ARTICLES

Platonic Passages

RICHARD P. JERRARD[1]
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

JOHN E. WETZEL[2]
1601 Lakeside Drive Unit A
Champaign, IL 61821
jewetz@comcast.net

LIPING YUAN
Hebei Normal University
Shijiazhuang 050016 P. R. China
lpyuan@hebtu.edu.cn

According to J. Wallis [**11**, pp. 470–71], more than three hundred years ago Prince Rupert of the Rhine[3] (1619–1682) won a wager that a hole could be cut through a cube large enough to permit another cube of the same size to slide through. In 1950, D. J. E. Schrek [**7**] published a detailed proof of this somewhat surprising fact together with a careful review of its history. C. J. Scriba [**8**] showed in 1968 that the regular tetrahedron and octahedron have this same property: each can transit through a suitable tunnel in another of the same size and type.

We show here that the remaining two platonic solids, the dodecahedron and icosahedron, also have this property (as announced in [**3**]).

Many convex bodies in R^3 share this Rupert property, but not all. It is easy to give examples of convex bodies that do not have the Rupert property, for example, the unit ball and the equilateral drum (a circular cylinder of unit diameter and height closed on each end by disks). But we know of no convex polyhedron that does not have the Rupert property. With a certain hesitancy, we even suggest that perhaps every convex polyhedron in R^3 has the Rupert property. In any case, an example of a convex polyhedron lacking the Rupert property would be of considerable interest.

Preliminaries

By a *convex body* in R^3 we mean a compact, convex set with nonempty interior. There seems to be little ambiguity in what is meant by the somewhat imprecise language, "a hole can be cut" in a convex body $\mathfrak{K}$; by a "hole" is meant simply a "straight tunnel."

Let π_n be a plane with unit normal vector n, and let $\varpi_n : R^3 \to \pi_n$ be the projection map of R^3 onto π_n. Let γ be a simple closed curve that lies in the plane π_n, and let I_γ be the domain in π_n interior to γ.

Math. Mag. **90** (2017) 87–98. doi:10.4169/math.mag.90.2.87.
MSC: Primary 51M16, Secondary 51M04; 52C15.

[1]Deceased July 23, 2014.

[2]Corresponding author.

[3]Count Palatine of the Rhine and Duke of Bavaria, son of Frederick V, the Winter King, Elector Palatine, and king of Bohemia, and Elizabeth, daughter of James I of England.

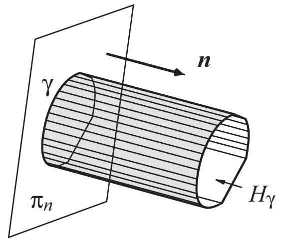


Figure 1 A right cylinder.

The *right cylinder* C_γ with *directrix* γ and *direction* n (Figure 1) is the set

$$C_\gamma = \{y + tn \in R^3\colon\ y \in \gamma,\ -\infty < t < \infty\},$$

and the *tunnel* H_γ defined by the right cylinder C_γ is just its interior region:

$$H_\gamma = \{y + tn \in R^3\colon\ y \in I_\gamma,\ -\infty < t < \infty\}.$$

Since $\mathfrak{K}$ is a convex body, its projection $\varpi_n(\mathfrak{K})$ on a plane π_n is a convex set with nonempty interior, and it follows that the boundary of the projection is a simple closed convex curve γ. The passage of $\mathfrak{K}$ through the tunnel H_γ determined by the right cylinder with directrix γ and direction n is described completely by

$$\mathfrak{K}_t = tn + \mathfrak{K} \subset H_\gamma, \quad -\infty < t < \infty.$$

Note that H_γ is an open set in R^3; so we demand in particular that $\mathfrak{K}_t$ not touch the bounding cylinder C_γ during the transit.

Now the following fundamental fact is virtually obvious.

Theorem 1. *Let $\mathfrak{K}$ be a convex body in R^3. If there are planes π_m and π_n so that the projection $P_i = \varpi_n(\mathfrak{K})$ of $\mathfrak{K}$ onto the plane π_n fits in the interior of the projection $P_o = \varpi_m(\mathfrak{K})$ of $\mathfrak{K}$ onto the plane π_m, then $\mathfrak{K}$ can be passed through the tunnel H_γ whose direction is m and whose directrix γ is the boundary of the projection P_o.*

We call P_i and P_o the *inner* and *outer* projections of $\mathfrak{K}$, respectively.

When the conditions of this theorem are met, we say that the convex body $\mathfrak{K}$ has the *Rupert property,* or is *Rupert*.

Nieuwland constants If a convex body $\mathfrak{K}$ has the Rupert property, a natural question to ask (and asked by Rupert in the case of the cube) is how large a body $\mathfrak{K}'$ similar to $\mathfrak{K}$ can be passed through a hole in $\mathfrak{K}$, i.e., for how large a positive scalar ν can the convex body $\nu\mathfrak{K}$ be passed through a suitable tunnel in $\mathfrak{K}$? We call this *Nieuwland's question* after P. Nieuwland (1764–1794), who answered this question for the cube. (Nieuwland's results for the cube were published posthumously by Swinden [**10**, pp. 512–513, 608–610].) Define the Nieuwland constant $\nu(\mathfrak{K})$ of a convex body $\mathfrak{K}$ by

$$\nu(\mathfrak{K}) = \sup\{\nu > 0 : \text{there is a tunnel in } \mathfrak{K} \text{ through which } \nu\mathfrak{K} \text{ can pass}\}.$$

Thus $\mathfrak{K}$ has the Rupert property if and only if $\nu(\mathfrak{K}) \geq 1$. Nieuwland showed that $\nu(\mathfrak{K}) = \frac{3}{4}\sqrt{2}$ if $\mathfrak{K}$ is a cube, so a cube of any edge $e < \frac{3}{4}\sqrt{2}$ can be passed through a suitable tunnel in a unit cube, but no cube of edge $e > \frac{3}{4}\sqrt{2}$ can be so passed. Determining the Nieuwland constant for a convex body $\mathfrak{K}$ is generally difficult, in part because there may be a multiplicity of tunnels to consider.

Table 1 collects the known estimates of the Nieuwland constants for the tetrahedron, cube, and octahedron, and it includes the new estimates for the dodecahedron and icosahedron given by Jerrard and described in the last section.

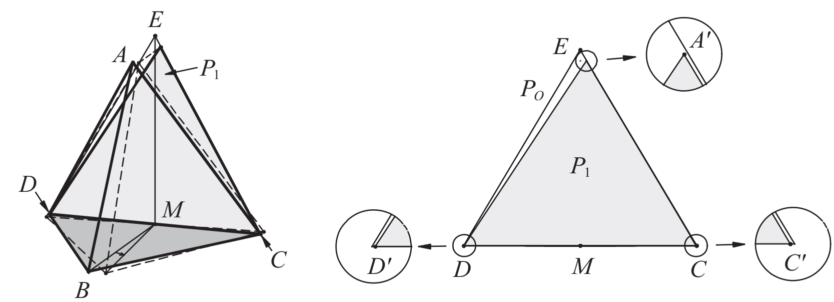
TABLE 1: Nieuwland constant estimates.

Platonic Solid	Nieuwland Estimate
Tetrahedron $\mathfrak{T}$	$\nu(\mathfrak{T}) \geq \frac{2}{5}\sqrt{3}(\sqrt{6}-1) > 1.004\,235$
Cube $\mathfrak{C}$	$\nu(\mathfrak{C}) = \frac{3}{4}\sqrt{2} \geq 1.060\,660$
Octahedron $\mathfrak{O}$	$\nu(\mathfrak{O}) \geq \frac{3}{4}\sqrt{2} \geq 1.060\,660$
Dodecahedron $\mathfrak{D}$	$\nu(\mathfrak{D}) \geq \frac{171}{170} > 1.005\,882$
Icosahedron $\mathfrak{I}$	$\nu(\mathfrak{I}) \geq 1108/1098 > 1.009\,107$

Platonic solids, I

According to Theorem 1, to show that a convex body $\mathfrak{K}$ has the Rupert property, we must exhibit the two projections $P_o(\mathfrak{K})$ and $P_i(\mathfrak{K})$ so that $P_i(\mathfrak{K})$ fits in the (relative) interior of $P_o(\mathfrak{K})$. The tetrahedron, cube, and octahedron can easily be handled geometrically, but the dodecahedron and icosahedron are a bit more difficult. We begin with the tetrahedron.

Tetrahedron Let $\mathfrak{T} = ABCD$ be a regular tetrahedron with unit edge labeled with the equilateral triangle BCD as base and apex A, and take the outer projection P_o to be an equilateral triangle CDE of unit side, placed as shown in Figure 2a with E in the plane π through the line CD perpendicular to the plane of the base of $\mathfrak{T}$. The orthogonal projection of $\mathfrak{T}$ into π nearly fits into the interior of P_o, the difficulty being at the vertices C and D. But clearly a small rotation of $\mathfrak{T}$ about the altitude EM



a. Initial placement. b. The projection.

Figure 2 The tetrahedron $\mathfrak{T}$.

of triangle CED moves the projections C' and D' of C and D on π into the edge CD of triangle CED keeping the projection A' of A inside P_o. It follows that $\mathfrak{T}$ has the Rupert property, because a small upward translation of the rotated $\mathfrak{T}$ moves the projections A', B', C', and D' of all four vertices inside P_o.

To estimate the Nieuwland constant $\nu(\mathfrak{T})$ we chose the angle ϑ of rotation so that the projected segment $A'C'$ lies inside P_o and parallel to the side EC (Figure 2b). One can show that

$$\vartheta = 60^\circ - \arcsin \tfrac{1}{3}\sqrt{6} \approx 5.264\,389^\circ. \tag{1}$$

Finally, since $A'E > C'C = D'D$, $CD = 1$, and $C'D' = \cos\vartheta$, the least upper bound of the ratio by which T can be expanded and still pass through a similarly situated tunnel in T is

$$\frac{CD}{C'D'} = \sec\vartheta = \frac{2}{5}\sqrt{3}(\sqrt{6}-1) > 1.004\,235, \tag{2}$$

where the surd expression follows from (1). It seems likely that $\lambda_{\mathfrak{T}} = \frac{2}{5}\sqrt{3}(\sqrt{6}-1)$, but in any case, $\nu(\mathfrak{T}) > 1.004\,235$.

Remark. A similar argument shows that every tetrahedron (not just the regular tetrahedron $\mathfrak{T}$) is Rupert. (Let P_o be the face with the greatest area, so that the projection of the fourth vertex into the plane of P_o lies in P_o.)

Cube The fact that the cube has the Rupert property is very well known and has become a staple of recreational mathematics. A careful and detailed development was given in 1950 by Schrek [**7**] (see [**4**]); here we settle for a drawing that shows the essential details. Let $\mathfrak{C}$ be the unit cube with opposite vertices A and B. The two points that lie on adjacent edges of $\mathfrak{C}$ at distance $3/4$ from A and the two points on the two parallel edges at distance $3/4$ from B (as pictured in Figure 3) are coplanar, and the plane on which they lie meets the cube in a square of side $\frac{3}{4}\sqrt{2} \approx 1.06 > 1$,

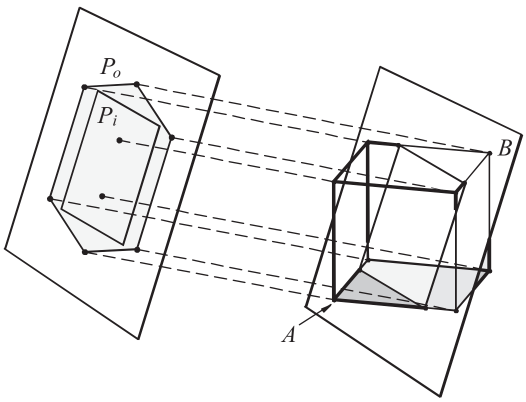


Figure 3 The cube $\mathfrak{C}$.

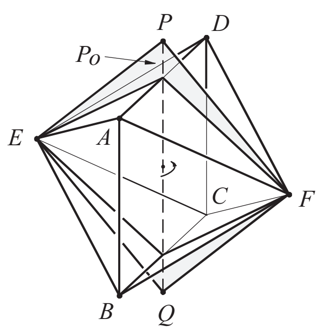
a square that is, in fact, the largest square that fits in the cube. Take P_o to be the projection of $\mathfrak{C}$ on that plane (pictured on a parallel plane in Figure 3 for clarity), and take the inner projection P_i to be the unit square. Since the inner projection P_i fits in the interior of the outer projection P_o, the cube has the Rupert property. Scaling the cube so that its largest square just touches the edges of P_o shows that $\nu(\mathfrak{C}) = \frac{3}{4}\sqrt{2}$ (for details, see [**7**]).

Remarks. Every box (i.e., every rectangular parallelepiped) is Rupert (see [**4**]). In fact, similar methods show that every parallelepiped is Rupert.

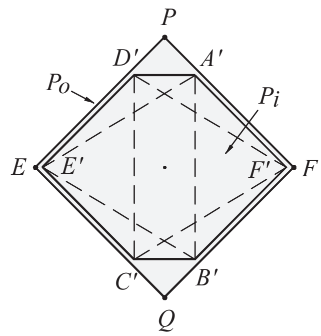
Determining the Nieuwland constant $\nu(\mathfrak{K})$ of the cube involves finding the largest square that fits in a unit cube. In his "Mathematical Games" column in the November 1966 issue of *Scientific American,* Martin Gardner asked for the largest cube that fits in a tesseract (a 4-cube) of unit edge (see [**1**, 172–73]). The question was answered in 1996, when Kay Shultz showed that the edge of the largest such cube is the square root of the smaller of the two real roots of the polynomial $4x^4 - 28x^3 - 7x^2 + 16x + 16$, approximately 1.0074348. (See the end of this article for a poem by Kay Schulz about solving this problem.)

More generally, what is the edge $f(m,n)$ of the largest m-cube that can fit in an n-cube of unit edge? Most of what is currently known about $f(m,n)$ is summarized by G. Huber in his preface to his reprinting of Shultz's 1996 notes, [**9**].

Octahedron The unit octahedron, $\mathfrak{O} = EABCDF$ in Figure 4a, is formed by eight equilateral faces formed into two congruent four-sided pyramids sharing a square base.



a. Projections.



b. P_i fits in P_o.

Figure 4 The octahedron $\mathfrak{O}$.

Its projection in the direction $\overrightarrow{EF}$ is a unit square; let P_o be the unit square $PEQF$ positioned as pictured, with one diagonal EF and the other diagonal passing through the midpoints of the sides AD and CD of the medial square $ABCD$. It is clear that a small rotation of $\mathfrak{O}$ about the axis PQ (leaving the square P_o in place) moves the projections E' and F' into the square while leaving the projections A', B', C', and D' of the vertices A, B, C, and D inside the square; in other words, the projection P_i of the rotated octahedron $\mathfrak{O}$ lies in the interior of P_o (Figure 4b). This establishes the claim that the octahedron has the Rupert property.

To bound the Nieuwland constant $\nu(\mathfrak{O})$, we choose the angle of rotation so that the projection $E'C'$ on P_o of the rotated edge EC is parallel to the edge EQ of the square. One can see that the appropriate angle ϑ of rotation is given by

$$\vartheta = \arccos\frac{1}{\sqrt{3}} - \arccos\sqrt{\frac{2}{3}} \approx 19.5^\circ. \tag{3}$$

Then the projections of the rotated edges BF, FA, and ED are all parallel to sides of the unit square, and the projection P_i of (the rotated octahedron) $\mathfrak{O}$ is as pictured in Figure 4b. A short calculation from (3) shows that

$$\cos\vartheta = \frac{2\sqrt{2}}{3},$$

and it follows that

$$E'F' = \sqrt{2}\cos\vartheta = \frac{4}{3}.$$

Consequently,

$$\nu(\mathfrak{O}) \geq \frac{EF}{E'F'} = \frac{3\sqrt{2}}{4} > 1.060\,660,$$

as shown in Table 1. It seems likely that the equality holds.

Remark. Similarly, the cuboctahedron (the solid formed by clipping each vertex of a cube by a plane through the midpoint of the three adjacent edges) is Rupert.

Platonic solids, II

We employ visual geometric reasoning to handle the dodecahedron, and then, for variety, we use coordinate methods to deal with the icosahedron. Many of these results were first worked out by Jerrard prior to 2005, and some of what follows has been assembled from notes he prepared more than a decade ago.

Dodecahedron Let O be the center of symmetry of the dodecahedron $\mathfrak{D}$, and label the 20 vertices of $\mathfrak{D}$ as illustrated in Figure 5. The vertices and edges visible "from above" are drawn bold, and those not visible are drawn in gray. Note that the vertices labeled k and $21 - k$ are symmetric with respect to the center O. Let P be the center of the face 1-2-3-4-5 and M the midpoint of the edge 7-12.

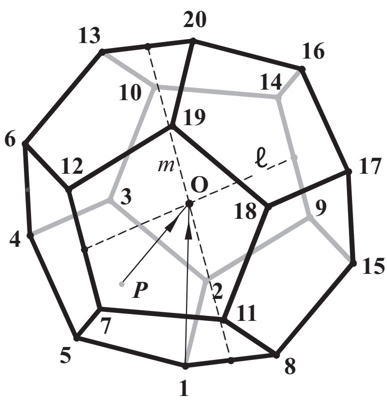


Figure 5 Labeled dodecahedron $\mathfrak{D}$.

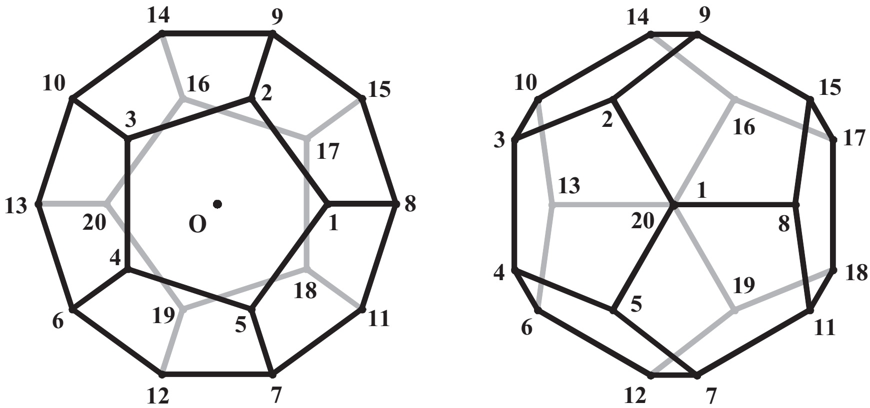
Let P_o be the projection of $\mathfrak{D}$ onto a faceplane, viz., in the direction of $\overrightarrow{P\mathrm{O}}$, pictured in Figure 6a. Its boundary is a regular decagon.

Let P_1 be the projection of $\mathfrak{D}$ from a vertex, viz., in the direction of $\overrightarrow{1\mathrm{O}}$, pictured in Figure 6b. Its boundary is an irregular dodecagon whose opposite edges are parallel.

In Figure 7a, the projection of the vertex labeled k of $\mathfrak{D}$ retains the label k in the projection P_1 but is underscored in the projection P_o.

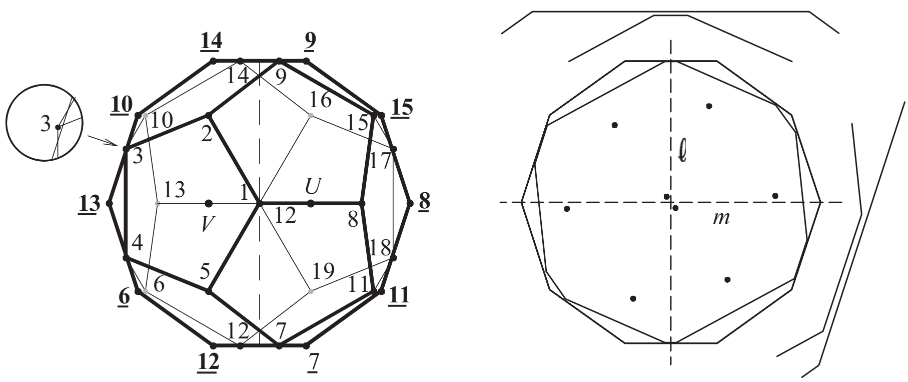
Placed as pictured with its center of symmetry at the center of P_o, the projection P_1 very nearly fits in P_o. The edge 14-9 of $\mathfrak{D}$ projects into the edge $\underline{14}$-$\underline{9}$ of the decagon boundary of P_o, the opposite edge 12-7 projects into the edge $\underline{12}$-$\underline{7}$, and the four vertices 3, 6, 17, and 18 project to points that lie just outside of P_o, as illustrated in the detail.

The dodecahedron can be rotated a little so that its projection P_i fits in the interior of P_o. We first make a small rotation of $\mathfrak{D}$ about the axis ℓ that joins the midpoints of the edges 14-9 and 12-7, moving the projections of both the edges 3-4 and 17-18 into the interior of P_o. This small rotation leaves the projected edges 14-9 and 12-7 on the edges $\underline{14}$-$\underline{9}$ and $\underline{12}$-$\underline{7}$. The axis m that joins the midpoints of the edges 8-1 and 13-20 (most easily seen in Figure 5), is parallel to and midway between the edges 7-12 and 9-14. It follows that a second small rotation of the rotated dodecahedron $\mathfrak{D}$ about m moves both the rotated edges 9-14 and 7-12 into the interior of P_o. Although this rotation moves the images of the four vertices 3, 4, 17, and 18 toward the boundary decagon of P_o, if the rotation is sufficiently small it leaves the projection of all of the vertices of the rotated dodecahedron (the 12 vertices of the interior irregular dodecagon



a. P_o. b. P_1.

Figure 6 Projections P_o and P_1 of $\mathfrak{D}$.



a. P_1 on P_o. b. P_i on P_o.

Figure 7 Fitting.

and the 8 marked points) in the interior of P_o (Figure 7b, in which P_i is centrally symmetric and concentric with P_o). So the projection P_i of the rotated dodecahedron lies in the interior of P_o.

Consequently $\mathfrak{D}$ is Rupert.

We find this intuitive geometric reasoning convincing. Jerrard introduced coordinates, determined suitable rotation angles (1.55° about m and 6.45° about ℓ), and verified numerically that the 20 vertices of P_i actually lie in the interior of P_o. (In Figure 7b the vertices are the 12 vertices of the projected dodecagon and the 8 dots. The fit is quite tight.) We omit the numerical details.

In 1776, Euler showed by an elementary geometric argument that the product of two rotations about nonparallel axes in R^3 is again a rotation. A contemporary discussion can be found, for example, in [**6**]. Consequently it is possible to rotate the dodecahedron $\mathfrak{D}$ about a suitable single axis in such a way that the rotated dodecahedron will pass through a suitable tunnel in a second dodecahedron of the same size.

Jerrard also determined the shortest distance from the inner projection P_i to the boundary of the outer projection P_o and showed that the inner projection can be expanded by about 1 part in 170, i.e., by a factor of at least $171/170 = 1.005\,882$, and still fit in P_o. This is the bound for $\nu(\mathfrak{D})$ included in Table 1.

Icosahedron Position the icosahedron $\mathfrak{I}$ with its center at the origin O, and label the 20 vertices as shown in Figure 8, with k and $13 - k$ symmetric with respect to O. The edges and vertices that are visible "from above" are drawn bold, and those hidden are in gray.

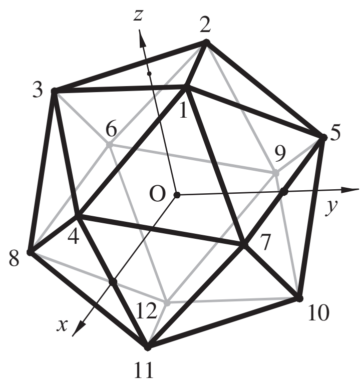


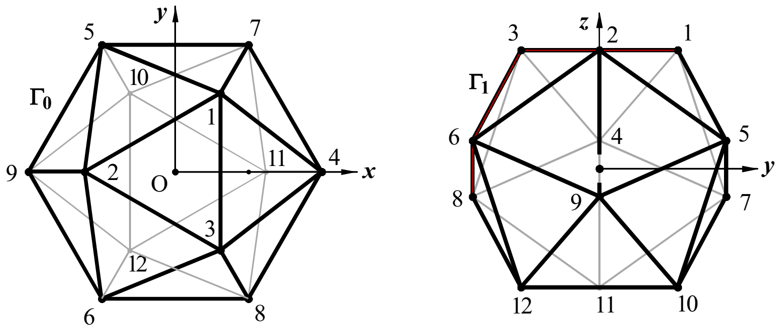
Figure 8 Labeled icosahedron $\mathfrak{I}$.

We choose an orthogonal coordinate system as follows: the origin is O, the positive x-axis passes through the midpoint of the edge 4-11, the positive y-axis passes through the midpoint of the edge 5-7, and the positive z-axis passes through the center of the face 1-2-3. The projection of $\mathfrak{I}$ into the xy-plane is a regular hexagon, and we chose the unit distance so that this hexagon has edge of length 2. The coordinates of the vertices are given in Table 2.

TABLE 2: Vertex coordinates of $\mathfrak{I}$.

Vertices	Coordinates	Values
1 and -12	$\left(1 + \frac{1}{\sqrt{5}},\ \sqrt{3}(1 + \frac{1}{\sqrt{5}}),\ 2(1 + \frac{2}{\sqrt{5}})\right)$	(1.4472, 2.5066, 3.7889)
2 and -11	$\left(-2(1 + \frac{1}{\sqrt{5}}),\ 0,\ 2(1 + \frac{2}{\sqrt{5}})\right)$	(−2.8944, 0, 3.7889)
3 and -10	$\left(1 + \frac{1}{\sqrt{5}},\ -\sqrt{3}(1 + \frac{1}{\sqrt{5}}),\ 2(1 + \frac{2}{\sqrt{5}})\right)$	(1.4472, −2.5066, 3.7889)
4 and -9	$\left(2(1 + \frac{3}{\sqrt{5}}),\ 0,\ \frac{2}{\sqrt{5}}\right)$	(4.6833, 0, 0.8944)
5 and −8	$\left(-(1 + \frac{3}{\sqrt{5}}),\ \sqrt{3}(1 + \frac{3}{\sqrt{5}}),\ \frac{2}{\sqrt{5}}\right)$	(−2.3416, 4.0558, 0.8944)
6 and −7	$\left(-(1 + \frac{3}{\sqrt{5}}),\ -\sqrt{3}(1 + \frac{3}{\sqrt{5}}),\ \frac{2}{\sqrt{5}}\right)$	(−2.3416, −4.0558, 0.8944)

Let P_o be the projection of $\mathfrak{I}$ onto the xy-plane, shown in Figure 9a. Its boundary Γ_o is a regular hexagon. To avoid confusion, we retain the same names for the projected vertices, so, for example, vertex 1 in this figure is the projection of vertex 1 into the xy-plane in Figure 8. The (x, y) coordinates of the points in this figure are obtained by setting the z-coordinate to zero in Table 2.



a. P_o, in the xy-plane. b. P_1, in the yz-plane.

Figure 9 Projections P_o and P_1.

Let P_1 be the projection of $\mathfrak{I}$ onto the yz-plane, shown in Figure 9b. Its boundary Γ_1 is an irregular octagon whose opposite sides are parallel. Since this projection is in the direction of the median from vertex 2 to the edge 1-3 in the face triangle 1-2-3 of $\mathfrak{I}$, it has been called the "face normal" projection (see "Regular icosahedron," *Wikipedia, the Free Encyclopedia* (accessed June 2015)). In this drawing edge 2-9 has been broken to show the origin and the edge 4-9, which lie behind it. The (y, z) coordinates of the points in this figure are obtained by setting the x-coordinate to zero in Table 2.

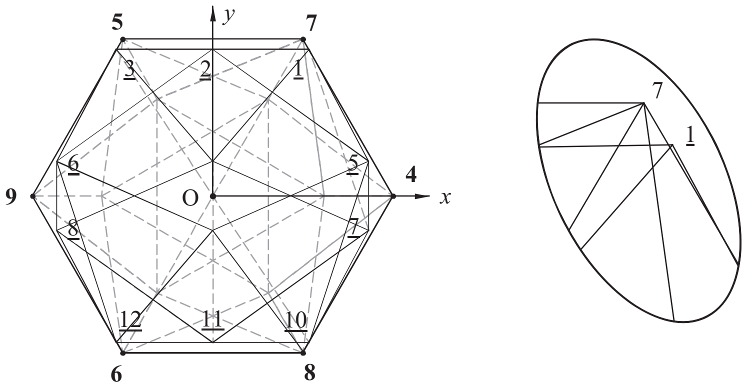
Placed with its center at the center of P_o, the projection P_1 very nearly but not quite fits in P_o (Figure 10a, in which the axes from P_1 have been replaced by those of P_o). In this figure the vertices on the boundary of P_o are named as in Figure 9a, but to avoid confusion the vertices of the projection P_1 are underscored. Figure 10b shows an enlarged detail of the fit near $\underline{1}$. The situation at $\underline{3}$, $\underline{10}$, and $\underline{12}$ is symmetrically the same.

Note that when P_1 is placed on P_o, the y-axis lands on the x-axis of P_o (Figure 10a). If $\mathfrak{I}$ is rotated by an angle φ about the y-axis, both the line segments $\underline{1}$-$\underline{3}$ and $\underline{10}$-$\underline{12}$ move into the interior of P_o, the segments $\underline{6}$-$\underline{8}$ and $\underline{5}$-$\underline{7}$ remain in the interior, but since the y-axis is parallel to the faceplane $\underline{1}$-$\underline{2}$-$\underline{3}$, the projections of the vertices $\underline{2}$ and $\underline{11}$ move closer to the boundary segments 5-7 and 6-8, respectively.

TABLE 3: Coordinates of projected vertices.

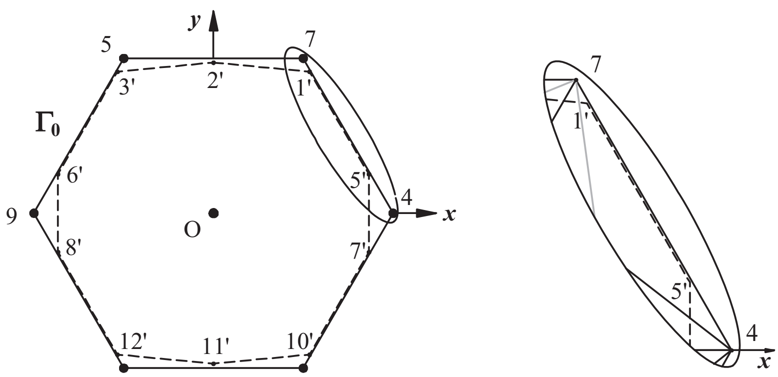
Vertices	(x, y) coordinates in P_o.	$\varphi = 3.1289^\circ$
$1'$ and $-12'$	$\left(\sqrt{3}(1+\frac{1}{\sqrt{5}}),\ (2+\frac{4}{\sqrt{5}})\cos\varphi - (1+\frac{1}{\sqrt{5}})\sin\varphi\right)$	$(2.5066,\ 3.7042)$
$2'$ and $-11'$	$\left(0,\ (2+\frac{4}{\sqrt{5}})\cos\varphi + (2+\frac{2}{\sqrt{5}})\sin\varphi\right)$	$(0.0,\ 3.9412)$
$3'$ and $-10'$	$\left(-\sqrt{3}(1+\frac{1}{\sqrt{5}}),\ (2+\frac{4}{\sqrt{5}})\cos\varphi - (1+\frac{1}{\sqrt{5}})\sin\varphi\right)$	$(-2.5066,\ 3.7042)$
$4'$ and $-9'$	$\left(0,\ \frac{2}{\sqrt{5}}\cos\varphi - (2+\frac{6}{\sqrt{5}})\sin\varphi\right)$	$(0.0,\ 0.6375)$
$5'$ and $-8'$	$\left(\sqrt{3}(1+\frac{3}{\sqrt{5}}),\ \frac{2}{\sqrt{5}}\cos\varphi + (1+\frac{3}{\sqrt{5}})\sin\varphi\right)$	$(4.0558,\ 1.0209)$
$6'$ and $-7'$	$\left(-\sqrt{3}(1+\frac{3}{\sqrt{5}}),\ \frac{2}{\sqrt{5}}\cos\varphi + (1+\frac{3}{\sqrt{5}})\sin\varphi\right)$	$(-4.0558,\ 1.0209)$

Because $\underline{3}$ and $\underline{1}$ are so much closer to the boundary segments than is $\underline{2}$, it seems clear that if φ is sufficiently small, all the projected vertices will be inside the boundary



a. P_1 on P_o. b. Detail.

Figure 10 P_1 nearly fits in P_o.



a. P_1 on P_o. b. Detail.

Figure 11 P_i fits inside P_o.

of P_o, as desired. Let $\mathfrak{I}$ be the rotated icosahedron, and denote the projection into P_o of the vertex k by k'. Table 3 shows the coordinates of the projected vertices k' in terms of φ, found from the familiar formulas for the rotation through the angle φ about the origin in the xz-plane. The coordinates are presented in the (x, y) axes in P_o, and to the same scale as the vertices given in Table 2.

Choosing φ so that the projection of the edge $\underline{1}$-$\underline{5}$ is inside and parallel to the edge 7-4 minimizes the maximum distance between these two segments, and a calculation shows that this is accomplished for $\varphi = 3.1289°$ (and similarly for $\underline{7}$-$\underline{10}$, $\underline{3}$-$\underline{6}$, and $\underline{12}$-$\underline{8}$), while leaving the image of vertices $\underline{2}$ and $\underline{11}$ inside P_o (Figure 11a).

The interior of P_o is described by the system of inequalities (4).

$$\begin{cases} -2\sqrt{3}(1+\frac{3}{\sqrt{5}}) - \sqrt{3}x < y < 2\sqrt{3}(1+\frac{3}{\sqrt{5}}) - \sqrt{3}x \\ -\sqrt{3}(1+\frac{3}{\sqrt{5}}) < y < \sqrt{3}(1+\frac{3}{\sqrt{5}}) \\ -2\sqrt{3}(1+\frac{3}{\sqrt{5}}) - \sqrt{3}x < y < 2\sqrt{3}(1+\frac{3}{\sqrt{5}}) + \sqrt{3}x \end{cases} \tag{4}$$

Verifying that these coordinates satisfy the constraints (4) establishes that the vertices of $\mathfrak{I}_i$ are in the interior of P_0, and this completes the coordinate proof that the icosahedron $\mathfrak{I}$ is Rupert. The tunnel cut in $\mathfrak{I}$ leaves but a thin shell behind, so making a physical model showing the interpenetration is likely to be difficult.

By computing the minimum distance between the boundary of P_i and P_o, one can see that the inner projection P_i can be enlarged by at least 1 part in 109.8, so the Nieuwland constant $\nu(\mathfrak{I})$ is greater than $1108/1098 > 1.009\,107\,47$, as shown in Table 1. We omit further details.

Remarks. There are interesting analogous questions in R^n that have not been studied. Which convex bodies in R^n can be passed through a hole in another?

The mathematics literature includes many "passage" problems related to Rupert's in which more general motions are permitted to accomplish the transit. In 1920 Zindler [**13**] described an affine cube that can pass through a circular ring of radius smaller than that of its smallest circumscribed cylinder, and this and related notions have been investigated by many authors since. See, for example, Zamfirescu [**12**] and the references included therein.

Acknowledgment We are indebted to Prof. Yuan's student Ying Chai for her interest in and contributions to these results, in particular for the numerical calculations for the icosahedron. We thank Dr. Daniel Lichtblau of Wolfram Research for his assistance with Mathematica. We are also indebted to several members of the DesignCAD User Forum, who helped with the preparation of some of the more difficult drawings. We want especially to thank Kevin Chapman (aka Droll Troll), IMSI/Designs, Novato, CA, for his insightful help and his patience. Finally, we express our gratitude to Prof. Richard Bishop and other participants in the UIUC Geometric Potpourri Seminar for their interest in this problem and their helpful suggestions.

REFERENCES

1. M. Gardner, *The Colossal Book of Mathematics*. W. W. Norton, New York, 2001.
2. R. P. Jerrard, J. E. Wetzel, Multipurpose stoppers are Rupert, *College. Math. J.* **39** (2008) 90–94.
3. ———, Platonic passages, 07T-51-2, *Abstracts Amer. Math. Soc.* **28** (2007) 351.
4. ———, Prince Rupert's rectangles, *Amer. Math. Monthly* **111** (2004) 22–31.
5. ———, Problem 11291, *Amer. Math. Monthly* **114** (2007) 451.
6. B. Palais, R. Palais, S. Rodi, A disorienting look at Euler's theorem on the axis of a rotation, *Amer. Math. Monthly* **116** (2009) 892–909, doi:10.4169/000298909X477014.
7. D. J. E. Schrek, Prince Rupert's problem and its extension by Pieter Nieuwland, *Scripta Math.* **16** (1950) 73–80, 261–267.
8. C. J. Scriba, Das problem des Prinzen Ruprecht von der Pfalz, *Praxis Math.* **10** (1968) 241–246.
9. K. R. Shultz, Largest m-cube in an n-cube: Partial solution, with a preface by G. Huber, KITP, NSF-ITP-13-142 (2013).
10. J. H. van Swinden, *Grondbeginsels der Meetkunde* (Elements of Geometry). Pieter den Hengst and Son, Amsterdam, 1816.
11. J. Wallis, *De Algebra Tractatus; Historicus & Practicus*, 1685, in Opera Mathematica, vol. 2, Oxoniae: E Theatro Sheldoniano, 1693.
12. T. Zamfirescu, Pushing convex and other bodies through rings and holes, *An. Univ. Vest Timiş. Ser. Mat.-Inform* **XLVIII** no. 3 (2010) 00–10.
13. K. Zindler, Über konvexe Gebilde, *Monatsh. Math. Phys.* 30 (1920) 87–102, 10.1007/BF01699908.

Summary. It is well known that a hole can be cut in a cube large enough to permit a second cube of equal size to pass through, a result attributed to Prince Rupert of the Rhine by J. Wallis more than three centuries ago. C. Scriba showed nearly 50 years ago that the tetrahedron and the octahedron have this same property. Somewhat surprisingly, the remaining two platonic solids, the dodecahedron and the icosahedron, also have this property: each can be passed through a suitable tunnel in another of the same size and kind. We supply the details.

RICHARD JERRARD (MR Author ID: 94525) received his bachelor's and master's degrees in mechanical engineering from the University of Wisconsin in 1950 and, after military service, his mathematics Ph.D. from the University of Michigan in 1957. He joined the mathematics faculty at the University of Illinois in 1958, spent three years at Cambridge University and the University of Warwick, and retired in 1995. Early in his career he

worked in applied mathematics, and later he became interested in topology and geometry. He died in 2014 after a long period of decline.

JOHN WETZEL (MR Author ID: 182185) did his undergraduate work at Purdue University, received his Ph.D. in mathematics from Stanford University in 1964, and spent his academic career (apart from sabbaticals) at the University of Illinois from which he retired in 1999. Always interested in classical geometry, he has studied the ways in which one shape fits in another—questions he regards as "fitting problems for retirement."

LIPING YUAN (MR Author ID: 689149) received her bachelor's, master's, and doctor's degrees in mathematics from Hebei Normal University (P.R. China) in 1993, 2000, and 2003, her second mathematics Ph.D. from Dortmund University (Germany) in 2006. She joined the mathematics faculty at Hebei Normal University in 1995. She is working in the area of discrete and combinatorial geometry.

Consider a cube for a minute,
And imagine the largest square in it.
Then if you're a math whiz,
Tell me how big it is;
It's tricky to even begin it!

Now let us move up one dimension:
Find the cube of the largest extension
That fit's (neatly packed)
Into one tesseract,
And, boy, will you have stress and tension!

Martin Gardner proposed this last question,
And I solved it, at no one's suggestion.
It took 15 years
Of blood, sweat and tears,
And gave me severe indigestion.

My proof fills up 100 pages;
Till I solved it, it stumped all the sages.
It was recently checked
And pronounced quite correct,
But it hasn't augmented my wages.

–Kay R. Pechenick DeVicci Shultz