DISQUISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\text{ etc.}$$

PARS PRIOR.

INTRODUCTIO.

1.

Series, quam in hac commentatione perscrutari suscipimus, tamquam functio quatuor quantitatum α, β, γ, x spectari potest, quas ipsius *elementa* vocabimus, ordine suo elementum primum α, secundum β, tertium γ, quartum x distinguentes. Manifesto elementum primum cum secundo permutare licet: quodsi itaque brevitatis caussa seriem nostram hoc signo $F(\alpha,\beta,\gamma,x)$ denotamus, habebimus $F(\beta,\alpha,\gamma,x) = F(\alpha,\beta,\gamma,x)$.

2.

Tribuendo elementis α,β,γ valores determinatos, series nostra in functionem unicae variabilis x transit, quae manifesto post terminum $1-\alpha^{\text{tum}}$ vel $1-\beta^{\text{tum}}$ abrumpitur, si $\alpha-1$ vel $\beta-1$ est numerus integer negativus, in casibus reliquis vero in infinitum excurrit. In casu priori series exhibet functionem algebraicam rationalem, in posteriori autem plerumque functionem transscendentem. Elementum tertium γ debet esse neque numerus negativus integer neque $=0$, ne ad terminos infinite magnos delabamur.

3.

Coëfficientes potestatum x^m, x^{m+1} in serie nostra sunt ut

$$1+\frac{\gamma+1}{m}+\frac{\gamma}{mm} : 1+\frac{\alpha+\beta}{m}+\frac{\alpha\beta}{mm}$$

adeoque ad rationem aequalitatis eo magis accedunt, quo maior assumitur m. Si itaque etiam elemento quarto x valor determinatus tribuitur, ab huius indole convergentia seu divergentia pendebit. Quoties scilicet ipsi x tribuitur valor realis positivus seu negativus, unitate minor, series certo, si non statim ab initio, tamen post certum intervallum, convergens erit, atque ad summam finitam ex asse determinatam perducet. Idem eveniet per valorem imaginarium ipsius x formae $a+b\sqrt{-1}$, quoties $aa+bb<1$. Contra pro valore ipsius x reali unitateque maiori, vel pro imaginario formae $a+b\sqrt{-1}$, quoties $aa+bb>1$, series si non statim tamen post certum intervallum necessario divergens erit, ita ut de ipsius *summa* sermo esse nequeat. Denique pro valore $x=1$ (seu generalius pro valore formae $a+b\sqrt{-1}$, quoties $aa+bb=1$) seriei convergentia seu divergentia ab ipsarum α, β, γ indole pendebit, de qua, atque in specie de summa seriei pro $x=1$, in Sect. tertia loquemur.

Patet itaque, quatenus functio nostra tamquam summa seriei definita sit, disquisitionem natura sua restrictam esse ad casus eos, ubi series revera convergat, adeoque quaestionem ineptam esse, quinam sit valor seriei pro valore ipsius x unitate maiori. Infra autem, inde a Sectione quarta, functionem nostram altiori principio superstruemus, quod applicationem generalissimam patiatur.

4.

Differentiatio seriei nostrae, considerando solum elementum quartum x tamquam variabile, ad functionem similem perducit, quum manifesto habeatur

$$\frac{\mathrm{d}F(\alpha,\beta,\gamma,x)}{\mathrm{d}x}=\frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1,\ \gamma+1,\ x)$$

Idem valet de differentiationibus repetitis.

5.

Operae pretium erit, quasdam functiones, quas ad seriem nostram reducere licet, quarumque usus in tota analysi est frequentissimus, hic apponere.

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I. $$(t+u)^n = t^n F(-n, \beta, \beta, -\tfrac{u}{t})$$

ubi elementum β est arbitrarium.

II. $$(t+u)^n+(t-u)^n = 2t^n F(-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{uu}{tt})$$

III. $$(t+u)^n+t^n = 2t^n F(-n, \omega, 2\omega, -\tfrac{u}{t})$$

denotante ω quantitatem infinite parvam.

IV. $$(t+u)^n-(t-u)^n = 2nt^{n-1}uF(-\tfrac{1}{2}n+\tfrac{1}{2}, -\tfrac{1}{2}n+1, \tfrac{3}{2}, \tfrac{uu}{tt})$$

V. $$(t+u)^n-t^n = nt^{n-1}uF(1-n, 1, 2, -\tfrac{u}{t})$$

VI. $$\log(1+t) = tF(1, 1, 2, -t)$$

VII. $$\log\tfrac{1+t}{1-t} = 2tF(\tfrac{1}{2}, 1, \tfrac{3}{2}, tt)$$

VIII. $$e^t = F(1, k, 1, \tfrac{t}{k}) = 1+tF(1, k, 2, \tfrac{t}{k}) = 1+t+\tfrac{1}{2}ttF(1, k, 3, \tfrac{t}{k}) \text{ etc.}$$

denotante e basin logarithmorum hyperbolicorum, k numerum infinite magnum.

IX. $$e^t+e^{-t} = 2F(k, k', \tfrac{1}{2}, \tfrac{tt}{4kk'})$$

denotantibus k, k' numeros infinite magnos.

X. $$e^t-e^{-t} = 2tF(k, k', \tfrac{3}{2}, \tfrac{tt}{4kk'})$$

XI. $$\sin t = tF(k, k', \tfrac{3}{2}, -\tfrac{tt}{4kk'})$$

XII. $$\cos t = F(k, k', \tfrac{1}{2}, -\tfrac{tt}{4kk'})$$

XIII. $$t = \sin t.\ F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \sin t^2)$$

XIV. $$t = \sin t.\ \cos t.\ F(1, 1, \tfrac{3}{2}, \sin t^2)$$

XV. $$t = \operatorname{tang} t.\ F(\tfrac{1}{2}, 1, \tfrac{3}{2}, -\operatorname{tang} t^2)$$

XVI. $$\sin nt = n\sin t.\ F(\tfrac{1}{2}n+\tfrac{1}{2}, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{3}{2}, \sin t^2)$$

XVII. $$\sin nt = n\sin t.\ \cos t.\ F(\tfrac{1}{2}n+1, -\tfrac{1}{2}n+1, \tfrac{3}{2}, \sin t^2)$$

XVIII. $$\sin nt = n\sin t.\ \cos t^{n-1} F(-\tfrac{1}{2}n+1, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{3}{2}, -\operatorname{tang} t^2)$$

XIX. $$\sin nt = n\sin t.\ \cos t^{-n-1} F(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{3}{2}, -\operatorname{tang} t^2)$$

XX. $$\cos nt = F(\tfrac{1}{2}n, -\tfrac{1}{2}n, \tfrac{1}{2}, \sin t^2)$$

XXI. $$\cos nt = \cos t.\ F(\tfrac{1}{2}n+\tfrac{1}{2}, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}, \sin t^2)$$

XXII. $$\cos nt = \cos t^n F(-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}, -\operatorname{tang} t^2)$$

XXIII. $$\cos nt = \cos t^{-n} F(\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}n, \tfrac{1}{2}, -\operatorname{tang} t^2)$$

6.

Functiones praecedentes sunt algebraicae atque transscendentes a logarithmis circuloque pendentes. Neutiquam vero *harum* caussa disquisitionem nostram *generalem* suscipimus, sed potius in gratiam theoriae functionum transscendentium altiorum promovendae, quarum genus amplissimum series nostra complectitur. Huc, inter infinita alia, pertinent coëfficientes ex evolutione functionis $(aa+bb-2ab\cos\varphi)^{-n}$ in seriem secundum cosinus angulorum φ, 2φ, 3φ etc. progredientem orti, de quibus *in specie* alia occasione fusius agemus. Ad formam seriei nostrae autem illi coëfficientes pluribus modis reduci possunt. Scilicet statuendo

$$(aa+bb-2ab\cos\varphi)^{-n} = \Omega = A+2A'\cos\varphi+2A''\cos 2\varphi+2A'''\cos 3\varphi+\text{ etc.}$$

habemus *primo*

$$\begin{aligned}
A &= a^{-2n}F(n, n, 1, \tfrac{bb}{aa})\\
A' &= na^{-2n-1}bF(n, n+1, 2, \tfrac{bb}{aa})\\
A'' &= \tfrac{n(n+1)}{1\,.\,2}a^{-2n-2}bbF(n, n+2, 3, \tfrac{bb}{aa})\\
A''' &= \tfrac{n(n+1)(n+2)}{1\,.\,2\,.\,3}a^{-2n-3}b^3F(n, n+3, 4, \tfrac{bb}{aa})\\
&\text{etc.}
\end{aligned}$$

Si enim $aa+bb-2ab\cos\varphi$ consideratur tamquam productum ex $a-br$ in $a-br^{-1}$ (designante r quantitatem $\cos\varphi+\sin\varphi.\sqrt{-1}$), fit Ω aequalis producto

ex a^{-2n}

$$\text{in}\qquad 1+n\frac{br}{a}+\frac{n(n+1)}{1\,.\,2}\cdot\frac{bbrr}{aa}+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\cdot\frac{b^3r^3}{a^3}+\text{ etc.}$$

$$\text{in}\qquad 1+n\frac{br^{-1}}{a}+\frac{n(n+1)}{1\,.\,2}\cdot\frac{bbr^{-2}}{aa}+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\cdot\frac{b^3r^{-3}}{a^3}+\text{ etc.}$$

Quod productum quum identicum esse debeat cum

$$A+A'(r+r^{-1})+A''(rr+r^{-2})+A'''(r^3+r^{-3})$$

valores supra dati sponte prodeunt.

Porro habemus *secundo*

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$$A\ = (aa+bb)^{-n}F(\tfrac{1}{2}n,\ \tfrac{1}{2}n+\tfrac{1}{2}.\ 1,\ \tfrac{4aabb}{(aa+bb)^2})$$
$$A'\ = n(aa+bb)^{-n-1}ab\,F(\tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{1}{2}n+1,\ 2,\ \tfrac{4aabb}{(aa+bb)^2})$$
$$A''= \tfrac{n(n+1)}{1\ .\ 2}(aa+bb)^{-n-2}aabb\,F(\tfrac{1}{2}n+1,\ \tfrac{1}{2}n+\tfrac{3}{2},\ 3,\ \tfrac{4aabb}{(aa+bb)^2})$$
$$A'''= \tfrac{n(n+1)(n+2)}{1\ .\ 2\ .\ 3}(aa+bb)^{-n-3}a^3b^3\,F(\tfrac{1}{2}n+\tfrac{3}{2},\ \tfrac{1}{2}n+2,\ 4,\ \tfrac{4aabb}{(aa+bb)^2})$$
etc.

qui valores facile deducuntur ex

$$\Omega(aa+bb)^n = 1+n(r+r^{-1})\frac{ab}{aa+bb}+\frac{n(n+1)}{1\ .\ 2}(r+r^{-1})^2\frac{aabb}{(aa+bb)^2}+\text{ etc.}$$

Tertio fit

$$A\ = (a+b)^{-2n}F(n,\ \tfrac{1}{2},\ 1,\ \tfrac{4ab}{(a+b)^2})$$
$$A'\ = n(a+b)^{-2n-2}ab\,F(n+1,\ \tfrac{3}{2},\ 3,\ \tfrac{4ab}{(a+b)^2})$$
$$A''= \tfrac{n(n+1)}{1\ .\ 2}(a+b)^{-2n-4}aabb\,F(n+2,\ \tfrac{5}{2},\ 5,\ \tfrac{4ab}{(a+b)^2})$$
$$A'''= \tfrac{n(n+1)(n+2)}{1\ .\ 2\ .\ 3}(a+b)^{-2n-6}a^3b^3\,F(n+3,\ \tfrac{7}{2},\ 7,\ \tfrac{4ab}{(a+b)^2})$$
etc.

Denique fit *quarto*

$$A\ = (a-b)^{-2n}F(n,\ \tfrac{1}{2},\ 1,\ -\tfrac{4ab}{(a-b)^2})$$
$$A'\ = n(a-b)^{-2n-2}ab\,F(n+1,\ \tfrac{3}{2},\ 3,\ -\tfrac{4ab}{(a-b)^2})$$
$$A''= \tfrac{n(n+1)}{1\ .\ 2}(a-b)^{-2n-4}aabb\,F(n+2,\ \tfrac{5}{2},\ 5,\ -\tfrac{4ab}{(a-b)^2})$$
$$A'''= \tfrac{n(n+1)(n+2)}{1\ .\ 2\ .\ 3}(a-b)^{2n-6}a^3b^3\,F(n+3,\ \tfrac{7}{2},\ 7,\ -\tfrac{4ab}{(a-b)^2})$$
etc.

Valores illi atque hi facile eruuntur ex

$$\Omega(a+b)^{2n} = \left(1-\frac{4ab\cos\frac{1}{2}\varphi^2}{(a+b)^2}\right)^{-n}$$
$$= 1+n\frac{ab}{(a+b)^2}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})^2+\frac{n(n+1)}{1\ .\ 2}\cdot\frac{aabb}{(a+b)^4}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})^4+\text{ etc.}$$
$$\Omega(a-b)^{2n} = \left(1+\frac{4ab\sin\frac{1}{2}\varphi^2}{(a-b)^2}\right)^{-n}$$
$$= 1+n\frac{ab}{(a-b)^2}(r^{\frac{1}{2}}-r^{-\frac{1}{2}})^2+\frac{n(n+1)}{1\ .\ 2}\cdot\frac{aabb}{(a-b)^4}(r^{\frac{1}{2}}-r^{-\frac{1}{2}})^4+\text{ etc.}$$

SECTIO PRIMA.

Relationes inter functiones contiguas.

7.

Functionem ipsi $F(\alpha, \beta, \gamma, x)$ *contiguam* vocamus, quae ex illa oritur, dum elementum primum, secundum, vel tertium unitate vel augetur vel diminuitur, manentibus tribus reliquis elementis. Functio itaque primaria $F(\alpha. \beta, \gamma, x)$ sex contiguas suppeditat, inter quarum binas ipsamque primariam aequatio persimplex linearis datur. Has aequationes, numero quindecim, hic in conspectum producimus, brevitatis gratia elementum quartum quod semper subintelligitur $= x$ omittentes, functionemque primariam simpliciter per F denotantes.

[1] $0 = (\gamma - 2\alpha - (\beta - \alpha)x)F + \alpha(1-x)F(\alpha+1, \beta, \gamma) - (\gamma-\alpha)F(\alpha-1, \beta, \gamma)$

[2] $0 = (\beta - \alpha)F + \alpha F(\alpha+1, \beta, \gamma) - \beta F(\alpha, \beta+1, \gamma)$

[3] $0 = (\gamma - \alpha - \beta)F + \alpha(1-x)F(\alpha+1, \beta, \gamma) - (\gamma-\beta)F(\alpha, \beta-1, \gamma)$

[4] $0 = \gamma(\alpha - (\gamma-\beta)x)F - \alpha\gamma(1-x)F(\alpha+1, \beta, \gamma) + (\gamma-\alpha)(\gamma-\beta)F(\alpha, \beta, \gamma+1)$

[5] $0 = (\gamma - \alpha - 1)F + \alpha F(\alpha+1, \beta, \gamma) - (\gamma-1)F(\alpha, \beta, \gamma-1)$

[6] $0 = (\gamma - \alpha - \beta)F - (\gamma-\alpha)F(\alpha-1, \beta, \gamma) + \beta(1-x)F(\alpha, \beta+1, \gamma)$

[7] $0 = (\beta - \alpha)(1-x)F - (\gamma-\alpha)F(\alpha-1, \beta, \gamma) + (\gamma-\beta)F(\alpha, \beta-1, \gamma)$

[8] $0 = \gamma(1-x)F - \gamma F(\alpha-1, \beta, \gamma) + (\gamma-\beta)xF(\alpha, \beta, \gamma+1)$

[9] $0 = (\alpha - 1 - (\gamma-\beta-1)x)F + (\gamma-\alpha)F(\alpha-1, \beta, \gamma) - (\gamma-1)(1-x)F(\alpha, \beta, \gamma-1)$

[10] $0 = (\gamma - 2\beta + (\beta-\alpha)x)F + \beta(1-x)F(\alpha, \beta+1, \gamma) - (\gamma-\beta)F(\alpha, \beta-1, \gamma)$

[11] $0 = \gamma(\beta - (\gamma-\alpha)x)F - \beta\gamma(1-x)F(\alpha, \beta+1, \gamma) - (\gamma-\alpha)(\gamma-\beta)F(\alpha, \beta, \gamma+1)$

[12] $0 = (\gamma - \beta - 1)F + \beta F(\alpha, \beta+1, \gamma) - (\gamma-1)F(\alpha, \beta, \gamma-1)$

[13] $0 = \gamma(1-x)F - \gamma F(\alpha, \beta-1, \gamma) + (\gamma-\alpha)xF(\alpha, \beta, \gamma+1)$

[14] $0 = (\beta - 1 - (\gamma-\alpha-1)x)F + (\gamma-\beta)F(\alpha, \beta-1, \gamma) - (\gamma-1)(1-x)F(\alpha, \beta, \gamma-1)$

[15] $0 = \gamma(\gamma - 1 - (2\gamma - \alpha - \beta - 1)x)F + (\gamma-\alpha)(\gamma-\beta)xF(\alpha, \beta, \gamma+1)$
$\qquad - \gamma(\gamma-1)(1-x)F(\alpha, \beta, \gamma-1)$

8.

Ecce iam demonstrationem harum formularum. Statuendo

$$\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+m-1)\beta(\beta+1)\ldots\ldots(\beta+m-2)}{1\,.\,2\,.\,3\,\ldots\ldots m\,.\,\gamma(\gamma+1)\ldots\ldots(\gamma+m-1)} = M$$

erit coëfficiens potestatis x^m

$$\begin{array}{lll} \text{in } F & \ldots\ldots & \alpha(\beta+m-1)M \\ \text{in } F(\alpha,\beta-1,\gamma) & \ldots & \alpha(\beta-1)M \\ \text{in } F(\alpha+1,\beta,\gamma) & \ldots & (\alpha+m)(\beta+m-1)M \\ \text{in } F(\alpha,\beta,\gamma-1) & \ldots & \frac{\alpha(\beta+m-1)(\gamma+m-1)M}{\gamma-1} \end{array}$$

coëfficiens autem potestatis x^{m-1} in $F(\alpha+1,\beta,\gamma)$, seu coëfficiens potestatis x^m in $xF(\alpha+1,\beta,\gamma)$

$$=m(\gamma+m-1)M$$

Hinc statim demanat veritas formularum 5 et 3; permutando α cum β, oritur ex 5 formula 12, atque ex his duabus per eliminationem 2. Perinde per eandem permutationem ex 3 oritur 6; ex 6 et 12 combinatis oritur 9, hinc per permutationem 14, quibus combinatis habetur 7; denique ex 2 et 6 eruitur 1, atque hinc permutando 10. Formula 8 simili modo ut supra formulae 5 et 3, e consideratione coëfficientium derivari potest (eodemque modo, si placeret, *omnes* 15 formulae erui possent), vel elegantius ex iam notis sequenti modo. Mutando in formula 5 elementum α in $\alpha-1$, atque γ in $\gamma+1$, prodit

$$0=(\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1)+(\alpha-1)F(\alpha,\beta,\gamma+1)-\gamma F(\alpha-1,\beta,\gamma)$$

Mutando vero in formula 9 tantummodo γ in $\gamma+1$, fit

$$0=(\alpha-1-(\gamma-\beta)x)F(\alpha,\beta,\gamma+1)+(\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1)-\gamma(1-x)F(\alpha,\beta,\gamma)$$

E subtractione harum formularum statim oritur 8, atque hinc per permutationem 13. Ex 1 et 8 prodit 4, hincque permutando 11. Denique ex 8 et 9 deducitur 15.

9.

Si $\alpha'-\alpha$, $\beta'-\beta$, $\gamma'-\gamma$, nec non $\alpha''-\alpha$, $\beta''-\beta$, $\gamma''-\gamma$ sunt numeri integri (positivi seu negativi), a functione $F(\alpha,\beta,\gamma)$ ad functionem $F(\alpha',\beta',\gamma')$, et perinde ab hac usque ad functionem $F(\alpha'',\beta'',\gamma'')$ transire licet per seriem similium functionum, ita ut quaelibet contigua sit antecedenti et consequenti, mutando scilicet primo elementum unum e.g. α continuo unitate, donec a $F(\alpha,\beta,\gamma)$ perventum sit ad $F(\alpha',\beta,\gamma)$, dein mutando elementum secundum, donec perven-

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tum sit ad $F(\alpha', \beta', \gamma)$, denique mutando elementum tertium, donec perventum sit ad $F(\alpha', \beta', \gamma')$, et perinde ab hac usque ad $F(\alpha'', \beta'', \gamma'')$. Quum itaque per art. 7 habeantur aequationes lineares inter functionem primam, secundam atque tertiam, et generaliter inter ternas quascunque consequentes huius seriei, facile perspicitur, hinc per eliminationem deduci posse aequationem linearem inter functiones $F(\alpha, \beta, \gamma)$, $F(\alpha', \beta', \gamma')$, $F(\alpha'', \beta'', \gamma'')$, ita ut generaliter loquendo e duabus functionibus, quarum tria elementa prima numeris integris differunt, quamlibet aliam functionem eadem proprietate gaudentem derivare liceat, siquidem elementum quartum idem maneat. Ceterum hic nobis sufficit, hanc veritatem insignem generaliter stabilivisse, neque hic compendiis immoramur, per quae operationes ad hunc finem necessariae quam brevissimae reddantur.

10.

Propositae sint e. g. functiones

$$F(\alpha, \beta, \gamma),\ F(\alpha+1, \beta+1, \gamma+1),\ F(\alpha+2, \beta+2, \gamma+2)$$

inter quas aequationem linearem invenire oporteat. Iungamus ipsas per functiones contiguas sequenti modo:

$$\begin{aligned}
&F(\alpha, \beta, \gamma) = F\\
&F(\alpha+1, \beta, \gamma) = F'\\
&F(\alpha+1, \beta+1, \gamma) = F''\\
&F(\alpha+1, \beta+1, \gamma+1) = F'''\\
&F(\alpha+2, \beta+1, \gamma+1) = F''''\\
&F(\alpha+2, \beta+2, \gamma+1) = F'''''\\
&F(\alpha+2, \beta+2, \gamma+2) = F''''''
\end{aligned}$$

Habemus itaque quinque aequationes lineares (e formulis 6, 13, 5 art. 7):

$$\begin{aligned}
&\text{I.} \quad 0 = (\gamma-\alpha-1)F - (\gamma-\alpha-1-\beta)F' - \beta(1-x)F''\\
&\text{II.} \quad 0 = \gamma F' - \gamma(1-x)F'' - (\gamma-\alpha-1)xF'''\\
&\text{III.} \quad 0 = \gamma F'' - (\gamma-\alpha-1)F''' - (\alpha+1)F''''\\
&\text{IV.} \quad 0 = (\gamma-\alpha-1)F''' - (\gamma-\alpha-2-\beta)F'''' - (\beta+1)(1-x)F'''''\\
&\text{V.} \quad 0 = (\gamma+1)F'''' - (\gamma+1)(1-x)F''''' - (\gamma-\alpha-1)xF''''''
\end{aligned}$$

Ex I et II prodit, eliminando F'

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VI. $\quad 0=\gamma F-\gamma(1-x)F''-(\gamma-\alpha-\beta-1)xF'''$

Hinc atque ex III, eliminando F''

VII. $\quad 0=\gamma F-(\gamma-\alpha-1-\beta x)F'''-(\alpha+1)(1-x)F''''$

Porro ex IV atque V, eliminando F'''''

VIII. $\quad 0=(\gamma+1)F'''-(\gamma+1)F''''+(\beta+1)xF''''''$

Hinc atque ex VII, eliminando F'''',

IX. $\quad 0=\gamma(\gamma+1)F-(\gamma+1)(\gamma-(\alpha+\beta+1)x)F'''-(\alpha+1)(\beta+1)x(1-x)F''''''$

11.

Si omnes relationes inter ternas functiones $F(\alpha,\beta,\gamma)$, $F(\alpha+\lambda,\beta+\mu,\gamma+\nu)$, $F(\alpha+\lambda',\beta+\mu',\gamma+\nu')$, in quibus $\lambda,\mu,\nu,\lambda',\mu',\nu'$ vel $=0$ vel $=+1$ vel $=-1$, exhaurire vellemus, formularum multitudo usque ad 325 ascenderet. Haud inutilis foret talis collectio, saltem simpliciorum ex his formulis: hoc vero loco sufficiat, paucas tantummodo apposuisse, quas vel ex formulis art. 7, vel si magis placet, simili modo ut duae priores ex illis in art. 8 erutae sunt, quivis nullo negotio sibi demonstrare poterit.

[16] $$F(\alpha,\beta,\gamma)-F(\alpha,\beta,\gamma-1)=-\frac{\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1,\gamma+1)$$

[17] $$F(\alpha,\beta+1,\gamma)-F(\alpha,\beta,\gamma)=\frac{\alpha x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

[18] $$F(\alpha+1,\beta,\gamma)-F(\alpha,\beta,\gamma)=\frac{\beta x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

[19] $$F(\alpha,\beta+1,\gamma+1)-F(\alpha,\beta,\gamma)=\frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2)$$

[20] $$F(\alpha+1,\beta,\gamma+1)-F(\alpha,\beta,\gamma)=\frac{\beta(\gamma-\alpha)x}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2)$$

[21] $$F(\alpha-1,\beta+1,\gamma)-F(\alpha,\beta,\gamma)=\frac{(\alpha-\beta-1)x}{\gamma}F(\alpha,\beta+1,\gamma+1)$$

[22] $$F(\alpha+1,\beta-1,\gamma)-F(\alpha,\beta,\gamma)=\frac{(\beta-\alpha-1)x}{\gamma}F(\alpha+1,\beta,\gamma+1)$$

[23] $$F(\alpha-1,\beta+1,\gamma)-F(\alpha+1,\beta-1,\gamma)=\frac{(\alpha-\beta)x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

SECTIO SECUNDA.

Fractiones continuae.

12.

Designando

$$\frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} \text{ per } G(\alpha,\beta,\gamma,x)$$

fit

$$\frac{F(\alpha+1,\beta,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \frac{F(\beta,\alpha+1,\gamma+1,x)}{F(\beta,\alpha,\gamma,x)} = G(\beta,\alpha,\gamma,x)$$

et proin, dividendo aequationem 19 per $F(\alpha,\beta+1,\gamma+1,x)$,

$$1-\frac{1}{G(\alpha,\beta,\gamma,x)} = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}x\,G(\beta+1,\alpha,\gamma+1,x)$$

sive

[24] $$G(\alpha,\beta,\gamma,x) = \cfrac{1}{1-\cfrac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}x\,G(\beta+1,\alpha,\gamma+1,x)}$$

et quum perinde fiat

$$G(\beta+1,\alpha,\gamma+1,x) = \cfrac{1}{1-\cfrac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}x\,G(\alpha+1,\beta+1,\gamma+2,x)}$$

etc., resultabit pro $G(\alpha,\beta,\gamma,x)$ fractio continua

[25] $$\frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}}$$

ubi

$$a = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} \qquad b = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}$$

$$c = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)} \qquad d = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}$$

$$e = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)} \qquad f = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)}$$

etc., cuius lex progressionis obvia est.

Porro ex aequationibus 17, 18, 21, 22 sequitur

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[26] $$\frac{F(\alpha,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)}=\frac{1}{1-\frac{\alpha x}{\gamma}G(\beta+1,\alpha,\gamma,x)}$$

[27] $$\frac{F(\alpha+1,\beta,\gamma,x)}{F(\alpha,\beta,\gamma,x)}=\frac{1}{1-\frac{\beta x}{\gamma}G(\alpha+1,\beta,\gamma,x)}$$

[28] $$\frac{F(\alpha-1,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)}=\frac{1}{1-\frac{(\alpha-\beta-1)x}{\gamma}G(\beta+1,\alpha-1,\gamma,x)}$$

[29] $$\frac{F(\alpha+1,\beta-1,\gamma,x)}{F(\alpha,\beta,\gamma,x)}=\frac{1}{1-\frac{(\beta-\alpha-1)x}{\gamma}G(\alpha+1,\beta-1,\gamma,x)}$$

unde, substitutis pro functione G eius valoribus in fractionibus continuis, totidem fractiones continuae novae prodeunt.

Ceterum sponte patet, fractionem continuam in formula 25 abrumpi, si e numeris $\alpha, \beta, \gamma-\alpha, \gamma-\beta$ aliquis fuerit integer negativus, alioquin vero in infinitum excurrere.

13.

Fractiones continuae in art. praec. erutae maximi sunt momenti, asserique potest, vix ullas fractiones continuas secundum legem obviam progredientes ab analystis hactenus erutas esse, quae sub nostris tamquam casus speciales non sint contentae. Imprimis memorabilis est casus is, ubi in formula 25 statuitur $\beta=0$, unde $F(\alpha,\beta,\gamma,x)=1$, adeoque, scribendo $\gamma-1$ pro γ

[30] $$F(\alpha,1,\gamma)=1+\frac{\alpha}{\gamma}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$
$$=\cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}}$$

ubi

$$a=\frac{\alpha}{\gamma} \qquad b=\frac{\gamma-\alpha}{\gamma(\gamma+1)}$$
$$c=\frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)} \qquad d=\frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)}$$
$$e=\frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)} \qquad f=\frac{3(\gamma+2-\alpha)}{(\gamma+4)(\gamma+5)}$$
etc.

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14.

Operae pretium erit, quosdam casus speciales huc adscripsisse. Ex formula I art. 5 sequitur, statuendo $t=1$, $\mathfrak{G}=1$

[31] $$(1+u)^n = \cfrac{1}{1-\cfrac{nu}{1+\cfrac{\frac{n+1}{2}u}{1-\cfrac{\frac{n-1}{2.3}u}{1+\cfrac{\frac{2(n+2)}{3.4}u}{1-\cfrac{\frac{2(n-2)}{4.5}u}{1+\text{ etc.}}}}}}}$$

E formulis VI, VII art. 5 sequitur

[32] $$\log(1+t) = \cfrac{t}{1+\cfrac{\frac{1}{2}t}{1+\cfrac{\frac{1}{6}t}{1+\cfrac{\frac{2}{6}t}{1+\cfrac{\frac{2}{10}t}{1+\cfrac{\frac{3}{10}t}{1+\cfrac{\frac{3}{14}t}{1+\text{ etc.}}}}}}}}$$

[33] $$\log\frac{1+t}{1-t} = \cfrac{2t}{1-\cfrac{\frac{1}{3}tt}{1-\cfrac{\frac{2.2}{3.5}tt}{1-\cfrac{\frac{3.3}{5.7}tt}{1-\cfrac{\frac{4.4}{7.9}tt}{1-\text{ etc.}}}}}}$$

Mutando hic signa — in + prodit fractio continua pro arc. tang t.

Porro habemus

[34] $$e^t = \cfrac{1}{1-\cfrac{t}{1+\cfrac{\frac{1}{2}t}{1-\cfrac{\frac{1}{6}t}{1+\cfrac{\frac{1}{6}t}{1-\cfrac{\frac{1}{10}t}{1+\cfrac{\frac{1}{10}t}{1-\text{ etc.}}}}}}}}$$

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[35]
$$t=\cfrac{\sin t\cos t}{1-\cfrac{\frac{1.2}{1.3}\sin t^2}{1-\cfrac{\frac{1.2}{3.5}\sin t^2}{1-\cfrac{\frac{3.4}{5.7}\sin t^2}{1-\cfrac{\frac{3.4}{7.9}\sin t^2}{1-\cfrac{\frac{5.6}{9.11}\sin t^2}{1-\text{ etc.}}}}}}}$$

Statuendo $\alpha=3$, $\gamma=\frac{5}{2}$, e formula 30 sponte sequitur fractio continua in *Theoria motus corporum coelestium* art. 90 proposita. Ibidem duae aliae fractiones continuae prolatae sunt, quarum evolutionem hacce occasione supplere visum est. Statuendo

$$Q=1-\cfrac{\frac{5.8}{7.9}x}{1-\cfrac{\frac{1.4}{9.11}x}{1-\frac{7.10}{11.13}x\text{ etc.}}}$$

fit l. c. $x-\xi=\dfrac{x}{1+\frac{2x}{35Q}}=\dfrac{xQ}{Q+\frac{2}{35}x}$, adeoque

$$\xi=\frac{\frac{2}{35}xx}{Q+\frac{2}{35}x}$$

quae est formula prior: posterior sequenti modo eruitur. Statuendo

$$R=1-\cfrac{\frac{1.4}{7.9}x}{1-\cfrac{\frac{5.8}{9.11}x}{1-\cfrac{\frac{3.6}{11.13}x}{1-\frac{7.10}{13.15}x\text{ etc.}}}}$$

erit per formulam 25

$$\frac{1}{R}=G(\tfrac{1}{2},\tfrac{3}{2},\tfrac{7}{2},x),\qquad\text{atque}\quad\frac{1}{Q}=G(\tfrac{5}{2},-\tfrac{1}{2},\tfrac{7}{2},x)$$

Hinc

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$$RF(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x)$$
$$QF(\tfrac{5}{2}, \tfrac{1}{2}, \tfrac{9}{2}, x) = F(\tfrac{5}{2}, -\tfrac{1}{2}, \tfrac{7}{2}, x)$$

sive permutando elementum primum cum secundo

$$QF(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x)$$

Sed per aequationem 21 habemus

$$F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x) - F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x) = -\tfrac{4}{7} x F(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x)$$

unde fit $Q = R - \frac{4}{7}x$, quo valore in formula supra data substituto prodit

$$\xi = \frac{\frac{2}{35}xx}{R - \frac{18}{35}x}$$

quae est formula posterior.

Statuendo in formula 30, $\alpha = \frac{m}{n}$, $x = -\gamma n t$, fit pro valore infinite magno ipsius γ

$$[36] \quad F(\tfrac{m}{n}, 1, \gamma, -\gamma n t) = 1 - mt + m(m+n)tt - m(m+n)(m+2n)t^3 + \text{etc.}$$

$$= \cfrac{1}{1 + \cfrac{mt}{1 + \cfrac{nt}{1 + \cfrac{(m+n)t}{1 + \cfrac{2nt}{1 + \cfrac{(m+2n)t}{1 + 3nt \text{ etc.}}}}}}}$$

SECTIO TERTIA.

De summa seriei nostrae statuendo elementum quartum $= 1$, ubi simul quaedam aliae functiones transscendentes discutiuntur.

15.

Quoties elementa α, β, γ omnia sunt quantitates positivae, omnes coëfficientes potestatum elementi quarti x positivi evadunt: quoties vero ex illis elementis unum alterumve negativum est, saltem inde ab aliqua potestate x^m omnes coëfficientes eodem signo affecti erunt, si modo m accipitur maior quam valor absolutus elementi negativi maximi. Porro hinc sponte patet, seriei summam pro

$x=1$ finitam esse non posse, nisi coëfficientes saltem post certum terminum in infinitum decrescant, vel, ut secundum morem analystarum loquamur, nisi coëfficiens termini x^∞ sit $=0$. Ostendemus autem, et quidem, in gratiam eorum, qui methodis rigorosis antiquorum geometrarum favent, omni rigore,

primo, coëfficientes (siquidem series non abrumpatur), in infinitum crescere, quoties fuerit $\alpha+\beta-\gamma-1$ quantitas positiva.

secundo, coëfficientes versus limitem finitum continuo convergere, quoties fuerit $\alpha+\beta-\gamma-1=0$.

tertio, coëfficientes in infinitum decrescere, quoties fuerit $\alpha+\beta-\gamma-1$ quantitas negativa.

quarto, summam seriei nostrae pro $x=1$, non obstante convergentia in casu tertio, infinitam esse, quoties fuerit $\alpha+\beta-\gamma$ quantitas positiva vel $=0$.

quinto, summam vero *finitam* esse, quoties $\alpha+\beta-\gamma$ fuerit quantitas negativa.

16.

Hanc disquisitionem generalius adaptabimus seriei infinitae M, M', M'', M''' etc. ita formatae, ut quotientes $\frac{M'}{M}, \frac{M''}{M'}, \frac{M'''}{M''}$ etc. resp. sint valores fractionis

$$\frac{t^\lambda+At^{\lambda-1}+Bt^{\lambda-2}+Ct^{\lambda-3}+\text{ etc.}}{t^\lambda+at^{\lambda-1}+bt^{\lambda-2}+ct^{\lambda-3}+\text{ etc.}}$$

pro $t=m$, $t=m+1$, $t=m+2$ etc. Brevitatis caussa huius fractionis numeratorem per P, denominatorem per p denotabimus: praeterea supponemus, P, p non esse identicas, sive differentias $A-a$, $B-b$, $C-c$ etc. non omnes simul evanescere.

I. Quoties e differentiis $A-a$, $B-b$, $C-c$ etc. prima quae non evanescit est positiva, assignari poterit limes aliquis l, quem simulac egressus est valor ipsius t, valores functionum P et p certo semper evadent positivi, atque $P>p$. Manifestum est, hoc evenire, si pro l accipiatur radix maxima realis aequationis $p(P-p)=0$; si vero haec aequatio nullas omnino radices reales habeat, proprietatem illam pro omnibus valoribus ipsius t locum habere. Quapropter in serie $\frac{M'}{M}, \frac{M''}{M'}, \frac{M'''}{M''}$ etc. saltem post certum intervallum (si non ab initio) omnes termini erunt positivi atque maiores unitate; quodsi itaque nullus neque $=0$ neque infinite magnus evadit, perspicuum est,

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seriem M, M', M'', M''' etc. *si non ab initio tamen post certum intervallum omnes suos terminos eodem signo affectos continuoque crescentes habituram esse.*

Eadem ratione, si e differentiis $A-a$, $B-b$, $C-c$ etc. prima quae non evanescit est negativa, series M, M', M'', M''' etc. si non ab initio tamen post certum intervallum omnes suos terminos eodem signo affectos continuoque decrescentes habebit.

II. Si iam coëfficientes A, a sunt inaequales, termini seriei M, M', M'', M''' etc. ultra omnes limites sive in infinitum vel crescent vel decrescent, prout differentia $A-a$ est positiva vel negativa: hoc ita demonstramus. Si $A-a$ est quantitas positiva, accipiatur numerus integer h ita, ut fiat $h(A-a)>1$, statuaturque $\frac{M^h}{m}=N$, $\frac{M'^h}{m+1}=N'$, $\frac{M''^h}{m+2}=N''$, $\frac{M'''^h}{m+3}=N'''$ etc., nec non $tP^h=Q$, $(t+1)p^h=q$. Tunc patet, $\frac{N'}{N}, \frac{N''}{N'}, \frac{N'''}{N''}$ etc. esse valores fractionis $\frac{Q}{q}$ ponendo $t=m$, $t=m+1$, $t=m+2$ etc., ipsas Q, q vero esse functiones algebraicas formae huius

$$Q = t^{\lambda h+1} + hAt^{\lambda h} + \text{ etc.}$$
$$q = t^{\lambda h+1} + (ha+1)t^{\lambda h} + \text{ etc.}$$

Quare quum per hyp. differentia $hA-(ha+1)$ sit quantitas positiva, termini seriei N, N', N'', N''' etc. si non ab initio tamen post certum intervallum continuo crescent (per 1); hinc termini seriei mN, $(m+1)N'$, $(m+2)N''$, $(m+3)N'''$ etc. necessario ultra omnes limites crescent, et proin etiam termini seriei M, M', M'', M''' etc., quippe quorum potestates exponente h illis sunt aequales. Q.E.P.

Si $A-a$ est quantitas negativa, accipere oportet integrum h ita, ut $h(a-A)$ fiat maior quam 1, unde per ratiocinia similia termini seriei

$$mM^h, \quad (m+1)M'^h, \quad (m+2)M''^h, \quad (m+3)M'''^h \quad \text{etc.}$$

post certum intervallum continuo decrescent. Quamobrem termini seriei M^h, M'^h, M''^h etc. adeoque etiam termini huius M, M', M'', M''' etc. necessario in infinitum decrescent. Q. E. S.

III. Si vero coëfficientes primi A, a sunt aequales, termini seriei M, M', M'', M''' etc. versus limitem finitum continuo convergent, quod ita demonstramus. Supponamus primo, terminos seriei post certum intervallum continuo crescere, sive e differentiis $B-b$, $C-c$ etc. primam quae non evanescat esse positivam. Sit h integer talis, ut $h+b-B$ fiat quantitas positiva, statuamusque

$$M(\tfrac{m}{m-1})^h = N,\quad M'(\tfrac{m+1}{m})^h = N',\quad M''(\tfrac{m+2}{m+1})^h = N'' \text{ etc.}$$

atque $(tt-1)^h P = Q$, $t^{2h}p = q$, ita ut $\frac{N'}{N}$, $\frac{N''}{N'}$ etc. sint valores fractionis $\frac{Q}{q}$ ponendo $t = m$, $t = m+1$ etc. Quum itaque habeatur

$$Q = t^{\lambda+2h} + At^{\lambda+2h-1} + (B-h)\,t^{\lambda+2h-2} \text{ etc.}$$
$$q = t^{\lambda+2h} + At^{\lambda+2h-1} + b\,t^{\lambda+2h-2} \text{ etc.}$$

atque per hyp. $B-h-b$ sit quantitas negativa, termini seriei N, N', N'', N''' etc. post certum saltem intervallum continuo decrescent, adeoque termini seriei M, M', M'', M''' etc., qui illis resp. semper sunt minores, dum continuo crescunt, tantummodo versus limitem finitum convergere possunt. Q. E. P.

Si termini seriei M, M', M'', M''' etc. post certum intervallum continuo decrescunt, accipere oportet pro h integrum talem, ut $h+B-b$ sit quantitas positiva, evinceturque per ratiocinia prorsus similia, terminos seriei

$$M(\tfrac{m-1}{m})^h,\quad M'(\tfrac{m}{m+1})^h,\quad M''(\tfrac{m+1}{m+2})^h \text{ etc.}$$

post certum intervallum continuo crescere, unde termini seriei M, M', M'' etc., qui illis resp. semper sunt maiores, dum continuo decrescunt, necessario tantummodo versus limitem finitum decrescere possunt. Q. E. S.

IV. Denique quod attinet ad *summam* seriei, cuius termini sunt M, M', M'' M'''etc., in casu eo, ubi hi in infinitum decrescunt, supponamus primo, $A-a$ cadere inter 0 et -1, sive $A+1-a$ esse vel quantitatem positivam vel $=0$. Sit h integer positivus, arbitrarius in casu eo, ubi $A+1-a$ est quantitas positiva, vel talis qui reddat quantitatem $h+m+A+B-b$ positivam in casu eo ubi $A+1-a=0$. Tunc erit

$$P(t-(m+h-1)) = t^{\lambda+1} + (A+1-m-h)\,t^{\lambda} + (B-A(m+h-1))\,t^{\lambda-1} \text{ etc.}$$
$$p(t-(m+h)) = t^{\lambda+1} + (a-m-h)\,t^{\lambda} + (b-a(m+h))\,t^{\lambda-1} \text{ etc.}$$

ubi vel $A+1-m-h-(a-m-h)$ erit quantitas positiva, vel, si haec fit $=0$, saltem $B-A(m+h-1)-(b-a(m+h))$ positiva erit. Hinc (per I) pro quantitate t assignari poterit valor aliquis l, quem simulac transgressa est, valores fractionis $\frac{P(t-(m+h-1))}{p(t-(m+h))}$ semper fient positivi atque unitate maiores. Sit n integer maior quam l simulque maior quam h, sintque termini seriei M, M', M'', M''' etc. qui respondent valoribus $t = m+n$, $t = m+n+1$, $t = m+n+2$ etc., hi N, N', N'', N''' etc. Erunt itaque

$$\frac{(n+1-h)N'}{(n-h)N}, \quad \frac{(n+2-h)N''}{(n+1-h)N'}, \quad \frac{(n+3-h)N'''}{(n+2-h)N''} \text{ etc.}$$

quantitates positivae unitate maiores, unde

$$N' > \frac{(n-h)N}{n+1-h}, \quad N'' > \frac{(n-h)N}{n+2-h}, \quad N''' > \frac{(n-h)N}{n+3-h} \text{ etc.}$$

adeoque summa seriei $N+N'+N''+N'''+$ etc. maior summa seriei

$$(n-h)N\left(\frac{1}{n-h}+\frac{1}{n+1-h}+\frac{1}{n+2-h}+\frac{1}{n+3-h}+\text{ etc.}\right)$$

quotcunque termini colligantur. Sed posterior series, crescente terminorum numero in infinitum, omnes limites egreditur, quum summa seriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ etc. quam infinitam esse constat etiam infinita maneat, si ab initio termini $1+\frac{1}{2}+\frac{1}{3}+$etc. $+\frac{1}{n-1-h}$ rescindantur. Quare summa seriei $N+N'+N''+N'''+$ etc., adeoque etiam summa huius $M+M'+M''+M'''+$ etc., cuius pars est illa, ultra omnes limites crescit.

V. Quoties autem $A-a$ est quantitas negativa absolute maior quam unitas, summa seriei $M+M'+M''+M'''+$ etc. in infinitum continuatae certo erit finita. Sit enim h quantitas positiva minor quam $a-A-1$, demonstrabiturque per ratiocinia similia, assignari posse valorem aliquem l quantitatis t, ultra quem fractio $\frac{Pt}{p(t-h-1)}$ semper adipiscatur valores positivos unitate minores. Quodsi iam pro n accipitur numerus integer ipsis $l, m, h+1$ maior, terminique seriei M, M', M'', M''' etc., valoribus $t=n$, $t=n+1$, $t=n+2$ etc. respondentes, designantur per N, N', N'' etc., erit

$$N' < \frac{n-h-1}{n} \cdot N, \quad N'' < \frac{(n-h-1)(n-h)}{n(n+1)} \cdot N' \text{ etc.}$$

adeoque summa seriei $N+N'+N''+$ etc, quotcunque termini colligantur, minor producto ex N in summam totidem terminorum seriei

$$1+\frac{n-h-1}{n}+\frac{(n-h-1)(n-h)}{n(n+1)}+\frac{(n-h-1)(n-h)(n-h+1)}{n(n+1)(n+2)} \text{ etc.}$$

Huius vero summa pro quolibet terminorum numero facile assignari potest; est scilicet

terminus primus $= \frac{n-1}{h} - \frac{n-h-1}{h}$

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$$\text{summa duorum terminorum} = \frac{n-1}{h} - \frac{(n-h-1)(n-h)}{hn}$$

$$\text{summa trium terminorum} = \frac{n-1}{h} - \frac{(n-h-1)(n-h)(n-h+1)}{hn(n+1)} \text{ etc.}$$

et quum pars altera (per II) formet seriem ultra omnes limites decrescentem, summa illa in infinitum continuata statui debet $=\frac{n-1}{h}$. Hinc $N+N'+N''$ etc. in infinitum continuata semper manebit minor quam $\frac{N(n-1)}{h}$, et proin $M+M'+M''$ etc. certo ad summam finitam converget. Q. E. D.

VI. Ut ea, quae generaliter de serie M, M', M'' etc. demonstravimus, ad coëfficientes potestatum x^m, x^{m+1}, x^{m+2} etc. in serie $F(\alpha, \beta, \gamma, x)$, applicentur, statuere oportebit $\lambda = 2$, $A = \alpha+\beta$, $B = \alpha\beta$, $a = \gamma+1$, $b = \gamma$, unde quinque assertiones in art. praec. propositae sponte demanant.

17.

Disquisitio itaque de indole summae seriei $F(\alpha, \beta, \gamma, 1)$ natura sua restringitur ad casum, quo $\gamma-\alpha-\beta$ est quantitas positiva, ubi illa summa semper exhibet quantitatem finitam. Praemittimus autem observationem sequentem. Si coëfficientes seriei $1+ax+bxx+cx^3+$ etc. $= S$ inde a certo termino ultra omnes limites decrescunt, productum

$$(1-x)S = 1+(a-1)x+(b-a)xx+(c-b)x^3+ \text{ etc.}$$

pro $x = 1$ statuere oportet $= 0$, etiamsi summa ipsius seriei S infinite magna evadat. Quoniam enim collectis duobus terminis summa fit $= a$, collectis tribus $= b$, collectis quatuor $= c$ etc., limes summae in infinitum continuatae est $= 0$. Quoties itaque $\gamma-\alpha-\beta$ est quantitas positiva, statuere oportet $(1-x)F(\alpha, \beta, \gamma-1, x) = 0$ pro $x = 1$, unde per aequationem 15 art. 7 habebimus

$$0 = \gamma(\alpha+\beta-\gamma)F(\alpha, \beta, \gamma, 1)+(\gamma-\alpha)(\gamma-\beta)F(\alpha, \beta, \gamma+1, 1), \text{ sive}$$

$$[37] \qquad F(\alpha, \beta, \gamma, 1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}F(\alpha, \beta, \gamma+1, 1)$$

Quare quum perinde fiat

$$F(\alpha, \beta, \gamma+1, 1) = \frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)}F(\alpha, \beta, \gamma+2, 1)$$

$$F(\alpha, \beta, \gamma+2, 1) = \frac{(\gamma+2-\alpha)(\gamma+2-\beta)}{(\gamma+2)(\gamma+2-\alpha-\beta)}F(\alpha, \beta, \gamma+3, 1)$$

et sic porro, erit generaliter, k denotante integrum positivum quemcunque

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$$F(\alpha.\beta,\gamma,1) \text{ aequalis producto ex } F(\alpha,\beta,\gamma+k,1)$$

in $(\gamma-\alpha)(\gamma+1-\alpha)(\gamma+2-\alpha)\ldots.(\gamma+k-1-\alpha)$

in $(\gamma-\beta)(\gamma+1-\beta)(\gamma+2-\beta)\ldots.(\gamma+k-1-\beta)$

diviso per productum

ex $\gamma(\gamma+1)(\gamma+2)\ldots.(\gamma+k-1)$

in $(\gamma-\alpha-\beta)(\gamma+1-\alpha-\beta)(\gamma+2-\alpha-\beta)\ldots(\gamma+k-1-\alpha-\beta)$

18.

Introducamus abhinc sequentem notationem:

[38] $$\Pi(k,z) = \frac{1\,.\,2\,.\,3\,.\ldots.\,k}{(z+1)(z+2)(z+3)\ldots.(z+k)}\,k^z$$

ubi k natura sua subintelligitur designare integrum positivum, qua restrictione $\Pi(k,z)$ exhibet functionem duarum quantitatum k, z prorsus determinatam. Hoc modo facile intelligetur, theorema in fine art. praec. propositum ita exhiberi posse

[39] $$F(\alpha,\beta,\gamma,1) = \frac{\Pi(k,\gamma-1)\,.\,\Pi(k,\gamma-\alpha-\beta-1)}{\Pi(k,\gamma-\alpha-1)\,.\,\Pi(k,\gamma-\beta-1)}\,.\,F(\alpha,\beta,\gamma+k,1)$$

19.

Operae pretium erit, indolem functionis $\Pi(k,z)$ accuratius perpendere. Quoties z est integer negativus, functio manifesto valorem infinite magnum obtinet, simulac ipsi k tribuitur valor satis magnus. Pro valoribus integris ipsius z non negativis autem habemus

$$\Pi(k,0) = 1$$

$$\Pi(k,1) = \frac{1}{1+\frac{1}{k}}$$

$$\Pi(k,2) = \frac{1\,.\,2}{(1+\frac{1}{k})(1+\frac{2}{k})}$$

$$\Pi(k,3) = \frac{1\,.\,2\,.\,3}{(1+\frac{1}{k})(1+\frac{2}{k})(1+\frac{3}{k})}$$

etc. sive generaliter

[40] $$\Pi(k,z) = \frac{1\,.\,2\,.\,3\,\ldots\ldots\,z}{(1+\frac{1}{k})(1+\frac{2}{k})(1+\frac{3}{k})\ldots.(1+\frac{z}{k})}$$

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Generaliter autem pro *quovis* valore ipsius z habemus

[41] $$\Pi(k,z+1)=\Pi(k,z)\,\frac{1+z}{1+\frac{1+z}{k}}$$

[42] $$\Pi(k+1,z)=\Pi(k,z)\left\{\frac{(1+\frac{1}{k})^{z+1}}{1+\frac{1+z}{k}}\right\}$$

adeoque, quum $\Pi(1,z)=\frac{1}{z+1}$,

[43] $$\Pi(k,z)=\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z.(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\cdots\frac{k^{z+1}}{(k-1)^z(k+z)}$$

20.

Imprimis vero attentione dignus est *limes*, ad quem pro valore dato ipsius z functio $\Pi(k,z)$ continuo converget, dum k in infinitum crescit. Sit primo h valor finitus ipsius k maior quam z, patetque, si k transire supponatur ex h in $h+1$, logarithmum ipsius $\Pi(k,z)$ accipere incrementum, quod per seriem convergentem sequentem exprimatur

$$\frac{z(1+z)}{2(h+1)^2}+\frac{z(1-zz)}{3(h+1)^3}+\frac{z(1+z^3)}{4(h+1)^4}+\frac{z(1-z^4)}{5(h+1)^5}+\text{ etc.}$$

Si itaque k e valore h transit in $h+n$, logarithmus ipsius $\Pi(k,z)$ accipiet incrementum

$$\begin{aligned}&\tfrac{1}{2}z(1+z)\left(\frac{1}{(h+1)^2}+\frac{1}{(h+2)^2}+\frac{1}{(h+3)^2}+\text{ etc. }+\frac{1}{(h+n)^2}\right)\\ &+\tfrac{1}{3}z(1-zz)\left(\frac{1}{(h+1)^3}+\frac{1}{(h+2)^3}+\frac{1}{(h+3)^3}+\text{ etc. }+\frac{1}{(h+n)^3}\right)\\ &+\tfrac{1}{4}z(1+z^3)\left(\frac{1}{(h+1)^4}+\frac{1}{(h+2)^4}+\frac{1}{(h+3)^4}+\text{ etc. }+\frac{1}{(h+n)^4}\right)\\ &+\text{ etc.}\end{aligned}$$

quod semper finitum manere, etiamsi n in infinitum crescat, facile demonstrari potest. Quare nisi iam in $\Pi(h,z)$ factor infinitus affuerit, i. e nisi z sit numerus integer negativus, limes ipsius $\Pi(k,z)$ pro $k=\infty$ certo erit quantitas finita. Manifesto itaque $\Pi(\infty,z)$ tantummodo a z pendet, sive functionem ipsius z ex asse determinatam exhibet, quae abhinc simpliciter per Πz denotabitur. Definimus itaque functionem Πz per valorem producti

$$\frac{1\;.\;2\;.\;3\;.\;.\;.\;.\;.\;k\;.\;k^z}{(z+1)(z+2)(z+3)\ldots.(z+k)}$$

pro $k = \infty$, aut si mavis per limitem producti infiniti

$$\frac{1}{z+1} \cdot \frac{2^{z+1}}{1^z(2+z)} \cdot \frac{3^{z+1}}{2^z(3+z)} \cdot \frac{4^{z+1}}{3^z(4+z)} \text{ etc.}$$

21.

Ex aequatione 41 statim sequitur aequatio fundamentalis

[44] $$\Pi(z+1) = (z+1)\Pi z$$

unde generaliter, designante n integrum positivum quemcunque

[45] $$\Pi(z+n) = (z+1)(z+2)(z+3)\ldots\ldots(z+n)\Pi z$$

Pro valore integro negativo ipsius z erit valor functionis Πz infinite magnus; pro valoribus integris non negativis habemus

$$\begin{aligned} \Pi 0 &= 1 \\ \Pi 1 &= 1 \\ \Pi 2 &= 2 \\ \Pi 3 &= 6 \\ \Pi 4 &= 24 \text{ etc.} \end{aligned}$$

atque generaliter

[46] $$\Pi z = 1.2.3\ldots.z$$

Sed male haec proprietas functionis nostrae tamquam ipsius definitio venditaretur, quippe quae natura sua ad valores integros restringitur, et praeter functionem nostram infinitis aliis (e. g. $\cos 2\pi z.\Pi z$, $\cos \pi z^{2n}\Pi z$ etc., denotante π semiperipheriam circuli, cuius radius $= 1$) communis est.

22.

Functio $\Pi(k,z)$, etiamsi generalior videatur quam Πz, tamen abhinc nobis superflua erit, quum facile ad posteriorem reducatur. Colligitur enim e combinatione aequationum 38, 45, 46

[47] $$\Pi(k,z) = \frac{k^z\Pi k.\Pi z}{\Pi(k+z)}$$

Ceterum nexus harum functionum cum iis, quas clar. KRAMP *facultates nu-*

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mericas nominavit, per se obvius est. Scilicet facultas numerica, quam hic auctor per $a^{b|c}$ designat, in signis nostris est

$$=\frac{c^b b^{\frac{a}{c}-1}\Pi b}{\Pi(b,\frac{a}{c}-1)}=\frac{c^b\Pi(\frac{a}{c}+b-1)}{\Pi(\frac{a}{c}-1)}$$

Sed consultius videtur, functionem *unius* variabilis in analysin introducere, quam functionem trium variabilium, praesertim quum hanc ad illam reducere liceat.

23.

Continuitas functionis Πz interrumpitur, quoties ipsius valor fit infinite magnus, i. e. pro valoribus integris negativis ipsius z. Erit itaque illa positiva a $z=-1$ usque ad $z=\infty$, et quum pro utroque limite Πz obtineat valorem infinite magnum, inter ipsos dabitur valor minimus, quem esse $=0{,}8856024$ atque respondere valori $z=0{,}4616321$ invenimus. Inter limites $z=-1$ et $z=-2$, valor functionis Πz fit negativus, inter $z=-2$ atque $z=-3$ iterum positivus et sic porro, uti ex aequ. 44 sponte sequitur. Porro patet, si omnes valores functionis Πz inter limites arbitrarios unitate differentes e. g. a $z=0$ usque ad $z=1$ pro notis habere liceat, valorem functionis pro quovis alio valore reali ipsius z adiumento aequationis 45 facile inde deduci posse. Ad hunc finem construximus *tabulam*, ad calcem huius sectionis annexam, quae ad figuras viginti exhibet logarithmos briggicos functionis Πz, pro $z=0$ usque ad $z=1$ per singulas partes centesimas summa cura computatos, ubi tamen monendum, figuram ultimam vigesimam interdum una duabusve unitatibus erroneam esse posse.

24.

Quum limes functionis $F(\alpha,\beta,\gamma+k,1)$, crescente k in infinitum, manifesto sit unitas, aequatio 39 transit in hanc

[48] $$F(\alpha,\beta,\gamma,1)=\frac{\Pi(\gamma-1).\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1).\Pi(\gamma-\beta-1)}$$

quae formula exhibet solutionem completam quaestionis, quae obiectum huius sectionis constituit. Sponte hinc sequuntur aequationes elegantes:

[49] $$F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma-\alpha-\beta, 1)$$

[50] $$F(\alpha, \beta, \gamma, 1) . F(-\alpha, \beta, \gamma-\alpha, 1) = 1$$

[51] $$F(\alpha, \beta, \gamma, 1) . F(\alpha, -\beta, \gamma-\beta, 1) = 1$$

in quarum prima γ, in secunda $\gamma-\beta$, in tertia $\gamma-\alpha$ debet esse quantitas positiva.

25.

Applicemus formulam 48 ad quasdam ex aequationibus art. 5. Formula XIII, statuendo $t = 90^0 = \frac{1}{2}\pi$, fit $\frac{1}{2}\pi = F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1)$, sive aequivalet aequationi notae

$$\tfrac{1}{2}\pi = 1 + \frac{1.1}{2.3} + \frac{1.1.3}{2.4.5} + \frac{1.1.3.5}{2.4.6.7} + \text{etc.}$$

Quare quum per formulam 48 habeatur $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1) = \frac{\Pi\frac{1}{2}.\Pi(-\frac{1}{2})}{\Pi 0.\Pi 0}$, atque sit $\Pi 0 = 1$, $\Pi\frac{1}{2} = \frac{1}{2}\Pi(-\frac{1}{2})$, fit $\pi = (\Pi(-\frac{1}{2}))^2$ sive

[52] $$\Pi(-\tfrac{1}{2}) = \sqrt{\pi}$$

[53] $$\Pi\tfrac{1}{2} = \tfrac{1}{2}\sqrt{\pi}$$

Formula XVI art. 5, quae aequivalet aequationi notae

$$\sin nt = n\sin t - \frac{n(nn-1)}{2\,.\,3}\sin t^3 + \frac{n(nn-1)(nn-9)}{2\,.\,3\,.\,4\,.\,5}\sin t^5 - \text{etc.}$$

atque generaliter pro quovis valore ipsius n locum habet, si modo t limites -90^0 et $+90^0$ non transgrediatur, dat pro $t = \frac{1}{2}\pi$

$$\sin\frac{n\pi}{2} = \frac{n\Pi\frac{1}{2}.\Pi(-\frac{1}{2})}{\Pi(-\frac{1}{2}n).\Pi\frac{1}{2}n}$$

unde deducitur formula elegans

$$\Pi\tfrac{1}{2}n.\Pi(-\tfrac{1}{2}n) = \frac{\frac{1}{2}n\pi}{\sin\frac{1}{2}n\pi},$$ sive statuendo $n = 2z$

[54] $$\Pi(-z).\Pi(+z) = \frac{z\pi}{\sin z\pi}$$

[55] $$\Pi(-z).\Pi(z-1) = \frac{\pi}{\sin z\pi}$$

nec non scribendo $z+\frac{1}{2}$ pro z

[56] $$\Pi(-\tfrac{1}{2}+z).\Pi(-\tfrac{1}{2}-z) = \frac{\pi}{\cos z\pi}$$

E combinatione formulae 54 cum definitione functionis Π sequitur, $\frac{z\pi}{\sin z\pi}$ esse limitem producti infiniti

$$\frac{(1\ .\ 2\ .\ 3\ .\ 4\ldots.k)^2}{(1-zz)(4-zz)(9-zz)\ldots.(kk-zz)}$$

crescente k in infinitum, adeoque

$$\sin z\pi = z\pi(1-zz)(1-\tfrac{zz}{4})(1-\tfrac{zz}{9}) \text{ etc. in inf.}$$

similique modo ex 56 deducitur

$$\cos z\pi = (1-4zz)(1-\tfrac{4zz}{9})(1-\tfrac{4zz}{25}) \text{ etc. in inf.}$$

formulae notissimae, quae ab analystis per methodos prorsus diversas erui solent.

26.

Designante n numerum integrum, valor expressionis

$$\frac{n^{nz}\Pi(k,z).\Pi(k,z-\frac{1}{n}).\Pi(k,z-\frac{2}{n})\ldots..\Pi(k,z-\frac{n-1}{n})}{\Pi(nk,nz)}$$

rite collectus invenitur

$$=\frac{(1.2.3\ldots.k)^n n^{nk}}{1.2.3\ldots.nk.k^{\frac{1}{2}(n-1)}}$$

adeoque a z est independens, sive idem manebit, quicunque valor ipsi z tribuatur. Exhiberi poterit itaque, quoniam $\Pi(k,0)=\Pi(nk,0)=1$, per productum

$$\Pi(k,-\tfrac{1}{n}).\Pi(k,-\tfrac{2}{n}).\Pi(k,-\tfrac{3}{n})\ldots.\Pi(k,-\tfrac{n-1}{n})$$

Crescente igitur k in infinitum, nanciscimur

$$\frac{n^{nz}\Pi z.\Pi(z-\frac{1}{n}).\Pi(z-\frac{2}{n})\ldots.\Pi(z-\frac{n-1}{n})}{\Pi nz}=\Pi(-\tfrac{1}{n}).\Pi(-\tfrac{2}{n}).\Pi(-\tfrac{3}{n})\ldots.\Pi(-\tfrac{n-1}{n})$$

Productum ad dextram, in se ipsum ordine factorum inverso multiplicatum, producit, per form. 55,

$$\frac{\pi}{\sin\frac{1}{n}\pi}.\frac{\pi}{\sin\frac{2}{n}\pi}.\frac{\pi}{\sin\frac{3}{n}\pi}\ldots.\frac{\pi}{\sin\frac{n-1}{n}\pi}=\frac{(2\pi)^{n-1}}{n}$$

Unde habemus theorema elegans

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[57] $$\frac{n^{nz}\Pi z.\Pi(z-\frac{1}{n}).\Pi(z-\frac{2}{n})\ldots.\Pi(z-\frac{n-1}{n})}{\Pi nz} = \frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}$$

27.

Integrale $\int x^{\lambda-1}(1-x^{\mu})^{\nu}\mathrm{d}x$, ita acceptum, ut evanescat pro $x=0$, exprimitur per seriem sequentem, siquidem λ, μ sunt quantitates positivae:

$$\frac{x^{\lambda}}{\lambda} - \frac{\nu x^{\mu+\lambda}}{\mu+\lambda} + \frac{\nu(\nu-1)x^{2\mu+\lambda}}{1.2.(2\mu+\lambda)} - \text{etc.} = \frac{x^{\lambda}}{\lambda}F(-\nu, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}+1, x^{\mu})$$

Hinc ipsius valor pro $x=1$ erit

$$= \frac{\Pi\frac{\lambda}{\mu}.\Pi\nu}{\lambda\Pi(\frac{\lambda}{\mu}+\nu)}.$$

Ex hoc theoremate omnes relationes, quas ill. EULER olim multo labore evolvit, sponte demanant. Ita e. g. statuendo

$$\int\frac{\mathrm{d}x}{\sqrt{(1-x^4)}} = A, \qquad \int\frac{xx\,dx}{\sqrt{(1-x^4)}} = B$$

erit $A = \frac{\Pi\frac{1}{4}.\Pi(-\frac{1}{2})}{\Pi(-\frac{1}{4})}$, $B = \frac{\Pi\frac{3}{4}.\Pi(-\frac{1}{2})}{3\Pi\frac{1}{4}} = \frac{\Pi(-\frac{1}{4}).\Pi(-\frac{1}{2})}{4\Pi\frac{1}{4}}$, adeoque $AB = \frac{1}{4}\pi$. Simul hinc sequitur, quoniam $\Pi\frac{1}{4}.\Pi(-\frac{1}{4}) = \frac{\frac{1}{4}\pi}{\sin\frac{1}{4}\pi} = \frac{\pi}{\sqrt{8}}$,

$$\Pi\tfrac{1}{4} = \sqrt[4]{(\tfrac{1}{8}\pi AA)} = \sqrt[4]{\frac{\pi^3}{128BB}}, \qquad \Pi(-\tfrac{1}{4}) = \sqrt[4]{\frac{\pi^3}{8AA}} = \sqrt[4]{(2\pi BB)}$$

Valor numericus ipsius A, computante STIRLING, habetur $= 1{,}3110287771\ 4605987$, valor ipsius B, secundum eundem auctorem, $= 0{,}5990701173\ 6779611$, ex nostro calculo, artificio peculiari innixo, $= 0{,}5990701173\ 6779610372$.

Generaliter facile ostendi potest, valorem functionis Πz, si z sit quantitas rationalis $= \frac{m}{\mu}$, denotantibus m, μ integros, ex $\mu-1$ valoribus determinatis talium integralium pro $x=1$ deduci posse, et quidem permultis modis diversis. Accipiendo enim pro λ numerum integrum atque pro ν fractionem, cuius denominator $=\mu$, valor illius integralis semper reducitur ad tres Πz, ubi z est fractio cum denominatore $=\mu$; quodvis vero huiusmodi Πz vel ad $\Pi(-\frac{1}{\mu})$, vel ad $\Pi(-\frac{2}{\mu})$, vel ad $\Pi(-\frac{3}{\mu})$ etc. vel ad $\Pi(-\frac{\mu-1}{\mu})$ reduci potest per formulam 45, siquidem z revera est fractio; si enim z est integer, Πz per se constat. Ex illis vero integralium valoribus, generaliter loquendo, quodvis $\Pi(-\frac{m}{\mu})$, si $m<\mu$, per eliminationem erui potest*). Quin adeo semissis talium integralium sufficiet, si formulam 54 simul in auxilium vocamus. Ita e. g. statuendo

$$\int\frac{dx}{\sqrt[5]{(1-x^5)}}=C,\quad \int\frac{dx}{\sqrt[5]{(1-x^5)^2}}=D,\quad \int\frac{dx}{\sqrt[5]{(1-x^5)^3}}=E,\quad \int\frac{dx}{\sqrt[5]{(1-x^5)^4}}=F,\quad \text{erit}$$

$$C=\Pi\tfrac{1}{5}.\Pi(-\tfrac{1}{5}),\quad D=\frac{\Pi\frac{1}{5}.\Pi(-\frac{2}{5})}{\Pi(-\frac{1}{5})},\quad E=\frac{\Pi\frac{1}{5}.\Pi(-\frac{3}{5})}{\Pi(-\frac{2}{5})},\quad F=\frac{\Pi\frac{1}{5}.\Pi(-\frac{4}{5})}{\Pi(-\frac{3}{5})}$$

Hinc propter $\Pi\frac{1}{5}=\frac{1}{5}\Pi(-\frac{4}{5})$, habemus

$$\Pi(-\tfrac{1}{5})=\sqrt[5]{\frac{5C^4}{DEF}},\quad \Pi(-\tfrac{2}{5})=\sqrt[5]{\frac{25C^3D^3}{EEFF}},\quad \Pi(-\tfrac{3}{5})=\sqrt[5]{\frac{125\,CCDDEE}{F^3}},$$
$$\Pi(-\tfrac{4}{5})=\sqrt[5]{(625\,CDEF)}$$

Formulae 54, 55 adhuc suppeditant

$$C=\frac{\pi}{\sin\frac{1}{5}\pi},\quad \frac{D}{F}=\frac{\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}$$

ita ut duo integralia D, E, vel E et F sufficiant, ad omnes valores $\Pi(-\frac{1}{5})$, $\Pi(-\frac{2}{5})$ etc. computandos.

28.

Statuendo $y=\nu x$, atque $\mu=1$, $\frac{\Pi\lambda.\Pi\nu}{\lambda\Pi(\lambda+\nu)}$ erit valor integralis $\int\frac{y^{\lambda-1}(1-\frac{y}{\nu})^\nu dy}{\nu^\lambda}$ ab $y=0$ usque ad $y=\nu$, sive valor integralis $\int y^{\lambda-1}(1-\frac{y}{\nu})^\nu dy$ inter eosdem limites $=\frac{\nu^\lambda\Pi\lambda.\Pi\nu}{\lambda\Pi(\lambda+\nu)}=\frac{\Pi(\nu,\lambda)}{\lambda}$ (form. 47), siquidem ν denotet integrum. Iam crescente ν in infinitum, limes ipsius $\Pi(\nu,\lambda)$ erit $=\Pi\lambda$, limes ipsius $(1-\frac{y}{\nu})^\nu$ autem e^{-y}, denotante e basin logarithmorum hyperbolicorum. Quamobrem si λ est positiva, $\frac{\Pi\lambda}{\lambda}$ sive $\Pi(\lambda-1)$ exprimet integrale $\int y^{\lambda-1}e^{-y}dy$ ab $y=0$ usque ad $y=\infty$, sive scribendo λ pro $\lambda-1$, $\Pi\lambda$ est valor integralis $\int y^\lambda e^{-y}dy$ ab $y=0$ usque ad $y=\infty$, si $\lambda+1$ est quantitas positiva.

Generalius statuendo $y=z^\alpha$, $\alpha\lambda+\alpha-1=\beta$, transit $\int y^\lambda e^{-y}dy$ in $\int\alpha z^\beta e^{-z^\alpha}dz$, quod itaque inter limites $z=0$ atque $z=\infty$ sumtum exprimetur per $\Pi(\frac{\beta+1}{\alpha}-1)$
sive

Valor integralis $\int z^\beta e^{-z^\alpha}dz$, a $z=0$ usque ad $z=\infty$ fit $=\frac{\Pi(\frac{\beta+1}{\alpha}-1)}{\alpha}=\frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$ si modo α atque $\beta+1$ sunt quantitates positivae (si utraque est negativa, in-

*) Haec eliminatio, si pro quantitatibus ipsis logarithmos introducimus, aequationibus tantummodo linearibus applicanda erit.

tegrale per $-\frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$ exprimetur). Ita e. g. pro $\beta = 0$, $\alpha = 2$, valor integralis $\int e^{-zz} dz$ invenitur $= \Pi\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

29.

Ill. EULER pro summa logarithmorum $\log 1 + \log 2 + \log 3 + \text{etc.} + \log z$ eruit seriem $(z+\frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} - \text{etc.}$ ubi $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$ etc. sunt numeri BERNOULLIANI. Per hanc itaque seriem exprimitur $\log \Pi z$; etiamsi enim primo aspectu haec conclusio ad valores integros restricta videatur, tamen rem propius contemplando invenietur, evolutionem ab EULERO adhibitam (Instit. Calc. Diff. Cap. VI. 159) saltem ad valores positivos fractos eodem iure applicari posse, quo ad integros: supponit enim tantummodo, functionem ipsius z, in seriem evolvendam, esse talem, ut ipsius diminutio, si z transeat in $z-1$, exhiberi possit per theorema TAYLORI, simulque ut eadem diminutio sit $= \log z$. Conditio prior innititur *continuitati* functionis, adeoque locum non habet pro valoribus negativis ipsius z, ad quos proin seriem illam extendere non licet: conditio posterior autem functioni $\log \Pi z$ generaliter competit sine restrictione ad valores integros ipsius z. Statuemus itaque

[58] $$\log \Pi z = (z+\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} - \frac{\mathfrak{D}}{7.8z^7} + \text{etc.}$$

Quum hinc quoque habeatur

$$\log \Pi 2z = (2z+\tfrac{1}{2})\log 2z - 2z + \tfrac{1}{2}\log 2\pi + \frac{\mathfrak{A}}{1.2.2z} - \frac{\mathfrak{B}}{3.4.8z^3} + \frac{\mathfrak{C}}{5.6.32z^5} - \frac{\mathfrak{D}}{7.8.128z^7} + \text{etc.}$$

atque per formulam 57, statuendo $n = 2$,

$$\log \Pi(z-\tfrac{1}{2}) = \log \Pi 2z - \log \Pi z - (2z+\tfrac{1}{2})\log 2 + \tfrac{1}{2}\log 2\pi, \text{ fit}$$

[59] $$\log \Pi(z-\tfrac{1}{2}) = z\log z - z + \tfrac{1}{2}\log 2\pi - \frac{\mathfrak{A}}{1.2.2z} + \frac{7\mathfrak{B}}{3.4.8z^3} - \frac{31\mathfrak{C}}{5.6.32z^5} + \frac{127\mathfrak{D}}{7.8.128z^7} - \text{etc.}$$

Hae duae series pro valoribus magnis ipsius z ab initio satis promte convergunt, ita ut summam approximatam commode satisque exacte colligere liceat: attamen probe notandum est, pro quovis valore dato ipsius z, quantumvis magno, praecisionem limitatam tantummodo obtineri posse, quum numeri BERNOULLIANI seriem hypergeometricam constituant, adeoque series illae, si modo satis longe extendantur, certo e convergentibus divergentes evadant. Ceterum negari nequit, theoriam talium serierum divergentium adhuc quibusdam difficultatibus premi, de quibus forsan alia occasione pluribus commentabimur.

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30.

E formula 38 sequitur

$$\frac{\Pi(k,z+\omega)}{\Pi(k,z)}=\frac{z+1}{z+1+\omega}\cdot\frac{z+2}{z+2+\omega}\cdot\frac{z+3}{z+3+\omega}\cdots\frac{z+k}{z+k+\omega}\cdot k^{\omega}$$

unde sumtis logarithmis, in series infinitas evolutis, prodit

$$[60]\quad \log\Pi(k,z+\omega)=\log\Pi(k,z)$$
$$+\omega(\log k-\frac{1}{z+1}-\frac{1}{z+2}-\frac{1}{z+3}-\text{etc.}-\frac{1}{z+k})$$
$$+\tfrac{1}{2}\omega\omega(\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\frac{1}{(z+3)^2}+\text{etc.}+\frac{1}{(z+k)^2})$$
$$-\tfrac{1}{3}\omega^3(\frac{1}{(z+1)^3}+\frac{1}{(z+2)^3}+\frac{1}{(z+3)^3}+\text{etc.}+\frac{1}{(z+k)^3})$$
$$+\text{ etc. in inf.}$$

Series, hic in ω multiplicata, quae, si magis placet, ita etiam exhiberi potest,

$$-\frac{1}{z+1}+\log 2-\frac{1}{z+2}+\log\tfrac{3}{2}-\frac{1}{z+3}+\log\tfrac{4}{3}-\frac{1}{z+4}+\log\tfrac{5}{4}-\text{etc.}+\log\frac{k}{k-1}-\frac{1}{z+k}$$

e terminorum multitudine finita constat, crescente autem k in infinitum, ad limitem certum converget, qui novam functionum transscendentium speciem nobis sistit, in posterum per Ψz denotandam.

Designando porro summas serierum sequentium, in *infinitum* extensarum,

$$\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\frac{1}{(z+3)^2}+\text{ etc.}$$
$$\frac{1}{(z+1)^3}+\frac{1}{(z+2)^3}+\frac{1}{(z+3)^3}+\text{ etc.}$$
$$\frac{1}{(z+1)^4}+\frac{1}{(z+2)^4}+\frac{1}{(z+3)^4}+\text{ etc.}$$
$$\text{etc.}$$

resp. per P, Q, R etc. (pro quibus signa functionalia introducere minus necessarium videtur), habebimus

$$[61]\qquad \log\Pi(z+\omega)=\log\Pi z+\omega\Psi z+\tfrac{1}{2}\omega\omega P-\tfrac{1}{3}\omega^3 Q+\tfrac{1}{4}\omega^4 R-\text{ etc.}$$

Manifesto functio Ψz erit functio derivata prima functionis $\log\Pi z$, adeoque

$$[62]\qquad \frac{d\Pi z}{dz}=\Pi z.\Psi z$$

Perinde erit $P=\frac{d\Psi z}{dz}$, $Q=-\frac{dd\Psi z}{2dz^2}$, $R=+\frac{d^3\Psi z}{2.3dz^3}$ etc.

31.

Functio Ψz aeque fere memorabilis est atque functio Πz, quapropter insigniores relationes ad illam spectantes hic colligemus. E differentiatione aequationis 44 fit

[63] $$\Psi(z+1) = \Psi z + \frac{1}{z+1}$$

unde

[64] $$\Psi(z+n) = \Psi z + \frac{1}{z+1} + \frac{1}{z+2} + \frac{1}{z+3} + \text{etc.} + \frac{1}{z+n}$$

Huius adiumento a valoribus minoribus ipsius z ad maiores progredi, vel a maioribus ad minores regredi licet: pro valoribus maioribus positivis ipsius z functionis valores numerici satis commode per formulas sequentes e differentiatione aequationum 58, 59 oriundas computantur, de quibus tamen eadem sunt tenenda, quae in art. 29 circa formulas 58 et 59 monuimus:

[65] $$\Psi z = \log z + \frac{1}{2z} - \frac{\mathfrak{A}}{2zz} + \frac{\mathfrak{B}}{4z^4} - \frac{\mathfrak{C}}{6z^6} + \text{etc.}$$

[66] $$\Psi(z-\tfrac{1}{2}) = \log z + \frac{\mathfrak{A}}{2.2zz} - \frac{7\mathfrak{B}}{4.8z^4} + \frac{31\mathfrak{C}}{6.32z^6} - \text{etc.}$$

Ita pro $z = 10$ computavimus

$$\Psi z = 2{,}3517525890\ 6672110764\ 743$$

unde regredimur ad

$$\Psi 0 = -0{,}5772156649\ 0153286060\ 653\,^{*)}$$

Pro valore integro positivo ipsius z fit generaliter

[67] $$\Psi z = \Psi 0 + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \text{etc.} + \frac{1}{z}$$

Pro valore integro negativo autem manifesto Ψz fit quantitas infinite magna.

---

*) Quum hic valor inde a figura vigesima discrepet ab eo quem computavit clar. Mascheroni in Adnotat. ad Euleri *Calculum Integr.*, adhortatus sum Fridericum Bernhardum Gothofredum Nicolai, iuvenem in calculo indefessum, ut computum illum repeteret ulteriusque extenderet. Invenit itaque per calculum duplicem, scilicet descendens tum a $z = 50$ tum a $z = 100$,

$$\Psi 0 = -0{,}5772156649\ 0153286060\ 6512090082\ 4024310421$$

Eidem calculatori exercitatissimo etiam debetur tabulae ad finem huius Sectionis annexae pars altera, exhibens valores functionis Ψz ad 18 figuras (quarum ultima haud certa), pro omnibus valoribus ipsius z a 0 usque ad 1 per singulas partes centesimas. Ceterum methodi, per quas utraque tabula constructa est, innituntur partim theorematibus quae hic traduntur, partim calculi artificiis singularibus, quae alia occasione proferemus.

32.

Formula 55 nobis suppeditat $\log\Pi(-z)+\log\Pi(z-1)=\log\pi-\log\sin z\pi$, unde fit per differentiationem

$$[68] \qquad \Psi(-z)-\Psi(z-1)=\pi\operatorname{cotang} z\pi$$

Et quum e definitione functionis Ψ generaliter habeatur

$$[69] \qquad \Psi x-\Psi y=-\frac{1}{x+1}+\frac{1}{y+1}-\frac{1}{x+2}+\frac{1}{y+2}-\frac{1}{x+3}+\text{ etc.}$$

oritur series nota

$$\pi\operatorname{cotang} z\pi=\frac{1}{z}-\frac{1}{1-z}+\frac{1}{1+z}-\frac{1}{2-z}+\frac{1}{2+z}-\frac{1}{3-z}+\text{ etc.}$$

Simili modo e differentiatione formulae 57 prodit

$$[70] \qquad \Psi z+\Psi(z-\tfrac{1}{n})+\Psi(z-\tfrac{2}{n})+\text{ etc. }+\Psi(z-\tfrac{n-1}{n})=n\Psi nz-n\log n$$

adeoque statuendo $z=0$

$$[71] \qquad \Psi(-\tfrac{1}{n})+\Psi(-\tfrac{2}{n})+\Psi(-\tfrac{3}{n})+\text{ etc. }+\Psi(-\tfrac{n-1}{n})=(n-1)\Psi 0-n\log n$$

Ita e. g. habetur

$\Psi(-\frac{1}{2})=\Psi 0-2\log 2=-1{,}9635100260\ 2142347944\ 099$, unde porro

$$\Psi\tfrac{1}{2}=+0{,}0364899739\ 7857652055\ 901.$$

33.

Sicuti in art. praec. $\Psi(-\frac{1}{2})$ ad $\Psi 0$ et logarithmum reduximus, ita generaliter $\Psi(-\frac{m}{n})$, designantibus m, n integros, quorum minor m, ad $\Psi 0$ et logarithmos reducemus. Statuamus $\frac{2\pi}{n}=\omega$, sitque φ alicui angulorum ω, 2ω, $3\omega \ldots. (n-1)\omega$ aequalis; unde $1=\cos n\varphi=\cos 2n\varphi=\cos 3n\varphi$ etc., $\cos\varphi=\cos(n+1)\varphi=\cos(n+2)\varphi$ etc., $\cos 2\varphi=\cos(n+2)\varphi$ etc., nec non $\cos\varphi+\cos 2\varphi+\cos 3\varphi+$ etc. $+\cos(n-1)\varphi+1=0$. Habemus itaque

$$\cos\varphi.\Psi\frac{1-n}{n}=-n\cos\varphi+\cos\varphi.\log 2-\frac{n}{n+1}\cos(n+1)\varphi+\cos\varphi.\log\tfrac{3}{2}-\text{ etc.}$$

$$\cos 2\varphi.\Psi\frac{2-n}{n}=-\frac{n}{2}\cos 2\varphi+\cos 2\varphi.\log 2-\frac{n}{n+2}\cos(n+2)\varphi+\cos 2\varphi.\log\tfrac{3}{2}-\text{ etc.}$$

$$\cos 3\varphi.\Psi\frac{3-n}{n}=-\frac{n}{3}\cos 3\varphi+\cos 3\varphi.\log 2-\frac{n}{n+3}\cos(n+3)\varphi+\cos 3\varphi.\log\tfrac{3}{2}-\text{ etc.}$$

etc. usque ad

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$$\cos(n-1)\varphi.\Psi(-\tfrac{1}{n}) = -\tfrac{n}{n-1}\cos(n-1)\varphi + \cos(n-1)\varphi.\log 2 - \tfrac{n}{2n-1}\cos(2n-1)\varphi$$
$$+\cos(n-1)\varphi.\log\tfrac{3}{2} - \text{ etc.}$$
$$\Psi 0 = -\tfrac{n}{n}\cos n\varphi + \log 2 - \tfrac{n}{2n}\cos 2n\varphi + \log\tfrac{3}{2} - \text{ etc.}$$

atque per *summationem*

$$\cos\varphi.\Psi\tfrac{1-n}{n} + \cos 2\varphi.\Psi\tfrac{2-n}{n} + \cos 3\varphi.\Psi\tfrac{3-n}{n} + \text{ etc. } + \cos(n-1)\varphi.\Psi(-\tfrac{1}{n}) + \Psi 0$$
$$= -n(\cos\varphi + \tfrac{1}{2}\cos 2\varphi + \tfrac{1}{3}\cos 3\varphi + \tfrac{1}{4}\cos 4\varphi + \text{ etc. in infin.})$$

Sed habetur generaliter, pro valore ipsius x unitate non maiori,

$$\log(1-2x\cos\varphi + xx) = -2(x\cos\varphi + \tfrac{1}{2}xx\cos 2\varphi + \tfrac{1}{3}x^3\cos 3\varphi + \text{ etc.})$$

quae quidem series facile sequitur ex evolutione $\log(1-rx)+\log(1-\frac{x}{r})$, denotante r quantitatem $\cos\varphi + \sqrt{-1}.\sin\varphi$. Hinc fit aequatio praecedens

$$[72]\quad \cos\varphi.\Psi\tfrac{1-n}{n} + \cos 2\varphi.\Psi\tfrac{2-n}{n} + \cos 3\varphi.\Psi\tfrac{3-n}{n} + \text{ etc. } + \cos(n-1)\varphi.\Psi(-\tfrac{1}{n})$$
$$= -\Psi 0 + \tfrac{1}{2}n\log(2-2\cos\varphi)$$

Statuatur in hac aequatione deinceps $\varphi = \omega$, $\varphi = 2\omega$, $\varphi = 3\omega$ etc. usque ad $\varphi = (n-1)\omega$, multiplicentur singulae hae aequationes ordine suo per $\cos m\omega$, $\cos 2m\omega$, $\cos 3m\omega$ etc. usque ad $\cos(n-1)m\omega$, productorumque aggregato adiičiatur aequatio 71

$$\Psi\tfrac{1-n}{n} + \Psi\tfrac{2-n}{n} + \Psi\tfrac{3-n}{n} + \text{ etc. } + \Psi(-\tfrac{1}{n}) = (n-1)\Psi 0 - n\log n$$

Quodsi iam perpenditur, esse

$$1 + \cos m\omega.\cos k\omega + \cos 2m\omega.\cos 2k\omega + \cos 3m\omega.\cos 3k\omega$$
$$+ \text{ etc. } + \cos(n-1)m\omega.\cos(n-1)k\omega = 0$$

denotante k aliquem numerorum $1, 2, 3 \ldots (n-1)$ exceptis his duobus m atque $n-m$, pro quibus summa illa fit $=\frac{1}{2}n$, patebit, ex summatione illarum aequationum prodire, post divisionem per $\frac{n}{2}$,

$$[73]\quad \Psi(-\tfrac{m}{n}) + \Psi(-\tfrac{n-m}{m}) =$$
$$2\Psi 0 - 2\log n + \cos m\omega.\log(2-2\cos\omega) + \cos 2m\omega.\log(2-2\cos 2\omega)$$
$$+\cos 3m\omega.\log(2-2\cos 3\omega) + \text{ etc. } + \cos(n-1)m\omega.\log(2-2\cos(n-1)\omega)$$

Manifesto terminus ultimus huius aequationis fit $=\cos m\omega.\log(2-2\cos\omega)$, pen-

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ultimus $=\cos 2m\omega.\log(2-2\cos 2\omega)$ etc., ita ut bini termini semper sint aequales, excepto, si n est par, termino singulari $\cos\frac{n}{2}.m\omega\log(2-2\cos\frac{n}{2}\omega)$, qui fit $=+2\log 2$ pro m pari, vel $=-2\log 2$ pro m impari. Combinando iam cum aequatione 73 hanc

$$\Psi(-\tfrac{m}{n})-\Psi(-\tfrac{n-m}{n})=\pi\operatorname{cotang}\tfrac{m}{n}\pi$$

habemus, pro valore impari ipsius n, siquidem m est integer positivus minor quam n

[74] $$\Psi(-\tfrac{m}{n})=\Psi 0+\tfrac{1}{2}\pi\operatorname{cotang}\tfrac{m\pi}{n}-\log n+\cos\tfrac{2m\pi}{n}.\log(2-2\cos\tfrac{2\pi}{n})$$
$$+\cos\tfrac{4m\pi}{n}.\log(2-2\cos\tfrac{4\pi}{n})+\cos\tfrac{6m\pi}{n}.\log(2-2\cos\tfrac{6\pi}{n})+\text{ etc.}$$
$$+\cos\tfrac{(n-1)m\pi}{n}.\log(2-2\cos\tfrac{(n-1)\pi}{n})$$

Pro valore pari ipsius n autem

[75] $$\Psi(-\tfrac{m}{n})=\Psi 0+\tfrac{1}{2}\pi\operatorname{cotang}\tfrac{m\pi}{n}-\log n+\cos\tfrac{2m\pi}{n}\log(2-2\cos\tfrac{2\pi}{n})$$
$$+\cos\tfrac{4m\pi}{n}\log(2-2\cos\tfrac{4\pi}{n})+\text{ etc. }+\cos\tfrac{(n-2)m\pi}{n}\log(2-2\cos\tfrac{(n-2)\pi}{n})$$
$$\pm\log 2$$

ubi signum superius valet pro m pari, inferius pro impari. Ita e. g. invenitur

$$\Psi(-\tfrac{1}{4})=\Psi 0+\tfrac{1}{2}\pi-3\log 2,\qquad \Psi(-\tfrac{3}{4})=\Psi 0-\tfrac{1}{2}\pi-3\log 2$$
$$\Psi(-\tfrac{1}{3})=\Psi 0+\tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}}-\tfrac{3}{2}\log 3,\quad \Psi(-\tfrac{2}{3})=\Psi 0-\tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}}-\tfrac{3}{2}\log 3$$

Ceterum combinatis his aequationibus cum aequatione 64 sponte patet, Ψz generaliter pro *quovis valore rationali* ipsius z, positivo seu negativo per $\Psi 0$ atque logarithmos determinari posse, quod theorema sane maxime est memorabile.

34.

Quum, per art. 28, $\Pi\lambda$ sit valor integralis $\int y^\lambda e^{-y}\mathrm{d}y$, ab $y=0$ usque ad $y=\infty$, siquidem $\lambda+1$ est quantitas positiva, fit differentiando secundum λ

$$\frac{\mathrm{d}\Pi\lambda}{\mathrm{d}\lambda}=\frac{\mathrm{d}\int y^\lambda e^{-y}\mathrm{d}y}{\mathrm{d}\lambda}=\int y^\lambda e^{-y}\log y\,\mathrm{d}y$$

sive

[76] $$\Pi\lambda.\Psi\lambda=\int y^\lambda e^{-y}\log y.\mathrm{d}y,\quad\text{ab } y=0 \text{ usque ad } y=\infty$$

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Generalius statuendo $y = z^\alpha$, $\alpha\lambda + \alpha - 1 = \beta$, valor integralis $\int z^\beta e^{-z^\alpha} \log z\,\mathrm{d}z$, a $z = 0$ usque ad $z = \infty$, fit

$$= \frac{1}{\alpha\alpha}\Pi\left(\frac{\beta+1}{\alpha} - 1\right).\Psi\left(\frac{\beta+1}{\alpha} - 1\right) = \frac{1}{\alpha(\beta+1)}\Pi\frac{\beta+1}{\alpha}.\Psi\frac{\beta+1}{\alpha} - \frac{1}{(\beta+1)^2}\Pi\frac{\beta+1}{\alpha}$$

siquidem simul $\beta+1$ atque α sunt quantitates positivae, vel aequalis eidem quantitati cum signo opposito, si utraque $\beta+1$, α est negativa.

35.

At non solum productum $\Pi\lambda.\Psi\lambda$, verum etiam ipsa functio $\Psi\lambda$ per integrale determinatum exhiberi potest. Designante k integrum positivum, patet valorem integralis $\int\frac{x^\lambda - x^{\lambda+k}}{1-x}.\mathrm{d}x$, ab $x = 0$ usque ad $x = 1$ esse

$$= \frac{1}{\lambda+1} + \frac{1}{\lambda+2} + \frac{1}{\lambda+3} + \text{etc.} + \frac{1}{\lambda+k}$$

Porro quum valor integralis $\int\left(\frac{1}{1-x} - \frac{kx^{k-1}}{1-x^k}\right)\mathrm{d}x$ generaliter sit $=$ Const. $+ \log\frac{1-x^k}{1-x}$, idem inter limites $x = 0$ atque $x = 1$ erit $= \log k$, unde patet, valorem integralis $S = \int\left(\frac{1 - x^\lambda + x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k}\right)\mathrm{d}x$ inter eosdem limites esse

$$= \log k - \frac{1}{\lambda+1} - \frac{1}{\lambda+2} - \frac{1}{\lambda+3} - \text{etc.} - \frac{1}{\lambda+k}$$

quam expressionem denotabimus per Ω. Discerpamus integrale S in duas partes

$$\int\left(\frac{1-x^\lambda}{1-x}\right)\mathrm{d}x + \int\left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k}\right)\mathrm{d}x$$

Pars prima $\int\frac{1-x^\lambda}{1-x}.\mathrm{d}x$, statuendo $x = y^k$ mutatur in

$$\int\frac{ky^{k-1} - ky^{\lambda k+k-1}}{1-y^k}\mathrm{d}y$$

unde sponte patet, illius valorem ab $x = 0$ usque ad $x = 1$, aequalem esse valori integralis

$$\int\frac{kx^{k-1} - kx^{\lambda k+k-1}}{1-x^k}\mathrm{d}x$$

inter eosdem limites, quum manifesto literam y sub hac restrictione in x mutare liceat. Hinc fit integrale S, inter eosdem limites

$$= \int\left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{\lambda k+k-1}}{1-x^k}\right)\mathrm{d}x$$

Hoc vero integrale, statuendo $x^k = z$, transit in

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$$\int\Big(\frac{z^{\frac{\lambda+1}{k}}}{k(1-z)^{\frac{1}{k}}}-\frac{z^\lambda}{1-z}\Big)\,\mathrm{d}z$$

quod itaque inter limites $z=0$ atque $z=1$ sumtum aequale est ipsi Ω. Sed crescente k in infinitum, limes ipsius Ω est $\Psi\lambda$, limes ipsius $\frac{\lambda+1}{k}$ est 0, limes ipsius $k(1-z)^{\frac{1}{k}}$ vero est $\log\frac{1}{z}$ sive $-\log z$. Quare habemus

[77] $$\Psi\lambda=\int\Big(\frac{1}{\log\frac{1}{z}}-\frac{z^\lambda}{1-z}\Big)\mathrm{d}z=\int\Big(-\frac{1}{\log z}-\frac{z^\lambda}{1-z}\Big)\mathrm{d}z$$

a $z=0$ usque ad $z=1$.

36.

Integralia determinata, per quae supra expressae sunt functiones $\Pi\lambda$, $\Pi\lambda.\Psi\lambda$, restringere oportuit ad valores ipsius λ tales, ut $\lambda+1$ evadat quantitas positiva: haec restrictio ex ipsa deductione demanavit, reveraque facile perspicitur, pro aliis valoribus ipsius λ illa integralia semper fieri infinita, etiamsi functiones $\Pi\lambda$, $\Pi\lambda.\Psi\lambda$ finitae manere possint. Veritati formula 77 certo eadem conditio subesse debet, ut $\lambda+1$ sit quantitas positiva (alioquin enim integrale certo infinitum evadit, etiamsi functio $\Psi\lambda$ maneat finita): sed deductio formulae primo aspectu generalis nullique restrictioni obnoxia esse videtur. Sed propius attendenti facile patebit, ipsi analysi, per quam formula eruta est, hanc restrictionem iam inesse. Scilicet tacite supposuimus, integrale $\int\frac{1-x^\lambda}{1-x}\mathrm{d}x$ cui aequale $\int\frac{kx^{k-1}-kx^{\lambda k+k-1}}{1-x^k}\mathrm{d}x$ substituimus, habere valorem *finitum*, quae conditio requirit, ut $\lambda+1$ sit quantitas positiva. Ex analysi nostra quidem sequitur, haec duo integralia semper esse aequalia, si hoc extendatur ab $x=0$ usque ad $x=1-\omega$, illud ab $x=0$ usque ad $x=(1-\omega)^k$, quantumvis parva sit quantitas ω, modo non sit $=0$: sed hoc non obstante in casu eo, ubi $\lambda+1$ non est quantitas positiva, duo integralia ab $x=0$ usque ad *eundem* terminum $x=1-\omega$ extensa neutiquam ad aequalitatem convergunt, sed potius tunc ipsorum differentia, decrescente ω in infinitum, in infinitum crescet. Hocce exemplum monstrat, quanta circumspicientia opus sit in tractandis quantitatibus infinitis, quae in ratiociniis analyticis nostro iudicio eatenus tantum sunt admittendae, quatenus ad theoriam limitum reduci possunt.

37.

Statuendo in formula 77, $z = e^{-u}$, patet, illam etiam ita exhiberi posse

$$\Psi\lambda = -\int\left(\frac{e^{-u}}{u} - \frac{e^{-u\lambda-u}}{1-e^{-u}}\right)du, \quad \text{ab } u = \infty \text{ usque ad } u = 0, \text{ i. e.}$$

[78] $$\Psi\lambda = \int\left(\frac{e^{-u}}{u} - \frac{e^{-\lambda u}}{e^{u}-1}\right)du, \quad \text{ab } u = 0 \text{ usque ad } u = \infty.$$

(Perinde valor ipsius $\Pi\lambda$ in art. 28 allatus, mutatur statuendo $e^{-y} = \upsilon$, in sequentem

$$\Pi\lambda = \int\left(\log\frac{1}{\upsilon}\right)^{\lambda} d\upsilon, \quad \text{a } \upsilon = 0 \text{ usque ad } \upsilon = 1)$$

Porro patet e formula 77, esse

[79] $$\Psi\lambda - \Psi\mu = \int\frac{z^{\mu}-z^{\lambda}}{1-z}dz, \quad \text{a } z = 0 \text{ usque ad } z = 1$$

ubi praeter $\lambda+1$ etiam $\mu+1$ debet esse quantitas positiva.

Statuendo in eadem formula 77, $z = u^{\alpha}$, designante α quantitatem positivam, fit

$$\Psi\lambda = \int\left(-\frac{u^{\alpha-1}}{\log u} - \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}}\right)du, \quad \text{ab } u = 0 \text{ usque ad } u = 1$$

et quum perinde statui possit, pro valore positivo ipsius β,

$$\Psi\lambda = \int\left(-\frac{u^{\beta-1}}{\log u} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}}\right)du$$

patet, fieri

$$0 = \int\left(\frac{u^{\alpha-1}-u^{\beta-1}}{\log u} + \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}}\right)du$$

sive

$$\int\frac{u^{\alpha-1}-u^{\beta-1}}{\log u}du = \int\left(\frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}} - \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}}\right)du$$

integralibus semper ab $u = 0$ usque ad $u = 1$ extensis. Sed ponendo $\lambda = 0$, integrale posterius *indefinite* assignari potest; est scilicet $= \log\frac{1-u^{\alpha}}{1-u^{\beta}}$, si evanescere debet pro $u = 0$; quare quum pro $u = 1$ statuere oporteat $\frac{1-u^{\alpha}}{1-u^{\beta}} = \frac{\alpha}{\beta}$, erit integrale $\log\frac{\alpha}{\beta} = \int\frac{u^{\alpha-1}-u^{\beta-1}}{\log u}du$, ab $u = 0$ usque ad $u = 1$, quod theorema olim ab ill. Euler per alias methodos erutum est.

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z	$\log \Pi z$	Ψz
0.00	0.0000000000 0000000000	0.5772156649 01532861
0.01	9.9975287306 5869172624	0.5608854578 68674498
0.02	9.9951278719 8879034144	0.5447893104 56179789
0.03	9.9927964208 8883589748	0.5289210872 85430502
0.04	9.9905334004 0842900595	0.5132748789 16830312
0.05	9.9883378587 9012046216	0.4978449912 99870371
0.06	9.9862088685 5581945437	0.4826259358 14825705
0.07	9.9841455256 3523567773	0.4676124198 67553632
0.08	9.9821469485 3403172902	0.4527993380 01712885
0.09	9.9802122775 3951136603	0.4381817634 95334764
0.10	9.9783406739 6180754713	0.4237549404 11076796
0.11	9.9765313194 0866250820	0.4095142760 71694248
0.12	9.9747834150 9201128963	0.3954553339 34292807
0.13	9.9730961811 6469083029	0.3815738268 38792064
0.14	9.9714688560 8569966779	0.3678656106 07749546
0.15	9.9699006960 1252903489	0.3543266779 76279272
0.16	9.9683909742 1917527943	0.3409531528 32261794
0.17	9.9669389805 3852656982	0.3277412847 48392299
0.18	9.9655440208 2789424567	0.3146874437 88860621
0.19	9.9642054164 5653136262	0.3017881155 74610030
0.20	9.9629225038 1404835193	0.2890398965 92188296
0.21	9.9616946338 3869862929	0.2764394897 32192051
0.22	9.9605211715 6456577252	0.2639837000 44220200
0.23	9.9594014956 8673884734	0.2516694306 96100107
0.24	9.9583349981 4361387302	0.2394936791 25936794
0.25	9.9573210837 1550754011	0.2274535333 76265408
0.26	9.9563591696 3881435774	0.2155461686 00265182
0.27	9.9554486852 3498063412	0.2037688437 30623157
0.28	9.9545890715 5360828076	0.1921188983 02221732
0.29	9.9537797810 2903856417	0.1805937494 20369178
0.30	9.9530202771 4980077695	0.1691908888 66799656
0.31	9.9523100341 4034352140	0.1579078803 36141874
0.32	9.9516485366 5449703876	0.1467423567 95996017
0.33	9.9510352794 8014390879	0.1356920179 64169332
0.34	9.9504697672 5460261315	0.1247546278 97003946
0.35	9.9499515141 9025401627	0.1139280126 83088296
0.36	9.9494800438 0996487612	0.1032100582 36977615
0.37	9.9490548886 9188515282	0.0925987081 87861259
0.38	9.9486755902 2321722697	0.0820919618 58406487
0.39	9.9483416983 6257525751	0.0716878723 29281510
0.40	9.9480527714 1057187897	0.0613845445 85116146
0.41	9.9478083757 8828733374	0.0511801337 37897756
0.42	9.9476080858 2329302469	0.0410728433 24024375
0.43	9.9474514835 4291742066	0.0310609236 71447052
0.44	9.9473381584 7445730981	0.0211426703 33530475
0.45	9.9472677074 5205163055	0.0113164225 86445845
0.46	9.9472397344 2994856529	0.0015805619 87083418
0.47	9.9472538503 0190930853	+ 0.0080664890 11364893
0.48	9.9473096727 2650396072	0.0176262683 88849468
0.49	9.9474068259 5806639475	0.0271002758 35486201
0.50	9.9475449406 8308573196	0.0364899739 78576520

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z	$\log \Pi z$	Ψz
0.50	9.9475449406 8308573196	+0.0364899739 78576520
0.51	9.9477236538 6182228429	0.0457967895 61914496
0.52	9.9479426085 7494550351	0.0550221145 79551622
0.53	9.9482014538 7500065798	0.0641673073 66077154
0.54	9.9484998446 4251966174	0.0732336936 45365776
0.55	9.9488374414 4659973817	0.0822225675 39644344
0.56	9.9492139104 0978143536	0.0911351925 40635189
0.57	9.9496289230 7706494873	0.0999728024 44444623
0.58	9.9500821562 8891076887	0.1087366022 51781439
0.59	9.9505732920 5807738191	0.1174277690 35011042
0.60	9.9511020174 5015512544	0.1260474527 73476253
0.61	9.9516680244 6766136244	0.1345967771 58445210
0.62	9.9522710099 3756789859	0.1430768403 68980212
0.63	9.9529106754 0213704917	0.1514887158 19958383
0.64	9.9535867270 1294797674	0.1598334528 83415463
0.65	9.9542988754 2799988466	0.1681120775 84327804
0.66	9.9550468357 1178337730	0.1763255932 71894293
0.67	9.9558303272 3821579829	0.1844749812 67329607
0.68	9.9566490735 9634064632	0.1925612014 89132418
0.69	9.9575028024 9869525351	0.2005851930 56747012
0.70	9.9583912456 9225480685	0.2085478748 73493948
0.71	9.9593141388 7186450668	0.2164501461 89604789
0.72	9.9602712215 9607519880	0.2242928871 46157521
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0.74	9.9622869327 4222223320	0.2398032061 35096466
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0.76	9.9644363698 1871920339	0.2550855103 23688336
0.77	9.9655606232 6853798084	0.2626431686 02762795
0.78	9.9667175803 2189101417	0.2701462043 14883540
0.79	9.9679070054 1227146665	0.2775953776 14168016
0.80	9.9691286662 4097614416	0.2849914332 93861542
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0.82	9.9716677818 6428658993	0.2996270965 64887544
0.83	9.9729847878 1655271065	0.3068681204 96501033
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0.87	9.9785640362 2467644771	0.3353360466 94485409
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0.89	9.9815374283 3339013630	0.3492814254 57135499
0.90	9.9830693440 8561111078	0.3561841611 64059720
0.91	9.9846311382 9969520321	0.3630410646 48881123
0.92	9.9862226132 1076437381	0.3698527244 06401469
0.93	9.9878435735 8573930651	0.3766197179 23498793
0.94	9.9894938266 7611664682	0.3833426119 46740214
0.95	9.9911731821 7189109803	0.3900219627 42043086
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