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# Siegel normality

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## Introduction

A famous theorem of Lindemann and Weierstrass (1885) states that for any set of **Q**-linearly independent algebraic numbers $\alpha_1, \ldots, \alpha_n$, the numbers $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over **Q**. In a classical paper [Si1] in 1929, C. L. Siegel developed a method for demonstrating the algebraic independence of values of $E$-functions satisfying linear differential equations over $\mathbf{C}(z)$. In particular, Siegel was able to show the algebraic independence of $e^{\alpha_1}, \ldots, e^{\alpha_n}$, $J_0(\xi)$, $J_0'(\xi)$ over **Q** if $\alpha_1, \ldots, \alpha_n$ are as above, $\xi$ is a nonzero algebraic number and $J_0(z)$ is the Bessel function of zero-th order. Another result is that $J_0(\xi_1), \ldots, J_0(\xi_n), J_0'(\xi_1), \ldots, J_0'(\xi_n)$ are algebraically independent if $\xi_1, \ldots, \xi_n$ are non-zero algebraic numbers such that $\xi_i/\xi_j \neq \pm 1$ for all $i \neq j$. When he formalised this approach in 1949, Siegel [Si2] introduced the notion of a normal homogeneous system of linear first order differential equations. Although this concept was introduced to guarantee the crucial non-vanishing of a certain determinant in the proof, it eluded verification for any further functions beyond those already considered in 1929.

In 1959 A. B. Šidlovskiĭ [Š2] showed in a fundamental advance, that for qualitative statements of algebraic independence the obviously necessary condition on the algebraic independence of the $E$-functions actually suffices to ensure the algebraic independence of their values at non-singular, non-zero algebraic points. For this reason many authors have developed techniques to obtain the algebraic independence of functions. For example V. A. Oleinikov [Ol] and V. Ch. Salichov [Sa2] define criteria implying the algebraic independence of components of solutions of certain systems of differential equations. We also refer to Mahler's book on transcendence [Ma, Ch. 7] for some very interesting examples. It should be mentioned however, that Šidlovskiĭ's method does not yield effective

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measures of algebraic independence. See [Br] for a detailed exposition. In order to obtain more effectivity, one can use zero-estimates such as in [Ne1], [Be-Be]. For complete effectivity, one can verify Siegel's normality criterion or Šidlovskiĭ's weaker irreducibility criterion [Š1], [Š3].

The purposes of establishing the requisite independence of new functions and of guaranteeing effective quantitative measures therefore motivate our re-examination of Siegel's original normality criterion in this paper. We now indicate briefly its contents.

Section 1 is mainly preliminary. However, the last result characterizes Siegel's original normality criterion in terms of the natural representation of the differential Galois group of the system of differential equations. In Section 2 we define Siegel normal systems. Then in Theorem 2.2 we characterize in three ways those Siegel normal systems which are irreducible. One of these characterizations is that the Galois group of our $n \times n$ system of first order linear differential equations contains either $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$. Another characterization is that the second symmetric square of our system is irreducible. Note that the latter characterization is also very practical in deciding whether a differential Galois group contains $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$. Using the irreducible systems as building blocks we then characterize general Siegel normal systems in Theorem 2.3. Here we use a theorem of Kolchin given in the appendix as Proposition A.3. In Section 3 we consider differential equations having only $z = 0$ and $z = \infty$ as singular points. We construct explicitly a set of elements of the Galois group of such differential equations. In Section 4 we apply the results from Sections 2 and 3 to deduce Siegel normality of families of equations for generalized hypergeometric functions ${}_pF_{q-1}$ with rational parameters. Consequently, we have the algebraic independence of the values of these functions and their derivatives at non-zero algebraic points. In addition, effective measures of algebraic independence of these numbers follow in the usual way. To be more concrete, we formulate Corollary 4.6 here, after introducing some notation.

A parameter set S is a set of real numbers of the form $\{\mu_1, \ldots, \mu_p; \nu_1, \ldots, \nu_q\}$ where $q > p \geq 0$, $q \geq 2$ and $\nu_q = 1$. The set S is called *admissible* if it satisfies at least one of the following conditions:

A) $\nu_j - \mu_i \notin \mathbf{Z}$ for $1 \leq i \leq p$, $1 \leq j \leq q$ and all sums $\nu_i + \nu_j$, $1 \leq i \leq j \leq q$ are distinct $\mathrm{mod}\, \mathbf{Z}$.

B) $p = 0$; $q$ is odd or $q = 2$; and the set $\{\nu_1, \ldots, \nu_q\}$ is modulo $\mathbf{Z}$ not a union of arithmetic sequences $\{\nu, \nu + 1/d, \ldots, \nu + (d-1)/d\}$ of a fixed length $d$, where $d | q$, $d > 1$.

Two parameter sets $S = \{\mu_1, \ldots, \mu_p; \nu_1, \ldots, \nu_q\}$, $S' = \{\mu'_1, \ldots, \mu'_{p'}; \nu'_1, \ldots, \nu'_{q'}\}$ are called *similar* if

a) $p' = p$, $q' = q$.

b) There exist $\mu, \nu \in \mathbf{R}$ and a choice of $\pm$ sign such that, on renumbering if necessary,

$$\mu_i' \equiv \mu \pm \mu_i \pmod 1, \qquad i = 1, \dots, p,$$
$$\nu_j' \equiv \nu \pm \nu_j \pmod 1, \qquad j = 1, \dots, q.$$

For a given parameter set $S = \{\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q\}$ consider the function

$$f_S(z) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_p)_n}{(\nu_1)_n \cdots (\nu_{q-1})_n} \frac{(-z)^{(q-p)n}}{n!}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. Hence $f_S(z)$ is a generalized hypergeometric function of type ${}_pF_{q-1}$ with $z$ replaced by $(-z)^{q-p}$. Let $S_1, \dots, S_r$ be (possibly equal) admissible parameter sets with rational $\mu$'s and $\nu$'s and $a_1, \dots, a_r \in \overline{\mathbf{Q}}^*$ (non-zero algebraic numbers). If the parameter sets $S_i$ and $S_j$ $(i \neq j)$ are similar of length $p;q$ then we assume that $\pm a_i/a_j$ are not $(q-p)$-th roots of unity. Let $b_1, \dots, b_s \in \overline{\mathbf{Q}}$ be $\mathbf{Q}$-linearly independent. Then the numbers

$$e^{b_1}, \dots, e^{b_s},$$
$$f_{S_1}(a_1), f_{S_1}'(a_1), \dots, f_{S_1}^{(q(1)-1)}(a_1),$$
$$\cdots\cdots$$
$$f_{S_r}(a_r), f_{S_r}'(a_r), \dots, f_{S_r}^{(q(r)-1)}(a_r),$$

where $q(i)$ is the $q$-parameter of $S_i$, are all algebraically independent over $\mathbf{Q}$. This result contains the results on hypergeometric functions obtained in [Sa2] if we take $\lambda_i = 1$ for at least one $i$ there.

In the appendix we collect relevant results on algebraic groups used in Section 2. Kostant's theorem on the quadratic generation of the ideal of polynomials vanishing on the orbit of the highest weight vector is central to Theorem 2.2. In the proof of Theorem A.5 we have essentially reproduced a proof of a simplified version of it.

In order to indicate the limits of applicability of our criteria we like to point out that there are many systems of equations which are not Siegel normal but to which Šidlovskiĭ's method can be applied successfully. These systems have a non-reductive differential Galois group and a typical example is the system of which $(1, \Sigma_1^\infty z^n/n!, \Sigma_1^\infty z^n/n \cdot n!, \dots, \Sigma_1^\infty z^n/n^k \cdot n!)^t$ is a solution. The differential field corresponding to this example is a Liouville extension which can be obtained by successive solution of inhomogeneous first order differential equations. More examples of equations with a non-reductive group are treated in [Fe-Sh], [Ma] or [Sa2] if we take $\lambda_i \neq 1$ for all $i$ in the latter paper.

Finally, we would like to express our gratitude to O. Gabber and N. M. Katz for pointing out an error in a previous version of this paper and for suggesting a simple way to prove a more general result on hypergeometric functions which is now stated as Proposition 4.4. Also, independently of our work, Katz and Pink gave a sufficient criterion for an $n \times n$ system of first order differential equations to have a Galois group containing $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{Sp}(n, \mathbf{C})$ or $\mathrm{SO}(n, \mathbf{C})$; see [Kat], [KaP]. Quite recently, Katz and Gabber have managed to give a complete description of the Galois groups of differential equations of hypergeometric functions of type ${}_pF_{q-1}$, $p < q$. Needless to say, these results are of great importance for the transcendence theory of such functions. In particular, using their results, it should be possible to extend the notion of admissibility in Cor. 4.6 considerably.

## 1. Linear systems of first order differential equations

Consider the $n \times n$ system of first order linear differential equations

$$(A) \qquad \frac{d}{dz}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where $a_{ij}(z) \in \mathbf{C}(z)$ for all $i$, $j$. Denote the coefficient matrix by $A$. Then we denote the corresponding system by $(A)$. Let $\sigma$ be a permutation of $1, \ldots, n$. A permutation of $(A)$ is an $n \times n$ system whose solutions are given by $(y_{\sigma(1)}, \ldots, y_{\sigma(n)})^t$ as $(y_1, \ldots, y_n)$ runs through the solutions of $(A)$.

Let $F$ be a differential field with constant field $\mathbf{C}$, which contains $n$ $\mathbf{C}$-linearly independent solutions of $(A)$. Let $F_0$ be the field obtained by adjoining to $\mathbf{C}(z)$ the components of all solutions of $(A)$. This field is called the Picard-Vessiot extension of $(A)$. Its differential isomorphism class is independent of the choice of $F$ [Ko3, Ch. VI, Prop. 13].

Let $G$ be the group of automorphisms of $F_0$ that fix $\mathbf{C}(z)$ and which commute with differentiation. It is called the differential Galois group of $F_0$, and we sometimes denote $G$ by $\mathrm{Gal}(F_0/\mathbf{C}(z))$. Standard references on differential fields and Galois groups are [Kap], [Ko 1, 3].

Let $V$ be the $\mathbf{C}$-vector space of solutions of $(A)$. Since any element of $G$ commutes with differentiation, it maps a solution vector of $(A)$ to another solution. Hence $G \hookrightarrow \mathrm{GL}(V)$, i.e. we have a faithful representation of $G$ acting on $V$. This will be our standard representation throughout the paper. Furthermore, representations of a group will be denoted by the corresponding representation spaces on which they act.

We have the following basic result of Kolchin [Kol] which has its roots in the work of Picard and Vessiot.

THEOREM 1.1. i) *The group* $\mathrm{Gal}(F_0/\mathbf{C}(z))$ *is an algebraic subgroup of* $\mathrm{GL}(V)$ *and hence has a finite number of connected components.*

ii) *The transcendence degree of* $F_0/\mathbf{C}(z)$ *equals the dimension of* $\mathrm{Gal}(F_0/\mathbf{C}(z))$.

iii) $\mathbf{C}(z)$ *is the maximal fixed field of* $\mathrm{Gal}(F_0/\mathbf{C}(z))$.

The system $(A)$ is called *linearly irreducible* or *irreducible over* $\mathbf{C}(z)$ if $G$ acts irreducibly on $V$.

Consider two systems $(A)$ and $(B)$ of size $n \times n$ and $m \times m$. Let $G$, $H$ be their corresponding Galois groups and $V$, $W$ their solution spaces. A *direct sum* of these systems is simply a permutation of the $(n + m) \times (n + m)$ system of first order linear equations, whose coefficient matrix is obtained by putting the coefficient matrix of each system on the diagonal of an $(n + m) \times (n + m)$ matrix and choosing all other coefficients zero. We denote such a system by $(A) \oplus (B)$. Its space of solutions is clearly isomorphic to $V \oplus W$. Its Galois group is a subgroup of $G \times H$ and it acts on the solutions via the direct sum representation $V \oplus W$ of $G \times H$. Similarly we can define direct sums of any number of systems.

Let $(x_1, \dots, x_n)^t$ be a solution of $(A)$ and $(y_1, \dots, y_m)^t$ a solution of $(B)$. Form the product columns consisting of all $nm$ products $x_i y_j$ $(i = 1, \dots, n;$ $j = 1, \dots, m)$. As we let $(x_1, \dots, x_n)^t$ and $(y_1, \dots, y_m)^t$ run through the solution spaces $V$ of $(A)$ and $W$ of $(B)$, their product columns span an $nm$ dimensional $\mathbf{C}$-vector space isomorphic to $V \otimes W$. The $nm \times nm$ system of first order linear equations having all these product columns as solution is called the tensor product of $(A)$ and $(B)$ and we denote it by $(A) \otimes (B)$. Its Galois group is a subgroup of $G \times H$ which acts on $V \otimes W$ by the tensor product representation of the standard representations of $G$ and $H$. Similarly we can define tensor products of any number of systems.

To every solution $Y = (y_1, \dots, y_n)^t$ of $(A)$ we associate the product column of length $N := \binom{m+n-1}{n-1}$ consisting of all monomials of degree $m$ in $y_1, \dots, y_n$. Let $V_{(m)}$ be the $\mathbf{C}$-linear span of such product columns as we let $Y$ run through $V$, the space of all solutions of $(A)$. It is not hard to show that $V_{(m)}$ is isomorphic to the $m$-th symmetric tensor product $S^m V$ of $V$. We identify $V_{(m)} \simeq S^m V$. The Galois group of $(A)$ acts on $S^m V$ via the $m$-th symmetric power of the standard representation of $G$ on $V$. Moreover, the elements of $S^m V$

satisfy an $N \times N$-system of linear first order equations, which we call the $m$-th symmetric power of $(A)$, and denote by $S^m(A)$. The one by one system $y' = 0$ will then be denoted by $S^0(A)$.

We have the following lemmas.

LEMMA 1.2. *Let $(A)$ and $(B)$ be two systems of linear equations such that their tensor product is a direct sum of smaller systems. Then at least one of the systems $(A)$ and $(B)$ is also a direct sum.*

*Proof.* Let $(A_{ij})$ $(i, j = 1, \dots, n)$ and $(B_{ij})$ $(i, j = 1, \dots, m)$ be the coefficient matrices of $(A)$ and $(B)$. Let $(u_1, \dots, u_n)^t$ be any solution of $(A)$ and $(v_1, \dots, v_m)^t$ any solution of $(B)$. From the identity $(u_i v_j)' = \sum_k (A_{ik} u_k v_j + B_{jk} u_i v_k)$ it is not hard to see that the coefficient matrix of $(A) \otimes (B)$ is given by $C_{ij,\,kl} = \delta_{jl} A_{ik} + \delta_{ik} B_{jl}$ where $\delta_{ij}$ is the Kronecker delta. The pairs $i, j$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ form the index set of the coefficient matrix of $(A) \otimes (B)$.

Since $(A) \otimes (B)$ is a direct sum, the index set of its coefficient matrix can be divided into two non-empty, disjoint sets $I$ and $J$ such that $C_{ij,\,kl} = 0$ if $i, j$ and $k, l$ do not both belong to $I$ or both to $J$.

Suppose there is an $l$ such that the set $\{(k, l) \mid k = 1, \dots, n\}$ has non-empty intersection with both $I$ and $J$. Then the set $\{1, \dots, n\}$ is divided into two non-empty sets $I_1, J_1$ such that $k \in I_1 \Leftrightarrow (k, l) \in I$. Suppose $i \neq k$; then $C_{il,\,kl} = A_{ik}$. Notice, $i$ and $k$ do not both belong to $I_1$ or both to $J_1 \Leftrightarrow (i, l)$ and $(k, l)$ do not both belong to $I$ or both to $J \Rightarrow C_{il,\,kl} = 0 \Rightarrow A_{ik} = 0$. Hence $(A)$ is a direct sum.

Suppose for every $l$ the set $\{(k, l) \mid k = 1, \dots, n\}$ belongs exclusively to $I$ or to $J$. Then there must be a $k$ such that $\{(k, l) \mid l = 1, \dots, m\}$ has non-empty intersection with both $I$ and $J$. By a similar argument, we now conclude that $(B)$ is a direct sum.

q.e.d.

LEMMA 1.3. *If the* $r$-th *symmetric power of a system $(A)$ is a direct sum of smaller systems, then $(A)$ itself is a direct sum.*

*Proof.* Let $(A_{ij})$ $(i, j = 1, \dots, n)$ be the coefficient matrix of $(A)$. Let $(u_1, \dots, u_n)^t$ be any solution of $(A)$ and notice

$$(u_{i_1} u_{i_2} \cdots u_{i_r})' = \sum_k \left( A_{i_1 k} u_k u_{i_2} \cdots u_{i_r} + \cdots + A_{i_r k} u_{i_1} \cdots u_{i_{r-1}} u_k \right).$$

The coefficient matrix of $S^r(A)$ is indexed by subsets $S$ of cardinality $r$ of $\{1, \dots, n\}$ where we allow for repetition of elements. Notice that $C_{S_1, S_2} = 0$ if $|S_1 \cap S_2| < r - 1$, $C_{i \cup S,\, j \cup S} = A_{ij}$ for any subset $S$ with $|S| = r - 1$ and $i \neq j$, $C_{S,S} = \sum_{i \in S} A_{ii}$ for any subset $S$ with $|S| = r$.

Since $S^r(A)$ is a direct sum, the index set of its coefficient matrix can be divided into two sets $I$ and $J$ such that $C_{S,T} = 0$ if $S$ and $T$ do not both belong to $I$ or both to $J$. There exists a subset $R$ of cardinality $r - 1$ such that the set $\{i \cup R | i = 1, \ldots, n\}$ has non-empty intersection with both $I$ and $J$. By exactly the same argument as in Lemma 1.3 we conclude that $(A)$ is a direct sum.

q.e.d.

We need two more concepts. Let $(A)$ and $(B)$ be two $n \times n$ systems of linear differential equations. Let $\mathbf{Y}$ be a complete solution matrix of $(A)$. Then $(A)$ and $(B)$ are said to be *cogredient* if there exists an $n \times n$ matrix $M$, $\det M \neq 0$, with entries in $\mathbf{C}(z)$ and a function $Q(z)$ whose logarithmic derivative is in $\mathbf{C}(z)$, such that $Q \cdot M\mathbf{Y}$ is a solution matrix of $(B)$. We call two cogredient systems *equivalent* if we can take $Q = 1$.

Let $(A)^d$ be the dual system obtained by taking minus the transpose of $A$ as coefficient matrix. Notice that

$$0 = \frac{d}{dz} Y \cdot Y^{-1} = Y \frac{d}{dz} Y^{-1} + \left( \frac{d}{dz} Y \right) \cdot Y^{-1}$$

$$= Y \frac{d}{dz} Y^{-1} + AY \cdot Y^{-1}.$$

Hence

$$\frac{d}{dz} Y^{-1} = - Y^{-1} A,$$

and on taking the transpose,

$$\frac{d}{dz} (Y^{-1})^t = - A^t (Y^{-1})^t.$$

So the dual system $(A)^d$ has $(\mathbf{Y}^{-1})^t$ as solution matrix. We call $(A)$ and $(B)$ *contragredient* if $(A)^d$ is cogredient with $B$. We have:

LEMMA 1.4. *Let $(A)$ and $(B)$ be two $n \times n$ systems of linear equations with solution spaces $V, W$. We denote equivalence of representations by $\simeq$. The symbol $\chi$ stands for a certain group character, and $V^d$ denotes the dual representation of $V$. Then,*

*$(A)$ and $(B)$ are equivalent, cogredient or contragredient $\Leftrightarrow$ The Galois group of $(A) \oplus (B)$ acts on $V$ and $W$ by representations such that $V \simeq W$, $V \simeq \chi \otimes W$, $V \simeq \chi \otimes W^d$ respectively.*

*Proof.* Here we prove only the statement concerning cogredient systems, the other two cases being similar. Suppose $(A)$ and $(B)$ are cogredient. Let $\mathbf{Y}$ be a solution matrix of $(A)$. Then there is a square matrix $M$ with entries in $\mathbf{C}(z)$ and a function $Q(z)$ with $Q'/Q \in \mathbf{C}(z)$ such that $Q \cdot M\mathbf{Y}$ is a solution matrix of

$(B)$. The Galois group of $(A) \oplus (B)$ acts on $\mathbf{Y}$ by right multiplication with constant matrices, and on $Q$ by multiplication with scalars. The matrix $M$ remains fixed. It is now obvious that $V \simeq \chi \otimes W$ for some group character $\chi$ on the Galois group.

Suppose conversely, that the Galois group of $(A) \oplus (B)$ acts on $V$ and $W$ by representations such that $V \simeq \chi \otimes W$. We can the find solution matrices $\mathbf{Y}$ of $(A)$ and $\mathbf{Z}$ of $(B)$ such that any $\sigma$ in the Galois group acts on $\mathbf{Y}$ by right multiplication with a constant matrix $\Sigma$ and on $\mathbf{Z}$ by right multiplication with $\Sigma$ times a scalar $\chi(\sigma)$. Hence the Galois group acts on $\mathbf{YZ}^{-1}$ by scalar multiplication with $\chi(\sigma)$, and $\mathbf{YZ}^{-1}$ must be of the form $Q \cdot M$ where $M$ has entries in $\mathbf{C}(z)$ and $Q$ is a function with $Q'/Q \in \mathbf{C}(z)$. q.e.d.

The last two propositions of this section establish a relation with the concepts developed here and the concepts as used by Siegel in [Si, Ch. II. 4, II. 8].

Proposition 1.5. *Let $(A)$ be an $n \times n$ system of first order linear differential equations over $\mathbf{C}(z)$, $V$ its space of solutions and $G$ its Galois group. Let $H$ be a subgroup of $G$ and $K/\mathbf{C}(z)$ its fixed field. Suppose there exists a proper $H$-stable subspace $W \subset V$ of dimension $m$. Then the $K$-rank of all $(p_1, \dots, p_n) \in K^n$ such that $p_1 y_1 + \cdots + p_n y_n = 0$, for any $(y_1, \dots, y_n)^t \in W$, equals $n - m$.*

*Proof.* Let $(Y_{r1}, \dots, Y_{rn})^t$, $r = 1, \dots, m$, be a basis of $W$. The matrix

$$\Omega = \begin{pmatrix} Y_{11} & \cdots & Y_{m1} \\ \vdots & & \\ Y_{1n} & \cdots & Y_{mn} \end{pmatrix}$$

contains an $m \times m$ non-singular submatrix $\omega$. The group $H$ acts on $\Omega$ and $\omega$ by matrix multiplication with constant $m \times m$ matrices on the right. Hence $\Omega\omega^{-1}$ is fixed under $H$, and thus has entries in $K$. Since the $K$-rank of $\Omega\omega^{-1}$ is $m$, the $K$-linear relations for the rows of $\Omega\omega^{-1}$ have $K$-rank $n - m$, and so the same holds for the $K$-linear relations for the rows of $\Omega$. q.e.d.

Proposition 1.6. *Let $(B) = (B_1) \oplus (B_2) \oplus \cdots \oplus (B_r)$ be the direct sum of $r$ systems of linear differential equations of size $n_i \times n_i$ $(i = 1, \dots, r)$. Then the following two statements are equivalent.*

i) *The systems $(B_i)$ are irreducible and non-equivalent.*

ii) *For any solution $\mathrm{y}(i) = (y_1(i), \dots, y_{n_i}(i))^t$ of $(B_i)$ $(i = 1, \dots, r)$ and any $\mathbf{p}_i \in \mathbf{C}(z)^{n_i}$ $(i = 1, \dots, r)$ the relation $\mathbf{p}_i \mathrm{y}(1) + \cdots + \mathbf{p}_r \mathrm{y}(r) = 0$ implies for each $i = 1, \dots, r$ that either $\mathbf{p}_i = 0$ or $\mathrm{y}(i) = 0$.*

*Proof.* i) $\Rightarrow$ ii). Let $\mathbf{p}_1, \dots, \mathbf{p}_r$ be given. Let $G$ be the Galois group of $(B)$ and $V$ its space of solutions. Let $V_i$ be the space of solutions of $(B_i)$. We have $V \simeq \oplus_{i=1}^{r} V_i$. Let $W$ be the subspace of $V_1 \oplus \cdots \oplus V_r$ of all $\mathrm{y}(1) \oplus \cdots \oplus \mathrm{y}(r)$ such that $\mathbf{p}_1 \cdot \mathrm{y}(1) + \cdots + \mathbf{p}_r \mathrm{y}(r) = 0$. Since $G$ fixes $\mathbf{C}(z)$, we see that $W$ is $G$-stable. By the assumption of our proposition, the direct sum $V \simeq V_1 \oplus \cdots \oplus V_r$ gives a reduction of the action of $G$ as a direct sum of irreducible representations. Since the $(B_i)$ are not equivalent, Lemma 1.4 implies that these irreducible representations are non-equivalent. This implies that the $G$-stable space $W$ is a direct subsum $\oplus_{i \in R} V_i$, where $R$ is a certain subset of $\{1, \dots, r\}$. If $i \notin R$, we have clearly $\mathrm{y}(i) = 0$. If $i \in R$, we have $\mathbf{p}_i \cdot \mathrm{y}(i) = 0$ for any $\mathrm{y}(i) \in V_i$. Since $V_i$ is the complete set of solutions of $(B_i)$, we conclude that $\mathbf{p}_i = 0$.

ii) $\Rightarrow$ i). Suppose that $(B_1)$ is reducible. Then, according to Proposition 1.5, there is a solution $\mathrm{y}_1$ of $(B_1)$ and a non-trivial vector $\mathbf{p}_1 \in \mathbf{C}(z)^{n_1}$ such that $\mathbf{p}_1 \cdot \mathrm{y}_1 = 0$, contradicting our assumptions. Hence every $(B_i)$ is irreducible.

Suppose that $(B_1)$ and $(B_2)$ say, are equivalent. By definition there exist a square matrix $M$ with entries in $\mathbf{C}(z)$, a solution $\mathrm{y}_2$ of $(B_2)$ and a solution $\mathrm{y}_1$ of $(B_1)$ such that $\mathrm{y}_2 = M \cdot \mathrm{y}_1$. This gives us relations between the components of a solution of $(B_1) \oplus (B_2)$ which are not of the form described in ii). Hence $(B_1)$ and $(B_2)$ are inequivalent and in general all $(B_i)$ are non-equivalent. q.e.d.

## 2. Siegel normal systems

In order to be able to apply his transcendence methods to values of functions satisfying a linear differential system of equations, Siegel had to assume a certain normality condition [Si, Ch. II. 4] which can be stated as follows:

*Definition.* A system $(A)$ of linear first order differential equations is called *Siegel normal* if, for each $m \geq 1$, the symmetric power $(B) = S^m(A)$ is a direct sum of systems $(B_1), \dots, (B_{r(m)})$ satisfying property ii) of Proposition 1.6.

Using Proposition 1.6 we now put forward the following equivalent definition.

*Definition.* A system $(A)$ of linear first order differential equations is called *Siegel normal* if each symmetric power $S^m(A)$ is a direct sum of irreducible, non-equivalent systems.

A direct consequence of Siegel normality is that the nonzero components of any solution of a Siegel normal system do not satisfy any homogeneous polynomial relation over $\mathbf{C}(z)$. The converse is not true, as is shown by the example

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/z \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

having the general solution $(\beta \log z + \alpha, \beta)^t$, $\alpha, \beta \in \mathbf{C}$. Clearly, the nonzero components of each such solution do not satisfy any homogeneous polynomial relation over $\mathbf{C}(z)$, but the system is not Siegel normal. In Theorem 2.3, we will see however, that we do have a converse if we require the Galois group to be reductive (see the appendix for definitions).

Before proving the main theorems, we need an observation.

LEMMA 2.1. *Let $(A) = (A_1) \oplus (A_2) \oplus \cdots \oplus (A_r)$ be a Siegel normal system, where $(A_i)$ is irreducible for $i = 1, \ldots, r$. Let $m \in \mathbf{N}$ and $m_i \in \mathbf{Z}_{\geq 0}$, $i = 1, \ldots, r$. Then the system $S^{m_1}(A_1) \otimes S^{m_2}(A_2) \otimes \cdots \otimes S^{m_r}(A_r)$ is irreducible and*

$$S^m(A) = \bigoplus_{\Sigma m_i = m} \{S^{m_1}(A_1) \otimes \cdots \otimes S^{m_r}(A_r)\}.$$

*Proof.* The decomposition of $S^m(A)$ is straightforward. Suppose, that $S^{m_1}(A_1) \otimes \cdots \otimes S^{m_r}(A_r)$ is reducible. Then, by definition of Siegel normality, it must be a direct sum of smaller systems. But then, according to Lemma 1.2, at least one of the $S^{m_i}(A_i)$ must be a direct sum system, and Lemma 1.3 then implies that $(A_i)$ itself is a direct sum, contradicting the irreducibility of $(A_i)$. Hence, each system $S^{m_1}(A_1) \otimes \cdots \otimes S^{m_r}(A_r)$ is irreducible.

THEOREM 2.2. *Let $(A)$ be an $n \times n$ system of first order linear differential equations, $V$ its space of solutions and $G$ the Galois group. Suppose* $\dim V = n \geq 2$. *Then the following properties are equivalent*:

i) *$(A)$ is irreducible over $\mathbf{C}(z)$ and Siegel normal.*

ii) *Every non-trivial solution of $(A)$ has components which do not satisfy any non-trivial homogeneous polynomial relation over $\mathbf{C}(z)$.*

iii) *$G$ divided out by its centre is infinite, and $G$ acts irreducibly on $S^2V$.*

iv) *$G$ is an extension by scalars of* $\mathrm{SL}(n, \mathbf{C})$ *or* $\mathrm{Sp}(n, \mathbf{C})$.

*Proof.* iv) $\Rightarrow$ i). It is a well known fact that the symmetric powers of the $n$-dimensional representations of $\mathrm{SL}(n, \mathbf{C})$ and $\mathrm{Sp}(n, \mathbf{C})$ are irreducible. Hence $(A)$ is irreducible and Siegel normal.

i) $\Rightarrow$ ii) follows from Proposition 1.6 applied to the symmetric powers of $(A)$ which, according to Lemma 2.1, are all irreducible.

iii) $\Rightarrow$ iv) follows from Theorems A.5 and A.7 of the appendix.

ii) $\Rightarrow$ iii). Let $Z(G)$ be the centre of $G$. Suppose for the moment that $[G{:}Z(G)]$ is finite. Then the fixed field of $Z(G)$ is an algebraic extension $K$ of $\mathbf{C}(z)$. Since $G$ acts irreducibly on $V$, $Z(G)$ acts on $V$ by scalars. Hence by Proposition 1.5, any solution $(y_1, \ldots, y_n)^t \in V$ satisfies a relation $L(y) =$

$a_1y_1 + \cdots + a_ny_n = 0$ with $a_i \in K$, for all $i$. Multiplying this relation with all its conjugates over $\mathbf{C}(z)$ we find a homogeneous polynomial relation for $y_1, \dots, y_n$ over $\mathbf{C}(z)$. This is contradicted by ii). Hence $[G:Z(G)]$ is infinite.

Let $G^0$ be the connected component of the identity in $G$. Let $K$ be the fixed field of $G^0$ and suppose now that $G^0$ acts reducibly on $V$. Then, according to Proposition 1.5 we can find a nontrivial solution $(y_1, \dots, y_n)^t$ and $a_i \in K$, not all zero, such that $a_1y_1 + \cdots + a_ny_n = 0$. Multiplying this relation with its conjugates over $\mathbf{C}(z)$ again yields a forbidden non-trivial homogeneous polynomial relation for $y_1, \dots, y_n$ over $\mathbf{C}(z)$. Hence $G^0$ acts irreducibly on $V$.

Let $G^0$ act on $V$ as the representation $V_\lambda$ with highest weight $\lambda$ (see the appendix for definitions). Let $v_\lambda \in V$ be a highest-weight vector. Then $v_\lambda \otimes v_\lambda$ is a highest-weight vector for the representation $V_{2\lambda} \subset S^2V$. Suppose finally $V_{2\lambda} \subsetneq S^2V$. Realising $S^2V$ as the $\mathbf{C}$-linear span of the second product columns of the solutions of $(A)$, and using Proposition 1.5, we see that $v_\lambda \otimes v_\lambda \in V_{2\lambda} \subsetneq S^2V$ implies that there is a non-trivial solution $(y_1, \dots, y_n)^t$ of $(A)$ and a homogeneous quadratic polynomial $P \in K[x_1, \dots, x_n]$ such that $P(y_1, \dots, y_n) = 0$. Multiplication of this relation by its conjugates would give another forbidden homogeneous polynomial relation for $y_1, \dots, y_n$ over $\mathbf{C}(z)$. Hence $V_{2\lambda} = S^2V$ and $G$ acts irreducibly on $S^2V$. q.e.d.

THEOREM 2.3. *Let $(A)$ be a system of first order linear differential equations. Then the following statements are equivalent*:

i) *$(A)$ is Siegel normal.*

ii) *$(A)$ has a reductive Galois group and the non-zero components of each solution do not satisfy any homogeneous polynomial relation over $\mathbf{C}(z)$.*

iii) *$(A)$ is a direct sum of $r$ non-contragredient and non-cogredient irreducible Siegel normal systems $(A_i)$ of size $n_i \times n_i$, $n_i \geq 2$ $(i = 1, \dots, r)$, and $s$ one-by-one systems $y' = B_jy$ $(j = 1, \dots, s)$ such that the coefficients $B_j \in \mathbf{C}(z)$ do not satisfy any relation of the form*

$$\sum_{j=1}^{s} a_jB_j = \frac{1}{f}\frac{d}{dz}f$$

*with $a_j \in \mathbf{Z}$, not all zero, $\sum_{j=1}^{s} a_j = 0$ and $f \in \mathbf{C}(z)^\times$.*

*Proof.* i) $\Rightarrow$ ii) follows from Proposition 1.6 applied to the symmetric powers of $(A)$.

ii) $\Rightarrow$ iii). If $(A)$ is irreducible then we are done according to Theorem 2.2. Now assume $(A)$ is reducible. Since the Galois group $G$ of $(A)$ is reductive the space of solutions $V$ of $(A)$ is a direct sum of, say, two proper $G$-stable subspaces $W_1 \oplus W_2$. Put $m_i = \dim W_i$ $(i = 1, 2)$, $\dim V = n = m_1 + m_2$. According to Proposition 1.5 there exist $n - m_1$ $\mathbf{C}(z)$-linear independent vectors $\mathbf{P}(i) \in \mathbf{C}(z)^n$

such that $\mathbf{P}(i) \cdot \mathrm{y} = 0$ for every solution $\mathrm{y} = (y_1, \ldots, y_n)^t$ of $(A)$ belonging to $W_1$ $(i = 1, \ldots, n - m_1)$. Since any relation between the components of a solution must arise trivially by assumption, we infer that for any solution $(y_1, \ldots, y_n)^t$ in $W_1$ at least $n - m_1$ components vanish. Since $\dim W_1 = m_1$ we conclude that there is a subset $R \subset \{1, \ldots, n\}$ such that $(y_1, \ldots, y_n)^t \in W_1 \Leftrightarrow y_i = 0$ for all $i \in R$. A similar argument holds for $W_2$, and since $W_1$ and $W_2$ span the space of solutions of $(A)$, we see $(y_1, \ldots, y_n)^t \in W_2 \Leftrightarrow y_i = 0$ for all $i \notin R$. Repeating the same arguments for $W_1$ and $W_2$ we conclude that $(A)$ is a direct sum of irreducible systems.

Write $(A) = (A_1) \oplus \cdots \oplus (A_r) \oplus (B_1) \oplus \cdots \oplus (B_s)$, where the $(A_i)$ have size $n_i \times n_i$, $n_i \geq 2$ and the $(B_i)$ are one-by-one systems. Suppose there is a relation between the $B_i$ of the form described in property ii). Let $y_j$ be a non-trivial solution of $y' = B_j y$; then the relation implies $\prod_j y_j^{a_j} = \gamma f$ for some $\gamma \in \mathbf{C}^\times$. This in turn implies a relation of the form $\prod y_j^{b_j} = \gamma f \prod y_j^{c_j}$ for $b_j, c_j \in \mathbf{Z} \geq 0$, $\sum b_j = \sum c_j$, contradicting the homogeneous algebraic independence of solutions over $\mathbf{C}(z)$.

From Theorem 2.2 follows that the $(A_i)$ are Siegel normal. Suppose two systems, say $(A_1)$ and $(A_2)$ are cogredient. The definition of cogredience implies the existence of a complete solution matrix $\mathbf{Y}$ of $(A_1)$, a complete solution matrix $\tilde{\mathbf{Y}}$ of $(A_2)$, a matrix $M$ with entries in $\mathbf{C}(z)$ and a function $Q(z)$ with $Q'/Q \in \mathbf{C}(z)$ such that $\mathbf{Y} = Q \cdot M\tilde{\mathbf{Y}}$. From this equality it is straightforward to derive homogeneous quadratic relations over $\mathbf{C}(z)$ for solutions $(y_1, \ldots, y_n, \tilde{y}_1, \ldots, \tilde{y}_n)$ of $(A_1) \oplus (A_2)$. Hence $(A_1)$ and $(A_2)$ are not cogredient. In the same way one verifies that $(A_1)$ and $(A_2)$ are not contragredient.

iii) $\Rightarrow$ i). Let $G_j$ be the Galois group of $(A_j)$ and $H_i$ the Galois group of $(B_i)$. By Theorem 2.2, $G_j$ is an extension by scalars of $K_j = \mathrm{SL}(n_j, \mathbf{C})$ or $K_j = \mathrm{Sp}(n_j, \mathbf{C})$. Let $G$ be the Galois group of $(A)$. Then $G \subset G_1 \times G_2 \times \cdots \times G_r \times H_1 \times \cdots \times H_s$. Let $Z_i$ be the centre of $K_i$ for $i = 1, \ldots, r$ and let $\pi_i$: $K_i \to K_i/Z_i$ be the canonical projection. Suppose $G \cap K_1 \times \cdots \times K_r \times (1) \times \cdots \times (1) = G'$ is a proper subgroup of $K_1 \times \cdots \times K_r$. Then, according to Proposition A.3 of Kolchin, there exist distinct indices $k, l \in \{1, \ldots, r\}$ and an isomorphism $f$: $K_k/Z_k \to K_l/Z_l$ such that $f(\pi_k(x_k)) = \pi_l(x_l)$ for every $(x_1, \ldots, x_r) \in G'$. Since every $K_i$ is simply connected, $f$ can be lifted to an isomorphism $\tilde{f}$: $K_k \to K_l$ such that $\tilde{f}(x_k) = x_l$ for every $(x_1, \ldots, x_r) \in G'$. By standard Lie-group theory any isomorphism $\tilde{f}$: $\mathrm{SL}(n, \mathbf{C}) \to \mathrm{SL}(n, \mathbf{C})$ is an inner automorphism or a dual map. Any isomorphism $\tilde{f}$: $\mathrm{Sp}(n, \mathbf{C}) \to \mathrm{Sp}(n, \mathbf{C})$ is an inner isomorphism. Lemma 1.4 now implies that the systems $(A_k)$ and $(A_l)$ are either cogredient or contragredient. Thus we have arrived at a contradiction and we must assume that $G$ contains $K_1 \times \cdots \times K_r \times \{1\} \times \cdots \times \{1\}$.

Let $V_j$ be the solution space of $(A_j)$ on which $G_j$ acts. Since $G_j$ contains $K_j$, each representation $S^m V_j$ of $K_j \subset G_j$ is irreducible and hence each representation $S^{m_1}V_1 \otimes \cdots \otimes S^{m_r}V_r$ of $K_1 \times \cdots \times K_r$ is irreducible. Let $W_i$ be the one-dimensional solution space of $(B_i)$ on which $H_i$ acts by a character $\chi_i$. Then the representation $S^{m_1}V_1 \otimes \cdots \otimes S^{m_r}V_r \otimes \chi_1^{k_1} \cdots \otimes \chi_s^{k_s}$ of $G$ is irreducible, since $G$ contains $K_1 \times \cdots \times K_r \times \{1\} \times \cdots \times \{1\}$. Suppose two representations $S^{n_1}V_1 \otimes \cdots \otimes S^{n_r}V_r \otimes \chi_1^{l_1} \otimes \cdots \otimes \chi_s^{l_s}$ and $S^{m_1}V_1 \otimes \cdots \otimes S^{m_r}V_r \otimes \chi_1^{k_1} \otimes \cdots \otimes \chi_s^{k_s}$ with $\Sigma n_i + \Sigma l_i = \Sigma m_i + \Sigma k_j$ are equivalent. Then it is obvious that $m_i = n_i$ $(i = 1, \ldots, r)$. This implies that the characters $\chi_1^{k_1} \cdots \chi_s^{k_s}$ and $\chi^{l_1} \cdots \chi^{l_s}$ of $G$ are equal. Hence the solutions $y_i$ of $y' = B_i y$ satisfy the relation $y_i^{k_1 - l_1} \cdots y_s^{k_s - l_s} \in \mathbf{C}(z)^x$ and $\Sigma(k_i - l_i) = 0$, contradicting our assumption on the $B_i$.

Thus we conclude that the representation $S^m V$ of $G$ is the direct sum of the representations $S^{m_1}V_1 \otimes \cdots \otimes S^{m_r}V_r \otimes \chi_1^{k_1} \otimes \cdots \otimes \chi_s^{k_s}$, $\Sigma m_i + \Sigma k_i = m$, which are inequivalent; hence $(A)$ is Siegel normal. q.e.d.

Proposition 2.4. *Let $y' = Ay$, $y' = By$ be two irreducible Siegel normal $n \times n$ systems of first order linear differential equations. Let $y' = A_m y$, $y' = B_m y$ be the systems obtained by the substitution $z \to z^m$. Then $(A_m), (B_m)$ are again irreducible and Siegel normal. Furthermore, if $(A_m)$ and $(B_m)$ are cogredient or contragredient then the same holds for $(A)$ and $(B)$.*

*Proof.* Let $y' = Cy$ be any system of first order linear equations and $y' = C_m y$ its transform by the substitution $z \to z^m$. Let $\mathscr{F}$ be the Picard-Vessiot extension corresponding to $(C)$. Let $d$ be the largest divisor of $m$ such that $z^{1/d} \in \mathscr{F}$. Then the Galois group of $(C_m)$ is isomorphic to $\mathrm{Gal}(\mathscr{F}/\mathbf{C}(z^{1/d}))$. Furthermore, notice that $\mathrm{Gal}(\mathscr{F}/\mathbf{C}(z^{1/d}))$ is a normal subgroup of $\mathrm{Gal}(\mathscr{F}/\mathbf{C}(z))$ of finite index.

If $(A)$ is irreducible and Siegel normal, then it has a Galois group containing $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$ according to Theorem 2.2. The above remark now implies that the Galois group of $(A_m)$ also contains $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$ since these groups are connected. Hence $(A_m)$ is again irreducible and Siegel normal.

For the second statement we will need the fact that any group $G \subset \mathrm{GL}(n, \mathbf{C})$ containing $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$ as a normal subgroup is an extension by scalars of $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{Sp}(n, \mathbf{C})$. Let $G$ be the Galois group of $(A_m) \oplus (B_m)$. Since $(A_m), (B_m)$ are cogredient, irreducible and Siegel normal, $G$ acts with representations on the solution spaces of $(A_m)$ and $(B_m)$ which differ only by a character (Proposition 1.4) and contain $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{Sp}(n, \mathbf{C})$. By the remark just made, the same should hold for $(A) \oplus (B)$ and hence $(A)$ and $(B)$ are also cogredient. A similar argument works in case $(A_m), (B_m)$ are contragredient. q.e.d.

## 3. Formal solutions and the Galois group

We prove a number of propositions on differential equations having their only singularities at $z = 0$ and $z = \infty$. The main results of this section are intermediate steps in the application of Theorems 2.2 and 2.3 to generalized hypergeometric functions. In the propositions of this section we have the following situations: $Ly = 0$, $L \in \mathbf{C}(z)[d/dz]$, is an $n$-th order differential equation over $\mathbf{C}(z)$ having a regular singularity at $z = 0$, an irregular one at $z = \infty$ and no others. Suppose that at $z = 0$ we have a $\mathbf{C}$-linearly independent set of solutions, $z^{\beta_i} h_i(z)$, $i = 1, \ldots, n$, where the $h_i(z)$ are power series in $z$ with infinite radius of convergence. Suppose that at $z = \infty$ we have a $\mathbf{C}$-linearly independent set of formal solutions, $e^{a_i t} t^{\alpha_i} g_i(1/t)$, $i = 1, \ldots, n$, where $\alpha_i \in \mathbf{R}$ and $g_i(1/t)$ is a formal power series in $1/t$, $t^q = z$.

Note that in any finite sum $\sum e^{at} t^{\alpha} g(1/t)$ we may combine terms until the $a$ are distinct and the $\alpha$ are distinct modulo $\mathbf{Z}$. Then we may adjust the $\alpha$ if necessary to obtain moreover that each $g$ is a power series with $g(0) \neq 0$. Lexicographical ordering of terms in this form with respect to $(\operatorname{Re} a, \operatorname{Im} a, \alpha)$ allows us to select a unique maximal summand. The product of maximal terms gives the unique maximal term of the product. So the formal solutions of $Ly = 0$ at $z = \infty$ and their derivatives generate an integral domain and thus a field $\mathscr{G}$ over $\mathbf{C}(z)$. Using this ordering it is straightforward to verify that the constants of $\mathscr{G}$ are exactly $\mathbf{C}$ and thus that $\mathscr{G}$ is a Picard-Vessiot extension of $\mathbf{C}(z)$ [Kap], [Ko1].

Let $\mathscr{F}$ be the field generated over $\mathbf{C}(z)$ by the solutions $z^{\beta_i} h_i(z)$ and their derivatives. Then, according to a theorem of Kolchin ([Ko3, Ch. VI, Prop. 13]), the fields $\mathscr{F}$ and $\mathscr{G}$ are differentially isomorphic.

PROPOSITION 3.1. *Let notation be as above. Then the following maps are elements of the Galois group* $\operatorname{Gal}(\mathscr{F}/\mathbf{C}(z)) \simeq \operatorname{Gal}(\mathscr{G}/\mathbf{C}(z))$:

$$A: \quad \mathscr{F} \to \mathscr{F} \text{ given by } z^{\beta_j} h_j(z) \to e^{2\pi i \beta_j} z^{\beta_j} h_j(z) \qquad (j = 1, \ldots, n);$$

$$B: \quad \mathscr{G} \to \mathscr{G} \text{ given by } e^{a_j t} t^{\alpha_j} g_j(1/t) \to e^{a_j w} e^{a_j t} t^{\alpha_j} g_j(1/t)$$

$$(j = 1, \ldots, n \text{ and any } w \in \mathbf{C}).$$

*Proof.* The map $A$: $\mathscr{F} \to \mathscr{F}$ is a monodromy substitution, hence an element of $\operatorname{Gal}(\mathscr{F}/\mathbf{C}(z))$.

Let $G_j = e^{\alpha_j t} t^{\alpha_j} g_j(1/t)$ and let $P$ be a polynomial over $\mathbf{C}(z)$ such that $P(G_i^{(j)}) \equiv 0$. Write $P(G_i^{(j)}) = \sum e^{d_r t} H_r(1/t)$, where the $d_r$ are all distinct and each $H_r(1/t)$ can be written as a sum of terms of the form $t^\kappa$ times a formal power series in $1/t$. Then $\exp(d_r t)H_r(1/t) = P_r(G_i^{(j)})$ for some polynomial $P_r$. Then $P(G_i^{(j)}) \equiv 0$ implies $P_r(G_i^{(j)}) \equiv 0$ for each $r$. Hence $P_r(e^{a_i w}G_i^{(j)}) = e^{d_r w}e^{d_r t}H_r(1/t) \equiv 0$ for each $r$ and each $w \in \mathbf{C}$ and so $P(e^{a_i w}G_i^{(j)}) \equiv 0$, as required. q.e.d.

The following lemma will be particularly useful for the question whether or not the connected component of the identity of a Galois group acts irreducibly.

LEMMA 3.2. *Let $G$ be a group acting irreducibly on a vector space $V$ of dimension $n$. Let $H$ be a normal subgroup of $G$ and suppose $G/H$ is cyclic with generator $A$. If $H$ acts reducibly on $V$, then there exists $d|n$, $d > 1$, such that the set of eigenvalues of $A$ has the form $\bigcup_j\{\lambda_j, \lambda_j\zeta, \dots, \lambda_j\zeta^{d-1}\}$, $\zeta = e^{2\pi i/d}$.*

*Proof.* Let $W \subset V$ be a proper irreducible $H$-stable subspace. For any $r \geq 1$ the space $A^rW$ is $H$-stable since $H$ is a normal subgroup of $G$. Hence $W \cap A^rW$ is $H$-stable and since $W$ is irreducible under $H$ we have either $W = A^rW$ or $W \cap A^rW = \{0\}$. Let $d \geq 1$ be the smallest integer such that $W = A^dW$. Then $V = \oplus_{i=1}^{d-1}A^iW$ since $G$ acts irreducibly on $V$. Moreover, $d > 1$. Let $\gamma_1, \dots, \gamma_{n/d}$ be the set of eigenvalues of $A^d$: $W \to W$. Hence $P(A^d) = \prod_{i=1}^{n/d}(A^d - \gamma_i) = 0$ as an endomorphism of $W$. However, $P(A^d)A^r = A^rP(A^d)$ for any $r \geq 0$ and thus $P(A^d) = 0$ as an endomorphism of each $A^rW$. Hence $P(A^d)$ acts trivially on all of $V$. The characteristic polynomial of $A$: $V \to V$ therefore reads $P(X^d) = 0$ and our statement follows by putting $\lambda_i = \gamma_i^{1/d}$ $(i = 1, \dots, n/d)$. q.e.d.

COROLLARY 3.3. *Let again notation be as above and let $G^0$ be the connected component of the identity of $G = \mathrm{Gal}(\mathscr{F}/\mathbf{C}(z))$. Suppose $G$ acts irreducibly on the space $V$ of solutions of $Ly = 0$. If $G^0$ acts reducibly on $V$ then there exists an integer $d > 1$, $d|n$, such that modulo $\mathbf{Z}$ the set $\{\beta_1, \dots, \beta_n\}$ equals a union of sets of the form $\{\mu, \mu + 1/d, \dots, \mu + (d-1)/d\}$.*

*Proof.* Let $M$ be the fixed field of $G^0$. Since $G/G^0$ is finite, $[M:\mathbf{C}(z)] < \infty$. The equation $Ly = 0$ has only singularities at $z = 0, \infty$; so elements of $\mathscr{F}$ can only have ramification at these points. Hence $M = \mathbf{C}(z^{1/f})$ for some $f > 1$. Thus it is clear that $G/G^0$ is cyclic of order $f$ and that the Galois element $A$ of Proposition 3.1 is a generator of $G/G^0$. Our assertion now follows from Lemma 3.2. q.e.d.

PROPOSITION 3.4. *Let the differential equation $Ly = 0$ and the parameters $\alpha_i, a_i, \beta_j$ satisfy the assumptions of this section. Let $\tilde{L}y = 0$ be another $n$-th order equation satisfying analogous conditions with parameters $\tilde{\alpha}_i, \tilde{a}_i, \tilde{\beta}_i$. If the* *systems corresponding to* $Ly = 0$ *and* $\tilde{L}y = 0$ *are cogredient, then*:

i) *There exists* $\lambda \in \mathbf{C}$ *such that on renumbering if necessary*,

$$\beta_i \equiv \tilde{\beta}_i + \lambda (\operatorname{mod} \mathbf{Z}), \qquad i = 1, \dots, n.$$

ii) *There exists* $\mu \in \mathbf{C}$, $\nu \in \mathbf{R}$ *such that on renumbering if necessary*,

$$a_i = \tilde{a}_i + \mu,\ \alpha_i \equiv \tilde{\alpha}_i + \nu (\operatorname{mod} \mathbf{Z}), \qquad i = 1, \dots, n.$$

*If the systems corresponding to* $Ly = 0$ *and* $\tilde{L}y = 0$ *are contragredient, then the conditions* (i) *and* (ii) *hold with* $-\alpha_i$, $-a_i$, $-\beta_i$ *instead of* $\alpha_i$, $a_i$, $\beta_i$.

*Proof.* The system of first order equations corresponding to $Ly = 0$ has at $z = 0$ a fundamental solution matrix of the form

$$\phi = Y_0(z) \begin{pmatrix} z^{\beta_1} & & 0 \\ & \ddots & \\ 0 & & z^{\beta_n} \end{pmatrix}$$

where $Y_0(z)$ is an invertible $n \times n$ matrix of Laurent expansions in $z$. Similarly the system corresponding to $\tilde{L}y = 0$ has at $z = 0$ the fundamental solution

$$\tilde{\phi} = \tilde{Y}_0(z) \begin{pmatrix} z^{\tilde{\beta}_1} & & 0 \\ & \ddots & \\ 0 & & z^{\tilde{\beta}_n} \end{pmatrix}.$$

Cogredience of the systems means that there is an invertible constant matrix $C = (c_{ij})$, a matrix $M$ with rational function entries, and a function $Q(z)$ with $Q'(z)/Q(z) \in \mathbf{C}(z)$ such that $QM\tilde{\phi} = \phi C$ or,

$$QY_0(z)^{-1} M \tilde{Y}_0(z) = \begin{pmatrix} z^{\beta_1} & & 0 \\ & \ddots & \\ 0 & & z^{\beta_n} \end{pmatrix} \begin{pmatrix} z^{-\tilde{\beta}_1} & & 0 \\ & \ddots & \\ 0 & & z^{-\tilde{\beta}_n} \end{pmatrix} = \left( z^{\beta_i} c_{ij} z^{-\tilde{\beta}_j} \right)_{i,j}.$$

Condition (i) now follows immediately on selection of a non-zero entry from each row and column of the last matrix.

Contragredience means that $QM\tilde{\phi} = (\phi^{-1})^t C$ and condition (i) follows in that case with $-\beta_i$ instead of $\beta_i$. Condition (ii) follows by similar consideration of the formal solutions at $z = \infty$. q.e.d.

## 4. Hypergeometric functions

For $0 \le p < q$ and $\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_{q-1} \in \mathbf{R}$, $\nu_i \ne 0, -1, -2, \dots$ for all $i$, we define the generalized hypergeometric function

$${}_pF_{q-1}(z) = {}_pF_{q-1}((\mu), (\nu); z) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_p)_n}{(\nu_1)_n \cdots (\nu_{q-1})_n} \frac{z^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ and $(\alpha)_0 = 1$. Note that if $\mu_i, \nu_j \in \mathbf{Q}$ for all $i, j$ then ${}_pF_{q-1}(z^{q-p})$ is an $E$-function in Siegel's sense [Si2, II. 9]. For example,

$${}_0F_1(1; z^2) = J_0(2iz),$$

where $J_0$ is the Bessel function of order zero. It is easy to verify that ${}_pF_{q-1}$ satisfies the differential equation

$$\left\{\prod_{j=1}^{q}(\theta+\nu_j-1) - z\prod_{i=1}^{p}(\theta+\mu_i)\right\}F = 0. \tag{4.1}$$

where $\theta = zd/dz$ and where we have put $\nu_q = 1$ for notational convenience. If all the $\nu_j$ are distinct modulo 1, then $q$ $\mathbf{C}$-linearly independent solutions of (4.1) are clearly given by

$$z^{1-\nu_j}{}_pF_{q-1}\big(1+(\mu)-\nu_j, 1+(\nu)'-\nu_j; z\big), \tag{4.2}$$

$j = 1,\dots,q$, where $1+(\mu)-\nu_j$ denotes the $p$-tuple $(1+\mu_i-\nu_j)$, $i = 1,\dots,p$, and $1+(\nu)'-\nu_j$ the analogous $(q-1)$-tuple $(1+\nu_k-\nu_j)$, $k = 1,\dots,q$, $k \neq j$.

When the $\nu_j$ are distinct modulo 1 and each $\nu_j - \mu_i \notin \mathbf{N}$, we consider the functions

$$G_0(z) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{q}\Gamma(1-\nu_j-s)}{\prod_{i=1}^{p}\Gamma(1-\mu_i-s)} z^s\,ds$$

$$G_k(z) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{q}\Gamma(1-\nu_j-s)}{\prod_{i=1}^{p}\Gamma(1-\mu_i-s)}\,\frac{\pi}{\sin\pi(\mu_k+s)} z^s\,ds \quad \text{for } k = 1,2,\dots,p$$

where the contour $C$ runs from $\infty - i\tau$ to $\infty + i\tau$ $(\tau > 0)$ and encloses all the poles $1-\nu_j, 2-\nu_j,\dots$ $j = 1,\dots,q$ but none of the poles $-\mu_k, -\mu_k-1, -\mu_k-2,\dots$ of the second integrand. These functions are special cases of Meijer's $G$-functions, which Meijer evaluates explicitly in terms of hypergeometric functions ([Me, (7)]).

$$G_k(z) = \sum_{h=1}^{q} c_{h,k} z^{1-\nu_h}{}_pF_{q-1}\big(1+(\mu)-\nu_h, 1+(\nu)'-\nu_h; (-1)^{q-p+\delta}z\big) \tag{4.3}$$

for $k = 0,1,\dots,p$ where $\delta = 0$ if $k = 0$ and $\delta = 1$ if $k > 0$ and

$$c_{h,0} = \prod_{j=1}^{q}{}'\Gamma(\nu_h-\nu_j)\Big/\prod_{i=1}^{p}\Gamma(\nu_h-\mu_i),$$

$$c_{h,k} = \frac{\pi c_{h,0}}{\sin\pi(\nu_h-\mu_k)}, \qquad k = 1,\dots,p.$$

The prime in the product indicates omission of the factor $\Gamma(\nu_h - \nu_h)$. Barnes' asymptotic expansions for the $G_k(z)$ can be stated as follows ([Me(18), (26)]),

$$(4.4)\qquad G_0(z) \sim \exp\left((p-q)z^{1/(q-p)}\right)z^{\lambda/(q-p)} \times\left\{\frac{(2\pi)^{1/2(q-p-1)}}{(q-p)^{1/2}} + \frac{M_1}{z^{1/(q-p)}} + \frac{M_2}{z^{2/(q-p)}} + \cdots\right\},$$

with $M_1, M_2, \ldots \in \mathbf{C}$ (depending on $(\mu), (\nu)$ in a complicated way), $\lambda = \{\sum_{i=1}^{p}\mu_i - \sum_{j=1}^{q}\nu_j - \frac{1}{2}(p-q-1)\}$ for $|\arg z| < (q-p+\varepsilon)\pi$ where $\varepsilon = \frac{1}{2}$ if $q = p+1$ and $\varepsilon = 1$ if $q \geq p+2$. For $k = 1, 2, \ldots, p$ we have,

$$(4.5)\qquad G_k(z) \sim \frac{\prod_{j=1}^{q}\Gamma(1-\nu_j+\mu_k)}{\prod_{\substack{i=1\\ i\neq k}}^{p}\Gamma(1-\mu_i+\mu_k)} z^{-\mu_k}{}_qF_{p-1}\left((1-\nu)+\mu_k,\ 1-(\mu)'+\mu_k; -\frac{1}{z}\right)$$

for $|\arg z| < (\frac{1}{2}q - \frac{1}{2}p + 1)\pi$. These expansions give us the following proposition.

PROPOSITION 4.1. *Let $\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q \in \mathbf{R}$ with $q > p$. Assume the $\mu_i$ (and $\nu_j$) to be distinct modulo 1 and that $(\nu_j - \mu_i) \notin \mathbf{Z}$ for each pair $i, j$. Consider the differential equation*

$$(4.6)\qquad \left\{(-1)^{q-p}z\prod_{i=1}^{p}(\theta+\mu_i) - \prod_{j=1}^{q}(\theta-1+\nu_j)\right\}F = 0, \qquad \theta = z\frac{d}{dz}.$$

*Then we have the following:*

i) *A complete set of solutions about $z = 0$ is provided by*

$$f_h(z) = z^{1-\nu_h}{}_pF_{q-1}\left(1+(\mu)-\nu_h, 1+(\nu)'-\nu_h; (-1)^{q-p}z\right), \quad h = 1, \ldots, q.$$

ii) *There are constants $M_0, M_1, \ldots \in \mathbf{C}$ such that a complete set of formal solutions at $z = \infty$ is given by the $p$ expansions*

$$(4.7)\qquad z^{-\mu_h}{}_qF_{p-1}\left(1-(\nu)+\mu_h, 1-(\mu)'+\mu_h; \frac{1}{z}\right), \qquad h = 1, \ldots, p$$

*and the $q-p$ expansions*

$$(4.8)\qquad \exp\left((p-q)\zeta^{1}t\right)t^{\lambda}\sum_{j=0}^{\infty}M_j\zeta^{-1j}t^{-j},$$

*$-(q-p)/2 \leq 1 < (q-p)/2$, where $t^{q-p} = z$, $\zeta = e^{2\pi i/(q-p)}$ and $\lambda$ is as above.*

*Proof.* It is straightforward to verify that the functions $f_h(z)$ satisfy (4.6). Since the $\nu_j$ are distinct modulo $\mathbf{Z}$, the $f_h(z)$ are $\mathbf{C}$-linearly independent and (i) holds.

It is similarly straightforward to check that the formal series (4.7) $h = 1, \ldots, p$ verify (4.6) and are $\mathbf{C}$-linearly independent. Equation (4.4) shows that the expansions (4.8) are formal solutions of (4.6). The $\mathbf{C}$-linear independence of the $q - p$ formal expansions is evident, and (ii) holds. q.e.d.

In order to show Siegel normality of the system of linear first order equations corresponding to (4.6) we need two lemmas.

LEMMA 4.2. *Let $\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q \in \mathbf{R}$, $0 \le p < q$. If $(\mu_i - \nu_j) \notin \mathbf{Z}$ for all $i$, $j$, then the differential equation* (4.6) *is irreducible over* $\mathbf{C}(z)$.

*Proof.* Suppose that (4.6) is reducible, that is, it can be written in the form $(L_1 \cdot L_2)F = 0$ where $L_1, L_2$ are elements of $\mathbf{C}(z)[\theta]$, $\theta = zd/dz$ of order less than $q$. Among the solutions of $L_2 F = 0$ there must be one of the form $z^{1-\nu_h}{}_pF_{q-1}((1 + \mu) - \nu_h, 1 + (\nu)' - \nu_h; z)$. This implies the existence of a generalized hypergeometric function ${}_pF_{q-1}((\alpha), (\beta); z)$, where $(\alpha) = 1 + (\mu) - \nu_h$, $(\beta) = 1 + (\nu)' - \nu_h$ satisfying a differential equation over $\mathbf{C}(z)$ of order $< q$. When ${}_pF_{q-1}((\alpha), (\beta); z) = \sum_{n=0}^{\infty} a_n z^n$, this implies the existence of a recurrence relation for the $a_n$ of the form

$$A_k(n)a_{n+k} + \cdots + A_1(n)a_{n+1} + A_0(n)a_n = 0, \qquad n \ge 0,$$

where the $A_i(n)$ are polynomials in $n$ of degree $< q$. Since $a_n = (\alpha_1)_n \cdots (\alpha_p)_n / n!(\beta_1)_n \cdots (\beta_{q-1})_n$ for all $n$, the quotients $a_{n+i}/a_n$ are rational functions in $n$. Thus we find,

$$(4.9) \qquad A_k(x)\frac{\prod_{i=1}^{p}(\alpha_i + x) \cdots (\alpha_i + x + k - 1)}{\prod_{j=1}^{q}(\beta_j + x) \cdots (\beta_j + x + k - 1)} + \cdots + A_0(x) = 0$$

for all $x \in \mathbf{C}$, and $\beta_q = 1$.

Since $\deg A_k(x) < q$ and $\alpha_i \not\equiv \beta_j \pmod 1$ for all $i$, $j$ the left-most term has a pole of the form $x = 1 - k - \beta_j$ which none of the other terms possesses. Hence (4.9) can never be satisfied and we have a contradiction. q.e.d.

LEMMA 4.3. *Let $G$ be an algebraic group acting on a vector space $V$ of dimension $n$. Suppose $G^0$ acts irreducibly on $V$. Let $g \in G$ act on a basis $v_1, \ldots, v_n$ of $V$ as $g$: $v_i \to \chi_i v_i$ $(i = 1, \ldots, n)$ and suppose $\chi_i \chi_j$ $(i \le j)$ are all distinct. Then $G$ acts irreducibly on $S^2V$.*

*Proof.* By assumption, $g$ acts on the vectors $v_i \otimes v_j + v_j \otimes v_i$ $(i \le j)$ with distinct characters. So any $G$-stable subspace $W \subset S^2V$ is spanned by vectors of the form $v_i \otimes v_j + v_j \otimes v_i$. Suppose $G$ acts reducibly on $S^2V$. Since $G$ acts completely reducibly on $S^2V$, $v_1 \otimes v_1$ is contained in a proper $G$-stable subspace $W \subset S^2V$. Let $\sigma \in G^0$ and $(a_{ij}) = (a_{ij}(\sigma))$ be its matrix with respect to the basis $v_1, \dots, v_n$. Then $\sigma\colon v_1 \to \sum_{i=1}^n a_{1i} v_i$ and

$$\sigma\colon v_1 \otimes v_1 \to \sum_{i=1}^{n} a_{1i}^2 (v_i \otimes v_i) + \sum_{i<j} a_{1i} a_{1j} (v_i \otimes v_j + v_j \otimes v_i).$$

As soon as $a_{1i} a_{1j} \neq 0$ for some $i, j$ and some $\sigma \in G^0$ we see that $v_i \otimes v_j + v_j \otimes v_i \in W$ by taking suitable linear combinations of $\sigma(v_1 \otimes v_1)$, $g\sigma(v_1 \otimes v_1), \dots$ . Since $W \subset S^2 V$ is a proper subspace there must exist $i, j$ such that $a_{1i}(\sigma) a_{1j}(\sigma) = 0$ for all $\sigma \in G^0$. However, $G^0$ is connected, which implies that at least one of $a_{1i}, a_{1j}$ is zero for all $\sigma \in G^0$. Say $a_{1i}(\sigma) = 0$, for all $\sigma \in G^0$. But then $\{\sigma v_1 | \sigma \in G^0\}$ is contained in the space spanned by $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$, contradicting the irreducibility of $G^0$ on $V$. q.e.d.

*Remark.* O. Gabber and N. M. Katz have shown that in fact $G^0$ contains $\mathrm{SL}(n, \mathbf{C})$. The above approach was inspired by their proof, which they generously placed at our disposal.

PROPOSITION 4.4. *Let $\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q \in \mathbf{R}$, $0 \leq p < q$. Suppose that $q \geq 2$ and that at least one of the following conditions* A), B) *holds.*

A) *$(\nu_j - \mu_i) \notin \mathbf{Z}$ for $1 \leq i \leq p$, $1 \leq j \leq q$ and the sums $\nu_i + \nu_j$ $(1 \leq i \leq j \leq q)$ are all distinct modulo $\mathbf{Z}$.*

B) *$p = 0$; $q$ is odd or $q = 2$; and the set $\{\nu_1, \dots, \nu_q\}$ is modulo $\mathbf{Z}$ not a union of arithmetic sequences $\{\nu, \nu + 1/d, \dots, \nu + (d-1)/d\}$ of a fixed length $d$, where $d | q$, $d > 1$.*

*Then the system of first order linear equations corresponding to*

$$\prod_{j=1}^{q} (\theta + \nu_j - 1) F = (-1)^{q-p} z \prod_{j=1}^{p} (\theta + \mu_j) F, \qquad \theta = z\frac{d}{dz} \tag{4.10}$$

*is irreducible and Siegel normal.*

*Remark.* Note that the functions satisfying (4.10) with condition B) extend the class of functions considered by Salichov in [Sa3] who restricted to the case where $q$ is a prime.

*Proof.* Note that (4.9) is a special case of the kind of equations discussed in Section 3. The parameters used are here replaced as follows,

$$q \to q - p,$$

$$\{a_i\} \to \{0, \dots, 0, p - q, (p - q)\zeta, \dots, (p - q)\zeta^{q-p-1}\}, \qquad \zeta = e^{2\pi i/(q-p)},$$

$$\{\alpha_i\} \to \{(p - q)\mu_1, \dots, (p - q)\mu_p, \lambda, \dots, \lambda\}, \qquad \lambda = \Sigma\mu_i - \Sigma\nu_j - \tfrac{1}{2}(p - q - 1),$$

$$\{\beta_i\} \to \{1 - \nu_i\}.$$

According to Theorem 2.2 it suffices to show that $G$ acts irreducibly on $S^2V$, where $V$ is the solution space of (4.10), and that $G^0$ is not a torus.

In both cases A) and B), Lemma 4.2 ensures that $G$ acts irreducibly on $V$. Furthermore the conditions on $\nu_i$ in both cases see to it that Corollary 3.3 implies that $G^0$ acts irreducibly on $V$. In particular, $G^0$ is not a torus.

In case A) we apply Lemma 4.3 with $g$ equal to the monodromy substitution $A$ of Proposition 3.1. Since $\chi_j = \exp(2\pi i(1-\nu_j))\ (j = 1, \dots, q)$ condition A) implies the distinctness of all products $\chi_i\chi_j\ (i \leq j)$.

In case B) we apply Lemma 4.3 with the Galois substitution $B$ of Proposition 3.1. Since $\chi_j = \exp(a_j w) = \exp(-q\zeta^j w)\ (\zeta = e^{2\pi i/q}, w \neq 0; j = 1, \dots, q)$ the distinctness of the products $\chi_i\chi_j\ (i \leq j)$ is equivalent to distinctness of the sums $\zeta^i + \zeta^j\ (i \leq j)$. The latter fact is easily verified in case $q$ is odd or if $q = 2$. q.e.d.

To state our final result concisely, we introduce the notion of *parameter sets* $S = \{\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q\}$, $q > p \geq 0$, $q \geq 2$, $\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q \in \mathbf{R}$, $\nu_q = 1$, and make the following definitions:

(I) $S$ is *admissible* if it satisfies conditions A) or B) of Proposition 4.4.

(II) Two parameter sets $S = \{\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q\}$, $S' = \{\mu'_q, \dots, \mu'_{p'}, \dots, \nu'_1, \dots, \nu'_{q'}\}$ are considered *similar* if

a) $p' = p$, $q' = q$ and

b) There exist $\mu_0, \nu_0 \in \mathbf{R}$ and $r \in \{0, 1\}$ such that on renumbering if necessary, we have

$$\mu'_i \equiv \mu_0 + (-1)^r \mu_i \text{ (modulo } \mathbf{Z}), \quad i = 1, \dots, p,$$

$$\nu'_j \equiv \nu_0 + (-1)^r \nu_j \text{ (modulo } \mathbf{Z}), \quad j = 1, \dots, q.$$

For a given parameter set $S = \{\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q\}$ we write

$$f_S(z) = {}_pF_{q-1}((\mu), (\nu); (-z)^{q-p}) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_p)_n}{(\nu_1)_n \cdots (\nu_{q-1})_n} \frac{(-z)^{(q-p)n}}{n!}.$$

THEOREM 4.5. *Let $S_1, \dots, S_r$ be (possibly equal) admissible parameter sets and $a_1, \dots, a_r \in \mathbf{C}^\times$. If the parameter sets $S_i$ and $S_j$, $i \neq j$ are similar of length $p; q$ then we assume that $\pm a_i/a_j$ are not $(q-p)$-th roots of unity. Let $b_1, \dots, b_s \in \mathbf{C}$ be $\mathbf{Q}$-linearly independent. Then the direct sum of the systems of first order linear equations corresponding to the differential equations for*

$$f_{S_1}(a_1 z), \dots, f_{S_r}(a_r z), 1, e^{b_1 z}, \dots, e^{b_s z}$$

*is Siegel normal.*

*Proof.* Let $S$ be an admissible parameter set and let $Ly = 0$ be the differential equation obtained from (4.10) with parameter set $S$ by the substitution $z \to a^{q-p}z$, $a \in \mathbf{C}$. The differential equation for $f_S(az)$ can then be obtained by the further substitution $z \to z^{q-p}$. According to Proposition 4.4 the first order system corresponding to $Ly = 0$ is irreducible and Siegel normal. Hence Proposition 2.4 implies that the first order system corresponding to $f_S(az)$ is also irreducible and Siegel normal.

Let $T$ be another admissible parameter set with the same $q, p$ and let $\tilde{L}y = 0$ be the differential equation obtained from (4.10) with parameter set $T$ by the substitution $z \to \tilde{a}^{q-p}z$, $\tilde{a} \in \mathbf{C}$. The differential equation for $f_T(\tilde{a}z)$ can again be obtained by $z \to z^{q-p}$. Suppose that the first order systems corresponding to $f_S(az)$, $f_T(\tilde{a}z)$ are cogredient. Then according to Proposition 2.4 the first order systems corresponding to $Ly = 0$, $\tilde{L}y = 0$ are cogredient. However, Proposition 3.3 then implies

(i) There exists $\nu \in \mathbf{R}$ such that, on renumbering if necessary,

$$\nu + \nu_i \equiv \tilde{\nu}_i \pmod 1, \qquad i = 1, \dots, q.$$

(ii) There exists $\mu \in \mathbf{R}$, $\lambda \in \mathbf{C}$ such that, on renumbering if necessary,

$$\mu + \mu_i \equiv \tilde{\mu}_i \pmod 1, \qquad i = 1, \dots, p$$

and for some permutation $\sigma$ of $\{1, \dots, q-p\}$,

$$\lambda + a\zeta^l = \tilde{a}\zeta^{\sigma(l)} \qquad (l = 1, \dots, q-p).$$

Note that (ii) implies $\lambda = 0$ and $(a/\tilde{a})^{q-p} = 1$. Similarly, contragredience of the systems corresponding to $f_S(az)$ and $f_T(\tilde{a}z)$ would imply either similarity of $S$ and $T$ or $(-a/\tilde{a})^{q-p} = 1$, both of which possibilities are excluded by our hypotheses. Hence the first part of Theorem 2.3 iii) is satisfied.

The functions $e^{b_i z}$ satisfy $y' = b_i y$, $i = 0, \dots, s$, if we set $b_0 = 0$. The $B_i(z)$ of Theorem 2.3 are our $b_i$. Notice that the condition there on the $B_i(z)$ is here equivalent to our condition that $b_1, \dots, b_s$ be $\mathbf{Q}$-linearly independent. Thus the hypotheses of Theorem 2.3 iii) are satisfied and our result is proved. q.e.d.

Corollary 4.6. *Let the parameter sets* $S_j$ *and the* $a_i$, $b_j$ *be as in Theorem* 4.5 *and assume in addition*:

(i) *The* $\mu$'s *and* $\nu$'s *all lie in* $\mathbf{Q}$,

(ii) $a_1, \dots, a_r, b_1, \dots, b_s \in \overline{\mathbf{Q}}^{\times}$ (*non-zero algebraic numbers*).

*Then the numbers*

$$e^{b_1}, \dots, e^{b_s};$$

$$f_{S_1}(a_1), f'_{S_1}(a_1), \dots, f_{S_1}^{(q(1)-1)}(a_1);$$

$$\dots\dots$$

$$f_{S_r}(a_r), f'_{S_r}(a_r), \dots, f_{S_r}^{(q(r)-1)}(a_r)$$

*are all algebraically independent over* $\mathbf{Q}$ *with an effective measure of algebraic independence.*

*Proof.* According to [Si2, II. 9] the functions $f_{S_i}(a_i z)$ and $e^{b_j z}$ are $E$-functions. According to Theorem 4.5 the conditions of Siegel's theorem in [Si2, II. 8] are therefore satisfied at $z = 1$, and the independence is demonstrated. The effective measure of algebraic independence is given by Theorem 1′ of [Sh3]. q.e.d.

## Appendix

Here we collect some facts on Lie groups. Standard references are [FV], [H1, 2].

Let $G \subset \mathrm{GL}(n, \mathbf{C})$ be a linear algebraic group. We write $G^0$ for the connected component of $G$ containing the identity, and $Z(G)$ for the centre of $G$. By definition, the largest connected normal solvable subgroup of $G$ is called the radical of $G$, denoted by $R(G)$. If $R(G)$ is trivial, $G$ is called semi-simple. If $R(G)$ consists entirely of semi-simple elements we call $G$ reductive. (This is independent of the embedding $G \subset \mathrm{GL}(n, \mathbf{C})$; cf. [H2, Section 15.3].)

Assume the representation of $G$ on $\mathbf{C}^n$ is completely reducible and let $\mathbf{C}^n = V_1 \oplus \cdots \oplus V_r$ be a decomposition of $\mathbf{C}^n$ into irreducible subspaces. By the Lie-Kolchin theorem [H2, Section 17.6] there exists for every $j$ a non-zero vector $v_j \in V_j$ such that $v_j$ is an eigenvector for all $g \in R(G)$. Since $R(G)$ is a normal subgroup of $G$ and $V_j$ is an irreducible representation of $G$, Lemma A.2 (see below) implies that $V_j$ is spanned by common eigenvectors for all $g \in R(G)$. In other words, $R(G)$ acts by diagonal matrices, and $G$ is reductive. The converse statement was proved by H. Weyl using the fact that a reductive group has a compact real form (the unitary trick; cf. [We]). Thus we have:

THEOREM A.1. *The linear algebraic group* $G \subset \mathrm{GL}(n, \mathbf{C})$ *is reductive if and only if the representation of* $G$ *on* $\mathbf{C}^n$ *is completely reducible.*

The following lemma is also known as Clifford's theorem and can be found in [Di, p. 25].

LEMMA A.2. *Let* $G \subset \mathrm{GL}(n, \mathbf{C})$ *be any group acting irreducibly on* $\mathbf{C}^n$. *Let* $H$ *be a normal subgroup of* $G$. *Then we can write* $\mathbf{C}^n = V_1 \oplus \cdots \oplus V_r$, *where* $V_1, \ldots, V_r$ *are* $H$*-stable subspaces of equal dimension, irreducible under* $H$. *Moreover, if* $h = [G : H] < \infty$, *then* $r \leq h$.

The following proposition was proved by Kolchin in [Ko2]. It is crucial for a theorem of Kolchin concerning the independence of Galois groups of differential fields.

PROPOSITION A.3. *Let $G_1, \dots, G_r$ be simple groups in the ordinary group theoretic sense. Let $G \subset G_1 \times \cdots \times G_r$ be a subgroup, such that the canonical projections $\pi_i$: $G \to G_i$ are surjective ($i = 1, \dots, r$). If $G$ is a proper subgroup of $G_1 \times \cdots \times G_r$, then there exist indices $k$, $l$ and an isomorphism $\sigma$: $G_k \to G_l$ such that $\pi_l = \sigma \cdot \pi_k$.*

Suppose $G$ is a connected reductive group. Fix a maximal torus $T$ in $G$, and denote by $\hat{T}$ the character ($=$ weight) lattice of $T$. The Weyl group $W := \mathrm{Norm}_G(T)/T$ acts on $T$ by conjugation, and the corresponding action of $W$ on $\hat{T}$ (or rather $\hat{T} \otimes_{\mathbf{Z}} \mathbf{C}$) realizes $W$ as a finite group generated by reflections. Fix a Weyl chamber, and call elements of $\hat{T}$ in this chamber dominant weights. Now the representation theory of $G$ can be described as follows: If $\pi$: $G \to \mathrm{GL}(V)$ is an irreducible representation of $G$ on a finite dimensional vector space $V$, we decompose $V = \bigoplus_{\mu \in \hat{T}} V_\mu$ as a direct sum of weight spaces for $T$. Here $V_\mu = \{v \in V;\ \pi(t)v = \mu(t)v \cdot \forall t \in T\}$ and $\mu$ is called a weight of the representation $(\pi, V)$ if $V_\mu \neq 0$. Then one can show that the extremal points of the set of weights of $(\pi, V)$ form a single Weyl group orbit $W \cdot \lambda$ for some dominant weight $\lambda \in \hat{T}$. The weight $\lambda$ is called the highest weight of the representation $(\pi, V)$ and we write $V = V_\lambda$ for the representation of $G$ with highest weight $\lambda$.

LEMMA A.4. *Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group whose connected component of the identity $G^0$ is non-scalar. Suppose $G$ acts irreducibly on the symmetric square $S^2V$. Then $G^0$ acts irreducibly on $V$.*

*Proof.* Notice that irreducibility of $G$ on $S^2V$ implies irreducibility of $G$ acting on $V$. According to Lemma A.2, $G^0$ acts on $V$ as a direct sum representation $V_1 \oplus \cdots \oplus V_r$ with $\dim V_1 = \cdots = \dim V_r = D$. Note that by Theorem A.1, $G^0$ is a connected reductive group.

Suppose $r \geq 2$. Again, by Lemma A.2, the representation of $G^0$ on $S^2V$ decomposes as a direct sum of irreducibles $S^2V = W_1 \oplus \cdots \oplus W_s$ with $\dim W_1 = \cdots = \dim W_s = d$. On the other hand, the representation of $G^0$ on $S^2V$ is equivalent to the representation $(\bigoplus_{i=1}^r S^2V_i) \oplus (\bigoplus_{i<j}(V_i \otimes V_j))$. Complete reduction of this representation will again give us a sum of $s$ irreducible representations of dimension $d$. This implies in particular that $d | \dim S^2V_1$; hence $d|(D+1)D$ and $d|\dim V_1 \otimes V_2$; hence $d|D^2$. We conclude $d|D$.

Suppose $V_\lambda$ is a summand in the action of $G^0$ on $V$. Then $V_{2\lambda}$ is a summand in $S^2V_1$ and we have $D = \dim V_\lambda$, $d = \dim V_{2\lambda}$. The fact that $d|D$ implies $\dim V_{2\lambda} \leq \dim V_\lambda$. However, from Weyl's dimension formula it is easy to see that $\dim V_{2\lambda} > \dim V_\lambda$ unless $\lambda$ is fixed under the Weyl group, or equivalently,

$d = \dim V_\lambda = 1$. This implies that $G^0$ is a torus. Now write $V$ as a direct sum $\oplus_{j=1}^{s} U_j$, where the $U_j$ are eigenspaces of $G^0$ with distinct characters. The group $G$ permutes the spaces $U_j$. Hence $\oplus_{j=1}^{s} S^2 U_j$ is an invariant subspace in the representation $S^2V$ of $G$. If $s > 1$, it is a proper subspace, contradicting the irreducibility of $S^2V$. So we conclude $s = 1$, and $G^0$ consists of scalar matrices.

This contradicts the assumption of our lemma and we are left with the case $r = 1$; i.e. $G^0$ acts irreducibly on $V$. q.e.d.

Let $G$ be a connected reductive group and $\mathfrak{g}$ its corresponding Lie algebra which we assume embedded in its universal enveloping algebra. For $X, Y \in \mathfrak{g}$ the Killing form $(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$ is an invariant non-degenerate symmetric bilinear form. Let $\{X_1, \dots, X_n\}$ and $\{X^1, \dots, X^n\}$ be dual bases of $\mathfrak{g}$ with respect to the Killing form. The *Casimir operator* $c = \sum_{i=1}^n X_i X^i$ is in fact independent of the choice of bases, and the important property is that $c$ commutes with all $X \in \mathfrak{g}$. Let $\pi\colon G \hookrightarrow \mathrm{GL}(V)$ be a finite dimensional faithful representation of $G$. It induces a representation $\pi$ of $\mathfrak{g}$ and its enveloping algebra. Now $c$ acts on $V$ by the operator $\sum_{i=1}^n \pi(X_i)\pi(X^i)$. However, in our notation we shall omit the $\pi$'s, assuming the representation to be known. If $V = V_\lambda$ is an irreducible representation of $G$ with highest weight $\lambda$, then $c$ acts on $V_\lambda$ as a scalar by Schur's lemma and this scalar can be computed from the action of $c$ on a highest weight vector: $c|V_\lambda = (\lambda, \lambda + 2\rho)\mathrm{Id}$. Here, as usual $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$ where $R_+$ is a system of positive roots of the adjoint representation of $G$.

Theorem A.5. *Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group whose connected component of the identity $G^0$ is non-scalar. Then the following statements are equivalent*:

i) *$S^m V$ is irreducible under $G$, for all $m \geq 1$,*
ii) *$S^2 V$ is irreducible under $G$,*
iii) *$S^m V$ is irreducible under $G^0$, for all $m \geq 1$.*

*Proof.* Since i) $\Rightarrow$ ii) and iii $\Rightarrow$ i) are trivial, we prove ii) $\Rightarrow$ iii). According to Lemma A.4, $G^0$ is a connected reductive group acting irreducibly on $V$. Say $V = V_\lambda$ is the representation of $G^0$ with highest weight $\lambda$.

First we note that $S^2 V_\lambda = V_{2\lambda}$; i.e. $G^0$ acts irreducibly on $S^2V$. Let $v_+$ be a highest weight vector in $V_\lambda$. Then $V_{2\lambda}$ is the $\mathbf{C}$-linear span of the vectors $hv_+ \otimes hv_+$, $h \in G^0$. Since $v_+$ is a highest weight vector, there exists for every $g \in G$ an $h \in G^0$ such that $gv_+ = hv_+$. Hence

$$\begin{aligned} S^2V &= \{\mathbf{C}\text{-span of } gv_+ \otimes gv_+ | g \in G\} \\ &= \{\mathbf{C}\text{-span of } hv_+ \otimes hv_+ | h \in G^0\} = V_{2\lambda}. \end{aligned}$$

In order to prove our theorem we study the action of the Casimir operator corresponding to $G^0$ on $S^mV$ for all $m \geq 2$. First, for any $v \in V_\lambda$,

$$c(v \otimes v) = cv \otimes v + v \otimes cv + \sum_{i=1}^{n} \left(X_i v \otimes X^i v\right) + \left(X^i v \otimes X_i v\right).$$

Using $c(v \otimes v) = (2\lambda, 2\lambda + 2\rho)(v \otimes v)$ and $cv = (\lambda, \lambda + 2\rho)v$ we find that

$$\text{(A.1)} \qquad \sum_{i=1}^{n} \left(X_i v \otimes X^i v\right) + \left(X^i v \otimes X_i v\right) = \left\{(2\lambda, 2\lambda + 2\rho) - 2(\lambda, \lambda + 2\rho)\right\}(v \otimes v).$$

Now consider $c$ acting on the $m$-th tensor product $v \otimes v \otimes \cdots \otimes v$,

$$\begin{aligned} c(v \otimes v \otimes \cdots \otimes v) = {} & \sum_{j=1}^{k} v \otimes \cdots \otimes cv \otimes \cdots \otimes v \\ & + \sum_{j_1 \neq j_2} \sum_{i=1}^{n} \big(v \otimes \cdots \otimes X_i v \otimes \cdots \otimes X^i v \otimes \cdots \otimes v \\ & \qquad + v \otimes \cdots \otimes X^i v \otimes \cdots \otimes X_i v \otimes \cdots \otimes v\big). \end{aligned}$$

Using (A.1) and $cv = (\lambda, \lambda + 2\rho)v$ we find

$$\begin{aligned} c(v \otimes v \otimes \cdots \otimes v) &= m(\lambda, \lambda + 2\rho)(v \otimes \cdots \otimes v) + \tfrac{1}{2}m(m-1) \\ &\quad \times \left\{(2\lambda, 2\lambda + 2\rho) - 2(\lambda, \lambda + 2\rho)\right\}(v \otimes \cdots \otimes v) \\ &= (m\lambda, m\lambda + 2\rho)(v \otimes \cdots \otimes v). \end{aligned}$$

Hence $c$ acts as the scalar $(m\lambda, m\lambda + 2\rho)$ on the $\mathbf{C}$-span of $v \otimes \cdots \otimes v$, $v \in V$, which is precisely $S^mV$. From this scalar action and the fact that $V_{m\lambda} \subset S^mV$ we conclude that $S^mV = V_{m\lambda}$. q.e.d.

We now rephrase our terminology in a language which is more appropriate to what will follow. Let again $v_+ \in V$ be a highest weight vector for the irreducible representation $V = V_\lambda$ of $G^0$. Let $\mathcal{O}$ be the $G^0$-orbit of $v_+$ and $\pi(\mathcal{O})$ the image of $\mathcal{O}$ under the projectivization $\pi$: $V \setminus \{0\} \to \mathbf{P}(V)$. Then $\pi(\mathcal{O})$ is a complete algebraic variety in $\mathbf{P}(V)$ and it equals $G^0/P$, where

$$P = \left\{g \in G^0 \; gv_+ = \alpha v_+ \text{ for some } \alpha \in \mathbf{C}\right\}.$$

If there is a homogeneous $m$-th degree polynomial vanishing on $\pi(\mathcal{O})$ then $G^0$ acts reducibly on the dual of $S^mV$ and hence on $S^mV$ itself. Thus we see:

$$S^mV \text{ irreducible under } G^0 \text{ for all } m \geq 1 \Leftrightarrow \pi(\mathcal{O}) = G^0/P = \mathbf{P}(V).$$

Theorem A.5 could also have been shown as follows. Suppose $S^2V$ is irreducible under $G^0$. Then the space of homogeneous quadratic polynomials vanishing on

$\pi(\mathcal{O})$ is trivial. By a theorem of Kostant the ideal of polynomials vanishing on $\pi(\mathcal{O})$ is generated by quadratic elements. The latter elements are only trivial, hence there are no polynomials vanishing on $\pi(\mathcal{O})$. So $\pi(\mathcal{O}) = \mathbf{P}(V)$ and $S^m V$ is irreducible under $G^0$, for all $m \geq 1$. The proof of Kostant's theorem exploits Casimir operators in a way similar to that in the proof of Theorem A.5.

Let $G$ be an arbitrary connected reductive group, and $T$ a maximal torus of $G$ with Weyl group $W = W(G)$. Put

$$P_W(t) = \sum_{w \in W} t^{2l(w)}$$

where $l(w)$ is the length function with respect to a set of simple reflections. We have

$$P_W(t) = \prod_{i=1}^{n} \frac{1 - t^{2d_i}}{1 - t^2}$$

where $n$ is the rank of $G$, and $d_1 \leq \cdots \leq d_n$ the primitive degrees of $W$. The number of $i$ such that $d_i = 1$ equals the dimension of the center $Z(G)$ of $G$. The number of $i$ such that $d_i = 2$ equals the number of simple factors of $G$.

THEOREM A.6. *Let $G$ be a connected semisimple group, and $P \subset G$ a parabolic subgroup which does not contain any simple factor of $G$. If the Poincaré polynomial*

$$P_{G/P}(t) = \sum_{k \geq 0} \dim H^k(G/P, \mathbf{C}) t^k$$

*has the form $(1 - t^{2d})/(1 - t^2)$ for some $d \in \mathbf{N}$, then we are in one of the following situations*:

| | *type of G* | *type of Levi factor of P* |
|---|---|---|
| i) | $A_n$ | $A_{n-1}$ |
| ii) | $B_n$ | $B_{n-1}$ |
| iii) | $C_n$ | $C_{n-1}$ |
| iv) | $G_2$ | $A_1$ |

*Proof.* From the Bruhat decomposition for $G/P$ follows

$$P_{G/P}(t) = \frac{P_{W(G)}(t)}{P_{W(P)}(t)}$$

where $W(G)$ is the Weyl group of $G$, and $W(P)$ the Weyl group of a Levi factor of $P$. Let $d_1, \ldots, d_n$ be the primitive degrees of $W(G)$ and $d_1', \ldots, d_n'$ the primitive degrees of $W(P)$, ordered such that $d_1 \leq \cdots \leq d_n$ and $d_1' \leq \cdots$

$\leq d_n'$. Then

$$\frac{1-t^{2d}}{1-t^2} = \frac{P_{W(G)}(t)}{P_{W(P)}(t)}$$

implies

$$\{d_1', \dots, d_n', d\} = \{1, d_1, \dots, d_n\}.$$

Since $G$ and $P$ have no simple factors in common, we have $d \geq 2$. Hence $d_1' = 1$, $d_2' = 2$ since $d_1 = 2$. This means that $P$ is a maximal parabolic subgroup, which in turn implies that $G$ is simple. Hence $d_2 \geq 3$ and therefore $d_3' \geq 3$. This means that the Dynkin diagram of a Levi factor of $P$ is connected. From [FV, p. 515, 516] where the exponents $m_i = d_i - 1$ are tabulated, we find the following list:

| Dynkin diagram | primitive degrees |
|---|---|
| $A_n$ | $2, 3, \dots, n+1$ |
| $B_n, C_n$ | $2, 4, \dots, 2n$ |
| $D_n$ | $2, 4, \dots, 2n-2, n$ |
| $E_6$ | $2, 5, 6, 8, 9$ |
| $E_7$ | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$ | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $F_4$ | $2, 6, 8, 12$ |
| $G_2$ | $2, 6$ |

Inspection of this table with respect to $\{d_2', \dots, d_n', d\} = \{d_1, \dots, d_n\}$ where $d_2', \dots, d_n'$ are the primitive degrees of a Levi factor of $P_1$ gives us the four cases of our theorem and $G = D_n$, a Levi factor of $P$ is $B_{n-1}, C_{n-1}$. The latter case can be excluded however, since the Dynkin diagram of $B_{n-1}, C_{n-1}$ cannot be obtained by deletion of a vertex from the diagram of $D_n$. q.e.d.

THEOREM A.7. *Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group. Then $S^mV$ is irreducible under $G$ for all $m \geq 1$ if and only if $G$ is an extension by scalars of $H$, where either*

i) $H = \mathrm{SL}(n, \mathbf{C})$ *and* $\dim(V) = n$

*or*

ii) $H = \mathrm{Sp}(2n, \mathbf{C})$ *and* $\dim(V) = 2n$.

*Proof.* Suppose $S^mV$ is irreducible under $G$ for all $m \geq 1$. If $\dim V = 1$, we are in case i) for $n = 1$.

Suppose $\dim V \geq 2$. If $G^0$ consists of scalars, then $G$ is a central extension of a finite group and $S^mV$ cannot be irreducible under $G$ for all $m \geq 1$. Hence, by Theorem A.5, $S^mV$ is irreducible under $G^0$ for all $m \geq 1$.

Write $H$ for the commutator subgroup of $G^0$. Then $H$ is a connected semi-simple group and an irreducible representation of $G^0$ remains irreducible under restriction to $H$. Hence $S^mV$ is an irreducible representation of $H$ for all $m \geq 1$. From the remarks following Theorem A.5 we know that this is equivalent to the fact that the projectivized orbit of the highest weight vector, being of the form $H/P$ for some parabolic subgroup $P$ of $H$, is equal to $\mathbf{P}(V)$. Moreover, since the representation $H \subset \mathrm{GL}(V)$ is faithful, $H$ and $P$ have no simple factors in common. Hence we can apply Theorem A.6 to conclude that $H$ and $P$ are of the form stated there. Also, the highest weight of $V$ as a representation of $H$ is a multiple of the fundamental weight corresponding to the maximal parabolic subgroup $P$ of $H$. We tabulate the dimension of $H/P$ and the relevant fundamental representations:

| $H$ | $\dim(H/P)$ | $\dim V$ | |
|---|---|---|---|
| $A_n$ | $n$ | $n+1$ | (standard representation of $\mathrm{SL}(n+1, \mathbf{C})$ or dual) |
| $B_n$ | $2n-1$ | $2n+1$ | (standard representation of $\mathrm{SO}(2n+1, \mathbf{C})$ |
| $C_n$ | $2n-1$ | $2n$ | (standard representation of $\mathrm{Sp}(2n, \mathbf{C})$ |
| $G_2$ | 5 | 7, 14 | |

The condition $H/P \simeq \mathbf{P}(V)$ implies $\dim V = 1 + \dim H/P$, and we observe that we are left with cases given in our statement. q.e.d.

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