13. GLESER L.J., On the distribution of the number of successes in independent trials, Ann. Probability, 3,1,182-188,1975.
14. HARARY F., Theory of graphs /Russian translation/, Mir, Moscow, 1973.
15. HOEFFDING W., Probability inequalities for sums of bounded random variables, J. Amer. Statist. Assoc., 58,301,13-30,1963.
16. BUKATOVA I.L. and MIKHASEV YU.I., Evolutionary synthesis of homogeneous recognition systems, Preprint In-ta radiotekhn. i elektroniki Akad. Nauk SSSR, 8 (311), Moscow,1981.
17. VAN DER WAERDEN B.L., Mathematical statistics /Russian translation/,IIL, Moscow, 1960.
18. FELLER W., Introduction to probability theory and its applications, 2, Wiley, 1968.
19. BERNSHTEIN S.N., Theory of probability (Teoriya veroyatnstei), OGIZ, Moscow-Leningrad, 1946.
20. SAMUELS S.M., On the number of successes in independent trials, Ann. Math. Statistics, 36,4,1272-1278, 1965.
21. CHOW C.K., Statistical independence and threshold functions, IEEE Trans. Electron. Comput., 14,1,66-68,1965.
22. ZUEV YU.A., A method of improving the reliability of classification when there are several classifiers, based on the monotonicity principle, Zh. vych. Mat. i mat. Fiz., 21,1,157-166, 1981.

Translated by D.E.B.

U.S.S.R. Comput.Maths.Math.Phys.,Vol.26,No.1,pp.179-180,1986 0041-5553/86 $10.00+0.00
Printed in Great Britain

SHORT COMMUNICATIONS

AN ALGORITHM FOR MULTIPLYING 3X3 MATRICES*

O.M. MAKAROV

An algorithm for multiplying two square 3X3 matrices with commutative variables is given, requiring 22 multiplications.

The algorithms for solving this problem - well-known in the literature - require no fewer than 23 multiplications to be completed (see /1-3/). If the elements of the multiplied matrices are non-commutative, the best of the known algorithms requires the same number of multiplication to be carried out /4/.

1. Consider the problem of multiplying square matrices with commutative variables:

$$\begin{Vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{Vmatrix} \begin{Vmatrix} a_1 & a_2 & a_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{Vmatrix} = \begin{Vmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{Vmatrix}. \tag{1}$$

We shall calculate (1) using the following algorithm which requires 22 multiplications to be carried out:

$$r_1=M_5+M_{10}+M_{11}+M_{12},\quad r_2=M_8+M_{10}-M_{14}+M_{17}-M_{18}+M_{19}-M_{22},$$
$$r_3=M_1-M_{11}-M_{12}-M_{16}+M_{17}-M_{18}+M_{19}-M_{22},\quad r_4=M_6-M_{10}+M_{11}+M_{13},$$
$$r_5=M_2-M_{10}+M_{13}+M_{14}+M_{15}+M_{17},\quad r_6=M_9-M_{11}+M_{15}-M_{16}+M_{17},$$
$$r_7=M_7-M_{10}-M_{11}+M_{20}-M_{21}+M_{22},\quad r_8=M_3-M_{10}+M_{14}-M_{17}+M_{18}+M_{20}+M_{22},$$
$$r_9=M_4+M_{11}+M_{16}-M_{17}+M_{18}+M_{21},$$

where

$$M_1=(a_3+c_1-c_2)(k_1+k_7-k_8+k_9),\quad M_2=(a_2+b_1+b_2)(k_2-k_4+k_5-k_6),$$
$$M_3=(a_2+b_1+b_3)(k_3-k_4+k_5-k_6),\quad M_4=(a_3-c_2-c_3)(k_3-k_7+k_8-k_9),$$
$$M_5=(a_1-c_1+c_2)k_1,\quad M_6=(a_1+b_1+b_2)k_2,\quad M_7=(a_1+b_1+b_3+c_2+c_3)k_3,$$
$$M_8=a_2(k_1+k_4-k_5+k_6),\quad M_9=a_3(k_2+k_7-k_8+k_9),\quad M_{10}=b_1k_4,\quad M_{11}=c_2k_7,$$
$$M_{12}=(c_1-c_2)(k_1+k_7),\quad M_{13}=(b_1+b_2)(k_4-k_2),\quad M_{14}=(a_2+b_1)(k_4-k_5+k_6),$$
$$M_{15}=b_2k_6,\quad M_{16}=(a_3-c_2)(k_7-k_8+k_9),\quad M_{17}=c_2k_8,\quad M_{18}=(b_3-c_2-c_3)k_6,$$
$$M_{19}=(c_1+c_3-b_1-b_3)k_8,\quad M_{20}=(b_1+b_3)(k_4-k_3+k_6+k_8),$$
$$M_{21}=(c_2+c_3)(k_3+k_6-k_7+k_8),\quad M_{22}=(c_2+c_3-b_1-b_3)(k_6+k_8).$$

Note that this algorithm does not require multiplications by constants (unlike the algorithms in /2, 3/).

2. When synthesizing the algorithm from Section. 1 the following propositions were used:

a) to obtain a commutative algoirthm of the multiplication of a vector by a matrix of the form

$$G_1=\begin{Vmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{Vmatrix},\quad \text{where } B=\begin{Vmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{Vmatrix},$$

it is sufficient to obtain a non-commutative algorithm for multiplying a vector by a matrix of the form (see /5, 6/)

*Zh.vychisl.Mat.mat.Fiz.,26,2,293-294,1986

$$G_2=\begin{Vmatrix} B^T & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{Vmatrix};$$

b) in /4/ when synthesizing an algorithm to multiply a vector by a matrix G_1 integral solutions were obtained of the system from 729 non-linear algebraic equations with 621 unknowns.

In this paper two basic algorithms were taken to multiply the vector by the matrices:

$$\left\Vert\begin{array}{cc|cc} 0 & a_1 & & \\ a_2 & a_3 & \multicolumn{2}{c}{0} \\ \hline & & 0 & a_1-a_3 \\ \multicolumn{2}{c|}{0} & a_2-a_3 & a_3 \end{array}\right\Vert, \quad \left\Vert\begin{array}{cc|cc} 0 & g_1 & & \\ g_2 & g_3 & \multicolumn{2}{c}{0} \\ \hline & & 0 & g_2-g_3 \\ \multicolumn{2}{c|}{0} & g_1-g_3 & g_3 \end{array}\right\Vert, \qquad (2)$$

each of which requires 5 multiplications to be completed. The algorithm from Section. 1 was obtained by separating submatrices of the form (2) in the matrix G_2 (5 such submatrices can be separated). At the first steps (a step is the separation of a submatrix of the form (2) in G_2) this separation can be carried out using many methods, most of which did not enable us to carry out more than 4 steps. The basic complication when synthesizing the algoirthm from Section. 1 was the determination of a way which would enable us to separate submatrices of the form (2) in G_2 5 times. For example, the separation method which reduces to the algorithm in /3/ turned out to be a dead-end, since it did not enable us to separate more than one submatrix (2) in the matrix in which the matrix G_2 was converted after separating 4 submatrices of the form (2) in it.

REFERENCES

1. WINOGRAD S., A new algorithm for inner product. IEEE Trans. Comput. 17,7,693-694,1968.
2. WAKSMAN A., On Winograd's algorithm for inner product. IEEE Trans. Comput. 19,4,360-361, 1970.
3. BROCKET R. and DOBKIN D., On the number of multiplication required for matrix multiplication. SIAM J. Comput. 5,4,624-628,1976.
4. LADERMAN J.D., A non-commutative algorithm for multiplying 3X3 matrices using 23 multiplications. Bull. Amer. Math. Soc., 82,1,126-128,1976.
5. MAKAROV O.M., Using duality for the synthesis of an optimal algorithm involving matrix multiplication. Inform. Proc. Letters, 13,2,48-49,1981.
6. JA' JA' J., On the complexicity of bilinear forms with commutativity. Communs. ACM, 179-208.1979.
7. MAKAROV O.M., Introduction tothe theory of optimizing calculations of bilinear forms. Kiev: Nauk. dumka, 1983.

Translated by H.Z.

U.S.S.R. Comput.Maths.Math.Phys.,Vol.26,No.1,pp.180-184, 1986 0041-5553/86 $10.00+0.00
Printed in Great Britain.

THE INTEGRAL FORM OF THE SOLUTION OF A PROBLEM ON RADIO PULSE PROPAGATION IN A NON-CONDUCTING MEDIUM *

N.I. KOZLOV and L.N. SOKOLOVA

The non-stationary solution of Maxwell's equations in a non-conducting medium with cylindrical symmetry is obtained by Fourier's method.

The mixed problem with zero initial and boundary conditions is considered for Maxwell's equations. The boundary conditions are specified for the tangential components of the electric field on the surface of an ideal conductor. The medium in which the fields are considered is assumed to be non-conducting.

The field sources are external currents specified as functions of the coordinates and time. The usual statement of the problem of finding the electromagnetic fields is then as follows, see /1/: it is required to find, with $\varepsilon=1$ and $\mu=1$, the solutions of Maxwell's equations

$$\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j}_{\text{ext}},\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$$

with the boundary and initial conditions

$$\mathbf{E}_{\text{tang}}|_S=0,\qquad \mathbf{E}=\mathbf{H}=0 \text{ for } t=0.$$

Here, $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{j}_{\text{ext}}$ are the vectors of the electric and magnetic fields and of the external currents, S is the surface of the ideal conductors, c is the velocity of light, and t is the time in the laboratory coordinate system. The vector $\mathbf{j}_{\text{ext}}$ is assumed to be a given function

**Zh.vychisl.Mat.mat.Fiz.*,26,2,294-299,1986.