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Current trends in asymptotics: some problems and some solutions [1]

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Abstract

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I discuss five topics of current interest in asymptotic analysis: the use of probabilistic methods to estimate the growth of combinatorial sequences, asymptotic methods in the theory of random walks, the asymptotic estimation of solutions of difference equations, asymptotic expansions in generalized scales, and the computation by asymptotic methods of distributions whose moments are known. I give a number of examples and mention some outstanding unsolved problems.

Keywords: Asymptotic analysis, central limit theorem, unimodality, random walks, convolution sequences, generating functions, hypergeometric functions, difference equations, Freud problem, asymptotic scales, orthogonal polynomials, moment problem, Bessel polynomials.

1. Introduction

Few disciplines are as important to the development of applied mathematics and engineering as are asymptotic methods. In the background of every hard fact that we know about physical reality is usually lurking some kind of asymptotic argument. This is dramatically true in quantum mechanics, where the technique of cancelling singularities, so well known to physicists, often involves an appeal to established methods such as the Euler–Maclaurin summation method. [2]

Three of the most popular general books on the subject of asymptotics are those of Erdélyi [24], Olver [75] and de Bruijn [16]. They have recently been joined by another one, that of Wong [101] which promises to become very popular indeed. In these and other books the presentation of the material usually proceeds along well-established lines: definitions; elementary methods, including integration by parts and the Euler–Maclaurin summation method; classical methods

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[2] This process is called renormalization. For a delightful discussion of how the Casimir effect can be derived using the Euler–Maclaurin summation formula, see [19]. Asymptotics is not the only approach to the problem. Renormalization can also be accomplished by using generalized functions, i.e., distribution theory, see [61].

applicable to integrals, including stationary phase, Laplace's method, Watson's lemma; methods applicable to sequences defined by generating functions, such as Darboux's method; contour integrals; differential equations containing a parameter; integrals with multiple stationary points. Each of the books above makes its unique contribution. Wong's book treats such topics as Mellin transform techniques, distributional methods, and the (difficult) problem of estimating higher-dimensional integrals, material which has not appeared in any previous book. De Bruijn's book contains an intriguing chapter on indirect asymptotics and the problem of estimating a sequence of iterates. Olver's book has the most extensive discussion of differential equations and turning point problems.

In this paper I wish to discuss some topics that have not yet been covered in general works on asymptotics. For instance, the central limit theorem is one of the keystones of probability theory, yet no readily available source on asymptotics has discussed its applications and ramifications. Certain problems in combinatorics, in particular the theory of random walks, can be elucidated by asymptotic methods; yet these have received no unified treatment. Most books have an extensive treatment of differential equations, but no book discusses difference equations, which are just as important, and for which a large body of methods have been developed. Even though other scales are briefly mentioned in some of the available books, seldom are any examples given to show how these scales arise, or to illustrate their usefulness in simplifying asymptotic analysis. Most often the scales involved in asymptotic expansions are Poincaré scales. Indeed, some authors would feel uncomfortable calling any other kind of expansion an asymptotic expansion. Finally, because of my private interest in the subject, I discuss the problem of using asymptotic methods to find a function on [0, 1] when its moments are known.

The discussion of the above topics is in no sense complete nor can it be said to have much depth. But I think I have provided enough references to allow the reader to satisfy the curiosity that I hope this paper has evoked.

2. Probabilistic methods in combinatorics

Probabilistic methods for obtaining asymptotic information have generated intense mathematical activity recently. The approach is useful not only in problems in combinatorics, but also in analysis, for instance in finding asymptotic formulas for polynomials orthogonal on the real line. Van Assche's book [91] contains many beautiful examples of this. The approach, generally, is to establish a sequence of probability distributions and then to invoke the appropriate limit theorems applicable to sums of independent random variables, for instance, the results in Petrov's book [77]. The most commonly used tool is some variant of the *local limit theorem* for lattice distributions [77, p.189]. One can recover this way, for instance, the root asymptotics of the Jacobi polynomials, i.e.,

$$P_n^{(\alpha,\beta)}(x) \sim \frac{1}{\sqrt{2\pi n}}(x-1)^{-\alpha/2}(x+1)^{-\beta/2}\left(\sqrt{x+1}+\sqrt{x-1}\right)^{\alpha+\beta}$$
$$\times(x^2-1)^{-1/4}\left(x+\sqrt{x^2-1}\right)^{n+1/2}, \quad n\to\infty,\ x>1,$$

when α, β are nonnegative integers. The technique in this case is to establish two independent sequences X_i, Y_i of independent Bernoulli random variables, to calculate the variance and expectation of certain sums of these variables (which turn out to be the Jacobi polynomials in question), and then apply the local limit theorem. The technique may be modified to obtain the complete asymptotic expansion of the polynomials.

Here I shall concentrate mostly on the use of probabilistic methods to describe asymptotically the "shape" of important combinatorial sequences. Recently, a large amount of work has been done on establishing a probabilistic framework for combinatorial asymptotics. Bender's paper [5] contains an excellent introduction to the main ideas. Recall the central limit theorem [1, p.173].

Theorem 2.1. *Let $f_1, f_2, \ldots$ be a sequence of bounded random variables on S that are independent and identically distributed with E their common expectation. Let $\sigma = V(f_1) = V(f_2) = \cdots$, and $S_n(x) := f_1(x) + \cdots + f_n(x) - nE$.*

Then for every pair of numbers a and b with $a < b$,

$$\mu\left\{x \in X\colon a < \frac{S_n(x)}{n^{1/2}} < b\right\} \to \frac{1}{\sqrt{2\pi\sigma}} \int_a^b e^{-t^2/2\sigma}\, dt.$$

The above theorem has a clear analogue when the quantities involved are of combinatorial rather than probabilistic interest. More precisely, let $a_n(k)$ be a doubly indexed sequence of numbers, usually with some combinatorial significance. One often wants to get a sense of the "distribution" of $a_n(k)$ as a function of k of the $a_n(k)$ for n large, or perhaps even an asymptotic formula for $a_n(k)$ as $n \to \infty$ for $k = k(n)$ in as large a set as possible, the size of which is a measure of the uniformity of the approximation.

The probabilistic approach is to set

$$p_n(k) := \frac{a_n(k)}{\Sigma_j a_n(j)},$$

and to interpret

$$F_n(x) := \sum_{k \leqslant \mu_n + \sigma_n x} p_n(k)$$

as a distribution function. Bender gave conditions on the generating function of the $a_n(k)$'s which are sufficient to guarantee that F_n converges to a standard normal distribution, i.e.,

$$\lim_{n\to\infty} \sup_x \left| F_n(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = 0.$$

If this is the case, one says that $a_n(k)$ is asymptotically normal (with mean μ_n and variance σ_n^2) or alternatively that $a_n(k)$ satisfies a central limit theorem. (Additional conditions lead to a local limit theorem, and hence an asymptotic formula for $a_n(k)$.)

As examples, Harper [40] showed in 1967 that the Stirling numbers of the second kind $S(n, k)$ are asymptotically normal, and the main theorem of Bender [5] gives for these numbers

$$\mu_n = \frac{n}{2 \ln 2} = 0.72135\, n, \qquad \sigma_n = \frac{\sqrt{n}\sqrt{1 - \ln 2}}{2 \ln 2} = 0.39959 \sqrt{n}\,.$$

Similarly, the Eulerian numbers $A_n(k)$, which are the number of permutations of $1, 2, \ldots, n$ having k rises, were established by David and Barton in 1962 [15] to be asymptotically normal, and Bender's theorem gives

$$\mu_n = \tfrac{1}{2}n, \qquad \sigma_n = \frac{\sqrt{n}}{2\sqrt{3}}.$$

This central limit theorem approach has been generalized by Bender and coauthors — as well as others — and applied to many combinatorial problems, see, for example, [13]. Bender's metatheorem is that convergence in distribution plus smoothness of $a_n(k)$ are sufficient to obtain asymptotic information. But this does not seem to depend in any essential way on the fact that the limiting distribution is Gaussian. In fact, Schmutz has found that a routine modification of Bender's argument yields the following theorem.

Theorem 2.2. *Let ρ be a uniformly continuous probability density with distribution F. Assume that $[p_n(k)]_k$ is log-concave and that*

$$\lim_{n\to\infty} \sup_x \left| \sum_{k \leqslant \mu_n + \sigma_n x} p_n(k) - F(x) \right| = 0,$$

for some σ_n and μ_n with $\sigma_n \to \infty$. Then

$$\lim_{n\to\infty} \sup_x | \sigma_n p_n(\langle \mu_n + \sigma_n x \rangle) - \rho(x) | = 0.$$

In the above $\langle \cdot \rangle$ denotes the greatest integer function. This result is certainly not the best possible. For example, log-concavity is just one way of establishing the "smoothness" of $a_n(k)$. There are other ways of doing this, see [7].

It is of interest to mention some problems where probabilistic methods have not yet been successful, but for which there is a great likelihood of success if the appropriate tools can be developed, i.e., theorems which require more easily verifiable hypotheses on the sequence $a_n(k)$. For instance, determining log-concavity is often not trivial. Let $b_n(k)$ denote the number of binary trees of height k with n internal nodes. Flajolet and Odlyzko [29] showed that the normalized height

$$h(T) := \frac{\text{height}(T)}{c\sqrt{n}}$$

has asymptotically a "theta" distribution with density

$$\rho(x) = 4x \sum_{k=1}^{\infty} k^2(2k^2x^2 - 3)\, \mathrm{e}^{-k^2x^2}.$$

The results of these authors extend easily to other families of trees. However in no case do they seem to help in finding an asymptotic formula for $b_n(k)$. See however [30] where the authors obtain an asymptotic formula for

$$B_n(k) := \sum_{j=1}^{k} b_n(j).$$

To use the methods described above it would suffice to demonstrate the log-concavity of $[b_n(k)]_k$. There is some numerical evidence to support this hypothesis. Even if it is wrong, there would still be some hope of obtaining an asymptotic formula since it is intuitively evident that

$b_n(k)$ varies smoothly with k. Incidentally, it would be of interest combinatorially to prove the weaker result that $[b_n(k)]_k$ is unimodal, and to locate the maximum. Unimodality of sequences has received a lot of attention lately. The sequence $\{a_n(k)\}_k$ is said to be *unimodal* if there is an integer r, $1 \leqslant r \leqslant n$, such that $a_n(1) \leqslant a_n(2) \leqslant \cdots \leqslant a_n(r) \geqslant a_n(r+1) \geqslant a_n(r+2) \geqslant \cdots \geqslant a_n(n)$. O'Hara recently proved by combinatorial methods a long-standing conjecture about the unimodality of the Gaussian integers [74].

Another problem is Kleitman and Winston's conjecture concerning the q-Catalan numbers. The q-Catalan number $c_n(q)$ is defined recursively by $c_1(q) = 1$ and

$$c_n(q) = \sum_{i=1}^{n-1} c_i(q) c_{n-i}(q) q^{i(n-i-1)}, \quad n > 1.$$

The numbers are a generalization of the well-known Catalan numbers (take $q \to 1$). For a good, simple discussion of these numbers and their history, see [22]. Kleitman and Winston posed the following [53].

Conjecture 2.3. *The q-Catalan numbers are*: (i) *unimodal polynomials, with* (ii) *coefficients that are no larger than* $c4^n/n^3$.

The O'Hara technique as improved by Zeilberger [103] may be relevant to the proof of (i), though it is probably not enough. Part (ii) is actually more important because it would yield a new upper bound for the number of tournament score sequences [37].

A number of other asymptotic results have a probabilistic flavor. As an example, I mention the result of Erdős and Turán on the order of members of the permutation group. Let σ be a permutation on n letters, $N_n(\sigma)$ the order of σ as a group element. Numerical evidence suggests that the "average" permutation N_n is roughly $n^{\ln n/2}$. The result referred to is stated in the next theorem.

Theorem 2.4.

$$\lim_{n\to\infty} \frac{\#\left\{\sigma \in S_n : \ln(N_n(\sigma)) < \frac{1}{2}\ln^2 n + \frac{x}{\sqrt{3}} \ln^{3/2} n\right\}}{n!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, \mathrm{d}t.$$

Other mathematicians have devised their own proofs of this theorem and have studied closely related problems, see [11] for a survey.

Another measure of the "shape" of $a_n(k)$ is the asymptotic behavior of the *average* of the sequence

$$A_n := \frac{\sum_{j=1}^{n} a_n(k)}{n}.$$

An analysis of A_n does not give the information that an asymptotic formula for $a_n(k)$ does, but there are very interesting problems in this area. Related to the result above is a conjecture concerning

$$\mu_n := \frac{1}{n!} \sum_{\sigma \in S_n} N_n(\sigma),$$

i.e., the average order of the elements in S_n. Erdős' conjecture about the size of μ_n was subsequently verified by Schmutz [86] and latter sharpened by Goh and Schmutz [34]. We have the following striking result.

Theorem 2.5.

$$\ln \mu_n \sim c\sqrt{\frac{n}{\ln n}}, \quad \textit{where } c = 2\left(2\int_0^\infty \ln \ln\left(\frac{e}{1-e^{-t}}\right) dt\right)^{1/2}.$$

These same authors have produced a number of other beautiful results all of which confirm the mystical fact that the average behavior of combinatorial numbers is intimately connected with the properties of the Gaussian distribution [35,36].

Theorem 2.6. *For $T \in GL_n(F_q)$ let $\Omega_n(T)$ be the number of irreducible factors of the characteristic polynomial of T. Then*

$$\lim_{n\to\infty} \frac{\#\{T\colon \Omega_n(T) < \ln n + x\sqrt{\ln n}\}}{\#GL_n(F_q)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Theorem 2.7. *Denote by $D_n(\lambda)$ the number of different part sizes of λ for λ one of the $P(n)$ partitions of n. (Each part size is to be counted only once, even though there may be many parts of a given size.) Then for any fixed x,*

$$\lim_{n\to\infty} \frac{\#\{\lambda\colon D_n(\lambda) < A_n + xB_n\}}{P(n)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt,$$

$$A_n = \frac{\sqrt{6}}{\pi} n^{1/2}, \qquad B_n = \left(\frac{\sqrt{6}}{2\pi} - \frac{\sqrt{54}}{\pi^3}\right)^{1/2} n^{1/4}.$$

The reader interested in a survey of results in this area up to 1974 should consult [6].

3. The drunkard's walk and convolution sequences

The idea of a random walk is an old one, going back at least to Pearson [76]. The classical *drunkard's walk* is a random walk in which the movements are restricted to vertical and horizontal movements and the walker must occupy the lattice points of the plane. In other words, the drunkard starts out and follows a cartography of city blocks. Other walks are, of course, possible, including continuous walks. These walks form the basis for various theories of diffusion and Brownian motion in the physical sciences. For some references, see [3,20,87,88].

There are a number of questions that may be asked about walks. What is the probability that the drunkard in the drunkard's walk returns to the origin (home or bar)? It is known that the probability is 1 in one and two dimensions. The concept has clear military applications, for instance, it is fatal (with probability 1) to stay at any fixed point in mined waters, but not necessarily so in mined space — a consequence of the Pólya effect. Pólya [78] addressed the problem of the drunkard's return to origin in an n-dimensional lattice. He showed that the probability of return in higher dimensions is *less* than 1. In three dimensions, in fact, it is 0.34053... . This phenomenon, which I will call the Pólya effect, has important implications, which result from the fact that when a walk is somehow constrained to pass from a three-dimensional lattice to a plane or line lattice, the probability of encounter with a fixed lattice point changes from something less than 1 to 1.

In almost all problems in the physical sciences, the walk is constrained (or *restricted*). Kac examined the properties of walks with absorbing barriers [51]. Sometimes the walk has restricted reversals [18] or the origin is excluded [82], or the size of the walk (there are several ways of measuring this) is limited [81], or the walk is self-avoiding [39,52,93]. In living organisms, the underlying explanation for many vital biochemical reactions is that a walk representing, say, the path of some enzyme system suddenly becomes constrained in a particular fashion. In membrane-associated reactions in cells [70,90], it has been found that the turnover number of certain membrane bound enzyme systems is enhanced if their substrates undergo two-dimensional diffusion along membrane surfaces. Eigen [23] formulated the principle that the reduction of dimensionality in a walk was nature's trick for overcoming the barrier of diffusion control and making multi-molecular reaction processes at low concentrations more efficient. (The reduction of dimensionality raises the probability of encounter in a given time with a fixed point.) A similar situation prevails in many other situations, for instance, chemical reactions in clays [56].

A walk model for the formation of cosmic strings in the early universe has recently been given [83]. It was found that the string network has the statistical properties of a set of Brownian walks, rather than self-avoiding walks, even in models in which string intersections in the initial configuration are impossible. Thus in certain physical systems there seems to be an innate preference for one (or no) constraint(s) versus another.

I consider first some known facts about a walk in the three-dimensional cubic lattice. Let a_n be the number of ways of going from the origin back to the origin in $2n$ steps. The expected number of returns to the origin is

$$m=\sum_{n=0}^{\infty}\frac{a_n}{6^{2n}},$$

the probability of returning to the origin being equal to $u=(m-1)/m$, and less than 1, in fact, $u=0.340\,537\,329\,544$, for instance, see [20]. The generating function for the a_n is

$$J_0(-2\mathrm{i}t)^3=\sum_{n=0}^{\infty}\frac{a_n t^{2n}}{(2n)!}.$$

Finally, I have

$$a_n \sim \frac{6^{2n}}{4}\left(\frac{3}{\pi}\right)^{3/2} n^{-3/2}, \quad n\to\infty.$$

Is is known [98] that a_n satisfies the following second-order homogeneous difference equation:

$$36(n+1)(2n+3)(2n+1)a_n-2(2n+3)(10n+30n+23)a_{n+1}+(n+2)^3 a_{n+2}=0.$$

In fact, the asymptotics for a_n (except for the lead constant) can be determined by applying known results from the asymptotic theory of difference equations [8,9] to this equation.

One of the oldest and most important constrained walks is the so-called *ballot problem* of Bertrand in which it is required to find the number of ways of walking from the origin to a typical point $(x_1, x_2, \ldots, x_n)$ such that the walk remains in the region $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. André [2] studied some aspects of the two-dimensional problem using reflection, and Zeilberger [102] extended the study to n dimensions. In a recent and pioneering study, Gessel and Zeilberger [33] extended these results to random walks over subsets of discrete lattices in $\mathbb{Z}^n$ which are symmetric with respect to the action of a Weyl group corresponding to a given root system.

The constrained three-dimensional Euclidean walk $x \geqslant y \geqslant z$ has importance in melting and wetting problems arising in physical chemistry, see [28,46,47]. A great deal of information is now available about the properties of this walk. Let b_n be the number of ways of returning to the origin in three-space after $2n$ steps and *always staying within* $x \geqslant y \geqslant z$, in other words, I assume the walls $x-y=-1$ and $y-z=-1$ are absorbing, and the drunkard dies if he bumps into a wall. The sequence b_n has the generating function

$$\left| J_{i-j}(-2\mathrm{i}t) \right|_{i,j=1,2,3} = \sum_{n=0}^{\infty} \frac{b_n t^{2n}}{(2n)!},$$

the asymptotic formula,

$$b_n \sim \frac{6^{2n}\sqrt{3}\,81}{16\pi^{3/2}} n^{-9/2},$$

and satisfies the *third-order* difference equation

$$\begin{aligned}&-72(n+2)(n+1)(2n+9)(2n+5)(2n+3)(2n+1)b_n + 4(n+2)(2n+5)(2n+3)\\&\times(38n^3+381n^2+1252n+1377)b_{n+1} - 2(n+3)(2n+5)(22n^2+145n+229)\\&\times(n+4)^2 b_{n+2} + (2n+7)(n+3)(n+5)^2(n+4)^2 b_{n+3} = 0.\end{aligned}$$

Let u^* be the probability that the drunkard will return to origin. A computation gives $u^* = 0.064\,844\,71\ldots$. (For all these statements, see [100].)

The rough asymptotics of b_n can be recovered from this recurrence. However, the lead constants cannot be determined (the connecting problem, see Section 4.) There is a way to determine the lead constants, namely, by an asymptotic analysis of certain *convolution sequences.*

Let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers. Define the convolution of $\{a_n\}$ and $\{b_n\}$ to be the sequence $\{c_n\}$ defined by

$$\{a_n * b_n\}_n := c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, 2, \ldots .$$

If the a_n are the coefficients of a formal power series $f(t)$ and b_n the coefficients of another formal power series $g(t)$, then, obviously, the quantities c_n are the coefficients of the formal series $f(t) \cdot g(t)$. Convolution sequences occur regularly in the construction of generating functions for quantities associated with walks. The sequence a_n in the previous section is a three-fold convolution of the Taylor series coefficients for $J_0(-2\mathrm{i}t)$, and the sequence b_n is a linear combination of three-fold convolutions of several Bessel functions of integer order.

An important problem is, if two sequences a_n and b_n have asymptotic representations, say, as generalized Birkhoff–Poincaré asymptotic series, does $\{a_n * b_n\}_n$ have a similar representation? Even if so, the convolution may not be represented by a single such series even though the original sequences are represented by series of the same type, see [98].

Many of the convolution sequences encountered in walks have the special form of the following theorem.

Theorem 3.1. *Let* λ, $\sigma > 0$, *and let the sequences* a_n, b_n *have the behavior*

$$a_n \sim \frac{(n+1)^{-\alpha}\lambda^n}{n!^2}\left\{1+\frac{c_1}{(n+1)}+\frac{c_2}{(n+1)^2}+\cdots\right\}, \quad n\to\infty,$$

$$b_n \sim \frac{(n+1)^{-\beta}\sigma^n}{n!^2}\left\{1+\frac{d_1}{(n+1)}+\frac{d_2}{(n+1)^2}+\cdots\right\}, \quad n\to\infty.$$

Then

$$\{a_n * b_n\}_n \sim \frac{(n+1)^{-\alpha-\beta-1/2}}{2\sqrt{\pi}\, n!^2}\sigma^{-\beta/2-1/4}\lambda^{-\alpha/2-1/4}\left(\sqrt{\sigma}+\sqrt{\lambda}\right)^{\alpha+\beta+1+2n}$$
$$\times\left\{1+\frac{e_1}{(n+1)}+\frac{e_2}{(n+1)^2}+\cdots\right\}.$$

Using this theorem one can find the asymptotics for the $(2n+1)$th Taylor series coefficient for $t^{-\nu r}J_\nu^r(t)$ (a special case ($\nu=0$) of which is related to the number of ways of going back to the origin in $2n$ steps in *a walk in r dimensions*, $r \geqslant 2$). Call this quantity A_n^r. I have the following theorem.

Theorem 3.2. (i) A_n^r *satisfies a recurrence relation of order* $\langle \frac{1}{2}r+1)\rangle$;

(ii) $$A_n^r \sim M_r(\nu)\frac{\left(-\frac{1}{4}r^2\right)^n}{n!^2}n^{1/2-(\nu+1/2)r}\left\{1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right\}, \quad n\to\infty,$$

where

$$M_r(\nu)=\frac{\left(\frac{1}{2}r\right)^{r(\nu+1/2)}}{2^{(r-2)/2}\pi^{(r-1)/2}}, \quad r=0,1,2,\ldots .$$

Determinants of the form

$$D_{\nu,r}=|a_{\nu+i-j}J_{\nu+i-j}(t)|, \quad i, j=1,2,3,\ldots,r,$$

occur frequently. Provided none of the determinants

$$\left|\frac{a_{\nu+i-j}}{\Gamma(\nu+1+i-j)}\right|, \quad r=1,2,3,\ldots, \tag{3.1}$$

is zero, one may write

$$D_{\nu,r}=t^{\nu r}\sum_{n=0}^{\infty}A_n^{\nu,r}t^{2n}, \quad A_0^{\nu,r}\neq 0.$$

Asymptotic estimates are possible for $A_n^{\nu,r}$, $n\to\infty$, under the assumption that none of certain auxiliary determinants vanishes, see [98].

Define the determinants $d_{\nu,r}$, $r = 1, 2, 3, \ldots$, by $d_{\nu,0} = 4^{-\nu}$,

$$d_{\nu,1} = \frac{\left(a_\nu^2 - a_{\nu-1}a_{\nu+1}\right)}{\sqrt{\pi}},$$

$$d_{\nu,r} = \frac{\left[d_{\nu,r-1}^2 - d_{\nu-1,r-1}d_{\nu+1,r-1}\right]}{d_{\nu,r-2}}, \quad r = 2, 3, 4, \ldots . \tag{3.2}$$

Theorem 3.3. *Let none of the determinants* (3.2) *vanish. Then*

$$A_n^{\nu,r} = \frac{d_{\nu,r} n^{-\nu r-(r-1)/2}}{n!}\left(-\tfrac{1}{4}r^2\right)^n\left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right), \quad n \to \infty.$$

In some important cases, for instance, b_n above, the determinants (3.2) *do* vanish. The analysis then becomes very difficult, for leading terms in the individual convolution sequences comprising the sequence cancel out. The case of the constrained walk above is such a case, with $r = 3$. Here $a_\nu = 1$. It is an interesting sidelight that one can obtain for this walk the explicit formula

$$\begin{aligned} b_n &= \frac{(2n)!}{n!(n+1)!}\,{}_3F_2\left(\begin{matrix}-n, -n-1, \tfrac{1}{2} \\ 2, 2\end{matrix}; 4\right) \\ &= (2n)! \sum_{k=0}^{n} \frac{(2k)!}{(n-k)!(n+1-k)!k!^2(k+1)!^2}. \end{aligned}$$

(For the notation of hypergeometric functions used above, see [25, vol. 1].) In this and many other cases, techniques for asymptotically estimating convolution sequences would enable one to estimate sums involving binomial coefficients, a significant class of problems in discrete mathematics.

4. Difference equations

It seems to be a general rule that the coefficients in the generating functions relating to walk problems satisfy linear recurrence relations, see the examples in the previous section. In fact, most combinatorial quantities satisfy recurrence relations, for some examples, see [6]. A number of combinatorial quantities satisfy nonlinear recurrence relations. For instance, let $f(n)$ be the number of distinct nth-order dendrites. (A dendrite is a tree in which each branch bifurcates into at most two other branches. The order n denotes the *level* of the bifurcations present.) It can be shown that f satisfies

$$f(n+2) = f(n+1)\left[\frac{f(n+1)}{f(n)} + \tfrac{1}{2}\{f(n+1) + f(n)\}\right], \quad n = 1, 2, 3, \ldots, \tag{4.1}$$

with $f(1) = f(2) = 1$, see [71, vol. 2, p.183]. Similar, but much more complicated nonlinear difference equations, are satisfied by trees in which each bifurcation is into three, four, etc., branches.

Another rich source of recurrence equations, many of them nonlinear, is the theory of orthogonal polynomials. For instance, denote by $p_n(x)$ the orthonormal polynomials orthogonal with respect to the weight function $w(x) = \exp(-x^4)$ on the real line. The polynomials satisfy the special recurrence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$

see [72], and the coefficients a_n satisfy the second-order nonlinear recurrence

$$n = 4a_n^2\left[a_{n+1}^2 + a_n^2 + a_{n-1}^2\right]. \tag{4.2}$$

The problem is to determine the asymptotic behavior of a_n from this equation.

One problem, which will not concern me here, is the actual construction of a recurrence for a given sequence. Sometimes the construction can be accomplished by finding a differential equation for the function generating the sequence. Even when such a differential equation can be found, carrying out the details may be very difficult. Zeilberger has described efficient algorithms for obtaining recurrence relations satisfied by combinatorial quantities [104–106]. Quite recently, Wilf and Zeilberger [95] presented a general method for proving and discovering combinatorial identities. The method proves, in a unified way, many known hypergeometric sum identities. It does this by means of certificates of proof, each of which consists of a pair of functions (F, G) that satisfy certain conditions whose verification is a simple mechanical task, and which involves the effect of difference operators on the two functions.

The next problem is the asymptotic analysis of the recurrence. An asymptotic analysis of the solutions of the recurrence is much easier when the recurrence is linear, especially when the coefficients are in the form of generalized Poincaré-type asymptotic series. (This approach constitutes the Birkhoff–Trjitzinsky theory, discussed in [97].) Recently, in [99], it was shown how the theory could be applied to several combinatorial problems, including finding the asymptotics of $u_r(n)$ = the number of permutations whose cycle structure consists of $1, 2, \ldots, r$ cycles but no cycle of length larger than r.

I now describe the form of the series yielded by the theory. Let $S(\rho, n)$ denote the series

$$\begin{aligned}
S(\rho, n) &:= e^{Q(\rho,n)}s(\rho, n),\\
Q(\rho, n) &:= \mu_0 n \ln n + \sum_{j=1}^{\rho} \mu_j n^{(\rho+1-j)/\rho},\\
s(\rho, n) &:= n^{\theta} \sum_{j=0}^{t} (\ln n)^j n^{r_{t-j}/\rho} q_j(\rho, n),\\
q_j(\rho, n) &:= \sum_{s=0}^{\infty} b_{s,j} n^{-s/\rho},
\end{aligned} \tag{4.3}$$

where ρ, r_j, $\mu_0\rho$ are integers, $\rho \geqslant 1$, μ_j, θ, $b_{i,j}$ are complex, $b_{0,j} \neq 0$ unless $b_{s,j} = 0$ for all s, $r_0 = 0$. Now let $y(n)$ satisfy the linear homogeneous difference equation of order σ,

$$\sum_{\nu=0}^{\sigma} A_\nu(n) y(n+\nu) = 0, \quad A_0 = 1, \ A_\sigma \neq 0, \ \sigma \geqslant 2, \ n = 0, 1, 2, \ldots, \tag{4.4}$$

where the $A_\nu(n)$ can be represented as asymptotic series,

$$A_\nu(n) \sim n^{K_\nu/\omega}\left[a_{0,\nu} + a_{1,\nu}n^{-1/\omega} + a_{2,\nu}n^{-2/\omega} + \cdots\right], \quad n \to \infty,$$

K_ν an integer, ω an integer $\geqslant 1$, and $a_{0,\nu} \neq 0$ unless $A_\nu \equiv 0$.

The Birkhoff–Trjitzinksy theory asserts that there exist σ formally linearly independent formal series solutions of the difference equation of the form (4.4) where $\rho = N\omega$ for some integer $N \geqslant 1$, and each represents asymptotically some solution of the equation. [3]

An extensive body of theory, due primarily to Harris and Sibuya [42,43], is available for the asymptotic analysis of certain types of nonlinear recurrences. It is interesting that this material does not seem to have received much attention from either analysts or combinatorists. While it is true that the asymptotic forms that are encompassed by Harris and Sibuya's results are more limited than those possible with the linear theory of Birkhoff and Trjitzinsky — being restricted to representations as simple Poincaré type asymptotic series — the theory has many interesting and useful applications, since it is appropriate to both linear and nonlinear problems. I quote one of their most interesting results for analytic difference equations, i.e., equations whose coefficients are analytic functions of the variable in some region.

Consider the system of nonlinear difference equations

$$y(x+1) = f(x, y(x)), \tag{4.5}$$

where x is a complex variable, y is an n-dimensional vector, and f is an n-dimensional vector with components holomorphic in the region

$$|x| \geqslant R, \qquad \| y \| = \sum_{i=1}^{n} |y_i| \leqslant \delta_0.$$

I assume $f(\infty, 0) = 0$ and f admits a uniformly asymptotic expansion in the region

$$f(x, y) \sim \sum_{m=0}^{\infty} x^{-m} f_m(y),$$

the f_m being analytic in the region. (Actually, I am simplifying things a little bit. The conditions of Harris and Sibuya are less restrictive.) Then there exists a unique solution of (4.5), $p(x)$, such that $p(x)$ is analytic in the region and admits the asymptotic expansion

$$p(x) \sim \sum_{m=1}^{\infty} x^{-m} p_m, \quad x \to \infty.$$

Equation (4.2) may be considered a nonlinear equation (in a_n^2) or a nonlinear equation (in a_n). Either way, the Harris–Sibuya theory can be used to estimate the growth of solutions. Let $n \to x$ and make the changes of variable

$$a_n^2 \to x^{1/2}[y_1(x) + c],$$

$$a_{n+1}^2 \to (x+1)^{1/2}[y_2(x) + c],$$

[3] Some now believe that the Birkhoff–Trjitzinsky theory has disabling gaps, see [48]. The alleged deficiencies are difficult to discern by a casual inspection of the original papers [8,9] since they are extremely long and their arguments are very laborious. My policy is not to use the theory unless its results can be substantiated by other arguments. Determining asymptotics of the associated Wilson polynomials is a typical problem where the results obtained by another method substantiate the theory, see [49].

c to be chosen to insure the condition $f(\infty, 0) = 0$. I find

$$\begin{bmatrix} y_1(x+1) \\ y_2(x+1) \end{bmatrix} = \begin{bmatrix} y_2(x) \\ -\left(\frac{x+1}{x+2}\right)^{1/2}[y_2(x)+c] - \left(\frac{x}{x+1}\right)^{1/2}[y_1(x)+c] + \frac{1}{4}\left(\frac{x+1}{x+2}\right)^{1/2} \\ \times[y_2(x)+c]^{-1} - c \end{bmatrix},$$

and the system fulfills the required conditions if $c = \pm 1/2\sqrt{3}$. I am interested in positive solutions, so I take the positive quantity. Putting everything together shows that the original recurrence (4.2) has a unique solution, call it $\hat{a}_n^2$, with the complete asymptotic expansion

$$\hat{a}_n^2 \sim n^{1/2}\left[\frac{1}{2\sqrt{3}} + \sum_{m=1}^{\infty} \frac{d_m}{n^m}\right], \quad n \to \infty. \tag{4.6}$$

But there remains a problem, the notorious connecting problem for asymptotics: how to relate $\hat{a}_n^2$ to the desired solution a_n^2? I shall say more about this later.

Because of the intense current interest in a problem concerning orthogonal polynomials, the asymptotics of nonlinear recurrences has received a lot of attention quite recently. The problem appears in a paper of Freud [31]. The generalized Freud conjecture is a generalization of the situation of the recurrence just studied, (4.2). In one formulation of the problem, one considers the weight function $w(x) = \exp(-Q(x))$, where $Q(x)$ is a polynomial of even degree. The problem is to find an asymptotic formula for the coefficients in the recurrence relation of the polynomials orthonormal with respect to this weight. A number of mathematicians have worked on this problem and its extensions using a variety of methods, see [57,62,63,65,72,79,80] and the references given there. Rahmanov, for example, [79,80], used potential theory. It can be shown that the coefficients in the recurrence relation for the polynomials satisfy a nonlinear difference equation. The asymptotics of the coefficients can be determined if the asymptotics of the difference equation can be resolved. Several interesting results are contained in [4,66–68], the most general application (to systems) being in [4]. The situations studied in these papers cover some of the same ground as the results of Harris and Sibuya, except there is a little better handle on the *connecting problem*, a crucial and extremely difficult problem in constructing asymptotics from a recurrence relation. It may be known that the quantity to be analyzed is a solution of a certain recurrence, and that the recurrence has certain solutions which can be represented as asymptotic series, but it may be difficult, or even impossible, to relate the desired solution to one of the available asymptotic series. The problem has not been solved in any generality even for linear recurrence relations, although there are certain growth theorems (the Olver theorems) which sometimes allow one to discard certain of the asymptotic series and thus to determine the solution being represented up to a constant factor, see [97, p.282ff.]. Often outside information is required to do this. For some Freud problems, Lew and Quarles [57] were able to show that the recurrence had a unique positive solution, and the asymptotic formula yielded positivity. Freud himself gave the lead term of the expansion (4.6), so I conclude that the expansion indeed represents a_n^2. One of the advantages of the main theorem in [4] is that it is only required to know that the desired solution goes to zero in order to identify it uniquely with the asymptotic expansion.

The connecting problem was also encountered in determining the asymptotics for A_n^r in Section 3. Although I could explicitly construct the difference equation satisfied by A_n^r and display an asymptotic basis of solutions for the equation, the connecting problem could not be solved. Eventually, the asymptotics were obtained via the theorem on convolution sequences.

For some recurrences, such as the one for ordered dendrites (4.1), the available methods fail. The reason reflects the fact that for nonlinear recurrences the possible varieties of asymptotic forms are far too great for solutions to be characterized as simple Poincaré type series. The solutions of the recurrence (4.1) grow very rapidly, for example $f(3)=2$, $f(4)=7$, $f(5)=56$, $f(6)=2212$, $f(7)=2\,595\,782$, etc. Matula [69] has shown that for some constant A,

$$f(n) \sim \langle 2A^{2^n} - 1 \rangle, \quad n \to \infty, \tag{4.7}$$

where $\langle \cdot \rangle$ denotes the greatest integer function. No scale present in the Birkhoff asymptotic forms (4.3) grows this rapidly. One may question whether (4.7) is really a legitimate asymptotic formula, since computationally it is more difficult to use this formula than to compute $f(n)$ from the defining recurrence. In fact it seems the only way to compute A is experimentally,

$$A := \lim_{n\to\infty} (f(n))^{2^{-n}}.$$

Unfortunately, the problem of computability in regard to asymptotic expansions has received little attention.

5. Asymptotic scales

For many problems, the traditional Poincaré type expansion is too restrictive; sometimes the analysis can be substantially simplified if one accepts an enlarged definition of an asymptotic expansion. The more general expansion is based on what is called a general asymptotic scale, introduced sometime ago (1937) by Schmidt [84]. The idea is discussed thoroughly in [24]. See also [75].

Here I will be interested in asymptotic expansions where the variable z is complex and $z \to z_0$ in some sector S in the complex plane (z_0 may be ∞) or when the variable $z=n$ is a nonnegative integer, $n \to \infty$. Any continuous function of z will be assumed to be defined for

$$z \in S \cap \{z \mid 0 < |z - z_0| < \delta\} \quad \text{or} \quad z \in S \cap \left\{z \mid |z| > \frac{1}{\delta}\right\},$$

some $\delta > 0$. A discrete function of z will be defined for $z = 0, 1, 2, \ldots$.

Definition 5.1. (i) The sequence of functions $\{\phi_k\}$ is called an *asymptotic scale* if $\phi_{k+1} = \mathrm{o}(\phi_k)$, $k = 0, 1, 2, \ldots$.

(ii) The series

$$\sum_{k=0}^{\infty} f_k$$

is an *asymptotic expansion* of the function f with respect to the scale $\{\phi_k\}$ if

$$f-\sum_{k=0}^{K} f_k = \mathrm{o}(\phi_k), \quad K=0,1,2,\dots . \tag{5.1}$$

I then write

$$f \sim \sum_{k=0}^{\infty} f_k;\ \{\phi_k\}.$$

Often, where there is no chance of confusion, I delete the $\{\phi_k\}$ after the sum above.

Examples 5.2. (i) As $z\to\infty$, $|\arg z|\leqslant \pi-\delta$, $0<\delta<\pi$,

$$\int_0^\infty \frac{e^{-t}}{(z+t)}\,dt \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{z^{k+1}};\ \left\{\frac{1}{z^k}\right\}.$$

(ii) As $n\to\infty$,

$$\frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2n!} \sim 2\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2n}};\ \left\{\frac{1}{(k+1)^{2n}}\right\}, \tag{5.2}$$

where B_{2n} is the Bernoulli number. The series on the right is also convergent and (an important distinction) converges to the quantity on the left-hand side.

The scales $\{z^k\}$ for $z\to 0$ in S, $\{z^{-k}\}$ for $z\to\infty$ in S, $\{n^{-k}\}$ for $n\to\infty$, are called *Poincaré scales*. Series associated with these scales are called *Poincaré series*.

Poincaré series are the traditional asymptotic series of analysis. But it does not seem to be generally recognized that problems that are difficult in one asymptotic scale may become quite easy if the scale is changed. In some situations series with scales $\{1/(k+1)^n\}$ may offer distinct advantages over series with the Poincaré scale $\{1/n^k\}$.

How do these more general series arise? A trivial modification of the method of Darboux covers many practical situations.

Theorem 5.3. *Let $f(z)$ be meromorphic with simple poles at the points $a_1, a_2, a_3,\dots$, $0<|a_1|<|a_2|<|a_3|<\cdots$, with residues $b_1, b_2, b_3,\dots$. Define the Taylor's series coefficients $\{c_n\}$ of f by*

$$f(z) := \sum_{n=0}^{\infty} c_n z^n, \quad |z|<|a_1|.$$

Then

$$c_n \sim -\sum_{k=0}^{\infty}\frac{b_{k+1}}{a_{k+1}^{n+1}};\ \left\{\frac{1}{a_{k+1}^n}\right\}, \quad n\to\infty.$$

Proof. Pick N and choose R so that $|a_N|<R<|a_{N+1}|$. The function

$$h(z) := f(z) - \sum_{k=1}^{N}\frac{b_k}{z-a_k}$$

is analytic in $|z| \leqslant R$. But

$$h(z)=\sum_{n=0}^{\infty} c_n z^n+\sum_{k=1}^{N} \frac{b_k/a_k}{1-z/a_k}=\sum_{n=0}^{\infty}\left\{c_n+\sum_{k=1}^{N} \frac{b_k}{a_k^{n+1}}\right\} z^n,$$

and the latter series converges at $z=R$. Thus, for this value of z the nth coefficient must go to zero, or

$$\lim_{n \to \infty} R^n\left\{c_n+\sum_{k=1}^{N} \frac{b_k}{a_k^{n+1}}\right\}=0,$$

or

$$c_n=-\sum_{k=1}^{N} \frac{b_k}{a_k^{n+1}}+\mathrm{O}(R^{-n}), \quad n \to \infty.$$

This is easily seen to be equivalent to the assertion of the theorem. □

Example 5.4. (i) Let

$$f(z)=\Gamma(z+1)=\sum_{n=0}^{\infty} c_n z^n,$$

so that $a_k=-k$, $b_k=(-1)^{k+1}/\Gamma(k)$, $k=1, 2, 3, \ldots$. I find immediately that

$$c_n(-1)^n \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{n+1}}, \quad n \to \infty. \tag{5.3}$$

Like Poincaré type series, series with general asymptotic scales are unique, but a little caution must be used with them. The above series is convergent, but, as I shall show, it does not converge to c_n.

To do this, I approach the problem a different way. By using the integral formula for $\Gamma(z+1)$ and differentiating repeatedly, one finds

$$\begin{aligned} c_n&=\frac{1}{n!}\int_0^{\infty} \mathrm{e}^{-t}(\ln t)^n \,\mathrm{d}t=a_n+b_n,\\ a_n&=\frac{1}{n!}\int_0^{1} \mathrm{e}^{-t}(\ln t)^n \,\mathrm{d}t, \qquad b_n=\frac{1}{n!}\int_1^{\infty} \mathrm{e}^{-t}(\ln t)^n \,\mathrm{d}t. \end{aligned} \tag{5.4}$$

Expanding e^{-t} in its Taylor series in the first integral and integrating termwise gives

$$a_n=(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{n+1}}. \tag{5.5}$$

The series on the right of (5.3) converges to $(-1)^n a_n$, not to $(-1)^n c_n$. This cannot happen with Poincaré series.

A trick serves to give an accurate estimate of b_n. Let

$$d_n:=\int_1^{\infty} \mathrm{e}^{-at} g(t)\,\mathrm{d}t, \quad g(t):=\mathrm{e}^{(a-1)t}(\ln t)^n.$$

The function $g(t)$ has a single maximum, call it t^*, in $[1, \infty]$, namely, where $t^* \ln t^* = n/(1-a)$. I can force this maximum to occur at $t^* = n$ by choosing $a = 1 - (\ln n)^{-1}$. Pulling the maximum value of g through the integral, integrating e^{-at}, and using Stirling's formula in the form

$$n! > \sqrt{2\pi}\, n^{n+1/2}\, \mathrm{e}^{-n},$$

to fill in the missing $n!$ gives the estimate

$$|b_n| < \frac{(\ln n)^{n+1}}{\sqrt{2\pi}(\ln n - 1)n^{n+1/2}}, \quad n > 1.$$

Thus

$$c_n = a_n + \mathrm{O}\left(\frac{(\ln n)^n}{n^{n+1/2}}\right),$$

conforming the legitimacy of the asymptotic expansion (5.3), since the scale above is subdominant asymptotically to $1/\{k+1\}^n$ for any value of k.

Another approach to obtaining an asymptotic expansion for c_n is to use Laplace's method, or some variant of it. Probably the most general formulation, which allows the position of the critical point to depend on the large parameter, is given in [26]. There are many misprints in this book, and at one point the proof seems to go wrong. But the errors can be corrected. [4]

Theorem 5.5. *Let*

$$F(x) := \int_a^b \phi(t)\, \mathrm{e}^{h(x,t)}\, \mathrm{d}t;$$

(a) *let $h(x, t)$ and its first two derivatives in t be continuous on (a, b);*
(b) *define*

$$w(x, t) := \frac{\partial^2 h(x, t)}{\partial t^2}, \qquad q(x) := \left(-w(x, t_0(x))\right)^{1/2};$$

for each fixed x, let $h(x, t)$ attain its maximum value at just one point $t = t_0(x)$, $a < t_0(x) < b$, while $w(x, t_0(x)) \to -\infty$ as $x \to \infty$;
(c) *assume for each $\epsilon > 0$ there is a $\chi_1(\epsilon)$ such that for $x > \chi_1(\epsilon)$ the set of points t that satisfy the inequality*

$$|\phi(t)|\, \mathrm{e}^{h(x,t)} > \frac{|\phi(t_0)|\, \mathrm{e}^{h(x,t_0(x))}}{q(x)^{1+\epsilon}}$$

form a connected interval lying entirely within the interval (a, b);
(d) *for all $\epsilon > 0$ on the interval I_ϵ determined by*

$$|t - t_0(x)| \leqslant \frac{\sqrt{2(1+\epsilon)\ln^+ q(x)}}{q(x)},$$

one has

[4] I am grateful to Bill Goh for this restatement of Evgrafov's result.

$$\lim_{x\to\infty} \frac{\phi(t)}{\phi(t_0(x))} = 1, \qquad \lim_{x\to\infty} \frac{w(x,\,t)}{w(x,\,t_0(x))} = 1,$$

uniformly for $t \in I_\varepsilon$; [5]
then

$$F(x) \sim \frac{\sqrt{2\pi}}{q(x)} \phi(t_0(x))\,\mathrm{e}^{h(x,t_0(x))}, \quad x \to \infty.$$

In the first integral in (5.4) let x in the above result be n. Then the theorem will give an asymptotic formula for the second integral, although one has to solve the transcendental equation $t \ln t = n$. [6] However, the major contribution to the integral is from the integral a_n, and for this integral the conditions of the theorem are violated.

Example 5.4. (ii) Using the expansion

$$f(z) := \tfrac{1}{2}\sqrt{z}\ \mathrm{ctnh}\ \tfrac{1}{2}\sqrt{z} = \sum_{n=0}^{\infty} \frac{B_{2n} z^n}{(2n)!},$$

with $a_k = -(2k\pi)2$, $b_k = -2(2k\pi)2$, I find the expansion (5.2). In this case the expansion converges to the quantity on the left, see [25, vol. 1, p.37 (14)], with $x = 0$.

Generalized asymptotic scales are of great importance in the problem of estimating asymptotically the Taylor's coefficients of analytic functions. There is a large body of results in the literature concerning such problems, which are extensions and generalizations of the method of Darboux, see [75, Chapter 8, Section 9] and the references there for some of the older literature. For certain choices of $P(z)$ in

$$f(z) := \mathrm{e}^{P(z)}$$

results of Odlyzko and Richmond [73] furnish asymptotic expansions for the Taylor coefficients for $f(z)$, but, unfortunately, not in the important case where $P(z)$ is a polynomial.

Others who have worked on this problem are Wilf [94] and Harris and Schoenfeld [41]. Let

$$f(z) := \sum_{n=0}^{\infty} c_n z^n.$$

The latter authors show that if f satisfies certain conditions (see the paper for definitions and conditions), then

$$c_n = \frac{f(u_n)}{2u_n^n\sqrt{\pi\beta(u_n)}} \left\{1 + \sum_{k=1}^{N} \frac{F_k(n)}{\beta(u_n)^k} + \mathrm{O}(R_N(n))\right\}, \tag{5.6}$$

where u_n is the largest positive solution of $zf'(z)/f(z) = n$, and [7]

$$\beta(z) = \tfrac{1}{2}\delta^2 \ln f(z), \quad \delta = \frac{\mathrm{d}}{\mathrm{d}z}.$$

[5] Define $\ln^+ x = 0$, $0 < x < 1$; $\ln^+ x = \ln x$, $x \geqslant 1$.
[6] While this equation may be solved asymptotically (see [16]), it is hardly possible to get more than a lead term for b_n.
[7] Unfortunately, the definition of the $F_k(n)$'s — they are linear combinations of higher derivatives of f evaluated at u_n — is rather technical, and I have to refer the reader to the cited paper.

Equation (5.6) is in general a series with a non-Poincaré type scale, but it is not necessarily an asymptotic series, satisfying the condition (5.1). In fact for $f(z)=e^{z^4-z^3+z^2}$, even though the conditions of Harris and Schoenfeld's theorem are satisfied, it is not an asymptotic series. Schmutz [85] has shown the following theorem.

Theorem 5.6. *Let*

$$P(z)=\sum_{t=0}^{m} b_t z^t, \quad b_m>0, \quad b_t \text{ real.}$$

Then (5.6) *is an asymptotic series if and only if* (i) $c_n>0$ *for* n *sufficiently large*; (ii) *for every* $d>1$ *there is an integer* s *such that* d *is not a factor of* s *and* $b_{s(d)}\neq 0$. *Further if* $\tau(d)$ *is the largest such integer, then* $b_\tau>0$.

Example 5.7.

$$f(z):=e^{z+z^2+z^3}.$$

An easy consequence of the theorem is that if P has real nonnegative coefficients, then the series is an asymptotic series.

It is interesting that for the proof of the theorem the author requires the following lemma.

Lemma 5.8. *Define* d_n *by*

$$\Gamma(z)\sim\frac{z^{z-1/2}e^{-z}}{\sqrt{2\pi}}\sum_{n=0}^{\infty} d_n z^{-n}, \quad z\to\infty, \ \operatorname{Re} z>0.$$

Then $d_n\neq 0$.

It is a lesson that must sometimes be relearned in asymptotics: not all series that look like asymptotic series *are* asymptotic series.

6. Computational asymptotics: the calculation of weight functions and the Bessel polynomials

I close this presentation by mentioning a problem which is rather old, but one which fascinates me, and to which I keep returning. Let $\mathscr{L}$ be a linear functional defined on the space $\mathscr{P}$ of real polynomials. Denote the *moments* of $\mathscr{L}$ by

$$c_n=\mathscr{L}[x^n], \quad n=0,1,2,\ldots .$$

Then it is known that $\mathscr{L}$ generates a system of polynomials $\{p_n(x)\}$ orthogonal with respect to this functional, i.e.,

$$\mathscr{L}[p_n(x)p_m(x)]=\begin{cases}0, & m\neq n,\\ h_n\neq 0, & m=n,\end{cases} \quad m,\ n=0,1,2,\ldots .$$

(For a complete discussion of these matters, see [12,14], [25, vol. 2].) Under certain mild assumptions p_n will be of exact degree n. Further, p_n can be written explicitly in terms of Gram determinants of the moments and satisfies a three-term recurrence relation of the form

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \quad n = 1, 2, \ldots . \tag{6.1}$$

If only the coefficient $C_n > 0$, it can be shown that there is a distribution function [8] $\mu(x)$ supported on the real line which generates the moments in the sense that

$$c_n = \int_{-\infty}^{\infty} c^n \, \mathrm{d}\mu.$$

Then

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) \, \mathrm{d}\mu = \begin{cases} 0, & m \neq n, \\ h_n \neq 0, & m = n, \end{cases} \quad m, n, = 0, 1, 2, \ldots .$$

The polynomial set $\{ p_n(x) \}$ is said to be *orthogonal* with respect to this distribution. This theorem is sometimes called Favard's theorem, see the 1935 paper [27]. The theorem is, in fact, older, and can be found — as can so many other wonderful things — in Stone's 1932 book [89, p.545, Theorem 10.27]. [9] Another result is that even if the positivity condition on C_n is violated, there will still be some measure on the real line with respect to which the polynomials $\{ p_n(x) \}$ are orthogonal, but in general the measure will be a signed (nonpositive) measure and *will not be unique.* [10] This is a consequence of the surprising result given in [10].

Theorem 6.1. *Any sequence* $\{ c_n \}$ *of real numbers can be represented in the form*

$$c_n = \int_0^{\infty} t^n \, \mathrm{d}\mu, \quad n = 0, 1, 2, \ldots,$$

where

$$\int_0^{\infty} |\mathrm{d}\mu| < \infty.$$

It is easy to see that this implies that the previous integral relationship of orthogonality still holds, only now, less restrictively, with respect to a function of bounded variation.

The question I want to pose is the following. Suppose we are given the moments c_n. How can the distribution μ be recovered? If positivity of the C_n holds and μ has some smoothness, there are many theoretical and approximate techniques for recovering μ. Littlejohn's interesting paper [60] reviews two theoretical (i.e., noncomputational) methods for recovering μ when the polynomials $\{ p_n(x) \}$ are the eigenfunctions of a linear differential operator with polynomial coefficients. One method is due to Krall and Morton [54], and the other to Littlejohn himself. Both employ distributional techniques. (See also [58,59].)

Also there is available an assortment of techniques for numerically inverting the Laplace transform. Most of them may be applied to this problem (just make the substitution $\mathrm{e}^t = u$ in the

[8] A distribution is a nondecreasing real-valued function with an infinite number of points of increase all of whose moments exist. See [44, vol. 2, pp. 580 and 585].

[9] I am grateful to Richard Askey for pointing out this reference to me. However, almost every mathematical result is older than almost everybody thinks. Chihara [14, p.21] states that a form of this theorem goes back to Stieltjes.

[10] The way measures in general and signed measures in particular are constructed from Riemann–Stieltjes integrals is discussed in [45, p.304ff.].

Laplace transform integral) provided that values of the *extended* moment generating functional

$$c_\beta = \int_{-\infty}^{\infty} x^\beta \, \mathrm{d}\mu, \quad \beta \geqslant 0,$$

are known — this is almost always the case — or provided the methods are interpolatory and sufficiently nonrestrictive in the choice of nodes in the interpolation process. For a review of many methods, see [55].

Most numerical methods for computing a distribution from its moments suffer from the defect that they are *nonlocal* in the sense that nonsmoothness of μ away from the point x where the numerical value of μ is desired adversely affects convergence. This may be a significant drawback because often μ' has jump discontinuities somewhere else in the interval, and the presence of these discontinuities is fatal to convergence. [11] However there are computational algorithms which are local: nonsmoothness of μ away from x does not affect the convergence at x. I gave a class of such methods in [96]. The methods depend on a parameter s which allows one to take advantage of the smoothness near x of

$$\mu'(x) := w(x).$$

The larger s is, the smoother μ is, and the more rapidly the process converges. The methods will not work unless μ is at least absolutely continuous in a neighborhood of x. Thus the methods can at best yield what is called the absolutely continuous component of the measure. (Actually, this is a drawback of any *general* method I am familiar with.)

We assume μ is supported (has its points of increase) on [0, 1]. (This does not constitute a problem if μ is supported on $(-\infty, \infty)$ since a change of scale can be made, via a rational transformation.) Let α, β be real and greater than -1, $\gamma := \alpha + \beta + 1$. Define

$$w_s(x) = \frac{\Gamma(s+\gamma+1)}{\Gamma(\beta+1)\Gamma(\alpha+s+1)} \sum_{m=0}^{\infty} \frac{c_{\beta+m}(\beta+m+1)_{s+1}(-\alpha-s)_m}{m!} K_{m,s}(x),$$

$$K_{m,s}(x) := \frac{1}{s!} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \frac{x^r(\gamma+s+1)_r}{(\beta+1)_r(\beta+m+r+1)}.$$

We use the traditional [25] notation $(a)_k = 1$, $k = 0$, $(a)_k = a(a+1)\cdots(a+k-1)$, $k = 1, 2, \ldots,$ for Pochhammer's symbol. If convergence problems are encountered above or if the general moment-generating functional $c_{\beta+m}$ is not known, then one just takes α, β to be integers. The case of most interest to us here is the $s = 0$ case.

Theorem 6.2. *Let* $x \in (0, 1)$, $w(x) \in L[0, 1]$, $\alpha = n\tau_1$, $\beta = n\tau_2$, $\tau_1 \equiv \tau_1(n, x)$, $\tau_2 \equiv \tau_2(n, x)$, $\lim_{n\to\infty} \tau_2/(\tau_1 + \tau_2) = x$. *Let*

(a) *x be a Lebesgue point of $w(x)$, or in a neighborhood of x let $w(x)$ be*

(b) *continuous, or*

(c) *of bounded variation, or*

(d) *satisfy a Lipschitz condition of order ν.*

[11] Nonlocal methods, however, may be attractive when one wants an analytical expression for the measure. For a nice survey of nonlocal methods, see [92].

Then, respectively,
(a), (b) $w_0(x) = w(x) + o(1)$, $n \to \infty$;
(c) $w_0(x) = \frac{1}{2}(w(x^+) + w(x^-)) + o(1)$, $n \to \infty$;
(d) $w_0(x) = w(x) + o(n^{-\nu/2})$, $n \to \infty$.

Example 6.3. Let

$$c_n := \frac{\Gamma(n + \frac{1}{2})}{n!}.$$

Then

$$\begin{aligned} w_0(x) &= \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \sum_{m=0}^{\infty} \frac{(-\alpha)_m \Gamma(m + \beta + \frac{1}{2})}{m! \Gamma(m + \beta + 1)} \\ &= \frac{\Gamma(\gamma + 1)\Gamma(\beta + \frac{1}{2})}{\Gamma(\alpha + 1)\Gamma(\beta + 1)^2} {}_2F_1\left(\begin{matrix} -\alpha, \beta + \frac{1}{2} \\ \beta + 1 \end{matrix} ; 1 \right) \\ &= \frac{\Gamma(\gamma + 1)\Gamma(\beta + \frac{1}{2})\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(\gamma)\Gamma(\frac{1}{2})}, \end{aligned}$$

where I use the notation of [25, vol. 1] for hypergeometric functions. Now let $\alpha = \langle n\tau_1 \rangle$, $\beta = \langle n\tau_2 \rangle$, $\langle \cdot \rangle$ the greatest integer function, and liberally employ Stirling's formula to find

$$\begin{aligned} w_0(x) &\sim \frac{(\alpha + \beta + 1)}{\sqrt{\alpha\beta}} = \frac{n(\tau_1 + \tau_2) + \epsilon_1}{\sqrt{n\tau_1 + \epsilon_2}\sqrt{n\tau_2 + \epsilon_3}}, \quad 0 \leqslant \epsilon_j < 1, \\ &\sim \frac{\tau_1 + \tau_2}{\sqrt{\tau_1}\sqrt{\tau_2}} = \frac{\tau_1/\tau_2 + 1}{\sqrt{\tau_1/\tau_2}}. \end{aligned}$$

But note that $\tau_1/\tau_2 \sim 1/x - 1$. Substituting this in the above expression shows that

$$w(x) = \lim_{n \to \infty} w_0(x) = \frac{1/x}{\sqrt{1/x - 1}} = [x(1 - x)]^{-1/2},$$

which is the weight for the Chebyshev polynomials $T_n^*(x) = T_n(2x - 1)$ shifted to the interval [0, 1].

The *Bessel polynomials* $P_n(a, x)$, $a > 0$, correspond to the moment-generating functional

$$\mathscr{L}[x^n] := \frac{1}{\Gamma(n + a + 1)}.$$

It is easy to show that the polynomials generated by $\mathscr{L}$ satisfy the recurrence (6.1) with

$$A_n = \frac{-(2n + a)(2n + a + 1)}{(n + a)}, \qquad B_n = \frac{(a - 1)(2n + a)}{(n + a)(2n + a - 1)},$$

$$C_n = \frac{-n(2n + a + 1)}{(n + a)(2n + a - 1)},$$

and that they have the explicit representation

$$P_n(a, x) := {}_2F_0(-n, n+a; x).$$

Because the positivity condition on C_n is violated, Favard's theorem — which is an if and only if statement — shows there is no distribution for P_n. However, there will exist a (nonunique) measure. The formal application of the method of Krall and Morton yields a measure whose moments do not exist. The method of Littlejohn, which requires the inversion of a Fourier transform, gives a function which has no inverse Fourier transform. My method is rather more interesting in that it does provide *some* information, namely, that $w(x) = 0$. What this means, of course, is that the measure μ of the Bessel polynomials is constant in any interval in which it is continuously differentiable. The guts of the measure is contained in other sets, and no one has any idea of what happens there. This is one of the outstanding problems in the theory of orthogonal polynomials.

It is interesting (but somewhat irrelevant to this discussion) that $\{P_n\}$ is an orthogonal set on paths in the *complex plane*. This follows immediately from the formula

$$\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^z z^{-a-n-1}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{\Gamma(n+a+1)}, \quad c>0.$$

Thus

$$\begin{aligned}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^z z^{-a-k-1} P_n\left(a, \frac{1}{z}\right)\mathrm{d}z &= 2\pi\mathrm{i}\sum_{r=0}^{n}\frac{(-n)_r(n+a)_r}{r!\Gamma(r+k+a+1)}\\ &= \frac{2\pi\mathrm{i}}{\Gamma(a+k+1)}\,{}_2F_1\left(\begin{matrix}-n,\ n+a\\ k+a+1\end{matrix};1\right)\\ &= \frac{2\pi\mathrm{i}k!}{\Gamma(n+k+a+1)\Gamma(k+1-n)}=0, \quad 0\leqslant k\leqslant n-1.\end{aligned}$$

This property of the Bessel polynomials makes them useful in the numerical inversion of Laplace transforms, see [64, vol. 2, 16.3.5].

Making a change of variable and multiplying by the coefficient of z^n in P_n gives

$$\oint_C \mathrm{e}^{1/z} z^{a-1} P_n(a, z) P_m(a, z)\,\mathrm{d}z = \frac{(-1)^n 2\pi\mathrm{i}n!\delta_{m,n}}{(2n+a)\Gamma(n+a)}, \quad m, n = 0, 1, 2, \ldots,$$

where C, for instance, is the path $z = 1 + \mathrm{e}^{\pi\mathrm{i}t}$, $-1 \leqslant t < 1$. It is interesting that $w(x) = x^{a-1}\,\mathrm{e}^{1/x}$ is essentially the function yielded by the formal application of the Krall–Morton technique. However a real orthogonality integral over $[0, \infty)$ cannot be derived from the above contour integral. Bessel polynomials can be expressed as limits of Jacobi polynomials [25, vol. 2, p.170],

$$P_n(a, x) = \lim_{\beta\to\infty} \beta^{-n} n!(-1)^n P_n^{(a-\beta-1,\beta)}(2x\beta - 1),$$

but an attempt to obtain a measure for P_n by taking this limit in the orthogonality integral for the Jacobi polynomials comes to nothing.

The book of Grosswald [38] provides a good introduction to the Bessel polynomials, giving many of their properties and applications; but it is now out of date. [12] De Bruin, Saff and Varga

[12] In an "added in proof" addendum to the book, dated August 30, 1978, Grosswald quotes from a handwritten manuscript sent to him by Krall containing a result which purports to be a distribution for the Bessel polynomials. But it is not a distribution in the commonly accepted sense.

have obtained important results on the location (in the complex plane) of the zeros of the polynomials, and their most recent paper [17] is a good source for recent references. The paper by Galvez and Dehesa [32] describes other open problems involving the polynomials and also provides a good bibliography. The polynomials are important in network synthesis and design, theory of Padé approximation, and numerical analysis. An exciting combinatorial model for the polynomials has recently been obtained by Dulucq and Favreau [21]. There is also a connection with statistics via de Student t-distribution, see [50].

There are many other intriguing problems associated with orthogonal polynomials whose recurrences violate the condition $C_n > 0$. A typical one is as follows. The *Hermite polynomials* $H_n(x)$ are well known to be orthogonal with respect to the weight function $\exp(-x^2)$ on $(-\infty, \infty)$. It is a consequence of the result [10] there is some (signed) measure supported on $[0, \infty)$ with respect to which they are orthogonal. Such a measure has not yet been found.

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