differentiation of L [28]) *For any $l \in \mathbb{N}$, let*

$$P_l(z) = \sum_{\mathrm{wgt}(\mathbf{r})=l} \sum_{w \in X^{\deg(\mathbf{r})}} \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i} \tau_{\mathbf{r}}(w) \in \mathcal{C}\langle X \rangle,$$

where, for any $w = x_{i_1} \cdots x_{i_k}$ and $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg(\mathbf{r}) = k$ and of weight $\mathrm{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$, the polynomial $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ is defined by

$$\forall r \in \mathbb{N}, \quad \tau_r(x_0) = \partial^r \frac{x_0}{z} = \frac{-r! x_0}{(-z)^{r+1}} \quad \textit{and} \quad \tau_r(x_1) = \partial^r \frac{x_1}{1-z} = \frac{r! x_1}{(1-z)^{r+1}}.$$

Denoting $\partial = d/dz$, we have $\partial^l \mathrm{L}(z) = P_l(z)\mathrm{L}(z)$.

Let $\{\check{S}_l\}_{l \in \mathcal{L}ynX}$ be the transcendence basis of the shuffle algebra $(\mathbb{Q}\langle X \rangle, ш)$ and $\{\check{S}_w\}_{w \in X^*}$ be the associated completed basis of the shuffle algebra $(\mathbb{Q}\langle X \rangle, ш)$ (viewed as a $\mathbb{Q}$-module). They are defined as follows [44]:

$$\check{S}_{1_{X^*}} = 1 \quad \text{for } l = 1_{X^*}, \tag{2.26}$$

$$\check{S}_l = x \check{S}_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \tag{2.27}$$

$$\check{S}_w = \frac{\check{S}_{l_1}^{ш i_1} ш \cdots ш \check{S}_{l_k}^{ш i_k}}{i_1! \cdots i_k!} \quad \text{for } w = l_1^{i_1} \cdots l_k^{i_k}, l_1 > \cdots > l_k. \tag{2.28}$$

Let $\{S_w\}_{w \in Y^*}$ be the PBW basis of the enveloping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ in duality with the basis $\{\check{S}_w\}_{w \in Y^*}$ and $\{S_l\}_{l \in \mathcal{L}ynX}$ is then the basis of the Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ [44].

Theorem 2.3 (Factorization of L [33, 35]) *Let*

$$\mathrm{L}_{\mathrm{reg}} := \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{\mathrm{Li}_{S_l}\, \check{S}_l}.$$

Then $\mathrm{L}(z) = e^{-x_1 \log(1-z)} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z}$.

For $l \in \mathcal{L}ynX - X$, the polynomial S_l is a finite combination of words in $x_0X^*x_1$. Then we can state the following.

Definition 2.2 [33, 35] We set $Z_{\sqcup\sqcup} := \mathrm{L}_{\mathrm{reg}}(1)$.

In Definitions 2.1 and 2.2 only *convergent* polyzetas arise and these noncommutative generating series will induce, in Sect. 3.1, two algebra morphisms of regularization as shown in Theorems 3.1 and 3.2, respectively. Hence, these power series are quite different from those given in [41] or in [42] (the latter is based on [6], see [8]) needing a regularization process.

2.2.2 *Asymptotic expansions by noncommutative generating series and regularized Chen generating series*

Let $\rho_{1-z}, \rho_{1-\frac{1}{z}}$ and $\rho_{\frac{1}{z}}$ [34, 35] be three monoid morphisms verifying

$$\rho_{1-z}(x_0) = -x_1 \quad \text{and} \quad \rho_{1-z}(x_1) = -x_0, \tag{2.29}$$

$$\rho_{1-1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1-1/z}(x_1) = -x_0, \tag{2.30}$$

$$\rho_{1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1/z}(x_1) = x_1. \tag{2.31}$$

Using homographic transformations belonging to the group $\mathcal{G}$, one has [34, 35]

$$\mathrm{L}(1-z) = e^{x_0 \log(1-z)} \rho_{1-z}\big[\mathrm{L}_{\mathrm{reg}}(z)\big] e^{-x_1 \log z} Z_{\sqcup\sqcup}, \tag{2.32}$$

$$\mathrm{L}(1-1/z) = e^{x_0 \log(1-z)} \rho_{1-\frac{1}{z}}\big[\mathrm{L}_{\mathrm{reg}}(z)\big] e^{-x_1 \log z} \rho_{1-1/z}\big(Z_{\sqcup\sqcup}^{-1}\big) e^{\mathrm{i}\pi x_0} \tag{2.33}$$

$$\mathrm{L}(1/z) = e^{-x_1 \log(1-z)} \rho_{1/z}\big[\mathrm{L}_{\mathrm{reg}}(z)\big] e^{(-x_0+x_1)\log z} \rho_{1/z}\big(Z_{\sqcup\sqcup}^{-1}\big) e^{\mathrm{i}\pi x_1} Z_{\sqcup\sqcup}. \tag{2.34}$$

Thus, (2.23) and (2.32) yield [34, 35]

$$\mathrm{L}(z) \underset{z \to 0}{\widetilde{}} \exp(x_0 \log z) \quad \text{and} \quad \mathrm{L}(z) \underset{z \to 1}{\widetilde{}} \exp\big(-x_1 \log(1-z)\big)\, Z_{\sqcup\sqcup}. \tag{2.35}$$

Let us call $\mathrm{LI}_{\mathcal{C}}$ the smallest algebra containing $\mathcal{C}$, closed under derivation and under integration with respect to ω_0 and ω_1. It is the $\mathcal{C}$-module generated by the polylogarithms $\{\mathrm{Li}_w\}_{w \in X^*}$.

Let $\pi_Y : \mathrm{LI}_{\mathcal{C}}\langle\langle X\rangle\rangle \longrightarrow \mathrm{LI}_{\mathcal{C}}\langle\langle Y\rangle\rangle$ be a projector such that for any $f \in \mathrm{LI}_{\mathcal{C}}$ and $w \in X^*$, $\pi_Y(f\, wx_0) = 0$. Then [29]

$$\Lambda(z) = \pi_Y \mathrm{L}(z) \underset{z \to 1}{\widetilde{}} \exp\left(y_1 \log \frac{1}{1-z}\right) \pi_Y Z_{\sqcup\sqcup}. \tag{2.36}$$

Since the coefficient of z^N in the ordinary Taylor expansion of $\mathrm{P}_{y_1^k}$ is $\mathrm{H}_{y_1^k}(N)$ then let

$$\mathrm{Mono}(z) := e^{-(x_1+1)\log(1-z)} = \sum_{k \geq 0} \mathrm{P}_{y_1^k}(z)\, y_1^k \tag{2.37}$$

$$\mathrm{Const} := \sum_{k \geq 0} \mathrm{H}_{y_1^k}\, y_1^k = \exp\left(-\sum_{k \geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right). \tag{2.38}$$

Proposition 2.4 [29] *We have*

$$\pi_Y \mathrm{P}(z) \underset{z\to 1}{\widetilde{\quad}} \mathrm{Mono}(z)\pi_Y Z_{\sqcup\!\sqcup} \quad \textit{and} \quad \mathrm{H}(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N)\pi_Y Z_{\sqcup\!\sqcup}.$$

Proof Let μ be the morphism verifying $\mu(x_0) = x_1$ and $\mu(x_1) = x_0$. Then, by Theorem 2.3, the noncommutative generating series of $\{\mathrm{P}_w\}_{w\in X^*}$ is given by

$$\begin{aligned}\mathrm{P}(z) = (1-z)^{-1}\mathrm{L}(z) &= e^{-(x_1+1)\log(1-z)}\mathrm{L}_{\mathrm{reg}}(z)e^{x_0\log z}\\ &= e^{x_0\log z}\mu\big[\mathrm{L}_{\mathrm{reg}}(1-z)\big]e^{-(x_1+1)\log(1-z)}Z_{\sqcup\!\sqcup}\\ &= e^{x_0\log z}\mu\big[\mathrm{L}_{\mathrm{reg}}(1-z)\big]\mathrm{Mono}(z)Z_{\sqcup\!\sqcup}.\end{aligned}$$

Thus, $\mathrm{P}(z) \underset{z\to 0}{\widetilde{\quad}} e^{x_0\log z}$ and $\mathrm{P}(z) \underset{z\to 1}{\widetilde{\quad}} \mathrm{Mono}(z)Z_{\sqcup\!\sqcup}$ leading to the expected results. □

As a consequence of (2.36)–(2.38) and of Proposition 2.4, one gets

Theorem 2.4 (à la Abel [29])

$$\lim_{z\to 1}\exp\left(y_1\log\frac{1}{1-z}\right)\Lambda(z) = \lim_{N\to\infty}\exp\left(\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = \pi_Y Z_{\sqcup\!\sqcup}.$$

Therefore, the knowledge of the ordinary Taylor expansion at 0 of the polylogarithmic functions $\{\mathrm{P}_w(1-z)\}_{w\in X^*}$ gives

Theorem 2.5 [11] *For all $g \in \mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, there exist algorithmically computable $c_j \in \mathbb{C}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N}$ and $b_i \in \mathbb{C}, \eta_i \in \mathbb{Z}, \kappa_i \in \mathbb{N}$ such that*

$$g(z) \underset{z\to 1}{\widetilde{\quad}} \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) \quad \textit{and} \quad \big[z^n\big]g(z) \underset{N\to+\infty}{\widetilde{\quad}} \sum_{i=0}^{+\infty} b_i n^{\eta_i}\log^{\kappa_i}(n).$$

Definition 2.3 Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzetas and let $\mathcal{Z}'$ be the[10] $\mathbb{Q}[\gamma]$-algebra generated by $\mathcal{Z}$.

Corollary 2.1 [11] *There exist algorithmically computable $c_j \in \mathcal{Z}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N}$ and $b_i \in \mathcal{Z}', \kappa_i \in \mathbb{N}, \eta_i \in \mathbb{Z}$ such that*

$$\forall w \in Y^*, \quad \mathrm{P}_w(z) \sim \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) \quad \textit{for } z \to 1,$$

$$\forall w \in Y^*, \quad \mathrm{H}_w(N) \sim \sum_{i=0}^{+\infty} b_i N^{\eta_i}\log^{\kappa_i}(N) \quad \textit{for } N \to +\infty.$$

The Chen generating series along the path $z_0 \rightsquigarrow z$, associated to ω_0, ω_1 is the following:

$$S_{z_0\rightsquigarrow z} := \sum_{w\in X^*}\langle S \mid w\rangle w \quad \text{with } \langle S \mid w\rangle = \alpha_{z_0}^z(w) \tag{2.39}$$

[10] Here, γ stands for the Euler constant $\gamma = 0.577215664901532860606512090082402431042 1\ldots$.

which solves the differential equation (2.22) with the initial condition

$$S_{z_0 \rightsquigarrow z_0} = 1. \tag{2.40}$$

Thus, $S_{z_0 \rightsquigarrow z}$ and $\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ satisfy the same differential equation taking the same value at z_0 and

$$S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}. \tag{2.41}$$

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group like [43] and depends only on the homotopy class of $z_0 \rightsquigarrow z$ [10]. The product of $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is the Chen generating series

$$S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}. \tag{2.42}$$

Let $\varepsilon \in]0, 1[$ and $z_i = \varepsilon \exp(\mathrm{i}\theta_i)$, for $i = 0$ or 1. We set $\theta = \theta_1 - \theta_0$. Let $\Gamma_0(\varepsilon, \theta)$ (resp. $\Gamma_1(\varepsilon, \theta)$) be the path turning around 0 (resp. 1) in the positive direction from z_0 to z_1. By induction on the length of w, one has

$$\left|\langle S_{\Gamma_i(\varepsilon,\theta)} \mid w\rangle\right| = (2\varepsilon)^{|w|_{x_i}} \frac{\theta^{|w|}}{|w|!}, \tag{2.43}$$

where $|w|$ denotes the length of w and $|w|_{x_i}$ denotes the number of occurrences of letter x_i in w, for $i = 0$ or 1.

For ε tends to 0^+, these estimations yield

$$S_{\Gamma_i(\varepsilon,\theta)} = e^{\mathrm{i}\theta x_i} + o(\varepsilon). \tag{2.44}$$

In particular, if $\Gamma_0(\varepsilon)$ (resp. $\Gamma_1(\varepsilon)$) is a circular path of radius ε turning around 0 (resp. 1) in the positive direction, starting at $z = \varepsilon$ (resp. $1 - \varepsilon$), then, by the noncommutative residue theorem [33, 35], we get

$$S_{\Gamma_0(\varepsilon)} = e^{2\mathrm{i}\pi x_0} + o(\varepsilon) \quad \text{and} \quad S_{\Gamma_1(\varepsilon)} = e^{-2\mathrm{i}\pi x_1} + o(\varepsilon). \tag{2.45}$$

Finally, the asymptotic behaviors of L on (2.35) give

Proposition 2.5 [33, 35] *We have*

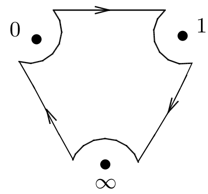
$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{-x_1 \log \varepsilon} Z_{\sqcup\!\sqcup} e^{-x_0 \log \varepsilon}.$$

In other terms, $Z_{\sqcup\!\sqcup}$ is the regularized Chen generating series $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ of differential forms ω_0 and ω_1: $Z_{\sqcup\!\sqcup}$ is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $e^{x_1 \log \varepsilon} S_{\varepsilon \rightsquigarrow 1-\varepsilon} e^{x_0 \log \varepsilon}$, i.e. the concatenation of $e^{x_0 \log \varepsilon}$ and then $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1 \log \varepsilon}$.

Proposition 2.6 *Let $\rho_{1-1/z}$ be the morphism given in Sect. 2.2.2. We have*

$$\prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\zeta(\check{l})l} = e^{\mathrm{i}\pi x_0} \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\zeta(\check{l})\rho_{1-1/z}(l)} e^{\mathrm{i}\pi(-x_0+x_1)} \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\zeta(\check{l})\rho^2_{1-1/z}(l)} e^{-\mathrm{i}\pi x_1}.$$

Fig. 1 Hexagonal path



Proof Following the hexagonal path given in Fig. 1, one has [34, 35]

$$\big(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0}\big)\rho_{1-1/z}\big(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0}\big)\rho^2_{1-1/z}\big(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0}\big) = 1 + O(\sqrt{\varepsilon}).$$

By Proposition 2.5, it implies the "hexagonal relation" (see [13, 14, 34, 35]) which is

$$Z_{\sqcup\!\sqcup}e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\!\sqcup})e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\!\sqcup})e^{-\mathrm{i}\pi x_1} = 1,$$

or equivalently,

$$e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\!\sqcup})e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\!\sqcup})e^{-\mathrm{i}\pi x_1} = Z^{-1}_{\sqcup\!\sqcup}.$$

Then the expected result follows. □

2.3 Indiscernibility over a class of formal power series

2.3.1 Residual calculus and representative series

Definition 2.4 Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$ and let $P \in \mathbb{Q}\langle X\rangle$.

The *left residual* (resp. *right residual*) of S by P, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X\rangle\rangle$ defined by

$$\langle P \triangleleft S \mid w\rangle = \langle S \mid wP\rangle \qquad \big(\text{resp.} \langle S \triangleright P \mid w\rangle = \langle S \mid Pw\rangle\big).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}\langle X\rangle$:

$$P \triangleleft (Q \triangleleft S) = PQ \triangleleft S, \qquad (S \triangleright P) \triangleright Q = S \triangleright PQ, \qquad (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \quad (2.46)$$

In case $x, y \in X$ and $w \in X^*$, we get:[11]

$$x \triangleleft (wy) = \delta_{x,y}w \quad \text{and} \quad xw \triangleright y = \delta_{x,y}w. \qquad (2.47)$$

Lemma 2.1 (Reconstruction lemma) *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$. Then*

$$S = \langle S \mid 1_{X^*}\rangle + \sum_{x\in X} x(S \triangleright x) = \langle S \mid 1_{X^*}\rangle + \sum_{x\in X}(x \triangleleft S)x.$$

Lemma 2.2 *The left* (*resp. right*) *residual by a letter x is a derivation of* $(\mathbb{Q}\langle\langle X\rangle\rangle, \sqcup\!\sqcup)$:

$$x \triangleleft (u \sqcup\!\sqcup v) = (x \triangleleft u) \sqcup\!\sqcup v + u \sqcup\!\sqcup (x \triangleleft v),$$
$$(u \sqcup\!\sqcup v) \triangleright x = (u \triangleright x) \sqcup\!\sqcup v + u \sqcup\!\sqcup (v \triangleright x).$$

[11] For any words u and $v \in X^*$, if $u = v$ then $\delta_{u,v} = 1$ else 0.

Proof Use the recursive definitions of the shuffle product. □

Consequently,

Lemma 2.3 *For any Lie polynomial $Q \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$, the linear maps "$Q\triangleleft$" and "$\triangleright Q$" are derivations on* $(\mathbb{Q}[\mathcal{L}ynX], ш)$.

Proof For any $l, l_1, l_2 \in \mathcal{L}ynX$, we have

$$\hat{l} \triangleleft (l_1 ш l_2) = l_1 ш (\hat{l} \triangleleft l_2) + (\hat{l} \triangleleft l_1) ш l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2,$$

$$(l_1 ш l_2) \triangleright \hat{l} = l_1 ш (l_2 \triangleright \hat{l}) + (l_1 \triangleright \hat{l}) ш l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2.$$ □

Lemma 2.4 *For any Lyndon word $l \in \mathcal{L}ynX$ and $\check{S}_l$ defined as in* (2.27), *one has*

$$x_1 \triangleleft l = l \triangleright x_0 = 0 \quad \textit{and} \quad x_1 \triangleleft \check{S}_l = \check{S}_l \triangleright x_0 = 0.$$

Proof Since $x_1\triangleleft$ and $\triangleright x_0$ are derivations and for any $l \in \mathcal{L}ynX - X$, the polynomial $\check{S}_l$ belongs to $x_0\mathbb{Q}\langle X\rangle x_1$ then the expected results follow. □

Theorem 2.6 (On representative series) *The following properties are equivalent for any series $S \in \mathbb{Q}\langle\langle X\rangle\rangle$:*

(1) *The left $\mathbb{C}$-module $Res_g(S) = span\{w \triangleleft S \mid w \in X^*\}$ is finite dimensional.*
(2) *The right $\mathbb{C}$-module $Res_d(S) = span\{S \triangleright w \mid w \in X^*\}$ is finite dimensional.*
(3) *There are matrices $\lambda \in \mathcal{M}_{1,n}(\mathbb{Q})$, $\eta \in \mathcal{M}_{n,1}(\mathbb{Q})$ and a representation of X^* in $\mathcal{M}_{n,n}$, such that*

$$S = \sum_{w \in X^*} [\lambda\mu(w)\eta]\, w = \lambda \left(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mu(S_l)\, \check{S}_l} \right) \eta.$$

A series that satisfies the items of this theorem will be called *representative series*. This concept can be found in [1, 16, 37]. The two first items are in [18, 21]. The third item can be deduced from [9, 15] for example and it was used to factorize first time, by Lyndon words, the output of bilinear and analytical dynamical systems, respectively, in [23, 32] and to study polylogarithms, hypergeometric functions and associated functions in [25, 28, 30]. The dimension of $Res_g(S)$ is equal to that of $Res_d(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 2.6. This rank is then equal to the rank of the Hankel matrix of S, that is, the infinite matrix $(\langle S \mid uv\rangle)_{u,v \in X}$ indexed by $X^* \times X^*$, and it is also called *Hankel rank* of S [18, 21]:

Definition 2.5 [18, 21] The *Hankel rank* of a formal power series $S \in \mathbb{C}\langle\langle X\rangle\rangle$ is the dimension of the vector space

$$\{S \triangleright \Pi \mid \Pi \in \mathbb{C}\langle X\rangle\}, \qquad (\text{resp. } \{\Pi \triangleleft S \mid \Pi \in \mathbb{C}\langle X\rangle\}).$$

The triplet (λ, μ, η) is called a *linear representation* of S. We define the minimal representation[12] of S as being a representation of S of minimal dimension.

[12] It can be shown that all minimal representations are isomorphic (see [2]).

For any proper series S, the following power series is called “star of S”:

$$S^* = 1 + S + S^2 + \cdots + S^n + \cdots. \tag{2.48}$$

Definition 2.6 [2, 46] A series S is called *rational* if it belongs to the closure in $\mathbb{Q}\langle\langle X\rangle\rangle$ of the noncommutative polynomial algebra by sum, product, and star operation of *proper*[13] elements. The set of rational power series will be denoted by $\mathbb{Q}^{\mathrm{rat}}\langle\langle X\rangle\rangle$.

Lemma 2.5 *For any noncommutative rational series* (*resp. polynomial*) R *and for any polynomial* P, *the left and right residuals of* R *by* P *are rational* (*resp. polynomial*).

Theorem 2.7 (Schützenberger [2, 46]) *Any noncommutative power series is representative if and only if it is rational.*

2.3.2 Continuity and indiscernibility

Definition 2.7 [22, 29] Let $\mathcal{H}$ be a class of formal power series over X and let $S \in \mathbb{C}\langle\langle X\rangle\rangle$.

(1) S is said to be *continuous*[14] over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the following sum, denoted by $\langle S \parallel \Phi\rangle$, is convergent in norm:

$$\sum_{w\in X^*} \langle S \mid w\rangle\langle \Phi \mid w\rangle.$$

The set of continuous power series over $\mathcal{H}$ will be denoted by $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$.

(2) S is said to be *indiscernible*[15] over $\mathcal{H}$ if and only if

$$\forall \Phi \in \mathcal{H}, \quad \langle S \parallel \Phi\rangle = 0.$$

Let ρ be the monoid morphism verifying $\rho(x_0) = x_1$ and $\rho(x_1) = x_0$ and let $\hat{w} = \rho(\tilde{w})$, where $\tilde{w}$ is the mirror of w.

Lemma 2.6 *Let* $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$. *If* $\langle S \parallel Z_{\sqcup\!\sqcup}\rangle = 0$ *then* $\langle \hat{S} \parallel Z_{\sqcup\!\sqcup}\rangle = 0$, *where*

$$\hat{S} := \sum_{w\in X^*} \langle S \mid w\rangle\, \hat{w}.$$

Proof For any $w \in x_0X^*x_1$, by “duality relation”, one has (see [34, 38, 49])

$$\zeta(\hat{w}) = \zeta(w), \quad \text{or equivalently} \quad Z_{\sqcup\!\sqcup} = \hat{Z}_{\sqcup\!\sqcup} := \sum_{w\in X^*} \langle Z_{\sqcup\!\sqcup} \mid w\rangle\, \hat{w}.$$

Using the fact

$$\langle \hat{S} \parallel Z_{\sqcup\!\sqcup}\rangle = \sum_{\hat{w}\in X^*} \langle S \mid \hat{w}\rangle\langle Z_{\sqcup\!\sqcup} \mid \hat{w}\rangle = \sum_{w\in X^*} \langle S \mid w\rangle\langle Z_{\sqcup\!\sqcup} \mid w\rangle,$$

one gets finally the expected result. □

[13] A series S is said to be proper if $\langle S \mid \epsilon\rangle = 0$.

[14] See [22, 29] for a convergence criterion and an example of continuous generating series.

[15] Here, we adapt this notion developed in [22] via the residual calculus.

Lemma 2.7 *Let $\mathcal{H}$ be a monoid containing $\{e^{tx}\}_{x\in X}^{t\in\mathbb{C}}$. Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ be indiscernible over $\mathcal{H}$. Then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ and they are indiscernible over $\mathcal{H}$.*

Proof Let us calculate $\langle x \triangleleft S \parallel \Phi\rangle = \langle S \parallel \Phi x\rangle$ and $\langle S \triangleright x \parallel \Phi\rangle = \langle S \parallel x\Phi\rangle$. Since

$$\lim_{t\to 0}\frac{e^{tx}-1}{t} = x \quad \text{and} \quad \lim_{t\to 0}\frac{e^{tx}-1}{t} = x$$

then, for any $\Phi \in \mathcal{H}$, by uniform convergence, one has

$$\langle S \parallel \Phi x\rangle = \left\langle S \,\middle\|\, \lim_{t\to 0}\Phi\frac{e^{tx}-1}{t}\right\rangle = \lim_{t\to 0}\left\langle S \,\middle\|\, \Phi\frac{e^{tx}-1}{t}\right\rangle,$$

$$\langle S \parallel x\Phi\rangle = \left\langle S \,\middle\|\, \lim_{t\to 0}\frac{e^{tx}-1}{t}\Phi\right\rangle = \lim_{t\to 0}\left\langle S \,\middle\|\, \frac{e^{tx}-1}{t}\Phi\right\rangle.$$

Since S is indiscernible over $\mathcal{H}$ then

$$\langle S \parallel \Phi x\rangle = \lim_{t\to 0}\frac{1}{t}\left\langle S \,\middle\|\, \Phi e^{tx}\right\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0,$$

$$\langle S \parallel x\Phi\rangle = \lim_{t\to 0}\frac{1}{t}\left\langle S \,\middle\|\, e^{tx}\Phi\right\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0. \qquad \square$$

Proposition 2.7 *Let $\mathcal{H}$ be a monoid containing $\{e^{tx}\}_{x\in X}^{t\in\mathbb{C}}$. The formal power series $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ is indiscernible over $\mathcal{H}$ if and only if $S = 0$.*

Proof If $S = 0$ then it is immediate that S is indiscernible over $\mathcal{H}$. Conversely, if S is indiscernible over $\mathcal{H}$ then by Lemma 2.7, for any word $w \in X^*$ and, by induction on the length of w, $w \triangleleft S$ is indiscernible over $\mathcal{H}$. In particular,

$$\langle w \triangleleft S \parallel \mathrm{Id}_{\mathcal{H}}\rangle = \langle S \mid w\rangle = 0.$$

In other words, $S = 0$. $\square$

3 Group of associators: polynomial relations among convergent polyzetas and identification of local coordinates

3.1 Generalized Euler constants and global regularization of polyzetas

3.1.1 Three regularizations of divergent polyzetas

Theorem 3.1 [36] *Let $\zeta_{⧢} : (\mathbb{Q}\langle Y\rangle, ⧢) \to (\mathbb{R}, .)$ be the morphism verifying the following properties:*

- *for $u, v \in Y^*$, $\zeta_{⧢}(u ⧢ v) = \zeta_{⧢}(u)\zeta_{⧢}(v)$,*
- *for all convergent word $w \in Y^* - y_1Y^*$, $\zeta_{⧢}(w) = \zeta(w)$,*
- *$\zeta_{⧢}(y_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{⧢}(w)\, w = Z_{⧢}.$$

Corollary 3.1 [36] *For any $w \in X^*$, $\zeta_{⧢}(w)$ belongs to the algebra $\mathcal{Z}$.*

Theorem 3.2 [36] *Let $\zeta_{⊔⊔} : (\mathbb{Q}\langle X\rangle, ⊔⊔) \to (\mathbb{R}, .)$ be the morphism verifying the following properties:*

- *for $u, v \in X^*$, $\zeta_{⊔⊔}(u ⊔⊔ v) = \zeta_{⊔⊔}(u)\zeta_{⊔⊔}(v)$,*
- *for all convergent word $w \in x_0X^*x_1$, $\zeta_{⊔⊔}(w) = \zeta(w)$,*
- *$\zeta_{⊔⊔}(x_0) = \zeta_{⊔⊔}(x_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{⊔⊔}(w)\, w = Z_{⊔⊔}.$$

Corollary 3.2 [36] *For any $w \in Y^*$, $\zeta_{⊔⊔}(w)$ belongs to the algebra $\mathcal{Z}$.*

Definition 3.1 For any $w \in Y^*$, let γ_w be the constant part[16] of the asymptotic expansion, on the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$, of $\mathrm{H}_w(n)$.

Let Z_γ be the noncommutative generating series of $\{\gamma_w\}_{w\in Y^*}$:

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w\, w.$$

Definition 3.2 We set

$$B(y_1) := \exp\left(-\sum_{k\geq 1} \gamma_{y_k} \frac{(-y_1)^k}{k}\right) \quad \text{and} \quad B'(y_1) := e^{-\gamma y_1} B(y_1).$$

The power series $B'(y_1)$ corresponds in fact to the mould[17] Mono in [17] and to the Φ_{corr} in [42] (see also [6, 8]). The power series $B(y_1)$ corresponds to the Gamma Euler function, with its product expansion,

$$B(y_1) = \Gamma(y_1 + 1), \qquad \frac{1}{\Gamma(y_1 + 1)} = e^{\gamma y_1} \prod_{n\geq 1} \left(1 + \frac{y_1}{n}\right) e^{-\gamma/n}. \tag{3.1}$$

Lemma 3.1 [29] *Let $b_{n,k}(t_1, \ldots, t_{n-k+1})$ be the (exponential) partial Bell polynomials in the variables $\{t_l\}_{l\geq 1}$ given by the exponential generating series*

$$\exp\left(u \sum_{l=0}^{\infty} t_l \frac{v^l}{l!}\right) = \sum_{n,k=0}^{\infty} b_{n,k}(t_1, \ldots, t_{n-k+1}) \frac{v^n u^k}{n!}.$$

[16] i.e. γ_w is the Euler–Mac Laurin constant of $\mathrm{H}_w(n)$.

[17] The readers can see why we have introduced the power series $\mathrm{Mono}(z)$ in Proposition 2.4.

For any $m \geq 1$, *let* $t_m = (-1)^m (m-1)! \gamma_{y_m}$. *Then*

$$B(y_1) = 1 + \sum_{n \geq 1} \left(\sum_{k=1}^{n} b_{n,k}\big(\gamma, -\zeta(2), 2\zeta(3), \ldots\big) \right) \frac{(-y_1)^n}{n!}.$$

Since the ordinary generating series of the finite parts of coefficients of Const(N) is nothing else but the power series $B(y_1)$, taking the constant part on either side of $\mathrm{H}(N) \underset{N \to \infty}{\widetilde{\quad}} \mathrm{Const}(N) \pi_Y Z_{\sqcup\!\sqcup}$ (see Proposition 2.4), we obtain

Theorem 3.3 [29] *We have* $Z_\gamma = B(y_1) \pi_Y Z_{\sqcup\!\sqcup}$.

Thus, identifying the coefficients of $y_1^k w$ on either side using the identity[18] (see [36])

$$\forall u \in X^* x_1, \quad x_1^k x_0 u = \sum_{l=0}^{k} x_1^l \sqcup\!\sqcup \big(x_0\big[(-x_1)^{k-l} \sqcup\!\sqcup u\big]\big). \tag{3.2}$$

Applying the morphism $\zeta_{\sqcup\!\sqcup}$ given in Theorem 3.2, we get [36]

$$\forall u \in X^* x_1, \quad \zeta_{\sqcup\!\sqcup}\big(x_1^k x_0 u\big) = \zeta\big(x_0\big[(-x_1)^k \sqcup\!\sqcup u\big]\big). \tag{3.3}$$

Corollary 3.3 [29] *For* $w \in x_0 X^* x_1$, *i.e.* $w = x_0 u$ *and* $\pi_Y w \in Y^* - y_1 Y^*$, *and for* $k \geq 0$, *the constant* $\gamma_{\sqcup\!\sqcup}(x_1^k w)$ *associated to the divergent polyzeta* $\zeta(x_1^k w)$ *is a polynomial of degree* k *in* γ *and with coefficients in* $\mathcal{Z}$:

$$\gamma_{x_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup u])}{i!} \left(\sum_{j=1}^{i} b_{i,j}\big(\gamma, -\zeta(2), 2\zeta(3), \ldots\big) \right).$$

Moreover, for $l = 0, \ldots, k$, *the coefficient of* γ^l *is of weight* $|w| + k - l$.

In particular, for $s > 1$, *the constant* $\gamma_{y_1 y_s}$ *associated to* $\zeta(y_1 y_s)$ *is linear in* γ *and with coefficients in* $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0 < i \leq (s-1)/2}$.

Corollary 3.4 [29] *The constant* $\gamma_{x_1^k}$ *associated to the divergent polyzeta* $\zeta(x_1^k)$ *is a polynomial of degree* k *in* γ *with coefficients in* $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0 < i \leq (k-1)/2}$:

$$\gamma_{x_1^k} = \sum_{\substack{s_1, \ldots, s_k \geq 0 \\ s_1 + \cdots + k s_k = k+1}} \frac{(-1)^k}{s_1! \cdots s_k!} (-\gamma)^{s_1} \left(-\frac{\zeta(2)}{2}\right)^{s_2} \cdots \left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Moreover, for $l = 0, \ldots, k$, *the coefficient of* γ^l *is of weight* $k - l$.

[18] By the Convolution Theorem [24], this is equivalent to

$$\forall u \in X^*, \quad \alpha_0^z\big(x_1^k x_0 u\big) = \int_0^z \frac{[\log(1-s) - \log(1-z)]^k}{k!} \alpha_0^s(u) \frac{ds}{s}$$
$$= \sum_{l=0}^{k} \frac{[-\log(1-z)]^l}{l!} \int_0^z \frac{\log^{k-l}(1-s)}{(k-l)!} \alpha_0^s(u) \frac{ds}{s}.$$

This theorem induces *de facto* the algebra morphism of regularization to 0 with respect to the shuffle product, as shown in Theorem 3.2.

We thereby obtain the following algebra morphism, denoted by $\gamma_\bullet$, for the regularization to γ with respect to the quasi-shuffle product *independently* to the regularization with respect to the shuffle product[19] and then by applying the tensor product of morphisms $\gamma_\bullet \otimes \mathrm{Id}$ on the diagonal series, over Y, we get (see Appendix 1)

Theorem 3.4 *The mapping $\gamma_\bullet$ realizes the morphism from $(\mathbb{Q}\langle Y\rangle, ⧢)$ to $(\mathbb{R}, .)$ verifying the following properties*:

- *for any words $u, v \in Y^*$, $\gamma_{u ⧢ v} = \gamma_u \gamma_v$,*
- *for any convergent word $w \in Y^* - y_1 Y^*$, $\gamma_w = \zeta(w)$,*
- *$\gamma_{y_1} = \gamma$.*

Then $Z_\gamma = e^{\gamma y_1} Z_{⧢}$.

3.1.2 Identities of noncommutative generating series of polyzetas

Corollary 3.5 *With the notations of Definition 3.2, we have*

$$\begin{aligned} Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup} \quad &\Longleftrightarrow \quad Z_{⧢} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}, \\ \pi_Y Z_{\sqcup\!\sqcup} = B^{-1}(x_1) Z_\gamma \quad &\Longleftrightarrow \quad Z_{\sqcup\!\sqcup} = B'^{-1}(x_1)\pi_X Z_{⧢}. \end{aligned}$$

Roughly speaking, for the quasi-shuffle product, the regularization to γ is "equivalent" to the regularization to 0.

Note also that the constant $\gamma_{y_1} = \gamma$ is obtained as the finite part of the asymptotic expansion of $\mathrm{H}_1(n)$ in the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

In the same way, since n and $\mathrm{H}_1(n)$ are algebraically independent, as arithmetical functions (see Proposition 2.1), then $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ constitutes a new comparison scale for asymptotic expansions.

Hence, the constants $\zeta_{\sqcup\!\sqcup}(x_1) = 0$ and $\zeta_{⧢}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of $\mathrm{Li}_1(z)$ and $\mathrm{H}_1(n)$, respectively, in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

Definition 3.3 [36] Let $C_1 := \mathbb{Q}1_{X^*} \oplus x_0 \mathbb{Q}\langle X\rangle x_1$, $C_2 := \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle$.

Lemma 3.2 [35, 36] *We get* $(C_1, \sqcup\!\sqcup) \cong (C_2, ⧢)$.

Using a theorem of Radford [44] and its analogue over Y (see Appendix 1), we get

Proposition 3.1 [35, 36]

$$\begin{aligned} \big(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup\big) \cong \big(\mathbb{Q}[\mathcal{L}ynX], \sqcup\!\sqcup\big) &= \quad C_1[x_0, x_1], \\ \big(\mathbb{Q}\langle Y\rangle, ⧢\big) \cong \quad \big(\mathbb{Q}[\mathcal{L}ynY], ⧢\big) &= \quad C_2[y_1]. \end{aligned}$$

This ensures the effective way to get the finite part of the asymptotic expansions, in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$, of $\{\mathrm{Li}_w(z)\}_{w\in Y^*}$ and $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ respectively.

[19] In [6, 8, 40, 47], the authors suggest the *simultaneous* regularizations, with respect to the shuffle product and the quasi-shuffle product, to the indeterminate T and then to set $T = 0$.

Proposition 3.2 [35, 36] *The restrictions of* $\zeta_{\sqcup\!\sqcup}$ *and* $\zeta_{\text{⧢}}$ *over* $(C_1, \sqcup\!\sqcup)$ *and* $(C_2, \text{⧢})$, *respectively, coincide with the following surjective algebra morphism*:

$$\zeta : \begin{matrix}(C_2, \text{⧢})\\(C_1, \sqcup\!\sqcup)\end{matrix} \longrightarrow (\mathbb{R}, .)$$
$$\begin{matrix} y_{r_1}\cdots y_{r_k} \\ x_0 x_1^{r_1-1}\cdots x_0 x_1^{r_k-1}\end{matrix} \longmapsto \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{r_1}\cdots n_k^{r_k}},$$

In Sect. 3.3 we will give the complete description of the kernel $\ker\zeta$.

With the double regularization[20] to zero [6, 8, 36, 42], the Drinfel'd associator Φ_{KZ} corresponds then to $Z_{\sqcup\!\sqcup}$ (obtained with only convergent polyzetas) as being the unique group-like element satisfying [33, 35]

$$\langle Z_{\sqcup\!\sqcup} \mid x_0\rangle = \langle Z_{\sqcup\!\sqcup} \mid x_1\rangle = 0 \quad \text{and} \quad \forall x \in x_0 X^* x_1, \quad \langle Z_{\sqcup\!\sqcup} \mid w\rangle = \zeta(w). \tag{3.4}$$

As a consequence of Proposition 2.2, one has

Proposition 3.3 [28]

$$\begin{aligned}\log Z_{\sqcup\!\sqcup} &= \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)\,\pi_1(w)\\ &= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in X^+} \zeta_{\sqcup\!\sqcup}(u_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k)\, u_1\cdots u_k.\end{aligned}$$

The associator Φ_{KZ} can be also graded in the adjoint basis of $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ as follows:

Proposition 3.4 [28] *For any* $l \in \mathbb{N}$ *and* $P \in \mathbb{C}\langle X\rangle$, *let* $\circ$ *denote the composite operation defined by* $x_1 x_0^l \circ P = x_1(x_0^l \sqcup\!\sqcup P)$. *Then*

$$Z_{\sqcup\!\sqcup} = \sum_{k\geq 0} \sum_{l_1,\ldots,l_k\geq 0} \zeta_{\sqcup\!\sqcup}\bigl(x_1 x_0^{l_1} \circ \cdots \circ x_1 x_0^{l_k}\bigr) \prod_{i=0}^{k} \mathrm{ad}_{x_0}^{l_i} x_1,$$

where $\mathrm{ad}_{x_0}^l x_1 = [x_0, \mathrm{ad}_{x_0}^{l-1} x_1]$ *is the iterated Lie bracket and* $\mathrm{ad}_{x_0}^0 x_1 = x_1$.

Using the following expansion [4]:

$$\mathrm{ad}_{x_0}^n x_1 = \sum_{i=0}^{n} \binom{i}{n} x_0^{n-i} x_1 x_0^i, \tag{3.5}$$

one deduces then, via the regularization process of Theorem 3.2, the expression of the Drinfel'd associator Φ_{KZ} given by Lê and Murakami [41].

[20]This double regularization is deduced from of the noncommutative generating series $Z_{\sqcup\!\sqcup}$ and $Z_{\text{⧢}}$ in Definitions 2.1 and 2.2 (see Theorems 3.1 and 3.2).

3.2 Action of differential Galois group of polylogarithms on their asymptotic expansions

3.2.1 *Group of associators theorem*

Let A be a commutative $\mathbb{Q}$-algebra.

Since the polyzetas satisfy (2.25), then by the Friedrichs criterion we can state the following.

Definition 3.4 Let $dm(A)$ be the set of $\Phi \in A\langle\langle X\rangle\rangle$ such that[21]

$$\langle \Phi \mid 1_{X^*}\rangle = 1, \qquad \langle \Phi \mid x_0\rangle = \langle \Phi \mid x_1\rangle = 0, \qquad \Delta_{⧢}\Phi = \Phi \otimes \Phi$$

and such that, for

$$\Psi = B'(y_1)\pi_Y \Phi \in A\langle\langle Y\rangle\rangle$$

then[22] $\Delta_{\uplus}\Psi = \Psi \otimes \Psi$.

Proposition 3.5 [28] *If $G(z)$ and $H(z)$ are exponential solutions of* (DE) *then there exists a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ such that $G(z) = H(z)\exp(C)$.*

Proof Since $H(z)H(z)^{-1} = 1$ then by differentiating, we have

$$d\big[H(z)\big]H(z)^{-1} = -H(z)d\big[H(z)^{-1}\big].$$

Therefore if $H(z)$ is a solution of Drinfel'd equation then

$$\begin{aligned} d\big[H(z)^{-1}\big] &= -H(z)^{-1}\big[dH(z)\big]H(z)^{-1} \\ &= -H(z)^{-1}\big[x_0\omega_0(z) + x_1\omega_1(z)\big], \\ d\big[H(z)^{-1}G(z)\big] &= H(z)^{-1}\big[dG(z)\big] + \big[dH(z)^{-1}\big]G(z) \\ &= H(z)^{-1}\big[x_0\omega_0(z) + x_1\omega_1(z)\big]G(z) \\ &\quad - H(z)^{-1}\big[x_0\omega_0(z) + x_1\omega_1(z)\big]G(z). \end{aligned}$$

By simplification, we deduce then $H(z)^{-1}G(z)$ is a constant formal power series. Since the inverse and the product of group like elements is group like then we get the expected result. □

The differential $\mathcal{C}$-module $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is the universal Picard–Vessiot extension of every linear differential equations, with coefficients in $\mathcal{C}$ and admitting $\{0, 1, \infty\}$ as regular singularities. The universal differential Galois group, denoted by $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$, is the set of differential $\mathcal{C}$-automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (i.e. the automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ that let $\mathcal{C}$ be point-wise fixed and that commute with derivation). The action of an automorphism of $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ can be determined by its action on Li_w, for $w \in X^*$. It can be resumed as its action on the noncommutative generating series L [28]:

[21] $\Delta_{⧢}$ denotes the co-product of the shuffle product.

[22] $\Delta_{\uplus}$ denotes the co-product of the quasi-shuffle product.

Let $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$. Then

$$\sum_{w \in X^*} \sigma \operatorname{Li}_w \; w = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\sigma \operatorname{Li}_{\check{S}_l} S_l}. \tag{3.6}$$

Since $d\sigma \operatorname{Li}_{x_i} = \sigma d \operatorname{Li}_{x_i} = \omega_i$ then by integrating the two members, we obtain $\sigma \operatorname{Li}_{x_i} = \operatorname{Li}_{x_i} + c_{x_i}$, where c_{x_i} is a constant of integration. More generally, for any Lyndon word $l = x_i l_1^{i_1} \cdots l_k^{i_k}$ with $l_1 > \cdots > l_k$, one has

$$\sigma \operatorname{Li}_{\check{S}_l} = \int \omega_{x_i} \frac{\sigma \operatorname{Li}_{\check{S}_{l_1}}^{i_1}}{i_1!} \cdots \frac{\sigma \operatorname{Li}_{\check{S}_{l_k}}^{i_k}}{i_k!} + c_{\check{S}_l}, \tag{3.7}$$

where $c_{\check{S}_l}$ is a constant of integration. For example,

$$\sigma \operatorname{Li}_{x_0 x_1} = \operatorname{Li}_{x_0 x_1} + c_{x_1} \operatorname{Li}_{x_0} + c_{x_0 x_1}, \tag{3.8}$$

$$\sigma \operatorname{Li}_{x_0^2 x_1} = \operatorname{Li}_{x_0^2 x_1} + \frac{c_{x_1}}{2} \operatorname{Li}_{x_0}^2 + c_{x_0 x_1} \operatorname{Li}_{x_0} + c_{x_0^2 x_1}, \tag{3.9}$$

$$\sigma \operatorname{Li}_{x_0 x_1^2} = \operatorname{Li}_{x_0 x_1^2} + c_{x_1} \operatorname{Li}_{x_0 x_1} + \frac{c_{x_1}^2}{2} \operatorname{Li}_{x_0} + c_{x_0 x_1^2}. \tag{3.10}$$

Consequently,

$$\sum_{w \in X^*} \sigma \operatorname{Li}_w \; w = \mathrm{L} e^{C_\sigma} \quad \text{where } e^{C_\sigma} := \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{c_{\check{S}_l} S_l}. \tag{3.11}$$

The action of $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ over $\{\operatorname{Li}_w\}_{w \in X^*}$ is then equivalent to the action of the Lie exponential $e^{C_\sigma} \in \mathrm{Gal}(\mathrm{DE})$ over the exponential solution L. So,

Theorem 3.5 [28] *We have* $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}}) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle\}$.

Typically, since $\mathrm{L}(z_0)^{-1}$ is group-like then $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ is another solution of (2.22) as already seen in (2.41).

Theorem 3.6 (Group of associators theorem) *Let $\Phi \in A\langle\langle X \rangle\rangle$ and $\Psi \in A\langle\langle Y \rangle\rangle$ be group-like elements, for the co-products $\Delta_{\sqcup\!\sqcup}$, $\Delta_{\uplus}$, respectively, such that*

$$\Psi = B(y_1)\pi_Y \Phi.$$

There exists a unique $C \in \mathcal{L}ie_A\langle\langle X \rangle\rangle$ such that $\Phi = Z_{\sqcup\!\sqcup} e^C$ and $\Psi = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup} e^C)$ and $\Psi' = B'(y_1)\pi_Y \Phi$.

Proof If $C \in \mathcal{L}ie_A\langle\langle X \rangle\rangle$ then $\mathrm{L}' = \mathrm{L}e^C$ is group-like, for the co-product $\Delta_{\sqcup\!\sqcup}$ and $e^C \in \mathrm{Gal}(\mathrm{DE})$. Let H' be the noncommutative generating series of the Taylor coefficients, belonging to the harmonic algebra, of $\{(1-z)^{-1}\langle \mathrm{L}' \mid w \rangle\}_{w \in Y^*}$. Then $\mathrm{H}'(N)$ is also group-like, for the co-product $\Delta_{\uplus}$. By the asymptotic expansion of L, we have [34, 35]

$$\mathrm{L}'(z) \underset{\varepsilon \to 1}{\widetilde{\quad}} e^{-x_1 \log(1-z)} Z_{\sqcup\!\sqcup} e^C.$$

We put then $\Phi := Z_{\sqcup\sqcup}e^C$ and we deduce that

$$\frac{\mathrm{L}'(z)}{1-z} \underset{z\to 1}{\widetilde{\quad}} \mathrm{Mono}(z)\Phi \quad \text{and} \quad \mathrm{H}'(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N)\pi_Y\Phi,$$

where the expressions of $\mathrm{Mono}(z)$ and $\mathrm{Const}(N)$ are given in (2.37) and (2.38), respectively. Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then

$$\sum_{w\in Y^*} \kappa_w\, w = B(y_1)\pi_Y\Phi.$$

We put then $\Psi := B(y_1)\pi_Y\Phi$ and $\Psi' := B'(y_1)\pi_Y\Phi$. □

Corollary 3.6 *If the commutative $\mathbb{Q}$-algebra A contains $\mathcal{Z}$ then $dm(A)$ is a group and*

$$dm(A) = \left\{Z_{\sqcup\sqcup}e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle \text{ and } \left\langle e^C \mid 1_{X^*}\right\rangle = 1, \left\langle e^C \mid x_0\right\rangle = \left\langle e^C \mid x_1\right\rangle = 0\right\}.$$

Proof On one hand, $\langle\Phi \mid x_0\rangle = \langle Z_{\sqcup\sqcup} \mid x_0\rangle = 0$, $\langle\Phi \mid x_1\rangle = \langle Z_{\sqcup\sqcup} \mid x_1\rangle = 0$ and on the other hand, $\langle\Phi \mid 1_{X^*}\rangle = \langle Z_{\sqcup\sqcup} \mid 1_{X^*}\rangle = 1$, the result follows. □

Corollary 3.7 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\sqcup}e^C$. Then, with the notations of Definition 3.2, we get*

$$\Psi = B(y_1)\pi_Y\Phi \quad \Longleftrightarrow \quad \Psi' = B'(y_1)\pi_Y\Phi.$$

Proof Since Ψ is group-like, for $\Delta_{⧢}$, and since $\langle\Phi \mid x_1\rangle = \langle\Psi' \mid y_1\rangle = 0$ and $\langle\Psi \mid y_1\rangle = \gamma$ then, using the factorization by Lyndon words, we get the expected result. □

Lemma 3.3 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\sqcup}e^C$ and $\Psi = B(y_1)\pi_Y\Phi$. The local coordinates (of second kind) of Φ (resp. Ψ) are polynomials on $\{\zeta_{\sqcup\sqcup}(\check{S}_l)\}_{l\in\mathcal{L}ynX}$ (resp. $\{\zeta_{⧢}(\check{\Sigma}_l)\}_{l\in\mathcal{L}ynY}$) of $\mathcal{Z}$ (resp. $\mathcal{Z}'$). While C describes $\mathcal{L}ie_A\langle\langle X\rangle\rangle$, these coordinates describe $A[\{\zeta_{\sqcup\sqcup}(\check{S}_l)\}_{l\in\mathcal{L}ynX}]$ (resp. $A[\{\zeta_{⧢}(\check{\Sigma}_l)\}_{l\in\mathcal{L}ynY}]$).*

Proof Using the factorization forms by Lyndon words, we get

$$\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\phi(\check{S}_l)\,S_l} = \left(\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(\check{S}_l)\,S_l}\right)\left(\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{p_{\check{S}_l}\,S_l}\right).$$

Expanding the Hausdorff product and identifying the local coordinates in the PBW-Lyndon basis there exist $I_l \subset \{\lambda \in \mathcal{L}ynX - X \text{ s.t. } |\lambda| \le |l|\}$, for $l \in \mathcal{L}ynX - X$, and the coefficients $\{c'_{\check{S}_u}\}_{u\in I_l}$ belonging to A such that

$$\phi(\check{S}_l) = \sum_{u\in I_l} c'_{\check{S}_u}\,\zeta(\check{S}_u).$$

This belongs to $A[\{\zeta(\check{S}_l)\}_{l\in\mathcal{L}ynX-X}]$ and holds for any $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$. □

With the notations of Definition 3.2 and by Corollary 3.7, we get in particular

Lemma 3.4 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\sqcup}e^C$. By identifying the local coordinates (of second kind) on two members of the identities $\Psi = B(y_1)\pi_Y\Phi$, or equivalently of $\Psi' = B'(y_1)\pi_Y\Phi$, we get polynomial relations, of coefficients in A, among generators of the A-algebra of convergent polyzetas.*

Therefore,

Theorem 3.7 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\sqcup}e^C$. The identities $\Psi = B(y_1)\pi_Y\Phi$ describe the ideal of polynomial relations, of coefficients in A, among generators of the A-algebra of convergent polyzetas. Moreover, if the Euler constant, γ, does not belong to A then these relations are algebraically independent of γ.*

Simplified computations in Sect. 3.3 are examples of such identities. Some consequences of Theorem 3.7 will be drawn in Sect. 4.2.

3.2.2 Concatenation of Chen generating series

As an example of the action of the differential Galois group of polylogarithms on their asymptotic expansions, we are interested in the action of their monodromy group which is contained in Gal(DE).

The monodromies at 0 and 1 of L are given, respectively, by [33, 35]

$$\mathcal{M}_0\mathrm{L} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_0} \quad \text{and} \quad \mathcal{M}_1\mathrm{L} = \mathrm{L}Z_{\sqcup\sqcup}^{-1}e^{-2\mathrm{i}\pi x_1}Z_{\sqcup\sqcup} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_1}, \tag{3.12}$$

$$\text{where} \quad \mathfrak{m}_0 = x_0 \quad \text{and} \quad \mathfrak{m}_1 = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{-\zeta(\check{S}_l)\,\mathrm{ad}_{S_l}}(-x_1). \tag{3.13}$$

- If $C = 2\mathrm{i}\pi\mathfrak{m}_0$ then

$$\Phi = Z_{\sqcup\sqcup}e^{2\mathrm{i}\pi x_0}, \tag{3.14}$$

$$\begin{aligned}\Psi &= \exp\Big(\gamma y_1 - \sum_{k\ge2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{\sqcup\sqcup}\\ &= Z_\gamma.\end{aligned} \tag{3.15}$$

  The monodromy at 0 consists in the multiplication on the right of $Z_{\sqcup\sqcup}$ by $e^{2\mathrm{i}\pi x_0}$ and does not modify $Z_{\text{⧢}}$.
- If $C = 2\mathrm{i}\pi\mathfrak{m}_1$ then

$$\Phi = e^{-2\mathrm{i}\pi x_1}Z_{\sqcup\sqcup}, \tag{3.16}$$

$$\begin{aligned}\Psi &= \exp\Big(\underbrace{(\gamma - 2\mathrm{i}\pi)}_{T:=}y_1 - \sum_{k\ge2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{\sqcup\sqcup}\\ &= e^{-2\mathrm{i}\pi y_1}Z_\gamma.\end{aligned} \tag{3.17}$$

  The monodromy at 1 consists in the multiplication on the left of $Z_{\sqcup\sqcup}$ and of Z_γ by $e^{-2\mathrm{i}\pi x_1}$ and $e^{-2\mathrm{i}\pi y_1}$, respectively.

Remark 3.1

(1) The monodromies around singularities of L could not allow, in this case, neither to introduce the factor $e^{\gamma x_1}$ on the left of $Z_{\sqcup\!\sqcup}$ nor to eliminate the left factor $e^{\gamma y_1}$ in Z_γ (by putting[23] $T = 0$, for example).

(2) By Proposition 2.5, we already saw that $Z_{\sqcup\!\sqcup}$ regularizes the concatenation of Chen generating series [10] $e^{x_0 \log \varepsilon}$ and then $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1 \log \varepsilon}$:

$$Z_{\sqcup\!\sqcup} \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{x_1 \log \varepsilon}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 \log \varepsilon}. \tag{3.18}$$

From (3.14) and (3.16), the action of the monodromy group gives

$$e^{x_1\, 2k_1 \mathrm{i}\pi} Z_{\sqcup\!\sqcup} e^{x_0\, 2k_0 \mathrm{i}\pi} \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{x_1 (\log \varepsilon + 2k_1 \mathrm{i}\pi)}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 (\log \varepsilon + 2k_0 \mathrm{i}\pi)}, \tag{3.19}$$

regularizing the concatenation of the Chen generating series $e^{x_0 (\log \varepsilon + 2k_0 \mathrm{i}\pi)}$, then the Chen generating series $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, the Chen generating series $e^{x_1 (\log \varepsilon + 2k_1 \mathrm{i}\pi)}$.

(3) More generally, by Corollary 3.6, the action of the Galois differential group of polylogarithms states, for any Lie series C, the associator $\Phi = Z_{\sqcup\!\sqcup} e^C$ regularizes the concatenation of some Chen generating series e^C and $e^{x_0 \log \varepsilon}$ and then the Chen generating series $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1 \log \varepsilon}$:

$$\Phi \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{x_1 \log \varepsilon}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 \log \varepsilon}\, e^C. \tag{3.20}$$

By construction (see Theorem 3.6) the associator Φ is then the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $S_{z_0 \rightsquigarrow 1-z_0} e^C$, for $z_0 = \varepsilon \to 0^+$. Hence,

Corollary 3.8 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\!\sqcup} e^C$. For any differential produced formal power series S over X, there exists*[24] *a* differential representation $(\mathcal{A}, f)$ *such that*

$$\langle \Phi \parallel S\rangle = \sum_{w \in X^*} \langle \Phi \mid w\rangle\, \mathcal{A}(w) \circ f_{|0} = \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{\langle \Phi \mid \check{S}_l\rangle\, \mathcal{A}(S_l)} \circ f_{|0}.$$

3.3 Algebraic combinatorial studies of polynomial relation among polyzêta via a group of associators

With the factorization of the monoids X^* and Y^* by Lyndon words, let $\{\hat{l}\}_{l \in \mathcal{L}ynX}$ and $\{\hat{l}\}_{l \in \mathcal{L}ynY}$ be the dual of the Lyndon basis over X and Y, respectively.

3.3.1 Preliminary study

As in Definition 3.3, let

$$A_1 = A1_{X^*} \oplus x_0 A\langle X\rangle x_1 \quad \text{and} \quad A_2 = A1_{Y^*} \oplus \big(Y - \{y_1\}\big) A\langle Y\rangle. \tag{3.21}$$

[23] Why?

[24] See Corollary 6.6 of Appendix 2.

For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{⧢}e^C$ and $\Psi = B'(y_1)\pi_Y\Phi$. Let us introduce two algebra morphisms

$$\begin{aligned} \phi : (A_1, ⧢) &\longrightarrow A, \\ u &\longmapsto \langle \Phi \mid u\rangle, \end{aligned} \qquad \begin{aligned} \psi : (A_2, \text{⧢}) &\longrightarrow A, \\ v &\longmapsto \langle \Psi \mid v\rangle, \end{aligned} \tag{3.22}$$

verifying, respectively, $\phi(1_{X^*}) = 1, \phi(x_0) = \phi(x_1) = 0$ and $\psi(1_{Y^*}) = 1, \psi(y_1) = 0$.

Lemma 3.5 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{⧢}e^C$ and $\Psi = B'(y_1)\pi_Y\Phi$. Then*

$$\forall w \in Y^* - y_1Y^*, \quad \psi(w) = \phi(\pi_X w),$$

$$\textit{or equivalently}, \quad \forall w \in x_0X^*x_1, \quad \phi(w) = \psi(\pi_{\bar{Y}} w).$$

Lemma 3.6 *We have*

$$\Phi = \sum_{u\in X^*} \phi(u)\, u = \prod_{l\in \mathcal{L}ynX - X}^{\searrow} e^{\phi(l)\,\hat{l}} \quad \textit{and} \quad \Psi = \sum_{v\in Y^*} \psi(u)\, u = \prod_{l\in \mathcal{L}ynY - \{y_1\}}^{\searrow} e^{\psi(l)\,\hat{l}}.$$

With the notations in Lemma 3.6, we can state the following.

Definition 3.5 We put

$$\mathcal{R} := \bigcap_{\Phi\in dm(A)} \ker\phi \qquad \Bigg(\text{resp.} \bigcap_{\substack{\Psi = B'(y_1)\pi_{\bar{Y}}\Phi \\ \Phi\in dm(A)}} \ker\psi\Bigg).$$

Lemma 3.7 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{⧢}e^C$. Let $Q \in \mathbb{Q}[\mathcal{L}ynX]$ (resp. $\mathbb{Q}[\mathcal{L}yn\bar{Y}]$). Then*

$$\langle Q \parallel \Phi\rangle = 0 \iff Q \in \ker\phi \qquad \big(\textit{resp.}\ \langle Q \parallel \Psi\rangle = 0 \iff Q \in \ker\psi\big).$$

Or equivalently (see Definition 2.7),

$$Q \in \mathcal{R} \iff Q \textit{ is indiscernible over } dm(A).$$

By Corollary 3.6, for $i = 1$ or 2, there exists $P_i \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that e^{-P_i} is well defined and let

$$\Phi_i = Z_{⧢}e^{P_i}, \quad \text{or equivalently,} \quad Z_{⧢} = \Phi_1 e^{-P_1} = \Phi_2 e^{-P_2}. \tag{3.23}$$

Then, we get

$$\Phi_1 = \Phi_2 e^{P_1 - P_2} \quad \text{and} \quad \Phi_2 = \Phi_1 e^{P_2 - P_1}. \tag{3.24}$$

By Lemma 3.3, we have

Lemma 3.8 *For any convergent Lyndon word, l, there exist a finite set $I_l \subset \{\lambda \in \mathcal{L}ynX - X$ s.t. $|\lambda| \leq |l|\}$ and the coefficients $\{p'_{i,u}\}_{u\in I_l}$ and $\{p''_{i,u}\}_{u\in I_l}$, for $i = 1$ or 2, belonging to A such that*

$$\phi_i(l) = \sum_{u\in I_l} p'_{i,u}\,\zeta(u), \quad \textit{or equivalently,} \quad \zeta(l) = \sum_{u\in I_l} p''_{i,u}\,\phi_i(u).$$

There also exist coefficients $\{p'_u\}_{u\in I_l}$ and $\{p''_u\}_{u\in I_l}$ belonging to A such that

$$\phi_1(l) = \sum_{u\in I_l} p'_u\,\phi_2(u), \quad \textit{or equivalently,} \quad \phi_2(l) = \sum_{u\in I_l} p''_u\,\phi_1(u).$$

Therefore, the $\{\phi_i(l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\psi_i(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$), for $i = 1$ or 2, are also generators of the A-algebra generated by convergent polyzetas.

3.3.2 Description of polynomial relations among coefficients of associator and irreducible polyzetas

Since the identities of Corollary 3.7 (see also Corollary 3.5) hold for any pair of bases in duality then, by Corollary 3.7, one gets

Theorem 3.8 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\!\sqcup}e^C$. We have*

$$\prod_{l\in\mathcal{L}ynY-y_1}^{\searrow} e^{\psi(l)\,\hat{l}} = \exp\left(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right)\pi_Y \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\phi(l)\,\hat{l}}.$$

If $\Phi = Z_{\sqcup\!\sqcup}$ and $\Psi = Z_{\text{ⱶ}}$ then, for $\ell \in \mathcal{L}ynX - X$ (resp. $\mathcal{L}ynY - y_1$), one has $\zeta(l) = \phi(l)$ (resp. $\psi(l)$). Hence, one obtains (see also Corollary 3.5)

Theorem 3.9 (Bis repetita)

$$\prod_{l\in\mathcal{L}ynY-y_1}^{\searrow} e^{\zeta(l)\,\hat{l}} = \exp\left(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right)\pi_Y \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(l)\,\hat{l}}.$$

Corollary 3.9 *For any $\ell \in \mathcal{L}ynY - y_1$ (resp. $\mathcal{L}ynX - X$), let $P_\ell \in \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ (resp. $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$) be the decomposition of the polynomial $\pi_X\hat{\ell} \in \mathbb{Q}\langle X\rangle$ (resp. $\pi_Y\hat{\ell} \in \mathbb{Q}\langle Y\rangle$) in the PBW basis, induced by $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}ynY}$), and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - X]$ (resp. $\mathbb{Q}[\mathcal{L}ynY - y_1]$) be its dual. Then one obtains*

$$\pi_X\ell - \check{P}_\ell \in \ker\phi \qquad (\textit{resp. } \pi_Y\ell - \check{P}_\ell \in \ker\psi).$$

In particular, for $\phi = \zeta$ (resp. $\psi = \zeta$) one also obtains

$$\pi_X\ell - \check{P}_\ell \in \ker\zeta \qquad (\textit{resp. } \pi_Y\ell - \check{P}_\ell \in \ker\zeta).$$

Moreover, for any $\ell \in \mathcal{L}ynY - y_1$ (resp. $\mathcal{L}ynX - X$), the homogeneous polynomial $\pi_X\ell - \check{P}_\ell \in \mathbb{Q}\langle X\rangle$ (resp. $\mathbb{Q}\langle Y\rangle$) is of degree $|\,\ell\,| \geq 2$.

Proof Since

$$\ell \in \mathcal{L}ynY \quad \Longleftrightarrow \quad \pi_X \ell \in \mathcal{L}ynX - \{x_0\}$$

then identifying the local coordinates (of second kind) on the two members of each identity in Theorem 3.8, one obtains

$$\forall \ell \in \mathcal{L}ynY - y_1 \subset Y^* - y_1 Y^*, \quad \psi(\ell) = \phi(\check{P}_\ell),$$
$$\big(\text{resp. } \forall \ell \in \mathcal{L}ynX - X \subset x_0 X^* x_1, \phi(\ell) = \psi(\check{P}_\ell)\big).$$

By Lemma 3.5, we get the expected result. □

With the notations of Corollary 3.9, we get the following.

Definition 3.6 Let Q_ℓ be the decomposition of the proper polynomial $\pi_Y \ell - \check{P}_\ell$ (resp. $\pi_X \ell - \check{P}_\ell$) in $\mathcal{L}ynY$ (resp. $\mathcal{L}ynX$). Let

$$\mathcal{R}_Y := \{Q_\ell\}_{\ell \in \mathcal{L}ynY - y_1} \quad \text{and} \quad \mathcal{R}_X := \{Q_\ell\}_{\ell \in \mathcal{L}ynX - X},$$
$$\mathcal{L}_{\text{irr}} Y := \{\ell \in \mathcal{L}ynY - y_1 \mid Q_\ell = 0\} \quad \text{and} \quad \mathcal{L}_{\text{irr}} X := \{\ell \in \mathcal{L}ynX - X \mid Q_\ell = 0\}.$$

It follows that

Lemma 3.9 *We have*

$$\big(\mathbb{Q}[\mathcal{L}ynY - y_1], ⧢\big) = (\mathcal{R}_Y, ⧢) \oplus \big(\mathbb{Q}[\mathcal{L}_{\text{irr}} Y], ⧢\big),$$
$$\big(\mathbb{Q}[\mathcal{L}ynX - X], ⧢\big) = (\mathcal{R}_X, ⧢) \oplus \big(\mathbb{Q}[\mathcal{L}_{\text{irr}} X], ⧢\big).$$

Then we can state the following.

Definition 3.7 Any word w is said to be *irreducible* if and only if w belongs to $\mathcal{L}_{\text{irr}} Y$ (resp. $\mathcal{L}_{\text{irr}} X$). In this case, the polyzeta $\zeta(w)$ is said to be A*-irreducible*.

For any $P \in \mathbb{Q}[\mathcal{L}_{\text{irr}} X]$, there exists[25] a *differential representation* $(\mathcal{A}, f)$ such that P can be *finitely* factorized (see also Corollary 3.8):

$$P = \sigma f_{|0} = \sum_{w \in X^*_{\text{irr}}} \mathcal{A}(w) \circ f \; w = \prod^{\searrow}_{\ell \in \mathcal{L}_{\text{irr}} X, \text{finite}} e^{\mathcal{A}(\hat{\ell})\, \ell} \circ f, \tag{3.25}$$

where X^*_{irr} denotes the set of words obtained by shuffling on $\mathcal{L}_{\text{irr}} X$.

Lemma 3.10 *Any proper polynomial* $P \in (\mathbb{Q}[\mathcal{L}_{\text{irr}} X], ⧢)$ (*resp.* $(\mathbb{Q}[\mathcal{L}_{\text{irr}} Y], ⧢)$) *is indiscernible over the Chen generating series* $\{e^{tx}\}^{t \in \mathbb{C}}_{x \in X}$:

$$\langle P \parallel e^{tx_0} \rangle = \langle P \parallel e^{tx_1} \rangle = 0 \qquad \big(\textit{resp.}\ \langle P \parallel e^{ty_1} \rangle = 0\big).$$

[25] See Corollary 6.6 of Appendix 2.

Proof By construction, x_0 and $x_1 \notin \mathcal{L}_{\mathrm{irr}}X$ (resp. $y_1 \notin \mathcal{L}_{\mathrm{irr}}X$). For any $n > 1$, x_0^n and x_1^n (resp. y_1^n) are not Lyndon words then they do not belong to $\mathcal{L}_{\mathrm{irr}}X$ (resp. $\mathcal{L}_{\mathrm{irr}}X$). Therefore, for any $n \geq 0$, one has

$$\langle P \mid x_0^n \rangle = \langle P \mid x_1^n \rangle = 0 \qquad \big(\text{resp. } \langle P \mid y_1^n \rangle = 0\big).$$

Using the expansion of the exponential, we find the expected result. □

Lemma 3.11 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\!\sqcup} e^C$ and let $t \in \mathbb{C}$, $x \in X$. For any proper polynomial $P \in (\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X], \sqcup\!\sqcup)$, if $\langle P \parallel \Phi\rangle = 0$ then*

$$\langle P \parallel \Phi e^{tx} \rangle = 0 \quad \textit{and} \quad \langle P \parallel e^{tx}\Phi \rangle = 0.$$

Proof Since $P \in (\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X], \sqcup\!\sqcup)$ and P is proper, then, by Lemma 3.10, for any $t \in \mathbb{C}$ and for any $x \in X$, we have $\langle P \parallel e^{tx}\rangle = 0$ and then $\langle P \parallel \Phi e^{tx}\rangle = 0$.

Since $\mathrm{supp}(P) \subset x_0 X^* x_1$ then $\langle P \parallel e^{tx_0}\Phi\rangle = \langle P \triangleright e^{tx_0} \parallel \Phi\rangle = 0$.

Next, since $e^{tx_1}\Phi = e^{tx_1} Z_{\sqcup\!\sqcup} e^C$ and, by Proposition 2.5, we get

$$e^{tx_1}\Phi \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{x_1(t+\log\varepsilon)}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 \log\varepsilon}\, e^C.$$

Hence, there exist a Chen generating series $C_{z \rightsquigarrow 1-z_0}$ and $S_{z_0 \rightsquigarrow 1-z_0}$ such that we get the following asymptotic behavior (see Sect. 3.2.2):

$$e^{tx_1}\Phi \underset{\varepsilon \to 0^+}{\widetilde{\quad}} C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z}\, e^C$$

and the following concatenation holds [10] (see formula (2.42)):

$$\begin{aligned} & C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} = S_{z_0 \rightsquigarrow 1-z_0} \\ \iff \quad & C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C = S_{z_0 \rightsquigarrow 1-z_0} e^C. \end{aligned}$$

Since $P \in \mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X]$ then by (3.25), applying $\langle \sigma f_{|0} \parallel \bullet\rangle$ to the two sides of the previous equality, one has

$$\langle \sigma f_{|0} \parallel C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C \rangle = \langle \sigma f_{|0} \parallel S_{z_0 \rightsquigarrow 1-z_0} e^C \rangle.$$

Thus, for $z_0 = \varepsilon$ it tends to 0^+, and one obtains

$$\langle \sigma f_{|0} \parallel e^{tx_1}\Phi \rangle \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \langle \sigma f_{|0} \parallel \Phi \rangle.$$

Since $\langle \sigma f_{|0} \parallel \Phi\rangle = \langle P \parallel \Phi\rangle = 0$ we get the expected result. □

Lemma 3.12 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\!\sqcup} e^C$ and let $\Psi = B'(y_1)\pi_Y \Phi$.*

We have $\mathcal{R}_Y \subseteq \ker\psi$ and $\mathcal{R}_X \subseteq \ker\phi$. In particular, $\mathcal{R}_Y \subseteq \ker\zeta$ and $\mathcal{R}_X \subseteq \ker\zeta$.

Proposition 3.6 *We have $\mathcal{R}_X \subseteq \mathcal{R}$ (resp. $\mathcal{R}_Y \subseteq \mathcal{R}$).*

Proposition 3.7 *For any proper polynomial $Q \in (\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X], \sqcup\!\sqcup)$ (or $(\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y], \uplus)$),*

$$Q \in \mathcal{R} \iff Q = 0.$$

Proof If $Q=0$ then since ϕ is an algebra homomorphism then $\phi(Q)=0$. Hence, $Q\in\ker\phi$ and then $Q\in\mathcal{R}$.

Conversely, if $Q\in\mathcal{R}$ then we get $\langle Q\parallel\Phi\rangle=0$. That means Q is indiscernible over $dm(A)$. Let $\mathcal{H}$ be the monoid generated by $dm(A)$ and by the Chen generating series $\{e^{tx}\}_{x\in X}^{t\in\mathbb{C}}$. By Lemma 6.2, Q is continuous over $\mathcal{H}$ and by Lemma 3.11, it is indiscernible over $\mathcal{H}$. By Proposition 2.7, the expected result follows. □

Therefore, by Propositions 3.6 and 3.7, we obtain

Theorem 3.10 *We have* $\mathcal{R}=\mathcal{R}_X$ (*resp.* $\mathcal{R}_Y$).

Proposition 3.8 *For any* $C\in\mathcal{L}ie_A\langle X\rangle$ *such that* $\langle e^C\mid x_0\rangle=\langle e^C\mid x_1\rangle=0$, *let* $\Phi=Z_{\sqcup\!\sqcup}e^C$ *and let* $\Psi=B'(y_1)\pi_Y\Phi$. *Let* $Q\in(\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X],\sqcup\!\sqcup)$ (*resp.* $(\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y],\,\uplus))$ *such that* $\langle\Phi\parallel Q\rangle=0$ (*resp.* $\langle\Psi\parallel Q\rangle=0$). *Then* $Q=0$.

Proof Let $\mathcal{H}$ be defined as the monoid generated by Φ and by Chen generating series $\{e^{tx}\}_{x\in X}^{t\in\mathbb{C}}$. By assumption, $\langle\Phi\parallel Q\rangle=0$ and by Lemma 3.11, Q is then indiscernible over $\mathcal{H}$. Finally, by Proposition 2.7, it follows that $Q=0$. □

Proposition 3.9 *For any* $C\in\mathcal{L}ie_A\langle X\rangle$ *such that* $\langle e^C\mid x_0\rangle=\langle e^C\mid x_1\rangle=0$, *let* $\Phi=Z_{\sqcup\!\sqcup}e^C$ *and let* $\Psi=B'(y_1)\pi_Y\Phi$. *We get* $\ker\phi=\mathcal{R}_X$ (*resp.* $\ker\psi=\mathcal{R}_Y$).

In particular, $\ker\zeta=\mathcal{R}_X$ (*resp.* $\ker\zeta=\mathcal{R}_Y$).

Proof By Lemma 3.12, $\mathcal{R}_X$ and $\mathcal{R}_Y$ are included in $\ker\phi$ and $\ker\psi$, respectively.

Conversely, two cases can occur (see Lemma 3.9):

(1) Case $Q\notin\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y]$). By Lemma 3.9, $Q\equiv_{\mathcal{R}_X}Q_1$ (resp. $Q\equiv_{\mathcal{R}_Y}Q_1$) such that $Q_1\in\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y]$) and $\phi(Q_1)=0$ (resp. $\psi(Q_1)=0$). This case is then reduced to the following.
(2) Case $Q\in\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y]$). Using Proposition 3.8, we have $Q\equiv_{\mathcal{R}_X}0$ (resp. $Q\equiv_{\mathcal{R}_Y}0$).

Then, $\mathcal{R}_X$ (resp. $\mathcal{R}_Y$) contains $\ker\phi$ (resp. $\ker\psi$). □

For any $Q\in(\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}X],\sqcup\!\sqcup)$ (resp. $(\mathbb{Q}[\mathcal{L}_{\mathrm{irr}}Y],\,\uplus))$, $\zeta(Q)$ is then a polynomial on $\mathbb{Q}$-irreducible polyzetas (see Definition 3.7). Moreover,

Proposition 3.10 *The* $\mathbb{Q}$-*algebra* $\mathcal{Z}$ *is generated by the family of* $\mathbb{Q}$-*irreducible polyzêtas* $\{\zeta(\ell)\}_{\ell\in\mathcal{L}_{\mathrm{irr}}Y}$ (*resp.* $\{\zeta(\ell)\}_{\ell\in\mathcal{L}_{\mathrm{irr}}X}$).

Proof By Radford's theorem [44], one just needs to prove it for Lyndon words.

Let $\ell\in\mathcal{L}ynY-y_1$. If $\pi_X\ell=\check{P}_\ell$ then the result follows; else one has $\pi_X\ell-\check{P}_\ell\in\ker\zeta$. Hence, $\zeta(\ell)=\zeta(\check{P}_\ell)$.

Since $\check{P}_\ell\in\mathbb{Q}[\mathcal{L}ynX-X]$ then $\check{P}_\ell$ is a polynomial on Lyndon words, over X, of degree less than or equal to $|\,\ell\,|$. For each Lyndon word which does appear in this decomposition of $\check{P}_\ell$, after applying π_Y, one uses the same recursive procedure until getting Lyndon words in $\mathcal{L}_{\mathrm{irr}}Y$.

The same treatment works for any $\ell'\in\mathcal{L}ynX-X$. □

By Proposition 3.9, one also has

$$\ker\phi = \ker\zeta = \mathcal{R}_X. \tag{3.26}$$

That means, for any irreducible Lyndon words $l \neq l'$,

$$\phi(l) = \phi(l') \iff \zeta(l) = \zeta(l'). \tag{3.27}$$

Let us state then the following.

Lemma 3.13 *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$, let $\Phi = Z_{\sqcup\!\sqcup} e^C$. Let us define the map $\varphi : \mathcal{Z} \longrightarrow A$ as follows*:

$$\forall l \in \mathcal{L}_{\mathrm{irr}}X, \quad \varphi\big(\zeta(l)\big) := \phi(l).$$

Then φ is an algebra homomorphism and $\{\varphi(\zeta(l))\}_{l\in\mathcal{L}_{\mathrm{irr}}X}$ are generators of A.

Thus, for any $\theta \in \mathcal{Z}$ there exist coefficients $\{\alpha_{l_1,\ldots,l_n}\}^{n\in\mathbb{N}}_{l_1,\ldots,l_n\in\mathcal{L}_{\mathrm{irr}}X}$ in A such that (see Proposition 3.10 and Lemma 3.13)

$$\theta = \sum_{n\geq 0}\sum_{l_1,\ldots,l_n\in\mathcal{L}_{\mathrm{irr}}X} \alpha_{l_1,\ldots,l_n}\,\zeta(l_1)\cdots\zeta(l_n), \tag{3.28}$$

$$\varphi(\theta) = \sum_{n\geq 0}\sum_{l_1,\ldots,l_n\in\mathcal{L}_{\mathrm{irr}}X} \alpha_{l_1,\ldots,l_n}\,\varphi\big(\zeta(l_1)\big)\cdots\varphi\big(\zeta(l_n)\big). \tag{3.29}$$

In particular, since for any $w \in X^*$, $\zeta_{\sqcup\!\sqcup}(w)$ belongs to $\mathcal{Z}$ (see Corollary 3.2) then $\varphi(\zeta_{\sqcup\!\sqcup}(w))$ is well defined and $\varphi(\zeta_{\sqcup\!\sqcup}(w))$ can be expressed as a polynomial on convergent polyzetas with coefficients in A:

Lemma 3.14 *With the notations in Lemma 3.13, one has*

$$\forall w \in X^*, \quad \varphi\big(\zeta_{\sqcup\!\sqcup}(w)\big) = \sum_{\substack{u,v\in X^*\\ uv=w}} \big\langle e^C \bigm| v\big\rangle\,\zeta_{\sqcup\!\sqcup}(u).$$

Proof The expected result follows by identifying the coefficients on $\Phi = Z_{\sqcup\!\sqcup} e^C$. □

Finally, we can state the following.

Theorem 3.11 *For any $\Phi \in dm(A)$, there exists a unique algebra homomorphism $\varphi : \mathcal{Z} \longrightarrow A$ such that Φ is computed from $Z_{\sqcup\!\sqcup}$ by applying φ to each coefficient*:

$$\Phi = \sum_{w\in X^*}\varphi\big(\zeta_{\sqcup\!\sqcup}(w)\big)\,w = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\varphi(\zeta(l))\,\hat{l}}.$$

Remark 3.2

(1) In this work, neither the question deciding any real number belongs to $\mathcal{Z}$ nor the question making explicit the coefficients $\{\alpha_{l_1,\ldots,l_n}\}^{n\in\mathbb{N}}_{l_1,\ldots,l_n\in\mathcal{L}_{\mathrm{irr}}X}$ in (3.29), are considered.

(2) Now, by considering the commutative indeterminates $t_1, t_2, t_3, \ldots$, let A be the $\mathbb{Q}$-algebra obtained by specializing $\mathbb{Q}[t_1, t_2, t_3, \ldots]$ at $t_1 = \mathrm{i}\pi$:

$$A = \mathbb{Q}[\mathrm{i}\pi][t_2, t_3, \ldots]. \tag{3.30}$$

Neither the Lie exponential series $e^{\mathrm{i}\pi x_0}$ nor $e^{\mathrm{i}\pi x_1}$ does belong to $dm(A)$ but it belongs to Gal(DE). In particular, it figures in the modromies (see Sect. 3.2.2) or in the functional relations (see (2.33) and (2.34)) of polylogarithms and in the hexagonal relation of polyzetas (see Proposition 2.6).

(3) Applying Baker–Campbell–Hausdorff formula [4] to Proposition 2.6 we get, at orders 2 and 3 as examples, the famous Euler formula saying $\zeta(2)$ is an algebraic number over $A = \mathbb{Q}[\mathrm{i}\pi]$:

$$\zeta(2) + \frac{(\mathrm{i}\pi)^2}{6} = 0 \quad \text{(order 2)}, \tag{3.31}$$

$$\zeta(3) - \zeta(2,1) = 0 \quad \text{(order 3, imaginary part)}. \tag{3.32}$$

Therefore, the first coming in mind homomorphism $\varphi : \mathcal{Z} \longrightarrow A$ maps, at least $\zeta(2)$ to $\varphi(\zeta(2)) = \pi^2/6$.

(4) For this reason, in [26], we have to consider the $\mathbb{Q}$-algebra generated by $\mathrm{i}\pi$ and by other A-irreducible polyzetas obtained in [3, 31, 35, 48] (and such algebra is denoted in this work by A).

This algebra came up from the studies of monodromies [33, 35], as already shown in (3.12), and the Kummer type functional equations of polylogarithms [34, 35], as already shown in (2.32)–(2.34). In particular, by (2.34), we get for example [34, 35]

$$\begin{aligned}\mathrm{Li}_{2,1}\frac{1}{t} = &-\frac{(\mathrm{i}\pi)^2}{2}\log t + \mathrm{i}\pi(\zeta(2) - \frac{\log^2 t}{2} - \mathrm{Li}_2 t \\ &- \mathrm{Li}_{2,1} t + \mathrm{Li}_3 t - \log t\,\mathrm{Li}_2 t + \zeta(3) - \frac{\log^3 t}{6}.\end{aligned}$$

Specializing $t = 1$, the real part of this leads again to the Euler identity in (3.32).

4 Concluding remarks: complete description of $\ker\zeta$ and structure of polyzetas

Let $t_1, t_2, t_3, \ldots$ be the commutative indeterminates and we suppose now A is the commutative $\mathbb{Q}$-algebra $\mathbb{Q}[t_1, t_2, t_3, \ldots]$.

4.1 A conjecture by Pierre Cartier

Definition 4.1 [8, 42] Let $DM(A)$ denote the set of $\Phi \in A\langle\langle X\rangle\rangle$ such that

$$\langle \Phi \mid 1_{X^*}\rangle = 1, \qquad \langle \Phi \mid x_0\rangle = \langle \Phi \mid x_1\rangle = 0, \qquad \Delta_{\sqcup\!\sqcup}\Phi = \Phi \otimes \Phi$$

and such that, for

$$\bar{\Psi} = \exp\biggl(-\sum_{n\geq 2}\langle \pi_Y\Phi \mid y_n\rangle\frac{(-y_1)^n}{n}\biggr)\pi_Y\Phi \in A\langle\langle Y\rangle\rangle,$$

we have $\Delta_{\boxplus}\bar{\Psi} = \bar{\Psi}\otimes\bar{\Psi}$.

Since $DM(A)$ contains already $Z_{ш}$ then for $\varPhi \in DM(A)$, by Theorem 3.6, there exists $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ verifying

$$\langle e^C \mid 1_{X^*}\rangle = 1 \quad \text{and} \quad \langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$$

such that

$$\varPhi = Z_{ш} e^C \tag{4.1}$$

and such that

$$\varPsi = B'(y_1)\pi_Y \varPhi = \exp\Bigl(-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Bigr)\pi_Y\varPhi, \tag{4.2}$$

$$\bar{\varPsi} = \exp\Bigl(-\sum_{n\geq 2}\langle \pi_Y\varPhi \mid y_n\rangle\frac{(-y_1)^n}{n}\Bigr)\pi_Y\varPhi. \tag{4.3}$$

By construction (see Definition 3.4 and Theorem 3.6), such $\varPhi$ and $\varPsi$ are group-like (for the co-products $\Delta_{ш}$ and $\Delta_{\text{ш}\!\!\!-}$, respectively) and here, $\bar{\varPsi}$ must be also group-like (for the co-product $\Delta_{\text{ш}\!\!\!-}$). If such a Lie series C exists then it is unique, due to the fact that $e^C = \varPhi Z_{ш}^{-1}$, and it is group-like (for the co-product $\Delta_{ш}$).

Corollary 4.1 (Conjectured by Cartier [8]) *For any $\varPhi \in DM(A)$, there exists a unique algebra homomorphism*[26] $\bar{\varphi} : \mathcal{Z} \longrightarrow A$ *such that $\varPhi$ is computed from $Z_{ш}$ by applying $\bar{\varphi}$ to each coefficient.*

Proof By Theorem 3.11, use the fact $DM(\mathbb{Q}) \subseteq DM(A) \subseteq dm(A)$. □

4.2 Arithmetical nature of γ

By Theorem 3.7, under the assumption that the Euler constant, γ, does not belong to a commutative $\mathbb{Q}$-algebra A then γ does not verify any polynomial with coefficients in A among the convergent polyzetas. It implies then,

Corollary 4.2 *If $\gamma \notin A$ then it is transcendental over the A-algebra generated by the convergent polyzetas.*

Or equivalently, by contraposition,

Corollary 4.3 *If there exists a polynomial relation with coefficients in A among the Euler constant, γ, and the convergent polyzetas then $\gamma \in A$.*

Therefore,

Corollary 4.4 *If the Euler constant, γ, does not belong to A then γ is not algebraic over A.*

Using Corollary 4.4, with $A = \mathbb{Q}$, it follows that

[26] See Remark 3.2 (3.2) for an example of $\bar{\varphi}$.

Corollary 4.5 *The Euler constant*, γ, *is not an algebraic irrational number.*

Corollary 4.6 *The Euler constant*, γ, *is a rational number.*

Proof Let us prove that in three steps:[27]

(1) Since γ verifies the equation $t^2 - \gamma^2 = 0$ then γ is algebraic over $\mathbb{Q}(\gamma^2)$.
(2) If γ is transcendental over $\mathbb{Q}$ then $\gamma \notin \mathbb{Q}(\gamma^2)$. Using Corollary 4.4, with $A = \mathbb{Q}(\gamma^2)$, γ is not algebraic over $A = \mathbb{Q}(\gamma^2)$. It contradicts the previous assertion (i.e. step (1)). Hence, γ is not transcendental over $\mathbb{Q}$.
(3) Thus, by Corollary 4.5, it remains that γ is rational over $\mathbb{Q}$.

□

Remark 4.1

(1) In the same spirit of Theorem 3.4, let $\zeta_{\text{⧢}}^T$ be the regularization morphism[28] from $(\mathbb{Q}\langle Y\rangle, \text{⧢})$ to $(\mathbb{R}, .)$ mapping y_1 to the symbol T.

Let $Z_{\text{⧢}}^T$ be the noncommutative generating series of regularized polyzetas with respect to $\zeta_{\text{⧢}}^T$. Thus, as in Theorem 3.4 and by infinite factorization by Lyndon words, we also get

$$Z_{\text{⧢}}^T := \sum_{w \in X^*} \zeta_{\text{⧢}}^T(w)\, w = e^{T y_1} Z_{\text{⧢}}. \tag{4.4}$$

(2) Now let us consider the regularization, for $N \to +\infty$ and with respect to $\zeta_{\text{⧢}}^T$, of the power series $\mathrm{Const}(N)$ given in (2.38) as

$$B^T(y_1) = e^{T y_1} B'(y_1). \tag{4.5}$$

As in Corollary 3.5, we always get

$$Z_{\text{⧢}}^T = B^T(y_1)\pi_Y Z_{\sqcup\sqcup} \quad \Longleftrightarrow \quad Z_{\text{⧢}} = B'(y_1)\pi_Y Z_{\sqcup\sqcup}. \tag{4.6}$$

Hence, roughly speaking, for the quasi-shuffle product, the symbolic regularization to T is also "equivalent" to the regularization to 0.
(3) Again, as in Corollary 4.2, if $T \notin A$ then $T \notin \bar{A}$.

A contrario, as in Corollary 4.3, if there exists a polynomial relation with coefficients in A among T and convergent polyzetas then $T \in A$.

4.3 Structure and arithmetical nature of polyzetas

Once again, let us consider (see Definition 3.3, Lemma 3.9, Definition 3.6)

$$\begin{aligned}(A_1, \sqcup\sqcup) &= (A 1_{X^*} \oplus x_0 A\langle X\rangle x_1, \sqcup\sqcup) \\ &\cong A[\mathcal{L}ynX - X], \sqcup\sqcup)\end{aligned}$$

[27] This part has been obtained after prolonged discussions with Michel Waldschmidt.

[28] This is a *symbolic* regularization and does not yet have an analytical justification as is done, separately, for $\zeta_{\sqcup\sqcup}$ and $\zeta_{\text{⧢}}$ in Sect. 3.1.2 as finite parts of the asymptotic expansions, in different scales of comparison, of $\mathrm{Li}_{x_1}(z)$, for $z \rightsquigarrow 1$, and $\mathrm{H}_{y_1}(N)$, for $N \to \infty$, respectively.

$$= (\mathcal{R}_X, ⧢) \oplus \big(A[\mathcal{L}_{\text{irr}} X], ⧢\big), \tag{4.7}$$

$$\begin{aligned}(A_2, ⧺) &= \big(A1_{Y^*} \oplus \big(Y - \{y_1\}\big)A\langle Y\rangle, ⧺\big)\\ &\cong \big(A[\mathcal{L}ynY - y_1], ⧺\big)\\ &= (\mathcal{R}_Y, ⧺) \oplus \big(A[\mathcal{L}_{\text{irr}} Y], ⧺\big).\end{aligned} \tag{4.8}$$

Then [35, 36]

$$(A_1, ⧢) \cong (A_2, ⧺). \tag{4.9}$$

Let us consider again the following algebra morphism (see Proposition 3.2):

$$\zeta : \begin{matrix}(A_2, ⧺)\\(A_1, ⧢)\end{matrix} \longrightarrow (\mathbb{R}, .) \tag{4.10}$$

$$\begin{matrix}y_{r_1}\cdots y_{r_k}\\ x_0 x_1^{r_1-1}\cdots x_0 x_1^{r_k-1}\end{matrix} \longmapsto \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{r_1}\cdots n_k^{r_k}}. \tag{4.11}$$

Lemma 4.1 *The image of the algebra morphism ζ is $\mathcal{Z}$.*

Let us make precise the structure of $\mathcal{Z}$ and the arithmetical nature of polyzetas: As consequences of Propositions 3.8, 3.9 and 3.10, by taking $\Phi = Z_{⧢}$, we have

$$\operatorname{Im}\zeta = \zeta\big(A[\mathcal{L}_{\text{irr}} Y]\big) \quad \text{and} \quad \ker\zeta = \mathcal{R}_Y \tag{4.12}$$

$$\big(\text{resp. } \operatorname{Im}\zeta = \zeta\big(A[\mathcal{L}_{\text{irr}} X]\big) \text{ and } \ker\zeta = \mathcal{R}_X\big). \tag{4.13}$$

By Corollary 3.9, $\ker\zeta$ is an ideal generated by the homogeneous polynomials of degree ≥ 2. Hence, the quotient $A_1/\mathcal{R}_X$ or $A_2/\mathcal{R}_Y$ (the source by the kernel of ζ) is graded [4] and it is isomorphic to $\operatorname{Im}\zeta$.

Therefore, by Lemma 4.1 and Proposition 3.10, we obtain, respectively, the following direct consequences.

Theorem 4.1 (Structure of polyzetas) *The A-algebra $\mathcal{Z}$ is*

(1) *isomorphic to the graded algebra $(A_1/\mathcal{R}_X, ⧢)$, or equivalently, $(A_2/\mathcal{R}_Y, ⧺)$.*
(2) *freely generated by the A-irreducible polyzetas $\{\zeta(l)\}_{l\in\mathcal{L}_{\text{irr}} Y}$ (resp. $\{\zeta(l)\}_{l\in\mathcal{L}_{\text{irr}} X}$).*

For any $p \geq 2$, let

$$\mathcal{Z}_p = \operatorname{span}_{\mathbb{Q}}\big\{\zeta(w) \,\big|\, w \in x_0 X^* x_1, |w| = p\big\}. \tag{4.14}$$

By the definition of graded algebra [4], Theorem 4.1 means also that (for $A = \mathbb{Q}$)

$$\mathcal{Z} = \mathbb{Q}1 \oplus \bigoplus_{p\geq 2} \mathcal{Z}_p \tag{4.15}$$

and there is no linear relation among elements of different $\mathcal{Z}_p$ ([8] conjecture C1 [47]).

Thus, if θ is a polyzeta verifying the following algebraic equation:

$$\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_0 = 0 \tag{4.16}$$

then $\theta = 0$ because $\mathcal{Z}_{p_1}\mathcal{Z}_{p_2} \subset \mathcal{Z}_{p_1+p_2}$, for $p_1, p_2 \geq 2$, and each monomial in (4.16) is then of different weight. By consequence,

Corollary 4.7 *Any* ($\mathbb{Q}$*-irreducible*) *polyzeta* θ *is transcendental over* $\mathbb{Q}$.

Remark 4.2 In this work, neither the study of dim $\mathcal{Z}_p$ [49] (see also [8], conjecture C2) nor the estimate of the number of A-irreducible polyzetas generating $\mathcal{Z}_p$, are discussed knowing the A-irreducible polyzetas form a transcendence basis of the A-algebra $\mathcal{Z}$.

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Appendix 1: Pair of bases in duality and proof of Theorem 2.2

5.1 Preliminary results

Let $\mathbb{Q}\langle Y\rangle$ be equipped with the concatenation and the quasi-shuffle, $\uplus$, defined by

$$\forall y_i, y_j \in Y = \{y_i\}_{i\geq 1},\ \forall u, v \in Y^*, \quad y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_{i+j}(y_i u \uplus v),$$
$$\forall w \in Y^*, \quad w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w,$$

or by its associated co-product, $\Delta_{\uplus}$, defined by

$$\forall y_k \in Y, \quad \Delta_{\uplus}(y_k) = y_k \otimes 1_{Y^*} + 1_{Y^*} \otimes y_k + \sum_{i+j=k} y_i \otimes y_j.$$

satisfying, for any $u, v, w \in Y^*$, $\langle u \otimes v \mid \Delta_{\uplus}(w)\rangle = \langle u \uplus v \mid w\rangle$.

Lemma 5.1 *Let* $S_1, \ldots, S_n$ *be* proper *formal power series in* $\mathbb{Q}\langle\langle Y\rangle\rangle$. *Let* $P_1, \ldots, P_m$ *be primitive elements*[29] *in* $\mathbb{Q}\langle Y\rangle$, *for the co-product* $\uplus$.

(1) *If* $n > m$ *then* $\langle S_1 \uplus \cdots \uplus S_n \mid P_1 \cdots P_m\rangle = 0$.
(2) *If* $n = m$ *then*

$$\langle S_1 \uplus \cdots \uplus S_n \mid P_1 \cdots P_n\rangle = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \langle S_i \mid P_{\sigma(i)}\rangle.$$

(3) *If* $n < m$ *then, by considering the language* $\mathcal{M}$ *over* $\mathcal{A} = \{P_1, \ldots, P_m\}$

$$\mathcal{M} = \left\{w \in \mathcal{A}^* | w = P_{j_1} \cdots P_{j_{|w|}}, j_1 < \cdots < j_{|w|}, |w| \geq 1\right\}$$

and the morphism $\mu : \mathbb{Q}\langle\mathcal{A}\rangle \longrightarrow \mathbb{Q}\langle Y\rangle$, *one has*

$$\langle S_1 \uplus \cdots \uplus S_n \mid P_1 \cdots P_m\rangle = \sum_{\substack{w_1,\ldots,w_m \in \mathcal{M} \\ \mathrm{supp}(w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_m) \ni P_1 \cdots P_m}} \prod_{i=1}^{n} \langle S_i \mid \mu(w_i)\rangle.$$

[29] *That is*, for any $i = 1, \ldots, m$, $\Delta_{\uplus}(P_i) = 1_{Y^*} \otimes P_i + P_i \otimes 1_{Y^*}$.

Proof On one hand, since the P_i's are primitive then

$$\Delta_{\text{⧢}}^{(n-1)}(P_i) = \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q}.$$

On the other hand, $\langle S_1 \text{⧢} \cdots \text{⧢} S_n \mid P_1 \cdots P_m\rangle = \langle S_1 \otimes \cdots \otimes S_n \mid \Delta_{\text{⧢}}^{(n-1)}(P_1 \cdots P_m)\rangle$ and $\Delta_{\text{⧢}}^{(n-1)}(P_1 \cdots P_m) = \Delta_{\text{⧢}}^{(n-1)}(P_1) \cdots \Delta_{\text{⧢}}^{(n-1)}(P_m)$. Hence,

$$\langle S_1 \text{⧢} \cdots \text{⧢} S_n \mid P_1 \cdots P_m\rangle = \left\langle \bigotimes_{i=1}^{n} S_i \,\middle|\, \prod_{i=1}^{m} \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q} \right\rangle.$$

(1) For $n > m$, by expanding $\Delta_{\text{⧢}}^{(n-1)}(P_1) \cdots \Delta_{\text{⧢}}^{(n-1)}(P_m)$, one obtains a sum of tensors containing at least one factor equal to 1. For $j = 1, \ldots, n$, the formal power series S_j is proper and the result follows immediately.

(2) For $n = m$, since

$$\prod_{i=1}^{n} \Delta_{\text{⧢}}^{(n-1)}(P_i) = \sum_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^{n} P_{\sigma(i)} + Q,$$

where Q is a sum of tensors containing at least one factor equal to 1 and the S_j's are proper then $\langle S_1 \otimes \cdots \otimes S_n \mid Q\rangle = 0$. Thus, the result follows.

(3) For $n < m$, noticing that, for $j = 1, \ldots, n$, the formal power series S_j is proper, the expected result follows by expanding the product

$$\prod_{i=1}^{m} \Delta_{\text{⧢}}^{(n-1)}(P_i) = \prod_{i=1}^{m} \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q}.$$

□

Proposition 5.1

(1) *We have*

$$\log\left(\sum_{w \in Y^*} w \otimes w\right) = \sum_{w \in Y^+} w \otimes \pi_1(w) = \sum_{w \in Y^+} \pi_1^*(w) \otimes w,$$

where π_1^* is the adjoint of π_1 and they are given by

$$\pi_1(w) = \sum_{k \ge 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \text{⧢} \cdots \text{⧢} u_k\rangle u_1 \cdots u_k,$$

$$\pi_1^*(w) = \sum_{k \ge 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \cdots u_k\rangle u_1 \text{⧢} \cdots \text{⧢} u_k.$$

In particular, for any $y_k \in Y$, one has

$$\pi_1(y_k) = y_k + \sum_{l \ge 2} \frac{(-1)^{l-1}}{l} \sum_{\substack{j_1, \ldots, j_l \ge 1 \\ j_1 + \cdots + j_l = k}} y_{j_1} \cdots y_{j_l},$$

$$\pi_1^*(y_k) = y_k.$$

(2) *For any $w \in Y^*$, we have*

$$w = \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k \in Y^*} \langle w \mid u_1 \shuffle \cdots \shuffle u_k\rangle \pi_1(u_1)\cdots\pi_1(u_k)$$
$$= \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k \in Y^*} \langle w \mid u_1 \cdots u_k\rangle \pi_1^*(u_1) \shuffle \cdots \shuffle \pi_1^*(u_k).$$

Proof

(1) Expanding the logarithm, we have

$$\log\Bigl(\sum_{w\in Y^*} w\otimes w\Bigr) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\Bigl(\sum_{w\in Y^+} w\otimes w\Bigr)^k$$
$$= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} (u_1 \shuffle \cdots \shuffle u_k)\otimes u_1\cdots u_k$$
$$= \sum_{w\in Y^+} w\otimes \Bigl(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 \shuffle \cdots \shuffle u_k\rangle u_1\cdots u_k\Bigr).$$

In the same way,

$$\log\Bigl(\sum_{w\in Y^*} w\otimes w\Bigr) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} (u_1 \shuffle \cdots \shuffle u_k)\otimes u_1\cdots u_k$$
$$= \sum_{w\in Y^+} \Bigl(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1\cdots u_k\rangle u_1 \shuffle \cdots \shuffle u_k\Bigr)\otimes w.$$

Thus, the expressions of $\pi_1(w)$ and $\pi_1^*(w)$ follow immediately.

(2) Since

$$\sum_{w\in Y^*} w\otimes w = \exp\Bigl(\log\Bigl(\sum_{w\in Y^*} w\otimes w\Bigr)\Bigr)$$

then, by the previous results, one has

$$\sum_{w\in Y^*} w\otimes w = \sum_{k\geq 0} \frac{1}{k!}\Bigl(\sum_{w\in Y^+} w\otimes \pi_1(w)\Bigr)^k$$
$$= \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} (u_1 \shuffle \cdots \shuffle u_k)\otimes \bigl(\pi_1(u_1)\cdots\pi_1(u_k)\bigr)$$
$$= \sum_{w\in Y^*} w\otimes \Bigl(\sum_{k\geq 1} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 \shuffle \cdots \shuffle u_k\rangle \pi_1(u_1)\cdots\pi_1(u_k)\Bigr).$$

In the same way,

$$\sum_{w\in Y^*} w\otimes w = \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} \bigl(\pi_1^*(u_1) \shuffle \cdots \shuffle \pi_1^*(u_k)\bigr)\otimes (u_1\cdots u_k)$$

$$= \sum_{w\in Y^*}\left(\sum_{k\ge 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1\cdots u_k\rangle \pi_1^*(u_1) ⧢ \cdots ⧢ \pi_1^*(u_k)\right)\otimes w.$$

Then the expected result follows. □

5.2 Pair of bases in duality

Definition 5.1 Let $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ be the family of $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$ obtained as follows:

$$\Sigma_{y_k} = \pi_1(y_k) \quad \text{for } k\ge 1,$$
$$\Sigma_l = [\Sigma_s, \Sigma_r] \quad \text{for } l\in \mathcal{L}ynX,\ \text{the standard factorization of } l=(s,r),$$

and the family $\{\Sigma_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$ (viewed as a $\mathbb{Q}$-module) obtained as follows:

$$\Sigma_l = 1 \quad \text{for } l = 1_{Y^*},$$
$$\Sigma_w = \Sigma_{l_1}^{i_1}\cdots\Sigma_{l_k}^{i_k} \quad \text{for } w = l_1^{i_1}\cdots l_k^{i_k}, l_1 > \cdots > l_k, l_1,\ldots,l_k\in\mathcal{L}ynY.$$

Let $\{\check{\Sigma}_w\}_{w\in Y^*}$ be the family of the quasi-shuffle algebra (viewed as a $\mathbb{Q}$-module) obtained by duality with $\{\Sigma_w\}_{w\in Y^*}$:

$$\forall u,v\in Y^*,\quad \langle\check{\Sigma}_v \mid \Sigma_u\rangle = \delta_{u,v}.$$

Proposition 5.2

(1) *For $l\in\mathcal{L}ynY$, the polynomial Σ_l is upper triangular*:

$$\Sigma_l = l + \sum_{v>w,(v)=(l)} c_v v.$$

(2) *The families $\{\Sigma_w\}_{w\in Y^*}$ and $\{\check{\Sigma}_w\}_{w\in Y^*}$ are upper and lower triangular, respectively. In other words, for any $w\in Y^+$, one has*

$$\Sigma_w = w + \sum_{v>w,(v)=(w)} c_v v \quad \textit{and} \quad \check{\Sigma}_w = w + \sum_{v<w,(v)=(w)} d_v v.$$

Here, for any $y_k\in Y$ and $w\in Y^*$, (w) denotes the degree of w and $(y_k)=\deg(y_k)=k$.

Proof

(1) Let us prove it by induction on the length of l:

- The result is immediate for $l\in Y$.
- The result is supposed verified for any $l\in\mathcal{L}ynY\cap Y^k$ and $0\le k\le N$.
- At $N+1$, by the standard factorization (l_1,l_2) of l, one has, by definition, $\Sigma_l = [\Sigma_{l_1},\Sigma_{l_2}]$ and $l_2l_1 > l_1l_2 = l$. By induction hypothesis,

$$\Sigma_{l_1} = l_1 + \sum_{v>l_1,(v)=(l_1)} c_v v \quad \text{and} \quad \Sigma_{l_2} = l_2 + \sum_{u>l_2,(v)=(l_2)} d_u u,$$
$$\Rightarrow \quad \Sigma_l = l + \sum_{w>l,(w)=(l)} e_w w,$$

getting e_w's from c_v's and d_u's. Actually, the Lie bracket gives

$$\begin{aligned}
\Sigma_l &= [l_1, l_2] + \sum_{\substack{u>l_2\\(v)=(l_2)}} d_u l_1 u + \sum_{\substack{v>l_1, u>l_2\\(v)=(l_1),(u)=(l_2)}} c_v d_u vu \\
&\quad - \sum_{\substack{v>l_1\\(v)=(l_1)}} c_v l_2 v - \sum_{\substack{v>l_1, u>l_2\\(v)=(l_2),(u)=(l_1)}} c_v d_u uv \\
&= [l_1, l_2] + \sum_{\substack{u>l_1 l_2\\(v)=(l_1 l_2)}} d'_u u + \sum_{\substack{vu>l_1 l_2\\(vu)=(l_1 l_2)}} c_v d_u vu \\
&\quad - \sum_{\substack{v>l_2 l_1\\(v)=(l_2 l_1)}} c'_v v - \sum_{\substack{uv>l_2 l_1\\(uv)=(l_2 l_1)}} c_v d_u uv \\
&= [l_1, l_2] + \sum_{\substack{u>l\\(v)=(l)}} d'_u u + \sum_{\substack{vu>l\\(vu)=(l)}} c_v d_u vu \\
&\quad - \sum_{\substack{v>l_2 l_1>l\\(v)=(l)}} c'_v v - \sum_{\substack{uv>l_2 l_1>l\\(uv)=(l)}} c_v d_u uv.
\end{aligned}$$

Hence, the conclusion follows.

(2) Let $w = l_1 \cdots l_k$, with $l_1 > \cdots > l_k$ and $l_1, \ldots, l_k \in \mathcal{L}ynY$. By (1), one has

$$\Sigma_{l_i} = l_i + \sum_{v>l_i,(v)=(l_i)} c_{i,v} v \quad \text{and} \quad \Sigma_w = l_1 \cdots l_k + \sum_{u>w,(v)=(w)} d_u u,$$

where the d_u's are obtained from the $c_{i,v}$'s. Hence, the family $\{\Sigma_w\}_{w\in Y^*}$ is upper triangular and, by duality, the family $\{\check{\Sigma}_w\}_{w\in Y^*}$ is lower triangular. □

Theorem 5.1

(1) *The family* $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ *forms a basis of the free Lie algebra.*
(2) *The family* $\{\Sigma_w\}_{w\in Y^*}$ *forms a basis of the free associative algebra* $\mathbb{Q}\langle Y\rangle$.
(3) *The family* $\{\check{\Sigma}_w\}_{w\in Y^*}$ *generates freely the quasi-shuffle algebra.*
(4) *The family* $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ *forms a transcendence basis of the quasi-shuffle algebra.*

Proof The family $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ of upper triangular polynomials is free and then, by a theorem of Viennot, we get the first result. The second one is a direct consequence of the Poincaré–Birkhoff–Witt theorem. By the Cartier–Quillen–Milnor–Moore theorem, we get the third one and the last one is also obtained as a consequence of the constructions of the families $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ and $\{\check{\Sigma}_w\}_{w\in Y^*}$ of lower triangular polynomials. □

Definition 5.2 Let $\pi_Y : (\mathbb{Q}1_{X^*} \oplus \mathbb{Q}\langle X\rangle x_1, .) \longrightarrow (\mathbb{Q}\langle Y\rangle, .)$ be the morphism mapping $x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1 \in X^* x_1$ to $y_{s_1} \cdots y_{s_r} \in Y^*$ and π_X be its inverse. Its extension over $\mathbb{Q}\langle X\rangle$ verifying $\pi_Y(p) = 0$, for $p \in \mathbb{Q}\langle X\rangle x_0$, is still denoted by π_Y.

Proposition 5.3

(1) *The homogeneous polynomials* $\{\pi_Y S_{\pi_X l}\}_{l\in\mathcal{L}ynY}$ *are upper triangular and linearly independent*[30] *and*

$$\pi_Y S_{\pi_X l} = \Sigma_l + \sum_{v>l,(v)=(l)} p_v v.$$

(2) *For any* $w \in Y^*$, *the following homogeneous polynomial*:

$$\pi_Y S_{\pi_X w} = \Sigma_w + \sum_{v>w,(v)=(w)} c_v v$$

is of degree (w) *and the family* $\{\pi_Y S_{\pi_X w}\}_{w\in Y^*}$ *forms a basis for* $\mathbb{Q}\langle Y\rangle$.

(3) *Let* $\{\Theta_w\}_{w\in Y^*}$ *be the family of homogeneous polynomials in duality with the family* $\{\pi_Y S_{\pi_X w}\}_{w\in Y^*}$:

$$\forall u, v \in Y^*, \quad \langle \pi_Y S_{\pi_X u} \mid \Theta_u\rangle = \delta_{u,v}.$$

Then, the family $\{\Theta_w\}_{w\in Y^*}$ *generate freely the quasi-shuffle algebra and, for any* $w \in Y^*$, Θ_w *is upper triangular of degree* (w):

$$\Theta_w = \check{\Sigma}_w + \sum_{v<w,(v)=(w)} d_v v.$$

(4) *The family* $\{\Theta_l\}_{l\in\mathcal{L}ynY}$ *does not form a transcendence basis of* $(\mathbb{Q}\langle Y\rangle, \uplus)$.

Proof

(1) For $l \in \mathcal{L}ynX$ (resp. $\mathcal{L}ynY$), one has $\deg(S_l) = |l|$ (resp. $\deg(\Sigma_l) = (l)$) and

$$S_l = l + \sum_{v>l,|v|=|l|} a_v v \qquad \left(\text{resp. } \Sigma_l = l + \sum_{v>l,(v)=(l)} c_v v\right),$$

Hence, for any $l \in \mathcal{L}ynY$, we have $\pi_X l \in \mathcal{L}ynX$ and

$$S_{\pi_X l} = \pi_X\left[\Sigma_l - \sum_{v>l,(v)=(l)} c_v v\right] + \sum_{v>\pi_X l,|v|=|l|} a_v v.$$

Thus,

$$\pi_Y S_{\pi_X l} = \Sigma_l + \sum_{u>l,(u)=(l)} \left(a'_u - c'_u\right) u.$$

Hence, we get the expected results by putting $p_u = a'_u - c'_u$, where the coefficients a'_u's (resp. c'_u's) are obtained from a_v's (resp. c_v's) by completing some null coefficients when it is necessary and by using the fact

$$\forall w_1, w_2 \in X^* x_1, \quad w_1 > w_2 \quad \Rightarrow \quad \pi_Y w_1 > \pi_Y w_2.$$

[30] For any $l \in \mathcal{L}ynY$, $S_{\pi_X l}$ and Σ_l are primitive but $\pi_Y S_{\pi_X l}$ is not necessarily primitive. For example, $S_{\pi_Y y_2} = [x_0, x_1]$ and $\Sigma_{y_2} = y_2 - \frac{1}{2} y_1^2$ are primitive but $\pi_Y S_{x_0 x_1} = y_2$ is not.

By Proposition 5.2, the polynomials $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ are upper triangular and are linearly independent then the $\{\pi_Y S_{\pi_X l}\}_{l\in\mathcal{L}ynY}$ are also.

(2) As in Proposition 5.2, let $w = l_1\cdots l_k$, with $l_1 > \cdots > l_k, l_1,\ldots,l_k \in \mathcal{L}ynY$. Firstly, one has $(\pi_X l_1)\cdots(\pi_X l_k) = \pi_X w$ and secondly,

$$S_{\pi_X l_i} = \pi_X l_i + \sum_{v>l_i,|v|=|l_i|} c_{i,v} v \quad \text{and} \quad S_{\pi_X w} = \pi_X w + \sum_{u>w,|u|=|w|} d_u u,$$

where the d_u's are obtained from the $c_{i,v}$'s. Hence, the family $\{S_{\pi_X w}\}_{w\in Y^*}$ is upper triangular. Using the restriction of π_Y, as being morphism from $(\mathbb{Q}1_{X^*} \oplus \mathbb{Q}\langle X\rangle x_1, .)$ to $(\mathbb{Q}\langle Y\rangle, .)$, we get the degree of the upper triangular homogeneous polynomial $\pi_Y S_{\pi_X w}$ as image of Σ_w is (see Proposition 5.2). The family $\{\pi_Y S_{\pi_X w}\}_{w\in Y^*}$ forms then a basis for the free algebra $\mathbb{Q}\langle Y\rangle$.

(3) It is a consequence of the Cartier–Quillen–Milnor–Moore theorem.

(4) If $\{\Theta_l\}_{l\in\mathcal{L}ynY}$ constitutes a transcendence basis of $(\mathbb{Q}\langle Y\rangle, ⊞)$ then, for any $l \in \mathcal{L}ynY$, $\pi_Y S_{\pi_X l}$ is primitive but it is false in general (see footnote 30). $\square$

Now, let us clarify the basis $\{\check{\Sigma}_w\}_{w\in Y^*}$ and then the transcendence basis $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ of the quasi-shuffle algebra $(\mathbb{Q}\langle Y\rangle, ⊞)$ as follows:

Theorem 5.2 *We have*

(1) *For* $w = 1_{Y^*}$, $\check{\Sigma}_w = 1$.

(2) *For any* $w = l_1^{i_1}\cdots l_k^{i_k}$, *with* $l_1,\ldots,l_k \in \mathcal{L}ynY$ *and* $l_1 > \cdots > l_k$,

$$\check{\Sigma}_w = \frac{1}{i_1!\cdots i_k!}\check{\Sigma}_{l_1}^{⊞ i_1} ⊞ \cdots ⊞ \check{\Sigma}_{l_k}^{⊞ i_k}.$$

(3) *For any* $y \in Y$, $\check{\Sigma}_y = \pi_1^*(y)$.

Proof

(1) Since $\Sigma_{1_{Y^*}} = 1$ then $\check{\Sigma}_{1_{Y^*}} = 1$.

(2) Let $u = u_1\cdots u_n = l_1^{i_1}\cdots l_k^{i_k}$, $v = v_1\cdots v_m = h_1^{j_1}\cdots h_p^{j_p}$ with $l_1,\ldots,l_k,h_1,\ldots,h_p,$ $u_1,\ldots,u_n,v_1,\ldots,v_m \in \mathcal{L}ynY, l_1 > \cdots > l_k, h_1 > \cdots > h_p, u_1 \geq \cdots \geq u_n, v_1 \geq \cdots \geq v_m$ and $i_1+\cdots+i_k = n$, $j_1+\cdots+j_p = m$.

Hence, if $m \geq 2$ (resp. $n \geq 2$) then $v \notin \mathcal{L}ynY$ (resp. $u \notin \mathcal{L}ynY$).

Since

$$\left\langle \check{\Sigma}_{u_1} ⊞ \cdots ⊞ \check{\Sigma}_{u_n} \,\middle|\, \prod_{i=1}^n \Sigma_{u_i}\right\rangle = \langle \check{\Sigma}_{u_1} \otimes \cdots \otimes \check{\Sigma}_{u_n} \mid \Delta_{⊞}^{(n-1)}(\Sigma_{v_1}\cdots\Sigma_{v_m})\rangle$$

then many cases occur:

(a) Case $n > m$. By Lemma 5.1(1), one has

$$\langle \check{\Sigma}_{u_1} ⊞ \cdots ⊞ \check{\Sigma}_{u_n} \mid \Sigma_{v_1}\cdots\Sigma_{v_m}\rangle = 0.$$

(b) Case $n = m$. By Lemma 5.1(2), one has

$$\left\langle \check{\Sigma}_{u_1} ⊞ \cdots ⊞ \check{\Sigma}_{u_n} \,\middle|\, \prod_{i=1}^n \Sigma_{v_i}\right\rangle = \sum_{\sigma\in\check{\Sigma}_n}\prod_{i=1}^n \langle \check{\Sigma}_{u_i} \mid \Sigma_{v_{\sigma(i)}}\rangle$$

$$= \sum_{\sigma \in \check{\Sigma}_n} \prod_{i=1}^{n} \delta_{u_i, v_{\sigma(i)}}.$$

Thus, if $u \neq v$ then $(u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)$, so the second member is vanishing else, i.e. $u = v$, the second member equals 1 because the factorization by Lyndon words is unique.

(c) Case $n < m$. By Lemma 5.1(3), let us consider the following language over the alphabet $\mathcal{A} = \{\Sigma_{v_1}, \ldots, \Sigma_{v_m}\}$:

$$\mathcal{M} = \left\{w \in \mathcal{A}^* | w = \Sigma_{v_{j_1}} \cdots \Sigma_{v_{j_{|w|}}}, j_1 < \cdots < j_{|w|}, |w| \geq 1\right\},$$

and the morphism $\mu : \mathbb{Q}\langle \mathcal{A} \rangle \longrightarrow \mathbb{Q}\langle Y \rangle$. We get

$$\left\langle \check{\Sigma}_{u_1} \uplus \cdots \uplus \check{\Sigma}_{u_n} \middle| \prod_{i=1}^{n} \Sigma_{u_i} \right\rangle = \sum_{\substack{w_1, \ldots, w_m \in \mathcal{M} \\ \mathrm{supp}(w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_m) \ni \Sigma_1 \cdots \Sigma_m}} \prod_{i=1}^{n} \langle \check{\Sigma}_{u_i} \mid \mu(w_i) \rangle$$
$$= 0.$$

Because in the right side of the first equality, on one hand, there exists at least one $w_i \in \mathcal{M}$, $|w_i| \geq 2$, corresponding to $\Sigma_{v_{j_1}} \cdots \Sigma_{v_{j_{|w_i|}}} = \mu(w_i)$ such that $v_{j_1} \geq \cdots \geq v_{j_{|w_i|}}$ and on the other hand, $v_i := v_{j_1} \cdots v_{j_{|w_i|}} \notin \mathcal{L}ynY$ and $u_i \in \mathcal{L}ynY$.

As a consequence,

$$\langle \check{\Sigma}_u \mid \Sigma_v \rangle = \frac{1}{i_1! \cdots i_k!} \langle \check{\Sigma}_{l_1}^{\uplus i_1} \uplus \cdots \uplus \check{\Sigma}_{l_k}^{\uplus i_k} \mid \Sigma_{h_1}^{j_1} \cdots \Sigma_{h_p}^{j_p} \rangle$$
$$= \delta_{u,v}.$$

(3) For any $l \in Y$, $\Sigma_l = \pi_1(l)$, $\check{\Sigma}_l = \pi_1^*(l)$ and π_1, π_1^* are mutually adjoint. Direct computation proves that

$$\forall w \in Y^+, \;\; y \in Y, \quad \langle \check{\Sigma}_w \mid \Sigma_y \rangle = \delta_{w,y}. \qquad \square$$

Corollary 5.1

(1) *For $w \in Y^+$, the polynomial $\check{\Sigma}_w$ is proper and homogeneous of degree $|w|$, for* $\deg(y_i) = i$, *and of rational positive coefficients.*

(2)

$$\sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \check{\Sigma}_w \otimes \Sigma_w = \prod_{l \in \mathcal{L}ynY}^{\searrow} \exp(\check{\Sigma}_l \otimes \Sigma_l).$$

(3) *The family $\mathcal{L}ynY$ forms a transcendence basis*[31] *of the quasi-shuffle algebra and the family of proper polynomials of rational positive coefficients defined by, for any $w =$*

[31] This result is an analogue of the Radford theorem (see [44]). Thus the bases $\mathcal{L}ynY$ and $\{\check{\Sigma}_l\}_{l \in \mathcal{L}ynY}$ belong to the class of Radford bases, i.e. the class of transcendence bases, of the quasi-shuffle algebra, as well as the bases $\mathcal{L}ynX$ and $\{S_l\}_{l \in \mathcal{L}ynX}$ belong to the class of Radford bases of the shuffle algebra.

$l_1^{i_1}\cdots l_k^{i_k}$ with $l_1>\cdots>l_k$ and $l_1,\ldots,l_k\in\mathcal{L}ynY$,

$$\chi_w=\frac{1}{i_1!\cdots i_k!}l_1^{⊎ i_1}⊎\cdots⊎ l_k^{⊎ i_k}$$

forms a basis of the quasi-shuffle algebra.

(4) *Let $\{\xi_w\}_{w\in Y^*}$ be the basis of the enveloping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ obtained by duality with the basis $\{\chi_w\}_{w\in Y^*}$:*

$$\forall u,v\in Y^*,\quad \langle\chi_v\mid\xi_u\rangle=\delta_{u,v}.$$

Then the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ forms a basis of the free Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$.

Proof

(1) The proof can be done by induction on the length of w using the fact that the product $⊎$ conserves the property, homogeneity, and rational positivity of the coefficients.
(2) Expressing w in the basis $\{\check{\Sigma}_w\}_{w\in Y^*}$ of the quasi-shuffle algebra and then in the basis $\{\Sigma_w\}_{w\in Y^*}$ of the enveloping algebra, we obtain successively

$$\begin{aligned}\sum_{w\in Y^*}w\otimes w&=\sum_{w\in Y^*}\left(\sum_{u\in X^*}\langle\Sigma_u\mid w\rangle\check{\Sigma}_u\right)\otimes w\\&=\sum_{u\in Y^*}\check{\Sigma}_u\otimes\left(\sum_{w\in X^*}\langle\Sigma_u\mid w\rangle w\right)\\&=\sum_{u\in Y^*}\check{\Sigma}_u\otimes\Sigma_u\\&=\sum_{\substack{l_1>\cdots>l_k\\ i_1,\ldots,i_k\geq 1}}\frac{1}{i_1!\cdots i_k!}\check{\Sigma}_{l_1}^{\sqcup\sqcup i_1}⊎\cdots⊎\check{\Sigma}_{l_k}^{\sqcup\sqcup i_k}\otimes\Sigma_{l_1}^{i_1}\cdots\Sigma_{l_k}^{i_k}\\&=\prod_{l\in\mathcal{L}ynY}^{\searrow}\sum_{i\geq 0}\frac{1}{i!}\check{\Sigma}_l^{⊎ i}\otimes\Sigma_l^i\\&=\prod_{l\in\mathcal{L}ynY}^{\searrow}\exp(\check{\Sigma}_l\otimes\Sigma_l).\end{aligned}$$

(3) For $w=l_1^{i_1}\cdots l_k^{i_k}$ with $l_1,\ldots,l_k\in\mathcal{L}ynY$ and $l_1>\cdots>l_k$, by Proposition 5.2, the proper polynomial of rational positive coefficients $\check{\Sigma}_w$ is lower triangular:

$$\begin{aligned}\check{\Sigma}_w&=\frac{1}{i_1!\cdots i_k!}\check{\Sigma}_{l_1}^{⊎ i_1}⊎\cdots⊎\check{\Sigma}_{l_k}^{⊎ i_k}\\&=w+\sum_{v<w,(v)=(w)}c_v v.\end{aligned}$$

In particular, for any $l_j\in\mathcal{L}ynY$, $\check{\Sigma}_{l_j}$ is lower triangular:

$$\check{\Sigma}_{l_j}=l_j+\sum_{v<l_j,(v)=(l_j)}c_v v.$$

Hence, $\check{\Sigma}_w = \chi_w + \chi'_w$, where χ'_w is a proper polynomial of $\mathbb{Q}\langle Y\rangle$ of rational positive coefficients. We deduce then the support of χ_w contains words which are less than w and $\langle \chi_w \mid w\rangle = 1$. Thus, the proper polynomial χ_w of rational positive coefficients is lower triangular:

$$\begin{aligned} \chi_w &= w + \sum_{v<w,(v)=(w)} c_v v, \\ \Rightarrow \quad \forall l \in \mathcal{L}ynY, \quad \chi_l &= l + \sum_{v<l,(v)=(l)} c_v v. \end{aligned}$$

Then the expected results follow.

(4) By duality, for $w \in Y^*$, the proper polynomial ξ_w is upper triangular. In particular, for any $l \in \mathcal{L}ynY$, the proper polynomial ξ_l is upper triangular:

$$\xi_l = l + \sum_{v>l,(v)=(l)} d_v v.$$

Hence, the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ is free and its elements verify an analogue of the generalized criterion of Friedrichs:

- for $w \in \mathcal{L}ynY$, one has $\langle \chi_w \mid \xi_l\rangle = \delta_{w,l}$,
- for $w \notin \mathcal{L}ynY$, $w = l_1 \cdots l_n$ with $l_1, \dots, l_n \in \mathcal{L}ynY$ and $l_1 > \cdots > l_n$, one has $\langle \chi_w \mid \xi_l\rangle = \langle \chi_{l_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \chi_{l_n} \mid \xi_l\rangle = 0$.

Moreover, the polynomials ξ_l's are primitive: by Corollary 5.1(3), one has

$$\begin{aligned} \Delta_{\sqcup\!\sqcup}(\xi_l) &= \sum_{u,v\in Y^*} \langle u \sqcup\!\sqcup v \mid \xi_l\rangle u \otimes v \\ &= \sum_{u\in Y^+} \langle u \sqcup\!\sqcup 1_{Y^*} \mid \xi_l\rangle u \otimes 1_{Y^*} + \sum_{v\in Y^+} \langle 1_{Y^*} \sqcup\!\sqcup v \mid \xi_l\rangle 1_{Y^*} \otimes v \\ &\quad + \sum_{u,v\in Y^+} \langle u \sqcup\!\sqcup v \mid \xi_l\rangle u \otimes v + \langle 1_{Y^*} \sqcup\!\sqcup 1_{Y^*} \mid \xi_l\rangle 1_{Y^*} \otimes 1_{Y^*} \\ &= \xi_l \otimes 1_{Y^*} + 1_{Y^*} \otimes \xi_l. \end{aligned}$$

Because, after decomposing u and v on the basis $\{\chi_l\}_{l\in\mathcal{L}ynY}$ and by the previous criterion, the third term is vanished. The last one is also vanished since the ξ_l's are proper. By a theorem of Viennot, we obtain then the expected result.

□

5.3 Proof of Theorem 2.2

Applying the tensor product of isomorphisms H ⊗ Id (Proposition 2.1) on the diagonal series (Corollary 5.1(ii)), the infinite factorization, by Lyndon words, of the noncommutative generating series of harmonic sums follows:[32]

$$\mathrm{H}(N) = \sum_{w \in Y^*} \mathrm{H}_w(N)\, w = \prod_{l \in \mathcal{L}ynY}^{\searrow} \exp\big(\mathrm{H}_{\check{\Sigma}_l}(N)\, \Sigma_l\big). \tag{5.1}$$

Appendix 2: Polysystem and differential realization

To facilitate reading, the following results are placed in this appendix which can be skipped by readers already familiar with the techniques developed by Fliess (and adapted by us for studies in this paper).

6.1 Polysystem and convergence criterion

6.1.1 Serial estimates from above

Here, generalizing a little, $\mathbb{K}$ is supposed a $\mathbb{C}$-algebra and a complete normed vector space equipped with a norm denoted by $\|.\|$.

For any $n \in \mathbb{N}$, $X^{\geq n}$ denotes the set of words over X of length greater than or equal to n. The set of formal power series (resp. polynomials) on X, is denoted by $\mathbb{K}\langle\langle X\rangle\rangle$ (resp. $\mathbb{K}\langle X\rangle$).

Definition 6.1 [22, 29] Let ξ, χ be real positive functions over X^*. Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$.

(1) S will be said to be ξ*-exponentially bounded from above* if it verifies

$$\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n}, \quad \|\langle S \mid w\rangle\| \leq K \frac{\xi(w)}{|w|!}.$$

We denote by $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ which are ξ-exponentially bounded from above.

(2) S verifies the χ*-growth condition* if it satisfies

$$\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n}, \quad \|\langle S \mid w\rangle\| \leq K \chi(w)\, |w|!.$$

We denote by $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ verifying the χ-growth condition.

Lemma 6.1 *We have*

$$R = \sum_{w \in X^*} |w|!\, w \quad \Rightarrow \quad \big\langle R^{\sqcup\!\sqcup 2} \,\big|\, w\big\rangle = \sum_{\substack{u,v \in X^* \\ \mathrm{supp}(u \sqcup\!\sqcup v) \ni w}} |u|!\, |v|! \leq 2^{|w|}\, |w|!.$$

[32] This proof omitted in previous versions uses mainly the results presented in this appendix that have not been published earlier but have already been presented at various workshops. It is an analogous way to obtain the infinite factorization, by Lyndon words over the alphabet X, of the noncommutative generating series of polylogarithms (see Theorem 2.3) by applying the tensor product of isomorphisms $\mathrm{Li} \otimes \mathrm{Id}$ (see Proposition 2.1) on the diagonal series, over X.

Proof One has

$$\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\sqcup\!\sqcup v)\ni w}} |u|!\,|v|! = \sum_{k=0}^{|w|} \sum_{\substack{|u|=k,\,|v|=|w|-k\\ \mathrm{supp}(u\sqcup\!\sqcup v)\ni w}} k!(|w|-k)!$$
$$= \sum_{k=0}^{|w|} \binom{|w|}{k} k!(|w|-k)!$$
$$= \sum_{k=0}^{|w|} |w|! = (1+|w|)\,|w|!.$$

By induction on the length of w, one has

$$1+|w| \leq 2^{|w|}.$$

The expected result follows. □

Proposition 6.1 *Let S_1 and S_2 verify the growth condition. Then S_1+S_2 and $S_1 \sqcup\!\sqcup S_2$ also verify the growth condition.*

Proof The proof for S_1+S_2 is immediate.

Next, since $\|\langle S_i \mid w\rangle\| \leq K_i\chi_i(w)\,|w|!$, for $i=1$ or 2 and for $w\in X^*$, then[33]

$$\langle S_1 \sqcup\!\sqcup S_2 \mid w\rangle = \sum_{\mathrm{supp}(u\sqcup\!\sqcup v)\ni w} \langle S_1 \mid u\rangle\langle S_2 \mid v\rangle,$$
$$\Rightarrow \quad \|\langle S_1 \sqcup\!\sqcup S_2 \mid w\rangle\| \leq K_1K_2 \sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\sqcup\!\sqcup v)\ni w}} \big(\chi_1(u)\,|u|!\big)\big(\chi_2(v)\,|v|!\big).$$

Let $K=K_1K_2$ and let χ be a real positive function over X^* such that

$$\forall w\in X^*,\quad \chi(w)=\max\big\{\chi_1(u)\chi_2(v) \mid u,v\in X^* \text{ and } \mathrm{supp}(u\sqcup\!\sqcup v)\ni w\big\}.$$

With the notations in Lemma 6.1, we get

$$\|\langle S_1 \sqcup\!\sqcup S_2 \mid w\rangle\| \leq K\chi(w)\big\langle R^{\sqcup\!\sqcup 2} \mid w\big\rangle.$$

Hence, $S_1 \sqcup\!\sqcup S_2$ verifies the χ'-growth condition with χ' defined as

$$\chi'(w) = 2^{|w|}\chi(w).$$

□

Definition 6.2 [22, 29] Let ξ be a real positive function defined over X^*, S will be said ξ-*exponentially continuous* if it is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$. The set of formal power series which are ξ-exponentially continuous is denoted by

$$\mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle.$$

[33] $\langle S_1 \sqcup\!\sqcup S_2 \mid w\rangle$ is the coefficient of the word w in the power series $S_1 \sqcup\!\sqcup S_2$.

Lemma 6.2 [22, 29] *For any real positive function ξ defined over X^*, we have $\mathbb{K}\langle X\rangle \subset \mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle$. Otherwise, for $\xi = 0$, we get $\mathbb{K}\langle X\rangle = \mathbb{K}^{0-ec}\langle\langle X\rangle\rangle$. Hence, any polynomial is 0-exponentially continuous.*

Proposition 6.2 [22, 29] *Let ξ, χ be real positive functions over X^* and let $P \in \mathbb{K}\langle X\rangle$.*

(1) *Let $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$. The right residual of S by P belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.*
(2) *Let $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$. The concatenation SR belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$.*

Proof

(1) Since $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n},\quad \|\langle S \mid w\rangle\| \leq K\frac{\xi(w)}{|w|!}.$$

If $u \in \mathrm{supp}(P) := \{w \in X^* \mid \langle P \mid w\rangle \neq 0\}$ then, for any $w \in X^*$, one has $\langle S \triangleright u \mid w\rangle = \langle S \mid uw\rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$:

$$\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n},\quad \|\langle S \triangleright u \mid w\rangle\| \leq \big[K\xi(u)\big]\frac{\xi(w)}{|w|!}.$$

It follows that $S \triangleright P$ is $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ by taking $K_1 = K\max_{u\in\mathrm{supp}(P)}\xi(u)$.
(2) Since $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n},\quad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

Let $v \in \mathrm{supp}(P)$ such that $v \neq \epsilon$. Since, for any $w \in X^*$, Rv belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ and one has $\langle Rv \mid w\rangle = \langle R \mid v \triangleleft w\rangle$:

$$\begin{aligned}\exists K \in \mathbb{R}_+,\ \exists n \in \mathbb{N},\ \forall w \in X^{\geq n},\quad \|\langle R \mid v \triangleleft w\rangle\| &\leq K\chi(v \triangleleft w)(|w| - |v|)!\\ &\leq K\,|w|\,\frac{\chi(w)}{\chi(v)}.\end{aligned}$$

Note that if $v \triangleleft w = 0$ then $\langle Rv \mid w\rangle = 0$ and the previous conclusion holds. It follows that RP is $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ by taking $K_2 = K\min_{v\in\mathrm{supp}(P)}\chi(v)^{-1}$. □

Proposition 6.3 [22, 29] *Two real positive morphisms over X^*, ξ and χ are assumed to verify the condition*

$$\sum_{x\in X}\chi(x)\xi(x) < 1.$$

Then for any $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$, F is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.

Proof If ξ, χ verify the upper bound condition then the following power series:

$$\sum_{w\in X^*}\chi(w)\xi(w) = \left(\sum_{x\in X}\chi(x)\xi(x)\right)^*$$

is well defined. If $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ and $C \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then there exist $K_i \in \mathbb{R}_+$ and $n_i \in \mathbb{N}$ such that for any $w \in X^{\geq n_i}$, $i = 1, 2$, one has

$$\|\langle F \mid w\rangle\| \leq K_1 \chi(w)\,|w|! \quad \text{and} \quad \|\langle C \mid w\rangle\| \leq K_2 \frac{\xi(w)}{|w|!}.$$

Hence,

$$\forall w \in X^*,\ |w| \geq \max\{n_1, n_2\}, \quad \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1 K_2 \chi(w)\xi(w),$$

$$\Rightarrow \sum_{w\in X^*} \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1 K_2 \sum_{w\in X^*} \chi(w)\xi(w) = K_1 K_2 \Big(\sum_{x\in X} \chi(x)\xi(x)\Big)^*. \qquad \square$$

6.1.2 Upper bounds à la Cauchy

Let $q_1, \ldots, q_n$ be commutative indeterminates over $\mathbb{C}$. The algebra of formal power series (resp. polynomials) over $\{q_1, \ldots, q_n\}$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[\![q_1, \ldots, q_n]\!]$ (resp. $\mathbb{C}[q_1, \ldots, q_n]$).

Definition 6.3 [22, 29] Let

$$f = \sum_{i_1,\ldots,i_n \geq 0} f_{i_1,\ldots,i_n} q_1^{i_1} \cdots q_n^{i_n} \in \mathbb{C}[\![q_1, \ldots, q_n]\!].$$

We set

$$E(f) := \big\{\rho \in \mathbb{R}_+^n : \exists C_f \in \mathbb{R}_+ \text{ s.t. } \forall i_1, \ldots, i_n \geq 0, |f_{i_1,\ldots,i_n}|\, \rho_1^{i_1} \cdots \rho_n^{i_n} \leq C_f\big\}.$$

$$\breve{E}(f): \quad \text{the interior of } E(f) \text{ in } \mathbb{R}^n.$$

$$\mathrm{CV}(f) := \big\{q \in \mathbb{C}^n : (|q_1|, \ldots, |q_n|) \in \breve{E}(f)\big\}: \quad \text{the convergence domain of } f.$$

The power series f is said to be *convergent* if $\mathrm{CV}(f) \neq \emptyset$. Let $\mathcal{U}$ be an open domain in $\mathbb{C}^n$ and let $q \in \mathbb{C}^n$. The power series f is said to be convergent on q (resp. over $\mathcal{U}$) if $q \in \mathrm{CV}(f)$ (resp. $\mathcal{U} \subset \mathrm{CV}(f)$). We set

$$\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!] = \big\{f \in \mathbb{C}[\![q_1, \ldots, q_n]\!] : \mathrm{CV}(f) \neq \emptyset\big\}.$$

Let $q \in \mathrm{CV}(f)$. There exist constants C_f, ρ and $\bar{\rho}$ such that

$$|q_1| < \bar{\rho} < \rho, \ldots, |q_n| < \bar{\rho} < \rho \quad \text{and} \quad |f_{i_1,\ldots,i_n}|\, \rho_1^{i_1} \cdots \rho_n^{i_n} \leq C_f,$$

for $i_1, \ldots, i_n \geq 0$. The *convergence modulus* of f at q is $(C_f, \rho, \bar{\rho})$.

Suppose that $\mathrm{CV}(f) \neq \emptyset$ and let $q \in \mathrm{CV}(f)$. If $(C_f, \rho, \bar{\rho})$ is a convergence modulus of f at q then $|f_{i_1,\ldots,i_n} q_1^{i_1} \cdots q_n^{i_n}| \leq C_f (\bar{\rho}_1/\rho_1)^{i_1} \cdots (\bar{\rho}_1/\rho_1)^{i_n}$. Hence, at q, the power series f is majored termwise by

$$C_f \prod_{k=0}^{m} \Big(1 - \frac{\bar{\rho}_k}{\rho_k}\Big)^{-1}. \tag{6.1}$$

Hence, f is uniformly absolutely convergent in $\{q \in \mathbb{C}^n : |q_1| < \bar{\rho}, \dots, |q_n| < \bar{\rho}\}$ which is an open domain in $\mathbb{C}^n$. Thus, $\mathrm{CV}(f)$ is an open domain in $\mathbb{C}^n$, since the partial derivation $D_1^{j_1} \cdots D_n^{j_n} f$ is estimated by

$$\|D_1^{j_1} \cdots D_n^{j_n} f\| \le C_f \frac{\partial^{j_1+\cdots+j_n}}{\partial \bar{\rho}^{j_1+\cdots+j_n}} \prod_{k=0}^{m} \left(1 - \frac{\bar{\rho}_k}{\rho_k}\right)^{-1}. \tag{6.2}$$

Proposition 6.4 [22] *We have*

$$\mathrm{CV}(f) \subset \mathrm{CV}\big(D_1^{j_1} \cdots D_n^{j_n} f\big).$$

Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]$. Let $\{A_i\}_{i=0,1}$ be a polysystem defined as follows:

$$A_i(q) = \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \tag{6.3}$$

where for any $j = 1, \dots, n$, $A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]$.

Lemma 6.3 [20] *For $i = 0, 1$ and $j = 1, \dots, n$, one has $A_i \circ q_j = A_i^j(q)$. Thus,*

$$\forall i = 0, 1, \quad A_i(q) = \sum_{j=1}^{n} (A_i \circ q_j) \frac{\partial}{\partial q_j}.$$

Let $(\rho, \bar{\rho}, C_f), \{(\rho, \bar{\rho}, C_i)\}_{i=0,1}$ be, respectively, the convergence modulus at

$$q \in \mathrm{CV}(f) \bigcap_{\substack{i=0,1 \\ j=1,\dots,n}} \mathrm{CV}\big(A_i^j\big) \tag{6.4}$$

of f and $\{A_i^j\}_{j=1,\dots,n}$. Let us consider the following monoid morphisms:

$$\mathcal{A}(\epsilon) = \text{identity} \quad \text{and} \quad C(\epsilon) = 1, \tag{6.5}$$

$$\forall w = v x_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v) A_i \quad \text{and} \quad C(w) = C(v) C_i. \tag{6.6}$$

Lemma 6.4 [19] *For any word w, $\mathcal{A}(w)$ is continuous over $\mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]$ and, for any $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]$, one has*

$$\mathcal{A}(w) \circ (fg) = \sum_{u,v \in X^*} \langle u \sqcup\!\sqcup v \mid w \rangle \big(\mathcal{A}(u) \circ f\big)\big(\mathcal{A}(v) \circ g\big).$$

These notations are extended, by linearity, to $\mathbb{K}\langle X \rangle$ and we will denote $\mathcal{A}(w) \circ f_{|q}$ the evaluation of $\mathcal{A}(w) \circ f$ at q.

Definition 6.4 [19] Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]$. The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f is given by

$$\sigma f := \sum_{w \in X^*} \mathcal{A}(w) \circ f \, w \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_n]\!]\langle\langle X \rangle\rangle.$$

Then the following generating series is called *Fliess generating series* of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q:

$$\sigma f_{|_q} := \sum_{w\in X^*} \mathcal{A}(w)\circ f_{|q}\, w \in \mathbb{C}\langle\langle X\rangle\rangle.$$

Lemma 6.5 [19] *Let $\{A_i\}_{i=0,1}$ be a polysystem. Then, the map*

$$\sigma : \left(\mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!], .\right) \longrightarrow \left(\mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!]\langle\langle X\rangle\rangle, ш\right),$$

is an algebra morphism, i.e. for any $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!]$ and $\mu,\nu\in\mathbb{C}$, one has

$$\sigma(\nu f+\mu h)=\nu\sigma f+\mu\sigma g \quad \text{and} \quad \sigma(fg)=\sigma f \,ш\, \sigma g.$$

Lemma 6.6 [20] *Let $\{A_i\}_{i=0,1}$ be a polysystem and let $f\in\mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!]$. Then*

$$\forall x_i\in X,\quad \sigma(A_i\circ f)=x_i \triangleleft \sigma f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!]\langle\langle X\rangle\rangle$$
$$\forall w\in X^*,\quad \sigma\big(\mathcal{A}(w)\circ f\big)=w \triangleleft \sigma f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\dots,q_n]\!]\langle\langle X\rangle\rangle.$$

Lemma 6.7 [22] *Let $\tau=\min_{1\le k\le n}\rho_k$ and $r=\max_{1\le k\le n}\bar\rho_k/\rho_k$. We have*

$$\|\mathcal{A}(w)\circ f\| \le C_f\frac{(n+1)}{(1-r)^n}\frac{C(w)\,|w|!}{\binom{n+|w|-1}{|w|}}\left[\frac{n}{\tau(1-r)^{n+1}}\right]^{|w|}$$
$$\le C_f\frac{(n+1)}{(1-r)^n}C(w)\left[\frac{n}{\tau(1-r)^{n+1}}\right]^{|w|}|w|!.$$

Theorem 6.1 [22] *Let $K=C_f(n+1)(1-r)^{-n}$ and let χ be the real positive function defined over X^* by*

$$\forall i=0,1,\quad \chi(x_i)=\frac{C_i n}{\tau(1-r)^{(n+1)}}.$$

Then the generating series σf of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f satisfies the χ-growth condition.

It is the same for the Fliess generating series $\sigma f_{|q}$ of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q.

6.2 Polysystems and nonlinear differential equation

6.2.3 Nonlinear differential equation (with three singularities)

Let us consider the following singular inputs:[34]

$$u_0(z):=z^{-1} \quad \text{and} \quad u_1(z):=(1-z)^{-1}, \tag{6.7}$$

[34] These singular inputs are not included in the studies of Fliess motivated, in particular, by the renormalization of $y(z)$ at $+\infty$ [19, 20].

and the following nonlinear dynamical system:[35]

$$\begin{cases} y(z) &= f(q(z)), \\ \dot{q}(z) &= A_0(q)\, u_0(z) + A_1(q)\, u_1(z), \\ q(z_0) = q_0, \end{cases} \tag{6.8}$$

where the state $q = (q_1, \ldots, q_n)$ belongs to the complex analytic manifold of dimension n, q_0 is the initial state, the observation f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (6.3).

Definition 6.5 [32] The following power series is called *transport operator* of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f:

$$\mathcal{T} := \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, \mathcal{A}(w).$$

By the factorization of the monoid by Lyndon words, we have [32]

$$\mathcal{T} = \left(\alpha_{z_0}^{z} \otimes \mathcal{A}\right)\left(\sum_{w \in X^*} w \otimes w\right) = \prod_{l \in \mathcal{L}ynX} \exp\left[\alpha_{z_0}^{z}(S_l)\, \mathcal{A}(\check{S}_l)\right]. \tag{6.9}$$

Let us consider again the Chen generating series $S_{z_0 \rightsquigarrow z}$ given in (2.39) of the differential forms involved in (DE) of Example 1, i.e.

$$\omega_0(z) = u_0(z)\, dz \quad \text{and} \quad \omega_1(z) = u_1(z)\, dz, \tag{6.10}$$

verifying the upper bound conditions given on (2.45).

6.2.4 Asymptotic behavior of the successive differentiation of the output via extended Fliess fundamental formula

The Fliess fundamental formula [19] can be then extended as follows:

Theorem 6.2 [29] *We have*

$$\begin{aligned} y(z) &= \mathcal{T} \circ f_{|q_0} \\ &= \sum_{w \in X^*} \langle S_{z_0 \rightsquigarrow z} \mid w \rangle \big\langle \mathcal{A}(w) \circ f_{|q_0} \bigm| w \big\rangle \\ &= \langle \sigma f_{|q_0} \parallel S_{z_0 \rightsquigarrow z} \rangle. \end{aligned}$$

By the factorization indexed by Lyndon words of the Lie exponential series L, the expansions of the output y of nonlinear dynamical system with singular inputs follow

[35] Any differential equation with three singularities in $\{a, b, c\}$, via homographic transformation

$$\frac{(z-a)(c-b)}{(z-b)(c-a)},$$

can be changed into a differential equation with three singularities in $\{0, 1, +\infty\}$ (the singularities of homographic transformations belong to the group $\mathcal{G}$).

Corollary 6.1 [29]

$$
\begin{aligned}
y(z) &= \sum_{w\in X^*} g_w(z)\,\mathcal{A}(w)\circ f_{|q_0}\\
&= \sum_{k\geq 0}\sum_{n_1,\dots,n_k\geq 0} g_{x_0^{n_1}x_1\cdots x_0^{n_k}x_1}(z)\ \mathrm{ad}_{A_0}^{n_1} A_1\cdots \mathrm{ad}_{A_0}^{n_k} A_1 e^{\log z A_0}\circ f_{|q_0}\\
&= \prod_{l\in \mathcal{L}ynX} \exp\bigl(g_{S_l}(z)\,\mathcal{A}(\check{S}_l)\circ f_{|q_0}\bigr)\\
&= \exp\Bigl(\sum_{w\in X^*} g_w(z)\,\mathcal{A}\bigl(\pi_1(w)\bigr)\circ f_{|q_0}\Bigr),
\end{aligned}
$$

where, for any word w in X^*, g_w belongs to the polylogarithm algebra.

Since $S_{z_0\rightsquigarrow z}=\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ and since $\sigma f_{|q_0}$ and $\mathrm{L}(z_0)^{-1}$ are invariant by $\partial = d/dz$ then, for any integer l, one has

$$\partial^l y(z)=\langle\sigma f_{|q_0}\parallel \partial^l S_{z_0\rightsquigarrow z}\rangle=\langle\sigma f_{|q_0}\parallel \partial^l \mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle. \tag{6.11}$$

With the notations of Proposition 2.3, we get

$$\partial^l y(z)=\langle\sigma f_{|q_0}\parallel \bigl[P_l(z)\mathrm{L}(z)\bigr]\mathrm{L}(z_0)^{-1}\rangle=\langle\sigma f_{|q_0}\triangleright P_l(z)\parallel \mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle. \tag{6.12}$$

For $z_0=\varepsilon\to 0^+$, the asymptotic behavior and the renormalization at $z=1$ of $\partial^l y(z)$ (or the asymptotic expansion and the renormalization of its Taylor coefficients at $+\infty$) are deduced from Proposition 2.5 and extend a little bit the results of [29] as follows:

Corollary 6.2 *For any integer l, we have*

$$
\begin{aligned}
\partial^l y(1) &\underset{\varepsilon\to 0^+}{\widetilde{\quad}} \langle\sigma f_{|q_0}\triangleright P_l(1-\varepsilon)\parallel e^{-x_1\log\varepsilon}\,Z_{\sqcup\!\sqcup}\,e^{-x_0\log\varepsilon}\rangle\\
&= \sum_{w\in X^*}\langle\mathcal{A}(w)\circ f_{|q_0}\mid w\rangle\langle P_l(1-\varepsilon)\,e^{-x_1\log\varepsilon}\,Z_{\sqcup\!\sqcup}\,e^{-x_0\log\varepsilon}\mid w\rangle.
\end{aligned}
$$

Corollary 6.3 *The differentiation of order $l\in\mathbb{N}$ of the output y of the dynamical system (6.8) is a $\mathcal{C}$-combination of the elements g belonging to the polylogarithm algebra. If its ordinary Taylor expansion exists then the coefficients of this expansion belong to the algebra of harmonic sums and there exist algorithmically computable coefficients $a_i\in\mathbb{Z}$, $b_i\in\mathbb{N}$ and c_i belonging to the $\mathbb{C}$-algebra generated by $\mathcal{Z}$ and by Euler's γ constant, such that*

$$\partial^l y(z)=\sum_{n\geq 0} y_n^{(l)} z^n,\qquad y_n^{(l)}\underset{n\to\infty}{\widetilde{\quad}}\textstyle\sum_{i\geq 0} c_i n^{a_i}\log^{b_i} n.$$

6.3 Differential realization

6.3.5 Differential realization

Definition 6.6 The *Lie rank* of a formal power series $S\in\mathbb{K}\langle\langle X\rangle\rangle$ is the dimension of the vector space generated by

$$\{S\triangleright \Pi\mid \Pi\in\mathcal{L}ie_{\mathbb{K}}\langle X\rangle\},\quad\text{ or respectively by }\{\Pi\triangleleft S\mid \Pi\in\mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}.$$

Definition 6.7 Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ and let us put

$$\mathrm{Ann}(S) := \{\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle \mid S \triangleright \Pi = 0\},$$
$$\mathrm{Ann}^{\perp}(S) := \left\{Q \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢) \mid Q \triangleright \mathrm{Ann}(S) = 0\right\}.$$

It is immediate that $\mathrm{Ann}^{\perp}(S) \ni S$ and it follows that (see [20, 45]).

Lemma 6.8 *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. If S is of finite Lie rank, d, then the dimension of* $\mathrm{Ann}^{\perp}(S)$ *equals d.*

By Lemma 2.3, the residuals are derivations for shuffle product. Then,

Lemma 6.9 *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. Then:*

(1) *For any Q_1 and $Q_2 \in \mathrm{Ann}^{\perp}(S)$, one has $Q_1 ⧢ Q_2 \in \mathrm{Ann}^{\perp}(S)$.*
(2) *For any $P \in \mathbb{K}\langle X\rangle$ and $Q_1 \in \mathrm{Ann}^{\perp}(S)$, one has $P \triangleleft Q_1 \in \mathrm{Ann}^{\perp}(S)$.*

Definition 6.8 [20] The formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is *differentially produced* if there exist

- an integer d,
- a power series $f \in \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$,
- a homomorphism $\mathcal{A}$ from X^* maps to the algebra of differential operators generated by

$$\mathcal{A}(x_i) = \sum_{j=1}^{d} A_i^j(\bar{q}_1, \ldots, \bar{q}_d)\frac{\partial}{\partial \bar{q}_j},$$

where, for any $j = 1, \ldots, d$, $A_i^j(\bar{q}_1, \ldots, \bar{q}_d)$ belongs to $\mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$ such that

$$\forall w \in X^*, \quad \langle S \mid w\rangle = \mathcal{A}(w) \circ f_{|0}.$$

The couple $(\mathcal{A}, f)$ is called the *differential representation* of S of dimension d.

Proposition 6.5 [45] *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. If S is differentially produced then it verifies the growth condition and its Lie rank is finite.*

Proof Let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then, by the notations of Definition 6.4, we get

$$\sigma f_{|0} = S = \sum_{w \in X^*} \big(\mathcal{A}(w) \circ f\big)_{|0}\, w.$$

For any $j = 1, \ldots, d$, we put

$$T_j = \sum_{w \in X^*} \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}\, w$$
$$\iff \quad \forall w \in X^*, \quad \langle T_j \mid w\rangle = \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}.$$

Firstly, by Theorem 6.1, the generating series σf verifies the growth condition. Secondly, for any $\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w\rangle = \langle \sigma f \mid \Pi w\rangle = \mathcal{A}(\Pi w) \circ f = \mathcal{A}(\Pi) \circ \big(\mathcal{A}(w) \circ f\big).$$

Since $\mathcal{A}(\Pi)$ is a derivation over $\mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$:

$$\mathcal{A}(\Pi) = \sum_{j=1}^{d} \big(\mathcal{A}(\Pi) \circ \bar{q}_j\big) \frac{\partial}{\partial \bar{q}_j},$$

$$\Rightarrow \quad \mathcal{A}(\Pi) \circ \big(\mathcal{A}(w) \circ f\big) = \sum_{j=1}^{d} \big(\mathcal{A}(\Pi) \circ \bar{q}_j\big) \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}$$

then we deduce that

$$\forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w\rangle = \sum_{j=1}^{d} \big(\mathcal{A}(\Pi) \circ \bar{q}_j\big) \langle T_j \mid w\rangle,$$

$$\iff \quad \sigma f \triangleright \Pi = \sum_{j=1}^{d} \big(\mathcal{A}(\Pi) \circ \bar{q}_j\big)\, T_j.$$

That means $\sigma f \triangleright \Pi$ is $\mathbb{K}$-linear combination of $\{T_j\}_{j=1,\ldots,d}$ and the dimension of the vector space $\operatorname{span}\{\sigma f \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$ is less than or equal to d. □

6.3.6 Fliess' local realization theorem

Proposition 6.6 [45] *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ be such that its Lie rank equals d. Then there exists a basis $S_1, \ldots, S_d \in \mathbb{K}\langle\langle X\rangle\rangle$ of $(\operatorname{Ann}^{\perp}(S), ⧢) \cong (\mathbb{K}[\![S_1, \ldots, S_d]\!], ⧢)$ such that the S_i's are proper and for any $R \in \operatorname{Ann}^{\perp}(S)$, one has*

$$R = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \cdots i_d!} S_1^{⧢ i_1} ⧢ \cdots ⧢ S_d^{⧢ i_d},$$

where the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$ belong to $\mathbb{K}$ and $r_{0,\ldots,0} = \langle R \mid 1_{X^*}\rangle$.

Proof By Lemma 6.8, such a basis exists. More precisely, since the Lie rank of S is d then there exist $P_1, \ldots, P_d \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ such that $S \triangleright P_1, \ldots, S \triangleright P_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ are $\mathbb{K}$-linearly independent. By duality, there exist $S_1, \ldots, S_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ such that

$$\forall i, j = 1, \ldots, d, \quad \langle S_i \mid P_j\rangle = \delta_{i,j}, \quad \text{and} \quad R = \prod_{i=1}^{d} \exp(S_i\, P_i).$$

Expending this product, one obtains, via Poincaré–Birkhoff–Witt theorem, the expected expression for the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$:

$$r_{i_1,\ldots,i_d} = \big\langle R \mid P_1^{i_1} \cdots P_d^{i_d}\big\rangle.$$

Hence, $(\operatorname{Ann}^{\perp}(S), ⧢)$ is generated by $S_1, \ldots, S_d$. □

With the notations of Proposition 6.6, one has, respectively, the following.

Corollary 6.4 *If $S \in \mathbb{K}[S_1, \dots, S_d]$ then, for any $i = 0, 1$ and for any $j = 1, \dots, d$, one has $x_i \triangleleft S \in \mathrm{Ann}^{\perp}(S) = \mathbb{K}[S_1, \dots, S_d]$.*

Corollary 6.5 *The power series S verifies the growth condition if and only if, for any $i = 1, \dots, d$, S_i also verifies the growth condition.*

Proof Assume there exists $j \in [1, \dots, d]$ such that S_j does not verify the growth condition. Since $S \in \mathrm{Ann}^{\perp}(S)$ then using the decomposition of S on $S_1, \dots, S_d$, one obtains a contradiction with the fact that S verifies the growth condition.

Conversely, using Proposition 6.1, we get the expected results. □

Theorem 6.3 [20] *The formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it verifies the χ-growth condition.*

Proof By Proposition 6.5, one gets a direct proof.

Conversely, since the Lie rank of S equals d then by Proposition 6.6, by putting $\sigma f_{|0} = S$ and, for any $j = 1, \dots, d$, $\sigma \bar{q}_i = S_i$,

(1) we choose the observation f as follows:

$$f(\bar{q}_1, \dots, \bar{q}_d) = \sum_{i_1, \dots, i_d \geq 0} \frac{r_{i_1, \dots, i_n}}{i_1! \cdots i_d!} \bar{q}_1^{i_1} \cdots \bar{q}_d^{i_d} \in \mathbb{K}[\![\bar{q}_1, \dots, \bar{q}_d]\!]$$

such that

$$\sigma f_{|0}(\bar{q}_1, \dots, \bar{q}_d) = \sum_{i_1, \dots, i_d \geq 0} \frac{r_{i_1, \dots, i_n}}{i_1! \cdots i_d!} (\sigma \bar{q}_1)^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (\sigma \bar{q}_d)^{\sqcup\!\sqcup i_d},$$

(2) it follows that, for $i = 0, 1$ and for $j = 1, \dots, d$, the residual $x_i \triangleleft \sigma \bar{q}_j$ belongs to $\mathrm{Ann}^{\perp}(\sigma f_{|0})$ (see also Lemma 6.9),
(3) since σf verifies the χ-growth condition then, by Corollary 6.5, the generating series $\sigma \bar{q}_j$ and $x_i \triangleleft \sigma \bar{q}_j$ (for $i = 0, 1$ and for $j = 1, \dots, d$) verify also the growth condition. We then take (see Lemma 6.6)

$$\forall i = 0, 1,\ \forall j = 1, \dots, d, \quad \sigma A_j^i(\bar{q}_1, \dots, \bar{q}_d) = x_i \triangleleft \sigma \bar{q}_j,$$

by expressing σA_j^i on the basis $\{\sigma \bar{q}_i\}_{i=1,\dots,d}$ of $\mathrm{Ann}^{\perp}(\sigma f_{|0})$,
(4) the homomorphism $\mathcal{A}$ is then determined as follows:

$$\forall i = 0, 1, \quad \mathcal{A}(x_i) = \sum_{j=0}^{d} A_j^i(\bar{q}_1, \dots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j},$$

where (see Lemma 6.3)

$$\forall i = 0, 1,\ \forall j = 1, \dots, d, \quad A_j^i(\bar{q}_1, \dots, \bar{q}_d) = \mathcal{A}(x_i) \circ \bar{q}_j.$$

Thus, $(\mathcal{A}, f)$ provides a differential representation[36] of dimension d of S. □

Moreover, one also has the following.

[36] In [20, 45], the reader can find the discussion on the *minimal* differential representation.

Theorem 6.4 [20] *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ be a differentially produced formal power series. If $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ are two differential representations of dimension n of S then there exists a continuous and convergent automorphism h of $\mathbb{K}$ such that*

$$\forall w \in X^*,\ \forall g \in \mathbb{K}, \quad h\big(\mathcal{A}(w) \circ g\big) = \mathcal{A}'(w) \circ \big(h(g)\big)$$

and

$$f' = h(f).$$

Since any rational power series (resp. polynomial), verifies the growth condition and its Lie rank is less than or equal to its Hankel rank which is finite [20] then

Corollary 6.6 *Any rational power series and any polynomial over X with coefficients in $\mathbb{K}$ are differentially produced.*

Remark 6.1

(1) Note that, by Corollary 6.4, if S is a polynomial over X then for any $j = 1, \ldots, d$, S_j is a polynomial. Therefore, for $i = 0, 1$ and $j = 1, \ldots, d$, $x_i \triangleleft S$ is also a polynomial over X. In this case, let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then f and $\{A_j^i\}_{j=1,\ldots,d}^{i=0,1}$ are obviously polynomials on $\bar{q}_1, \ldots, \bar{q}_d$ and the Lie algebra generated by $\{\mathcal{A}(x_i)\}^{i=0,1}$ is nilpotent.

(2) Note also that, by Theorem 2.6, if S is rational over X of linear representation (λ, μ, η) then the observation $f(q_1, \ldots, q_n)$ equals $\lambda_1 q_1 + \cdots + \lambda_n q_n$ and the polysystem $\{\mathcal{A}(x)\}_{x\in X}$ obtained by putting

$$\forall x_i \in X, \quad \mathcal{A}(x_i) = \sum_{j=1}^{n} \big(\mu(x_i)\big)_j^i \frac{\partial}{\partial q_j}$$

yields a *linear* differential representation not necessarily of minimal dimension [20].

(3) Assume $S \in \mathbb{K}\epsilon \oplus x_0\mathbb{K}\langle\langle X\rangle\rangle x_1$ and S is a differentially produced. If there exists a basis $S_1, \ldots, S_d$ of $(\mathrm{Ann}^{\perp}(S), ⧢) \cong (x_0\mathbb{K}\langle\langle X\rangle\rangle x_1, ⧢)$ such that

$$S = \sum_{i_1,\ldots,i_d \ge 0} r_{i_1,\ldots,i_n} \frac{S_1^{⧢ i_1}}{i_1!} ⧢ \cdots ⧢ \frac{S_d^{⧢ i_d}}{i_d!} \in \big(\mathbb{K}[S_1, \ldots, S_d], ⧢\big), \tag{6.13}$$

then

$$\Sigma := \sum_{i_1,\ldots,i_d \ge 0} r_{i_1,\ldots,i_n} \frac{\Sigma_1^{\uplus i_1}}{i_1!} \uplus \cdots \uplus \frac{\Sigma_d^{\uplus i_d}}{i_d!} \in \big(\mathbb{K}[\Sigma_1, \ldots, \Sigma_d], \uplus\big), \tag{6.14}$$

where

$$\Sigma_i := \pi_Y S_i, \quad \text{for } i = 1, \ldots, d \tag{6.15}$$

It is a generalization of Radford's theorem because [25, 26]:

- If $S \in \mathbb{Q}X$ then (6.13) and (6.14) are decompositions on Radford bases.

- If S is rational then these are *noncommutative partial decompositions*.
  In this case, one has in general $\pi_Y S \neq \Sigma$ but

$$\zeta(S) = \zeta(\Sigma) = \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{\zeta(S_1)^{i_1}}{i_1!} \cdots \frac{\zeta(S_d)^{i_d}}{i_d!} \tag{6.16}$$

$$\text{and} \quad \zeta(S_i) = \zeta(\Sigma_i). \tag{6.17}$$

Thus, these yield also identities on polyzetas at arbitrary weight [27].

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