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ON DIRICHLET'S DIVISOR PROBLEM

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I.

Introduction.

1.1. It was proved by Dirichlet* that, if $d(n)$ denotes the number of divisors of n, unity and n itself included, then

$$D(x) = \sum_1^x d(n) = x \log x + (2\gamma - 1)\, x + O(\sqrt{x}), \tag{1.11}$$

where γ is Euler's constant. We may therefore write

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.12}$$

where

$$\Delta(x) = O(\sqrt{x}). \tag{1.13}$$

The problem which I call "Dirichlet's Divisor Problem" is that of determining as precisely as possible the maximum order of the error term

* "Über die Bestimmung der mittleren Werthe in der Zahlentheorie", *Berliner Sitzungsberichte*, 1849, pp. 69-83; *Werke*, Vol. 2, pp. 49-66.

$\Delta(x)$. That the relation (1.13) could be replaced by some more precise relation was asserted by Dirichlet himself* ; but there is nothing to show what were the results at which he had arrived.

It was proved by Voronoï† in 1903 that

$$\Delta(x) = O(x^{\frac{1}{3}} \log x), \tag{1.14}$$

and this result is still the best that is known. It is remarkable that Voronoï's proof, though difficult and long, is "elementary": it does not, that is to say, involve any appeal to the theory of functions of a complex variable, as do all the principal results in the theory of the distribution of primes. In fact no one has as yet succeeded in deducing from the theory of the Zeta-function a result even as good as (1.14), though Landau‡ has developed a powerful and important method which not only leads to the slightly less precise result

$$\Delta(x) = O(x^{\frac{1}{3}+\epsilon}),\S \tag{1.15}$$

where ϵ is any positive number, but is capable of application to a whole class of problems of which Dirichlet's problem is one.

1.2. My object in this paper is to attack the problem from the other end, and to prove, not an "O" result, but what Mr. Littlewood and I‖

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* "Seit unserm neulichen Gespräch . . . . ist es mir gelungen, die Summe $\sum_{1}^{n} \left[\frac{n}{s}\right]$, wo [ ] nach Gauss dass grösste Ganze bezeichnet und die ich bisher nur mit einem Fehler der Ordnung $\sqrt{n}$ angeben konnte, bedeutend in die Enge zu treiben. Die Auffindung des hiezu dienenden Mittels, welches aller Wahrscheinlichkeit nach auch auf die folgenden Fälle anwendbar seyn wird, macht mir zwar grosses Vergnügen, kommt mir aber in sofern zu ungelegener Zeit als ich . . . ." (Dirichlet to Kronecker, 23 July 1858, *Werke*, Vol. 2, p. 407).

† "Sur un problème du calcul des fonctions asymptotiques", *Journal für Math.*, Vol. 126, pp. 241–282.

‡ "Über die Anzahl der Gitterpunkte in gewissen Bereichen", *Göttinger Nachrichten*, 1912, pp. 687–771.

§ This formula had been given by Pfeiffer ("Über die Periodicität in der Teilbarkeit der Zahlen und über die Verteilung der Klassen positiver quadratischer Formen auf ihre Determinanten", *Jahresbericht der Pfeiffer'schen Lehr- und Erziehungs-Anstalt zu Jena*, 1886, pp. 1–21) long before the appearance of Voronoï's memoir. The proof given by Pfeiffer was not entirely satisfactory, but Landau has shown that the ideas on which Pfeiffer bases his proof are sound and lead to a proof not only of (1.15) but of Voronoï's more precise formula (1.14). See Landau, "Die Bedeutung der Pfeiffer'schen Methode für die analytische Zahlentheorie", *Wiener Sitzungsberichte*, Vol. 121, Part 2*a*, pp. 2195–2332.

‖ "Some Problems of Diophantine Approximation (II)", *Acta Mathematica*, Vol. 37, pp. 193–238 (p. 225).

have called an " Ω " result; viz., that

$$\Delta(x) = \Omega_R(x^{\frac{1}{4}}), \quad \Delta(x) = \Omega_L(x^{\frac{1}{4}}), \tag{1.16}$$

or, in other words, that a constant K exists such that each of the inequalities

$$\Delta(x) > Kx^{\frac{1}{4}}, \quad \Delta(x) < -Kx^{\frac{1}{4}} \tag{1.161}$$

holds for an infinity of values of x surpassing all limit.* It follows that the function $x^{\frac{1}{3}} \log x$ of Voronoï's relation (1.14) certainly cannot be replaced by any function of lower order than $x^{\frac{1}{4}}$. So far as I am aware, nothing is known in this direction except the relation

$$\Delta(x) = \Omega\left\{2^{(1-\epsilon)\frac{\log x}{\log\log x}}\right\}, \tag{1.17}$$

which follows at once from the fact that

$$d(n) > 2^{(1-\epsilon)\frac{\log n}{\log\log n}}, \tag{1.171}$$

for an infinity of values of n.†

1.3. The theorem expressed by the relations (1.16) or (1.161) is analogous to a well known theorem of Schmidt. It was proved by Schmidt‡ that

$$\begin{cases} \Pi(x) - Li\, x + \frac{1}{2} Li \sqrt{x} = \Omega_R\left(\dfrac{\sqrt{x}}{\log x}\right), \\ \Pi(x) - Li\, x + \frac{1}{2} Li \sqrt{x} = \Omega_L\left(\dfrac{\sqrt{x}}{\log x}\right), \end{cases} \tag{1.31}$$

where $\Pi(x)$ is the number of primes not exceeding x. A simplified

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* I have already published this result, with summary indications of the proof, under the title " Sur le problème des diviseurs de Dirichlet " (*Comptes Rendus*, 10 May 1915).

† Wigert, " Sur l'ordre de grandeur du nombre des diviseurs d'un entier", *Arkiv för Math.*, Vol. 3, No. 3, pp. 1–18; see also Landau, *Handbuch*, p. 219. A number of more precise results concerning the maximum order of $d(n)$ will be found in Mr. Ramanujan's paper "On Highly Composite Numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14, pp. 347–409. It should be observed that the function $2^{\log n/\log\log n}$ is of higher order than any power of $\log n$, but of lower order than any power of n.

‡ "Über die Anzahl der Primzahlen unter gegebener Grenze", *Math. Annalen*, Vol. 57, pp. 195–204.

form of Schmidt's proof will be found in Landau's *Handbuch**. My proof of (1.16) does not differ in principle from Landau's proof of (1.31); its difficulty lies not in its principle but in the construction of the new formulæ required for the application of the principle to the divisor problem. It will therefore probably make the argument clearer if I recapitulate shortly the proof, not exactly of the relations (1.31), but of the substantially equivalent† relations

(1.311) $$\psi(x)-x = \Omega_R(\sqrt{x}), \quad \psi(x)-x = \Omega_L(\sqrt{x}),$$

where $\psi(x)$ is Tschebyschef's function

$$\psi(x) = \sum_{p^m \leqslant x} \log p.$$

1.4. In proving (1.311) we may assume the Riemann hypothesis‡. It is then easy to show that the series

$$\sum \frac{\psi(n)-n}{n^{1+s}}$$

is convergent for $\sigma > \frac{1}{2}$, and represents a function

$$F(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \zeta(s) + g(s),$$

where $g(s)$ is regular for $\sigma > 0$, so that $F(s)$ is regular for $\sigma > 0$ except for simple poles at the zeros of $\zeta(s)$. It follows that, if $s = \frac{1}{2}+i\gamma$ is a zero of $\zeta(s)$, then

$$\sum \frac{\psi(n)-n}{n^{\frac{3}{2}+i\gamma+\delta}} \sim -\frac{\kappa}{(\frac{1}{2}+i\gamma)\delta},$$

as $\delta \to 0$, where κ is the multiplicity of the zero.

Suppose now that (*e.g.*)

(1.41) $$\psi(n)-n < K\sqrt{n}$$

for all sufficiently large values of n. Then

(1.42) $$\sum \frac{\psi(n)-n-K\sqrt{n}}{n^{\frac{3}{2}+i\gamma+\delta}} \sim -\frac{\kappa}{(\frac{1}{2}+i\gamma)\delta},$$

and

(1.43) $$\sum \frac{\psi(n)-n-K\sqrt{n}}{n^{\frac{3}{2}+\delta}} \sim -\frac{K}{\delta},$$

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* Pp. 711 *et seq.*; see also Landau, "Über einen Satz von Tschebyschef", *Math. Annalen*, Vol. 61, pp. 527–550. It has been proved recently by Littlewood that more is true than is stated by the relations (1.31), in fact that

$$\Pi(x) - Li\,x = \Omega_R\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right), \quad \Pi(x) - Li\,x = \Omega_L\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right).$$

See Littlewood, "Sur la distribution des nombres premiers", *Comptes Rendus*, 22 June 1914; Hardy and Littlewood, "Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes", *Acta Mathematica* (as yet unpublished).

† Cf. Hardy and Littlewood, *l.c. supra*.

‡ If this is false then *more* is true. See Landau, *Handbuch*, p. 713.

since $F(s)$ is regular for $s=\frac{1}{2}$, and $\zeta(s)$ is regular for $s=1+i\gamma$ but has a simple pole for $s=1$. But $\psi(n)-n-K\sqrt{n}$ is ultimately of constant sign, and $|n^{-i\gamma}|=1$; so that these results are contradictory if

$$K<\frac{\kappa}{\sqrt{(\frac{1}{4}+\gamma^2)}}.$$

Thus (1.41) is certainly false for sufficiently small values of K and an infinity of values of n.

II.

The Function $F(s)=\Sigma\dfrac{d(n)}{\sqrt{n}}e^{-s\sqrt{n}}$.

2.1. I write

(2.11) $$s=\sigma+it,\quad z=x+iy.$$

From the formulæ

(2.12) $$e^{-s}=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-z}\Gamma(z)\,dz\quad(\sigma>0,\ \kappa>0),^*$$

(2.13) $$\{\zeta(z)\}^2=\Sigma\frac{d(n)}{n^z}\quad(x>1),$$

we deduce

(2.14) $$F(s)=\Sigma\frac{d(n)}{\sqrt{n}}e^{-s\sqrt{n}}=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-z}\Gamma(z)\{\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\}^2dz,$$

where now $\sigma>0$, $\kappa>1$. From (2.14) it follows, by a simple application of Cauchy's theorem,† that

(2.15) $$F(s)=-\frac{4\log s}{s}+\sum_0^p\frac{(-s)^n}{n!}\{\zeta(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2+J_p,$$

where p is any positive integer, and

(2.16) $$J_p=\frac{1}{2\pi i}\int_{-p-\frac{1}{2}-i\infty}^{-p-\frac{1}{2}+i\infty}s^{-z}\Gamma(z)\{\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\}^2dz.$$

Let us suppose for the moment that s is real and $0\leqslant s<4\pi$. Then

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* Mellin, "Om definiter integraler", *Acta Soc. Fennicae*, Vol. 20, No. 7, pp. 1-39 (p. 6), and "Abriss einer einheitlichen Theorie der Gamma- und hypergeometrischen Funktionen", *Math. Annalen*, Vol. 68, pp. 305-337.

† We have $|\Gamma(z)|=O(|y|^K e^{-\frac{1}{2}\pi|y|}),\quad |\zeta(\frac{1}{2}+\frac{1}{2}z)|=O(|y|^K),$

where K is independent of y, uniformly for $-p-\frac{1}{2}\leqslant x\leqslant\kappa$. There is a double pole at $z=1$, and simple poles at $z=0,\ z=-1,\ \ldots,\ z=-p$.

$J_p \to 0$ as $p \to \infty$. For, if

$$z = -p-\tfrac{1}{2}+iy,$$

we have

$$|s^{-z}| = s^{p+\frac{1}{2}},$$

$$|\Gamma(z)| = |\Gamma(-p-\tfrac{1}{2}+iy)| = \frac{\pi}{\cosh \pi y\, |\Gamma(p+\frac{3}{2}-iy)|},$$

$$|\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)|^2 = |\zeta(-\tfrac{1}{2}p+\tfrac{1}{4}+\tfrac{1}{2}iy)|^2$$

$$= |2(2\pi)^{-\frac{1}{2}p-\frac{3}{4}+\frac{1}{2}iy} \cos(\tfrac{1}{4}p+\tfrac{3}{8}-\tfrac{1}{4}iy)\pi\, \Gamma(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)\, \zeta(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)|^2$$

$$< K(2\pi)^{-p} e^{\frac{1}{2}\pi|y|}\, |\Gamma(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)|^2;$$

and so

$$|s^{-z}\Gamma(z)\{\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\}|^2 < K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \left|\frac{\{\Gamma(\frac{1}{2}p+\frac{3}{4}-\frac{1}{2}iy)\}^2}{\Gamma(p+\frac{3}{2}-iy)}\right|$$

$$= K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \left|\int_0^1 \{w(1-w)\}^{\frac{1}{2}p-\frac{1}{4}-\frac{1}{2}iy}\, dw\right|$$

$$< K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \int_0^1 \{w(1-w)\}^{\frac{1}{2}p-\frac{1}{4}}\, dw$$

$$< K\left(\frac{s}{4\pi}\right)^p e^{-\frac{1}{2}\pi|y|}.*$$

Hence

$$(2.17) \qquad |J_p| < K\left(\frac{s}{4\pi}\right)^p \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi|y|}\, dy < K\left(\frac{s}{4\pi}\right)^p \to 0.$$

2.2. We have therefore

$$(2.21) \qquad F(s) = -\frac{4\log s}{s} + \sum_0^\infty \frac{(-s)^n}{n!}\{\zeta(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2,$$

if $0 < s < 4\pi$, or

$$(2.22) \qquad F(s) = -\frac{4\log s}{s} + G(s),$$

say. The power series $G(s)$ is convergent for $|s| < 4\pi$, and represents a branch of an analytic function regular at the origin; and the equation (2.22), considered as a relation between analytic functions, holds through-

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* Because $0 \leqslant w(1-w) \leqslant \frac{1}{4}$. In this series of equations and inequalities the K's are of course independent both of p and of y.

out their region of existence. The study of the singularities of $F(s)$ is therefore reduced to the study of those of $G(s)$.*

III.

The Function $G(s) = \Sigma \frac{(-s)^n}{n!} \{\zeta(\frac{1}{2}-\frac{1}{2}n)\}^2$. *The Singularities of* $G(s)$ *and* $F(s)$.

3.1. We proceed now to the discussion of the singularities of the function

$$G(s) = \sum_0^\infty \frac{(-s)^n}{n!} \{\zeta(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2. \tag{3.11}$$

Suppose first that $|s| < 4\pi$, so that the power series is convergent. We have

$$\zeta(\tfrac{1}{2}-\tfrac{1}{2}n) = 2(2\pi)^{-\frac{1}{2}n-\frac{1}{2}} \cos\{\tfrac{1}{4}(n+1)\pi\}\, \Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\, \zeta(\tfrac{1}{2}n+\tfrac{1}{2}),$$

$$\cos^2\{\tfrac{1}{4}(n+1)\pi\} = \tfrac{1}{2}+\tfrac{1}{4}i^{n+1}+\tfrac{1}{4}(-i)^{n+1},$$

and so

$$G(s) = A+Bs+\frac{1}{\pi} M\left(-\frac{s}{2\pi}\right)+\frac{i}{2\pi} M\left(-\frac{is}{2\pi}\right)-\frac{i}{2\pi} M\left(\frac{is}{2\pi}\right), \tag{3.12}$$

where $A = \{\zeta(\tfrac{1}{2})\}^2, \quad B = -\{\zeta(0)\}^2 = -\tfrac{1}{4},$

$$M(s) = \sum_2^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} \{\zeta(\tfrac{1}{2}n+\tfrac{1}{2})\}^2 s^n \tag{3.13}$$

$$= \sum_2^\infty c_n \{\zeta(\tfrac{1}{2}n+\tfrac{1}{2})\}^2 s^n,$$

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* It can be shown by similar reasoning that the function

$$F_{a,\beta}(s) = \Sigma \frac{d(n)}{n^a} e^{-sn^\beta} \quad (0 < \beta < \tfrac{1}{2})$$

is the sum of a certain elementary algebraic or algebraico-logarithmic function and an *integral* function $G_{a,\beta}(s)$. This has been observed already by Mellin, "Die Dirichletschen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichen Geschlecht", *Acta Societatis Fennicae*, Vol. 31, No. 2, pp. 1–48 (p. 12), and *Acta Mathematica*, Vol. 28, pp. 37–64. This memoir contains a large variety of formulæ of the highest interest for possible applications in the Analytic Theory of Numbers; but Mellin does not seem to have considered the case in which $\beta = \frac{1}{2}$, which is plainly the critical case in the theory of $d(n)$.

say. Also

$$(3.14)\qquad M(s) = \sum_2^\infty c_n s^n \sum_1^\infty \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}}$$

$$= \sum_1^\infty \frac{d(q)}{\sqrt{q}} \sum_2^\infty c_n \left(\frac{s}{\sqrt{q}}\right)^n = \sum_1^\infty \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right),$$

where

$$(3.15)\qquad L(s) = \sum_2^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n.$$

3.2. The function $L(s)$ is an elementary function. For suppose s real and $0 < s < 2$. Then

$$(3.21)\qquad L(s) = -\pi - s + \sum_0^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n,$$

and

$$(3.22)\quad \sum_0^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n = \sum_0^\infty s^n \int_0^1 \{w(1-w)\}^{\frac{1}{2}n-\frac{1}{2}} dw$$

$$= 2\sum_0^\infty s^n \int_0^{\frac{1}{2}\pi} (\cos\theta \sin\theta)^n d\theta = 2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 - s\cos\theta\sin\theta}$$

$$= \int_0^\pi \frac{d\phi}{1-\frac{1}{2}s\sin\phi} = \frac{8}{\sqrt{(4-s^2)}} \arctan \sqrt{\left(\frac{2+s}{2-s}\right)},$$

where the many valued functions have their principal values, so that $L(s)$ is positive for $0 < s < 2$ and

$$L(s) \sim \frac{2\pi}{\sqrt{(2-s)}}$$

as $s \to 2$. We have thus proved that if $|s| < 2$ then

$$(3.23)\qquad M(s) = \sum_1^\infty \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right),$$

where

$$(3.24)\qquad L(s) = -\pi - s + \frac{8}{\sqrt{(4-s^2)}} \arctan \sqrt{\left(\frac{2+s}{2-s}\right)}.$$

The principal branch of $L(s)$ has, as its sole singularity, an algebraic infinity of order $\frac{1}{2}$ at the point $s = 2$,* the principal part of $L(s)$ near

* Other branches have a singularity at $s = -2$. Compare Hardy, "On the Singularities of Functions defined by Taylor's Series", *Proc. London Math. Soc.*, Ser. 2, Vol. 5, pp. 197–205.

$s = 2$, being

$$\frac{2\pi}{\sqrt{(2-s)}}.$$

3.3. It is easy to deduce from the preceding analysis that the singularities of $M(s)$ are those of

$$L\left(\frac{s}{\sqrt{q}}\right) \quad (q = 1, 2, 3, \ldots).$$

We have in fact, if p is any positive integer,

$$(3.31)\quad M(s) - \sum_{1}^{p-1} \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right) = \sum_{2}^{\infty} c_n s^n \left[\{\zeta(\tfrac{1}{2}n+\tfrac{1}{2})\}^2 - \sum_{1}^{p-1} \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}}\right]$$

$$= \sum_{2}^{\infty} c_n s^n \sum_{p}^{\infty} \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}} = M_p(s),$$

where $M_p(s)$ is regular for $|s| < 2\sqrt{p}$. We are thus enabled to specify completely the singularities of $G(s)$, and so of $F(s)$. In particular we see that $F(s)$ *has algebraical infinities of order* $\frac{1}{2}$ *at each of the points*

$$(3.32)\qquad s = 4\pi i\sqrt{q}, \quad -4\pi i\sqrt{q} \quad (q = 1, 2, 3, \ldots),$$

and that

$$(3.33)\qquad F(s) \sim \frac{i}{2\pi} M\left(-\frac{is}{2\pi}\right) \sim \frac{i}{2\pi} \frac{d(q)}{\sqrt{q}} L\left(-\frac{is}{2\pi\sqrt{q}}\right)$$

$$\sim \frac{e^{\frac{1}{4}\pi i} d(q)}{q^{\frac{1}{4}}} \sqrt{\left(\frac{2\pi}{s - 4\pi i\sqrt{q}}\right)},$$

when $s = \sigma + 4\pi i\sqrt{q}$ *and* $\sigma \to 0$.

IV.

Application to the Divisor Problem.

4.1. Let

$$(4.11)\qquad f(s) = \Sigma d(n)\, e^{-s\sqrt{n}} = -F'(s),$$

when $\sigma > 0$. Then it follows from what precedes

(i) that $f(s)$ is regular at all points of the imaginary axis except the origin and the points (3.32);

(ii) that

(4.12) $$f(s)+\frac{4\log s}{s^2}-\frac{4}{s^2}$$

is regular at the origin ;

(iii) that $f(s)$ has an algebraical infinity of order $\frac{3}{2}$ at each of the points (3.32) ; and

(iv) that

(4.13) $$f(s)\sim\frac{e^{\frac{1}{4}\pi i}d(q)}{2q^{\frac{1}{4}}}\frac{\sqrt{(2\pi)}}{\sigma^{\frac{3}{2}}}$$

when $s=\sigma+4\pi i\sqrt{q}$ and $\sigma\to 0$. We shall also require the corresponding results for the simpler functions

(4.141) $$\phi(s)=(\log n+2\gamma)\,e^{-s\sqrt{n}},$$

(4.142) $$\psi(s)=\Sigma n^{-\frac{1}{4}}e^{-s\sqrt{n}}.$$

We have, in the first place,

(4.15) $$\phi(s)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-z}\,\Gamma(z)\,Z(z)\,dz,$$

if $\sigma>0$, $\kappa>2$, and

(4.151) $$Z(z)=-\zeta'(\tfrac{1}{2}z)+2\gamma\zeta(\tfrac{1}{2}z).$$

Applying to (4.15) a transformation similar to that of 2.1,* we find that

(4.152) $$\phi(s)=-\frac{4\log s}{s^2}+\frac{4}{s^2}+g(s),$$

where $g(s)$ is an integral function of s.

In the second place we have

(4.161) $$\psi(s)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-s}\,\Gamma(z)\,\zeta(\tfrac{1}{4}+\tfrac{1}{2}z)\,dz,$$

if $\sigma>0$ and $\kappa>\frac{3}{2}$; from which, by a repetition of our arguments, we deduce that

(4.162) $$\psi(s)=\sqrt{\pi}\,s^{-\frac{3}{2}}+h(s),$$

where $h(s)$ is an integral function of s.

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* The details of the analysis are now much simpler.

4.2. From the results of the last paragraph we deduce that the function $\mathfrak{F}(s)$ defined, when $\sigma > 0$, by the equation

$$\mathfrak{F}(s) = \Sigma d_n e^{-s\sqrt{n}}, \tag{4.21}$$

where

$$d_n = d(n) - \log n - 2\gamma, \tag{4.211}$$

is regular at all points of the imaginary axis except the points (3.32), at each of which it has the same singular part as $f(s)$.

Now let

$$D_n = \sum_1^n d_\nu, \tag{4.22}$$

and

$$\mathfrak{G}(s) = \sum_1^\infty \frac{D_n}{\sqrt{n}} e^{-s\sqrt{n}}; \tag{4.23}$$

and let us consider first the behaviour of $\mathfrak{G}(s)$ when $s = \sigma + 4\pi i\sqrt{q}$ and $\sigma \to 0$. We have

$$e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}} = \frac{se^{-s\sqrt{n}}}{2\sqrt{n}} + O\left(\frac{e^{-\sigma\sqrt{n}}}{n}\right), \tag{4.24}$$

and so

$$\mathfrak{F}(s) = \sum_1^\infty d_n e^{-s\sqrt{n}} = \sum_1^\infty D_n \{e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}}\} \tag{4.25}$$

$$= \tfrac{1}{2}s\mathfrak{G}(s) + O\sum_1^\infty \frac{e^{-\sigma\sqrt{n}}}{\sqrt{n}} * = \tfrac{1}{2}s\mathfrak{G}(s) + O(1/\sigma).$$

Hence

$$\mathfrak{G}(s) = \frac{2}{s}\mathfrak{F}(s) + O\left(\frac{1}{\sigma}\right) \sim \frac{(1+i)\, d(q)}{4i\sqrt{\pi}\, q^{\frac{1}{4}}\sigma^{\frac{3}{2}}}. \tag{4.26}$$

We must also consider how $\mathfrak{G}(s)$ behaves when $s \to 0$ by positive values. We have now

$$e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}} = \frac{se^{-s\sqrt{n}}}{2\sqrt{n}} + O\left(\frac{s^2 e^{-s\sqrt{n}}}{n}\right) + O\left(\frac{se^{-s\sqrt{n}}}{n\sqrt{n}}\right), \tag{4.27}$$

where the constants implied by the O's are independent of both s and n. Hence, by (4.25), we have

$$\mathfrak{F}(s) = \tfrac{1}{2}s\mathfrak{G}(s) + O\left(s^2 \Sigma \frac{e^{-s\sqrt{n}}}{n}\right) + O\left(s\Sigma\frac{e^{-s\sqrt{n}}}{n}\right) = \tfrac{1}{2}s\mathfrak{G}(s) + O(1),$$

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* Here we use Dirichlet's result $\Delta(n) = O(\sqrt{n})$, which is sufficient for our purpose.

so that

(4.28) $$\mathfrak{G}(s) = O(1/s).*$$

4.3. The relations (4.162), (4.26), and (4.28) enable us to complete the proof of our main theorem. We have, in fact,

(4.311) $$\Sigma \frac{D_n}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim \frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}},$$

(4.312) $$\Sigma \frac{D_n}{\sqrt{n}} e^{-\sigma\sqrt{n}} = o\left(\frac{1}{\sigma^{\frac{3}{2}}}\right),$$

(4.313) $$\Sigma n^{-\frac{1}{4}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} = o\left(\frac{1}{\sigma^{\frac{3}{2}}}\right),$$

(4.314) $$\Sigma n^{-\frac{1}{4}} e^{-\sigma\sqrt{n}} \sim \frac{\sqrt{\pi}}{\sigma^{\frac{3}{2}}},$$

when $\sigma \to 0$. It follows that

(4.321) $$\Sigma \frac{D_n - Kn^{\frac{1}{4}}}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim \frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}},$$

(4.322) $$\Sigma \frac{D_n - Kn^{\frac{1}{4}}}{\sqrt{n}} e^{-\sigma\sqrt{n}} \sim -\frac{K\sqrt{\pi}}{\sigma^{\frac{3}{2}}}.$$

Suppose now that

(4.33) $$D_n - Kn^{\frac{1}{4}} \leqslant 0,$$

for all sufficiently large values of n. Then

$$\Sigma \frac{|D_n - Kn^{\frac{1}{4}}|}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim -\frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}},$$

$$\Sigma \frac{|D_n - Kn^{\frac{1}{4}}|}{\sqrt{n}} e^{-\sigma\sqrt{n}} \sim \frac{K\sqrt{\pi}}{\sigma^{\frac{3}{2}}}.$$

Since $|e^{-4\pi i\sqrt{q}}| = 1$, these results are contradictory if

(4.34) $$K < \frac{1}{2\pi\sqrt{2}}\,\frac{d(q)}{q^{\frac{3}{4}}}.$$

Similarly we can show that it is impossible that

(4.35) $$D_n + Kn^{\frac{1}{4}} \geqslant 0,$$

for all sufficiently large values of n.

---

* It is easy to obtain a much more precise equation, but this very crude one is sufficient.

Since

$$D_n = \sum_1^n \{d(\nu)-\log\nu-2\gamma\} = \Delta(n)+O(\log n), \tag{4.36}$$

it follows that *neither of the inequalities*

$$\Delta(n)-Kn^{\frac{1}{4}} \leqslant 0, \qquad \Delta(n)+Kn^{\frac{1}{4}} \geqslant 0 \tag{4.37}$$

can hold for all sufficiently large values of n. We have therefore proved our main theorem.

The value of q which gives the best value of K is $q=2$. We have then

$$\frac{1}{2\pi\sqrt{2}}\frac{d(2)}{2^{\frac{3}{4}}} = \frac{1}{2^{\frac{5}{4}}\pi} = .13\ldots > \tfrac{1}{8};$$

so that we may take

$$K = \tfrac{1}{8}. \tag{4.38}$$

The actual value of K is however of little importance.*

V.

Applications of the Method to other Problems. The Problems of Piltz and Sierpinski.

5.1. Dirichlet's problem was generalised by Piltz†, who considered instead of $d(n)$ the number $d_k(n)$ of decompositions of n into k factors‡. Piltz proved that

$$D_k(n) = \sum_1^n d_k(n) = x\,\{a_{k,1}(\log x)^{k-1}+\ldots+a_{k,k}\}+\Delta_k(x), \tag{5.11}$$

---

* In the first place, $\frac{1}{4}$ may not be the best possible index; there may be a number α such that $\frac{1}{4} < \alpha \leqslant \frac{1}{3}$ and

$$\Delta(x) = \Omega_R(x^\alpha), \qquad \Delta(x) = \Omega_L(x^\alpha).$$

And even if $\frac{1}{4}$ is the best possible index, it is, in the light of Mr. Littlewood's result concerning $\Pi(x)$, highly probable that $x^{\frac{1}{4}}$ can be replaced by some such function as $x^{\frac{1}{4}}(\log x)^\beta$, where $\beta > 0$. See the additional note at the end of the paper (p. 23).

† "Über das Gesetz, nach welchem die mittlere Darstellbarkeit der natürlichen Zahlen als Produkte einer gegebenen Anzahl Faktoren mit der Grösse der Zahlen wächst", *Dissertation*, Berlin, 1881.

‡ Thus $d_3(1)=1$, $d_3(2)=3$, $d_3(4)=6$: the decompositions of 4 being 1.1.4, 1.4.1, 4.1.1, 1.2.2, 2.1.2, 2.2.1.

where the a's are constants, and

(5.111) $$\Delta_k(x) = O\{x^{\frac{k-1}{k}}(\log x)^{k-2}\}.$$

Landau* has shown that

(5.12) $$\Delta_k(x) = O(x^{\frac{k-1}{k+1}+\epsilon}),$$

for all positive values of ϵ; and this is the best result known. Landau† has, however, pointed out that it follows from a theorem of Littlewood that, *if the Riemann hypothesis concerning the roots of the Zeta-function is true*‡, then

(5.121) $$\Delta_k(x) = O(x^{\frac{1}{2}+\epsilon}),$$

for every k and every positive ϵ.

The application of the method of this paper to Piltz's problem presents no fresh difficulty of principle. I find that

(5.13) $$\Delta_k(x) = \Omega_R(x^{\frac{k-1}{2k}}), \quad \Delta_k(x) = \Omega_L(x^{\frac{k-1}{2k}}).$$

It should be observed that Landau's O-index $\dfrac{k-1}{k+1}$ takes for $k = 2, 3, 4, \ldots$ the values

$$\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{7}{9}, \ldots,$$

tending to 1, while my Ω-index $\dfrac{k-1}{2k}$ takes the values

$$\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{5}{12}, \tfrac{3}{7}, \tfrac{7}{16}, \ldots,$$

tending to $\frac{1}{2}$. It is rash to hazard a conjecture on such a subject, but the fact that Landau's indices exceed $\frac{1}{2}$ from $k = 4$ onwards suggest that the real order of $\Delta_k(x)$ may be nearer to my result than to his.

5.2. Let $r(n)$ denote the number of representations of n as the sum of two squares, *i.e.*, the number of solutions of the equation

(5.21) $$u^2+v^2 = n,$$

---

* *L.c. supra* (p. 2, footnote ‡), p. 723.

† *Ibid.*, p. 728. See also Littlewood, "Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\sigma > \frac{1}{2}$", *Comptes Rendus*, 29 Jan. 1912.

‡ Indeed if only it be true that

$$\zeta(\tfrac{1}{2}+ti) = O(|t|^{\epsilon})$$

for every positive ϵ.

u and v being integers, positive, negative, or zero. It is well known that

$$r(n) = 4\{d_1(n) - d_3(n)\},$$

where $d_1(n)$ and $d_3(n)$ denote the numbers of the divisors of n of the forms $4k+1$ and $4k+3$ respectively. It is also well known that

$$R(x) = \sum_{1}^{x} r(n) = \pi x + \mathrm{P}(x),^{*} \tag{5.22}$$

where

$$\mathrm{P}(x) = O(\sqrt{x}). \tag{5.221}$$

The problems which arise in connection with $\mathrm{P}(x)$ are wholly analogous to those connected with $\Delta(x)$, and the theory is in much the same state, the best known result as to the order of $\mathrm{P}(x)$ being

$$\mathrm{P}(x) = O(x^{\frac{1}{3}}).\dagger \tag{5.222}$$

The question arises as to whether the method of this paper is applicable to this problem also; and the answer is in the affirmative. It shows that a constant K exists such that each of the inequalities

$$\mathrm{P}(x) > Kx^{\frac{1}{4}}, \qquad \mathrm{P}(x) < -Kx^{\frac{1}{4}} \tag{5.23}$$

is satisfied for values of x which surpass all limit. The true maximum order of $\mathrm{P}(x)$ therefore lies somewhere between $x^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$.

The proof of this result is in principle entirely similar to that of the corresponding result for $\Delta(n)$, and is formally a little simpler. It starts from the equation

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{-s\sqrt{(m^2+n^2)}} = 2\pi s \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \frac{1}{\{s^2+4\pi^2(m^2+n^2)\}^{\frac{3}{2}}}, \tag{5.24}$$

which holds when $\sigma > 0$. I proved this formula in 1908‡, by a method

* $R(x)$ is the number of points with integral coordinates which lie inside or on the circle

$$u^2+v^2 = x.$$

† This result is due to Sierpinski, "O pewnem zagadnieniu z rachunku funkcyj asymptotycznych", *Prace matematyczno fizyczne*, Vol. 17, pp. 77–118. Another proof has been given by Landau, "Über die Zerlegung der Zahlen in zwei Quadrate", *Annali di Matematica*, Vol. 20, p. 28. See also Landau, "Über einen Satz des Herrn Sierpinski", *Giornale di Matematiche di Battaglini*, Vol. 51, pp. 73-81, where it is shown that Landau's method alluded to on p. 2 leads to the slightly less precise relation

$$\mathrm{P}(n) = O(n^{\frac{1}{3}+\epsilon}).$$

‡ "Some multiple integrals", *Quarterly Journal*, Vol. 39, pp. 357–375 (p. 373). At the time when this paper was written I had no thought of any application of its results in the analytic theory of numbers.

depending on the theory of the linear transformation of the double Theta-functions: it may also be proved by the methods of Sections 2 and 3. The equation (5 . 24) may be written in the form

$$(5\,.\,241)\qquad 1+\sum_1^{\infty} r(n)e^{-s\sqrt{n}} = \frac{2\pi}{s^2}+2\pi s\sum_1^{\infty}\frac{r(n)}{(s^2+4\pi^2 n)^{\frac{3}{2}}};$$

and reveals at once the existence of the algebraical infinities of the function

$$\Sigma\, r(n)e^{-s\sqrt{n}},$$

on the imaginary axis, which play the chief part in the argument.

I also proved the more general formula

$$(5\,.\,25)\quad \Sigma\Sigma e^{-s\sqrt{(\alpha m^2+\beta mn+\gamma n^2)}} = \frac{4\pi s}{\Delta}\Sigma\Sigma\left\{s^2+\frac{16\pi^2}{\Delta^2}(\alpha m^2+\beta mn+\gamma n^2)\right\}^{-\frac{3}{2}},$$

where α, β, γ are integers, and α and $\Delta^2 = 4\alpha\gamma-\beta^2$ positive. This formula enables us to obtain a corresponding result for the function $\mathrm{E}(x)$ which bears to the ellipse

$$\alpha u^2+\beta uv+\gamma u^2 = x$$

the same relation that $\mathrm{P}(x)$ bears to the circle $u^2+v^2 = x$.*

VI.

The Series $\Sigma\dfrac{d(n)}{\sqrt{n}}e^{-it\sqrt{n}}$.

6 . 1. The function $L(s)$ of 3 . 2 is, near the singular point $x = 2$, of the form

$$\frac{2\pi}{\sqrt{(2-x)}}+P\{\sqrt{(2-x)}\},$$

where P denotes a power series convergent for sufficiently small values of

* If $E(x)$ is the number of points with integral coordinates which lie in or on the ellipse, then

$$E(x) = \frac{2\pi x}{\Delta}+\mathrm{E}(x).$$

Landau has proved that $\qquad \mathrm{E}(x) = O(x^{\frac{1}{3}})$:

see his memoirs already quoted.

A full proof of the results stated in 5 . 2 will be found in my paper "On the representation of a number as the sum of two squares", *Quarterly Journal*, Vol. 46, pp. 263–283. This paper was written after the present one. See also the additional note at the end.

the variable. It follows that $F(s)$ is regular for $s = it$, or of the form

$$(6.11) \qquad \frac{e^{\frac{1}{4}\pi i} d(q)}{q^{\frac{1}{4}}} \sqrt{\left(\frac{2\pi}{s-it}\right)} + P\{\sqrt{(s-it)}\},$$

according as t is not or is of the form $\pm 4\pi\sqrt{q}$, where q is an integer.

It has been proved by Dr. Marcel Riesz* that, *if*

$$(6.12) \qquad a_1 + a_2 + \ldots + a_n = o(e^{\lambda_n c}) \quad (c > 0),$$

then the Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$, *obviously convergent for* $\sigma > c$, *is convergent at every regular point of the line* $\sigma = c$.

The condition (6.12) is certainly satisfied if

$$(6.121) \qquad a_n = o\{(\lambda_n - \lambda_{n-1}) e^{\lambda_{n-1} c}\},$$

and if $\lambda_n - \lambda_{n-1} = O(1)$, we may take $c = 0$.† If, in particular we take $\lambda_n = \sqrt{n}$, we obtain the theorem: *if*

$$(6.122) \qquad a_n = o\left(\frac{1}{\sqrt{n}}\right)$$

then the series $\Sigma a_n e^{-s\sqrt{n}}$ *is convergent at every regular point of the line* $\sigma = 0$. The actual theorem which I wish to use is however not quite this, but a slightly less precise theorem, which can be proved in substantially the same way, viz. *if*

$$(6.123) \qquad a_n = o(n^{-\frac{1}{2}+\delta}),$$

for every positive δ, *then*

$$(6.124) \qquad a_1 e^{-s\sqrt{1}} + a_2 e^{-s\sqrt{2}} + \ldots + a_n e^{-s\sqrt{n}} = o(n^{\delta})$$

at all regular points on the imaginary axis.

Observing that

$$(6.13) \qquad d(n) = o(n^{\delta}),$$

---

* "Sur les séries de Dirichlet et les séries entières", *Comptes Rendus*, 22 Nov. 1909. This theorem is Theorem 37 of our joint tract "The general theory of Dirichlet's series". Detailed proofs have been published only in the special cases in which $\lambda_n = n$ (Riesz, "Über einen Satz des Herrn Fatou", *Journal für Math.*, Vol. 140, pp. 89-99) and $\lambda_n = \log n$ (Landau, "Über die Bedeutung einiger neuen Grenzwertsätze der Herrn Hardy und Axer", *Prace matematyczno fizyczne*, Vol. 21, pp. 97-177, and in particular pp. 151-167).

† Cf. Hardy and Riesz, *loc. cit.* The theorem then reduces, when $\lambda_n = n$, to a well known theorem of Fatou.

we see that

$$(6.14)\qquad \sum_{1}^{n} \frac{d(\nu)}{\sqrt{\nu}} e^{-it\sqrt{\nu}} = o(n^{\delta})$$

*for all values of t which are not of the form $\pm 4\pi\sqrt{q}$.** It follows that *the series*

$$(6.15)\qquad \sum \frac{d(n)}{n^{a}} e^{-it\sqrt{n}},$$

where $t \neq \pm 4\pi\sqrt{q}$, q being an integer, is convergent for $a > \frac{1}{2}$.

Riesz's theorems all hold uniformly in any interval of regularity, and the series (6.15) is, for any value of a greater than $\frac{1}{2}$, uniformly convergent in any interval free from exceptional points.

6.2. Suppose now that $t = 4\pi\sqrt{q}$. Then $F(s)$ is, in the neighbourhood of the point

$$s = s_0 = 4\pi i\sqrt{q},$$

of the form

$$\frac{H}{\sqrt{(s-s_0)}} + P\{\sqrt{(s-s_0)}\},$$

where H is a constant whose value is given by (3.33). Also the function defined, for $\sigma > 0$, by the series

$$\sum n^{-\frac{3}{4}} e^{-s\sqrt{n}},$$

is, near $s = 0$, of the form

$$2\sqrt{\left(\frac{\pi}{s}\right)} + P(s).\dagger$$

It follows that the function $\Phi(s)$, defined when $\sigma > 0$ by the series

$$(6.21)\qquad \sum \left\{ \frac{d(n)}{\sqrt{n}} e^{-4\pi i\sqrt{(qn)}} - \frac{H}{2\sqrt{\pi}} n^{-\frac{3}{4}} \right\} e^{-s\sqrt{n}},$$

is, in the neighbourhood of the origin, of the form

$$(6.22)\qquad P(\sqrt{s}).$$

---

* We can, in fact, replace n^{δ} by a definite function of increase less than that of any power of n, viz.,

$$2^{(1+\epsilon)\frac{\log n}{\log\log n}},$$

where ϵ is any positive number.

† The function is substantially the integral of the function $\psi(s)$ of 4.1.

We cannot apply Riesz's theorem, as we have stated it, to the series (6.21), since $P(\sqrt{s})$ is not regular for $s = 0$. But Riesz has shown that the hypothesis of regularity in his theorem may be replaced by much more general hypotheses. In particular all the results stated in 6.1 hold, for $s = 0$, whenever the function represented by the series satisfies the condition

$$|f(s)| < K|s|^a \quad (a > 0)$$

in a region
$$0 < \sigma \leqslant \sigma_1, \quad -t_1 \leqslant t \leqslant t_1;$$

a condition obviously satisfied in this case. It follows that

$$\text{(6.23)} \qquad \sum_1^n \left\{ \frac{d(\nu)}{\sqrt{\nu}} e^{-4\pi i \sqrt{(q\nu)}} - \frac{H}{2\sqrt{\pi}} \nu^{-\frac{3}{4}} \right\} = o(n^\delta),$$

and so that

$$\text{(6.24)} \qquad \sum_1^n \frac{d(\nu)}{\sqrt{\nu}} e^{-4\pi i \sqrt{(q\nu)}} = \frac{2H}{\sqrt{\pi}} n^{\frac{1}{4}} + o(n^\delta) = \frac{2(1+i)\, d(q)}{q^{\frac{1}{4}}} n^{\frac{1}{4}} + o(n^\delta).$$

We have thus proved the following theorem: *if*

$$S_n = \sum_1^n \frac{d(\nu)}{\sqrt{\nu}} e^{-it\sqrt{\nu}} \quad (t > 0),$$

then
$$S = o(n^\delta)$$

or
$$S = \frac{2(1+i)\, d(q)}{q^{\frac{1}{4}}} n^{\frac{1}{4}} + o(n^\delta),$$

according as t is not or is of the form $4\pi\sqrt{q}$, *where q is an integer. The series*

$$\sum \frac{d(\nu)}{\nu^a} e^{-it\sqrt{\nu}}$$

is convergent for $a > \frac{1}{2}$, *or for* $a > \frac{3}{4}$, *accordingly.*

The series
$$\sum \frac{\sigma(\nu)}{\nu^a} e^{-it\sqrt{\nu}},$$

where $\nu\sigma(\nu)$ is the sum of the divisors of ν, behaves more simply: it is *always* convergent for $a > \frac{1}{2}$. This has been shown by Wigert* by means of general theorems belonging to the arithmetic theory of series. The corresponding result for $d(n)$, given above, seems to lie rather deeper.

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* "Sur quelques fonctions arithmétiques", *Acta Mathematica*, Vol. 37, pp. 113–140.

6.3. The series

$$\chi(t) = \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} e^{-it\sqrt{\nu}} \tag{6.31}$$

may be discussed in the same way. It is convergent except for

$$t = \pm 4\pi\sqrt{q},$$

where q is an integer, and uniformly convergent throughout any interval of values of t free from these exceptional values.

The series

$$\frac{\lambda^{\frac{1}{4}}}{\pi\sqrt{2}} \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} \cos\{4\pi\sqrt{(\lambda\nu)} - \tfrac{1}{4}\pi\} = \frac{\lambda^{\frac{1}{4}}}{\pi\sqrt{2}} \mathfrak{R}\{e^{\frac{1}{4}\pi i}\chi(4\pi\sqrt{\lambda})\} \tag{6.32}$$

is of quite exceptional interest. It is convergent for all real values of λ, and uniformly convergent throughout any interval of values of λ which includes no integral value. At the point $\lambda = n$ it has a finite discontinuity equal to

$$d(n).$$

This circumstance at once suggests that the series must play a fundamental part in the problem of the determination of an explicit formula for

$$d(1) + d(2) + \ldots + d(n),$$

analogous to Riemann's formula for the number of primes less than x. Such a formula was first found by Voronoï*. The formula is†

$$\begin{aligned} \Delta(\lambda) - \tfrac{1}{4} &= \sum_{n \leqslant \lambda} d(n) - \lambda\log\lambda - (2\gamma - 1)\lambda - \tfrac{1}{4} \\ &= \sqrt{\lambda} \sum_{1}^{\infty} \frac{d(\nu)}{\sqrt{\nu}} [H_1\{4\pi\sqrt{(\lambda\nu)}\} - Y_1\{4\pi\sqrt{(\lambda\nu)}\}], \end{aligned} \tag{6.33}$$

where $Y_1(x)$ is the ordinary second solution of Bessel's equation, and

$$H_1(x) = \frac{2}{\pi} \int_1^{\infty} \frac{w e^{-xw}\, dw}{\sqrt{(w^2 - 1)}}$$

is one of Hankel's cylinder-functions. If we make use of the known asymptotic expansion of $Y_1(x)$, and observe that $H_1(x)$ tends exponentially

* "Sur une fonction transcendante et ses applications à la sommation de quelques séries", *Annales de l'École Normale*, Ser. 3, Vol. 21, pp. 207-268, and 459-534. See the note at the end of the paper.

† I have altered Voronoï's notation.

to zero as $x \to \infty$, we find at once that

$$(6.34)\qquad \Delta(\lambda) = \frac{\lambda^{\frac{1}{4}}}{\pi\sqrt{2}} \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} \cos\{4\pi\sqrt{(\lambda\nu)} - \tfrac{1}{4}\pi\} + R(\lambda),$$

where $R(\lambda)$ is a series absolutely and uniformly convergent for all positive values of λ. The "non-trivial" part of $\Delta(\lambda)$ is therefore precisely the series (6.32).

There are analogous results for the function $P(\lambda)$ of 5.2.*

Additional Note (23 *Aug.* 1915).

It was only while this paper was being printed that I discovered that the formula (6.33) had been proved already by Voronoï. It is owing to this that Mr. Ramanujan, in the introduction to his paper "Highly composite numbers"† attributes the formula to me. I owe my knowledge of Voronoï's memoir, quoted on p. 20, to a reference in a paper by Landau‡ of which I received a copy about a month ago.

Voronoï's proof of the formula is exceedingly elaborate and difficult, and I cannot yet claim to have mastered all its details. I have myself constructed two independent proofs. One of these is not much less difficult than Voronoï's, which it resembles in its general character. Voronoï bases his proof on a study of the function

$$g(x) = -\tfrac{1}{4}\log x - \tfrac{1}{2}\gamma - \frac{\log(4\pi^2 x)}{4\pi^2 x} + \frac{1}{2\pi^2}\sum_1^\infty d(n)\log\frac{x}{n}\left(\frac{1}{x-n} + \frac{1}{x+n}\right),$$

and I base mine on a study of the function

$$h(x) = x\sum_1^\infty \frac{d(n)}{n(\sqrt{n}-x)}.$$

If my proof is shorter than his, it is because I have at my disposal a number of theorems in the general theory of Dirichlet's series proved only since he wrote, and in particular Riesz's generalisations of Fatou's theorem, and am so able to prove the convergence of the critical series (6.32) in a comparatively simple manner.

* The proofs of these are given in my paper in the *Quarterly Journal* referred to on p. 16.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, pp. 347-409.

‡ "Zur analytischen Zahlentheorie der definiten quadratischen Formen (Über die Gitterpunkte in einen mehrdimensionalen Ellipsoid)", *Berliner Sitzungsberichte*, 20 May 1915, pp. 458-476.

My second proof is much simpler, and is in all essentials similar to that of the corresponding formula in the "Sierpinski" problem (5.2), a proof given in full in my paper in the *Quarterly Journal* referred to on p. 16. A formal deduction of this latter formula is given in another paper of Voronoï*, which also has come to my knowledge through a reference by Landau. Voronoï states explicitly that his proof is not rigorous, and mine is, so far as I can discover, the first that is.

It is evident in the light of these facts that my statement† that it was Wigert "who was the first to recognise the importance in the Analytic Theory of Numbers of series of Bessel's functions" is inaccurate. This in no way diminishes my own indebtedness to Wigert's work.

I begin my second proof of (6.33) by proving the formula

$$f(s) = \Sigma d(n)\, e^{-s\sqrt{n}} = -\frac{4}{s^2}(\log s - 1) + \tfrac{1}{4}$$
$$+\frac{1}{2\pi^2}\Sigma\frac{d(q)}{q}\left\{ l\left(-\frac{s}{2\pi\sqrt{q}}\right) - \tfrac{1}{2} l\left(-\frac{is}{2\pi\sqrt{q}}\right) - \tfrac{1}{2} l\left(\frac{is}{2\pi\sqrt{q}}\right)\right\},$$

where
$$l(x) = \int_0^{\frac{1}{2}\pi} \frac{\sin\theta\, d\theta}{(1-\frac{1}{2}x\sin\theta)^2}.$$

This formula, which is substantially equivalent to (2.21) and (3.12) of this paper, then assumes the *rôle* of (5.241), or (2.12) of my paper in the *Quarterly Journal;* and the rest of the argument differs from that of Section III of the latter paper only in formal detail, the kernel of the proof consisting, in the one case as in the other, in an application of Riesz's theorems.

So much as regards the explicit formulæ: I must also add a few remarks concerning the "Ω" results. I find from another paper of Landau's‡ which I have also received recently that he has attacked this side of the "two square" problem simultaneously with myself. My result

* "Sur le développement, à l'aide des fonctions cylindriques, des sommes doubles $\Sigma f(pm^2+2qmn+rn^2)$, où $pm^2+2qmn+rn^2$ est une forme positive à coefficients entiers", *Verhandlungen des dritten Internationalen Math.-Kongresses in Heidelberg*, 1904, pp. 241–245. Further reference to the formulæ given by Voronoï is made by Sierspinski in a paper "O sumowaniu szeregu $\Sigma\tau(n)f(n)$, gdzie $\tau(n)$ oznacza liczbę rozkładów liczby n na sumę kwadratów dwóch liczb całkowitych", *Prace matematyczno fizyczne*, Vol. 18, pp. 1–59: but it appears to me (so far as I can judge from the Polish) that Sierpinski merely quotes Voronoï's work.

† *Loc. cit.*, p. 266.

‡ "Über die Gitterpunkte in einem Kreise (Zweite Mitteilung)", *Göttinger Nachrichten*, 5 June 1915.

states a little more than his, viz.,

$$\mathrm{P}(x) = \Omega_R(x^{\frac{1}{4}}), \qquad \mathrm{P}(x) = \Omega_L(x^{\frac{1}{4}}),$$

instead of

$$\mathrm{P}(x) = \Omega(x^{\frac{1}{4}-\epsilon})$$

for every positive ϵ. But the essential part of the result, viz. that Sierpinski's index $\frac{1}{3}$ cannot be replaced by any index less than $\frac{1}{4}$, is precisely the same. I have no evidence that Landau has considered the Ω-problem for $\Delta(n)$, which is the main subject of this paper.

In the footnote at the end of Section IV of this paper I implied that the exact value of the constant K furnished by the argument was unimportant, because $x^{\frac{1}{4}}$ could in all probability be replaced by some function of slightly more rapid increase. I have since succeeded in proving that this conjecture is correct. I find in fact

$$\Delta(x) = \Omega_R\{(x \log x)^{\frac{1}{4}} \log\log x\}, \qquad \Delta(x) = \Omega_L\{(x \log x)^{\frac{1}{4}} \log\log x\};$$

and similarly

$$\mathrm{P}(x) = \Omega_R\{(x \log x)^{\frac{1}{4}}\}, \qquad \mathrm{P}(x) = \Omega_L\{(x \log x)^{\frac{1}{4}}\}.$$

The proofs are based on the explicit formulæ and an application of the theory of Diophantine approximation after the manner of Bohr and Landau* and Littlewood.†

To prove, for example, that

(1) $$\Delta(x) = \Omega_R\{(x \log x)^{\frac{1}{4}} \log\log x\}$$

is equivalent to proving that

$$\Sigma \frac{d(n)}{n^{\frac{3}{4}}} \cos\{4\pi\sqrt{(n\lambda)} - \tfrac{1}{4}\pi\} = \Omega_R\{(\log\lambda)^{\frac{1}{4}} \log\log\lambda\},$$

or that

(2) $$\mathfrak{R}F(it) = \Omega_R\{(\log t)^{\frac{1}{4}} \log\log t\},$$

where

$$F(s) = e^{\frac{1}{4}\pi i} \Sigma \frac{d(n)}{n^{\frac{3}{4}}} e^{-s\sqrt{n}}.$$

I prove first that

(3) $$\mathfrak{R}F(s) = \Omega_R\{(\log t)^{\frac{1}{4}} \log\log t\}$$

* "Über das Verhalten von $\zeta(s)$ und $\zeta_\kappa(s)$ in der Nähe der Geraden $\sigma = 1$", *Göttinger Nachrichten*, 1910, pp. 303–320.

† *L.c.* p. 4.

in the quarter-plane $\sigma > 0$, $t > 1$, *i.e.* that a positive K exists such that the inequality

$$\mathfrak{R}F(s) > K(\log t)^{\frac{1}{4}} \log\log t \tag{4}$$

has solutions in every quarter-plane $\sigma > 0$, $t > T \geqslant 1$.

I take $\sigma = 1/g$ and $N = (Hg)^2$, where g is a large positive number and H a constant to be fixed later, and I write

$$\mathfrak{R}F(s) = \Sigma \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sigma\sqrt{n}} \cos(t\sqrt{n} - \tfrac{1}{4}\pi) = \sum_1^N + \sum_{N+1}^{\infty} = S_1 + S_2. \tag{5}$$

Suppose now that η is an arbitrarily small positive number. Then it follows from Dirichlet's fundamental theorem on Diophantine approximation* that a t exists such that

$$1 < t < (1/\eta)^N,$$

and

$$|\{t\sqrt{n}\}| < \eta \quad (n = 1, 2, \ldots, N),$$

where $\{x\}$ denotes the difference of x from the nearest integer. Choosing an appropriate η, and a corresponding value of t, we have

$$S_1 > K \sum_1^N \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sqrt{n}/g}.$$

Transforming the right-hand side by partial summation, and using the formula

$$D(n) = d(1) + d(2) + \ldots + d(n) \sim n \log n,$$

we find that

$$S_1 > K\sqrt{g} \log g. \tag{6}$$

In these inequalities *the K's are independent of H.*

On the other hand

$$|S_2| < \sum_{N+1}^{\infty} \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sqrt{n}/g},$$

and similar transformations, applied to the series on the right-hand side, show that

$$|S_2| < \kappa_H \sqrt{g} \log g, \tag{7}$$

where κ_H is a number which may be made as small as we please by in-

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* See, *e.g.*, Hardy and Littlewood, "Some problems of Diophantine approximation (I)", *Acta Mathematica*, Vol. 37, pp. 155–190 (p. 159).

creasing H sufficiently. From (5), (6), and (7) it follows that

$$\mathfrak{R}F(s) > K\sqrt{g}\log g. \tag{8}$$

This inequality holds for $\sigma = 1/g$ and some corresponding value of t. But it is not difficult to show that $\mathfrak{R}F(s)$ is bounded in any strip such as $\sigma > 0$, $1 \leqslant t \leqslant T$, and so that the values of t for which (8) holds must tend to infinity with g. Since $N = (Hg)^2$ and $t < (1/\eta)^N$, we obtain (4) and so (3).

The deduction of (1) from (3) depends on the same modification of a theorem of Lindelöf that was used by Littlewood in his investigations quoted on p. 4 (footnote *). This theorem will be proved in full in our forthcoming memoir in the *Acta Mathematica*.

The corresponding result for $\Delta_k(x)$ is

$$\Delta_k(x) = \Omega_R\left\{(t\log t)^{\frac{k-1}{2k}}(\log\log t)^{k-1}\right\}.^* \tag{9}$$

It is highly probable, in the light of all that is now known, that

$$\Delta(x) = O(x^{\frac{1}{4}+\epsilon}), \qquad \Delta_k(x) = O(x^{\frac{k-1}{2k}+\epsilon}), \tag{10}$$

for every positive ϵ. All that I am able to prove in this direction, however, is that

$$\{\Delta(1)\}^2 + \{\Delta(2)\}^2 + \ldots + \{\Delta(n)\}^2 = O(n^{\frac{3}{2}+\epsilon}) \tag{11}$$

and so

$$|\Delta(1)| + |\Delta(2)| + \ldots + |\Delta(n)| = O(n^{\frac{5}{4}+\epsilon}). \tag{12}$$

If I could prove the analogous result for every k, I could deduce that

$$\zeta(\tfrac{1}{2}+ti) = O(t^{\epsilon}):$$

in particular this would follow from (10).

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* I have not yet constructed a complete proof of this result.