LECTURE 1. ARTIN APPROXIMATION

DORIN POPESCU

We start with some preliminaries.

A local ring (A, m) is called *Henselian* if the following property holds:

"Let f be a polynomial in one variable Y over A. If $\tilde{y} \in A$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $(\partial f/\partial Y)(\tilde{y}) \notin m$ then there exists a solution $y \in A$ of f in A such that $y \equiv \tilde{y} \mod m$."

One can show that this property is equivalent with the following somehow stronger property:

"Let $f = (f_1, \ldots, f_n)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over A, and D the determinant of the Jacobian matrix $(\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $D(\tilde{y}) \notin m$ then there exists a solution $y \in A^n$ of f in A such that $y \equiv \tilde{y} \mod m$."

Moreover, we may write the above property even in another stronger form:

"Let $f = (f_1, \ldots, f_r)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over A, $r \leq n$ and M be an $r \times r$-minor of the Jacobian matrix $(\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $M(\tilde{y}) \notin m$ then there exists a solution $y \in A^n$ of f in A such that $y \equiv \tilde{y} \mod m$."

Actually the above property follows easily from the previous one because if for example M is the determinant of $(\partial f_i/\partial Y_j)_{1 \leq i,j \leq r}$ then we may add the polynomials $f_i = Y_i - \tilde{y}_i$ for $r < i \leq n$ obtaining a system of n polynomials. Now the determinant of the Jacobian matrix of the new bigger system is M and we may apply the second property.

These are properties which you studied in a previous lecture here, I will remind you the Implicit Function Theorem from Differential or Analytic Geometry.

Theorem 1. *Let $f_i(x_1, \ldots, x_n, Y_1, \ldots, Y_n) = 0$, $1 \leq i \leq n$ be some analytic equations, where $f_i : \mathbf{C}^{2n} \to \mathbf{C}$ are analytic maps in a neighborhood of $(0, 0) \in \mathbf{C}^{2n}$ with the property that $f_i(0, 0) = 0$, $1 \leq i \leq n$, and $det(\partial f_i/\partial Y_j)(0, 0) \neq 0$. Then there exist some unique maps $y_j : \mathbf{C}^n \to \mathbf{C}$, $1 \leq i \leq n$, which are analytic in a neighborhood of $0 \in \mathbf{C}^n$ such that $y_j(0) = 0$ and $f(x, y) = 0$ in a neighborhood of 0.*

The above theorem says in particular that the local ring of all analytic germs of maps defined in a neighborhood of $0 \in \mathbf{C}^n$ in other words the local ring $\mathbf{C}\{x\}$ of convergent power series in some variables x over $\mathbf{C}$, $x = (x_1, \ldots, x_n)$ is Henselian. Other important Henselian rings are the formal power series ring $K[[x]]$, $x = (x_1, \ldots, x_n)$ over a field K, or its subring the algebraic power series ring $K < x >$, whose elements are those formal power series, which satisfy polynomial equations in one variable over $K[x]$.

The Henselian property is very important. For example with its help we may find in $\mathbf{C}[[x]]$ a root of the polynomial $f = Y^s - u$ for all s and all formal power series u with a free term, that is $u \notin (x)$, of $\mathbf{C}[[x]]$. Indeed, let $a \in \mathbf{C}$ be such that $a^s = u(0)$, ($\mathbf{C}$ is algebraically closed field). Then we have $f(a) \in (x)$ and $(\partial f/\partial Y)(a) = sa^{s-1} \notin (x)$ and so f must have a solution in $\mathbf{C}[[x]]$. In particular you may see that the polynomial $g = y^2 - x^2 - x^3$ is irreducible in $\mathbf{C}[x, y]$ but reducible in $\mathbf{C}[[x, y]]$ since there $g = (y - x\rho)(y + x\rho)$, where ρ is the square root of $1 + x$.

A very important property of Henselian rings is the following lemma.

Lemma 2. *(Newton Lemma) Let (A, m) be a Henselian ring, c a positive integer, $f = (f_1, \ldots, f_r)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over A, $r \leq n$, and M an $r \times r$-minor of the Jacobian matrix $J = (\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$ satisfies $f(\tilde{y}) \equiv 0$ mod $M(\tilde{y})^2 m^c$ then there exists a solution $y \in A^n$ of f in A such that $y \equiv \tilde{y}$ mod $M(\tilde{y})m^c$.*

Proof. As above we may reduce our problem to the case $r = n$ adding some new polynomials of the form $Y_i - \tilde{y}_i$. Just to get the idea we consider the case $r = n = 1$, $M = \partial f/\partial Y$. We try to find $z \in A$ such that $f(\tilde{y} + M(\tilde{y})z) = 0$. We may suppose $M(\tilde{y}) \neq 0$, because otherwise $\tilde{y}$ is already a solution of f. Using Taylor's formula we have

$$f(\tilde{y} + M(\tilde{y})Z) = f(\tilde{y}) + (\partial f/\partial Y)(\tilde{y})(M(\tilde{y})Z) + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})Z)^j = 0.$$

By hypothesis we have $f(\tilde{y}) = M(\tilde{y})^2 a$ for some $a \in m^c$. Dividing the above equations by $M(\tilde{y})^2$ we get

$$a + Z + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} Z^j) = 0,$$

where we may apply the Henselian property. Thus there exists $z \in m$ such that

$$a + z + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} z^j) = 0,$$

and so $y = \tilde{y} + M(\tilde{y})z$ is a solution of f. Remains to show that z is in fact in m^c. Suppose that $z \in m^j$ for some $j \geq 1$. Then from the above equation we get

$$z = -a - \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} z^j) \in m^{\min\{c, 2j\}}$$

and it follows by recurrence $z \in m^c$.

Next we give an idea about the case $r = n > 1$. In this case by Taylor's formula we get for $z \in A^n$ as above the system of equations

$$f_k(\tilde{y} + M(\tilde{y})Z) = f_k(\tilde{y}) + \sum_{i=1}^{n} (\partial f_k/\partial Y_i(\tilde{y})(M(\tilde{y})Z_i) +$$

$$\sum_{j>1} (\sum_{j_1, \ldots, j_n \geq 0, j = j_1 + \ldots + j_n} (\partial^j f_k/\partial Y_{j_1} \cdots \partial Y_{j_n})(\tilde{y})(M(\tilde{y})^j Z_1^{j_1} \cdots Z_n^{j_n}) = 0,$$

$k=1,\dots n$. By hypothesis we have $f(\tilde{y}) = M(\tilde{y})^2 a$ for some $a \in m^c A^n$. Using linear algebra there exists a $n \times n$ matrix C over $A[Y]$ such that $JC = MI_n$, I_n being the $n \times n$ unit matrix. Then the above system of equations becomes

$$M(\tilde{y})J(\tilde{y})[C(\tilde{y})a + Z + (\text{terms in } Z \text{ of degree } \geq 2)] = 0.$$

Then it is enough to find z satisfying

$$C(\tilde{y})a + Z + (\text{terms in } Z \text{ of degree } \geq 2) = 0,$$

which follows from the Henselian property. □

Lemma 3. *Let $A = K < x >$, $x = (x_1, \dots, x_n)$, $f = (f_1, \dots, f_r)$ be a system of polynomials in $Y = (Y_1, \dots, Y_n)$ over A, $r \leq n$, and M an $r \times r$-minor of the Jacobian matrix $(\partial f_i / \partial Y_j)$. Let $\hat{y}$ be a solution of f in $\hat{A} = K[[x]]$ such that $M(\hat{y}) \not\equiv 0$ mod (x). Then for any $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in A such that $y^{(c)} \equiv \hat{y}$ mod $(x)^c$.*

Proof. Choose c and an element $\tilde{y}$ in A^n such that $\tilde{y} \equiv \hat{y}$ mod $(x)^c$. Then $f(\tilde{y}) \equiv f(\hat{y}) = 0$ mod $(x)^c$, $M(\tilde{y}) \equiv M(\hat{y}) \neq 0$ mod (x). It follows $M(\tilde{y})$ invertible and so $f(\tilde{y}) \equiv 0$ mod $M^2(\tilde{y})(x)^c$. Now it is enough to apply Newton Lemma. □

The above lemma says in fact that a special solution $\hat{y}$ of a special system of polynomial equations f can be approximated as well we want in the (x)-adic topology of $\hat{A}$ by solutions in A, that is the algebraic ones. This is a preliminary form of Artin approximation.

The following lemma is a consequence of the so called Jacobian criterion.

Lemma 4. *If A is a domain, $q \subset A[Y]$ a prime ideal, and the field extension $Q(A) \subset Q(A[Y]/q)$ is separable then there exist some polynomials $f = (f_1, \dots, f_r)$ of q such that $qA[Y]_q = (f_1, \dots, f_r)A[Y]_q$ and the Jacobian matrix $(\partial f_i / \partial Y_j)$ has an $r \times r$-minor which is not in q.*

Since $Q(A) \subset Q(A[Y]/q)$ is separable of finite type one can find $u_1, \dots, u_s \in A[Y]$ algebraically independent over $Q(A)$ and $v \in A[Y]$ such that $Q(A[Y]/q) = Q(A)(u_1, \dots, u_s, v)$ and v is algebraic separable over $Q(A)(u_1, \dots, u_s)$. The irreducible polynomial associated to v over $Q(A)(u_1, \dots, u_s)$ should have the derivation non-zero in $Q(A[Y]/q)$. This is the idea behind the above lemma.

A Noetherian local ring (A, m) has the *property of approximation* if every finite system of polynomial equations f over A in $Y = (Y_1, \dots, Y_N)$ has its solutions in A dense with respect to the m-adic topology in the set of its solutions in the completion $\hat{A}$ of A; that is, for every solution $\hat{y}$ of f in $\hat{A}$ and every positive integer c there exists a solution y of f in A such that $y \equiv \hat{y} \mod m^c \hat{A}$.

Example 5. Let $k := \mathbf{F}_p(T_1^p, \dots, T_s^p, \dots)$, where $\mathbf{F}_p$ is the field with p elements, and $K := k(T_1, \dots, T_s, \dots)$, p being a prime number and $(T)_i$ a countable set of variables. Then $[K : k] = \infty$ and the discrete valuation ring $R := k[[x]][K]$ is Henselian but not complete because for example the formal power series $f := \sum_{i=1}^{\infty} T_i x^i$ in the variable x over K is not in R. In fact one can show that a formal power series $g = \sum_i g_i x^i$

in x over K is in R if and only if $[k((g_i)_i) : k] < \infty$. Since the polynomial $Y^p - f^p$ has no solutions in R it follows that R has not the property of approximation.

Next lemma shows how one can apply the property of approximation in an easy algebraic problem.

Lemma 6. *Suppose that A is a domain and it has the property of approximation. Then $\hat{A}$ is a domain too.*

Proof. Suppose that $\hat{A}$ is not a domain, that is there exist two nonzero elements $\hat{y}, \hat{z} \in \hat{A}$ such that $\hat{y}\hat{z} = 0$. Choose a positive integer c such that $\hat{y}, \hat{z} \notin m^c\hat{A}$. Take $f = YZ$. Then there exists $y, z \in A$ a solution of f such that $y \equiv \hat{y} \mod m^c\hat{A}$, $z \equiv \hat{z} \mod m^c\hat{A}$. It follows $yz = 0$ and $y, z \notin m^c$. Contradiction! $\square$

As in a lemma above we can see that if A has the property of approximation then it share with its completion $\hat{A}$ many algebraic properties. The following example is an exception.

Example 7. *Let $\hat{R} = \mathbf{C}[[x]]$ and $h = e^{(e^x - 1)}$, where $e^x = \sum_{i=0}^{\infty} x^i/(i!)$. Let R be the algebraic closure of $\mathbf{C}[x, h]$ in $\hat{R}$. Then R is an excellent Henselian discrete valuation ring. Moreover, by Artin's Theorem it has the property of approximation. One can show that the identity is the only* $\mathbf{C}$*-automorphism of R by G. Pfister, which is not the case of $\hat{R}$.*

Example 8. Let K be a field of characteristic $p > 0$ and $g_0, \ldots, g_{p-1} \in \hat{R} = K[[x]]$ be some formal power series which are algebraically independent over $K[x]$. Set $g = \sum_{i=0}^{p-1} g_i^p x^i$ and let R be the algebraic closure of $K[x, g]$ in $\hat{R}$. Then R is a Henselian discrete valuation ring but it has not the property of approximation because the polynomial $g - \sum_{i=0}^{p-1} Y_i^p x^i$ has a solution in $\hat{R}$ but none in R since the transcendental degree of R over $K[x]$ is 1.

Let K be a field and $S = K[[x]]$, $x = (x_1, \ldots, x_n)$. A formal power series $f \in S$ is called *regular* in x_n of order t if $f(0, \ldots, 0, x_n) \neq 0$ has the order t.

Theorem 9. *(Weierstrass Preparation Theorem) If f is regular in x_n of order t then there exist a unique polynomial $g = x_n^t + \sum_{i=0}^{t-1} a_i x_n^i$, with $a_i \in (x_1, \ldots, x_{n-1})K[[x_1, \ldots, x_{n-1}]]$ and a unique unit $u \in S$ such that $f = ug$ (g is called the* Weierstrass polynomial *of f) .*

Theorem 10. *(Weierstrass Division Theorem) Let $f, h \in S$ be two formal power series. If f is regular of order t then there exist unique formal power series $q \in S$, $w_0, \ldots, w_{t-1} \in K[[x_1, \ldots, x_{n-1}]]$ such that $h = qf + \sum_{i=1}^{t-1} w_i x_n^i$.*

Lemma 11. *If K is infinite and $f \in S$ is non-zero then there exists a K-automorphism τ of S of the form $x_i \to x_i + c_i x_n$, for $i < n$, $x_n \to x_n$, $c_i \in K$ such that $\tau(f)$ is regular in x_n of a certain order t.*

Example 12. Let h be a polynomial of degree $>> 0$ in Y_1 over $\mathbf{C}$ and $w = e^x = \sum_{i=o}^{\infty} x^i/(i!)$. Let $f = h - h(x) + (Y_1 - x)Y_2$ be a polynomial equation in $Y = (Y_1, Y_2)$ over $R := \mathbf{C} < x >$. Set $y_1 = x + w$. By Taylor's formula $h(y_1) - h(x) \in (y_1 - x) =$

(w) and so there exists $y_2 \in \hat{R} = \mathbf{C}[[x]]$ such that (y_1, y_2) is a solution of f in $\hat{R}$. Let $c \in \mathbf{N}$ and set $w_c = \sum_{i=o}^{c} x^i/(i!)$. Then for $\tilde{y}_1 = x + w_c$ there exists $\tilde{y}_2 \in R$ using Taylor's formula such that $(\tilde{y}_1, \tilde{y}_2)$ is a solution of f with $\tilde{y}_1 \equiv y_1 \bmod x^c$. It follows that $w_c y_2 \equiv w y_2 = h(y_1) - h(x) \equiv h(\tilde{y}_1) - h(x) = w_c \tilde{y}_2 \bmod x^c$. Thus $\tilde{y}_2 \equiv y_2 \bmod x^c$ and so the solution $(\tilde{y}_1, \tilde{y}_2)$ of f in R coincides modulo x^c with the previous one. This is in fact what our next theorem states.

Let $R = K < x >$, $x = (x_1, \ldots, x_n)$ be the ring of algebraic power series in x over K, that is the algebraic closure of the polynomial ring $K[x]$ in the formal power series ring $\hat{R} = K[[x]]$. Let $f = (f_1, \ldots, f_q)$ in $Y = (Y_1, \ldots, Y_N)$ over R and $\hat{y}$ a solution of f in $\hat{R}$.

Theorem 13. *(M. Artin [2]) For any $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in R such that $y^{(c)} \equiv \hat{y}$ mod $(x)^c$.*

Proof. We restrict to the case when the characteristic of K is zero, because we do not want to have separability problems and we believe it is enough for the purpose of our lectures. Apply induction on n, the case $n = 0$ being trivial. Let $h : R[Y] \to \hat{R}$ be the morphism of R-algebras given by $Y \to \hat{y}$. Since $\hat{R}$ is a domain we see that Ker h is a prime ideal. It is enough to consider the case when f generates Ker h. There is an argument to reduce the problem to the case when f generates $P :=$ Ker $h \cap K[x, Y]$ but we prefer to skip it. However we will suppose that f generates P.

Set $r = \text{height}(P)$. As the fraction field extension $Q(R) \subset Q(\hat{R})$ is separable it follows that $Q(R) \subset Q(R[Y]/P)$ is separable and we may suppose after renumbering of (f_i) that there exists an $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)_{i \in [r], j \in [N]}$ which is not in P, that is $M(\hat{y}) \neq 0$.

Applying an automorphism of R of type $x_i \to x_i + a_i x_n$ for $i < n$ and $x_n \to x_n$ for some $a_i \in K$ we may suppose that $M(0, \ldots, 0, x_n) \neq 0$, and so M^2 is regular in x_n of order a certain $u \in \mathbf{N}$. Set $R' = K < x_1, \ldots, x_{n-1} >$, $\hat{R}' = K[[x_1, \ldots, x_{n-1}]]$ and $m' = (x_1, \ldots, x_{n-1})$. By Weierstrass Preparation Theorem (see [3, Section 3.2]) there exist $\hat{a}_i$, $0 \leq i < u$ in $m'\hat{R}'$ such that $M^2(\hat{y})$ is associated in divisibility with the polynomial

$$\hat{a} = x_n^u + \hat{a}_{u-1} x_n^{u-1} + \ldots + \hat{a}_0$$

from $\hat{R}'[x_n]$.

Let Y_{sj}, A_k, $1 \leq s \leq N$, $0 \leq j, k < u$ be some new variables. Substitute in f, M^2 the variable Y_s by

$$Y_s^+ = \sum_{j=0}^{u-1} Y_{sj} x_n^j$$

and divide the result by the monic polynomial

$$A = x_n^u + \sum_{j=0}^{u-1} A_j x_n^j.$$

We obtain

$$M^2(Y^+) = AH((Y_{sj}),(A_k)) + \sum_{j=0}^{u-1} G_j((Y_{sj}),(A_k))x_n^j,$$

$$f_i(Y^+) = AH_i((Y_{sj}),(A_k)) + \sum_{j=0}^{u-1} F_{ij}((Y_{sj}),(A_k))x_n^j,$$

for some polynomials $H, H_i, G_j, F_{ij} \in K[x_1,\ldots,x_{n-1}][(Y_{sj}), A_k]$, $1 \le i \le r$, $0 \le j, k < u$. Using Weierstrass Division Theorem we get

$$\hat{y}_s = \sum_{j=0}^{u-1} \hat{y}_{sj} x_n^j + \hat{a}\hat{b}_s,$$

where $\hat{y}_{sj} \in \hat{R}'$, $\hat{b}_s \in \hat{R}$. Set $\hat{y}_s^+ = \sum_{j=0}^{u-1} \hat{y}_{sj} x_n^j$. Substituting Y^+ by $\hat{y}^+$ above we obtain

$$M^2(\hat{y}^+) = \hat{a}H((\hat{y}_{sj}),(\hat{a}_k)) + \sum_{j=0}^{u-1} G_j((\hat{y}_{sj}),(\hat{a}_k))x_n^j,$$

$$f_i(\hat{y}^+) = \hat{a}H_i((\hat{y}_{sj}),(\hat{a}_k)) + \sum_{j=0}^{u-1} F_{ij}((\hat{y}_{sj}),(\hat{a}_k))x_n^j,$$

$1 \le i \le r$. But

$$M^2(\hat{y}^+) \equiv M^2(\hat{y}) \equiv 0, \quad f_i(\hat{y}^+) \equiv f_i(\hat{y}) = 0 \mod \hat{a}$$

and using the unicity from Weierstrass Division Theorem, it follows from the above system of equations that

$$G_j((\hat{y}_{sj}),(\hat{a}_k)) = 0, \quad F_{ij}((\hat{y}_{sj}),(\hat{a}_k)) = 0,$$

$1 \le i \le r$, $0 \le j < u$, that is $(\hat{y}_{sj}),(\hat{a}_k)$ is a solution of the system of polynomials $G = (G_j)$, $F = (F_{ij})$ in $\hat{R}'$. Then by induction hypothesis for $c' = c + u + 1$ there exists a solution $(y_{sj}^{(c')}, a_k^{(c')})$ of F, G in R' such that

$$y_{sj}^{(c')} \equiv \hat{y}_{sj}, \quad a_k^{(c')} \equiv \hat{a}_k \mod m'^{c'}\hat{R}'.$$

Choose $b_s \in K[x]$ such that $b_s \equiv \hat{b}_s \bmod (x)^{c'}$ and set $\tilde{a}^{(c')} = x_n^u + \sum_{k=0}^{u-1} a_k^{(c')} x_n^k$, $\tilde{y}_s^{(c')} = \tilde{a}^{(c')} b_s + \sum_{j=0}^{u-1} y_{sj}^{(c')} x_n^j$. Clearly $\tilde{y}_s^{(c')} \equiv \hat{y}_s \mod (x)^{c'}$ and

$$M^2(\tilde{y}^{(c')}) \equiv 0, \quad f_i(\tilde{y}^{(c')}) \equiv 0 \mod (\tilde{a}^{(c')}).$$

On the other hand $M^2(\tilde{y}^{(c')}) \equiv M^2(\hat{y}) \bmod (x)^{(c')}$. As $c' > u$ it follows that $M^2(\tilde{y}_s^{(c')})$ is regular in x_n of order u and so $M^2(\tilde{y}_s^{(c')})$ is associated in divisibility with $\tilde{a}^{(c')}$ by Weierstrass Preparation Theorem. Thus $f_i(\tilde{y}^{(c')}) \equiv 0 \mod M^2(\tilde{y}_s^{(c')})$, $1 \le i \le r$.

Now note that $M^2(\tilde{y}_s^{(c')})$ does not belong to $(x)^{u+1}$ because it is regular of order u, but $f_i(\tilde{y}^{(c')}) \equiv f_i(\hat{y}) = 0 \bmod (x)^{c'}$ and we get $f_i(\tilde{y}^{(c')}) \equiv 0 \mod M^2(\tilde{y}_s^{(c')})(x)^c$. By Newton Lemma there exists a solution $y^{(c)}$ in R of f_i, $1 \le i \le r$ such that $y^{(c)} \equiv \tilde{y}^{(c')} \equiv \hat{y} \bmod (x)^c$.

It remains to show that the solution $y^{(c)}$ is a solution also of f_i with $r < i \leq q$ if c is big enough. In other words, a solution of f_i, $1 \leq i \leq r$ which is closed to $\hat{y}$ is a solution of f_i for all $i > r$. Let $I := \sqrt{(f_1, \ldots, f_r)} = \cap_{i=1}^e p_i$ be the irreducible primary decomposition of I, p_i being prime ideals of $K[x, Y]$ and $h_c : K[x, Y] \to \hat{R}$ be the morphism given by $Y \to y^{(c)}$. Clearly $\operatorname{Ker} h_c \supset I$. Since $P \supset (f_1, \ldots, f_r)$ we see that $P \supset p_i$ for some i, let us say $P \supset p_1$. But height $p_1 \geq r$ since the jacobian matrix $(\partial f_i / \partial Y_j)_{i \in [r], j \in [N]}$ has a $r \times r$-minor M which is not in P and so not in p_1. Thus $P = p_1$, which ends the problem when $e = 1$.

Suppose that $e > 1$. Choose a polynomial $g \in (\cap_{i=2}^e p_i) \setminus p_1$. Then $g(\hat{y}) \neq 0$ and for c big enough we get $g(y^{(c)}) \neq 0$, that is $\cap_{i=2}^e p_i \not\subset \operatorname{Ker} h_c$. As $(\cap_{i=2}^e p_i) p_1 \subset I \subset \operatorname{Ker} h_c$ we get $P = p_1 \subset \operatorname{Ker} h_c$, which is enough. $\square$

Let (A, m) be a Noetherian local ring. A has the property of *strong approximation* if for every finite system of polynomial equations f in $Y = (Y_1, \ldots, Y_N)$ over A there exist a map $\nu : \mathbf{N} \to \mathbf{N}$ with the following property: If $\tilde{y} \in A^N$ satisfies $f(\tilde{y}) \equiv 0 \mod m^{\nu(c)}$, $c \in \mathbf{N}$ then there exists a solution $y \in A^N$ of f in A with $y \equiv \tilde{y} \mod m^c$. The function ν is called the *Artin function* of A. By Artin the Henselization of a local ring, which is essentially of finite type over a field, has the property of strong approximation. If A is complete then it has the property of strong approximation by G. Pfister and myself.

Next lemma shows how one can apply the property of strong approximation in an easy algebraic problem.

Lemma 14. *Suppose that A is a domain and it has the property of strong approximation. Let $(a_n)_{n \in \mathbf{N}}$ be a sequence of elements of A, which converges in the m-adic topology to an element $a \in A$. If a is irreducible then there exists a positive integer t such that a_n is irreducible for all $n \geq t$.*

Proof. Let ν be the Artin function associated to the polynomial $f = YZ - a$ over A. Let $t > \nu(1)$ be such that $a_n \equiv a \mod m^{\nu(1)}$ for all $n \geq t$. If a_n is reducible for some $n \geq t$ then there exist $\tilde{y}$, $\tilde{z}$ in m such that $\tilde{y}\tilde{z} = a_n \equiv a \mod m^{\nu(1)}$. In particular, $f(\tilde{y}, \tilde{z}) \equiv 0 \mod m^{\nu(1)}$ and so there exists $y, z \in A$ such that $f(y, z) = 0$ and $y \equiv \tilde{y},\ z \equiv \tilde{z} \mod m$. Thus $a = yz$ and $y, z \in m$ which is impossible. $\square$

Proposition 15. *A Noetherian local ring has the property of approximation if and only if it has the property of strong approximation.*

Proof. Suppose that (A, m) has the property of strong approximation. Let f be a finite system of polynomial equations in $Y = (Y_1, \ldots, Y_N)$ over A and $\hat{y}$ a solution of f in the completion $\hat{A}$. Let ν be the Artin function associated to f, c be a positive integer and choose $y \in A^N$ such that $y \equiv \hat{y} \bmod m^{\nu(c)}\hat{A}$. Then $f(y) \equiv f(\hat{y}) = 0 \bmod m^{\nu(c)}\hat{A}$ and so there exists a solution $\tilde{y}$ of f in A such that $\tilde{y} \equiv y \bmod m^c$. Clearly, $\tilde{y} \equiv \hat{y} \bmod m^c\hat{A}$.

Conversely, suppose that A has the property of approximation. Let f, c be as above. Since the completion $\hat{A}$ has the strong approximation, let ν be the Artin function of f over $\hat{A}$. We claim that this function works for f over A too. Indeed, let $\tilde{y}$ be in A^N such that $f(\tilde{y}) \equiv 0 \bmod m^{\nu(c)}$. Then there exists a solution $\hat{y}$ of f

in $\hat{A}$ such that $\hat{y} \equiv \tilde{y} \bmod m^c\hat{A}$. Since A has the property of approximation there exists a solution y of f in A such that $y \equiv \hat{y} \equiv \tilde{y} \bmod m^c\hat{A}$. □

References

[1] M. Artin, *On the solution of analytic equations*, Inventiones Math., 5, (1968), 277-291.

[2] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. IHES, 36 (1969), 23-58.

[3] T. De Jong, G. Pfister, *Local Analytic Geometry*, Vieweg (2000).

Dorin Popescu, "Simion Stoilow" Institute of Mathematics , Research unit 5, University of Bucharest, P.O.Box 1-764, Bucharest 014700, Romania

E-mail address: dorin.popescu@imar.ro