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Many odd zeta values are irrational

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Abstract

Building upon ideas of the second and third authors, we prove that at least $2^{(1-\varepsilon)(\log s)/(\log\log s)}$ values of the Riemann zeta function at odd integers between 3 and s are irrational, where ε is any positive real number and s is large enough in terms of ε. This lower bound is asymptotically larger than any power of $\log s$; it improves on the bound $(1-\varepsilon)(\log s)/(1+\log 2)$ that follows from the Ball–Rivoal theorem. The proof is based on construction of several linear forms in odd zeta values with related coefficients.

Introduction

When $s \geqslant 2$ is an even integer, the value $\zeta(s)$ of the Riemann zeta function is a non-zero rational multiple of π^s and, therefore, a transcendental number. On the other hand, no such relation is expected to hold for $\zeta(s)$ when $s \geqslant 3$ is odd; a folklore conjecture states that the numbers π, $\zeta(3)$, $\zeta(5)$, $\zeta(7), \dots$ are algebraically independent over the rationals. This conjecture is predicted by Grothendieck's period conjecture for mixed Tate motives. But both conjectures are far out of reach and we do not even know the transcendence of a single odd zeta value.

It was only in 1978 that Apéry astonished the mathematics community with his proof [Apé79] of the irrationality of $\zeta(3)$ (see [Fis04] for a survey). The next breakthrough was made in 2000 by Ball and Rivoal [BR01, Riv00] who proved the following result.

Theorem 1 (Ball–Rivoal). *Let $\varepsilon > 0$. Then for any $s \geqslant 3$ odd and sufficiently large with respect to ε, we have*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \zeta(7), \dots, \zeta(s)) \geqslant \frac{1-\varepsilon}{1+\log 2}\log s.$$

Their corresponding result for small s has been refined several times [Zud02, FZ10], but the question whether $\zeta(5)$ is irrational remains open. The proof of Theorem 1 involves the well-poised hypergeometric series

$$n!^{s-2r}\sum_{t=1}^{\infty}\frac{\prod_{j=0}^{(2r+1)n}(t-rn+j)}{\prod_{j=0}^{n}(t+j)^{s+1}}, \tag{0.1}$$

which happens to be a $\mathbb{Q}$-linear combination of 1 and odd zeta values when s is odd and n is even, and Nesterenko's linear independence criterion [Nes85]. The bound $(1-\varepsilon)(\log s)/(1+\log 2)$ follows from comparison of how small the linear combination is with respect to the size of its

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coefficients, after multiplying by a common denominator to make them integers. To improve on this bound using the same strategy, one has to find linear combinations that are considerably smaller, with coefficients not too large; it turns out to be a rather difficult task. This may be viewed as an informal explanation of why the lower bound in Theorem 1 has never been improved for large values of s, whereas the theorem itself has been generalized to several other families of numbers.

Using (with $s = 20$) the series

$$n!^{s-6} \sum_{k=1}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\left(t + \frac{n}{2} \right) \frac{\prod_{j=0}^{3n} (t - n + j)^3}{\prod_{j=0}^{n} (t + j)^{s+3}} \right) \bigg|_{t=k},$$

which is a $\mathbb{Q}$-linear combination of 1 and odd zeta values starting from $\zeta(5)$, Rivoal has proved [Riv02] that among the numbers $\zeta(5)$, $\zeta(7), \dots, \zeta(21)$, at least one is irrational. This result has been improved by the third author [Zud01]: among the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$, at least one is irrational; and he also showed [Zud02] that, for any odd $\ell \geqslant 1$, there is an irrational number among $\zeta(\ell+2), \zeta(\ell+4), \dots, \zeta(8\ell-1)$. Proofs of these results do not require use of linear independence criteria: if a sequence of $\mathbb{Z}$-linear combinations of real numbers from a given (fixed) collection tends to 0, and is non-zero infinitely often, then at least one of these numbers is irrational. A drawback of this approach is that it only allows one to prove that *one* number in a family is irrational.

The situation drastically changed when the third author introduced [Zud18] a new method (see also [KZ19]). He casts (with $s = 25$) the rational function in the form

$$R(t) = 2^{6n} n!^{s-5} \frac{\prod_{j=0}^{6n} (t - n + j/2)}{\prod_{j=0}^{n} (t + j)^{s+1}}$$

and proves that both series

$$\sum_{t=1}^{\infty} R(t) \quad \text{and} \quad \sum_{t=1}^{\infty} R\left(t + \frac{1}{2} \right)$$

are $\mathbb{Q}$-linear combinations of $1, \zeta(3), \zeta(5), \dots, \zeta(s)$ with *related* coefficients. This allows him to eliminate one odd zeta value, and to prove that *at least two* zeta values among $\zeta(3), \zeta(5), \dots, \zeta(25)$ are irrational. In view of Apéry's theorem, the result means that one number among $\zeta(5), \dots,$ $\zeta(25)$ is irrational, nothing really novel, but the method of proof is new and more elementary than those in [Riv02, Zud01] as it avoids use of the saddle-point method. More importantly, the method allows one to prove the irrationality of at least two zeta values in a family without having to produce very small linear forms. The same strategy has been adopted by Rivoal and the third author [RZ18] to prove that among $\zeta(5), \zeta(7), \dots, \zeta(69)$, at least two numbers are irrational.

The method in [Zud18] has been generalized by the second author [Spr18], who introduces another integer parameter $D > 1$ and considers the rational function

$$R(t) = D^{6(D-1)n} n!^{s-3D-1} \frac{\prod_{j=0}^{3Dn} (t - n + j/D)}{\prod_{j=0}^{n} (t + j)^{s+1}}. \tag{0.2}$$

He proves that for any divisor d of D the series

$$\sum_{j=1}^{d} \sum_{t=1}^{\infty} R\left(t + \frac{j}{d} \right)$$

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is a $\mathbb{Q}$-linear combination of $1, \zeta(3), \zeta(5), \ldots, \zeta(s)$. The crucial point of this construction is that each $\zeta(i)$ appears in this $\mathbb{Q}$-linear combination with a coefficient that depends on d in a very simple way. This makes it possible to eliminate from the entire collection of these linear combinations as many odd zeta values as the number of divisors of D. Finally, taking D equal to a power of 2 and s sufficiently large with respect to D, the second author proves that at least $\log D/\log 2$ numbers are irrational among $\zeta(3), \zeta(5), \ldots, \zeta(s)$. This strategy represents a new proof that $\zeta(i)$ is irrational for infinitely many odd integers i.

Building upon the approach in [Zud18, Spr18], we prove the following result (announced in [FSZ18]).

Theorem 2. *Let $\varepsilon > 0$, and let $s \geqslant 3$ be an odd integer sufficiently large with respect to ε. Then among the numbers*

$$\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s),$$

at least

$$2^{(1-\varepsilon)(\log s)/(\log\log s)}$$

are irrational.

In this result, the lower bound is asymptotically greater than $\exp(\sqrt{\log s})$, and than any power of $\log s$; 'to put it roughly, [it is] much more like a power of s than a power of $\log s$' [HW79, ch. XVIII, § 1].

In comparison, Theorem 1 gives only $(1-\varepsilon)(\log s)/(1+\log 2)$ irrational odd zeta values, but they are linearly independent over the rationals, whereas Theorem 2 ends up only with their irrationality.

Our proof of Theorem 2 follows the above-mentioned strategy of the second and third authors. The main new ingredient, compared to the proof in [Spr18], is to take D large (about $s^{1-2\varepsilon}$) and equal to the product of the first prime numbers (the so-called primorial); such a number has asymptotically the largest possible number of divisors with respect to its size (see [HW79, ch. XVIII, § 1]). To perform the required elimination of a prescribed set of odd zeta values, we need to establish that a certain auxiliary matrix is invertible. Whereas the second author's choice of D in [Spr18] allows him to deal with elementary properties of a Vandermonde matrix, we use at this step a generalization of the corresponding result. We give three different proofs of the latter, based on arguments from combinatorics of partitions, from linear algebra accompanied with a lemma of Fekete, and from analysis using Rolle's theorem.

The structure of this paper is as follows. In § 1 we construct linear forms in values of the Hurwitz zeta function. Denominators of the coefficients are studied in § 2; and the asymptotics of the linear forms are dealt with in § 3. Section 4 is devoted to the proof that an auxiliary matrix is invertible. Finally, we establish Theorem 2 in § 5.

1. Construction of linear forms

From now on we let s, D be positive integers such that $s \geqslant 3D$; we assume that s is odd. Let n be a positive integer, such that Dn is even. Consider the rational function

$$R_n(t) = D^{3Dn} n!^{s+1-3D} \frac{\prod_{j=0}^{3Dn}(t-n+j/D)}{\prod_{j=0}^{n}(t+j)^{s+1}}$$

which, of course, depends also on s and D. Notice that the difference between $R_n(t)$ and the corresponding function in [Spr18] is in the factor D^{3Dn} instead of $D^{6(D-1)n}$ (see Equation (0.2)).

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Similar rational functions have already been considered; see [RZ03] for the case $D = 2$ and [Nas04, Nis11, Fis18] for general D. However, the 'central' factors $t-n+j/D$ with $Dn < j < 2Dn$ are missing, and (as the second author noticed [Spr18]) they play a central role in the arithmetic estimates (see Lemma 2 below).

Remark 1. Though one can implement an additional parameter r in the definition of the rational function $R_n(t)$, in a way similar to that for the Ball–Rivoal series (0.1), we have verified that this does not bring any improvement to the result of Theorem 2.

The rational function $R_n(t)$ has a partial fraction expansion

$$R_n(t) = \sum_{i=1}^{s} \sum_{k=0}^{n} \frac{a_{i,k}}{(t+k)^i}. \tag{1.1}$$

For any $j \in \{1, \ldots, D\}$, take

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\biggl(m + \frac{j}{D}\biggr).$$

This series converges because of the following estimate for the degree of the rational function:

$$\deg R_n(t) = (3Dn+1) - (n+1)(s+1) \leqslant -2.$$

We recall that the Lerch zeta function is defined by the convergent series

$$\Phi(z, i, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^i}$$

for $\alpha \in \mathbb{R}_{>0}$, $z \in \mathbb{C}$ and $i \in \mathbb{Z}$ with either $|z| < 1$, or $|z| = 1$ and $i \geqslant 2$; the Hurwitz zeta function is its special case

$$\zeta(i, \alpha) = \Phi(1, i, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^i}.$$

The following lemma is precisely [Spr18, Lemma 1.5]; the change of the normalizing factor D^{3Dn} does not affect the statement.

Lemma 1. *For each $j \in \{1, \ldots, D\}$, we have*

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3 \leqslant i \leqslant s \\ i \text{ odd}}} \rho_i \zeta\biggl(i, \frac{j}{D}\biggr),$$

where

$$\rho_i = \sum_{k=0}^{n} a_{i,k} \quad \text{for } 3 \leqslant i \leqslant s, i \text{ odd},$$

does not depend on j, and

$$\rho_{0,j} = -\sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{i=1}^{s} \frac{a_{i,k}}{(\ell + j/D)^i}. \tag{1.2}$$

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Proof. We follow the strategy of proofs in [Zud18, Lemma 3] and [Spr18, Lemma 1.5]. Let z be a real number such that $0 < z < 1$. We have

$$\begin{aligned}\sum_{m=1}^{\infty} R_n\left(m+\frac{j}{D}\right)z^m &= \sum_{m=1}^{\infty}\sum_{i=1}^{s}\sum_{k=0}^{n}\frac{a_{i,k}z^m}{(m+k+j/D)^i}\\ &= \sum_{i=1}^{s}\sum_{k=0}^{n} a_{i,k}z^{-k}\sum_{m=1}^{\infty}\frac{z^{m+k}}{(m+k+j/D)^i}\\ &= \sum_{i=1}^{s}\sum_{k=0}^{n} a_{i,k}z^{-k}\left(\Phi\left(z,i,\frac{j}{D}\right)-\sum_{\ell=0}^{k}\frac{z^\ell}{(\ell+j/D)^i}\right).\end{aligned}$$

Now we let z tend to 1 in the equality we have obtained; the left-hand side tends to $r_{n,j}$. On the right-hand side, the term involving the Lerch function with $i = 1$ has coefficient $\sum_{k=0}^{n} a_{1,k}z^{-k}$. Since $\Phi(z, 1, j/D)$ has only a logarithmic divergence as $z \to 1$ and

$$\sum_{k=0}^{n} a_{1,k} = \lim_{t\to\infty} tR_n(t) = 0,$$

this term tends to 0 as $z \to 1$. All other terms have finite limits as $z \to 1$, so that

$$r_{n,j} = \rho_{0,j} + \sum_{i=2}^{s}\rho_i\zeta\left(i,\frac{j}{D}\right),$$

where $\rho_{0,j}$ is given by Equation (1.2), and $\rho_i = \sum_{k=0}^{n} a_{i,k}$ for any $i \in \{2, \dots, s\}$.

To complete the proof, we apply the symmetry phenomenon of [BR01, Riv00]. Since s is odd and Dn is even we have $R_n(-n-t) = -R_n(t)$. Now the partial fraction expansion (1.1) is unique, so that $a_{i,n-k} = (-1)^{i+1}a_{i,k}$ for any i and k. This implies that $\rho_i = 0$ when i is even, and Lemma 1 follows. □

2. Arithmetic estimates

As usual, we let $d_n = \operatorname{lcm}(1, 2, \dots, n)$.

Lemma 2. *We have*

$$d_n^{s+1-i}\rho_i \in \mathbb{Z} \quad \text{for } i = 3, 5, \dots, s, \tag{2.1}$$

and

$$d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z} \quad \text{for any } j \in \{1, \dots, D\}. \tag{2.2}$$

For part (2.1) we use the strategy of the proof of [Fis18, Lemma 4.5]; note that [Spr18, Lemma 1.3] does not apply in our present situation because of the different normalization of the rational function $R_n(t)$ compared to that in (0.2). To establish (2.2) we follow the proof of [Spr18, Lemma 1.4]; we use d_{n+1} here instead of d_n to include the case corresponding to $j = D$.

Proof of Lemma 2. For any $\alpha \in (1/D)\mathbb{Z}$ we introduce

$$F_\alpha(t) = D^n\frac{\prod_{j=1}^{n}(t+\alpha+j/D)}{\prod_{j=0}^{n}(t+j)} = \sum_{k=0}^{n}\frac{A_{\alpha,k}}{t+k},$$

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where $A_{\alpha,k}$ is an integer in view of the explicit formulas

$$(-1)^k A_{\alpha,k} = \binom{n}{k} \frac{\prod_{j=1}^{n}(D(\alpha-k)+j)}{n!} = \begin{cases} \binom{n}{k}\binom{D(\alpha-k)+n}{n} & \text{if } \alpha-k \geqslant 0, \\ 0 & \text{if } \frac{-n}{D} \leqslant \alpha-k < 0, \\ (-1)^n\binom{n}{k}\binom{D(k-\alpha)-1}{n} & \text{if } \alpha-k < \frac{-n}{D}. \end{cases}$$

We also consider

$$G(t) = \frac{n!}{\prod_{j=0}^{n}(t+j)} = \sum_{k=0}^{n} \frac{(-1)^k\binom{n}{k}}{t+k},$$

so that

$$R_n(t) = (t-n)G(t)^{s+1-3D} \prod_{\ell=0}^{3D-1} F_{-n+\ell n/D}(t). \tag{2.3}$$

From this expression we compute the partial fraction expansion of $R_n(t)$ using the rules

$$\frac{t-n}{t+k} = 1 - \frac{k+n}{t+k} \quad \text{and} \quad \frac{1}{(t+k)(t+k')} = \frac{1}{(k'-k)(t+k)} + \frac{1}{(k-k')(t+k')} \quad \text{for } k \neq k'.$$

A denominator appears each time the second rule is applied, and the denominator is always a divisor of d_n (see [Col03] or [Zud18, Lemma 1]). This happens $s+1-i$ times in each term that contributes to $a_{i,k}$ because there are $s+1$ factors in the product (2.3) (apart from $t-n$). Therefore,

$$d_n^{s+1-i} a_{i,k} \in \mathbb{Z} \quad \text{for any } i \text{ and } k,$$

implying (2.1).

We now proceed with the second part of Lemma 2, that is, with demonstrating the inclusions (2.2). Recall from Lemma 1 that

$$d_{n+1}^{s+1}\rho_{0,j} = -\sum_{k=0}^{n}\sum_{\ell=0}^{k}\left(\sum_{i=1}^{s}\frac{d_{n+1}^{s+1}a_{i,k}}{(\ell+j/D)^i}\right). \tag{2.4}$$

If $j = D$ then

$$d_{n+1}^{s+1-i}a_{i,k} \quad \text{and} \quad \frac{d_{n+1}^{i}}{(\ell+j/D)^i}$$

are integers for any k, ℓ and i, so that $d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z}$. From now on, we assume that $1 \leqslant j \leqslant D-1$ and we prove that for any k and any ℓ the internal sum over i in Equation (2.4) is an integer. With this aim in mind, fix integers k_0 and ℓ_0, with $0 \leqslant \ell_0 \leqslant k_0 \leqslant n$, and assume that the corresponding sum is not an integer. Recall that R_n was defined at the beginning of § 1, and vanishes at all non-integer values of $n - j'/D$ with $0 \leqslant j' \leqslant 3Dn$. Since $1 \leqslant j \leqslant D-1$ we have $R_n(\ell_0 - k_0 + j/D) = 0$; it follows from (1.1) that

$$\sum_{i=1}^{s}\frac{d_{n+1}^{s+1}a_{i,k_0}}{(\ell_0+j/D)^i} = -\sum_{\substack{k=0\\k\neq k_0}}^{n}\sum_{i=1}^{s}\frac{d_{n+1}^{s+1}a_{i,k}}{(\ell_0-k_0+k+j/D)^i}. \tag{2.5}$$

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By our assumption this rational number is not an integer: it has negative p-adic valuation for at least one prime number p. Therefore, on either side of (2.5) there is at least one term with negative p-adic valuation: there exist $i_0, i_1 \in \{1, \ldots, s\}$ and $k_1 \in \{0, \ldots, n\}$, $k_1 \neq k_0$, such that

$$v_p\biggl(\frac{d_{n+1}^{s+1} a_{i_0,k_0}}{(\ell_0 + j/D)^{i_0}}\biggr) < 0 \quad \text{and} \quad v_p\biggl(\frac{d_{n+1}^{s+1} a_{i_1,k_1}}{(\ell_0 - k_0 + k_1 + j/D)^{i_1}}\biggr) < 0.$$

Since $d_{n+1}^{s+1-i} a_{i,k} \in \mathbb{Z}$ for any i and k, this leads to

$$v_p\biggl(\frac{d_{n+1}^{i_0}}{(\ell_0 + j/D)^{i_0}}\biggr) < 0 \quad \text{and} \quad v_p\biggl(\frac{d_{n+1}^{i_1}}{(\ell_0 - k_0 + k_1 + j/D)^{i_1}}\biggr) < 0,$$

implying

$$\min\biggl(v_p\biggl(\ell_0 + \frac{j}{D}\biggr), v_p\biggl(\ell_0 - k_0 + k_1 + \frac{j}{D}\biggr)\biggr) > v_p(d_{n+1}).$$

As $k_0 - k_1 = (\ell_0 + j/D) - (\ell_0 - k_0 + k_1 + j/D)$, we deduce that $v_p(k_0 - k_1) > v_p(d_{n+1})$, which is impossible in view of the inequalities $0 < |k_0 - k_1| \leqslant n$. The contradiction completes the proof of Lemma 2. □

Remark 2. It is made explicit in [RZ18] (though for a particular situation considered there), that the inclusions in Lemma 2 can be sharpened into the form

$$\Phi_n^{-1} d_n^{s+1-i} \rho_i \in \mathbb{Z} \quad \text{for } i = 3, 5, \ldots, s,$$

and

$$\Phi_n^{-1} d_{n+1}^{s+1} \rho_{0,j} \in \mathbb{Z} \quad \text{for any } j \in \{1, \ldots, D\},$$

where $\Phi_n = \Phi_n(D)$ is a certain product over primes in the range $2 \leqslant p \leqslant n$, whose asymptotic behaviour

$$\phi = \phi(D) = \lim_{n\to\infty} \frac{\log \Phi_n}{n}$$

can be controlled by means of the prime number theorem. It is possible to show that the quantity $\phi(D)/D$ increases to ∞ and at the same time $\phi(D)/(D \log^{\varepsilon} D) \to 0$ as $D \to \infty$, for any choice of $\varepsilon > 0$. Later, we choose D such that $D \log D < s$, implying that the arithmetic gain coming from the factors Φ_n^{-1} is asymptotically negligible as $s \to \infty$. For the same reason, any arithmetic improvement like the one established in [KR07] would have no influence on the statement of Theorem 2.

3. Asymptotic estimates of the linear forms

The following lemma is proved along the same lines as [Spr18, Lemma 2.1] (see also [Zud18, Lemma 4] and the second proof of [BR01, Lemme 3]). The difference is that here we only assume $s/(D \log D)$ to be sufficiently large, whereas in [Spr18] parameter D is fixed and $s \to \infty$.

Lemma 3. *Assume that $D \geqslant 2$ and*

$$\frac{s}{D \log D} \text{ is larger than some (effectively computable) absolute constant.} \tag{3.1}$$

Then

$$\lim_{n\to\infty} r_{n,j}^{1/n} = g(x_0) < 4^{-(s+1)} \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{n,j'}}{r_{n,j}} = 1 \quad \text{for any } j, j' \in \{1, \ldots, D\}, \tag{3.2}$$

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where

$$g(x) = D^{3D}\frac{(x+3)^{3D}(x+1)^{s+1}}{(x+2)^{2(s+1)}}$$

and x_0 is the unique positive root of the polynomial

$$(X+3)^D(X+1)^{s+1} - X^D(X+2)^{s+1}.$$

Proof. For $j \in \{1,\dots,D\}$ and $k \geqslant 0$, let

$$c_{k,j} = R_n\left(n+k+\frac{j}{D}\right) = D^{3Dn}n!^{s+1-3D}\frac{\prod_{\ell=0}^{3Dn}(k+(j+\ell)/D)}{\prod_{\ell=0}^{n}(n+k+\ell+j/D)^{s+1}},$$

so that

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\left(m+\frac{j}{D}\right) = \sum_{k=0}^{\infty} c_{k,j}$$

is a sum of positive terms. We have

$$\frac{c_{k+1,j}}{c_{k,j}} = \left(\prod_{\ell=1}^{D}\frac{k+3n+(j+\ell)/D}{k+(j+\ell-1)/D}\right)\left(\frac{k+n+j/D}{k+2n+1+j/D}\right)^{s+1} \tag{3.3}$$

implying that, for any j, the quotient $c_{k+1,j}/c_{k,j}$ tends to $f(\kappa)$ as $n \to \infty$ assuming $k \sim \kappa n$ for $\kappa > 0$ fixed, where

$$f(x) = \left(\frac{x+3}{x}\right)^D\left(\frac{x+1}{x+2}\right)^{s+1}.$$

For the logarithmic derivative of this function we have

$$\frac{f'(x)}{f(x)} = \frac{D}{x+3} - \frac{D}{x} + \frac{s+1}{x+1} - \frac{s+1}{x+2} = \frac{ax^2+bx+c}{x(x+1)(x+2)(x+3)}$$

with $a = s+1-3D > 0$ and $c = -6D < 0$, hence the derivative $f'(x)$ vanishes at exactly one positive real number x_1. This means that the function $f(x)$ decreases on $(0, x_1]$ and increases on $[x_1, +\infty)$. Since $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = 1$, we deduce that there exists a unique positive real number x_0 such that $f(x_0) = 1$.

Let us now prove (3.2). As in [dBru81, § 3.4] we wish to demonstrate that the asymptotic behaviour of $r_{n,j}$ is governed by the terms $c_{k,j}$ with k close to $x_0 n$ (see Equation (3.8) below). To begin with, notice that

$$\begin{aligned} c_{k,j} &= D^{-1}n!^{s+1-3D}\frac{\prod_{\ell=0}^{3Dn}(Dk+j+\ell)}{\prod_{\ell=0}^{n}(n+k+\ell+j/D)^{s+1}} \\ &= D^{-1}n!^{s+1-3D}\frac{(3Dn+Dk+j)!}{(Dk+j-1)!}\frac{\Gamma(n+k+j/D)^{s+1}}{\Gamma(2n+k+1+j/D)^{s+1}}. \end{aligned}$$

Denoting by $k_0(n)$ the integer part of $x_0 n$ and applying the Stirling formula to the factorial and gamma factors we obtain, as $n \to \infty$,

$$\begin{aligned} c_{k_0(n),j}^{1/n} \sim \left(\frac{n}{e}\right)^{s+1-3D}&\left(\frac{3Dn+Dk_0(n)+j}{e}\right)^{3D+Dx_0}\left(\frac{e}{Dk_0(n)+j-1}\right)^{Dx_0} \\ &\times\left(\frac{n+k_0(n)+j/D}{e}\right)^{(s+1)(x_0+1)}\left(\frac{e}{2n+k_0(n)+j/D+1}\right)^{(s+1)(x_0+2)} \end{aligned}$$

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$$\begin{aligned}&\sim \frac{((x_0+3)D)^{(x_0+3)D}}{(x_0D)^{x_0D}}\frac{(x_0+1)^{(s+1)(x_0+1)}}{(x_0+2)^{(s+1)(x_0+2)}}\\&= g(x_0)f(x_0)^{x_0}=g(x_0).\end{aligned} \tag{3.4}$$

We now show that the asymptotic behaviour of $r_{n,j}$ as $n\to\infty$ is determined by the terms $c_{k,j}$ with k close to x_0n. Given D and s, we take $\varepsilon>0$ sufficiently small to accommodate the condition

$$b(\varepsilon)=\max\biggl(f(x_0+\varepsilon),\frac{1}{f(x_0-\varepsilon)}\biggr)<1.$$

Then there exists $A(\varepsilon)>x_1$, where x_1 is the unique positive root of $f'(x)=0$, such that $f(A(\varepsilon))=b(\varepsilon)$. We have $f(x)\geqslant 1/b(\varepsilon)$ for any $x\in(0,x_0-\varepsilon]$ and $f(x)\leqslant b(\varepsilon)$ for any $x\in[x_0+\varepsilon,A(\varepsilon)]$. For any k such that $(x_0+2\varepsilon)n\leqslant k\leqslant (A(\varepsilon)-\varepsilon)n$, Equation (3.3) and the sentence after imply that $c_{k,j}\leqslant b(\varepsilon)c_{k-1,j}$ provided n is large (in terms of D, s and ε), so that, taking $k_1=\lfloor(x_0+2\varepsilon)n\rfloor$ and $k_2=\lfloor(x_0+3\varepsilon)n\rfloor$, we obtain

$$\sum_{k_2\leqslant k\leqslant(A(\varepsilon)-\varepsilon)n} c_{k,j}\leqslant c_{k_1,j}\sum_{k=k_2}^{+\infty} b(\varepsilon)^{k-k_1}\leqslant c_{k_1,j}\frac{b(\varepsilon)^{k_2-k_1}}{1-b(\varepsilon)}\leqslant \varepsilon c_{k_1,j} \tag{3.5}$$

for all n sufficiently large. In the same way, we get the estimate

$$\sum_{0\leqslant k\leqslant\lfloor(x_0-3\varepsilon)n\rfloor} c_{k,j}\leqslant \varepsilon c_{\lfloor(x_0-2\varepsilon)n\rfloor,j} \tag{3.6}$$

for all n large (in terms of D, s and ε). Finally, choosing ε small, we can assume that $A(\varepsilon)$ is sufficiently large (in terms of D and s), so that for $k\geqslant(A(\varepsilon)-\varepsilon)n$ we have

$$c_{k,j}\leqslant(2D)^{3Dn}\biggl(\frac{n!}{k^{n+1}}\biggr)^{s+1-3D}$$

for n large. Combining this result with the estimate

$$\begin{aligned}\sum_{k=\lceil(A(\varepsilon)-\varepsilon)n\rceil}^{+\infty}\frac{1}{k^{(n+1)(s+1-3D)}}&\leqslant\frac{1}{\lceil(A(\varepsilon)-\varepsilon)n\rceil^{(n+1)(s+1-3D)-2}}\sum_{k=\lceil(A(\varepsilon)-\varepsilon)n\rceil}^{+\infty}\frac{1}{k^2}\\&\leqslant\frac{2}{\lceil(A(\varepsilon)-\varepsilon)n\rceil^{(n+1)(s+1-3D)-2}}\end{aligned}$$

gives

$$\sum_{k=\lceil(A(\varepsilon)-\varepsilon)n\rceil}^{+\infty}c_{k,j}\leqslant 2(2D)^{3Dn}\frac{n!^{s+1-3D}}{\lceil(A(\varepsilon)-\varepsilon)n\rceil^{(n+1)(s+1-3D)-2}}.$$

Using hypothesis (3.1) and the Stirling formula, the latter estimate implies

$$\sum_{k=\lceil(A(\varepsilon)-\varepsilon)n\rceil}^{+\infty}c_{k,j}\leqslant\biggl(\frac{2D}{e(A(\varepsilon)-\varepsilon)}\biggr)^{sn/2}$$

provided n is sufficiently large. For ε small, we can assume that $A(\varepsilon)$ is sufficiently large (in terms of D and s) to obtain

$$\sum_{k=\lceil(A(\varepsilon)-\varepsilon)n\rceil}^{+\infty}c_{k,j}\leqslant\biggl(\frac{1}{2}g(x_0)\biggr)^{n}. \tag{3.7}$$

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Combining Equations (3.4)–(3.7), we deduce that

$$(1-3\varepsilon)r_{n,j} \leqslant \sum_{(x_0-3\varepsilon)n\leqslant k\leqslant(x_0+3\varepsilon)n} c_{k,j} \leqslant r_{n,j}. \tag{3.8}$$

Now for any k in the range $(x_0-3\varepsilon)n \leqslant k \leqslant (x_0+3\varepsilon)n$ it follows from the proof of Equation (3.4) that

$$g(x_0)-h(\varepsilon) \leqslant c_{k,j}^{1/n} \leqslant g(x_0)+h(\varepsilon)$$

for n large (in terms of D, s and ε), where h is a positive function of ε such that $\lim_{\varepsilon\to 0^+} h(\varepsilon)=0$. Using (3.8), this implies

$$(g(x_0)-2h(\varepsilon))^n \leqslant 5\varepsilon n(g(x_0)-h(\varepsilon))^n \leqslant r_{n,j} \leqslant \frac{7\varepsilon n}{1-3\varepsilon}(g(x_0)+h(\varepsilon))^n \leqslant (g(x_0)+2h(\varepsilon))^n$$

for n sufficiently large, and finishes the proof of $\lim_{n\to\infty} r_{n,j}^{1/n} = g(x_0)$ for any j.

To establish

$$\lim_{n\to\infty}\frac{r_{n,j'}}{r_{n,j}} = 1$$

for any $j, j' \in \{1,\ldots,D\}$, we can assume that $1\leqslant j\leqslant D-1$ and $j'=j+1$. For any k we have

$$\frac{c_{k,j+1}}{c_{k,j}} = \frac{k+3n+(j+1)/D}{k+j/D}\left(\frac{\Gamma(n+k+(j+1)/D)}{\Gamma(n+k+j/D)}\frac{\Gamma(2n+k+1+j/D)}{\Gamma(2n+k+1+(j+1)/D)}\right)^{s+1}.$$

It follows from the Stirling formula that $\Gamma(x+1/D) \sim x^{1/D}\Gamma(x)$ as $x\to\infty$, so that for $k=\lfloor x_0 n\rfloor$ we have, as $n\to\infty$,

$$\frac{c_{k,j+1}}{c_{k,j}} \sim \frac{x_0+3}{x_0}\left(\frac{(x_0+1)^{1/D}}{(x_0+2)^{1/D}}\right)^{s+1} = f(x_0)^{1/D} = 1.$$

More generally, for k in the range $(x_0-3\varepsilon)n\leqslant k\leqslant(x_0+3\varepsilon)n$ and n sufficiently large we have

$$1-\tilde h(\varepsilon) \leqslant \frac{c_{k,j+1}}{c_{k,j}} \leqslant 1+\tilde h(\varepsilon)$$

with $\lim_{\varepsilon\to 0^+}\tilde h(\varepsilon)=0$. Using Equation (3.8) twice, we deduce that

$$\begin{aligned} r_{n,j+1} &\leqslant \frac{1}{1-3\varepsilon}\sum_{(x_0-3\varepsilon)n\leqslant k\leqslant(x_0+3\varepsilon)n} c_{k,j+1} \\ &\leqslant \frac{1+\tilde h(\varepsilon)}{1-3\varepsilon}\sum_{(x_0-3\varepsilon)n\leqslant k\leqslant(x_0+3\varepsilon)n} c_{k,j} \\ &\leqslant \frac{1+\tilde h(\varepsilon)}{1-3\varepsilon} r_{n,j}. \end{aligned}$$

In the same way we obtain

$$r_{n,j+1} \geqslant (1-3\varepsilon)(1-\tilde h(\varepsilon))r_{n,j}.$$

Combining these inequalities and using that $\varepsilon>0$ is arbitrary results in

$$\lim_{n\to\infty}\frac{r_{n,j+1}}{r_{n,j}} = 1.$$

This concludes the proof of (3.2), except for the upper bound on $g(x_0)$ which we verify now.

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To estimate $g(x_0)$ from above, we can assume by (3.1) that the value

$$f(\tfrac{1}{2}) = 7^D(\tfrac{3}{5})^{s+1}$$

is smaller than 1, so that $x_0 < \frac{1}{2}$. The logarithmic derivative of $g(x)$ is given by

$$\frac{g'(x)}{g(x)} = \frac{3D}{x+3} + \frac{s+1}{x+1} - \frac{2(s+1)}{x+2}.$$

By the hypothesis (3.1), the function $g(x)$ is decreasing on the interval $[0, \frac{1}{2}]$. We deduce that

$$g(x_0) < g(\tfrac{1}{2}) = D^{3D}(\tfrac{7}{2})^{3D}(\tfrac{24}{25})^{s+1}(\tfrac{1}{4})^{s+1} < (\tfrac{1}{4})^{s+1},$$

where (3.1) was used again. This completes our proof of Lemma 3. □

Remark 3. For $s = 77$ and $D = 4$ one computes $g(x_0) < \exp(-78)$. Thus, the suitable linear combinations (cf. § 5)

$$\hat{r}_{n,1} = r_{n,4}, \quad \hat{r}_{n,2} = r_{n,2} + r_{n,4} \quad \text{and} \quad \hat{r}_{n,4} = r_{n,1} + r_{n,2} + r_{n,3} + r_{n,4}$$

of the corresponding linear forms allow us to eliminate three of the odd zeta values from the list

$$\{\zeta(3), \zeta(5), \dots, \zeta(77)\}.$$

In particular, we obtain that two out of $\{\zeta(5), \zeta(7), \dots, \zeta(77)\}$ are irrational. This result is slightly weaker than the result of Rivoal and the third author [RZ18], but it emerges as a byproduct of the construction above. The arithmetic gain given by $\Phi_n(4)$ for $\Phi_n(D)$ defined in Remark 2 can be used to slightly reduce the bound of 77 to 73, still weaker than that in [RZ18].

4. A non-vanishing determinant

The following lemma is used to eliminate irrational zeta values in § 5 below.

Lemma 4. *For $t \geqslant 1$, let $x_1 < \cdots < x_t$ be positive real numbers and $\alpha_1 < \cdots < \alpha_t$ non-negative integers. Then the generalized Vandermonde matrix $[x_j^{\alpha_i}]_{1 \leqslant i,j \leqslant t}$ has positive determinant.*

We remark that, subject to the hypothesis that $x_1, \dots, x_t$ are real and positive, Lemma 4 is a stronger version of [LMN95, Lemme 1] and, therefore, has potential applications to the zero estimates for linear forms in two logarithms.

The above result is quite classical and known to many people. While writing this paper we found various proofs of rather different nature, three given below. We leave it to the reader to choose their favourite proof.

Combinatorial proof of Lemma 4. As pointed out in [Kra99, § 2.1], the generalized Vandermonde determinant in question is closely related to Schur polynomials. Let $\Delta := \det[x_j^{\alpha_i}]_{1 \leqslant i,j \leqslant t}$, and

$$V = \det[x_j^{i-1}]_{1 \leqslant i,j \leqslant t} = \prod_{1 \leqslant i < j \leqslant t} (x_j - x_i) > 0$$

be the Vandermonde determinant of $x_1, \dots, x_t$. For any $i \in \{1, \dots, t\}$, we take $\lambda_i = \alpha_{t+1-i} + i - t$, so that $\lambda_1 \geqslant \cdots \geqslant \lambda_t \geqslant 0$; then $\lambda = (\lambda_1, \dots, \lambda_t)$ is a partition of the integer $\lambda_1 + \cdots + \lambda_t$. The associated Schur polynomial

$$s_\lambda = s_\lambda(x_1, \dots, x_t) = \frac{\Delta}{V}$$

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possesses the expression

$$s_\lambda = \sum_T x_1^{m_1(T)} \cdots x_t^{m_t(T)},$$

with the sum over all column-strict Young tableaux T of shape λ (see, for instance, [FH91, Appendix A, A.31] or [Mac79, I.3], and [Pro89] for a direct proof). Here, $m_i(T)$ denotes the number of occurrences of i in the tableau T. From this we deduce that s_λ is a positive real number, thus $\Delta = s_\lambda \cdot V > 0$. □

Linear algebra proof of Lemma 4. Write $A_{J,K}$ for the minor of an $n \times m$ matrix A, where $n \leqslant m$, determined by ordered index sets J and K. A classical result due to Fekete [FP12] asserts that if all $(n-1)$-minors

$$A_{(1,2,\dots,n-1),K}, \quad K = (k_1, \dots, k_{n-1}) \quad \text{with } 1 \leqslant k_1 < \cdots < k_{n-1} \leqslant m$$

are positive, and all minors of size n with consecutive columns are positive, then all n-minors of A are positive. Thus, Lemma 4 follows by induction on t from Fekete's result applied to the matrix $[x_j^k]_{1 \leqslant j \leqslant t,\, 0 \leqslant k < m}$, using the positivity of the Vandermonde determinant. □

Analytical proof of Lemma 4 (see [GK02, pp. 76–77]). By induction on t one proves the following claim. *A non-zero function*

$$f(x) = \sum_{i=1}^{t} c_i x^{\alpha_i},$$

with $c_i, \alpha_i \in \mathbb{R}$, has at most $t-1$ positive zeros. Indeed, if f has t positive zeros then Rolle's theorem provides $t-1$ positive zeros of the derivative $(\mathrm{d}/\mathrm{d}x)(x^{-\alpha_1} f(x))$. The non-vanishing of the determinant in Lemma 4 is an immediate consequence of this claim. Since the determinant depends continuously on the parameters α_i, we deduce the required positivity from the positivity of the Vandermonde determinant. □

5. Elimination of odd zeta values

Let $0 < \varepsilon < \frac{1}{3}$, and let s be odd and sufficiently large with respect to ε. We take D to be the product of all primes less than or equal to $(1-2\varepsilon)\log s$ (such a product has asymptotically the largest possible number of divisors with respect to its size; see [HW79, ch. XVIII, § 1]). We have

$$\log D = \sum_{\substack{p \text{ prime} \\ p \leqslant (1-2\varepsilon)\log s}} \log p \leqslant (1-\varepsilon)\log s$$

by the prime number theorem, that is, $D \leqslant s^{1-\varepsilon}$. Then $D \log D \leqslant s^{1-\varepsilon} \log s$: the assumption of Lemma 3 holds.

Notice that D has precisely $\delta = 2^{\pi((1-2\varepsilon)\log s)}$ divisors, with

$$\log \delta = \pi((1-2\varepsilon)\log s)\log 2 \geqslant (1-3\varepsilon)(\log 2)\frac{\log s}{\log\log s}.$$

Assume that the number of irrational odd zeta values between $\zeta(3)$ and $\zeta(s)$ is less than or equal to $\delta - 1$. Let $3 = i_1 < i_2 < \cdots < i_{\delta-1} \leqslant s$ be odd integers such that if $\zeta(i) \notin \mathbb{Q}$ and i is odd, $3 \leqslant i \leqslant s$, then $i = i_j$ for some j. We set $i_0 = 1$, and consider the set $\mathcal{D}$ of all divisors of D,

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so that $\operatorname{Card}\mathcal{D} = \delta$. Lemma 4 implies that the matrix $[d^{i_j}]_{d\in\mathcal{D},0\leqslant j\leqslant\delta-1}$ is invertible. Therefore, there exist integers $w_d \in \mathbb{Z}$, where $d \in \mathcal{D}$, such that

$$\sum_{d\in\mathcal{D}} w_d d^{i_j} = 0 \quad \text{for any } j \in \{1,\ldots,\delta-1\} \tag{5.1}$$

and

$$\sum_{d\in\mathcal{D}} w_d d^{i_0} = \sum_{d\in\mathcal{D}} w_d d \neq 0. \tag{5.2}$$

With the help of Lemma 1 we construct the linear forms

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3\leqslant i\leqslant s \\ i \text{ odd}}} \rho_i \zeta\Big(i, \frac{j}{D}\Big)$$

for $n \geqslant 1$ and $1 \leqslant j \leqslant D$. The crucial point (as in [Spr18, § 3]) is that for any $d \in \mathcal{D}$ and any $i \geqslant 2$,

$$\sum_{j=1}^{d} \zeta\Big(i, \frac{jD/d}{D}\Big) = \sum_{j=1}^{d} \zeta\Big(i, \frac{j}{d}\Big) = \sum_{n=0}^{\infty}\sum_{j=1}^{d} \frac{d^i}{(dn+j)^i} = d^i\zeta(i),$$

implying that

$$\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,jD/d} = \sum_{j=1}^{d} \rho_{0,jD/d} + \sum_{\substack{3\leqslant i\leqslant s \\ i \text{ odd}}} \rho_i d^i \zeta(i)$$

are linear forms in the odd zeta values with asymptotic behaviour

$$\widehat{r}_{n,d} = (d + o(1)) r_{n,1} \quad \text{as } n \to \infty \text{ where } \lim_{n\to\infty} r_{n,1}^{1/n} = g(x_0) < 4^{-(s+1)},$$

by Lemma 3.

We now use the integers w_d to eliminate the odd zeta values $\zeta(i_j)$ for $j = 1,\ldots,\delta-1$, including all irrational ones, as in [Zud18, Spr18]. For that purpose, consider

$$\widetilde{r}_n = \sum_{d\in\mathcal{D}} w_d \widehat{r}_{n,d}.$$

Equations (5.1) imply that

$$\widetilde{r}_n = \sum_{d\in\mathcal{D}} w_d \sum_{j=1}^{d} \rho_{0,jD/d} + \sum_{i\in I} \rho_i \Big(\sum_{d\in\mathcal{D}} w_d\, d^i\Big) \zeta(i),$$

where $I = \{3,5,7,\ldots,s\}\backslash\{i_1,\ldots,i_{\delta-1}\}$; in particular, no irrational zeta value $\zeta(i)$, where $3 \leqslant i \leqslant s$, appears in this linear combination. Using Equation (5.2), we obtain

$$\widetilde{r}_n = \Big(\sum_{d\in\mathcal{D}} w_d d + o(1)\Big) r_{n,1} \quad \text{with } \sum_{d\in\mathcal{D}} w_d d \neq 0,$$

so that

$$\lim_{n\to\infty} |\widetilde{r}_n|^{1/n} = g(x_0) < 4^{-(s+1)}.$$

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Now all $\zeta(i)$, $i \in I$, are assumed to be rational. Denoting by A their common denominator, we deduce from Lemma 2 that $Ad_{n+1}^{s+1}\widetilde{r}_n$ is an integer. From the prime number theorem we have $\lim_{n\to\infty} d_{n+1}^{1/n} = e$, hence the sequence of integers satisfies

$$0 < \lim_{n\to\infty} |Ad_{n+1}^{s+1}\widetilde{r}_n|^{1/n} = e^{s+1}g(x_0) < \left(\frac{e}{4}\right)^{s+1} < 1.$$

This contradiction concludes the proof of Theorem 2.

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