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An Asymptotic Formula for Binomial Sums

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We obtain complete asymptotic expansions for certain binomial sums, including the Apéry numbers. In general, binomial sums cannot be expressed by closed formulae, but they do satisfy polynomial recurrence relations. We use the asymptotic expansion of a binomial sum to calculate a lower bound for the number of terms in its recurrence relation.

1. INTRODUCTION

In his celebrated proof of the irrationality of $\zeta(3)$ Apéry [12] shows that the binomial sum

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the three term polynomial recurrence

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)\, a_{n-1} + (n-1)^3\, a_{n-2} = 0.$$

Since the characteristic polynomial $x^2 - 34x + 1$ has roots $(1 \pm \sqrt{2})^4$, it follows that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = (1 + \sqrt{2})^4$$

is irrational. Thus a_n cannot satisfy a two term recurrence. Cohen [12] has calculated that

$$a_n = \frac{(1+\sqrt{2})^{4n+2}}{(2\pi n\sqrt{2})^{3/2}} \left(1 - \frac{48 - 15\sqrt{2}}{64n} + O(n^{-2})\right).$$

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The existence of an asymptotic expansion of this type for polynomially recursive functions was established Birkhoff and Trjitzinsky [1] (also see [8]). This justifies the method of subsituting the asymptotic series

$$K(n)\, n^{\theta} \left(1+\frac{A_1}{n}+\frac{A_2}{n^2}+\cdots\right),$$

where $K(n)$ is a simple polynomially recursive leading term, into the recurrence and solving for $K(n)$, θ, A_1, A_2, ... Since $K(n)$ often contains a transcendental term (in the Apéry case a power of π) other techniques must be employed to obtain an exact expression for $K(n)$.

Without using a recurrence relation we will show more precisely that

$$a_n=\frac{(1+\sqrt{2})^{4n+2}}{(2\pi n\sqrt{2})^{3/2}}\left(1-\frac{48-15\sqrt{2}}{64n}+\frac{2057-1200\sqrt{2}}{4096n^2}\right.$$
$$\left.-\frac{87024-62917\sqrt{2}}{262144n^3}+O(n^{-4})\right). \tag{1.1}$$

Wilf, Zeilberger [14] and others have shown that certain hypergeometric sums, including the binomial sum

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\cdots\binom{n+mk}{k}^{r_m}, \tag{1.2}$$

where $r_0, r_1, r_2, ..., r_m$ are nonnegative integers, satisfy polynomial recurrences. It is not difficult to see that if

$$\mu=\lim_{n\to\infty}\frac{S(n+1)}{S(n)},$$

then μ is a root of the characteristic polynomial of a recurrence for $S(n)$. Furthermore, if μ is an algebraic number of degree d, then $S(n)$ cannot satisfy a polynomial recurrence with fewer that $d+1$ terms. Thus a lower bound for the number of terms in a polynomial recurrence for $S(n)$ can be obtained from its asymptotic expansion.

In this article we give a complete asymptotic expansion for the binomial sum $S(n)$. It is not necessary that the exponents $r_0, r_1, r_2, ..., r_m$ be nonnegative integers. We assume that $r_0, r_1, r_2, ..., r_m$ are nonnegative real numbers with $r_0>0$. Unless otherwise stated $r=r_0+r_1+r_2+\cdots+r_m$ throughout this article.

Main Theorem. *For each nonnegative integer p,*

$$S(n)=\frac{\mu^{n+1/2}}{\sqrt{v(2\pi\lambda n)^{r-1}}}\left(1+\sum_{k=1}^{p}\frac{R_k}{n^k}+O(n^{-p-1})\right),$$

where $0<\lambda<1$ *is defined by*

$$\prod_{j=0}^{m}\left(\frac{(1+j\lambda)^{j}}{\lambda(1+j\lambda-\lambda)^{j-1}}\right)^{r_j}=1,$$

$$\mu=\prod_{j=0}^{m}\left(\frac{1+j\lambda}{1+j\lambda-\lambda}\right)^{r_j},$$

$$\nu=\sum_{j=0}^{m}\frac{r_j}{(1+j\lambda-\lambda)(1+j\lambda)},$$

and each R_k *is rational function of the exponents* $r_0, r_1, r_2, ..., r_m$ *and* λ.

2. Proof of the Main Theorem

Let $f(n,k)$ be the summand in (1.2). Thus

$$S(n)=\sum_{k=0}^{n} f(n,k).$$

Setting $k=nt$, where $0\leqslant t\leqslant 1$ is a continuous variable, we obtain

$$f(n,nt)=\binom{n}{nt}^{r_0}\binom{n+nt}{nt}^{r_1}\binom{n+2nt}{nt}^{r_2}\cdots\binom{n+mnt}{nt}^{r_m}, \tag{2.1}$$

where

$$\binom{n+jnt}{nt}=\frac{\Gamma(n+jnt+1)}{\Gamma(nt+1)\,\Gamma(n+jnt-nt+1)}.$$

Let p be any nonnegative integer. From Stirling's formula

$$\log\Gamma(x+1)=x\log x-x+\frac{1}{2}\log x+\frac{1}{2}\log 2\pi$$
$$+\sum_{k=1}^{p}\frac{B_{2k}}{2k(2k-1)}x^{-2k+1}+O(x^{-2p-1}),$$

(see for example [13, pp. 26–28]) we obtain

$$\log\binom{n+jnt}{nt}=n\log\frac{(1+jt)^{1+jt}}{t^{t}(1+jt-t)^{1+jt-t}}-\frac{1}{2}\log n-\frac{1}{2}\log 2\pi$$
$$+\frac{1}{2}\log\frac{1+jt}{t(1+jt-t)}+\sum_{k=1}^{p}\frac{B_{2k}}{2k(2k-1)}$$

$$\times \left\{ \frac{1}{(1+jt)^{2k-1}} - \frac{1}{t^{2k-1}} - \frac{1}{(1+jt-t)^{2k-1}} \right\} n^{-2k+1}$$
$$+ O(n^{-2p-1}) \tag{2.2}$$

uniformly for $0 < \delta \leqslant t \leqslant 1 - \delta < 1$. This implies that

$$\log f(n, nt) = n \log \prod_{j=0}^{m} \left(\frac{(1+jt)^{1+jt}}{t^t (1+jt-t)^{1+jt-t}} \right)^{r_j} - \frac{r}{2} \log n - \frac{r}{2} \log 2\pi$$
$$+ \frac{1}{2} \log \prod_{j=0}^{m} \left(\frac{1+jt}{t(1+jt-t)} \right)^{r_j} + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)}$$
$$\times \sum_{j=0}^{m} \left\{ \frac{r_j}{(1+jt)^{2k-1}} - \frac{r_j}{t^{2k-1}} - \frac{r_j}{(1+jt-t)^{2k-1}} \right\} n^{-2k+1}$$
$$+ O(n^{-2p-1}) \tag{2.3}$$

uniformly for $0 < \delta \leqslant t \leqslant 1 - \delta < 1$. From (2.3) we see that

$$\lim_{n \to \infty} f(n, nt)^{1/n} = \prod_{j=0}^{m} \left(\frac{(1+jt)^{1+jt}}{t^t (1+jt-t)^{1+jt-t}} \right)^{r_j}, \tag{2.4}$$

and the convergence is uniform $0 < \delta \leqslant t \leqslant 1 - \delta < 1$.

To show that the convergence in (2.4) is uniform on $0 \leqslant t \leqslant 1$ it suffices to show that for $j = 0, 1, 2, ..., m$,

$$\binom{n + jnt}{nt} \to \frac{(1+jt)^{1+jt}}{t^t (1+jt-t)^{1+jt-t}} \tag{2.5}$$

uniformly for $0 \leqslant t \leqslant 1$. Pointwise convergence of (2.5) follows from (2.2). We now use the fact that pointwise convergence of monotone functions implies uniform convergence. When $j \geqslant 1$ both sides of (2.5) are increasing functions of t, and therefore the convergence is uniform on $0 \leqslant t \leqslant 1$. When $j = 0$ the above argument shows that the convergence of (2.5) is uniform on $0 \leqslant t \leqslant 1/2$, and hence by symmetry the convergence is uniform on $0 \leqslant t \leqslant 1$.

Let

$$f(t) = \prod_{j=0}^{m} \left(\frac{(1+jt)^{1+jt}}{t^t (1+jt-t)^{1+jt-t}} \right)^{r_j}.$$

We have shown that

$$\lim_{n \to \infty} f(n, nt)^{1/n} = f(t) \tag{2.6}$$

uniformly for $0 \leqslant t \leqslant 1$.

By direct calculation

$$\frac{d}{dt}\log f(t)=\log\prod_{j=0}^{m}\left(\frac{(1+jt)^{j}}{t(1+jt-t)^{j-1}}\right)^{r_j} \tag{2.7}$$

and

$$\frac{d^2}{dt^2}\log f(t)=-\sum_{j=0}^{m}\frac{r_j}{t(1+jt-t)(1+jt)}<0. \tag{2.8}$$

Since $r_0>0$, we find that

$$\lim_{t\to 1^-}\frac{d}{dt}\log f(t)=-\infty,$$

and since f is increasing on $0\leqslant t\leqslant 1/2$, it follows that f has a unique maximum in the interval $1/2\leqslant t<1$. From (2.7) we see that this maximum occurs at $t=\lambda$, where

$$\prod_{j=0}^{m}\left(\frac{(1+j\lambda)^{j}}{\lambda(1+j\lambda-\lambda)^{j-1}}\right)^{r_j}=1. \tag{2.9}$$

The maximum value of f, denoted by μ, is given by

$$\begin{aligned}\mu=f(\lambda)&=\prod_{j=0}^{m}\left(\frac{(1+j\lambda)^{1+j\lambda}}{\lambda^{\lambda}(1+j\lambda-\lambda)^{1+j\lambda-\lambda}}\right)^{r_j}\\&=\prod_{j=0}^{m}\left(\frac{(1+j\lambda)^{j}}{\lambda(1+j\lambda-\lambda)^{j-1}}\right)^{r_j\lambda}\prod_{j=0}^{m}\left(\frac{1+j\lambda}{1+j\lambda-\lambda}\right)^{r_j}\\&=\prod_{j=0}^{m}\left(\frac{1+j\lambda}{1+j\lambda-\lambda}\right)^{r_j}\end{aligned} \tag{2.10}$$

by (2.9).

Let $\varepsilon>0$ be fixed. We will prove that

$$S(n)=\sum_{k\in T}f(n,k)\{1+o(e^{-\delta n})\} \tag{2.11}$$

as n approaches ∞, where $T=\{k\colon \lambda-\varepsilon\leqslant k/n\leqslant\lambda+\varepsilon\}$ and δ is a positive constant depending upon ε. Thus, as n tends to ∞, the primary contribution to $S(n)$ arises from those k which are close to λn.

Let $u=\sup\{f(t)\colon |t-\lambda|\geqslant\varepsilon\}$. Since $f(t)$ attains its maximum at $t=\lambda$ we have $u<f(\lambda)$. Choose v and w so that $u<v<w<f(\lambda)$. Since $f(n,nt)^{1/n}$ tends to $f(t)$ uniformly for $0\leqslant t\leqslant 1$ as n tends to ∞, it follows that for n sufficiently large

$$f(n,k)<v^n$$

for all $k \notin T$. Also, for each n sufficiently large there exists k such that

$$f(n, k) > w^n.$$

Therefore

$$\frac{\sum_{k=0,\, k\notin T}^{n} f(n, k)}{\sum_{k=0}^{\infty} f(n, k)} < \frac{\sum_{k=0,\, k\notin T}^{n} v^n}{w^n} \leqslant \frac{(n+1)\, v^n}{w^n} < e^{-\delta n}$$

from some positive constant δ and n sufficiently large, since $v < w$. This establishes (2.11).

Returning to (2.3), put

$$\begin{aligned} h(t) &= n \log \prod_{j=0}^{m} \left(\frac{(1+jt)^{1+jt}}{t^t (1+jt-t)^{1+jt-t}} \right)^{r_j} - \frac{r}{2} \log 2\pi \\ &\quad + \frac{1}{2} \log \prod_{j=0}^{m} \left(\frac{1+jt}{t(1+jt-t)} \right)^{r_j} + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)} \\ &\quad \times \sum_{j=0}^{m} \left\{ \frac{r_j}{(1+jt)^{2k-1}} - \frac{r_j}{t^{2k-1}} - \frac{r_j}{(1+jt-t)^{2k-1}} \right\} n^{-2k+1} \\ &= h_0(t)\, n + h_1(t) + \sum_{k=1}^{p} h_{2k}(t)\, n^{-2k+1}. \end{aligned} \tag{2.12}$$

Thus, by (2.3), we see that

$$\log f(n, nt) = h(t) - \frac{r}{2} \log n + O(n^{-2p-1}).$$

Hence from (2.8)

$$S(n) = n^{-r/2} \sum_{k\in T} e^{h(k/n)} \{1 + O(n^{-2p-1})\}, \tag{2.13}$$

where $T = \{k : \lambda - \varepsilon \leqslant k/n \leqslant \lambda + \varepsilon\}$.

Anticipating an application of the Euler–Maclaurin sum formula, we examine

$$\frac{1}{\sqrt{n}} \int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} e^{h(x/n)}\, dx = \int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h(\lambda + x/\sqrt{n})}\, dx.$$

Since $h(t)$ is analytic in a neighborhood of $t = \lambda$, by Taylor's Theorem and (2.12) we have for $|t - \lambda| < \varepsilon$,

$$h(t) = \sum_{j=0}^{\infty} \left\{ h_0^{(j)}(\lambda)\, n + h_1^{(j)}(\lambda) + \sum_{k=1}^{p} h_{2k}^{(j)}(\lambda)\, n^{-2k+1} \right\} \frac{(t-\lambda)^j}{j!}$$

or

$$h(\lambda + x/\sqrt{n}) = \sum_{j=0}^{\infty} \left\{ h_0^{(j)}(\lambda)\, n^{-j/2+1} + h_1^{(j)}(\lambda)\, n^{-j/2} + \sum_{k=1}^{p} h_{2k}^{(j)}(\lambda)\, n^{-j/2-2k+1} \right\} \frac{x^j}{j!}.$$

From the definitions of $h(t)$ and $f(t)$ we see that

$$h_0(t) = \log f(t), \tag{2.14}$$

and since $f'(\lambda)=0$, it follows that $h_0'(\lambda)=0$. Thus

$$h(\lambda + x/\sqrt{n}) = h_0(\lambda)\, n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\, n^{-2k+1} + \frac{h_0''(\lambda)}{2} x^2 + \sum_{k=1}^{\infty} u_k(x)\, n^{-k/2},$$

where $u_k(x)$, $k \geqslant 1$, is a polynomial in x. The polynomial $u_k(x)$, $k \geqslant 1$, is an even or odd polynomial according as k is even or odd, respectively. Exponentiating, we find that

$$e^{h(\lambda + x/\sqrt{n})} = \exp\left(h_0(\lambda)\, n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\, n^{-2k+1} \right) \times e^{h_0''(\lambda)\, x^2/2} \left(1 + \sum_{k=1}^{\infty} p_k(x)\, n^{-k/2} \right),$$

where $p_k(x)$, $k \geqslant 1$, is a polynomial in x that is even or odd, according as k is even or odd, respectively. Hence

$$\int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h(\lambda + x/\sqrt{n})}\, dx = \exp\left(h_0(\lambda)\, n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\, n^{-2k+1} \right) \times \int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h_0''(\lambda)\, x^2/2} \left(1 + \sum_{k=1}^{\infty} p_k(x)\, n^{-k/2} \right) dx. \tag{2.15}$$

By symmetry the contributions of the terms involving $p_{2k+1}(x)$, $k \geqslant 0$, equal 0.

From (2.14) and (2.8) we get

$$h_0''(\lambda) = -\sum_{j=0}^{m} \frac{r_j}{\lambda(1 + j\lambda - \lambda)(1 + j\lambda)} = -\frac{\nu}{\lambda} < 0, \tag{2.16}$$

where v is given in the statement of our theorem. Thus for each nonnegative integer j,

$$\int_{-\infty}^{+\infty} e^{h_0''(\lambda)\,x^2/2} x^{2j}\,dx = \left(-\frac{2}{h_0''(\lambda)}\right)^{j+1/2} \Gamma\left(j+\frac{1}{2}\right), \tag{2.17}$$

and therefore

$$\int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h_0''(\lambda)\,x^2/2} p_{2j}(x)\,n^{-j}\,dx = O(n^{-j}).$$

Thus from (2.15), we have

$$\begin{aligned}
\frac{1}{\sqrt{n}} \int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} & e^{h(x/n)}\,dx \\
&= \int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h(\lambda+x/\sqrt{n})}\,dx \\
&= \exp\left(h_0(\lambda)\,n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\,n^{-2k+1}\right) \\
&\quad\times \left\{\int_{-\varepsilon\sqrt{n}}^{+\varepsilon\sqrt{n}} e^{h_0''(\lambda)\,x^2/2} \left(1+\sum_{k=1}^{2p} p_{2k}(x)\,n^{-k}\right) dx + O(n^{-2p-1})\right\} \\
&= \exp\left(h_0(\lambda)\,n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\,n^{-2k+1}\right) \\
&\quad\times \left\{\int_{-\infty}^{+\infty} e^{h_0''(\lambda)\,x^2/2} \left(1+\sum_{k=1}^{2p} p_{2k}(x)\,n^{-k}\right) dx + O(n^{-2p-1})\right\},
\end{aligned} \tag{2.18}$$

since the integrals over the intervals $(-\infty, -\varepsilon\sqrt{n})$ and $(+\varepsilon\sqrt{n}, +\infty)$ are $o(e^{-\delta n})$ for some positive δ as n tends to ∞. By using (2.17) in (2.18) we can develop an asymptotic series for the integral on the left side of (2.18).

We now apply the Euler–Maclaurin formula (see for example [13, pp. 14–15]) to the sum on the right side of (2.13). For brevity set $H(t) = e^{h(t)}$. Then for every nonnegative integer p,

$$\begin{aligned}
\sum_{j\in T} e^{h(j/n)} &= \int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} e^{h(x/n)}\,dx + \frac{1}{2} H(\lambda-\varepsilon) + \frac{1}{2} H(\lambda+\varepsilon) \\
&\quad + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \{H^{(2k-1)}(\lambda+\varepsilon) - H^{(2k-1)}(\lambda-\varepsilon)\}\, n^{-2k+1} \\
&\quad + \frac{1}{(2p+1)!} \left\{\int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} B_{2p+1}(x-[x])\, H^{(2p+1)}(x/n)\,dx\right\} n^{-p-1},
\end{aligned} \tag{2.19}$$

where for simplicity we have made the assumption that $n\varepsilon$ is a positive integer.

$$\int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} B_{2p+1}(x-[x])\, H^{(2p+1)}(x/n)\, dx$$

$$= n \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} B_{2p+1}(ns-[ns])\, H^{(2p+1)}(s)\, ds$$

$$= O(n)$$

as n tends to ∞. From the definition of $h(t)$ we see that

$$h(t) = h_0(t)\, n + O(1)$$

or

$$H(t) = e^{h(t)} = O(e^{h_0(t)\, n}),$$

where $h_0(t) = \log f(t)$. As demonstrated earlier, $f(t)$ attains its maximum at $t=\lambda$. Therefore the terms in (2.19) involving $H(\lambda \pm \varepsilon)$ and $H^{(2k-1)}(\lambda \pm \varepsilon)$, $1 \leqslant k \leqslant p$, are exponentially small in comparison with the leading term in (2.18). Thus from (2.19)

$$\sum_{k \in T} e^{h(k/n)} = \int_{n(\lambda-\varepsilon)}^{n(\lambda+\varepsilon)} e^{h(x/n)}\, dx \{1 + o(e^{-\delta n})\} \tag{2.20}$$

for some positive δ as $n \to \infty$. Using (2.20) and (2.18) in (2.13) we obtain the asymptotic formula

$$S(n) = \exp\left(h_0(\lambda)\, n - \frac{(r-1)}{2} \log n + h_1(\lambda) + \sum_{k=1}^{p} h_{2k}(\lambda)\, n^{-2k+1}\right)$$
$$\times \int_{-\infty}^{+\infty} e^{h_0''(\lambda)\, x^2/2} \left(1 + \sum_{k=1}^{2p} p_{2k}(x)\, n^{-k}\right) dx \{1 + O(n^{-2p-1})\}. \tag{2.21}$$

3. The First Few Terms of Our Asymptotic Formula

We now calculate the first few terms of the right side of (2.21). From (2.14) and (2.10) we get

$$\exp\left(h_0(\lambda)\, n - \frac{(r-1)}{2} \log n\right) = \frac{\mu^n}{n^{(r-1)/2}}. \tag{3.1}$$

From the definition of $h_1(t)$, given in (2.12), we obtain

$$h_1(\lambda) = -\frac{r}{2}\log 2\pi + \frac{1}{2}\log \prod_{j=0}^{m} \left(\frac{1+j\lambda}{\lambda(1+j\lambda-\lambda)}\right)^{r_j} = \log \sqrt{\frac{\mu}{(2\pi\lambda)^r}}.$$

Thus

$$e^{h_1(\lambda)} \int_{-\infty}^{+\infty} e^{h_0''(\lambda)\, x^2/2}\, dx = \sqrt{\frac{\mu}{v(2\pi\lambda)^{r-1}}}, \tag{3.2}$$

since $h_0''(\lambda) = -v/\lambda$ by (2.16). Combining (3.1) and (3.2) gives

$$S(n) \sim \frac{\mu^{n+1/2}}{\sqrt{v(2\pi\lambda n)^{r-1}}}.$$

Taking further terms of (2.21) into consideration it follows that $S(n)$ has an asymptotic expansion of the type

$$S(n) = \frac{\mu^{n+1/2}}{\sqrt{v(2\pi\lambda n)^{r-1}}}\left(1 + \sum_{k=1}^{p} \frac{R_k}{n^k} + O(n^{-p-1})\right)$$

for each nonnegative integer p, where each R_k is a rational function of the exponents $r_0, r_1, r_2, ..., r_m$ and λ. This completes the proof of our main theorem.

In the next section we use the computer algebra system MAPLE [9] to obtain explicit asymptotic expansions for some particular binomial sums that appear in the literature.

4. Some Special Binomial Sums

One of the most common binomial sums is

$$S_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r.$$

We have the well-known identities $S_1(n) = 2^n$ and $S_2(n) = \binom{2n}{n}$. One might well expect similar formulae for $S_r(n)$ for larger values of r, but no such formulae are known. For positive integers r, Cusick [3] gives an elementary method for finding a polynomial recurrence for $S_r(n)$ with $[(r+3)/2]$ terms. When r is a positive real number we get $\lambda = 1/2$, $\mu = 2^r$, $v = 2r$ and

$$S_r(n) = \frac{2^{rn}}{\sqrt{r(\pi n/2)^{r-1}}}\left(1 - \frac{(r-1)^2}{4rn} + \frac{(r-1)(r+1)(3r^2-12r+13)}{96r^2n^2} + O(n^{-3})\right).$$

For $n \geqslant 0$ define

$$S_{r,s}(n) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s.$$

When r and s are positive real numbers, λ is given by

$$\lambda^{r+s} = (1-\lambda)^r (1+\lambda)^s, \qquad 0 < \lambda < 1,$$

$$\mu = \frac{(1+\lambda)^s}{(1-\lambda)^r},$$

$$\nu = \frac{r}{1-\lambda} + \frac{s}{1+\lambda}$$

and

$$S_{r,s}(n) = \frac{\mu^{n+1/2}}{\sqrt{\nu(2\pi\lambda n)^{r+s-1}}} \left(1 + \frac{\alpha}{n} + O(n^{-2})\right),$$

where

$$\begin{aligned}
\alpha = \{ & (10r^4 - 16r^3 s - 12r^3 + 24r^2 s + 2r^2 \\
& + 8rs^3 - 12rs^2 - 4rs - 2s^4 + 2s^2)\,\lambda^6 \\
& + (30r^4 - 36r^3 - 36r^2 s^2 + 6r^2 + 36rs^2 + 6s^4 - 6s^2)\,\lambda^5 \\
& + (21r^4 + 60r^3 s - 24r^3 + 6r^2 s^2 - 72r^2 s + 3r^2 \\
& \quad - 36rs^3 + 30rs - 3s^4 + 3s^2)\,\lambda^4 \\
& - (16r^4 - 24r^3 s - 24r^3 - 72r^2 s^2 + 24r^2 s + 8r^2 \\
& \quad - 24rs^3 + 48rs^2 + 8s^4 - 8s^2)\,\lambda^3 \\
& - (24r^4 + 48r^3 s - 36r^3 + 12r^2 s^2 - 48r^2 s + 12r^2 - 24rs^3 - 12rs^2 \\
& \quad + 24rs - 12s^4 + 12s^2)\,\lambda^2 \\
& - (6r^4 + 24r^3 s - 12r^3 + 36r^2 s^2 - 24r^2 s + 6r^2 + 24rs^3 \\
& \quad - 12rs^2 + 6s^4 - 6s^2)\,\lambda \\
& + r^4 + 4r^3 s + 6r^2 s^2 - r^2 + 4rs^3 - 2rs + s^4 - s^2 \}/ \\
& \{24\lambda(1-\lambda)(1+\lambda)(r\lambda - s\lambda + r + s)^3\}.
\end{aligned}$$

When $r=s$ or $r=2s$ we find that λ is a quadratic irrational. In these cases we obtain

$$S_{r,r}(n)=\frac{(1+\sqrt{2})^{2nr+r}}{\sqrt{4r(\pi n\sqrt{2})^{2r-1}}}\times\left(1-\frac{48r^2-24r-(4r^2+24r-19)\sqrt{2}}{96rn}+O(n^{-2})\right) \tag{4.1}$$

and

$$S_{2r,r}(n)=\frac{((1+\sqrt{5})/2)^{5rn+4r}}{\sqrt{(5+2\sqrt{5})\,r(2\pi n)^{3r-1}}}\times\left(1-\frac{25r^2+96r-61+(26r^2-72r+34)\sqrt{5}}{120rn}+O(n^{-2})\right).$$

The Apéry numbers a_n are defined by $a_n=S_{2,2}(n)$. Setting $r=2$ in (4.1) we obtain (1.1). The sum $S_{2,1}(n)$ is associated with $\zeta(2)$ in the same way that $S_{2,2}(n)$ is associated with $\zeta(3)$ (see [12]). Let

$$p_n(x)=\frac{1}{n!}\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$$

be the "shifted" Legendre polynomials ([2], p. 366) and note that

$$p_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}x^k.$$

Then $p_n(x)$ is related to $S_{1,1}(n)$ by $p_n(-1)=S_{1,1}(n)$.

When all of the exponents r_i in $S(n)$ equal 1,

$$\mu=\lim_{n\to\infty}\frac{S(n+1)}{S(n)}$$

enjoys some interesting properties. We end with a short discussion of this case.

For $n\geqslant 0$ define

$$B_m(n)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{n+2k}{k}\cdots\binom{n+mk}{k}.$$

By our main theorem we see that $\mu=(1+m\lambda)/(1-\lambda)$, where $0<\lambda<1$ satifies $\lambda^{m+1}=(1-\lambda)(1+m\lambda)^m$. It follows that $\mu>1$ is a zero of the polynomial

$$p(z)=(z-1)^{m+1}-(m+1)^{m+1}z^m.$$

The following binomial series for λ and μ can be deduced from Gould's list of combinatorial identities [7].

$$\lambda=\sum_{k=0}^{\infty}\frac{(-1)^k\binom{kM}{k}}{M^{kM}}, \qquad M\geqslant 2$$

and

$$\mu=M^M+1+(M-1)\sum_{k=0}^{\infty}\frac{(-1)^k\binom{kM}{k}}{(k+1)M^{kM}}, \qquad M\geqslant 1,$$

where $M=m+1$. By Rouché's Theorem we see that $p(z)$ has one real zero $\mu>1$ and has $M-1$ complex zeros in the disk $|z|<1/m$. Therefore $p(z)$ is irreducible and μ is an algebraic number of degree $m+1$. Thus the minimum number of terms in a polynomial recurrence for $B_m(n)$ is $m+2$. In [10] the author proved that for each $m\geqslant 0$, $B_m(n)$ satisfies a polynomial recurence with $m+2$ terms.

In 1945, Fasenmyer [4, 5, 6] developed a general method for obtaining pure recurrence relations for hypergeometric polynomials. This method was overlooked by workers in combinatorics who developed various ad hoc methods for evaluating, either explicitly or inductively, sums involving products of binomial coefficients. The method of "creative telescoping" is often used to prove recurrences, but does not help to find them. Zeilberger [15] showed how Sister Celine's technique could be used to obtain recurrences for hypergeometric sums. Although the general message of [15] is valid it's author has discovered that many details, including some complete proofs and algorithms are incorrect. The errors are corrected in [14]. Fortunately, the method given in [15] works for the binomial sum $B_m(n)$ and with the aid of computer algebra explicit recurrences for $B_m(n)$ with $0\leqslant m\leqslant 6$ have been obtained [10].

5. Concluding Remarks

Asymptotic expansions for a wide variety of unimodal sums can be obtained by our method provided that the terms in sum satisfy a uniform limit similar to (2.6).

In [11] we use this method to obtain asymptotic expansions for q-hypergeometric series, including those occurring in the Rogers-Ramanujan identities. In particular, when $q=e^{-t}$ and $t \to 0^+$,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+\mu)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$
$$= \sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{\sqrt{5}-1}{2}\right)^{\mu} \exp\left\{\frac{\pi^2}{15t} + \left(\frac{15\mu^2-3\mu-1}{60} - \frac{\mu(\mu-1)}{20}\sqrt{5}\right)t\right.$$
$$- \mu(\mu-1)\left(\frac{1}{50}+\frac{2\mu-1}{300}\sqrt{5}\right)t^2$$
$$- \mu(\mu-1)\left(\frac{2\mu-1}{500}+\frac{\mu^2-\mu+6}{3000}\sqrt{5}\right)t^3$$
$$\left.- \mu(\mu-1)\left(\frac{\mu^2-\mu+26}{15000}-\frac{(2\mu-1)(3\mu^2-3\mu-31)}{90000}\sqrt{5}\right)t^4 + O(t^5)\right\},$$

where μ is any real number. Comparing asymptotic expansions associated with different q-series we discovered the new identity

$$\sum_{n=0}^{\infty} \frac{q^{(2n+\mu)(2n+\mu+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}$$
$$= \prod_{m=1}^{\infty}(1+q^m) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2-\mu n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})},$$

where μ is any integer.

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