THE IRRATIONALITY OF $\zeta(3)$ AND APÉRY NUMBERS

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To

Maa and Dadi

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for an undergraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

Sneha Chaubey
Date:

The thesis work reported in the thesis entitled "**The Irrationality of $\zeta(3)$ and Apéry Numbers**" was carried out under my supervision, in the School of Mathematical Sciences at NISER, Bhubaneswar, India.

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Contents

Chapter 1

Introduction

The Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is one of the most important objects in modern number theory and the role played by their special values at integral arguments is the theme of many interesting mathematical questions. The prototypical result on special values of Riemann zeta function is the theorem that

$$\zeta(s) = \frac{-(2\pi is)^s B_s}{2s!}$$

for $s > 0$ and s is even and B_s are the Bernoulli numbers. This was proved by Euler in 1735. Using this identity and the transcendence of π proved by Lindemann, the value of Riemann zeta function at even integers was proved to be transcendental. The motivation behind this thesis is to investigate the arithmetic nature of the value of Riemann zeta function at odd integers.

The most affirmative result on the arithmetic nature of $\zeta(s)$ for odd s is attributed to Apéry who showed $\zeta(3)$ is irrational. In this thesis we have studied about the irrationality of $\zeta(3)$ through two perspectives. First, Apéry's proof of irrationality of $\zeta(3)$ and then Beuker's proof of irrationality using modular forms. Though both seem to be entirely different at a glance, they are indeed related. We have seen that their connection is justified using some special modular forms of the congruence group $\Gamma_1(6)$. The sequences a_n and b_n used to prove $\zeta(3)$ irrational are called Apéry numbers. These numbers satisfy congruence relation modulo prime powers. The congruence properties satisfied by the Apéry numbers are important in a way as these relations are similar to the Atkin Swinnerton-Dyer congruence relations for certain congruence subgroups [T2]. Apéry's proof is also significant as their relation with modular forms can be explained with the help of some differential equations satisfied by the generating function of the Apéry numbers. This is more vividly explained at the end of Chapter 6.

Chapter 2 of the thesis gives the basic results on algebraicity and transcendence and we give Hermite's and Lindemann's proofs of transcendence of e and π respectively. The chapter ends with a discussion on few results in diophantine approximation theory, most prominent being Dirichlet's theorem.

Chapter 3 contains the proof of irrationality of $\zeta(3)$ from Van der Poorten's informal report on Apéry's proof of the irrationality of $\zeta(3)$ written in 1978.

Chapter 4 introduces modular forms. It also discusses the Dedekind eta function and the Eisenstein series. It gives the associated Dirichlet series for a modular from, its L-series. The chapter ends with a discussion on Mellin Transforms and the Hecke's lemma by Weil and Razar.

Chapter 5 discusses Beuker's proof of irrationality of $\zeta(3)$ using modular forms. This proof is taken from F. Beukers paper on irrationality proofs using modular forms. The last section of the chapter proves the irrationality of certain combinations of L-functions using Hecke's Lemma and the results obtained in previous two sections.

The sequences obtained in the course of Apéry's proof of irrationality of $\zeta(3)$ satisfy some beautiful congruence properties. Chapter 6 gives proofs of two of these properties.

The thesis ends by discussing some of the observations made from the proof of Beukers of irrationality of $\zeta(3)$ and the relation of irrationality proof of $\zeta(3)$ with the theory of differential equations.

Chapter 2

Algebraic and Transcendental numbers

2.1 Algebraic numbers

Definition 2.1.1. (**Algebraic numbers**) A complex number α is called an algebraic number if there exists $p(x) \in \mathbb{Z}[x]$, $p \neq 0$ with $p(\alpha) = 0$.

Definition 2.1.2. (**Transcendental numbers**) A complex number α is called transcendental if it is not algebraic.

Remark 2.1.1. The property of a number being rational, irrational, algebraic or transcendental is called its arithmetic property. Many interesting properties of numbers depend on whether or not a number is algebraic or transcendental. Transcendental numbers are not scarce, in fact Cantor showed that the set of algebraic numbers form a countable set, so transcendental numbers exist and are a measure 1 set in $[0, 1]$ and hence essentially all numbers are transcendental.

Example 2.1.1. (**Algebraic numbers**) All rational numbers, square root of any rational number, the cube root of any rational number, numbers of the form $r^{\frac{p}{q}}$ where $r, p, q \in \mathbb{Q}$, $i = \sqrt{-1}$, $\sqrt{3\sqrt{2} - 5}$.

Example 2.1.2. (**Transcendental numbers**)

$$e, \pi, e^{\pi}, \sqrt{2}^{\sqrt{2}}, \zeta(3)$$

Remark 2.1.2. The transcendence of the number

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log_e(n)\right)$$

(called Euler's constant) is a well known open question in transcendental number theory. It is not even known if γ is rational or not.

2.1.1 Properties of Algebraic numbers

Theorem 2.1.1. *The set of algebraic numbers forms a countable set.*

Theorem 2.1.2. *(Cantor) The set of all real numbers is uncountable.*

The proof of the above theorem is a consequence of the following result.

Lemma 2.1.3. *Let S be the set of all binary sequences, i.e. sequences $\{y_i\}_{i \in \mathbb{Z}}$ with $y_i \in \{0, 1\}$. Then S is uncountable.*

The proof of Cantor's theorem follows from this.

Proof. (Proof of Cantor's Theorem) Consider all numbers in the interval $[0, 1]$ whose decimal expansion consists entirely of 0's and 1's. Then there is a bijection between this subset of real numbers and the set S defined in the previous lemma. But, from the Lemma 2.2.3, S is uncountable. Consequently $\mathbb{R}$ has an uncountable subset and thus $\mathbb{R}$ is uncountable. $\square$

As a consequence of Cantor's theorem we have,

Corollary 2.1.4. *The set of transcendental numbers is uncountable.*

Proof. As $\mathbb{R} = A \cup T$ where A is the set of algebraic numbers in $\mathbb{R}$ and T is the set of transcendental numbers in $\mathbb{R}$. As A is the subset of all algebraic numbers, A is countable and since $\mathbb{R}$ is uncountable from above hence T is uncountable. Now set of all transcendental numbers in $\mathbb{R}$ forms a subset of set of all transcendental numbers, thus the set of transcendental numbers is uncountable. $\square$

Lemma 2.1.5. *If α and β are algebraic numbers, then $\alpha \pm \beta$ and $\alpha\beta$ are also algebraic numbers. Also, if α, β are algebraic integers, then $\alpha \pm \beta$ and $\alpha\beta$ are also algebraic integers.*

Lemma 2.1.6. *If α is an algebraic number with minimal polynomial $g(x) \in \mathbb{Z}$ with b as the leading coefficient of $g(x)$, then $b\alpha$ is an algebraic integer.*

Lemma 2.1.7. *If α is an algebraic integer and α is rational, then α is an integer.*

2.2 Transcendence of e and π

We know there are more transcendental numbers than algebraic; we finally show the numbers e and π to be transcendental. Before that we will see a brief history on the development of proofs of transcendence of e and π. Liouville proved the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

to be transcendental, and this was one of the first numbers proven to be transcendental. Lambert (1761), Legendre (1794), Hermite (1873) and others proved π is irrational. Lindemann (1882) proved π transcendental. What about e? Euler(1737) proved that e and e^2 are irrational. Liouville(1844) proved e is not an algebraic number of degree 2, and Hermite(1873) proved e is transcendental. Liouville (1851) gave a construction for an infinite (in fact uncountable) family of transcendental numbers. In this section we will give Lindemann'a and Hermite's proof of the transcendence of π and e respectively. To prove the transcendence of both e and π we shall make use of the function,

$$I(t) = \int_0^t e^{t-u} f(u) du$$

where t is a complex number and $f(x)$ is a polynomial with complex coefficients which will be specified as per our need. Integration by parts of the expression on the right gives:

$$\begin{aligned} I(t) &= -f(u)e^{t-u} \mid_0^t + \int_0^t f'(u)e^{t-u}du \\ &= -f(t) + f(0)e^{-t}(-f'(t) + f'(0)e^{-t}) + \int_0^t f''(u)e^{t-u}du \\ &= e^t \sum_{j=0}^{d_f} f^j(0) - \sum_{j=0}^{d_f} f^j(t) \end{aligned}$$

where d_f is the degree of f also note that if $f(x) - \sum_{j=0}^{d_f} a_j x^j$, we set $\bar{f}(x) = \sum_{j=0}^{d_f} \mid a_j \mid x^j$. Then,

$$\mid I(t) \mid \leq \mid \int_0^t \mid e^{t-u} f(u) \mid du \mid \leq \mid t \mid \max \mid e^{t-u} \mid\mid \max(\mid f(u) \mid) \leq \mid t \mid e^{|t|} \bar{f}(\mid t \mid).$$

Hence we get an upper bound for $I(t)$.

2.2.1 Transcendence of e

Theorem 2.2.1. *The number e is transcendental*

Proof. Assume e is algebraic. There is a minimal polynomial

$$g(x) = \sum_{k=0}^{r} b_k x^k \in \mathbb{Z}[x],$$

with

$$g(e) = \sum_{k=0}^{r} b_k e^k = 0.$$

Let p be a large prime greater than max $\{r, |b_0|\}$ to be chosen suitably and define

$$f(x) = x^{p-1} \prod_{j=1}^{r} (x-j)^p.$$

The degree of f is $n = (r+1)p - 1$. Consider

$$\begin{aligned}
J &= b_0 I(0) + b_1 I(1) + ... + b_r I(r) = \sum_{k=0}^{r} b_k I(k) \\
&= \sum_{k=0}^{r} b_k \left(e^k \sum_{j=0}^{n} f^j(0) - \sum_{j=0}^{n} f^j(k) \right) \\
&= \sum_{j=0}^{n} f^j(0) \sum_{k=0}^{r} b_k e^k - \sum_{k=0}^{r} \sum_{j=0}^{n} b_k f^j(k) \\
&= -\sum_{k=0}^{r} \sum_{j=0}^{n} b_k f^j(k) \\
&= -b_0 \sum_{j=0}^{n} f^j(0) - \sum_{k=1}^{r} \left(\sum_{j=0}^{p-1} b_k f^j(k) - \sum_{j=p}^{n} b_k f^j(k) \right).
\end{aligned}$$

Here we have used the fact that the degree of f is $(r+1)p-1$. With a simple computation we get,

$$f^j(0) = \begin{cases} 0 & \text{if } j \leq p-2 \\ (p-1)!(-1)^p ... (-n)^p & \text{if } j = p-1 \\ \equiv 0 \pmod{p!} & \text{if } j \geq p. \end{cases}$$

For $1 \leq k \leq r$,

$$f^j(k) = \begin{cases} 0 & \text{if } j \leq p-1 \\ \equiv 0 \pmod{p!} & \text{if } j \geq p. \end{cases}$$

Now since $p >\mid b_0 \mid$ the first summand is completely divisible by $(p-1)!$ and the second summand is divisible by $p!$ hence by $(p-1)!$ thus we find that J is an integer which is divisible by $(p-1)!$ but not p and

$$\mid J \mid \geq (p-1)!.$$

Since $f(x) = x^{p-1} \prod_{j=1}^{r} (x-j)^p$ we get

$$\bar{f}(k) \leq k^{p-1} \prod_{j=1}^{r} (k+j)^p \leq (2r)^n < (2r)^{2rp}$$

for $0 \leq k \leq r$. From this we deduce that,

$$\mid J \mid \leq \sum_{j=0}^{r} \mid b_j \mid\mid I(j) \mid \leq \sum_{j=0}^{r} \mid b_j \mid j e^j \bar{f}(j) \leq r(r+1) C e^r (2r)^{2rp} < K^p,$$

where $C = \max_{1\leq k\leq r}\{\mid b_k \mid\}$, and $K = r(r+1)Ce^r(2r)^{2r}$, which is a constant not depending on p. But $\mid J \mid$ is bounded below by $(p-1)!$, thus

$$(p-1)! \leq\mid J \mid\leq K^p,$$

but p has been chosen arbitrarily large and this inequality bounds p hence we get a contradiction. Thus, e is transcendental. □

2.2.2 Transcendence of π

Theorem 2.2.2. *The number π is transcendental.*

Proof. Observe if π is algebraic, then $i\pi$ is also algebraic. Therefore it suffices to show that $\theta = i\pi$ is transcendental. Assume θ is algebraic. Let r be the degree of the minimal polynomial $g(x)$ for θ. For some $r \in \mathbb{N}$ let $\theta = \theta_1, \theta_2, ..., \theta_r$ denote the algebraic conjugates of θ. Let b be the leading coefficient of $g(x)$ (so $b\theta_j$ is an algebraic integer). Since $e^\theta = e^{i\pi} = -1$, called the Euler's identity for e,
so $(1 + e^{\theta_1})(1 + e^{\theta_2})...(1 + e^{\theta_r}) = 0$. Rewriting this expression into an expression containing 2^r number of terms we get,

$$\prod_{j=1}^{r}(1+e^{\theta_j}) = \sum_{\substack{\rho=\sum_{j=1}^{r} b_j\theta_j \\ b_j\in\{0,1\}}} e^\rho.$$

Let $\theta_1, ..., \theta_n$ denote the non-zero expressions of the form e^ρ so that

$$2^r - n + e^{\theta_1} + e^{\theta_2} + ... + e^{\theta_n} = 0.$$

Let

$$f(x) = b^{np}x^{p-1}\prod_{i=1}^{n}(x-\theta_i)^p$$

where p is an arbitrarily chosen large prime. Consider $\theta_1, \theta_2, ..., \theta_{2^r}$ first n of which are non zero. Use

$$\prod_{j=1}^{2^r}(x-\theta_j) = x^{2^r-n}\prod_{j=1}^{n}(x-\theta_j)$$

is symmetric in $\theta_1, ..., \theta_n$ thus by the fundamental theorem of elementary symmetric functions and with the use of Lemmas 2.1.5 and 2.1.6 we can check $f(x) \in \mathbb{Z}$. Define

$$J = \sum_{k=1}^{n} I(\theta_k)$$

where

$$I(t) = e^t \sum_{j=0}^{m} f^j(0) - \sum_{j=0}^{m} f^j(t)$$

where $m = (n+1)p - 1$ is the degree of $f(x)$. Substituting the value of $I(\theta_k)$ in J we get

$$\begin{aligned} J &= \sum_{k=1}^{n} \left(e^{\theta(k)} \sum_{j=0}^{m} f^j(0) - \sum_{j=0}^{m} f^j(\theta_k) \right) \\ &= -\sum_{j=0}^{m} \sum_{k=1}^{n} f^j(\theta_k) + \sum_{j=0}^{m} f^j(0) \sum_{k=1}^{n} e^{\theta_k} \\ &= -(2^r - n) \sum_{j=0}^{m} f^j(0) - \sum_{j=0}^{m} \sum_{k=1}^{n} f^j(\theta_k). \end{aligned}$$

Observe that $\sum_{k=1}^{n} f^j(\theta_k)$ is a symmetric polynomial in $b\theta_1, ..., b\theta_n$ with integer coefficients and thus a symmetric polynomial with integer coefficients in the 2^r numbers, $b\theta_i$. By the fundamental theorem of symmetric functions, this sum is a rational number and so is an integer. Since, $f^j(\theta_k) = 0$ for $j < p$. The double sum in the expression for J above is an integer divisible by $p!$ for $j \geq p$. Also,

$$f^{p-1}(0) = b^{np}(-1)^{np}(p-1)!(\theta_1\theta_2...\theta_n)^p.$$

From the fundamental theorem of elementary symmetric functions and Lemmas 2.1.5 and 2.1.6, we find $f^{p-1}(0)$ is an integer divisible by $(p-1)!$, if p is sufficiently large, then $f^{p-1}(0)$ is not divisible by p. If $p > 2^r - n$, then $\mid J \mid \geq (p-1)!$ On the other hand using the upper bound for $\mid I(t) \mid$, we get

$$\mid J \mid \leq \sum_{k=1}^{n} \mid \theta_k \mid e^{|\theta_k|} \bar{f}(\theta_k) \leq C^p$$

where C is a constant independent of p. Hence we obtain a contradiction since p was arbitrary. Thus, π is transcendental. $\square$

2.3 Rational Approximations

We discuss the problem of approximating a given number ξ (usually irrational), by a rational fraction $r = \frac{p}{q}$ (we suppose throughout that $0 < \xi < 1$ and that $\frac{p}{q}$ is irreducible, i.e. it is expressed in its lowest terms) i.e.,

given ξ and $\epsilon > 0$, there exists $r = \frac{p}{q}$ such that

$$|r - \xi| = \left|\frac{p}{q} - \xi\right| \leq \epsilon.$$

Dirichlet (1842) proved that for any real irrational number, there are infinitely many rational numbers approximating it very closely.

Theorem 2.3.1. *(Dirichlet's Theorem) Given ξ be a real number and η, a positive integer. If ξ is irrational, then there exists an infinite set of integers p, q where $0 < q \leq n$, such that $\left|\xi - \frac{p}{q}\right| \leq \frac{1}{q^2}$.*

Remark 2.3.1. If ξ is a rational and is equal to $\frac{a}{b}$ where a and b are integers. If $r \neq \xi$, then

$$|r - \xi| = \left|\frac{p}{q} - \frac{a}{b}\right| = \left|\frac{bp - aq}{bq}\right| \geq \frac{1}{bq}.$$

From the theorem we get,

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}$$

hence

$$\frac{1}{bq} \leq \left|\frac{p}{q} - \frac{a}{b}\right| < \frac{1}{q^2}.$$

From this we get, $q < b$ and hence there are only finite number of solutions to the inequality,

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof. (Dirichlet's Theorem) Denote $[x]$ as the integral part of a real number x and $\{x\}$ as the fractional part of x. Let Q be a fixed integer greater than 1. Consider the $Q+1$ numbers, $0, \{\xi\}, \{2\xi\}, \{3\xi\}, ..., \{Q\xi\}$ and the Q intervals, $\left[\frac{i}{Q}, \frac{i+1}{Q}\right]$ where $i \in \{0, 1, ..., Q-1\}$. Hence there must be an interval, say $[\frac{j}{Q}, \frac{j+1}{Q}]$ containing two of these $Q+1$ numbers, say $a\xi$ and $b\xi$ with $0 \leq a < b \leq Q$. Hence,

$$|b\xi - a\xi| \leq \frac{1}{Q}.$$

Write $a\xi = \alpha + \{a\xi\}$ and $b\xi = \beta + \{b\xi\}$ where $\alpha, \beta \in \mathbb{Z}$ are the integral parts of $a\xi$ and $b\xi$ respectively. Then,

$$\{b\xi\} - \{a\xi\} = \alpha - \beta + (b-a)\xi.$$

Put $q = b - a$ so, $0 \leq q \leq Q$. Let $p = \beta - \alpha$ then,

$$|q\xi - p| \leq \frac{1}{Q} \Rightarrow \left|\xi - \frac{p}{q}\right| \leq \frac{1}{pQ}.$$

This is true for any Q and as $q \leq Q$, $\dfrac{1}{qQ} \leq \dfrac{1}{q^2}$, we get an infinite number of solutions to the inequality,

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}$$

thus proving the theorem. □

¿From the Dirichlet's theorem we get a criterion for proving a number irrational.

Corollary 2.3.2. *A real number is irrational if there exists a $\delta > 0$ and infinite pairs $p, q \in \mathbb{Z}$ and $(p, q) = 1$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{1+\delta}}.$$

Chapter 3

Apéry's Proof of Irrationality of $\zeta(3)$

In this chapter we will discuss Apéry's proof [J6] of irrationality of the Riemann zeta function at odd integer 3. We begin with giving some preliminary results required for our proof.

3.1 Preliminaries

We begin with stating an identity involving the lowest common multiple of first n natural numbers

Lemma 3.1.1.

$$d_n = \prod_{\substack{p \le n \\ p \text{ is prime}}} p^{\left\lceil \frac{\log n}{\log p} \right\rceil}$$

where $d_n = lcm\ \{1, 2, ..., n\}$.

Proof. Suppose p is a prime number and x is a real number. Then, $n = p^x \Rightarrow x = \frac{\log n}{\log p}$ so, $\lceil x \rceil = \left\lceil \frac{\log n}{\log p} \right\rceil$ is the highest integer power of p for which $p^{\lceil x \rceil} < n$. Now suppose $\{A_1, A_2, ..., A_n\}$ is a set of positive integers where

$$A_i = {p_1}^{y_{i1}} {p_2}^{y_{i2}} ... {p_\alpha}^{y_{i\alpha}} \quad \text{for } i = 1, ..., n \quad \text{and } p_i \text{ is prime and}$$

(y_i will be 0 for many i). As $lcm\ \{A_1, A_2, ..., A_n\} = {p_1}^{m_1} {p_2}^{m_2} ... {p_\alpha}^{m_\alpha}$ where $m_j = \max\{y_i, i = 1, ..., n\}$. In the special case where $A_i = i$,

$$m_j = \left\lceil \frac{\log n}{\log p_j} \right\rceil.$$

□

Lemma 3.1.2. *Given n,*

$$\prod_{\substack{p\leq n \\ p \text{ is prime}}} p^{\lceil \frac{\log n}{\log p} \rceil} = \prod_{\substack{p\leq n \\ p \text{ is prime}}} n.$$

Proof. To prove the lemma, it suffices to show

$$\prod_{p\leq n} p^{\frac{\log n}{\log p}} = \prod_{p\leq n} n.$$

Let $p^{\dfrac{\log n}{\log p}} = y \Rightarrow \dfrac{\log n}{\log p} \log p = \log y \Rightarrow \log n = \log y \Rightarrow n = y$. Thus,

$$p^{\frac{\log n}{\log p}} = n.$$

□

The next lemma expresses $\zeta(3)$ in terms of binomial coefficients. Using this expression a double sequence converging to $\zeta(3)$ is obtained.

Lemma 3.1.3.

$$\zeta(3) =: \sum_{n-1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Proof. We will prove the lemma step by step.

Step 1:

$$\sum_{k=1}^{K} \frac{a_1 a_2 .... a_{k-1}}{(x+a_1)(x+a_2)...(x+a_k)} = \frac{1}{x} - \frac{a_1 a_2 ... a_K}{x(x+a_1)...(x+a_K)}$$

where $a_k's$ are integers. Put

$$A_0 = \frac{1}{x} \text{ and } A_K = \frac{a_1 a_2 ... a_K}{x(x+a_1)...(x+a_K)}.$$

Now,

$$\begin{aligned} \frac{a_1 a_2 .... a_{k-1}}{(x+a_1)(x+a_2)...(x+a_k)} &= \frac{a_1 a_2 .... a_{k-1}}{x(x+a_1)(x+a_2)...(x+a_{k-1})} - \frac{a_1 a_2 .... a_k}{(x+a_1)(x+a_2)...(x+a_k)} \\ &= A_{k-1} - A_k \end{aligned}$$

Hence,

$$\sum_{k=1}^{K} \frac{a_1 a_2 ....a_{k-1}}{(x+a_1)(x+a_2)...(x+a_k)} = \sum_{k=1}^{K} A_{k-1} - A_k = A_0 - A_K.$$

Step 2:

Putting $x = n^2; a_k = -k^2$ and letting $1 \leq K \leq n-1$ we have

$$\sum_{k=1}^{K} \frac{-1^2. - 2^2... - (k-1)^2}{(n^2-1)...(n^2-k^2)} = \sum_{k=1}^{K} \frac{(-1)^{k-1} 1^2 2^2 ....(k-1)^2}{(n^2-1^2)...(n^2-k^2)}.$$

$$\begin{aligned} \Rightarrow \sum_{k=1}^{n-1} \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)...(n^2-k^2)} &= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2 n 2n}{n^2(2n)!^2} \\ &= \frac{1}{n^2} - \frac{(-1)^{n-1} 2n! n!}{n^2(2n)!} \\ &= \frac{1}{n^2} - \frac{(-1)^{n-1} 2}{n^2 \binom{2n}{n}}. \end{aligned}$$

Write

$$\epsilon_{n,k} = \frac{1}{2} \frac{(k!)^2 (n-k)!}{k^3 (n+k)!}.$$

Since

$$(-1)^k n(\epsilon_{n,k} - \epsilon_{n-1,k}) = \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)...(n^2-k^2)},$$

upon expanding we get,

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} (-1)^k (\epsilon_{n,k} - \epsilon_{n-1,k}) = \sum_{k=1}^{N} (-1)^k (\epsilon_{N,k} - \epsilon_{k,k}) = \sum_{k=1}^{N} \frac{(-1)^k}{2k^3 \binom{N+k}{k} \binom{N}{k}} + \frac{1}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}.$$

From above, the right hand side of the equation

$$= \sum_{k=1}^{N} \frac{1}{n^3} - 2 \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Hence

$$\sum_{k=1}^{N} \frac{1}{n^3} - 2 \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \sum_{k=1}^{N} \frac{(-1)^k}{2k^3 \binom{N+k}{k} \binom{N}{k}} + \frac{1}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}.$$

Take $N \longrightarrow \infty$, the first term vanishes and we get

$$\sum_{k=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}.$$

$\square$

Thus, define the double sequence,

$$c_{n,k} := \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

One may remark that $c_{n,k} \to \zeta(3)$ as $n \to \infty$, uniformly in k i.e., the sequence $c_{n,n} \to \zeta(3)$ but the following lemma will show that our conclusion will be false and we cannot apply the irrationality criterion.

Lemma 3.1.4.

$$2{d_n}^3 c_{n,k} \binom{n+k}{k} \text{is an integer}$$

where $d_n = \text{lcm}(1, 2, ..., n)$.

Proof. It suffices to show,

$$2c_{n,k} \binom{n+k}{k}$$

is a rational number whose denominator expressed in lowest terms divides ${d_n}^3$. Furthermore to prove this it suffices to show that the number of times any given prime p divides the denominator of $2c_{n,k}\binom{n+k}{k}$is less then the number of times that prime divides ${d_n}^3$. Let $\text{ord}_p(x)$ mean the highest exponent of p that divides x. Observe,

$$\frac{\binom{n+k}{k}}{\binom{n+m}{m}} = \frac{\binom{n+k}{k-m}}{\binom{k}{m}}.$$

Now,

$$\begin{aligned} ord_p \binom{n}{m} &= ord_p \left[\frac{n(n+1)....(n-m-1)}{m!} \right] \leq \text{ord}_p \left(\frac{n!}{m!} \right) \\ &= ord_p(n!) - ord_p(m!) \leq ord_p(d_n) - ord_p(m) \\ &= \left\lceil \frac{\log n}{\log p} \right\rceil - ord_p(m). \end{aligned}$$

Next, we check that the denominator of each term in the expression for $2c_{n,k}\binom{n+k}{k}$ divides ${d_n}^3$. Now,

$$2c_{n,k}\binom{n+k}{k} = 2\binom{n+k}{k}\sum_{m=1}^{n}\frac{1}{m^3} + 2\binom{n+k}{k}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

The denominator of

$$2\binom{n+k}{k}\sum_{m=1}^{n}\frac{1}{m^3}$$

divides ${d_n}^3$. Hence we will check for the denominator of the second term,

$$2\binom{n+k}{k}\sum_{m=1} k\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$$

whose denominator is

$$\frac{2m^3\binom{n}{m}\binom{n+m}{m}}{\binom{n+k}{k}} = \frac{2m^3\binom{n}{m}\binom{k}{m}}{\binom{n+k}{k-m}}.$$

Hence,

$$\begin{aligned}
ord_p\left(\frac{m^3\binom{n}{m}\binom{n+m}{m}}{\binom{n+k}{k}}\right) &= ord_p\left(\frac{m^3\binom{n}{m}\binom{k}{m}}{\binom{n+k}{k-m}}\right) \\
&\leq 3ord_p(m) + \left\lceil\frac{\log n}{\log p}\right\rceil - ord_p(m) \\
&\quad + \left\lceil\frac{\log k}{\log p}\right\rceil - ord_p(m) - ord_p\binom{n+k}{k-m} \\
&\leq \left\lceil\frac{\log n}{\log p}\right\rceil + \left\lceil\frac{\log k}{\log p}\right\rceil + ord_p(m).
\end{aligned}$$

As $m \leq k \leq n$,

$$ord_p(m) \leq ord_p(d_m) \leq ord_p(d_k) \leq ord_p(d_n)$$

and therefore

$$\left\lceil\frac{\log n}{\log p}\right\rceil + \left\lceil\frac{\log k}{\log p}\right\rceil + ord_p(m) \leq \left\lceil\frac{\log n}{\log p}\right\rceil + \left\lceil\frac{\log k}{\log p}\right\rceil + ord_p(d_m).$$

But $ord_p(d_m) = \left\lceil\frac{\log m}{\log p}\right\rceil \leq 3\left\lceil\frac{\log n}{\log p}\right\rceil = ord_p({d_n}^3)$. $\square$

Corollary 3.1.5. *The Denominator of a_n divides ${d_n}^3$.*

Proof. From the definition of a_n,

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}$$

where

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$$

and from the above lemma, denominator of $c_{n,k}$ divides ${d_n}^3$ and thus denominator of a_n divides ${d_n}^3$. □

3.2 Irrationality of $\zeta(3)$

Lemma 3.2.1. *The equation,*

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2,$$

is equivalent to

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0, \quad n \geq 1.$$

Proof. The recurrence relation,

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0$$

can be obtained by replacing n by $n-1$ in

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}.$$

And hence we get

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0, \quad n \geq 1.$$

□

Lemma 3.2.2. *$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}$ and $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfy the recurrence relation,*

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2.$$

Proof. In order to show that the sequence a_n satisfies the recurrence relation, we define a double sequence,

$$B_{n,k} = 4(2n+1)[k(2k+1) - (2n+1)^2]\binom{n}{k}^2 \binom{n+k}{k}^2.$$

Step 1:

We observe

$$\begin{aligned} B_{n,k} - B_{n,k-1} &= (n+1)^3 \binom{n+1}{k}^2 \binom{n+1+k}{k}^2 - (34n^3 + 51n^2 + 27n + 5)\binom{n}{k}^2 \binom{n+k}{k}^2 \\ &\quad + n^3 \binom{n-1}{k}^2 \binom{n-1+k}{k}^2. \end{aligned}$$

Step 2:

Using the definition of $c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$ then

$$c_{n,k} - c_{n-1,k} = \frac{(-1)^k k!^2 (n-k-1)!}{n^2(n+k)!}.$$

This can be obtained as follows:

$$\begin{aligned} c_{n,k} - c_{n-1,k} &= \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} - \sum_{m=1}^{n-1} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n-1}{m}\binom{n-1+m}{m}} \\ &= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m (m-1)!^2 (n-m-1)!}{(n+m)!}. \end{aligned}$$

The fraction $\frac{(-1)^m (m-1)!^2 (n-m-1)!}{(n+m)!}$ can be split into two parts,

$$\begin{aligned} \frac{(-1)^m (m-1)!^2 (n-m-1)!}{(n+m)!} &= \frac{(-1)^m (m-1)!^2 (n-m-1)!(m^2+n^2-m^2)}{n^2(n+m)!} \\ &= \frac{(-1)^m m!^2 (n-m-1)! - (-1)^{m+1}(m-1)!^2 (n-m)!(n+m)}{n^2(n+m)!} \\ &= \frac{(-1)^m (m!)^2 (n-m-1)!}{n^2(n+m)!} - \frac{(-1)^{m+1}(m-1)!^2 (n-m)!}{n^2(n+m-1)!}. \end{aligned}$$

Thus,

$$\begin{aligned} c_{n,k} - c_{n-1,k} &= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m (m!)^2 (n-m-1)!}{n^2 (n+m)!} - \sum_{m=1}^{k} \frac{(-1)^{m-1} (m-1)!^2 (n-m)!}{n^2 (n+m-1)!} \\ &= \frac{1}{n^3} - \left(\frac{(n-1)!}{n^2 n!} \right) + \left(\frac{(-1)^k (k!)^2 (n-k-1)!}{n^2 (n+k)!} \right). \end{aligned}$$

<u>Step 3</u>:

If $b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$, define $B_{n,k} = 4(2n-1)(k(2k+1) - (2n+1)^2) b_{n,k}$ then

$$(B_{n,k} - B_{n,k-1}) c_{n,k} + (n+1)^3 b_{n-1,k} (c_{n-1,k} - c_{n,k}) - n^3 b_{n-1,k} (c_{n,k} - c_{n-1,k}) = A_{n,k} - A_{n,k-1}$$

where

$$A_{n,k} = B_{n,k} c_{n,k} + \frac{5(2n+1)(-1)^{k-1} k}{n(n+1)} \binom{n}{k} \binom{n+k}{k}.$$

<u>Step 4</u>:

Using the previous definition of a_n and $c_{n,k}$ we will show a_n satisfies:

$$n^3 u_n + (n-1)^3 u_{n-2} = 34n^3 - 51n^2 + 27n - 5) u_{n-1}, \quad n \geq 2.$$

From Lemma 3.2.1, the expression

$$n^3 u_n + (n-1)^3 u_{n-2} = 34n^3 - 51n^2 + 27n - 5) u_{n-1}, \quad n \geq 2,$$

is equivalent to

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5) u_n + n^3 u_{n-1} = 0, \quad n \geq 1.$$

Substituting a_n for u_n in the above identity,

$$(n+1)^3 \sum_{k=1}^{n} b_{n+1,k} c_{n+1,k} - P(n) \sum_{k=1}^{n} b_{n,k} c_{n,k} + n^3 \sum_{k=1}^{n} b_{n-1,k} c_{n-1,k} = 0, \quad n \geq 1,$$

where $b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ and $P(n) = 34n^3 + 51n^2 + 27n + 5$. Since for $r > n$, $\binom{n}{r} = 0$,

$$(n+1)^3 \sum_{k=1}^{n} b_{n+1,k} c_{n+1,k} - P(n) \sum_{k=1}^{n} b_{n,k} c_{n,k} + n^3 \sum_{k=1}^{n} b_{n-1,k} c_{n-1,k} = 0, \quad n \geq 1,$$

can be written as

$$\sum_{k=0}^{n+1}\left\{(n+1)^3 b_{n+1,k}c_{n+1,k} - P(n)b_{n,k}c_{n,k} + n^3 b_{n-1,k}c_{n-1,k}\right\} = 0, \quad n \geq 1. \quad (1)$$

We will now prove this expression. From Step 1,

$$B_{n,k} - B_{n,k-1} = (n+1)^3 b_{n+1,k} - P(n)b_{n,k} + n^3 b_{n-1,k}$$

where $B_{n,k} = 4(2n-1)[k(2k+1)-(2n+1)^2]b_{n,k}$ and therefore

$$-P(n)b_{n,k}c_{n,k} = (B_{n,k} - B_{n,k-1})c_{n,k} - (n+1)^3 b_{n+1,k}c_{n,k} - n^3 b_{n-1,k}c_{n,k}.$$

Substituting in (1)

$$\sum_{k=0}^{n+1}\left\{(n+1)^3 b_{n+1,k}c_{n+1,k} + (B_{n,k} - B_{n,k-1})c_{n,k} - (n+1)^3 b_{n+1,k}c_{n,k} - n^3 b_{n-1,k}c_{n,k} + n^3 b_{n-1,k}c_{n-1,k}\right\}$$

$$= \sum_{k=0}^{n+1}\left\{(n+1)^3 b_{n+1,k}(c_{n+1,k} - c_{n,k}) + (B_{n,k} - B_{n,k-1})c_{n,k} + n^3 b_{n-1,k}(c_{n-1,k} - c_{n,k})\right\}.$$

From the previous lemma this becomes

$$\sum_{k=0}^{n+1}(A_{n,k} - A_{n,k-1})$$

where

$$A_{n,k} = B_{n,k}c_{n,k} + \frac{5(2n+1)9-1)^{k-1}k\binom{n}{k}\binom{n+k}{k}}{n(n+1)}.$$

Now

$$\sum_{k=0}^{n+1} A_{n,k} - A_{n,k-1} = A_{n,n+1} - A_{n,-1}.$$

But

$$A_{n,n+1} = B_{n,n+1}c_{n,n+1} + \frac{5(2n+1)(-1)^n(n+1)\binom{n}{n+1}\binom{2n+1}{n+1}}{n(n+1)} = 0$$

(as $B_{n,n+1}$ has a factor $\binom{n}{n+1}$). Similarly $A_{n,-1} = 0$. Thus, a_n satisfies the recurrence

$$n^3 u_n + (n-1)^3 u_{n-2} = 34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2.$$

Step 5:

Finally we will show

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} b_{n,k}$$

satisfies the recurrence relation,

$$n^3 u_n + (n-1)^3 u_{n-2} = 34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2.$$

From earlier,

$$B_{n,k} = 4(2n+1)(k(2k+1)-(2n+1)^2)\binom{n}{k}^2 \binom{n+k}{k}^2 = 4(2n+1)(k(2k+1)-(2n+1)^2)b_{n,k}.$$

From Step (1),

$$B_{n,k} - B_{n,k-1} = (n+1)^3 b_{n+1,k} - P(n) b_{n,k} - n^3 b_{n-1,k} := G_{n,k}.$$

So,

$$B_{n,k} = G_{n,k} + B_{n,k-1} = G_{n,k} + G_{n,k-1} + B_{n,k-2} = G_{n,k} + G_{n,k-1} + G_{n,k-2} + \ldots.$$

Continuing in this manner and using the fact that $B_{n,r} = 0$ for $r < 0$, we get

$$B_{n,k} = \sum_{i=0}^{k} G_{n,i}.$$

Thus,

$$\begin{aligned}
B_{n,n+1} &= \sum_{i=0}^{n+1} G_{n,i} \\
&= \sum_{i=0}^{n+1} \left((n+1)^3 b_{n+1,i} - P(n) b_{n,i} - n^3 b_{n-1,i}\right) \\
&= (n+1)^3 \sum_{i=0}^{n+1} b_{n+1,i} - P(n) \sum_{i=0}^{n+1} b_{n,i} - n^3 \sum_{i=0}^{n+1} b_{n-1,i} \\
&= (n+1)^3 \sum_{i=0}^{n+1} b_{n+1,i} - P(n) \sum_{i=0}^{n} b_{n,i} - n^3 \sum_{i=0}^{n-1} b_{n-1,i}.
\end{aligned}$$

Now since $b_{n,n+1} = 0$, $b_{n-1,n+1} = 0$, $b_{n-1,n} = 0$,

$$B_{n,n+1} = (n+1)^3 b_{n+1} - P(n) b_n - n^3 b_{n-1}.$$

But $B_{n,r} = 0$ for $r > n$. Thus,

$$(n+1)^3 b_{n+1} - P(n) b_n - n^3 b_{n-1} = 0.$$

And hence b_n satisfies the recurrence relation. □

Lemma 3.2.3. *Given that a_n and b_n satisfy the recurrence relation*

$$n^3 u_n - (n-1)^3 u_{n-2} = P(n) u_{n-1}, \quad n \geq 2$$

where $P(n) = 34n^3 - 51n^2 + 27n - 5$ then

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}$$

for a_n and b_n as defined.

Proof. From the above lemma since a_n and b_n satisfy the recurrence relation, we substitute a_n and b_n in the recurrence relation to get,

$$n^3 a_n + (n-1)^3 a_{n-2} = P(n) a_{n-1},$$

$$n^3 b_n + (n-1)^3 b_{n-2} = P(n) b_{n-1}.$$

Multiplying the first equation by b_{n-1} and the second equation by a_{n-1} and subtracting gives,

$$n^3 (a_n b_{n-1} - b_n a_{n-1}) = -(n-1)^3 (a_{n-2} b_{n-1} - b_{n-2} a_{n-1}).$$

$$
\begin{aligned}
a_n b_{n-1} - b_n a_{n-1} &= \frac{(n-1)^3}{n^3}\left(a_{n-1}b_{n-2} - b_{n-1}a_{n-2}\right) \\
&= \frac{(n-1)^3}{n^3}\frac{(n-2)^3}{(n-1)^3}\left(a_{n-2}b_{n-3} - a_{n-3}b_{n-2}\right) \\
&= \vdots \\
&= \frac{(n-(n-1))^3}{n^3}\left(a_{n-(n-1)}b_{n-n} - a_{n-n}b_{n-(n-1)}\right) \\
&= \frac{1}{n^3}(a_1 b_0 - a_0 b_1) \\
&= \frac{1}{n^3}.
\end{aligned}
$$

Now from definition $a_1 = 6$, $b_0 = 1$, $a_0 = 0$, $b_1 = 5$ and so, $a_1 b_0 - a_0 b_1 = 6$ hence we get

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}.$$

□

Lemma 3.2.4. *If*

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \quad k \leq n$$

then $c_{n,k} \to \zeta(3)$ as $n \to \infty$ uniformly in k.

Proof.

$$|c_{n,k} - \zeta(3)| = \left| -\sum_{m=n+1}^{\infty} \frac{1}{m^3} - \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \right| \leq \left| \sum_{m=n+1}^{\infty} \frac{1}{m^3} \right| + \left| \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \right|.$$

Since $\sum_{m=1}^{\infty} \dfrac{1}{m^3}$ is convergent, it follows that given $\epsilon > 0$, $\forall\ n > N_1$,

$$\left| \sum_{m=n+1}^{\infty} \frac{1}{m^3} \right| < \epsilon.$$

Next we see that

$$\left| \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \right| \leq \frac{k}{2n(n+1)} \leq \frac{n}{2n(n+1)} = \frac{1}{2(n+1)} \leq \frac{1}{2n} < \epsilon, \quad \forall\ n > \frac{1}{\epsilon}.$$

Let $N_2 = \frac{1}{2\epsilon}$, $N = max(N_1, N_2)$ then for $n > N$,

$$| \, c_{n,k} - \zeta(3) \, | \leq 2\epsilon + \epsilon = 3\epsilon$$

since this convergence is independent of k so the convergence is uniform. □

Corollary 3.2.5.

$$\frac{a_n}{b_n} \to \zeta(3) \text{ as } n \to \infty.$$

Proof. Using the fact that

$$a_n = \sum_{k=1}^{n} b_{n,k} c_{n,k},$$

$$b_n = \sum_{k=1}^{n} b_{n,k}$$

and the result that if $x_{n,k}, y_{n,k} \in \mathbb{R}$ and $y_{n,k} \to L$ as $n \to \infty$ (uniformly in n) then

$$\frac{\sum_{k=1}^{n} x_{n,k} y_{n,k}}{\sum_{k=1}^{n} x_{n,k}} \to L \text{ as } \; n \to \infty.$$

We get

$$\frac{a_n}{b_n} \to \zeta(3) \text{ as } n \to \infty.$$

□

Lemma 3.2.6. *If* $a_n b_{n-1} - a_{n-1} b_n = \dfrac{6}{n^3}$, *then*

$$\zeta(3) - \frac{a_n}{b_n} = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}.$$

Proof. We have

$$\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{6}{n^3 b_n b_{n-1}}.$$

Let $\zeta(3) - \frac{a_n}{b_n} = x_n$, then

$$x_n - x_{n+1} = \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{6}{(n+1)^3 b_{n+1} b_n}.$$

Hence

$$\zeta(3) - \frac{a_n}{b_n} = \frac{6}{(n+1)^3 b_{n+1} b_n} + x_{n+1},$$

and proceeding in this manner we get,

$$\begin{aligned}\zeta(3) - \frac{a_n}{b_n} &= \frac{6}{(n+1)^3 b_{n+1} b_n} + \frac{6}{(n+2)^3 b_{n+2} b_{n+1}} + \cdots + \frac{6}{(n+m)^3 b_{n+m} b_{n+m-1}} + x_{m+n} \\ &= \sum_{k=n+1}^{m} \frac{6}{k^3 b_k b_{k-1}} + x_{n+m}.\end{aligned}$$

Now, $\zeta(3) - \frac{a_n}{b_n} \to 0$ as $n \to \infty$ and thus,

$$\mid \zeta(3) - \frac{a_n}{b_n} \mid = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}.$$

□

Theorem 3.2.7. $\zeta(3)$ *is irrational*

Proof. From the above estimation,

$$\mid \zeta(3) - \frac{a_n}{b_n} \mid = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}$$

we have

$$\zeta(3) - \frac{a_n}{b_n} = O({b_n}^{-2}).$$

The recurrence relation satisfied by the $b_n's$ can be rewritten as

$$b_n - (34 - 51n^{-1} + 27n^{-2} - 5n^{-3})b_{n-1} + (1 - 3n^{-1} + 3n^{-2} - n^{-3})b_{n-2} = 0.$$

From this relation we get the generating polynomial for b_n which is $x^2 - 34x + 1$ and it has zeros $17 \pm 12\sqrt{2} = (1 \pm \sqrt{2})^4$. From this we get $b_n = O(\alpha_n)$, where $\alpha = (1 + \sqrt{2})^4$ Now, we have already seen that $a_n's$ are rationals with denominator dividing ${d_n}^3$ where $d_n = \text{lcm}(1, 2, .., n)$ we define two new sequences p_n and q_n such that $\zeta(3) \to \dfrac{p_n}{q_n}$ as $n \to \infty$. Define

$$p_n = 2{d_n}^3 a_n,\ q_n = 2{d_n}^3 b_n.$$

Clearly $p_n, q_n \in \mathbb{Z}$ and

$$q_n = O(\alpha^n e^{3n}), \quad \zeta(3) - \frac{p_n}{q_n} = \zeta(3) - \frac{a_n}{b_n} = O({b_n}^{-2}) = O(\alpha^{-2n}).$$

Choosing

$$\delta = \frac{\log\alpha - 3}{\log\alpha + 3} = 0.080.. > 0,$$

we get

$$\log\alpha = \frac{3(1+\delta)}{(1-\delta)} \Rightarrow \alpha^{-1+\delta} = e^{-3(1+\delta)}$$

$$\Rightarrow \alpha^{-2}\alpha^{1+\delta} = e^{-3(1+\delta)}$$

$$\Rightarrow \alpha^{-2} = \alpha^{-1-\delta}e^{-3(1+\delta)}$$

$$\Rightarrow \alpha^{-2n} = (\alpha^n e^3 n)^{-(1+\delta)}.$$

Thus, $O(\alpha^{-2n}) = O({q_n}^{-(1+\delta)})$. Hence we have

$$\zeta(3) - \frac{p_n}{q_n} = O({q_n}^{-(1+\delta)}).$$

This implies $\zeta(3)$ is irrational. □

Chapter 4

Theory of Modular forms

In this chapter we discuss the theory on modular forms, in particular Eisenstein series, Dadekind eta function and L- functions associated to modular forms.

4.1 $SL(2, \mathbb{Z})$ and its congruence subgroups

Let

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

We give an action of $SL_2(\mathbb{Z})$ on the extended complex plane $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$\gamma z = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The upper half plane $H = \{z \in \mathbb{C}|\ \text{Im}\ z > 0\}$ is stable under the given action of $SL_2(\mathbb{Z})$. Let S and T be two elements of $SL(2,\mathbb{Z})$ given by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that $SL(2,\mathbb{Z})$ is generated by S and T. For any $z \in H$ we have, $Sz = \frac{-1}{z}$, $Tz = z + 1$.

A fundamental domain D is a subset of H satisfying the properties:

1. $\forall\ z \in H, \exists\ g \in G$ s.t. $gz \in D$,
2. If $z_1, z_2 \in int(D)$, then $\nexists\ g \in G$ s.t. $gz_1 = z_2$.

Theorem 4.1.1. *The region* $D = \{z \in H : |z| \geq 1, -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}\}$ *is a fundamental domain for the action of modular group* $SL(2,\mathbb{Z})$ *on* H.

Definition 4.1.1. The principal congruence subgroup of $SL(2,\mathbb{Z})$ of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Definition 4.1.2. A subgroup Γ of $SL(2,\mathbb{Z})$ is called congruence subgroup if $\Gamma \supset \Gamma(N)$ for some $N \in \mathbb{N}$, then Γ is called congruence subgroup of level N.

Besides the principal congruence subgroups, the important congruence subgroups are:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

(where " * " means "unspecified")and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

satisfying

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL(2,\mathbb{Z}).$$

4.2 Modular functions and modular forms

Let $f(z)$ be a meromorphic function on the upper half plane H and let k be an integer. Suppose that $f(z)$ satisfies the relation

$$f(\gamma z) = (cz+d)^k f(z) \ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

In particular, for the elements $\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ above condition gives,

$$f(z+1) = f(z);\ f\left(\frac{-1}{z}\right) = (-z)^k f(z).$$

Hence f is a 1-periodic function. $f(z)$ is called a modular function of weight k for $SL(2,\mathbb{Z})$ if there are finitely many non zero coefficients a_n with $n < 0$ in the fourier series expansion of f,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n, \quad \text{where} \quad q = e^{2\pi i z}.$$

Definition 4.2.1. For $k \in \mathbb{Z}$, a function f is a modular form of weight k for $SL(2,\mathbb{Z})$ if f is holomorphic on H and holomorphic at infinity ($a_n = 0 \ \forall \ n < 0$) and satisfies

$$f(\gamma z) = (cz+d)^k f(z) \ \ \forall \ \gamma \in SL_(2,\mathbb{Z}), z \in H.$$

The space of all modular forms of weight k is denoted by $M_k(SL(2,\mathbb{Z}))$.

Definition 4.2.2. A cusp form is a modular form with the constant term $a_0 = 0$ in its fourier series expansion. The space of all cusp forms of weight k are denoted by $S_k(SL(2,\mathbb{Z}))$.

Remark 4.2.1. As the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{Z})$, $f(z) = (-1)^k f(z)$. Hence if k is odd, there are no nonzero modular functions of weight k for $SL(2,\mathbb{Z})$.

Definition 4.2.3. Let $\gamma \in SL_2(\mathbb{Z})$, then slash operator of weight k on functions on $\mathbb{H}$ is defined as

$$(f|_k\gamma)(z) := (cz+d)^{-k} f(\gamma z).$$

Definition 4.2.4. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ of level N and let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if

1. f is holomorphic on $\mathbb{H}$,
2. $f|_k\alpha = f \ \forall \ \ f \in \Gamma$,
3. $f|_k\alpha$ is holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

If in addition, we have $a_0 = 0$ in the Fourier expansion of $f|_k\alpha$ for all $\alpha \in SL_2(\mathbb{Z})$, then we call f a cusp form of weight k with respect to Γ.

Example 4.2.1. Now we will explicitly construct a class of modular forms of all even weights $k > 2$, the Eisenstein series of weight k.

Definition 4.2.5. Let V be a real vector space. A subgroup Γ of V is a lattice if there exists a $\mathbb{R}-$basis $\{e_1, ...e_n\}$ of V, which is a $\mathbb{Z}-$basis for Γ, i.e. $\Gamma = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$.

Lemma 4.2.1. *Let Γ be a lattice in $\mathbb{C}$. Then the series $\sum_{\gamma\in\Gamma,\gamma\neq 0} \frac{1}{|\gamma|^\sigma}$ is convergent for $\sigma > 2$.*

Definition 4.2.6. The Eisenstein series of index k is defined by

$$G_k(z) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^{2k}}, \quad z \in H.$$

Proposition 4.2.2. *G_k is a modular form of weight k. Also $G_k(\infty) = 2\zeta(k)$.*

Proof. First we see that

$$G_k(z+1) = G_k(z)$$
$$G_k\left(\frac{-1}{z}\right) = z^{2k}G_k(z)$$

Next we show G_k is holomorphic in H. For this it is enough to show that the series converges uniformly in all compact subsets of H. Let w be a complex number. For $z \in D$ since $\mid z \mid > 1$ and $Re(z) > \frac{-1}{2}$,

$$\begin{aligned}
\mid mz+n \mid^2 &= (mz+n)(m\bar{z}+n) \\
&= m^2 \mid z \mid^2 + 2mnRe(z) + n^2 \\
&\geq m^2 - mn + n^2 \\
&= \mid m\omega + n \mid^2 \\
&\Rightarrow \frac{1}{\mid mz+n \mid^2} \leq \frac{1}{\mid mw+n \mid^2} \\
&\Rightarrow \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{|mz+n|^{2k}} \leq \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{|m\omega+n|^{2k}}.
\end{aligned}$$

But the latter series runs over all the lattice points of the lattice generated by ω and 1, hence by Lemma 4.2.1 it is convergent for $2k > 2$ i.e. $k > 1$ and hence the

Eisenstein series G_k. Since for any other point $z \in H$ there exists $g \in G$ such that $gz \in D$. From the relation $G_k(gz) = z^{2k} G_k(z)$, we show G_k converges for all $z \in H$ and hence it is holomorphic in H.

$$\begin{aligned}\lim_{z\to i\infty} G_k(z) &= \lim_{z\to i\infty} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^{2k}} \\ &= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \lim_{z\to i\infty} \frac{1}{(mz+n)^{2k}}.\end{aligned}$$

For $m \neq 0$ it vanishes but for $m = 0$ it contributes $\sum_{n\in\mathbb{Z}} \frac{1}{n^{2k}} = 2\zeta(2k)$. Thus $G_k(\infty) = 2\zeta(2k)$. Hence $G_k(z)$ is holomorphic in H including infinity and so it is a modular form. □

4.3 q-expansion of a modular form

We recall, the q-expansion of a modular form f is

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \text{ where } q = e^{i2\pi z}.$$

We compute the q-expansion coefficients for G_k.

Proposition 4.3.1. *For even integer $k > 2, z \in H$. Then,*

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where, $\sigma_k(n) = \sum_{d|n} d^k$ and the Bernoulli numbers are defined by the expansion,

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

We normalize the value of G_k at infinity by defining E_k i.e.,

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)} = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where

$$c_k = \frac{(2i\pi)^k}{(k-1)!}\frac{1}{\zeta(k)} = \frac{-2k}{B_k}.$$

We shall be mainly interested in the standard Eisenstein series,

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

4.3.1 Estimates for the coefficients in q-expansion

In this section, we give an estimate on the growth of the coefficients a_n in q-expansion of modular form f.

Proposition 4.3.2. *If $f = G_k$, the $a_n = O(n^{2k-1})$.*

Proof. From the q-expansion of G_k in Proposition 4.3.1, for some constant A,

$$a_n = A(-1)^k\sigma_{2k-1}(n)$$

$$\Rightarrow \mid a_n \mid = A\sigma_{2k-1}(n) \geq An^{2k-1}$$

$$\Rightarrow \frac{\mid a_n \mid}{n^{2k-1}} = A\sum_{d|n}\frac{d^{2k-1}}{n^{2k-1}} = A\sum_{d|n}\frac{1}{d^{2k-1}} \leq A\sum_{d=1}^{\infty}\frac{1}{d^{2k-1}} = B < \infty$$

where $B = A\zeta(2k-1)$, we have

$$An^{2k-1} \leq \mid a_n \mid \leq Bn^{2k-1}$$

Thus, $a_n = O(n^{2k-1})$. □

The next theorem is on estimates of coefficients of cusp forms which is due to Hecke.

Theorem 4.3.3. *If f is a cusp form of weight $2k$, then*

$$a_n = O(n^k).$$

Proof. Since for a cusp form $a_0 = 0$,

$$f(z) = q\left(\sum_{n=1}^{\infty} a_n q^{n-1}\right).$$

Hence, if $q \to 0$, $|f(z)| = O(q) = O(e^{-2\pi y})$, where $y = Im(z)$. Define a function $\phi(z) = |f(z)|y^k$. Note that ϕ is invariant under G and continuous in the fundamental domain D. Also as $y \to \infty$, $\phi(z) \to 0 \Rightarrow \phi(z)$ is bounded in H, i.e. $\exists$ M s.t.

$$\phi(z) \leq M \quad \forall \; z \in H,$$

$$\Rightarrow \mid f(z) \mid \leq My^{-k} \;\; \forall \;\; z \in H.$$

Using the formula to compute Fourier coefficients

$$a_n = \int_0^1 f(z)e^{-i2\pi nz}dx.$$

$$\Rightarrow |a_n| \leq My^{-k}e^{2\pi ny} \;\; \forall \;\; y > 0.$$

Putting $y = \frac{1}{n}$ we get,

$$|a_n| \leq Mn^k e^{2\pi} \Rightarrow a_n = O(n^k).$$

□

4.3.2 The Eisenstein series of weight 2

Define the Eisenstein series G_2,

$$G_2(z) = \sum_{c\in\mathbb{Z}} \sum_{d\in\mathbb{Z}'_c} \frac{1}{(cz+d)^2}$$

where $\mathbb{Z}'_c = \mathbb{Z} - \{0\}$ if $c = 0$ and $\mathbb{Z}'_c = \mathbb{Z}$ otherwise. The normalized Eisenstein series is,

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24q - 72q^2 - \ldots.$$

E_2 satisfies all the properties of a modular form except the transformation law $E_k(-1/z) = z^2E_k(z)$. In fact the Eisenstein series of weight 2 satisfies the following transformation law:

Proposition 4.3.4. *For $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

Definition 4.3.1. The Dedekind eta function $\eta : H \longrightarrow \mathbb{C}$ is defined by an infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \quad \text{for} \quad q = e^{2\pi i z}.$$

The transformation law for the Dedekind eta function is

Theorem 4.3.5. *Let $\sqrt{}$ denote the branch of the square root having non negative real part. Then*

$$\eta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z).$$

Proof. Compute the derivative of the logarithm of the eta function.

$$\begin{aligned}
\frac{d}{d\tau}\log(\eta(\tau)) &= \frac{\pi i}{12} + 2\pi i \sum_{d=1}^{\infty} \frac{dq^d}{1-q^d} \\
&= \frac{\pi i}{12} + 2\pi i \sum_{d=1}^{\infty} d \sum_{m=1}^{\infty} q^{dm} \\
&= \frac{\pi i}{12} + 2\pi i \sum_{m=1}^{\infty} \sum_{d=1}^{m} dq^{dm} \\
&= \frac{\pi i}{12} + 2\pi i \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n \\
&= \frac{\pi i}{12} E_2(\tau).
\end{aligned}$$

Thus,

$$\frac{d}{d\tau}\log(\eta(\tau)) = \frac{\pi i}{12}\tau^{-2} E_2\left(\frac{-1}{\tau}\right)$$

and

$$\frac{d}{d\tau}\log(\sqrt{-i\tau}\eta(\tau)) = \frac{1}{2\tau} + \frac{\pi i}{12}E_2(\tau) = \frac{\pi i}{12}\left(E_2(\tau) + \frac{12}{2\pi i \tau}\right).$$

Now

$$\frac{\pi i}{12}\tau^{-2}E_2(\frac{-1}{\tau}) = \frac{\pi i}{12}\left(E_2(\tau) + \frac{12}{2\pi i\tau}\right).$$

Hence the desired relation holds up to a multiplicative constant. Setting $\tau = i$ shows that the constant is 1. $\square$

Remark 4.3.1. $\eta(\tau)^{24}$ is a cusp form of weight 12 of $SL(2,\mathbb{Z})$ and also of $\Gamma_1(6)$ and so $\eta(\tau)^{24} \in S_{12}(\Gamma_1(6))$.

4.4 L-functions and Modular forms

Each modular form $f \in M_k(\Gamma_1(N))$ has an associated Dirichlet series, its L-function.

Definition 4.4.1. The L-function associated to a modular form, $f(z) = \sum_{n=0}^{\infty} a_n q^n$ for a complex variable $s \in \mathbb{C}$ is

$$L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Theorem 4.4.1. *If $f \in M_k(\Gamma_1(N))$ is a cusp form then $L(s,f)$ converges absolutely for all s with $Re(s) > \frac{k}{2}+1$. If f is not a cusp form then $L(s,f)$ converges absolutely for all s with $Re(s) > k$.*

$L(s,f)$ satisfies the functional equation,

$$(2\pi)^{-s}\Gamma(s)L(s,f) = (-1)^{\frac{k}{2}}(2\pi)^{-(k-s)}\Gamma(k-s)L(k-s,f).$$

4.4.1 Mellin Transform

If $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N)$. As $T \in \Gamma_1(N)$, we have a q-expansion rather than q_N, we set

$$g(s) := \int_0^{i\infty} f(z)z^{s-1}dz.$$

This integral is known as the Mellin transform of the function f. It converges for $Re\ s > c+1$ since for $f(z) = \sum_{n=1}^{\infty} a_n q^n$, if $|a_n| = O(n^c)$ then $|\frac{a_n}{n^c}| < \infty \quad \text{as} \quad n \to \infty$. We conclude this section by stating the Hecke's Lemma for Dirichlet L-functions.

Theorem 4.4.2 (J2). *(Hecke's Lemma by Weil's and Razar) Let* $F(\tau) = \sum_{n=0}^{\infty} a_n \frac{q^n}{\lambda}$, $G(\tau) = \sum_{n=1}^{\infty} b_n \frac{q^n}{\lambda}$,

$$f(\tau) = \frac{a_0 z^{k-1}}{(k-1)!} + (\frac{2\pi i}{\lambda})^{-(k-1)} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{\frac{qn}{\lambda}},$$

$$g(\tau) = \frac{b_0 \tau^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{b_n}{n^{k-1}} e^{\frac{qn}{\lambda}}.$$

Let $L(F,s) = \sum_{n=1}^{\infty} \left(\frac{a_n}{n^s}\right)$ *be the associated* $L-$ *function for* F. *Here* k *is a positive integer and* $\gamma \in \mathbb{C}$ *such that* $F(z) = \gamma(\tau)^{-k} G(\frac{-1}{\tau})$, *then*

$$f(\tau) - \gamma \tau^{k-2} g\left(\frac{-1}{\tau}\right) = \sum_{j=0}^{k-2} \frac{L(F, k-1-j)}{j!} \left(\frac{2\pi i}{\lambda}\right)^{-(k-1-j)} \tau^j.$$

Chapter 5

Irrationality Proofs using Modular Forms

In this chapter we discuss Beuker's proof of irrationality of $\zeta(3)$ using modular forms. Let $t(q) = \sum_{n=0}^{\infty} t_n q^n$ be a power series convergent for all $\mid q \mid < 1$. Let $w(q)$ be another analytic function on $|q| < 1$. Our aim is to study w as a function of t. In order to make it a single-valued function, assume $t_0 = 0, t_1 \neq 0$. Let $q(t)$ be the inverse of $t(q)$ in a neighbourhood of 0 and $q(0) = 0$. Let $w(q(t))$ be defined for the value of w around $t = 0$. We will be interested in determining the radius of convergence of the power series $w(q(t)) = \sum_{n=0}^{\infty} w_n t^n$.

5.1 Preliminaries

Definition 5.1.1. Let $t(q) = \sum_{n=0}^{\infty} t_n q^n$. x is a branching value of t, if either x is not in the image of t, or if $t'(y) = 0$ for some y with $t(y) = x$.

Let us assume, that t has a discrete set of branching values, say $t_1, t_2, ...$ where $t_i \neq 0 \ \forall \ i$ and $|t_1| < |t_2| < ...$ and so the radius of convergence is $|t_1|$. We shall now extend $w(q(t)$ by analytic continuation to get a bigger radius of convergence. If we continue $w(q(t))$ analytically to the disc $|t| < |t_2|$ with exception of the possible isolated singularity t_1 and it remains bounded around t_1 we can conclude that the radius of convergence is at least $|t_2|$. Thus, we can continue doing so until we have a large radius of convergence. The point of having a large radius of convergence is used in the following theorem.

Theorem 5.1.1. *Let $f_0(t), f_1(t), ..., f_k(t)$ be power series in t. Suppose that for any $n \in \mathbb{N}$, $i = 0, 1, ..., k$ the n-th coefficient in the Taylor series of f_i is rational and has denominator dividing $d^n {d_n}^r$ where r, d are certain fixed positive integers and d_n is the lowest common multiple of $1, 2, ..., n$. Suppose there exist real numbers $\theta_1, \theta_2, ..., \theta_k$ such that $f_0(t) + \theta_1 f_1(t) + \theta_2 f_2(t) + ... + \theta_k f_k(t)$ has radius of convergence ρ and infinitely many non zero Taylor coefficients. If $\rho > de^r$, then at least one of $\theta_1, ..., \theta_k$ is irrational.*

Proof. As $\rho > de^r$, there exists an $\epsilon > 0$ such that $\rho - \epsilon > de^{r(1+\epsilon)}$. Let $f_i(t) = \sum_{n=0}^{\infty} a_{in} t^n$. From the hypothesis, if $a_{in} = \dfrac{p_{in}}{q_{in}}$, then $q_{in} | d^n {d_n}^r$. Since the radius of convergence of $f_o(t) + \theta_1 f_1(t) + ... + \theta_k f_k(t)$ is ρ, we have for sufficiently large n,

$$|a_{0n} + a_{1n}\theta_1 + ... + a_{kn}\theta_k| < (\rho - \epsilon)^{-n}.$$

Suppose $\theta_1, ..., \theta_k$ are all rational and have common denominator D. Then,

$$A_n = Dd^n {d_n}^r |a_{0n} + a_{1n}\theta_1 + ... + a_{kn}\theta_k|$$

is an integer and

$$|A_n| < Dd^n {d_n}^r (\rho - \epsilon)^{-n}$$

for sufficiently large n. But for such n $d_n < e^{(1+\epsilon)n}$. Hence we get,

$$| \ A_n \ | < Dd^n e^{(1+\epsilon)nr} (\rho - \epsilon)^{-n}.$$

Since,

$$de^{r(1+\epsilon)r} (\rho - \epsilon)^{-1} < 1$$

implies that $A_n = 0$ for sufficiently large n, which is a contradiction that $A_n \neq 0$ for infinitely many n. Thus, at least one of $\theta_1, ..., \theta_k$ is irrational. □

The values for which irrationality results are obtained are integral points of Dirichlet series associated to modular forms.

Proposition 5.1.2. *Let* $F(\tau) = \sum_{n=1}^{\infty} a_n q^n$ *be a Fourier series convergent for* $\mid q \mid < 1$ *such that for some* $k, N \in \mathbb{N}$,

$$F\left(\frac{-1}{N\tau}\right) = \epsilon(-i\tau\sqrt{N})^k F(\tau),$$

where $\epsilon = \pm 1$. *Let* $f(\tau)$ *be the Fourier series*

$$f(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} q^n.$$

Let

$$L(F, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$h(\tau) = f(\tau) - \sum_{0 \le r \le \frac{k-2}{2}} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r.$$

Then

$$h(\tau) - D = (-1)^{k-1}\epsilon(-i\tau\sqrt{N})^{k-2} h\left(\frac{-1}{N\tau}\right)$$

where $D = 0$ *if* k *is odd and* $D = L(F, \frac{k}{2})\frac{(2\pi i\tau)^{(\frac{k}{2}-1)}}{(\frac{k}{2}-1)!}$ *if* k *is even. Moreover* $L(F, \frac{k}{2}) = 0$ *if* $\epsilon = -1$.

Proof. Applying the lemma of Hecke, with $G(\tau) = \epsilon\frac{F(\tau)}{(i\sqrt{N})^k}$ to obtain,

$$f(\tau) - \epsilon(-1)^{k-1}(-i\tau\sqrt{N})^{k-2} f\left(\frac{-1}{N\tau}\right) = \sum_{r=0}^{k-2} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r.$$

Splitting the summation on the right hand side into summations over $r < \frac{k}{2} - 1$, $r > \frac{k}{2} - 1$ and $r = \frac{k}{2} - 1$. We thus write,

$$\begin{aligned}\sum_{r=0}^{k-2} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r &= \sum_{r=0}^{\frac{k-4}{2}} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r + \frac{L(F, \frac{k}{2})}{(\frac{k-2}{2})!} \\ &\quad + \sum_{r=\frac{k}{2}}^{k-2} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r.\end{aligned}$$

Now using the functional equation of the Dirichlet L-function for F

$$\Lambda(s) = C\Lambda(k-s), \text{ where } \Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s).$$

For the region $r > \frac{k}{2} - 1$, putting $s = r+1$ we get,

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{r+1} r!L(F,r+1) = C\left(\frac{\sqrt{N}}{2\pi}\right)^{k-r-1} (k-r-1)!L(F,k-r-1).$$

$$\Rightarrow \frac{L(F,k-r-1)}{r!} = \epsilon(-1)^k(-i\sqrt{N})^{k-2}\left(\frac{-1}{N}\right)^{k-r-2}(2\pi i)^{k-2r-2}\frac{L(F,r+1)}{(k-r-2)!}.$$

Now substitute r by $k-2-r$. □

5.2 The group $\Gamma_1(6)$ and the irrationality of $\zeta(3)$

Define a modular function on $\Gamma_1(6)$,

$$y(\tau) = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4}$$

where,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

Using

$$y(0) = \frac{1}{9},\ y(\frac{1}{3}) = 1,\ y(\frac{1}{2}) = \infty,\ y(\infty) = 0,$$

we find the function $y(\frac{-1}{6\tau})$ is invariant on $\Gamma_1(6)$. From this one derives

$$y\left(\frac{-1}{6\tau}\right) = \frac{y(\tau)-\frac{1}{9}}{y(\tau)-1}. \tag{5.2.1}$$

Remark 5.2.1. $t(\tau) = y(\tau)\frac{1-9y(\tau)}{1-y(\tau)}$ is invariant under the involution $\tau \to \frac{-1}{6\tau}$.

Since

$$t\left(\frac{-1}{6\tau}\right) = y\left(\frac{-1}{6\tau}\right)\frac{1-9y(\frac{-1}{6\tau})}{1-y(\frac{-1}{6\tau})}.$$

Now replace $y(\frac{-1}{6\tau})$ by the identity described to get the stated result.

Remark 5.2.2.

$$t(\tau) = \left(\frac{\Delta(6\tau)\Delta(\tau)}{\Delta(3\tau)\Delta(2\tau)}\right)^{\frac{1}{2}}.$$

Proof. Write,

$$t(\tau) = 9y(\tau)y\left(\frac{-1}{6\tau}\right) = 9\left(\frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4}\right)\left(\frac{\eta(\frac{-1}{\tau})^8\eta(\frac{-1}{6\tau})^4}{\eta(\frac{-2}{6\tau})^8\eta(\frac{-3}{6\tau})^4}\right).$$

Use, $\eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$. Hence, we have

$$y\left(\frac{-1}{6\tau}\right) = \frac{1}{9}\frac{\eta(\tau)^8\eta(6\tau)^4}{\eta(3\tau)^8\eta(2\tau)^4}.$$

Now, $\Delta(\tau) = \eta(\tau)^{24}$ so

$$t(\tau) = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4} \times \frac{\eta(\tau)^8\eta(6\tau)^4}{\eta(3\tau)^8\eta(2\tau)^4} = \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}.$$

$\square$

Remark 5.2.3. The function, defined above is modular with respect to $\Gamma_1(6)$, invariant under $\tau \rightarrow \frac{1}{6\tau}$ and its zeros and poles coincide with those of $t(\tau)$.

Proposition 5.2.1. *The function* $t(i\infty) = 0,\ t(i\sqrt{6}) = (\sqrt{2}-1)^4,\ t\left(\frac{2}{5} + \frac{i}{5\sqrt{6}}\right) = (\sqrt{2}+1)^4,\ t(\frac{1}{2}) = \infty.$

Proof. By definition $t(\tau) = y(\tau)\frac{1-9y(\tau)}{1-y(\tau)}$. So, $y(\infty) = 0 \Rightarrow t(i\infty) = 0$. And, $y\left(\frac{1}{2}\right) = \infty \Rightarrow t\left(\frac{1}{2}\right) = \infty$. For $\tau = \frac{i}{\sqrt{6}}$ and $y_0 = y\left(\frac{i}{\sqrt{6}}\right)$ we find,

$$y\left(\frac{i}{\sqrt{6}}\right) = \frac{y\left(\frac{i}{\sqrt{6}}\right) - \frac{1}{9}}{y\left(\frac{i}{\sqrt{6}}\right) - 1}$$

hence $y_0 = 1 \pm \frac{2\sqrt{2}}{3}$ and correspondingly $t\left(\frac{i}{\sqrt{6}}\right) = (\sqrt{2} \pm 1)^4$. To decide which sign should be taken, one can estimate by obtaining the values numerically. $\square$

Using the above functions we show the irrationality of $\zeta(3)$.

Theorem 5.2.2. *$\zeta(3)$ is irrational.*

Proof. Construct functions

$$F(\tau) = \frac{1}{40}\left(E_4(\tau) - 36E_4(36\tau) - 7\left(4E_4(2\tau) - 9E_4(3\tau)\right)\right),$$
$$E(\tau) = \frac{1}{24}\left(-5\left(E_2(\tau) - 6E_2(6\tau)\right) + 2E_2(2\tau) - 3E_2(3\tau)\right)$$

where

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \quad ; \quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_n q^n.$$

We note that $F(\tau) \in M_4(\tau(6))$ and

$$F\left(\frac{-1}{N\tau}\right) = \epsilon\left(-i\tau\sqrt{N}\right)^k F(\tau) \Rightarrow F\left(\frac{-1}{6\tau}\right) = -36\tau^4 F(\tau).$$

Also

$$E(\tau) \in M_2(\Gamma_1(6)), \quad E\left(\frac{-1}{6\tau}\right) = -6\tau^2 E(\tau).$$

The Dirichlet series corresponding to $F(\tau)$ is

$$L(F,s) = \sum_{n=1}^{\infty}\left(\frac{6\sigma_3(n)}{n^s} - 36\frac{6\sigma_3(n)}{(6n)^s} - 28\frac{6\sigma_3(n)}{(2n)^s} + 63\frac{6\sigma_3(n)}{(3n)^s}\right).$$

$$\begin{aligned}
F(\tau) &= \frac{1}{40}\left(E_4(\tau) - 36E_4(36\tau) - 7\left(4E_4(2\tau) - 9E_4(3\tau)\right)\right) \\
&= \frac{1}{40}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n - 36\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{36n}\right)\right) \\
&\quad \frac{-7}{40}\left(4\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right) - 9\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}\right)\right) \\
&= 6\sum_{n=1}^{\infty}\sigma_3(n)q^n - 36\times 6\sum_{n=1}^{\infty}\sigma_3(n)q^{36n} - 28\times 6\sum_{n=1}^{\infty}\sigma_3(n)q^{2n} + 63\times 6\sum_{n=1}^{\infty}\sigma_3(n)q^{3n}.
\end{aligned}$$

Dirichlet series in the form of zeta function is

$$\begin{aligned}
L(F,s) &= 6(1 - 6^{2-s} - 7.2^{2-s} + 7.3^{2-s})\sum_{1}^{\infty}\frac{\sigma_3(n)}{n^s} \\
&= 6(1 - 6^{2-s} - 7.2^{2-s} + 7.3^{2-s})\zeta(s)\zeta(s-3).
\end{aligned}$$

Define $f(\tau)$ by

$$\left(\frac{d}{d\tau}\right)^3 f(\tau) = (2\pi i)^3 F(\tau) \;\; ; \;\; f(i\infty) = 0.$$

Comparing with the hypothesis of Proposition 5.1.4 here $k = 4$ and $r = 0$.

$$\begin{aligned} h(\tau) &= f(\tau) - L(F,3) - L(F,2)(2\pi i\tau) \\ &= -\epsilon(-i\tau\sqrt{6})^2 \left(f\left(\frac{-1}{6\tau}\right) - L(F,3)\right). \end{aligned}$$

$$\Rightarrow f(\tau) - L(F,3) = -6\tau^2 \left(f\left(\frac{-1}{6\tau}\right) - L(F,3)\right).$$

Since $L(F,3) = 6 \times \left(\frac{-1}{3}\right)\zeta(3)\zeta(0) = \zeta(3)$

$$6\tau^2 \left(f\left(\frac{-1}{6\tau}\right) - \zeta(3)\right) = -\left(f(\tau) - \zeta(3)\right).$$

Multiplying the above equation by $E\left(\frac{-1}{6\tau}\right) = -6\tau^2 E(\tau)$ we get,

$$E\left(\frac{-1}{6\tau}\right)\left(f\left(\frac{-1}{6\tau}\right) - \zeta(3)\right) = E(\tau)\left(f(\tau) - \zeta(3)\right). \qquad (1)$$

In general, the function $E(\tau)(f(\tau) - \zeta(3))$ can be a multivalued function of $t = t(\tau)$.

$$t(\tau) = q\prod_{n=1}^{\infty}(1-q^{6n+1})^{12}(1-q^{6n+5})^{-12} = q - 12q^2 + 66q^3 - 220q^4 + 495q^5 - \cdots$$

The inverse function expansion is

$$q = t + 12t^2 + 222t^3 + \cdots$$

We get

$$E(t) = 1 + 5t + 73t^2 + 1445t^3 + \cdots.$$

and $E(t) \in \mathbb{Z}[t]$. We also have $E(t)f(t) = \sum_{n=1}^{\infty} a_n t^n$ where $a_n \in \mathbb{Z}/{d_n}^3$. The function $t \to \tau$ has a branch point at $t = (\sqrt{2}-1)^4$ since $t\left(\frac{i}{\sqrt{6}}\right) = (\sqrt{2}-1)^4$ and $t'\left(\frac{i}{\sqrt{6}}\right) = 0$. We can expect the radius of convergence of $E(t)(f(t) - \zeta(3))$ to be $(\sqrt{2}-1)^4$ but by equation (1), the function $t \to E(t)(f(t) - \zeta(3))$ has no branch point at $(1+\sqrt{2})^4$, and thus its radius of convergence, ρ equals at least the next branching value, which is $t = (1+\sqrt{2})^4$ (since $t\left(\frac{2}{5}+\frac{i}{5\sqrt{6}}\right) = (1+\sqrt{2})^4$). Thus the radius of convergence $\rho > e^3$.

Remark 5.2.4. The function,

$$E(t)(f(t)-\zeta(3))$$

is not a polynomial since otherwise weight of $E(\tau)$ being 2, weight of $f(\tau)-\zeta(3)$ would be -2 which is not possible.

Apply Theorem 5.1.3 with the two Taylor series,

$$f_0(t) = E(t)f(t) \text{ and } f_1(t) = \zeta(3)f(t).$$

The n-th coefficient in the Taylor series of f_i is rational with denominator dividing ${d_n}^3$. Also, the radius of convergence is larger than e^3 and since $E(t)(f(t)-\zeta(3))$ is not a polynomial, it has infinitely many nonzero Taylor coefficients. Therefore from Theorem 5.1.3 we have at least one of $\theta_1 = 1$ or $\theta_2 = \zeta(3)$ is irrational. But 1 is rational, hence $\zeta(3)$ is irrational. □

5.3 Irrationality of L-functions using modular forms

In this section we prove the irrationality of some combinations of L-functions.

Theorem 5.3.1. *Let $F(\tau) = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$ and $L(F,s)$ the corresponding Dirichlet series. Then at least one of the two numbers,*

$$\pi^{-2}L(F,2) \quad \text{and} \quad L(F,3) + \frac{47L(F,2)\zeta(3)}{48\pi^2}$$

is irrational.

Proof. The function $F(\tau) \in M_4(\Gamma_1(6))$ and it is a cusp form. From the functional equation satisfied by the Dadekind-eta function we get,

$$F\left(\frac{-1}{6\tau}\right) = 36\tau^4 F(\tau).$$

Let $f(\tau)$ be defined as the Fourier series such that $\left(\frac{d}{d\tau}\right)^3 f(\tau) = (2\pi i)^3 F(\tau)$ and $f(i\infty) = 0$. Apply Proposition 5.1.4 for $h(\tau) = f(\tau) - L(F,3)$. Observe that

$$f(\tau) - L(F,3) - L(F,2)(2\pi i\tau) = (-i\tau\sqrt{6})^2 \left(f\left(\frac{-1}{6\tau}\right) - L(F,3)\right).$$

$$\Rightarrow 6\tau^2 \left(f\left(\frac{-1}{6\tau}\right) - L(F,3)\right) = f(\tau) - L(F,3) - L(F,2)(2\pi i\tau).$$

Define the function $G(\tau)$ such that

$$240G(\tau) = 13(E_4(\tau) + 36E_4(6\tau) - 37(4E_4(2\tau) + 9E_4(3\tau)).$$

Note that this function has the properties $G(i\infty) = 0,\ G\left(\frac{-1}{6\tau}\right) = 36\tau^4 G(\tau)$. In order to evaluate the corresponding Dirichlet series for $G(\tau)$ write the q-expansion of $G(\tau)$.

$$G(\tau) = 13\sum_{n=1}^{\infty} \sigma_3(n)q^n + 13 \times 36 \sum_{n=1}^{\infty} \sigma_3(n)q^{6n} - 148 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n} - 333 \sum_{n=1}^{\infty} \sigma_3(n)q^{3n}.$$

Thus the L-series is

$$L(G,s) = \left(13 + 13.6^{2-s} - 37.2^{2-s} - 37.3^{2-s}\right) \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^s}.$$

$$\Rightarrow L(G,s) = \left(13 + 13.6^{2-s} - 37.2^{2-s} - 37.3^{2-s}\right) \zeta(s)\zeta(s-3)$$

Hence,

$$L(G,3) = \frac{47}{6}\zeta(3)\ ; \quad L(G,2) = -48\zeta(2).$$

From Proposition 5.0.4

$$6\tau^2(g\left(\frac{-1}{6\tau}\right) - L(G,3)) = g(\tau) - L(G,3) - L(G,2)(2\pi i\tau), \qquad (2)$$

and so

$$6\tau^2\left(g\left(\frac{-1}{6\tau}\right)-\frac{47}{6}\zeta(3)\right)=g(\tau)-\frac{47}{6}\zeta(3)+48\zeta(2)(2\pi i\tau). \qquad (3)$$

The function,

$$h(\tau)=48\zeta(2)(f(\tau)-L(F,3))+L(F,2)\left(g(\tau)-\frac{47}{6}\zeta(3)\right)$$

satisfies the equation,

$$6\tau^2 h\left(\frac{-1}{6\tau}\right)=h(\tau).$$

This is obtained by eliminating $2\pi i\tau$ from equations (2) and (3). Consider the function

$$E(\tau)=E_2(\tau)-2E_2(2\tau)+6E_2(6\tau)-3E_2(3\tau).$$

We see, $E(\tau)\in M_2(\Gamma_1(6))$ and $E\left(\frac{-1}{6\tau}\right)=6\tau^2E(\tau)$. Consequently,

$$E\left(\frac{-1}{6\tau}\right)h\left(\frac{-1}{6\tau}\right)=E(\tau)h(\tau).$$

Repeating the arguments of the previous theorem we find

$$48\zeta(2)f(t)E(t)+L(F,2)g(t)E(t)-(48\zeta(2)L(F,3)+L(F,2)\frac{47}{6}\zeta(3))E(t)$$

is a power series in t with radius of convergence $(\sqrt{2}+1)^4$.

In order to apply Theorem 5.1.3 we have the Taylor series, $f_0(t)=E(t)f(t),\ \ f_1(t)=E(t)g(t),\ \ f_2(t)=E(t)$ where the n-th coefficient divides ${d_n}^3$. Thus the hypothesis of Theorem 5.1.3 are satisfied. On substituting the value of $\zeta(2)$ we get, at least one of the numbers

$$\pi^{-2}L(F,2)\ \ ;\ \ L(F,3)+\left(\frac{47L(F,2)\zeta(3)}{48\pi^2}\right)$$

is irrational. □

Chapter 6

Congruence Properties of Apéry numbers

The Apéry numbers a_n, b_n are defined by the finite sums,

$$a_n = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k}^2, \quad b_n = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

for every integer $n \geq 0$. These numbers were introduced by R. Apéry[1978] in proving irrationality of $\zeta(3)$ and $\zeta(2)$. Hence these numbers are called Apéry numbers. The first few Apéry numbers, $a_n{}'s$ and $b_n{}'s$ are

$$1, 5, 73, 1445, 33001 \cdots$$

and

$$1, 3, 19, 147, 1251 \cdots$$

respectively. The Apéry numbers satisfy some congruence properties. Chowla, Cowles and Cowles[J4] were the first to consider congruences for $a_n' s$. They proved

1. for $n \geq 0$, $a_{5n+1} \equiv 0 \pmod 5$ and $a_{5n+3} \equiv 0 \pmod 5$.

2. for all primes p, $a_p \equiv a_1 \pmod{p^3}$.

Beuker further found some congruences for $a_n' s$. The significance of these congruence properties and the reason for the Apéry numbers to satisfy these congruences is still unknown and is believed to be due to the interplay between the Apéry numbers and the ζ function of a certain algebraic K-3 surface. In this chapter we will discuss the following congruence properties of the Apéry numbers.

1. For $m, r \in \mathbb{N}$ and $p > 3$ prime. Then for $p \geq 3$ prime,

$$a_{mp^r-1} \equiv a_{mp^{r-1}-1} \pmod{p^{3r}}, \quad b_{mp^r-1} \equiv b_{mp^{r-1}-1} \pmod{p^{3r}}.$$

2. Let

$$f(\tau) = \eta(2\tau)^4 \eta(2\tau)^4 = \sum_{n=1}^{\infty} \gamma_n q^n$$

where $\eta(\tau)$ is the Dedekind eta function. $f(\tau)$ is a modular form of weight 4 for $\Gamma_0(4)$. Then

$$a_{\frac{(p-1)}{2}} \equiv \gamma_p \pmod{p}.$$

Remark 6.0.1. Beuker conjectured that the congruence

$$a_{\frac{(p-1)}{2}} \equiv \gamma_p \pmod{p^2}$$

also holds true for all primes $p \geq 2$. This conjecture is true and is proved by Ahlgren and Ono[J9].

6.1 Congruences of Apéry numbers

Theorem 6.1.1. *Let $m, r \in \mathbb{N}$ and $p > 3$ prime. Then*

$$a_{mp^r-1} \equiv a_{mp^{r-1}-1} \pmod{p^{3r}}.$$

Before proving this we will prove a weaker version of Theorem 6.1.1 which holds for all primes.

Theorem 6.1.2. *Let $m, r \in \mathbb{N}$ and let p be any prime. Then*

$$b_{mp^r-1} \equiv b_{mp^{r-1}-1} \pmod{p^r}, \quad a_{mp^r-1} \equiv a_{mp^{r-1}-1} \pmod{p^r}.$$

In order to prove this we require the congruences

Lemma 6.1.3. *1.* $\binom{mp^r}{k} \equiv 0 \pmod{p^r}$ $\forall m \in \mathbb{Z}, k, r \in \mathbb{N}, k \not\equiv 0 \pmod{p}$,

2. $\binom{mp^r}{kp} \equiv \binom{mp^{r-1}}{k}(-1)^{k(p-1)} \pmod{p^r}$ $\forall m \in \mathbb{Z}, k, r \in \mathbb{N}$,

3. $\binom{mp^r-1}{k} \equiv \binom{mp^{r-1}-1}{[k/p]}(-1)^{k-[k/p]} \pmod{p^r}$ $\forall m \in \mathbb{Z}, k, r \in \mathbb{N}$.

Proof. Congruence (1) is easy to show since $\binom{mp^r}{k} = \frac{mp^r}{k}\binom{mp^r-1}{k-1}$. Now $\binom{mp^r}{k}$ and $\binom{mp^r-1}{k-1}$ are integers thus $\frac{mp^r}{k}$ is also an integer but p does not divide k and so $k|m$ hence $(p^r k)|(mp^r)$. Finally we get

$$\binom{mp^r}{k} \equiv 0 \pmod{p^r}.$$

To prove the second congruence one proceeds by proving first

$$(1-X)^{mp^r} \equiv (1-X^p)^{mp^{r-1}} \pmod{p^r}$$

by induction on r. For $r = 1$ it is trivial since

$$\begin{aligned} (1-X)^{mp} &= ((1-X)^p)^m \\ &= \left(1 - pX + \frac{p(p+1)}{2}X^2 - ... - X^p\right)^m \\ &\equiv (1-X^p)^m \pmod{p}. \end{aligned}$$

Let the congruence be true for $r = k$. Now for $r = k+1$,

$$\begin{aligned} (1-X)^{mp^{k+1}} &= ((1-X)^{mp^k})^p \\ &\equiv \left((1-X^p)^{mp^{k-1}}\right)^p \pmod{p^{k+1}} \\ &= (1-X^p)^{mp^k} \pmod{p^{k+1}}, \end{aligned}$$

where the equivalence in Step 2 follows from induction. Thus we have,

$$(1-X)^{mp^r} \equiv (1-X^p)^{mp^{r-1}} \pmod{p^r}.$$

We then equate the coefficient of X^{kp} and get

$$(-1)^{kp}\binom{mp^r}{kp} \equiv \binom{mp^{r-1}}{k}(-1)^k \pmod{p^r}.$$

Hence we get congruence (2). In order to prove congruence (3), we compare the coefficients of X^k on both sides

$$\begin{aligned}(1-X)^{mp^r-1} &\equiv (1-X^p)^{mp^{r-1}-1}\frac{1-X^p}{1-X} \\ &\equiv (1-X^p)^{mp^{r-1}-1}(1+X+...+X^{p-1}) \pmod{p^r}.\end{aligned}$$

We get

$$\begin{aligned}(-1)^k\binom{mp^r-1}{k} &\equiv \binom{mp^{r-1}-1}{[\frac{k}{p}]} \pmod{p^r} \\ \Rightarrow \binom{mp^r-1}{k} &\equiv \binom{mp^{r-1}-1}{[\frac{k}{p}]}(-1)^{k-[\frac{k}{p}]} \pmod{p^r}.\end{aligned}$$

□

Now we shall prove the congruence established in Theorem 6.1.2 using the lemma above.

Proof. Notice that $\binom{n+k}{k} = (-1)^k\binom{-n-1}{k}$ we get,

$$\begin{aligned}a_{mp^r-1} &= \sum_{k=0}^{mp^r-1}\binom{mp^r-1+k}{k}\binom{mp^r-1}{k}^2 \\ &= \sum_{k=0}^{mp^r-1}\binom{mp^r-1}{k}^2\binom{-mp^r}{k}(-1)^k.\end{aligned}$$

Since by congruence(1) for $p \nmid k$, $\binom{-mp^r}{k} \equiv 0 \pmod{p^r}$, we have for $p \nmid k$,

$$a_{mp^r-1} \equiv 0 \pmod{p^r}.$$

The remaining sum is over all k from 0 to $mp^r - 1$ for which $p|k$ and so putting $k = lp$ we get

$$a_{mp^r-1} \equiv \sum_{l=0}^{mp^{r-1}-1}\binom{mp^r-1}{lp}^2\binom{-mp^r}{lp}(-1)^{lp} \pmod{p^r}.$$

Using congruences (2) and (3) we obtain

$$\begin{aligned}
a_{mp^r-1} &\equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^{r-1}-1}{l}^2 \binom{-mp^{r-1}}{l} (-1)^{lp}(-1)^{l(p-1)} \pmod{p^r} \\
&\equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^{r-1}-1}{l}^2 \binom{-mp^{r-1}}{l} (-1)^{l} \pmod{p^r} \\
&\equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^{r-1}-1}{l}^2 \binom{mp^{r-1}-1+l}{l} \pmod{p^r} \\
&\equiv a_{mp^{r-1}-1} \pmod{p^r}.
\end{aligned}$$

Similarly we prove congruences for b_n's. □

In order to prove Theorem 6.1.1, we shall make use of the following congruences which we list down in the form of lemmas and propositions. The notations used in the lemma are:

p is a fixed prime ≥ 5

r is a fixed natural number

$[x]$ is the largest integer not exceeding x

$\{x\}$ is $x - [x]$

$\prod'_{k\in V}$ or $\sum'_{k\in V}$ means that we take the product or sum over those values $k \in V$ for which $p \nmid k$

$v_p(n)$ is the number of prime factors p in n.

Lemma 6.1.4. *For any $l \in \mathbb{Z}$ we have*

$$\sum_{[\lambda p^{-r}]=l}' \frac{1}{\lambda} \equiv 0 \pmod{p^{2r}}.$$

Proof. Observe that

$$\frac{1}{2}\sum_{[\lambda p^{-r}]=l}' \left(\frac{1}{\lambda} + \frac{1}{(2l+1)p^r - \lambda}\right) = \frac{1}{2}\sum_{\substack{\lambda=p^r l \\ p\nmid\lambda}}^{p^r(l+1)-1} \left(\frac{1}{\lambda} + \frac{1}{(2l+1)p^r - \lambda}\right).$$

On expanding the summand term by term we get

$$\sum_{\substack{\lambda=p^r l\\ p\nmid\lambda}}^{p^r(l+1)-1} \frac{1}{\lambda} = \sum_{\substack{\lambda=p^r l\\ p\nmid\lambda}}^{p^r(l+1)-1} \frac{1}{(2l+1)p^r-\lambda}.$$

But

$$\sum_{[\lambda p^r]=l}{}^{'} \frac{1}{\lambda} = \sum_{\substack{\lambda=p^r l\\ p\nmid\lambda}}^{p^r(l+1)-1} \frac{1}{\lambda}.$$

Thus we have,

$$\begin{aligned}
\sum_{[\lambda p^r]=l}{}^{'} \frac{1}{\lambda} &= \frac{1}{2}\sum_{[\lambda p^{-r}]=l}{}^{'} \left(\frac{1}{\lambda}+\frac{1}{(2l+1)p^r-\lambda}\right)\\
&= \frac{1}{2}\sum_{[\lambda p^r]=l}{}^{'} \left(\frac{1}{\lambda}-\frac{1}{\lambda}\left(\frac{1}{1-\frac{(2l+1)p^r}{\lambda}}\right)\right)\\
&= \frac{1}{2}\sum_{[\lambda p^r]=l}{}^{'} \left(\frac{1}{\lambda}-\frac{1}{\lambda}\left(1+\frac{(2l+1)p^r}{\lambda}+...\right)\right)\\
&= \frac{1}{2}\sum_{[\lambda p^r]=l}{}^{'} \left(\frac{1}{\lambda}-\frac{1}{\lambda}-\frac{(2l+1)p^r}{\lambda^2}-\frac{(2l+1)^2p^{2r}}{\lambda^3}.....\right)\\
&\equiv -\frac{(2l+1)p^r}{2}\sum_{\substack{\lambda=1\\ p\nmid\lambda}}^{p^r-1}\frac{1}{\lambda^2} \equiv 0 \pmod{p^{2r}}.
\end{aligned}$$

Hence

$$\sum_{[\lambda p^{-r}]=l}{}^{'} \frac{1}{\lambda} \equiv 0 \pmod{p^{2r}}.$$

□

Lemma 6.1.5. *For any $m \in \mathbb{Z}, k \in \mathbb{N}$ we have*

1.

$$\binom{mp^r-1}{k} = \binom{mp^{r-1}-1}{[k/p]}\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{k}\frac{mp^r-\lambda}{\lambda}$$

2. *If $p \mid k$, then*
$$\binom{mp^r}{k} = \binom{mp^{r-1}}{\frac{k}{p}} \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \frac{mp^r - \lambda}{\lambda}.$$

Proof. (Proof of (1)) The idea is to write the binomial coefficient $\binom{mp^r-1}{k}$ in terms of products and then split the product into products with $p \nmid \lambda$ and one with $\lambda = \mu p$, i.e.,

$$\begin{aligned}
\binom{mp^r-1}{k} &= \frac{(mp^r-1)(mp^r-2)...(mp^r-k)}{k!} \\
&= \prod_{\lambda=1}^{k} \frac{mp^r-\lambda}{\lambda} \\
&= \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \frac{mp^r-\lambda}{\lambda} \prod_{\substack{\mu=1 \\ \lambda=\mu p}}^{[\frac{k}{p}]} \frac{mp^r-\mu p}{\mu p} \\
&= \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \frac{mp^r-\lambda}{\lambda} \prod_{\mu=1}^{[\frac{k}{p}]} \frac{mp^{r-1}-\mu}{\mu} \\
&= \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \frac{mp^r-\lambda}{\lambda} \binom{mp^{r-1}-1}{[\frac{k}{p}]}.
\end{aligned}$$

Hence part (1) is proved. □

Proof. (Proof of (2)) We will again proceed by a similar method as in part (1) by first breaking the summand into products

$$\begin{aligned}
\binom{mp^r}{k} &= \frac{mp^r}{k}\binom{mp^r-1}{k-1} \\
&= \frac{mp^{r-1}}{\frac{k}{p}}\binom{mp^{r-1}-1}{[\frac{k-1}{p}]} \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k-1} \frac{mp^r-\lambda}{\lambda}.
\end{aligned}$$

Note that $[\frac{k-1}{p}] = \frac{k}{p} - 1$ and
$$\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k-1} = \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k}.$$

Hence we get,

$$\frac{mp^{r-1}}{\frac{k}{p}}\binom{mp^{r-1}-1}{[\frac{k-1}{p}]}\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{k-1}\frac{mp^r-\lambda}{\lambda}=\binom{mp^{r-1}}{\frac{k}{p}}\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{k}\frac{mp^r-\lambda}{\lambda}.$$

Thus we have proved part (2) of the lemma

$$\binom{mp^r}{k}=\binom{mp^{r-1}}{\frac{k}{p}}\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{k}\frac{mp^r-\lambda}{\lambda}.$$

□

Lemma 6.1.6. *Let $a_k \in \mathbb{Z}_p (k = 0, 1, 2, 3...)$ be such that*

$$\sum_{[kp^{-s}]=l} a_k \equiv 0 \pmod{p^s} \text{ for any } s, l \in \mathbb{Z}_{\geq 0}.$$

Let $e \in \mathbb{N}$. Then,

1.

$$\sum_{[kp^{-r}]=l} a_k \binom{mp^r-1}{k}^2 \binom{-mp^r-1}{k}^e (-1)^{ke} \equiv 0 \pmod{p^r}\ \forall\ l, m \in \mathbb{Z}_{\geq 0}. \tag{1}$$

If in addition $a_k = 0$ for all $k \equiv 0 \pmod p$, then

2.

$$\sum_{[kp^{-r}]=l} a_k \binom{mp^r-1}{k}^2 \binom{-mp^r-1}{k-1}^e (-1)^{ke} \equiv 0 \pmod{p^r}\ \forall\ l, m \in \mathbb{Z}_{\geq 0}. \tag{2}$$

Proof. (Proof of (1)) We prove using induction on r. For $r = 0$ it is trivial. Assume it to be true for $r-1$. We then show that the right hand side of (1) which we denote by A, is zero mod p^r. To prove this we will apply

$$\binom{mp^r-1}{k}=\binom{mp^{r-1}-1}{[\frac{k}{p}]}(-1)^{\{\frac{k}{p}\}p}, \qquad m \in \mathbb{Z}.$$

This equality can be derived from Lemma (1.2.3) part (3). Using this we get,

$$\sum_{[kp^{-r}]=l} a_k \binom{mp^r-1}{k}^2 \binom{-mp^r-1}{k}^e (-1)^{ke}$$

$$= \sum_{[kp^{-r}]=l} a_k \binom{mp^r-1}{[\frac{k}{p}]}^2 \binom{-mp^{r-1}-1}{[\frac{k}{p}]}^e (-1)^{ke}(-1)^{(k-[\frac{k}{p}])e} \pmod{p^r}$$

Collecting all such terms which have the same $[\frac{k}{p}]$ from the summand above we get,

$$A \equiv \sum_{[np^{-r+1}]=l} \left(\sum_{[kp^{-1}]=n} a_k \right) \binom{mp^{r-1}-1}{n}^2 \binom{-mp^{r-1}-1}{n}^e (-1)^{en} \pmod{p^r}$$

Applying induction hypothesis for $r-1$ with the new coefficients

$$\bar{a_n} = \frac{1}{p} \sum_{[kp^{-1}]=n} a_k.$$

$\bar{a_n}$ satisfy the hypothesis of our lemma since

$$\sum_{[np^{-s}]=l} \bar{a_n} = \frac{1}{p} \sum_{[kp^{-s-1}]=l} a_k \equiv 0 \pmod{p^s} \ \forall \ s, l \in \mathbb{Z}_{\geq 0}.$$

We have

$$\sum_{[np^{-s}]=l} a_n = \frac{1}{p} \sum_{[kp^{-1}]=n} a_k = \frac{1}{p} \sum_{[\frac{k}{p^{s+1}}]=l} a_k = \frac{1}{p} \sum_{[kp^{-s-1}]=l} \equiv 0 \pmod{p^s} \ \forall \ s, l \in \mathbb{Z}$$

and so we obtain,

$$A = \sum_{[np^{-r+1}]=l} p\bar{a_n} \binom{mp^{r-1}-1}{n}^2 \binom{-mp^{r-1}-1}{n}^e (-1)^{en} \equiv \pmod{p^r}.$$

□

Proof. (Proof of (2)) To prove part (2) we apply induction to the left hand side of the congruence which is

$$\sum_{[kp^{-r}]=l} a_k \binom{mp^r-1}{k}^2 \binom{-mp^r-1}{k-1}^e (-1)^{ke}$$

and put $[\frac{k-1}{p}] = [\frac{k}{p}]$ since the sum is over $(k,p)=1$. Then we can proceed similarly as in the proof of Lemma 6.1.6 (part 1). □

We state some results used to prove Theorem 6.1.1 in the form of a lemma.

Lemma 6.1.7. *1. Let $s, t \in \mathbb{Z}_{\geq 0}$. Then,*

(a)

$$\sum_{[np^{-t}]=i} \left(\sum_{[kp^{-s}]=n}^{\prime} \frac{1}{k} \right) \left(\sum_{\lambda=1'}^{n} \frac{1}{\lambda} \right) \equiv 0 \pmod{p^{t+2s}} \ \forall \ \ i \in \mathbb{Z}.$$

(b)

$$\sum_{[kp^{-t}]=i}^{\prime} \frac{1}{k} \sum_{\lambda=1'}^{kp^s} \frac{1}{\lambda} \equiv 0 \pmod{p^{t+s}} \ \forall \ \ i \in \mathbb{Z}.$$

2. For any $l \in \mathbb{N} \cup 0, m \in \mathbb{N}$ we have,

$$\sum_{[kp^{-r}]=l}^{\prime} \frac{1}{k} \binom{mp^r - 1}{k}^2 \binom{-mp^r - 1}{k - 1} (-1)^k \equiv 0 \pmod{p^{2r}}.$$

With the aid of the above lemmas we will now prove Theorem 6.1.1 on congruences satisfied by Apéry numbers:

$$a_{mp^r - 1} \equiv a_{mp^{r-1} - 1} \pmod{p^{3r}}.$$

Proof. (Proof of Theorem 6.1.1)

We begin by writing

$$a_{mp^r-1} = \sum_{l=0}^{mp^{r-1}-1} \binom{mp^r - 1}{lp}^2 \binom{-mp^r}{lp}^2 + \sum_{\substack{k=0 \\ p \nmid k}}^{mp^r - 1} \binom{mp^r - 1}{k}^2 \binom{-mp^r}{k}^2.$$

Then we observe

$$\sum_{\substack{k=0 \\ p \nmid k}}^{mp^r-1} \binom{mp^r - 1}{k}^2 \binom{-mp^r}{k}^2 = \sum_{\substack{k=0 \\ p \nmid k}}^{mp^r-1} \frac{m^2 p^{2r}}{k^2} \binom{mp^r - 1}{k}^2 \binom{-mp^r - 1}{k - 1}^2. \tag{3}$$

Since,

$$\sum_{[kp^{-s}]=n}^{\prime} \frac{1}{k^2} \equiv 0 \pmod{p^s} \ \ \forall \ n \in \mathbb{Z}, \ \ s \in \mathbb{N},$$

we apply Lemma 6.1.6 part (2) with $a_k = k^{-2}$ if $p \nmid k$ and $a_k = 0$ otherwise and conclude that expression (3) is zero mod p^{3r}. Hence

$$a_{mp^r-1} \equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^r-1}{lp}^2 \binom{-mp^r}{lp}^2 \pmod{p^{3r}}.$$

We then apply Lemma 6.1.3 part (2) to get,

$$b_{mp^r-1} \equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^{r-1}-1}{l}^2 \binom{-mp^{r-1}}{l}^2 \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{lp} \left(1-\frac{m^2p^{2r}}{\lambda}\right)^2 \pmod{p^{3r}}.$$

Notice that

$$\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{lp} \left(1-\frac{m^2p^{2r}}{\lambda}\right)^2 \equiv 1-2m^2p^{2r}\sum_{\substack{\lambda=1 \\ p\nmid\lambda}}^{lp} \frac{1}{\lambda^2} \pmod{p^{3r}} \equiv 1 \pmod{p^{min(3r,2r+v_p(lp))}}$$

and

$$\binom{-mp^{r-1}}{l}^2 = m^2\frac{p^{2r-2}}{l^2}\binom{-mp^{r-1}-1}{l-1}^2 \equiv 0 \pmod{p^{2\max(r-v_p(lp),0)}}.$$

Thus

$$a_{mp^r-1} \equiv \sum_{l=0}^{mp^{r-1}-1} \binom{mp^{r-1}-1}{l}^2 \binom{-mp^{r-1}}{l}^2 \equiv a_{mp^{r-1}-1} \pmod{p^{3r}}.$$

The proof of

$$b_{mp^r-1} \equiv b_{mp^{r-1}-1} \pmod{p^{3r}}$$

is also established likewise. □

6.2 Congruences of Apéry Numbers relating modular forms

The congruences proved in the preceding section appears to be proven by a brute force method. The congruences in this section appears to be more interesting as the proof relates the generating function of $a_n's$ to a certain modular form.

Proposition 6.2.1. *Let p be a prime and*

$$w(t) = \sum_{n=1}^{\infty} f_n t^{n-1} dt$$

a differential form with $f_n \in \mathbb{Z}_p$, the p-adic integers, $\forall n$. Let $t(u) = \sum_{n=1}^{\infty} A_n u^n, A_n \in \mathbb{Z}_p$, $\forall n$ and suppose

$$w(t(u)) = \sum_{n=1}^{\infty} c_n u^{n-1} du.$$

Suppose there exist $\alpha_p, \beta_p \in \mathbb{Z}_p$ with $p \mid \beta_p$ such that

$$f_{mp^r} - \alpha_p f_{mp^{r-1}} + \beta_p f_{mp^{r-2}} \equiv 0 \pmod{p^r} \ \forall m, r \in \mathbb{N} \qquad (4)$$

Then

$$c_{mp^r} - \alpha_p c_{mp^{r-1}} + \beta_p c_{mp^{r-2}} \equiv 0 \quad (\text{mod } p^{)} \ \forall m, r \in \mathbb{N}. \qquad (5)$$

Moreover, if A_1 is a p-adic unit then the congruences (5) *imply* (4).

Remark 6.2.1. We will use the convention $b_{mp^s} = 0$ if $mp^s \notin \mathbb{Z}$.

Proof. We first observe that the congruence (2) is equivalent to

$$w(t) - \frac{\alpha_p}{p} w(t^p) + \frac{\beta_p}{p^2} w(t(u^{p^2})) = dv(t), \ dv(t) \in \mathbb{Z}_p[[t]] \qquad (6)$$

Since,

$$w(t) = \sum_{n=1}^{\infty} f_n t^{n-1} dt$$

$$\Rightarrow w(t^p) = \sum_{n=1}^{\infty} f_n (t^p)^{(n-1)} d(t^p)$$

$$\Rightarrow \frac{1}{p} w(t^p) = \sum_{n=1}^{\infty} \frac{f_n}{p} t^{np-p} d(t^p) = \sum_{n=1}^{\infty} \frac{b_n}{np} d(t^{pn}).$$

But we also have

$$t(u)^{np} = t(u^p)^n + nph_n(u) \qquad h_n(u) \in \mathbb{Z}_p[[u]],$$

so substituting $t(u)^{np}$ in the expression for $\frac{1}{p}w(t^p)$ we get,

$$\begin{aligned}\frac{1}{p}w(t^p) &= \sum_{n=1}^{\infty}\frac{f_n}{np}d(t(u)^{np}) \\ &= \sum_{n=1}^{\infty}\frac{f_n}{np}d(t(u^p)^n) + f_n dh_n(u) \\ &= \frac{1}{p}w(t(u^p)) + d\psi(u), \quad \psi(u) \in \mathbb{Z}_p[[u]].\end{aligned}$$

So now substituting the expression of $\frac{1}{p}w(t^p)$ in (6) we have,

$$w(t(u)) - \frac{\alpha_p}{p}w(t(u^p)) + \frac{\beta_p}{p^2}w(t(u^{p^2})) = dv(t) + \alpha_p d\psi(u) := d\varphi(u).$$

And hence equating the coefficient of mp^r from both sides we get,

$$f_{mp^r} - \alpha_p f_{mp^{r-1}} + \beta_p f_{mp^{r-2}} \equiv 0 \pmod{p^r} \ \forall m, r \in \mathbb{N}$$

Thus we showed that congruence (4) is equivalent to (6). □

The generating function of the Apéry numbers, a_n satisfies an identity in terms of products of $\eta(\tau)$.

Proposition 6.2.2. *Let*

$$\lambda(\tau) = \left(\frac{\eta(2\tau)\eta(12\tau)}{\eta(4\tau)\eta(6\tau)}\right)^6$$

Then,

$$A(\lambda^2)d(\lambda) = (\eta(2\tau))^4(\eta(4\tau))^4 - 9(\eta(6\tau))^4(\eta(12\tau))^4 dq.$$

where $A(\lambda)$ is the generating function of the Apéry numbers a_n.

Theorem 6.2.3. *Let*

$$\sum_{n=1}^{\infty}\gamma_n q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4.$$

Then for any odd prime p and any $m, r \in \mathbb{N}, m$ odd, we have

$$a_{\frac{mp^r-1}{2}} - \gamma_p a_{\frac{mp^{r-1}-1}{2}} + p^3 a_{\frac{mp^{r-2}-1}{2}} \equiv 0 \pmod{p^r}.$$

Proof. The function

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4$$

is the unique cusp form in $S_4(\Gamma_0(8))$, its corresponding Dirichlet series has an Euler product,

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^s} = \prod_{p \text{ odd}} \frac{1}{1-\gamma_p p^{-s} + p^{3-2s}}.$$

Let

$$\eta(2\tau)^4 \eta(4\tau)^4 - 9\eta(6\tau)^4 \eta(12\tau)^4 = \sum_{n=1}^{\infty} \tilde{\gamma_n} q^n.$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\tilde{\gamma_n}}{n^s} &= \sum_{n=1}^{\infty} \frac{\gamma_n - 9\gamma_{n/3}}{n^s} \\ &= \left(1 - \frac{9}{3^s}\right) \sum_{n=1}^{\infty} \frac{\gamma_n}{n^s} \\ &= \left(1 - \frac{9}{3^s}\right) \prod_{p \text{ odd}} \frac{1}{1-\gamma_p p^{-s} + p^{3-2s}}. \end{aligned}$$

Multiplying both sides of the above equation by $(1-\gamma_p p^{-s} + p^{3-2s})$ and then equating mp^r-th coefficient we get,

For any prime $p \geq 5$,

$$\tilde{\gamma}_{mp^r} - \gamma_p \tilde{\gamma}_{mp^{r-1}} + p^3 \tilde{\gamma}_{mp^{r-2}} = 0 \;\; \forall\; m, r \in \mathbb{N}.$$

For $p = 3$,

$$\tilde{\gamma}_{m3^r} - \gamma_3 \tilde{\gamma}_{mp^{r-1}} + 27 \tilde{\gamma}_{m3^{r-2}} = \begin{cases} 0 & \text{if } 9 \parallel m3^r \\ -9\gamma_m & \text{if } 3 \parallel m3^r, \;\forall\;\; m, r \in \mathbb{N} \end{cases}$$

Hence for all odd primes p,

$$\tilde{\gamma}_{mp^r} - \gamma_p \tilde{\gamma}_{mp^{r-1}} + p^3 \tilde{\gamma}_{mp^{r-2}} \equiv 0 \pmod{p^r} \;\forall m, r \in \mathbb{N}$$

From Proposition 6.1.9,

$$\sum_{n=0}^{\infty} a_n \lambda^{2n}(q) d\lambda(q) = \sum_{n=1}^{\infty} \tilde{\gamma}_n q^{n-1} dq, \quad \text{where} \quad \lambda(q) = q \prod_{n=0}^{\infty} \left(\frac{1-q^{12n+2}}{1-q^{12n+10}} \right)^6.$$

and Proposition 6.1.8 implies the congruence.

$$a_{\frac{mp^r-1}{2}} - \gamma_p a_{\frac{mp^{r-1}-1}{2}} + p^3 a_{\frac{mp^{r-2}-1}{2}} \equiv 0 \pmod{p^r}.$$

□

Chapter 7

Conclusion

The n-th Apéry numbers

$$a_n = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

satisfy the recurrence relation

$$(n+1)^3 u_{n+1} = 34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}$$

with initial values $u_0 = 1, u_1 = 5$. Consider the generating function of the Apéry numbers, $A(t) = \sum_{n=0}^{\infty} a_n t^n$. From the recurrence relation for $a_n's$, the generating function $A(t)$ satisfies the third order differential equation.

$$(t^4 - 34t^3 + t^2)y^{'''} + (6t^3 - 153t^2 + 3t)y^{''} + (7t^2 - 112t + 1)y^{'} + (t-5)y = 0.$$

One observes that this differential equation is the symmetric square of a second order differential equation with coefficients in $\mathbb{C}(t)$. A third order differential equation is the symmetric square of a second order differential equation if the third order equation is spanned by squares of solutions of second order differential equation.

To test whether a third differential equation is a symmetric square or not, we can find the second order differential equation whose symmetric square is the third order equation. This can be achieved by taking $a_1 = \frac{c_2}{3}$ and $a_0 = \frac{c_1 - {a_1}' - 2{a_1}^2}{4}$ and test if $4a_0 a_1 + 2{a_0}' = c_0$.

Using this we find that the differential equation of the generating function of the Apéry numbers,

$$(t^4 - 34t^3 + t^2)y^{'''} + (6t^3 - 153t^2 + 3t)y^{''} + (7t^2 - 112t + 1)y^{'} + (t-5)y = 0$$

is the symmetric square of a second order differential equation given by

$$(t^3 - 34t^2 + t)y'' + (2t^2 - 51t + 1)y' + \frac{1}{4}(t - 10)y = 0. \qquad (1)$$

Thus, $\sqrt{A(t)}$ is the solution of (1). Equation (1) has four singularities at $t = 0, (1 \pm \sqrt{2})^4, \infty$. Notice that all these singular points are regular singular points and hence the second order differential equation is a Fuchsian equation which is a linear ordinary differential equation whose coefficients are rational functions having only regular singular points. The local exponents of a singular point is the solution of the indicial equation corresponding to the singular point. For the singular point $t = 0$, the exponents are $0, 0$, for $(1 \pm \sqrt{2})^4$, the exponents are $0, \frac{1}{2}$ and for the singular point $t = \infty$, the local exponents are $\frac{1}{2}, \frac{1}{2}$. Consider the rational function $f(x) = \dfrac{x(1-9x)}{(1-x)}$. Now, $f(x)$ takes the value $(1 \pm \sqrt{2})^4$ at $x = \dfrac{3 \pm 2\sqrt{2}}{3}$. Moreover every point except the points $(1 \pm \sqrt{2})^4$ has two pre images. Thus, we can conclude that the rational function has degree two and ramifies above the points $(1 \pm \sqrt{2})^4$ at $x = \dfrac{(3 \pm 2\sqrt{2})}{3}$. Replacing t in equation (2) by $\dfrac{x(1-9x)}{1-x}$ and then a replacing y by $(1-x^2)^{\frac{1}{2}}y$ we get,

$$x(x-1)(9x-1)y'' + (27x^2 - 20x + 1)y' + (9x - 3)y = 0.$$

This equation is the Picard-Fuchs equation of the modular family of elliptic curves associated to $\Gamma_1(6)$[J3]. This gives a relation between the proof of irrationality of $\zeta(3)$ and modular forms on $\Gamma_1(6)$. The relation of irrationality proof with modular forms is also visible from some of the observations we make from Beukers proof of irratinality of $\zeta(3)$ using modular forms. The coefficients of the Taylor series expansion of $E(t)$ are indeed the Apéry numbers themselves. Moreover the radius of convergence of the Taylor series expansion of $E(t)f(t) - \zeta(3)E(t)$ is $(\sqrt{2}+1)^4 = \alpha$ but recall that the order of the Apéry number $b_n = O(\alpha^n)$. Some of these observations have found an explanation in Beuker's and Steinstra's paper [J7].

The most remarkable results for the arithmetic nature of Riemann zeta function at odd integers greater than 3 has been obtained by Tanguay Rivoal and others where it is proved that there are infinitely many irrational values of the Riemann zeta function at odd positive integers. Infact it was proved by Rivoal that at least one of the nine numbers $\zeta(5), \zeta(7), ..., \zeta(21)$ is irrational. Later Zudilin improved upon this and obtained that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational. The example of $\zeta(3)$ in irrationality proofs of Riemann zeta function at other odd integer values using modular forms has been used extensively without much success. It will be challenging and interesting to investigate more on the relation of modular forms with irrationality of $\zeta(3)$ and $\zeta(2)$ in particular and use it to say something on the arithmetic nature of other zeta values, $\zeta(5), \zeta(7)\cdots$ in general.

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