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On Power Series of Algebraic and Rational Functions in C^n

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1. INTRODUCTION

We consider a power series

$$a(z) = \sum_{\alpha \in I_1} a_\alpha z^\alpha, \tag{1}$$

$$\alpha = (\alpha_1, \dots, \alpha_n), \qquad \alpha_j \geq 0,\, j = 1, \dots, n,$$

$$I_1 = \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_1 \geq 1\}, \qquad a_\alpha z^\alpha = a_{\alpha_1, \dots, \alpha_n} z^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}.$$

Suppose the power series (1) converges in a neighborhood of the origin and let us consider the coefficients of the power series under the assumption that its sum is a branch of an algebraic function.

Let us also suppose that a power series in $n + 1$ variables

$$R(z_0, z) = \sum_{\substack{\alpha_0 \geq 1 \\ \alpha \in I_1}} R_{\alpha_0, \alpha} z_0^{\alpha_0} z^\alpha \tag{2}$$

converges in a neighborhood of the origin; here

$$R_{\alpha_0, \alpha} z_0^{\alpha_0} z^\alpha = R_{\alpha_0, \alpha_1, \dots, \alpha_n} z_0^{\alpha_0} z_1^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}, \qquad z_0 \in C^1.$$

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Then the power series in n variables

$$\sum_{\alpha\in I_1} R_{\alpha_1,\,\alpha} z^{\alpha} \tag{3}$$

is called the *diagonal* of the power series (2).

Furstenberg [1] considered the diagonal of the power series of a rational function. He had a different definition of the diagonal of a multi-variable power series (his definition coincides with ours only in the case $n = 1$). In particular, Furstenberg proved the diagonal of the power series of a rational function of two variables to be an algebraic univariable function. Hessel showed [2] the same for the Laurent series. The same problems for algebraic and rational functions over finite fields have been studied by some other authors (e.g., Deligne [3]).

The author proved [4, 5] that the converse is valid; namely, for any algebraic univariable function

$$a(z) = \sum_{k\geq 0} a_k z^k$$

there is a rational function

$$r(z_1, z_2) = \sum_{k_1, k_2\geq 0} r_{k_1, k_2} z_1^{k_1} z_2^{k_2}$$

in C^2 such that

$$\sum_{k\geq 0} a_k z^k = \sum_{k\geq 0} r_{k,k} z^k. \tag{4}$$

Thus a necessary and sufficient condition for $a(z)$ to be algebraic is the equality (4), where $r(z_1, z_2)$ is some rational function.

In the present paper this result is generalized for algebraic functions of n variables. Namely, we will prove the following main theorem.

THEOREM 1. *Suppose the function* (2) *is rational. Then the holomorphic function defined in a neighborhood of the origin by* (3) *is algebraic. Conversely, if the function* (1) *is a branch of an algebraic function which is holomorphic near the origin, then there is a rational function* (2) *and a unimodular $n \times n$ matrix with non-negative integral elements such that for all α*

$$a_{\alpha} = R_{\beta_1,\,\beta}\big|_{\beta=\alpha A}, \tag{5}$$

where $\beta = (\beta_1, \ldots, \beta_n)$.

Remark. If A is the identity matrix, then the condition (5) is $a_\alpha = R_{\alpha_1,\alpha}$; i.e., the function (1) is the diagonal of the series (2). If A is an arbitrary unimodular matrix, then it would appear natural that the series (1) connected with the series (2) by the relation (5) is the A-diagonal of the series (2).

We note that the structure of the coefficients of a rational univariable function is completely described by the Kronecker criterion [6]. By applying the Kronecker criterion with respect to every variable in the multiple power series of a rational function, one can obtain a certain description of those coefficients.

Further, in this paper we consider some conditions of separate algebraicity and applications of Theorem 1 to, e.g., Eisenstein's theorem on power series of algebraic functions.

2. PROOF OF THEOREM 1

Let $R(z_0, z)$ be a holomorphic function in the polydisk $\{|z_j| \le \rho;\ j = 0, 1, \dots, n\}$ and have the power series expansion (2). Since this series converges absolutely, the series $\sum_\alpha R_{\alpha_1,\alpha} z^\alpha$ converges absolutely, too.

We choose ε, $0 < \varepsilon < \rho$. Then the function $R(w, z_1/w, z_2, \dots, z_n)$ is holomorphic for $w \in \Gamma(\varepsilon) = \{|w| = \varepsilon\}$, $|z_1| \le \delta = \min\{\rho\varepsilon, \rho\}$, and evidently we have that

$$\sum_\alpha R_{\alpha_1,\alpha} z^\alpha = \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} R\left(w, \frac{z_1}{w}, z_2, \dots, z_n\right)\frac{dw}{w}. \tag{6}$$

Now let the integrand be rational. Then evaluating it by residues leads (e.g., Proposition 10.2 in [10]) to an algebraic function of variables $z_1, \dots, z_n$. Thus the function $\sum_\alpha R_{\alpha_1,\alpha} z^\alpha$ is algebraic. Note that its coefficients are connected with the coefficients of the function $R(z_0, z)$ by the equality (5), where A is the identity matrix. The first part of Theorem 1 is proved.

To prove the converse we have to give two lemmas.

Let $P(w, z)$ be a polynomial, such that for the algebraic function (1) $P(a(z), z) = 0$, $a(0) = 0$, $P(0, 0) = 0$. We denote by O the ring of all germs of holomorphic functions at the origin.

LEMMA 2. *Let $a(z) \in O$ be a branch of an algebraic function represented by the series* (1) *and let the polynomial defining the function $a(z)$ be of the form*

$$P(w, z) = (w - a(z))^k u(w, z)$$

in a neighborhood of zero, where $u(w, z)$ is an invertible germ of O. Then there exists a rational function (2), holomorphic at zero, such that the equality

$$a_\alpha = R_{\alpha_1, \alpha}$$

holds for all α.

Proof. We denote

$$\tilde{R} = \frac{1}{k} \frac{w^2 P'_w(w, z)}{P(w, z)}$$

and consider the rational function

$$R(z_0, z) = \tilde{R}(z_0, z_0 z_1, z_2, \ldots, z_n).$$

Then $R(z_0, z) \in O$. In fact, since $a(z) = z_1 a_1(z)$, where $a_1(z) \in O$, the denominator of the rational function $R(z_0, z)$ is the polynomial $P(z_0, z_0 z_1, z_2, \ldots, z_n) = (z_0 - z_0 z_1 a_1(z_0 z_1, z_2, \ldots, z_n))^k\ u = z_0^k (1 - z_1 a_1)^k u$.

The numerator of function $R(z_0, z)$ is of the form

$$\begin{aligned} &z_0^2 \Big(k(z_0 - z_0 z_1 a_1)^{k-1} u + (z_0 - z_0 z_1 a_1)^k v \Big) \\ &\qquad = z_0^{k+1} \Big(k(1 - z_1 a_1)^{k-1} u + z_0 (1 - z_1 a_1)^k v \Big), \end{aligned}$$

$v \in O$. Now since $u(0, 0) \neq 0$, it is clear that the function $R(z_0, z)$ is holomorphic in some polydisk $\{|z_j| \leq \rho;\ j = 1, \ldots, n\}$ and has the expansion (2) there. Hence we obtain the equality (6); on the other hand, according to the logarithmic residue formula we have that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma(\varepsilon)} R\left(w, \frac{z_1}{w}, z_2, \ldots, z_n\right) \frac{dw}{w} &= \frac{1}{2\pi i} \int_{\Gamma(\varepsilon)} \tilde{R}(w, z) \frac{dw}{w} \\ &= \frac{1}{2\pi i k} \int_{\Gamma(\varepsilon)} \frac{w P'_w(w, z)}{P(w, z)}\, dw = a(z). \end{aligned}$$

Thus $a(z) = \sum_\alpha R_{\alpha_1, \alpha} z^\alpha$; hence $a_\alpha = R_{\alpha_1, \alpha}$ for all α. Lemma 2 is proved.

Now we give a corollary of Lemma 2 which is not relevant to the proof of Theorem 1 but is of an independent significance. For this, we divide the variables $z = (z_1, \ldots, z_n)$ into two non-empty groups: $z = (z', z'')$. Then $\alpha = (\alpha', \alpha'')$, $a_\alpha z^\alpha = a_{\alpha' \alpha''} (z')^{\alpha'} (z'')^{\alpha''}$.

COROLLARY 2*. *Let the function*

$$w = a(z) = \sum_{\alpha \in I_1} g_{\alpha'}(z'')(z')^{\alpha'}$$

be a holomorphic solution of a polynomial equation $P(w, z) = 0$ *in a neighborhood of zero and let at least one of the coefficients of the Hartogs series be an irrational algebraic function. Then in any neighborhood of the point* $(0, 0', 0'')$ *there is at least another solution* $w = w(z)$ *of the equation* $P(w, z) = 0$.

We prove the corollary by contradiction. Namely, let the polynomial $P(w, z)$ satisfy the condition of Lemma 2. Then there exists a rational function (2) such that $a_{\alpha'\alpha''} = R_{\alpha_1, \alpha'\alpha''}$. Hence

$$\begin{aligned} g_{\alpha'}(z'') &= \sum_{\alpha''} a_{\alpha'\alpha''}(z'')^{\alpha''} = \sum_{\alpha''} R_{\alpha_1, \alpha'\alpha''}(z'')^{\alpha''} = G_{\alpha_1, \alpha'}(z'') \\ &= \frac{1}{\alpha_1! \alpha'!} \frac{\partial^{\alpha_1 + |\alpha'|} R(0, 0', z'')}{\partial z_1^{\alpha_1} (\partial z')^{\alpha'}}. \end{aligned}$$

Since $R(z_0, z)$ is rational, all the functions $g_{\alpha'}(z'') = G_{\alpha_1, \alpha'}(z'')$ are rational. This contradiction proves the corollary.

In the case where the condition of Lemma 2 does not hold, several branches of the implicit function defined by the equation $P(w, z) = 0$ may pass through the point $(0, 0) \in C^{n+1}$ and the logarithmic residue

$$\frac{1}{2\pi i} \int_{\Gamma(\varepsilon)} \frac{wP'(w, z)}{P(w, z)}\, dw$$

equals the sum of these branches. Therefore we will do a partial resolution of the singularity of the hypersurface $\{P(w, z) = 0\}$ at zero by choosing some suitable birational change of variables separating the germ of the graph $\{w = a(z)\}$ from other germs of the analytic set $\{P(w, z) = 0\}$.

We prove the following lemma.

LEMMA 3. *Let* $P(w, z) \in O$, $a(z) \in O$, $P(a(z), z) = 0$, $a(0) = 0$. *Then there exists a polynomial mapping of the form*

$$\begin{aligned} w &= S_q(r_1(\zeta), \ldots, r_n(\zeta)) + v\zeta_1^{k_1} \cdot \cdots \cdot \zeta_n^{k_n} = r_0(v, \zeta) \\ z_j &= \zeta_1^{v_{1j}} \cdot \cdots \cdot \zeta_n^{v_{nj}} = r_j(\zeta), \qquad j = 1, \ldots, n; \end{aligned} \tag{7}$$

here $v \in C^1$, $\zeta = (\zeta_1, \dots, \zeta_n) \in C^n$, $S_q(z)$ *is a polynomial of the degree* q *in* n *variables* $(z_1, \dots, z_n) = z$, $S_q(0) = 0$, *such that*:

(i) *the inverse mapping*

$$v = \rho_0(w, z)$$
$$\zeta_j = \rho_j(z), \qquad j = 1, \dots, n,$$

is rational;

(ii) $P(r_0(v, \zeta), r(\zeta)) = \zeta_1^{m_1} \cdot \dots \cdot \zeta^{m_n}(v - h(\zeta))^k u(v, \zeta)$, *where* $h \in O$, $u \in O$, $h(0) = 0$, $u(0,0) \neq 0$, $r(\zeta) = (r_1(\zeta), \dots, r_n(\zeta))$.

The mapping (7) which has the properties (i) and (ii) from Lemma 3, will be called the *resolving* mapping for the branch $w = a(z)$, holomorphic at zero, of the implicit function with respect to the equation $P(w, z) = 0$.

PROPOSITION 4. *Let the mapping* (7) *be a resolving mapping for the holomorphic branch* $w = a(z)$ *of the implicit function with respect to the equation* $P(w, z) = 0$. *Then* $a(z) = r_0(h(\rho(z)), \rho(z))$, *where* $\rho(z) = (\rho_1(z), \dots, \rho_n(z))$ *is the inverse mapping and for every* $\zeta \in \{|\zeta_j| < \delta;\ j = 1, \dots, n\}$ *we have the equality*

$$h(\zeta) = \frac{1}{2\pi i k} \int_{|v| = \varepsilon} \frac{v\left[P(r_0(v, \zeta), r(\zeta))\right]'_v \, dv}{P(r_0(v, \zeta), r(\zeta))}, \tag{8}$$

$0 < \delta \ll \varepsilon \ll 1$.

The formula (8) follows from the equality (ii) of Lemma 3 and the logarithmic residue formula. The equality $a(z) = r_0(h(\rho(z)), \rho(z))$ follows from the equalities $w = r_0(v, \zeta)$, $w = a(z)$, $v = h(\zeta)$, $\zeta = \rho(z)$.

Proof of Lemma 3. We denote by $\operatorname{ord} f$ the order of the zero of a holomorphic function f at the point 0, assuming moreover $\operatorname{ord} f = 0$ if $f(0) \neq 0$ and $\operatorname{ord} f = +\infty$ if $f \equiv 0$, and let $(f)_*$ be the lowest homogeneous polynomial not identically zero of the function f at the point 0 [7].

From the condition of Lemma 3 it follows that $P(w, z) = (w - a(z))^k \times G(w, z)$, $G \in O$. $G(a(z), z) \not\equiv 0$, i.e., $\mu = \operatorname{ord} G(a(z), z) < +\infty$. If $\mu = 0$, then $G(a(0), 0) \neq 0$; hence the resolving mapping is the identity mapping. And now we consider the case $\mu > 0$.

We denote by $S_q(z) = \sum_{\|\alpha\| \le q} a_\alpha z^\alpha$ a partial sum of the power series of the function $a(z)$. Clearly, if $q \geq \mu$, then $\operatorname{ord} G(S_q(z), z) = \operatorname{ord} G(a(z), z) = \mu$ because monomials of a degree more than μ do not influence the order. For example, we choose $q = \mu(\mu + 1)/2$. Let us do the change of variables $w \mapsto v + S_q(\zeta)$, $z \mapsto \zeta$; after that the function $P(w, z)$ goes into a function

$$P\big(v+S_q(\zeta),\zeta\big)=\Big(v-\sum_{\|\alpha\|>q}a_\alpha\zeta^\alpha\Big)^k G\big(v+S_q(\zeta),\zeta\big).$$

Let $G(v+S_q(\zeta),\zeta)=\Sigma d_j(\zeta)v^j$ be the Hartogs series expansion. Then

$$\operatorname{ord} d_0(\zeta)=\operatorname{ord} G\big(S_q(\zeta),\zeta\big)=\mu.$$

We will write a change of variables after which the function $f(v,\zeta)$ takes the form $f(\varphi(v,\zeta),\psi(v,\zeta))$ in the following way: $v\mapsto\varphi(v,\zeta)$, $\zeta\mapsto\psi(v,\zeta)$. Let us do the monomial change of variables

$$\begin{aligned} v &\mapsto v\zeta_i^\mu\\ \zeta_i &\mapsto \zeta_i\\ \zeta_j &\mapsto \zeta_i\zeta_j, \qquad j=1,\dots,[i],\dots,n, \end{aligned}\tag{9}$$

where the variable ζ_i is chosen by the condition

$$\deg_{\zeta_i}(d_0(\zeta))_*=\max_j\big\{\deg_{\zeta_j}(d_0(\zeta))_*\big\}.\tag{10}$$

The transformation (9) leads the function $G(v+S_q(\zeta),\zeta)$ to the form

$$\zeta_i^\mu\left[\frac{d_0(\zeta_i\zeta_1,\dots,\zeta_i,\dots,\zeta_i\zeta_n)}{\zeta_i^\mu}+vd_1(\zeta_i\zeta_1,\dots,\zeta_i,\dots,\zeta_i\zeta_n)+\cdots\right]$$
$$=\zeta_i^\mu\left[\frac{d_0}{\zeta_i^\mu}+vT(v,\zeta_1,\dots,\zeta_n)\right];$$

moreover $d_0(\zeta_i\zeta_1,\dots,\zeta_i,\dots,\zeta_i\zeta_n)/\zeta_i^\mu\in O$ and the inequality

$$\operatorname{ord}\frac{d_0(\zeta_i\zeta_1,\dots,\zeta_i,\dots,\zeta_i\zeta_n)}{\zeta_i^\mu}\le\mu-1\tag{11}$$

is valid. Indeed, let $d_0(z)=\Sigma_{m\ge\mu}(d_0)_m(z)$ be the expansion in homogeneous polynomials, $(d_0)_\mu(z)=(d_0(z))_*$. Then

$$d_0(\zeta_i\zeta_1,\dots,\zeta_i,\dots,\zeta_i\zeta_n)/\zeta_i^\mu=(d_0)_\mu(\zeta_1,\dots,1,\dots,\zeta_n)$$
$$+\zeta_i(d_0)_{\mu+1}(\zeta_1,\dots,1,\dots,\zeta_n)+\cdots.\tag{12}$$

Since the polynomial $(d_0)_\mu(\zeta_1, \dots, 1, \dots, \zeta_n)$ does not depend on the variable ζ_i while all other monomials of the series (12) contain a non-zero power of ζ_i, the order of the zero of the sum (12) does not exceed $\operatorname{ord}(d_0)_\mu(\zeta_1, \dots, 1, \dots, \zeta_n) < \mu$. Hence, the inequality (11) is valid.

Thus, the transformation (9) leads the function to the form

$$\zeta_i^\mu\left[d_0/\zeta_i^\mu + vT(v,\zeta)\right] =: \zeta_i^\mu G^{(1)}(v,\zeta), G^{(1)}(v,\zeta) \in O$$

and besides $\operatorname{ord} G^{(1)}(0,\zeta) = \operatorname{ord} d_0(\zeta_i\zeta_1, \dots, \zeta_i, \dots, \zeta_i\zeta_n)/\zeta_i^\mu \le \mu - 1$.

Simultaneously, after the change of the variable (9) the function $(v - \sum_{\|\alpha\| \ge q+1} a_\alpha z^\alpha)^k$ takes the form

$$\zeta_i^{k\mu}\left(v - \zeta_i^{q+1-\mu} \sum_{\|\alpha\|>q} a_\alpha(\zeta[i])^\alpha \zeta_i^{\|\alpha\|-q-1}\right)^k,$$

where $(\zeta[i])^\alpha$ is a monomial ζ^α in which the variable ζ_i is omitted. Thus this function can be written in the form $\zeta_i^{k\mu}(v - a^{(1)}(\zeta))^k$, $a^{(1)} \in O$, $\operatorname{ord} a^{(1)}(\zeta) \ge q + 1 - \mu$.

Finally, the function $P(v + S_q(\zeta), \zeta)$ goes into the function

$$\zeta_i^{\gamma_i}\left(v - a^{(1)}(\zeta)\right)^k G^{(1)}(v,\zeta),$$

where $\operatorname{ord} G^{(1)}(0,\zeta) \le \mu - 1$, $\operatorname{ord} a^{(1)}(\zeta) > q - \mu$, $\gamma_i = k\mu + \mu$.

Now we use the change of variables (9) where the number $\mu_1 = \operatorname{ord} G^{(1)}(0,\zeta)$ is chosen instead of μ and the variable ζ_i is defined by the condition

$$\deg_{\zeta_i} G^{(1)}(0,\zeta) = \max_j\left\{\deg_{\zeta_j} G^{(1)}(0,\zeta)\right\}.$$

Applying this change of variables to the function $(v - a^{(1)}(\zeta))^k G^{(1)}(v,\zeta)$, we get the function

$$\zeta_i^{\gamma_i}\zeta_j^{\delta_j}\left(v - a^{(2)}(\zeta)\right)^k G^{(2)}(v,\zeta),$$

where $\operatorname{ord} G^{(1)}(0,\zeta) \le \mu_1 - 1 \le \mu - 2$, $\operatorname{ord} a^{(2)}(\zeta) > q - \mu - \mu_1 \ge q - \mu - (\mu - 1)$.

Continuing this process, after a finite number of steps we get a function

$$\zeta_1^{m_1} \cdot \dots \cdot \zeta_n^{m_n}\left(v - \tilde{a}(\zeta)\right)^k \tilde{G}(v,\zeta),$$

where $\operatorname{ord} \tilde{a}(\zeta) > q - \mu - (\mu - 1) - \dots - 1 = q - \mu(\mu + 1)/2 = 0$ and $\operatorname{ord} \tilde{G}(0,\zeta) = 0$; moreover

$$\tilde{G}(v,\zeta) = g(\zeta) + v\tilde{T}(v,\zeta), \qquad g(\zeta) \in O,$$

$$\tilde{T}(v,\zeta) \in O, \qquad g(0) \ne 0.$$

The desired resolving mapping (7) is defined in the following way: sequentially performing the change of variables $w \mapsto v + S_q(z)$, $z \mapsto \zeta$ and the transformations defined above, we get a final transformation of the variables $w \mapsto r_0(v, \zeta)$, $z \mapsto r(\zeta)$ which is the resolving mapping

$$\begin{aligned} w &= r_0(v, \zeta) \\ z_j &= r_j(\zeta), \qquad j = 1, \dots, n. \end{aligned}$$

The resolving mapping leads the function $P(w, z)$ to the form $\zeta^m(v - h(\zeta))u(v, \zeta)$, where

$$h(\zeta) = \tilde{a}(\zeta), \qquad u(\zeta) = g(\zeta) + v\tilde{T}(v, \zeta),$$
$$h(0) = 0, \qquad u(0,0) \neq 0.$$

Since the resolving mapping is constructed as a superposition of polynomial mappings having rational inverse mappings, the inverse mapping for the resolving mapping is rational. Lemma 3 is proved.

Proof of Theorem 1. Let $a(z)$ be an algebraic function represented by the series (1) and defined by the polynomial $P(w, z)$. By Lemma 3 there exists the resolving mapping (7) for the function $a(z)$, such that (ii) holds. One can assume that the function $h(\zeta)$ is divisible by ζ_1 in the ring O; otherwise, constructing the resolving mapping, we do one more change of variables,

$$\begin{aligned} \zeta_1 &\mapsto \zeta_1 \\ \zeta_j &\mapsto \zeta_1 \zeta_j, \qquad j = 2, \dots, n. \end{aligned}$$

Note that the matrix $A_1 = (\nu_{ij})_{i,j=1}^n$ of the exponents of the monoidal mapping (7) is unimodular, $\nu_{ij} \geq 0$ because it is the product of the unimodular matrices corresponding to the monoidal transformations (σ-processes),

$$\begin{aligned} \zeta_1 &\to \zeta_i \\ \zeta_j &\mapsto \zeta_i \zeta_j, \qquad j = 1, \dots, [i], \dots, n \end{aligned}$$

(cf., for example, [8]).

The power series of the functions $h(\zeta)$ and $a(z)$ are connected as

$$h(\zeta) =: \sum h_\beta \zeta^\beta = \zeta^{-l} \sum_{\|\alpha\| > q} a_\alpha (\zeta^\nu)^\alpha,$$

where

$$(\zeta^{v})^{\alpha} = (\zeta_1^{\nu_{11}} \cdot \dots \cdot \zeta_n^{\nu_{n1}})^{\alpha_1} \cdot \dots \cdot (\zeta_1^{\nu_{n1}} \cdot \dots \cdot \zeta_n^{\nu_{nn}})^{\alpha_n},$$
$$\beta = (\beta_1, \dots, \beta_n), \quad \zeta^{-l} = \zeta_1^{-l_1} \cdot \dots \cdot \zeta_n^{-l_n}, \qquad l_j \geq 0,\ j = 1, \dots, n.$$

From the condition (ii) of Lemma 3 it follows that the polynomial

$$\zeta_1^{-m_1} \cdot \dots \cdot \zeta_n^{-m_n} P\big(r_0(v, \zeta), r(\zeta)\big)$$

satisfies the condition of Lemma 2. Hence for the function $h(\zeta)$ there exists a rational function, holomorphic at the origin,

$$R^{(1)}(z_0, z) = \sum R^{(1)}_{\beta_0, \beta} z_0^{\beta_0} z^{\beta},$$

such that $h_\beta = R^{(1)}_{\beta_1, \beta}$ for all multi-indices $\beta = (\beta_1, \dots, \beta_n)$. Hence for the function

$$h_1(\zeta) = \zeta^{l} h(\zeta) =: \sum h^{(1)}_{\beta} \zeta^{\beta} = \sum_{\|\alpha\| > q} a_\alpha (\zeta^{\nu})^{\alpha}$$

we have the rational function

$$R^{(1)}(z_0, z) = (z_0 z_1)^{l_1} z_2^{l_2} \cdot \dots \cdot z_n^{l_2} R^{(1)}(z_0, z) =: \sum R^{(2)}_{\beta_0, \beta} z_0^{\beta_0} z^{\beta},$$

such that $h^{(1)}_{\beta} = R^{(2)}_{\beta_1, \beta}$ for all β.

Since

$$a(\zeta^{v}) = \sum_{\alpha \in I_1} a_\alpha (\zeta^{\nu})^{\alpha} = S_q(\zeta^{\nu}) + h^{(1)}(\zeta),$$

where S_q is a polynomial, for the function $a(\zeta^{\nu}) =: \sum \tilde{a}_\beta \zeta^{\beta}$ there exists a rational function $R(z_0, z)$, holomorphic at the origin and represented by the series (2), such that $\tilde{a}_\beta = R_{\beta_1, \beta}$ for all $\beta = (\beta_1, \dots, \beta_n)$. The coefficients a_α and $\tilde{a}_\beta$ are connected in the following way: $\tilde{a}_\beta = a_\alpha$, if $\beta = \alpha A_1^t$, where $A_1^t =: A$ is the transposed matrix A_1, i.e., if

$$\begin{aligned} \beta_1 &= \nu_{11}\alpha_1 + \dots + \nu_{1n}\alpha_n \\ &\vdots \\ \beta_n &= \nu_{n1}\alpha_n + \dots + \nu_{nn}\alpha_n. \end{aligned}$$

Hence if $\beta = \alpha A$ then $a_\alpha = R_{\beta_1, \beta}$. Thus

$$\sum_{a \in I_1} a_\alpha z^\alpha = \sum_{\substack{\beta = \alpha A \\ \alpha \in I_1}} R_{\beta_1, \beta} z^\beta$$

and Theorem 1 is completely proved.

3. EXAMPLES

In this section we consider a few examples connected with Theorem 1.

EXAMPLE 1. The univariable function

$$f_{pq}^r(z) = \sum_{k \geq 0} \frac{(rk)!}{(pk)!(pk)!} z^k,$$

where p and q are mutually disjoint, is algebraic if and only if $r = p + q$.

Indeed, if $r > p + q$, then the convergence radius equals 0; if $r < p + q$, then the convergence radius equals $+\infty$; hence the function $f_{pq}^r(z)$ is an entire function which differs from a polynomial. Thus, $f_{pq}^r(z)$ is not an algebraic function.

The function $f_{pq}^{p+q}(z^{pq})$ is the diagonal of the power series of the rational function

$$R(z_0, z) = \sum_{m, n \geq 0} \frac{(m+n)!}{m!n!} z_0^{pm} z^{qn} = \sum_{j \geq 0} \left(z_0^p + z^q\right)^j$$
$$= \left(1 - z_0^p - z^q\right)^{-1}.$$

Let $p = 2$, $q = 1$. The formula (6) gives that

$$f_{21}^3(z^2) = \frac{-1}{2\pi i} \int_{|w| = \varepsilon} \frac{dw}{w^3 - w + z},$$

where z varies near zero. As is well known, the last integral can have singularities on the discriminant set of the denominator only, i.e., if $z = \pm 2/3\sqrt{3}$. It follows from Cardano's formula that the algebraic function $f_{21}^3(z^2)$ has branch points of order 2 at the points $z = \pm 2/3\sqrt{3}$.

The following example shows that the unimodular matrix A from Theorem 1, in general, cannot be considered as the identity matrix.

EXAMPLE 2. Let $a(z_1, z_2) = z_1(1 - z_1 - z_2)^{1/2} = \sum a_{\alpha_1\alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}$. Assume by way of contradiction that there exists a rational function

$$R(z_0, z_1, z_2) = \sum R_{\alpha_0, \alpha_1, \alpha_2} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2},$$

such that $R_{\alpha_1, \alpha_1, \alpha_2} = a_{\alpha_1, \alpha_2}$ for all (α_1, α_2). This is the condition (5) for $n = 2$, $A = I$.

Then for the function

$$g_{\alpha_1}(z_2) := \sum_{\alpha_2 \geq 0} R_{\alpha_1, \alpha_1, \alpha_2} z_2^{\alpha_2} = \sum_{\alpha_2 \geq 0} a_{\alpha_1, \alpha_2} z_2^{\alpha_2}$$

we have the equality

$$g_{\alpha_1}(z_2) = \frac{1}{(\alpha_1!)^2} \frac{\partial^{\alpha_1 + \alpha_2}}{\partial z_0^{\alpha_1}\, \partial z_1^{\alpha_1}} R(0, 0, z_2),$$

which implies that the function $g_{\alpha_1}(z_2)$ is rational. On the other hand we have that

$$g_{\alpha_1}(z_2) = \frac{1}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} a(0, z_2);$$

in particular, $g_1(z_2) = (1 - z_2)^{1/2}$ is an irrational function. It is a contradiction.

Let us construct the function $R(z_0, z_1, z_2)$ and the matrix A for the branch $a(z_1, z_2) = z_1(1 - z_1 - z_2)^{1/2}$, $a(1, -1) = 1$, in the following way.

The defining polynomial for $a(z_1, z_2)$ is $P(w, z_1, z_2) = w^2 - z_1^2(1 - z_1 - z_2)$. Take the resolving mapping $w = r_0(v, \zeta_1, \zeta_2) = \zeta_1 + \zeta_1 v$, $z_1 = \zeta_1$, $z_2 = \zeta_1 \zeta_2$. Then

$$P(r_0(v, \zeta_1, \zeta_2), \zeta_1, \zeta_1 \zeta_2) = \zeta_1^2 (v^2 + 2v + \zeta_1 + \zeta_1 \zeta_2)$$

and the branch $a(z_1, z_2)$ goes into the function $h(\zeta_1, \zeta_2) = -1 + (1 - \zeta_1 - \zeta_1 \zeta_2)^{1/2}$, $h(0, 0) = 0$. Continuing the construction as in the proof of Theorem 1, we get the rational function

$$R(z_0, z_1, z_2) = z_0 z_1 + \frac{z_0 z_1 (2 z_0^2 + 2 z_0)}{2 + z_0 + z_1 + z_1 z_2}$$

which Taylor coefficients are connected with the coefficients of $a(z_1, z_2)$ by means of the equality

$$a_{\alpha_1 \alpha_2} = R_{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2},$$

i.e., the function $a(z_1, z_2)$ is the A-diagonal of the function $R(z_0, z_1, z_2)$, where the matrix $A = (a_{ij})$, $a_{11} = 1$, $a_{12} = 0$, $a_{21} = 1$, $a_{22} = 1$.

As shown in [3], the diagonal of an algebraic function of two variables over a finite field is algebraic. Our next example shows that it is not valid for the series over the field of complex numbers.

EXAMPLE 3. Let us consider the algebraic function [9, 10]

$$\left((1 - z_1 - z_2)^2 - 4z_1 z_2\right)^{-1/2} = \sum_{m,n\geq 0} \binom{m+n}{n}^2 z_1^m z_2^n;$$

its diagonal equal

$$a(z) = \sum_{k\geq 0} \binom{2k}{k}^2 z^k.$$

We have [9, 10] that $a(z) = F(\frac{1}{2}, \frac{1}{2}, 1; 16z)$, where $F(a, b, c; z)$ is Gauss's hypergeometric function and for $a = b = \frac{1}{2}$, $c = 1$ this function is given by the complete elliptic integral

$$a(z) = \frac{2}{\pi}\int_0^1 \left((1 - x^2)(1 - zx^2)\right)^{-1/2} dx$$

and is not algebraic (has a logarithmic singularity).

4. SOME CONDITIONS FOR ALGEBRAICITY

Now we apply Theorem 1 to generalize the famous Eisenstein theorem [6] for algebraic functions in several variables.

THEOREM 5. *If the sum of an n-multiple power series*

$$\sum_{\|\alpha\|>0} a_\alpha z^\alpha \tag{13}$$

is an algebraic function and a_α are rational numbers for all α, then there exists an integer $b \neq 0$, such that the numbers $a_\alpha b^{\|\alpha\|}$ are integers for all α.

First we prove the lemma.

LEMMA 6. *The conclusion of Theorem* 5 *is valid if the function* (13) *is rational.*

Proof. Let

$$R(z_0, z) = M(z_0, z)/N(z_0, z) = \sum R_{\alpha_0, \alpha} z_0^{\alpha_0} z^{\alpha} \tag{14}$$

be a rational function and let $R_{\alpha_0, \alpha}$ be rational numbers. We may assume without loss of generality that the coefficients of the polynomials M and N are integers. Denote $c = N(0,0)$, $N_0(z_0, z) = N(z_0, z) - c$. If z_0, z are close to zero, we have

$$M/N = M/(c + N_0) = (M/c) \sum_{j \geq 0} (-1)^j (N_0/c)^j.$$

Since $M(0,0) = N_0(0,0) = 0$, we have the function

$$R(cz_0, cz) = \big(M(cz_0, cz)/c\big) \sum_{j \geq 0} (-1)^j \big(N_0(cz_0, cz)/c\big)^j$$

which Taylor coefficients are integers; i.e., the numbers $R_{\alpha_0, \alpha} c^{\alpha_0 + \|\alpha\|}$ are integers. Lemma 6 is proved.

Proof of Theorem 5. Let $a(z)$ be an algebraic function represented by (13). We will consider the function

$$a(z_1, z_1 z_2, \ldots, z_1 z_n) = \sum_{\alpha \in I_1} a_\alpha^{(1)} z^{\alpha},$$

which satisfies the hypothesis of Theorem 1. Then there exists a unimodular matrix A and a rational function (14), such that $a_\alpha^{(1)} = R_{\beta_1, \beta}|_{\beta = \alpha A}$ for all α.

We may assume without loss of generality that the coefficients of the defining polynomial $P(w, z)$ for $a(z)$ are integers. Then by the construction in the proof of Theorem 1 we can consider the coefficients of the polynomials M and N to be integers. Hence, the coefficients of the series (14) are rational numbers. By Lemma 6 there exists an integer $c \neq 0$, such that $R_{\alpha_0, \alpha} c^{\alpha_0 + \|\alpha\|}$ are integers; in particular,

$$R_{\beta_1, \beta} c^{2\beta_1 + \beta_2 + \cdots + \beta_n}$$

are integers; hence $R_{\beta_1, \beta} (c^2)^{\|\beta\|}$ are integers. Since $\beta = \alpha A$, $A = (\nu_{ij})$, we have $\|\beta\| = \alpha_1(\nu_{11} + \cdots + \nu_{1n}) + \cdots + \alpha_n(\nu_{n1} + \cdots + \nu_{nn})$. Let $s = \nu_{11} + \cdots + \nu_{1n} + \cdots + \nu_{nn}$ be the sum of all the elements of the matrix A. Then it is clear that $a_\alpha^{(1)} = (c^2)^{s\|\alpha\|}$ are integers. Since

$$a_{\alpha_1, \ldots, \alpha_n} = a^{(1)}_{\|\alpha\|, \alpha_2, \ldots, \alpha_n},$$

we obtain that $a_\alpha(c^{2s})^{2\|\alpha\|}$ are integers. Thus $a_\alpha b^{\|\alpha\|}$ are integers if $b = c^{4s}$. This completes the proof.

Now we consider a Hartogs series

$$f(z,w) = \sum_{\|\alpha\|>0} g_\alpha(w) z^\alpha, \tag{15}$$

where $z \in C^n$, $w \in C^m$ and let its sum be holomorphic in a neighborhood of the origin. Is there any analog of Eisenstein's theorem for Hartogs series? For example, we may put the following question.

QUESTION. *Is the following statement true?—If the sum of the Hartogs series* (15) *is an algebraic function with respect to z and the functions $g_\alpha(w)$ are rational, then there exists a non-trivial polynomial $Q(w)$, such that $g_\alpha(w)(Q(w))^{\|\alpha\|}$ are polynomials for all α.*

The author has proved this statement in the case when the defining polynomial for $f(z,w)$ satisfies the condition $P_f'(0,0,0) \neq 0$. However, a complete answer is still unknown to us.

If a function $f(z,w)$, holomorphic at the origin, is algebraic, then the coefficients of its Hartogs series

$$g_\alpha(w) = \frac{1}{\alpha!} \frac{\partial^{\|\alpha\|} f(0,w)}{\partial z^\alpha}$$

are algebraic. We can prove the following inverse statement.

PROPOSITION 7. *Let the function* (15) *be holomorphic in the polydisk $\{\|z\| < r, \|w\| < r\}$ and be algebraic with respect to z for each w, $\|w\| < r$, and let the coefficients $g_\alpha(w)$ be algebraic, $\|\alpha\| > 0$. Then $f(z,w)$ is an algebraic function.*

COROLLARY 7*. *Let the function $f(z)$ be holomorphic in a neighborhood of the origin and let it be algebraic in every complex straight line passing through $0 \in C^n$. Then $f(z)$ is an algebraic function.*

Proof of Proposition 7. Let the function $f(z,w)$ be algebraic with respect to z and be defined for each w, $\|w\| < r$, by the polynomial equation

$$P_w(f,z) = \sum_{k+\|\beta\| \le q(w)} C_{k,\beta}(w) f^k z^\beta = 0,$$

where $C_{k,\beta}(w)$ are some functions. Following a method from [11], we will show that the degree $q(w)$ is a constant for all w from some polydisk $\{\|w - w^0\| < \varepsilon\}$. Denote by B_m the set of those z for which the degree of the polynomial P_w is equal to m. Then B_m is closed. Indeed, without loss of generality one can consider that $\Sigma|C_{k,\beta}(w)|^2 = 1$. Let $w_j \to \tilde{w}$, $\{w_j\} \subset B_m$. Since the set $\{C_{k,\beta} : \Sigma|C_{k,\beta}|^2 = 1\}$ is compact, one can choose a subsequence $w'_j \to \tilde{w}$, such that $C_{k,\beta}(w'_j) \to \tilde{C}_{k,\beta}$. Since for every w'_j the equality

$$P_{w'_j}\big(f(z, w'_j), z\big) \equiv 0$$

holds and the function $f(z,w)$ is continuous at the point $(z,\tilde{w})$, we have the equality

$$\sum_{k+\|\beta\|\le m} \tilde{C}_{k,\beta}[f(z,\tilde{w})]^k z^\beta = 0$$

for all z, thus $\tilde{w} \in B_m$.

Since $\{\|w\| < r\} \subset \bigcup_{m\ge 0} B_m$, by Baire's theorem we have that some set B_m contains a polydisk $\{\|w - w^0\| < \varepsilon\}$ in which $q(w) = m$.

Further, the polynomial P_w can be written in the form

$$P_w = a_1T_1 + \cdots + a_lT_l + T_0,$$

where $T_0, \dots, T_l$ are polynomials with algebraic coefficients and the functions $a_1, \dots, a_l$ are linearly independent over the field of algebraic functions. It follows that the series (15) with algebraic coefficients satisfies every equation $T_j = 0$, in particular, $T_1 = 0$. Let the polynomial T_1 be of the form

$$T_1 = \sum_{k+\|\beta\|\le m} d_{k,\beta}(w) f^k z^\beta.$$

The function $F(z, y)$ of the variables

$$z = (z_1, \dots, z_n), \qquad y = (y_{k,\beta})_{k+\|\beta\|\le m}$$

satisfying the equation

$$\sum_{k+\|\beta\|\le m} y_{k,\beta} f^k z^\beta = 0$$

is an algebraic function of the variables z, y.

Since the functions $d_{k,\beta}(w)$ are algebraic, we get that the function $f(z,w) = F(z, d(w))$, where $d(w) = (d_{k,\beta}(w))$ is an algebraic function of z, w. Proposition 7 is proved.

Corollary 7* is derived in the following way. Let $f(z) = \Sigma(f)_k(z)$ be the series in homogeneous polynomials. Hence, the Hartogs series $\Sigma(f)_k(z)t^k$

of the function $f(tz_1, \dots, tz_n)$ which is algebraic with respect to the variable t satisfies the hypothesis of Proposition 7. Hence $f(z)$ is algebraic.

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