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On Rational Solutions of Systems of Linear Differential Equations

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Let K be a field of characteristic zero and $\mathcal{M}(Y) = N$ a system of linear differential equations with coefficients in $K(x)$. We propose a new algorithm to compute the set of rational solutions of such a system. This algorithm does not require the use of cyclic vectors. It has been implemented in Maple V and it turns out to be faster than cyclic vector computations. We show how one can use this algorithm to give a method to find the set of solutions with entries in $K(x)[\log x]$ of $\mathcal{M}(Y) = N$.

1. Introduction

In the study of certain class of solutions of linear differential equations, particularly in differential Galois theory (see Singer, 1991, 1996; van Hoeij and Weil, 1996), there appears the following problem. Let K be a field of characteristic zero. Let $\mathcal{M} = \frac{d}{dx} - M$, $M \in Mat_n(K(x))$ be a matrix differential operator and N a column-vector with n components in $K(x)$.

(P) Determine the set of all the rational solutions $Y \in K(x)^n$ of the differential system:

$$\mathcal{M}(Y) = \frac{dY}{dx} - MY = N. \tag{1.1}$$

The main purpose of the present paper is to develop an algorithm which solves the problem above in an efficient way.

First of all, let us recall that by means of the so-called "cyclic vector" method (see, for example, Barkatou, 1993, Section 5, pp. 193–195) any matrix differential equation of the type (1.1) can be reduced to an equivalent scalar nth differential equation with coefficients in $K(x)$. So, in theory our problem **(P)** is equivalent to the one of finding the rational solutions of a scalar differential equation. On the other hand this last problem is much easier to solve than the former and there exist a number of algorithms, more or less satisfying, for solving it (Abramov, 1989; Schwarz, 1989; Abramov and Kvashenko, 1991; Abramov *et al.*, 1995; Becken, 1995; Barkatou, 1997). In view of all this, one has a first method (as far as we know, the only one which is proposed in the literature) to solve the problem **(P)**. However it is to be noted that computing an equivalent scalar equation from a given matrix equation can be very costly, especially when one deals with systems of "large" dimension n (see Section 4.6). Hence, from an algorithmic point of view, "direct methods" are to be preferred.

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In this paper we propose a new algorithm to solve problem **(P)** directly (avoiding the use of cyclic vectors). It is divided into two steps. In the first step we construct a (scalar) rational function R such that: for all $Y \in K(x)^n$, if Y is a solution of the equation $\mathcal{M}(Y) = N$, then RY is a polynomial vector. The substitution $Y = R^{-1}Z$ into (1.1) then reduces the problem to finding polynomial solutions of a matrix differential equation in Z of the same type as (1.1). The second step of our algorithm deals with this last problem. In fact, we will develop an algorithm which solves the more general problem where the right-hand side of (1.1) depends (linearly) on some given parameters.

We will now give a brief summary of these two steps. Let us start by the first step and suppose, for the sake of clarity, that $N = 0$. Recall that any irreducible factor p of the denominator of a (possible) rational solution of $\mathcal{M}(Y) = 0$ divides the denominator of the matrix system M. More precisely, one can prove that if p^{-m} is the largest power of p dividing the denominator of Y then m must satisfy the so-called *indicial equation* at p (this is the polynomial equation over $K[x]/(p)$). The smallest integer root m_p of this equation gives a lower bound on the possible m. Thus, if the m_p's are known then $\prod p^{-m_p}$ gives the desired function R (here the product is taken over the set of the irreducible factors of the denominator of M). It is then clear that we need a method for computing the indicial equation and the bound m_p for each *singularity* p. The problem, however, is that the indicial equation (at a given point p) is not immediately obtained for a given system. This led us to introduce the concept of *simple* systems. Roughly speaking, a system is simple (at p) if the indicial equation (at p) can be obtained immediately from the matrix system. We prove that any system can be reduced to an equivalent simple one. This reduction can be achieved using an adapted version of the *super-reduction* algorithm of Hilali and Wazner (1987) (see the Appendix). This yields an algorithm for computing indicial equations associated with a given system.

We will now explain how we proceed in the second step of our algorithm. Here we look for polynomial solutions of a system $\mathcal{M}(Y) = N$. First of all the system is reduced to a simple form at ∞. This yields, in particular, the indicial equation at infinity which plays an important role in this step. Our idea is to compute, one after another, the different monomials of the (possible) general polynomial solution. To do this we proceed as follows.

First we substitute $Y = cx^\mu + Z$ where $c \in K^n$ and $\mu \in \mathbb{N}$ into the equation $\mathcal{M}(Y) = N$; then Z satisfies a similar equation $\mathcal{M}(Z) = N - \mathcal{M}(x^\mu c)$. Then we try to determine μ and c so that $\mu = \mathrm{d}^\circ\, Y > \mathrm{d}^\circ\, Z$. By inspecting at each step the degree of the right-hand side of the system and the integer roots of the indicial equation at ∞, one can decide whether the (μ, c) exists and compute it if the case arises. When such a couple (μ, c) is found we restart with the equation satisfied by Z and so on. This process can be repeated until we obtain an equation of the form $\mathcal{M}(W) = F = N - \mathcal{M}(c_1 x^{\mu_1} + \cdots + c_\ell x^{\mu_\ell})$ for which there is no non-zero polynomial solution with degree $< \mu_\ell$. Hence, the original equation $\mathcal{M}(Y) = N$ has a polynomial solution iff $F = 0$. This yields a system of linear algebraic equations for the components of the c_i's. Then, the general solution of this system gives the general polynomial solution of the differential equation $\mathcal{M}(Y) = N$.

The rest of the paper is organized as follows. The first step of our algorithm will be discussed in Section 4 while Section 3 concerns the second step. In Section 5 we will consider the following more general problem.

Determine the set of solutions $Y \in K(x)[\log x]^n$ for a given differential system (1.1). We will show that this problem reduces to problem **(P)**. In Section 6 we will show how our algorithm can be adapted in order to to compute meromorphic formal series solutions of system linear differential equations. We have implemented[†] our algorithm under MAPLE V. Examples of computation will be given in Section 4.5. Finally, we have added an appendix on the notion of super-irreducibility.

2. Notation

We begin by setting up some notation. If a rational function $a = u/v$, $u, v \in K[x]$ is not 0, we set $\mathrm{ord}_\infty(a) = -\mathrm{d}^\circ a = -\mathrm{d}^\circ u + \mathrm{d}^\circ v$, and denote by $\ell c_\infty(a)$ the coefficient of $x^{-\mathrm{ord}_\infty a}$ in the expansion of a as a power series in x^{-1}:

$$a = \ell c_\infty(a) x^{-\mathrm{ord}_\infty a} + \text{ terms of order } > \mathrm{ord}_\infty(a).$$

We set $\mathrm{ord}_\infty(0) = -\mathrm{d}^\circ 0 = +\infty$.

If U is a matrix (or a vector) of rational functions then we define its order at infinity (notation $\mathrm{ord}_\infty U$) to be the minimum of the orders of its entries. We define $\mathrm{d}^\circ U := -\mathrm{ord}_\infty U$ and denote by $\ell c_\infty(U)$ the coefficient of $x^{-\mathrm{ord}_\infty(U)}$ in the series expansion of U at infinity.

Let p be a finite "point" of $K(x)$, that is an irreducible polynomial in $K[x]$.

If f is a non-zero element of $K(x)$, we define $\mathrm{ord}_p(f)$ (read order of f at p) to be the unique integer n such that:

$$f = \frac{a}{b} p^n, \qquad \text{with } a, b \in K[x] \setminus \{0\}, \qquad p \nmid a \qquad \text{and} \qquad p \nmid b.$$

We set $\mathrm{ord}_p(0) = +\infty$. For $f, g \in K(x)$, one has

(i) $\mathrm{ord}_p(f + g) \geq \min(\mathrm{ord}_p(f), \mathrm{ord}_p(g))$, and equality holds if $\mathrm{ord}_p(f) \neq \mathrm{ord}_p(g)$.
(ii) $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$.

Recall that each element $f \in K(x)$ has a unique p-adic expansion:

$$f = f_n p^n + f_{n+1} p^{n+1} + \cdots$$

where $n = \mathrm{ord}_p(f)$, the f_i's are polynomial of degree $< \deg p$, with $f_n \neq 0$ (when $f \neq 0$). This coefficient f_n will be called the *leading coefficient* of f at p and will be denoted by $\ell c_p(f)$.

If $A = (a_{i,j})$ is a matrix (or a vector) with entries in $K(x)$, we define its order at p by $\mathrm{ord}_p A = \min(\mathrm{ord}_p(a_{i,j}))$. Each matrix (or vector) A with entries in $K(x)$ has a unique p-adic expansion of A:

$$A = p^{\mathrm{ord}_p A}(A_{0,p} + pA_{1,p} + \cdots),$$

here the $A_{i,p}$ are matrices (or vectors) with entries in the set $\{a \in K[x] \mid \deg a < \deg p\}$.

We will denote by $\ell c_p(A)$ the coefficient of $p^{\mathrm{ord}_p A}$ in the p-adic expansion of A at p.

If $A = (\frac{a_{i,j}}{b_{i,j}})$ is a matrix (or a vector) of reduced rational functions then we will denote by $\mathrm{denom}(A)$ (the *denominator* of A) the least common multiple of the $b_{i,j}$'s.

We will say that a matrix (or a vector) A has a pole at p if $\mathrm{ord}_p(A) < 0$ (this holds iff p divides $\mathrm{denom}(A)$).

For a commutative ring R with unit element and n a positive integer, the ring of $n \times n$ matrices with entries in R will be denoted by $\mathrm{Mat}_n(R)$. We write $GL(n, R)$ for the group of invertible matrices in $\mathrm{Mat}_n(R)$.

[†]The program is contained in the package ISOLDE at `http://www-lmc.imag.fr/CF/logiciel.html`

By I_n we denote the identity matrix of order n. By $\mathrm{diag}\,(a, b, c, \ldots)$ we denote the square diagonal matrix whose diagonal elements are $a, b, c, \ldots$.

Let $T \in GL(n, K(x))$. The substitution $Y = TZ$ transforms a differential system

$$\mathcal{M}(Y) = \frac{dY}{dx} - MY = N, \qquad M \in \mathrm{Mat}_n(K(x)), N \in K(x)^n$$

into a system

$$\tilde{\mathcal{M}}(Z) = \frac{dZ}{dx} - \tilde{M}Z = \tilde{N},$$

where

$$\tilde{M} = T[M] := T^{-1}MT - T^{-1}\frac{dT}{dx} \qquad \text{and} \qquad \tilde{N} = T^{-1}N.$$

These two differential systems (resp. the matrices M and $\tilde{M}$) are said to be *equivalent*.

3. Search of the Polynomial Solutions

Given a matrix differential equation (1.1), the problem is to determine the set of all $Y \in K[x]^n$ which are solutions of (1.1). Our method is based on the following idea.

Write $Y = cx^{\mu} + Z$ where $c \in K^n$ and $\mu \in \mathbb{N}$; if $\mathcal{M}(Y) = N$ then Z satisfies the equation:

$$\mathcal{M}(Z) = N - \mathcal{M}(x^{\mu}c) \tag{3.2}$$

which is of the same type as equation (1.1).

Try to determine $\mu \in \mathbb{N}$ and $c \in K^n$ in such a way that $\mu = \mathrm{d}^\circ Y > \mathrm{d}^\circ Z$. If such a μ and such a c are found then restart with Z and the equation (3.2) and so on.

This process can be repeated until we obtain an equation of the form:

$$\mathcal{M}(W) = F = N - \mathcal{M}(c_1x^{\mu_1} + \cdots + c_\ell x^{\mu_\ell})$$

which has no non-zero polynomial solutions with degree $< \mu_\ell$. Hence, the original equation $\mathcal{M}(Y) = N$ has a polynomial solution iff $F = 0$. This yields a system of linear algebraic equations for the components of the c_i's. Then, the general solution of this system gives the general polynomial solution of the differential equation $\mathcal{M}(Y) = N$.

We will show in the following how the couple (μ, c) can be determined.

3.1. Simple operators and the indicial equation at $x = \infty$

In this section we will introduce some useful definitions and notation.

Consider a matrix differential equation of the form

$$\mathcal{M}(Y) = x\frac{dY}{dx} - MY = N, \qquad M \in \mathrm{Mat}_n(K(x)), N \in K(x)^n. \tag{3.3}$$

Let us study the action of the operator $\mathcal{M}$ on a monomial $x^{-\lambda}c$, $c \in K^n$. For this, we need first to rewrite the equation $\mathcal{M}(Y) = N$ in a suitable form. Let M_i denote the ith row of the matrix M. Put $\alpha_i = -\min\,(\mathrm{ord}_\infty(M_i), 0)$, for $1 \le i \le n$, and define the matrix D by $D = \mathrm{diag}\,(x^{-\alpha_1}, \ldots, x^{-\alpha_n})$. Then equation (3.3) can be rewritten as

$$\mathcal{L}(Y) = Dx\frac{dY}{dx} - AY = B, \tag{3.4}$$

where $A = DM$ and $B = DN$. Let us note that the matrix $D \in \mathrm{Mat}_n(K[x^{-1}])$ and the matrix A satisfies $\mathrm{ord}_\infty(A) \geq 0$. So, these two matrices can be written respectively as:

$$D = D_0 + \frac{D_1}{x} + \cdots$$
$$A = A_0 + \frac{A_1}{x} + \cdots,$$

where the dots represent terms of higher order at ∞.

We will now describe the action of the operator $\mathcal{L}$ given by (3.4) on a monomial $x^{-\lambda}c$. One has $\mathcal{L}(x^{-\lambda}c) = -\lambda x^{-\lambda}Dc - x^{-\lambda}Ac$. It then follows that

$$\mathcal{L}(x^{-\lambda}c) = -x^{-\lambda}((\lambda D_0 + A_0)c + \cdots) \tag{3.5}$$

where the dots represent terms of order > 0.

In consequence we have the following property:

$\mathrm{ord}_\infty(\mathcal{L}(x^{-\lambda}c)) \geq \lambda$, for all $\lambda \in \mathbb{Z}$ and for all $c \in K^n$ and equality holds iff $c \notin \ker(A_0 + \lambda D_0)$.

In view of this last property, it is natural to expect that the values of λ for which the determinant $\det(A_0 + \lambda D_0)$ is zero, will play an important and a particular role for our problem. But, it may happen that this last determinant vanishes identically in λ, in which case it is quite useless to us. This motivates the following definition:

Definition 3.1. An operator $\mathcal{M}$ is said to be *simple* at ∞ if $\det(A_0 + \lambda D_0) \neq 0$ (as a polynomial in λ). In this case the polynomial $E_\infty(\lambda) := \det(A_0 + \lambda D_0)$ will be called the indicial polynomial of $\mathcal{M}$ at ∞.

As a first example of simple operators, let us consider a matrix differential operator $\mathcal{M} = x\frac{d}{dx} - M$, with $\mathrm{ord}_\infty(M) \geq 0$ (in this case the point $x = \infty$ is at worst a singularity of the first kind for the operator $\mathcal{M}$). Then for each row M_i of M one has $\mathrm{ord}_\infty M_i \geq 0$. So, $\alpha_i = 0$ for all i and hence D is the identity matrix I_n. It then follows that the matrices D_0 and A_0 (as defined above) are respectively I_n and the matrix $M(\infty)$. So, $\det(A_0 + \lambda D_0) = \det(M(\infty) + \lambda I_n) \neq 0$. Consequently, the operator $\mathcal{M}$ is simple and its indicial polynomial is the characteristic polynomial of the matrix $-M(\infty)$ which is a polynomial in λ of degree n.

Another example of simple operators is given by the class of companion matrix differential operators. That is, operators $\mathcal{M} = x\frac{d}{dx} - M$, with a matrix of the form

$$M = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & & \\ & & & & 1 \\ a_0 & a_1 & & & a_{n-1} \end{pmatrix}.$$

Indeed, if one puts $a_n := 1$ and $h := -\min_{0 \leq i \leq n} \mathrm{ord}_\infty a_i$, then one has $\alpha_n = h$ and $\alpha_i = 0$ for $i = 0, \ldots, n-1$. Thus, $D_0 = \mathrm{diag}(1, \ldots, 1, \epsilon)$ where $\epsilon = 1$ or 0 according to whether $h = 0$ or not. The matrix A_0 is given by

$$A_0 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & & \\ & & & & 1 \\ \bar{a}_0 & \bar{a}_1 & & & \bar{a}_{n-1} \end{pmatrix}$$

with $\bar{a}_i = \ell c_\infty(a_i)$ if $\mathrm{ord}_\infty a_i = -h$ and 0 otherwise. It then follows that

$$\det(A_0 + \lambda D_0) = \sum_{\mathrm{ord}_\infty a_i = -h} \bar{a}_j (-\lambda)^j,$$

which is a non-zero polynomial in λ. Thus any companion matrix differential operator is simple at ∞.

Remark 3.1. As a consequence of this last result, one sees that any matrix differential operator can be reduced to an equivalent operator which is simple at ∞. This is because one knows (from the cyclic vector lemma) that any matrix differential operator is equivalent to a companion matrix operator. However, this result will not be used in this paper since our strategy is to avoid the use of cyclic vector method.

Now let us give an example of an operator which is not simple. Consider the differential system

$$\mathcal{M}(Y) = x\frac{dY}{dx} - MY = 0, \qquad M = \begin{bmatrix} 1 & x^3 \\ \frac{2}{x} & 1 \end{bmatrix}.$$

One has

$$D = \begin{bmatrix} x^{-3} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A = DM = \begin{bmatrix} x^{-3} & 1 \\ \frac{2}{x} & 1 \end{bmatrix}.$$

Thus

$$D_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It then follows that the matrix $A_0 - \lambda D_0$ is singular for all values of λ. Consequently, the operator $\mathcal{M}$ is not simple at ∞. Now consider the matrix

$$T = \begin{bmatrix} 0 & x^2 \\ 1 & 0 \end{bmatrix}$$

and put $X = TY$ then X satisfies the equivalent differential system $\tilde{\mathcal{M}}(X) = x\frac{dX}{dx} - \tilde{M}X = 0$ where

$$\tilde{M} = \left(TM + x\frac{dT}{dx}\right)T^{-1} = \begin{bmatrix} 3 & 2x \\ x & 1 \end{bmatrix}.$$

One can readily verify that the operator $\tilde{\mathcal{M}}$ is simple at ∞ and that its indicial polynomial at ∞ is the constant polynomial -2.

This situation above is, in fact, general. In other words, any differential operator can be reduced (without using the cyclic vector method) to an equivalent operator which is simple at ∞. This is contained in the following proposition:

Proposition 3.1. *Given a differential system $\mathcal{M}(Y) = N$ then one can construct a non-singular matrix T such that if one puts $Y = TX$ then X satisfies an equivalent differential system $\tilde{\mathcal{M}}(X) = \tilde{N}$ which is simple at ∞. Moreover the matrix T^{-1} is polynomial in x.*

We will prove (see the appendix) that such a transformation T can be constructed using the so-called *super-reduction* algorithm due to Hilali and Wazner (1987).

It is to be noted that the fact that the inverse transformation T^{-1} (in the above proposition) can be chosen to be polynomial is important when one is interested in the polynomial solutions of $\mathcal{M}(Y) = N$, since in this case, if Y is a polynomial solution of the last system then $X = T^{-1}Y$ is a polynomial solution of the equivalent system $\tilde{\mathcal{M}}(X) = \tilde{N}$. Thus without any loss of generality we may suppose (and we will assume in the following) that $\mathcal{M}$ is *simple* at ∞.

Now we prove a useful lemma.

LEMMA 3.1. *Consider a matrix differential equation of the form (3.4). Then, for all $Z \in K(x)^n$ (or more generally $Z \in (K[[x^{-1}]][x])^n$ a column vector of meromorphic formal power series in x^{-1}) $\operatorname{ord}_\infty \mathcal{L}(Z) \geq \operatorname{ord}_\infty Z$ and equality holds iff $\ell c_\infty(Z) \notin \ker(A_0 + (\operatorname{ord}_\infty Z)D_0)$. This is the case, for instance, when $Z \neq 0$ and $\operatorname{ord}_\infty Z$ is not a root of the indicial equation of $\mathcal{L}$ at infinity. In particular, if $Z \neq 0$ and $\mathcal{L}(Z) = 0$ then $E_\infty(\operatorname{ord}_\infty Z) = 0$ and $\ell c_\infty(Z) \in \ker(A_0 + (\operatorname{ord}_\infty)D_0)$.*

PROOF. Write $Z = \ell c_\infty(Z)x^{-\operatorname{ord}_\infty Z} +$ terms of higher order, then using the linearity of $\mathcal{L}$ and (3.5) one sees that:

$$\mathcal{L}(Z) = -x^{-\operatorname{ord}_\infty Z}((\operatorname{ord}_\infty Z)D_0 + A_0)\ell c_\infty(Z) + \text{ terms of higher order.}$$

It then follows that $\operatorname{ord}_\infty \mathcal{L}(Z) \geq \operatorname{ord}_\infty Z$ and equality holds iff $(\operatorname{ord}_\infty Z D_0 + A_0)\ell c_\infty(Z) \neq 0$.

Now, if $\mathcal{L}(Z) = 0$, with $Z \neq 0$ then $((\operatorname{ord}_\infty Z)D_0 + A_0)\ell c_\infty(Z) = 0$ with $\ell c_\infty(Z) \neq 0$, and therefore, $\det(A_0 + (\operatorname{ord}_\infty Z)D_0) = 0$.□

3.2. HOW TO CHOOSE μ AND c?

Let us keep the notation of the previous section. Consider a matrix differential equation of the form (3.4). We want to search for the polynomial solutions of the equation $\mathcal{L}(Y) = B$. In fact, we will develop a method which solves the more general problem where B depends on some parameters. More precisely, let B be of the form:

$$B = B_0 + \sum_{i=1}^{m} p_i B_i,$$

where the B_i are column vectors with rational function coefficients and the p_i's are given parameters. Our purpose is to find the set of all parameters p_i for which the equation $\mathcal{L}(Y) = B$ has polynomial solutions and to give these solutions.

As mentioned above, we may assume, without any loss of generality, that the operator $\mathcal{L}$ is simple at ∞. This means that the polynomial $E_\infty(\lambda) = \det(A_0 + \lambda D_0) \neq 0$. If B is not zero, we set $\delta = \mathrm{d}^\circ B$ and $B = \ell c_\infty(B)x^\delta +$ terms of degree $< \delta$. If $B = 0$, we set $\delta = -\infty$ and $\ell c_\infty(B) = 0$. Note that the components of $\ell c_\infty(B)$ are polynomials of degree ≤ 1 in the parameters $p_1, \ldots, p_m$.

Now write $Y = x^\mu c + Z$, with $c \in K^n$ and $\mu \in \mathbb{N}$. Then $\mathcal{L}(Y) = B$ gives $\mathcal{L}(Z) = B - \mathcal{L}(x^\mu c)$ and hence, using (3.5), one finds

$$\mathcal{L}(Z) = x^\delta \ell c_\infty(B) + x^\mu(-\mu D_0 + A_0)c + \text{ terms of degree } < \max(\mu, \delta). \tag{3.6}$$

The question is: can we find a vector $c \in K^n$ and a $\mu \in \mathbb{N}$ such that $\mathcal{L}(Y) = B$ and $\mathrm{d}^\circ Z < \mu$? By Lemma 3.1 we know that $\mathrm{d}^\circ \mathcal{L}(Z) \leq \mathrm{d}^\circ Z$. It then follows that a necessary condition that μ and c exist is that the degree of the right-hand side of (3.6) must be $<\mu$.

Let $\mathcal{R} := \{\lambda \in \mathbb{N}, \text{ s.t. } E_\infty(-\lambda) = 0\}$. Then several possibilities may occur:

(1) If $\mathcal{R} = \emptyset$ and $\delta < 0$ then the degree of the right-hand side of (3.6) is equal to μ for all $\mu \in \mathbb{N}$. So, in this case there is no couple (μ, c) answering the question.
(2) If $\mathcal{R} \neq \emptyset$ and $\max \mathcal{R} > \delta$ then one may take $\mu = \max \mathcal{R}$ and c an arbitrary element in $\ker(A_0 - (\max \mathcal{R}) D_0)$.
(3) If $\mathcal{R} \neq \emptyset$ and $\delta \geq \max \mathcal{R}$ or $\mathcal{R} = \emptyset$ and $\delta \geq 0$ then the only possible choice for μ is $\mu = \delta$. Indeed if one takes $\mu \neq \delta$ the degree of the right-hand side of (3.6) is $\geq \mu$ and hence $\mathrm{d}^\circ Z \geq \mathrm{d}^\circ \mathcal{L}(Z) \geq \mu$. So, we must choose $\mu = \delta$.
Now if one takes $\mu = \delta$ then (3.6) reduces to

$$\mathcal{L}(Z) = x^\delta(\ell c_\infty(B) - (\delta D_0 - A_0)c) + \text{ terms of degree } < \delta.$$

Consequently, in order to have $\mathrm{d}^\circ \mathcal{L}(Z) < \mu = \delta$, one must choose c as a solution of the linear equation

$$\ell c_\infty(B) - (\delta D_0 - A_0)c = 0. \tag{3.7}$$

This last equation has solutions iff $\ell c_\infty(B)$ belongs to the *range* of $(A_0 - \delta D_0)$ (that is the space generated by the columns). Thus, one has to consider two cases:

(3.1) If one can choose the p_i's so that $\ell c_\infty(B) \in range(A_0 - \delta D_0)$ then one can take $\mu = \delta$ and c any solution of the equation (3.7).
For instance, if $E_\infty(-\delta) \neq 0$ then $\ell c_\infty(B) \in range(A_0 - \delta D_0)$, for all values of the parameters, and in this case c is uniquely determined by $c = (\delta D_0 - A_0)^{-1} \ell c_\infty(B)$.
(3.2) If $\ell c_\infty(B) \notin range(A_0 - \delta D_0)$ for all $p_1, \ldots, p_m$ then there is no couple (μ, c) answering our question.

3.3. Algorithm for finding polynomial solutions

The above discussion leads to the following algorithm *next-term* which will be used later in the description of the main algorithm for polynomial solutions. It takes as input a rational function B, a list $\mathcal{R}$ of integers, a list $\mathcal{P}$ of parameters, a list $\mathcal{C}$ of linear relations on $\mathcal{P}$ (the constraints on the parameters) and a polynomial *sol*. The first call to this algorithm is done with B (the right-hand side of the given equation), $\mathcal{R}$ and $\mathcal{P}$ as defined above, $\mathcal{C} = \emptyset$ and $sol = 0$. It produces a new set of parameters $\mathcal{P}$, a set $\mathcal{C}$ of linear constraints on these parameters and a polynomial *sol*, parametrized by the elements of $\mathcal{P}$, which represents the possible general solution of the given equation.

Algorithm next-term$(B, \mathcal{R}, \mathcal{P}, \mathcal{C}, sol)$

0. Let $\delta := \deg B$ and $\ell c_\infty(B) :=$ the leading-coefficient of B.
(Recall that when $\ell c_\infty(B) \neq 0$, it is a polynomial of degree ≤ 1 in the elements of $\mathcal{P}$).
1. If $\mathcal{R} = \emptyset$ and $\delta < 0$ then return$(sol, \mathcal{P}, \mathcal{C})$.
2. If $\mathcal{R} \neq \emptyset$ and $\max \mathcal{R} > \delta$ then
take $\mu = \max \mathcal{R}$ and $c \in \ker(A_0 - \mu D_0)$ *(note that c depends on* $\dim \ker(A_0 - \mu D_0)$ *arbitrary constants c_i);*
and call **next-term** with $B := B - \mathcal{L}(x^\mu c)$, $\mathcal{R} := \mathcal{R} \setminus \{\mu\}$, $\mathcal{P} := \mathcal{P} \cup \{c_i' s\}$, *$\mathcal{C}$ is not changed* and $sol := sol + x^\mu c$.
Note that in this case the number of elements of $\mathcal{R}$ decreases.
3. If $(\mathcal{R} \neq \emptyset$ and $\delta \geq \max \mathcal{R})$ or $(\mathcal{R} = \emptyset$ and $\delta \geq 0)$ then

(3.1) If $E_\infty(-\delta) \neq 0$ then set $c := (\delta D_0 - A_0)^{-1} \ell c_\infty(B)$ and call **next-term** with: $B := B - \mathcal{L}(x^\delta c)$, $sol := sol + x^\delta c$, $\mathcal{R}, \mathcal{P}, \mathcal{C}$ *are not changed;*
Note that in this case the degree of B decreases;

(3.2) if $E_\infty(-\delta) = 0$ then one has to know whether $\ell c_\infty(B)$ belongs to $range(A_0 - \delta D_0)$ or not; this condition is equivalent to a system, say $\mathcal{G}$, of linear equations in the parameters.

(a) If the relations $\mathcal{G}$ are compatible with the set $\mathcal{C}$ of constraints then solve $\ell c_\infty(B) = (A_0 - \delta D_0)c$, let c be the general solution of this system, *(it depends on some arbitrary constants c_i)*, then call **next-term** with $B := B - \mathcal{L}(x^\delta c)$, $\mathcal{R} := \mathcal{R} \setminus \{\delta\}$, $\mathcal{P} := \mathcal{P} \cup \{c_i\}$, $\mathcal{C} := \mathcal{C} \cup \mathcal{G}$, and $sol := sol + x^\delta c$.
Note that in this case the degree of the right-hand side B and the number of elements of $\mathcal{R}$ decrease.

(b) If the conditions $\mathcal{G}$ are not compatible with the constraints $\mathcal{C}$ then return $(sol, \mathcal{P}, \mathcal{C})$.

The reason why the above algorithm works is that at each step either the degree of the right-hand side B decreases or the number of elements of $\mathcal{R}$ decreases. So, after a finite number of steps, one has $\mathcal{R} = \emptyset$ and $\delta < 0$, unless the situation in 3.2 (b) occurs in which case one stops.

Remark 3.2. The above algorithm computes, in fact, the *singular part* of the general meromorphic formal series solution at ∞ of the given system (that is solution Y with entries in $K[[x^{-1}]][x]$). Note that only the monomials which really occur in this singular part are computed. Thus, the number of necessary steps for computing the candidate polynomial solution of a given system depends only on the number of the (non-zero) monomials occurring in the singular part of its general meromorphic formal series solution at ∞. So, in case of sparse solutions our algorithm could be very fast.

We proceed now with a description of our algorithm for searching for polynomial solutions with a system $\mathcal{M}(Y) = x\frac{dY}{dx} - MY = N$ as our starting point. Here $M \in \mathrm{Mat}_n(K(x))$ and $N = N_0 + \sum_{i=1}^m p_i N_i$ where the N_i's are in $K(x)^n$ and the p_i's are some parameters. The output is a triplet $(\mathcal{P}, \mathcal{C}, Y)$ where $\mathcal{P}$ is a set of parameters, $\mathcal{C}$ is a set of linear relations on the elements of $\mathcal{P}$ and Y is a rational function parametrized by the entries of $\mathcal{P}$ which is solution of the given system when the constraints $\mathcal{C}$ hold.

(1) Apply, if necessary, the algorithm of super-irreducibility (see the Appendix) to make the given system simple at ∞. Let $\mathcal{L}(X) = B$ denote the resulting system and T the matrix which achieves the transformation (one has $Y = TX$).
Note that the components of the new right-hand side B are (as the components of N) linear in the parameters p_i's.

(2) Let $E_\infty(\lambda) = \det(A_0 + \lambda D_0)$ be the indicial polynomial of $\mathcal{L}$ at ∞. Set $\mathcal{R} := \{\lambda \in \mathbb{N}, E_\infty(-\lambda) = 0\}$, $\mathcal{P} := \{p_1, \ldots, p_m\}$ *($\mathcal{P}$ is the set of free parameters, it may be empty)*, $\mathcal{C} := \emptyset$ *(the set of constraints on these parameters)*, and $sol := 0$.

(3) Call **next-term** with $B, \mathcal{R}, \mathcal{P}, \mathcal{C}$, and sol.
One then obtains a new set of parameters $\mathcal{P}$, a set $\mathcal{C}$ of linear constraints on these parameters and a polynomial $sol = \sum x^\mu c_\mu$ where the c_μ's are column vectors, the components of which are linear in the elements of $\mathcal{P}$.

(4) Substituting $Y = T * sol$ in the equation $\mathcal{M}(Y) = N$ yields a system, say $\mathcal{F}$, of linear equations in the parameters $\mathcal{P}$.
(5) If the system $\mathcal{F}$ is compatible with the constraints $\mathcal{C}$ then $Y = T * sol$ gives the general solution of our problem, otherwise, there is no polynomial solution.

4. Rational Solutions

Given a system (1.1), the problem is to construct an $R \in K(x)$ such that RY is a polynomial for all possible rational solutions Y of (1.1). When such an R is known then substituting $Y = R^{-1}Z$ into $\mathcal{M}(Y) = N$, reduces the problem (**P**) to finding polynomial solutions of $\tilde{\mathcal{M}}(Z) = \mathcal{M}(RY) = N$. In this section we will explain how to construct such a function R.

First, let us recall that the finite singularities of a system (1.1) are the poles of M and the poles of N in $\bar{K}$ (an algebraic closure of K). It is well-known that the singularities of the solutions of (1.1) are among the singularities of the system (1.1). In other words, if $Y \in K(x)^n$ satisfies $\mathcal{M}(Y) = N$, then each pole of Y in $\bar{K}$ is a pole of M or a pole of N. It then follows that any irreducible factor p of the denominator of a solution $Y \in K(x)^n$ must divide the denominator of M or N. Let us define the *finite singularities in* $K(x)$ of (1.1) to be the irreducible factors of the denominators of M and N. Suppose that we can associate with any singularity p an integer ℓ_p such that $\mathrm{ord}_p(Y) \geq \ell_p$ for all solution $Y \in K(x)^n$ of (1.1). Then the rational function R given by

$$R = \prod_{p \text{ singularity of } (1.1)} p^{-\ell_p}$$

has the property mentioned above.

Thus, it remains to show how the bounds ℓ_p can be determined. For this we need first to define the *indicial polynomial* of a system (1.1) at a point p. This is the subject of the following section.

4.1. The indicial equation at a finite point

Consider a matrix differential operator $\mathcal{M}$ with rational function coefficients in $K(x)$:

$$\mathcal{M} = \frac{d}{dx} - M, \qquad M \in \mathrm{Mat}_n(K(x))$$

Let p be an irreducible polynomial in $K[x]$. Let M_i denote the ith row of the matrix M. Put $\alpha_i = -\min(1 + \mathrm{ord}_p M_i, 0)$, for $1 \leq i \leq n$, and let $D = \mathrm{diag}(p^{\alpha_1}, \ldots, p^{\alpha_n})$. Put

$$\mathcal{L} = D\frac{p}{p'}\frac{d}{dx} - A, \qquad \text{with } A = \frac{p}{p'}DM, \tag{4.8}$$

here p' designates the derivative of p with respect to x. It is clear that a column vector Y satisfies an equation of the form $\mathcal{M}(Y) = N$, with $N \in K(x)^n$ iff it satisfies the equation $\mathcal{L}(Y) = B$ where $B = \frac{p}{p'}DN$.

Note that $\mathrm{ord}_p A \geq 0$ and that $D \in \mathrm{Mat}_n(K[x])$. More precisely D can be written as:

$$D = D_0 + pD_1 + \cdots + p^q D_q,$$

where the D_i's are diagonal constant matrices such that $\sum D_i = I_n$. The matrix A has a $p-$adic expansions of the form $A = A_{0,p} + pA_{1,p} + \cdots$ where the $A_{i,p}$'s are polynomial matrices of degree less than $\deg p$.

Let us now study the action of $\mathcal{L}$ on a monomial $p^\lambda c$ where $c \in (K[x]/(p))^n$.

One has $\mathcal{L}(p^\lambda c) = p^\lambda(\lambda D - A)c$ and then

$$\mathcal{L}(p^\lambda c) = p^\lambda((\lambda D_0 - A_{0,p})c + \cdots). \tag{4.9}$$

Here the dots represent terms of positive order at p.

It then follows that $\mathrm{ord}_p\,(p^{-\lambda}\mathcal{L}(p^\lambda c)) \geq 0$ and equality holds iff $c \notin \ker\,(A_{0,p} - \lambda D_0)$. This motivates the following definition:

Definition 4.1. The operator $\mathcal{M}$ is said to be *simple* at p if $\det\,(A_{0,p} - \lambda D_0) \neq 0$ (as a polynomial in λ). When $\mathcal{M}$ is *simple* at p then $E_p(\lambda) := \det\,(A_{0,p} - \lambda D_0)$ will be called the indicial polynomial of $\mathcal{M}$ at p and the equation $E_p(\lambda) = 0$ the indicial equation of $\mathcal{M}$ at p.

When an operator $\mathcal{M}$ is *simple* at p we will say that the differential system $\mathcal{M}(Y) = N, N \in K(x)^n$ is *simple* at p.

Remark 4.1. A matrix differential operator $\mathcal{M} = \frac{d}{dx} - M$, $M \in \mathrm{Mat}_n(K(x))$, is *simple* at all points p which are *ordinary points* (i.e. $\mathrm{ord}_p\,M \geq 0$), or *singular points of the first kind* (i.e. $\mathrm{ord}_p\,M = -1$). Moreover, the indicial polynomial of $\mathcal{M}$ at a such point has degree n.

Indeed, let p be an irreducible polynomial in $K[x]$ such that $\mathrm{ord}_p\,M \geq -1$, then for each row M_i of M one has $\mathrm{ord}_p\,M_i \geq -1$. So, $\alpha_i = \min\,(1 + \mathrm{ord}_p\,M_i, 0) = 0$ for all i and hence $p^\alpha = I_n$. It follows that the matrix D_0 (defined above) is equal to the identity matrix I_n. Hence $\det\,(A_{0,p} - \lambda D_0)$ is the characteristic polynomial of the matrix $A_{0,p}$ which is a polynomial in λ of degree n.

Note that when $\mathrm{ord}_p\,M \geq 0$, the matrix $A_{0,p}$ is zero and then $E_p(\lambda) = (-\lambda)^n$. Thus the indicial polynomial at an ordinary point reduces to $(-\lambda)^n$.

In Barkatou and Pflügel (1997) the following proposition is proved (see the appendix):

Proposition 4.1. *Given a system of the form (1.1) and an irreducible polynomial $p \in K[x]$, then one can construct a nonsingular matrix T which is polynomial in x and satisfies* $\det T = \gamma p^{h_p}$ *(for some positive integer h_p and some constant $\gamma \in K$), such that the change of variables $Y = TZ$ transforms (1.1) into an equivalent system $\tilde{\mathcal{M}}(Y) = \tilde{N}$ which is simple at p.*

4.2. removing the denominator

Consider a matrix differential equation of the form (1.1). The problem is to find $R \in K(x)$ such that the product of R by any rational solution of (1.1) is a polynomial. In other words we are looking for a rational function R such that $\mathrm{ord}_p(RY) \geq 0$ for all irreducible polynomial p and all rational solutions Y of (1.1). In this section we will show how to construct such a function R.

At first we will prove some useful lemmas.

Lemma 4.1. *Let $p \in K[x]$ be an irreducible polynomial and $\mathcal{L}$ a differential operator of* *the form (4.8). Then for all* $Y \in K(x)^n$ *one has* $\operatorname{ord}_p \mathcal{L}(Y) \geq \operatorname{ord}_p Y$ *and equality holds iff* $\ell c_p(Y) \notin \ker(A_{0,p} - (\operatorname{ord}_p Y)D_0)$.

PROOF. Let Y be a non-zero element in $(K(x))^n$ and write:

$$Y = \ell c_p(Y) p^{\operatorname{ord}_p Y} + \text{ terms of order } > \operatorname{ord}_p Y.$$

Using (4.9), one sees that

$$\mathcal{L}(Y) = -p^{\operatorname{ord}_p Y}(A_{0,p} - (\operatorname{ord}_p Y)D_0)\ell c_p(Y) + \text{ terms of order } > \operatorname{ord}_p Y.$$

It follows that $\operatorname{ord}_p \mathcal{L}(Y) \geq \operatorname{ord}_p Y$ and equality holds iff

$$\ell c_p(Y) \notin \ker(A_{0,p} - (\operatorname{ord}_p Y)D_0).$$

In particular, if $Y \neq 0$ and $\mathcal{L}(Y) = 0$ then $\ell c_p(Y) \in \ker(A_{0,p} - (\operatorname{ord}_p Y)D_0)$ and $E_p(\operatorname{ord}_p Y) = 0$.□

COROLLARY 4.1. *Let* $B \in (K(x))^n$ *and* p, $\mathcal{L}$ *as in Lemma 4.1. Suppose that there exists* $Y \in K(x)^n$ *such that* $\mathcal{L}(Y) = B$. *Then either* $E_p(\operatorname{ord}_p Y) \neq 0$ *and* $\operatorname{ord}_p Y = \operatorname{ord}_p B$ *or* $E_p(\operatorname{ord}_p Y) = 0$ *and* $\operatorname{ord}_p Y \leq \operatorname{ord}_p B$.

LEMMA 4.2. *Consider a differential system of the form (1.1). Let* $p \in K[x]$ *be irreducible such that* $\mathcal{M}$ *be simple at* p. *Suppose that there exists* $Y \in K(x)^n$ *such that* $\mathcal{M}(Y) = N$, *then*

$$\operatorname{ord}_p Y \geq \min(1 + \operatorname{ord}_p N, m_p) \tag{4.10}$$

where

$$m_p := \min\{\lambda \in \mathbb{Z} | E_p(\lambda) = 0\}$$

with $m_p = +\infty$ *if* $E_p(\lambda) = 0$ *has no integer root.*

PROOF. Write the system $\mathcal{M}(Y) = N$ in the form $\mathcal{L}(Y) = B$ where $\mathcal{L}$ is given by (4.8), and $B = \frac{p}{p'}DN$. Then (4.10) follows from Corollary 4.1 and the fact that $\operatorname{ord}_p B = 1 + \operatorname{ord}_p(DN) \geq 1 + \operatorname{ord}_p D + \operatorname{ord}_p N \geq 1 + \operatorname{ord}_p N$.□

REMARK 4.2. If p is not a pole of M (i.e. $\operatorname{ord}_p M \geq 0$) then the right-hand side of (4.10) may be replaced by $\min(0, 1 + \operatorname{ord}_p N)$. Indeed, in this case p is an ordinary "point" of the homogeneous differential system $\mathcal{M}(Y) = 0$ and then according to Remark 4.1 $\mathcal{M}$ is simple at p and $E_p(\lambda) = (-\lambda)^n$. This implies that $m_p = 0$.

When the differential system $\mathcal{M}(Y) = N$ is not simple at p then, by Proposition 4.1, it can be reduced to an equivalent system $\tilde{\mathcal{M}}(Z) = \tilde{N}$ which is simple at p. Let T be the matrix which achieves this transformation. One has $Y = TZ$, and $\tilde{N} = T^{-1}N$. Since $T \in \operatorname{Mat}_n(K[x])$ and $\det T = \gamma p^{h_p}$ (with $\gamma \in K$), it follows that $\operatorname{ord}_p T^{-1} \geq -h_p$ and then $\operatorname{ord}_p \tilde{N} \geq \operatorname{ord}_p T^{-1} + \operatorname{ord}_p N \geq \operatorname{ord}_p N - h_p$. On the other hand $\operatorname{ord}_p Y \geq$

$\mathrm{ord}_p\, T + \mathrm{ord}_p\, Z \geq \mathrm{ord}_p\, Z$ (for T is polynomial). Now, since the operator $\tilde{\mathcal{M}}$ is simple at p, one has by the previous lemma:

$$\mathrm{ord}_p\, Z \geq \min\,(1 + \mathrm{ord}_p\, \tilde{N}, m_p).$$

Hence

$$\mathrm{ord}_p\, Y \geq \min\,(1 - h_p + \mathrm{ord}_p\, N, m_p).$$

We have therefore proved the following:

Lemma 4.3. *Consider a differential system of the form (1.1). Let $p \in K[x]$ be irreducible. Suppose that there exists $Y \in K(x)^n$ such that $\mathcal{M}(Y) = N$. Then*

$$\mathrm{ord}_p\, Y \geq \ell_p := \min\,(1 - h_p + \mathrm{ord}_p\, N, m_p), \tag{4.11}$$

where m_p and h_p are defined as above (with $h_p = 0$ if $\mathcal{M}$ is simple at p).

From the results above one can easily prove the following proposition:

Proposition 4.2. *Consider a differential system of the form (1.1). Let $p_1, \ldots, p_r$ be the irreducible factors in $K[x]$ of* $\mathrm{denom}(M)$*:*

$$\mathrm{denom}(M) = p_1^{\nu_1} \cdots p_r^{\nu_r}, \nu_i \geq 1, \qquad \text{for } i = 1, \ldots, r.$$

Write $\mathrm{denom}(N)$ *as a product of irreducible polynomials:*

$$\mathrm{denom}(N) = p_1^{\gamma_1} \cdots p_r^{\gamma_r} q_1^{\delta_1} \cdots q_s^{\delta_s},$$

here $\gamma_i \geq 0$, for $i = 1, \ldots, r$, and $\delta_j \geq 1$, for $j = 1, \ldots, s$. Consider the rational function

$$R := \prod_{j=1}^{s} q_j^{-\delta_j + 1} \prod_{i=1}^{r} p_i^{-\ell_{p_i}} \tag{4.12}$$

where ℓ_{p_i} is defined as in (4.11). Then for all $Y \in K(x)^n$, if $\mathcal{M}(Y) = N$ then RY is a polynomial vector.

4.3. a method for computing m_p

We shall now indicate how to compute m_p. Given an irreducible polynomial p and the corresponding indicial polynomial $E_p(\lambda)$, the problem is to find the minimal integer root of the equation $E_p(\lambda) = 0$. Recall that $E_p(\lambda)$ is a non-zero polynomial in λ (of degree $\leq n$) with coefficients in the field $K[x]/(p)$. It can be represented by a polynomial (in λ and x) of the form $\sum_{i=0}^{n} \alpha_i(x)\lambda^i$ with $\alpha_i \in K[x]$ and $\mathrm{d}^\circ\, \alpha_i < \mathrm{d}^\circ\, p$. So, $E_p(\lambda)$ can be rewritten in the form

$$E_p(\lambda) = \sum_{j=0}^{\mathrm{d}^\circ p - 1} \beta_j(\lambda) x^j, \qquad \text{with } \beta_j \in K[\lambda].$$

Let $\nu \in \mathbb{Z}$ then $E_p(\nu) = \sum_{j=0}^{\mathrm{d}^\circ p - 1} \beta_j(\nu) x^j \in K[x]/(p)$. It then follows that $E_p(\nu) = 0$ iff $\beta_j(\nu) = 0$ for $j = 0, \ldots, \mathrm{d}^\circ p - 1$. Hence the set of the integer roots of E_p is equal to the set of integer roots that the β_j's have in common. Thus the problem is reduced to the one of finding integer roots of a given polynomial with coefficients in K.

Now to determine m_p one can proceed as follows: Choose a j_0 such that $\mathrm{d}^\circ\, \beta_{j_0} \leq \mathrm{d}^\circ\, \beta_j$

for all $j = 0, \ldots, \mathrm{d}^\circ p - 1$. Compute the set S of integer roots of β_{j_0} and then the set $C = \{\nu \in S \text{ such that } E_p(\nu) = 0\}$. If $C = \emptyset$ then E_p has no integer roots and $m_p := +\infty$, else $m_p := \min C$.

4.4. The algorithm for searching rational solutions

We now give a sketch of our algorithm for searching rational solutions. Consider a differential system $\mathcal{M}(Y) = \frac{dY}{dx} - MY = N$, with $M \in \mathrm{Mat}_n(K(x)), N \in (K(x))^n$.

(1) Compute the rational function R given by (4.12).
(2) Perform, in the equation $\mathcal{M}(Y) = N$, the substitution $Y = R^{-1}Z$. Let $\tilde{\mathcal{M}}(Z) = \tilde{N}$ be the resulting equation.
(3) Compute the set P of polynomial solutions of this last system. Then $R^{-1}P$ gives the general rational solution of our system unless P is empty in which case there is no rational solution for our system.

Note that the above algorithm solves also *parametrized* linear differential systems. That is systems with a right-hand side depending (linearly) on some given parameters. The output is then a triplet $(\mathcal{P}, \mathcal{C}, Y)$ where $\mathcal{P}$ is a set of parameters, $\mathcal{C}$ is a set of linear relations on the elements of $\mathcal{P}$ and Y is a rational function parametrized by the entries of $\mathcal{P}$ which is solution of the given system when the constraints $\mathcal{C}$ hold.

4.5. Examples of computation

We have implemented the above algorithm in the computer algebra system Maple V. The main procedure is called *Mratsolde*. It takes as arguments a matrix M with rational function coefficients, the independent variable x and a vector N of rational functions. It returns the general rational solution of the matrix differential equation $Y' - MY = N$ or the empty set. We will now show some examples of computations using *Mratsolde*.

At first consider the non-homogeneous differential system $Y' - MY = N$ where

$$M = \begin{bmatrix} -\frac{6+2x}{-1+x^2} & 0 & -\frac{6}{-1+x^2} & \frac{6}{-1+x^2} \\ 0 & \frac{4+x}{2x} & -\frac{2}{x} & 0 \\ 0 & \frac{4+3x}{2x} & -\frac{2}{x} & 0 \\ \frac{-7+x^2-2x}{-1+x^2} & \frac{4+3x}{2x} & -\frac{6x+2x^2-2}{x(-1+x^2)} & \frac{6}{-1+x^2} \end{bmatrix}$$

$$N = \left[0, -\frac{2x}{(1+x^2)^2} - \frac{4+x}{2x(1+x^2)}, -\frac{4+3x}{2x(1+x^2)}, -\frac{4+3x}{2x(1+x^2)}\right].$$

The general rational solution of the system $Y' - MY = N$ computed by our program *Mratsolde* is

$$\left[(c_2 - 72c_1)x, \frac{2x^5c_1 - 36x^4c_1 + 146x^3c_1 - 132x^2c_1 + 144c_1x - 96c_1 + 1}{1+x^2}, \right.$$
$$\left. \frac{(x^4 - 20x^3 + 72x^2 + 96x - 192)c_1}{2}, \frac{x^4c_1}{2} - 10x^3c_1 + \frac{x^2c_2}{2} + xc_2 - 24c_1x - 84c_1 - \frac{c_2}{6}\right],$$

here c_1 and c_2 are arbitrary constants. This computation took 5 s (on an IBM/RS6000 7030-3AT).

Now consider the homogeneous differential system

$$\mathcal{L}_m(Y) = Y' - Sym^m(M)Y = 0$$

where $Sym^m(M)$ designates the mth symmetric power of the above matrix M (for a definition of symmetric powers of a matrix, see for example, van Hoeij and Weil (1996)). We applied our procedure *Mratsolde* to solve the system $\mathcal{L}_m(Y) = 0$ for $m = 1, 2, 3$. The rational solutions spaces obtained are of respective dimension $2, 3$ and 4. The following table gives Maple cpu seconds to solve $\mathcal{L}_m(Y) = 0$ for $m = 1, 2, 3$ using our procedure *Mratsolde* (third column), and the time for computing a scalar equivalent equation using the cyclic vector approach (fourth column). Note that the time for solving this scalar equation is not taken into account. The second column gives the dimension n of the considered system. A $*$ indicates that the computation of a cyclic vector did not terminate after 6 h.

m	n	Mratsolde	Cyclic vector
1	4	4	1
2	10	27	75
3	20	247	*

This example shows that our method is faster than computing cyclic vector. This suggests that our method can be much faster than the cyclic vector approach no matter which method for the scalar case is used.

4.6. NOTE ON THE USE OF THE CYCLIC VECTORS METHOD

Let us recall that by means of the so-called "cyclic vector" method one can reduce any matrix differential equation (1.1) to an equivalent scalar differential equation of the form:

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = b \tag{4.13}$$

where b and the a_i's are in $K(x)$. To say that equations (1.1) and (4.13) are equivalent means that there exists a non-singular matrix $T \in \mathrm{Mat}_n(K(x))$ such that if Y and y are solutions respectively of (1.1) and (4.13), then $Y = TZ$ where Z is the column vector whose components are $y, y', \ldots, y^{(n-1)}$. Thus an alternative way to get the rational solutions of a system of linear differential equations may be the following:

(i) convert the given system to a single scalar differential equation by a cyclic vector process,
(ii) solve the resulting scalar differential equation using any algorithm for scalar equations,
(iii) and then build the solutions of the original system from those of the scalar equation. This method turns out to be, generally, quite unsatisfactory.

The difficulties which one meets with are of two types:

(a) for systems with "large" dimension n (in practice $n \geq 10$), the construction of an equivalent scalar equation may take a "long time";

(b) the scalar equation (when it can be computed) has often "too complicated" coefficients compared with the entries of the matrix system (even for small dimensions) and in consequence solving this equation can be costly.

5. Solution in $K(x)[\log x]^n$

In this section we consider the problem of finding solutions Y of a system (1.1) which have the following form

$$Y = Y_s + Y_{s-1}\frac{\log x}{1!} + \cdots + Y_0\frac{\log^s x}{s!}, \tag{5.14}$$

where s is a non-negative integer and the Y_i's are rational functions over K. Here Y_0 is assumed to be not zero when $Y \neq 0$.

5.1. the homogeneous case

Let us start with a homogeneous system, i.e. $N = 0$ in (1.1). The non-homogeneous case will be discussed in the next section.

Consider a homogeneous differential system of the type (3.4):

$$\mathcal{L}(Y) = Dx\frac{dY}{dx} - AY = 0.$$

We may suppose, without loss of generality, that $\mathcal{L}$ is simple at ∞.

It is important to note that the largest possible integer s in (5.14) is bounded by the sum of the multiplicities of the integer roots of the indicial equation of $\mathcal{L}$ at ∞ whose degree is at most n.

We will now show that the problem above can be reduced to solving several differential systems in $K(x)^n$. First let us remark that substituting $Y = R^{-1}Z$, where R is given by (4.12), reduces the problem to finding solutions (5.14) with the Y_i's in $K[x]^n$. We will now explain how to solve this last problem. Let Y be a function of the form (5.14) then one easily verifies that

$$\mathcal{L}(Y) = \sum_{i=0}^{s}(\mathcal{L}(Y_{s-i}) + DY_{s-i-1})\frac{\log^i x}{i!},$$

where $Y_{-1} = 0$. It then follows that Y is a solution of the system $\mathcal{L}(Y) = 0$ iff the following conditions hold

$$\mathcal{L}(Y_0) = 0 \qquad \text{and} \qquad \mathcal{L}(Y_i) = -DY_{i-1} \qquad \text{for } i = 1, \ldots, s. \tag{5.15}$$

Thus, to solve our problem we may proceed as follows. At first we apply our algorithm for polynomial solutions to determine the non zero polynomial solutions of $\mathcal{L}(Y_0) = 0$. After Y_0 has been determined then solve the next system $\mathcal{L}(Y_1) = -DY_0$ and so on. Since s is bounded then we we will obtain, after a finite number of steps, a system for which there is no polynomial solution. This implies that the maximal (possible) s has been reached and then the general solution (5.14) has been found.

Let us give an example illustrating the method described above. Consider the differential system $\mathcal{L}(Y) = x\frac{dY}{dx} - AY = 0$ with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its general solution is given by

$$Y = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \log x + \begin{pmatrix} \beta \\ \alpha \end{pmatrix},$$

where α and β are arbitrary constants.

We will now apply our method to compute for this example all the solutions (5.14). First of all one verifies easily that the operator is simple at infinity and its indicial polynomial is λ^2. This implies that $s < 2$. Note that this last fact will not be used in the following.

First solve in $K[x]^2 \setminus \{0\}$ the system $\mathcal{L}(Y_0) = 0$. We find that $Y_0 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ where α is a <u>non-zero</u> arbitrary constant.

Next, solve in $K[x]^2$ the system $\mathcal{L}(Y_1) = -Y_0$. We find that $Y_1 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ where β is an arbitrary constant.

Next, solve in $K[x]^2$ the system $\mathcal{L}(Y_2) = -Y_1$. One can see that the degree of the possible polynomial solution is bounded by 0. Now when we substitute $Y_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ we find that $\begin{pmatrix} \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. This implies $\alpha = 0$. But this last condition is in contradiction with $Y_0 \neq 0$. So, Y_2 does not exist. Consequently, $s = 1$ and the general solution in $K(x)[\log x]^2$ is given by $Y_0 \log x + Y_1$.

5.2. THE NON-HOMOGENEOUS CASE

The problem of solving a non-homogeneous system (1.1) can be transformed into the problem of solving certain homogeneous system. This can be done by setting

$$X = (Y^t, 1)^t.$$

If Y solves (1.1) then X solves the homogeneous system $X' = AX$ where A is the $(n+1) \times (n+1)$ matrix given by

$$A = \begin{pmatrix} M & N \\ 0 & 0 \end{pmatrix}.$$

Conversely, if $X = (x_1, \ldots, x_n, x_{n+1})^t$ is a solution of the system $X' = AX$ then x_{n+1} is an arbitrary constant in K and $Y = (x_1/x_{n+1}, \ldots, x_n/x_{n+1})^t$, with $x_{n+1} \in K \setminus \{0\}$, satisfies the non-homogeneous system (1.1).

6. Meromorphic Formal Series Solutions

With slight modifications, our algorithm *polynomial solutions* can also be used to compute meromorphic formal series solutions at ∞ of equations of the form (1.1) with meromorphic formal series coefficients. Consider such an equation and let $Y = \sum_{-\nu}^{i=+\infty} Y_i x^{-i} \in K[[x^{-1}]][x]^n$ be a meromorphic formal series such that $\mathcal{M}(Y) = N$. The problem is to compute Y up to some order $m \in \mathbb{Z}$; this means compute the finite sum $\sum_{i=-\nu}^{m} Y_i x^{-i}$. To do this, take our algorithm for finding polynomial solutions and make the following modifications: in step 2, take $\mathcal{R} := \{\lambda \in \mathbb{Z} | \lambda \leq -m \text{ and } E_\infty(-\lambda) = 0\}$; in step 4, replace the condition $M(Y) = N$ by $M(Y) - N = O(x^{-m})$; and finally, replace in step 2 (respectively, in step 1) of the procedure *next-term*, the condition $\delta \geq 0$ (respectively, $\delta < 0$) by $\delta \geq m$ (respectively, by $\delta < m$).

The resulting algorithm will produce the meromorphic formal series solutions at ∞ of an equation of the form $\mathcal{M}(Y) = N$ (for more details see Barkatou and Pflügel, 1997).

7. Solving Other Linear Equations

To conclude this article let us note that one can adapt our method to solve systems of linear ($q-$)difference equations. In fact, the notion of super-irreducibility at ∞ has been generalized to difference systems (see Chapter 7 of Barkatou, 1989). Hence, one can define the indicial equation at ∞ for matrix difference operators in a very similar way as for matrix differential operators. This provides an algorithm for searching for polynomial solutions of matrix difference equations. It remains to solve the problem of finding a multiple of the denominator of the possible rational solutions of a given ($q-$)difference system. We are working on this question.

Appendix A. The Notion of Super-irreducibility

Appendix A.1. Definition and Properties

The notion of super-irreduciblity has been introduced in a joint work of Hilali and Wazner (1987), and is used there to study linear homogeneous differential systems near an irregular singularity. In this appendix, we will give the definition of super-irreduciblity. Furthermore we will show the connexion between the super-irreducibility and the notion of simplicity introduced in Section 3.1.

Consider a differential system of the form:

$$x\frac{dY}{dx} = M(x)Y, \qquad M \in \mathrm{Mat}_n(K[[x^{-1}]][x]). \tag{7.16}$$

Put $q := -\operatorname{ord}_\infty(M)$ and define the rational number $m_\infty(M)$ by

$$m_\infty(M) = \begin{cases} q + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{q-1}}{n^q} & \text{if } q > 0 \\ 1 & \text{if } q \leq 0 \end{cases}$$

where n_i denotes the number of rows (see the Remark 7.1) of M of order $-q + i$ for $i = 0, \ldots, q-1$.

Finally, define the rational number $\mu_\infty(M)$ by

$$\mu_\infty(M) = \min\{m_\infty(T[M]) | T \in GL_n(K[[x^{-1}]][x])\}.$$

Definition 7.1. The system (7.16) (or the matrix M) is called *super-irreduc-ible* (at ∞) iff $m_\infty(M) = \mu_\infty(M)$.

Remark 7.1. The above definition of super-irreducibility is not exactly the same as the one given in Hilali and Wazner (1987). Indeed, firstly Hilali and Wazner consider meromorphic differential systems $\frac{dY}{dx} = A(x)Y$ at $x = 0$ (i.e. $A \in \mathrm{Mat}_n(K[[x]][x^{-1}])$)

with respect to the derivation $\frac{d}{dx}$ while we work with meromorphic differential operators $xd/dx - M(x)$ at $x = \infty$ (i.e. $M \in \mathrm{Mat}_n(K[[x^{-1}]][x])$) w.r.t. the derivation $x\frac{d}{dx}$. Secondly, we work with the rows of the matrix system while in Hilali–Wazner's definition the columns are used instead. In view of these remarks, it is not difficult to see that a matrix $M \in \mathrm{Mat}_n(K[[x]][x^{-1}])$ is super-irreducible (at $x = \infty$) by our definition iff the matrix $-x^{-1}\,{}^tM(x^{-1})$ is super-irreducible (at $x = 0$) in the sense of Hilali–Wazner.

In Hilali and Wazner (1987) a criterion to decide whether a system (7.16) is super-irreducible is given. We will repeat this criterion here since it will be used later.

Let us keep the notation above. Suppose $q > 0$ and define the integers $r_1, \ldots, r_q$ by

$$r_k = kn_0 + (k-1)n_1 + \cdots + n_{k-1}.$$

For $1 \le k \le q$ define

$$\theta_k(M, \lambda) = x^{-r_k} \det(\lambda I_n - x^{-q+k} M(x))_{|x=\infty}.$$

Then $\theta_k(M, \lambda) \in K[\lambda]$ for all $1 \le k \le q$. In Hilali and Wazner (1987) the following proposition is proved.

Proposition 7.1. (Hilali and Wazner, 1987) *The system (7.16) is* super-irreducible *(at ∞) iff the polynomials $\theta_k(M, \lambda)$ do not vanish identically in λ, for $k = 1, \ldots, q$.*

We will now prove the following.

Proposition 7.2. *If a system (7.16) is super-irreducible (at $x = \infty$) then it is simple (at $x = \infty$).*

Proof. Consider a system of the form (7.16) and put $q = -\mathrm{ord}_\infty(M)$. If $q \le 0$ then (as we have mentioned in Section 3.1) the system is simple. Suppose that $q > 0$ and define the matrix α by $\alpha = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = -\min(0, \mathrm{ord}_\infty(M_i))$, where M_i, $0 \le i \le n$, denotes the ith row of M. Then the matrix $D(x) := x^{-\alpha} \in \mathrm{Mat}_n(K[x^{-1}])$ and the matrix $A(x) := x^{-\alpha}M(x) \in \mathrm{Mat}_n(K[[x^{-1}]])$. Put $D_0 := D(\infty)$ and $A_0 := A(\infty)$, then one has

$$\det(A_0 - \lambda D_0) = \theta_q(M, \lambda).$$

Indeed, one easily verifies that $\det(x^{-\alpha}) = x^{-r_q}$ and then

$$x^{-r_q} \det(\lambda I_n - M(x)) = \det(x^{-\alpha}) \det(\lambda I_n - M(x)) = \det(\lambda x^{-\alpha} - x^{-\alpha}M(x)).$$

Hence

$$\theta_q(M, \lambda) = x^{-r_q} \det(\lambda I_n - M(x))_{|x=\infty} = \det(\lambda D(x) - A(x))_{|x=\infty} = \det(\lambda D_0 - A_0).$$

Now if (7.16) is super-irreducible then, by Proposition 7.1, the polynomial $\theta_q(M, \lambda)$ is not zero and (7.16) is simple.□

Remark 7.2. Note that a system may be simple without being super-irreducible. Since super-irreducibility requires that $\theta_k \ne 0$ for all $1 \le k \le q$ while simplicity requires only that $\theta_q \neq 0$ (as was mentioned in the proof above). As an example consider the differential system

$$x\frac{dY}{dx} = M(x)Y, \qquad M(x) = \begin{bmatrix} 0 & x^2 \\ 1 & 1 \end{bmatrix}.$$

It is simple but not super-irreducible at ∞. Indeed, one has $q = 2$, $n_0 = 1$, $n_1 = 0$, $r_1 = n_0 = 1$, $r_2 = 2n_0 + n_1 = 2$, $\theta_1(M,\lambda) = x^{-1}\det(\lambda I_2 - x^{-1}M(x))_{|x=\infty} = 0$, and $\theta_2(M,\lambda) = x^{-2}\det(\lambda I_2 - M(x))_{|x=\infty} = -1$.

In Hilali and Wazner (1987) an algorithm is presented which transforms a given system (7.16) into an equivalent system which is super-irreducible. This algorithm has been implemented in Maple V by E. Pflügel and the author of this paper. More precisely, given a matrix $M \in \mathrm{Mat}_n(K[[x^{-1}]][x])$ this algorithm produces a nonsingular matrix S which is polynomial in x^{-1} such that the equivalent matrix $\tilde{M} := S[M]$ is super-irreducible at ∞. Moreover, S satisfies $\det S = \gamma x^h$, for some integer h and $\gamma \in K \setminus \{0\}$.

This last result implies that the matrix S^{-1} is of the form $S^{-1} = x^{-\nu}(S_0 + S_1 x + \cdots + S_1 x^d)$ for some integers ν and d. Put $T := x^{-\nu}S$ then T^{-1} is polynomial in x and the matrix $T[M] = x^{-\nu}[\tilde{M}] = \tilde{M} + \nu I_n$ is super-irreducible at ∞. Thus we have proved Proposition 3.1.

Appendix A.2. A Generalization of Super-Irreducibility

In this section, we will consider differential system with coefficients in $K(x)$ (instead of $K[[x^{-1}]][x]$). Of course it is understood that all the results stated in the above section are still valid for this type of system. We will now show briefly how these results, which are relative to the point at infinity, can be extended to any point p of $K(x)$ (for more details see Barkatou and Pflügel, 1997). Let us start with the analogue of Definition 7.1. Let p be an irreducible polynomial in $K[x]$. Consider a differential system with coefficients in $K(x)$. Then it can be written as

$$\frac{p}{p'}\frac{dY}{dx} = M(x)Y, \qquad M \in \mathrm{Mat}_n(K(x)). \tag{7.17}$$

Put $q := -\mathrm{ord}_p(M)$ and define the rational number $m_p(M)$ by

$$m_p(M) = \begin{cases} q + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{q-1}}{n^q} & \text{if } q > 0 \\ 1 & \text{if } q \leq 0 \end{cases}$$

where n_i denotes the number of rows of M of order $-q+i$ at p for $i = 0, \ldots, q-1$. Finally define the rational number $\mu_p(M)$ by

$$\mu_p(M) = \min\{m_p(T[M]) | T \in GL_n(K(x))\}.$$

Definition 7.2. The system (7.17) (or the matrix M) is called *super-irreducible* (at p) iff $m_p(M) = \mu_p(M)$.

When $q > 0$ we associate with the system (7.17) the polynomials

$$\theta_k(M,\lambda) = p^{-r_k}\det(\lambda I_n - p^{q-k}M(x)) mod p,$$

where $r_k = kn_0 + (k-1)n_1 + \cdots + n_{k-1}$, for $k = 1, \ldots, q$. In Barkatou and Pflügel (1997) the following proposition is proved.

Proposition 7.3. *The system (7.17) is* super-irreducible *(at p) iff the polynomials* $\theta_k(M, \lambda)$ *do not vanish identically in* λ*, for* $k = 1, \ldots, q$.

One can prove in the same manner as in the above section the following result:

Proposition 7.4. *If a system (7.17) is super-irreducible (at p) then it is simple (at p).*

We have given in Barkatou and Pflügel (1997) an algorithm which reduces any differential system to an equivalent system which is super-irreducible at a given point p. Moreover, the transformation T which achieves the reduction is polynomial and has a determinant of the form $\det T = \gamma p^{h_p}$ (for some positive integer h_p and some constant $\gamma \in K$).

It is to be noted that this algorithm is a generalization of our *rational Moser algorithm* presented in Barkatou (1995).

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