ARITHMETIC THEORY OF DIFFERENTIAL EQUATIONS (*)

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Introduction.

Let

$$y = \sum_{j=0}^{\infty} b_j x^j \in \mathbb{Q}[\![x]\!] \tag{0.1}$$

be a formal power series solution of the ordinary linear differential equation

$$Ly = 0 \tag{0.2}$$

where

$$L = A_0 D^n + A_1 D^{2-1} + \ldots + A_n \in \mathbb{Q}[x][D]$$

with $D = d/dx$. If the origin is not an irregular singular point of L then we know that the series (0.1) converges (in the complex sense) up to the nearest singularity. This gives an estimate for the archimedean magnitude of b_j but gives no indication of the magnitude of the denominator of b_j when written as a ratio of integers.

A general and elementary estimate of Popkin [25, p. 206] gives

$$\log(\text{denominator } b_j) = O(j \log j)\,. \tag{0.3}$$

The remarkable results of Apery on the irrationality of § (3) has stimulated interest in questions of this type. F. Bombieri in lectures at the Institute for Advanced Study (Princeton) in the fall of 1978 and Čudnovsky in lectures at the College de France in the spring of 1979 have raised the question of determining under what conditions, the

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series (0.1) are G functions. This question reduces (cf. 5. 6. 4 below) to that of the p-adic nature of solution of (2). (Def: The series (0.1) is said to represent a G-function if there exists a real number C such that

(i) $|b_j| < C^j$

(ii) $[d_1, \ldots, d_j] < C^j$

where for each s, d_s denotes the denominator of b_s. If the b_j are algebraic numbers (instead of rational numbers) then we just change $|b_j|$ in (i) to the maximal archimedean value of b_j.)

The origin may be an ordinary or singular point of L. In either case the indicial polynomial has algebraic roots. It follows [2] that the p-adic radius of convergence of (0.1) is not trivial. Our object is to report on the status of the open question:

(0.4) What is the p-adic radius of convergence of the series (0.1)? In particular how does this radius vary with p?

1. Eisenstein's theorem.

This question is well understood in the case of algebraic functions.

1.1 Theorem [4]: If the series (0.1) represents a function algebraic over $\mathbb{Q}(x)$ then there exists an integer m such that $m^j b_j \in \mathbb{Z}$ for all j.

Following work of S. Brown [1] we had in joint work with Robba made the above result more precise.

1.2 Theorem [16]: If the series (0.1) represents an algebraic function y, then after multiplication by an integer, the only primes appearing in the denominators of the b_j are either those primes not exceeding the degree of y or those primes p for which the singularities of y (*i.e.* the zeros of the discriminant of the polynomial equation over $\mathbb{Q}[x]$ satisfied by y together with the zeros of the coefficient of the leading term of this equation excluding the origin) have p-adic distance from the origin less than unity. *Note*: At this conference H. Hironaka explained to us a direct proof of this result based on the fact that if $f \in \mathbb{Z}[x, y]$ is the polynomial irreducible over $\mathbb{Q}(x)$ satisfied by y then the Puiseaux expansions of the solutions of the mod p reduction of f can be lifted to characteristic zero provided the discriminant of f and of its mod p reduction have the same degrees as polynomials in x. This explains one set of special primes in 1.2, the other set involves the possible nonexistence of Puiseaux expansions for solutions of polynomial equations of degree p over field of characteristic p.

Eisenstein's Theorem 1.1 may be written in view of question 0.4.

1.3 THEOREM: If (0.1) represents an algebraic function then for almost all primes p the series converges and is bounded by unity in the open p-adic disk $D(0, 1^-)$ of radius unity and center at the origin. For each prime p the series converges in a non-trivial disk $D(0, r_p^-)$.

2. Equations with Frobenius.

2.1 In the 1960's it was realized [5, 6, 8, 9, 10, 23] that for certain classes of differential equations there exists an action of Frobenius. Roughly speaking this would mean (say with the origin an ordinary point of L) that for almost all primes p there exists a transformation of solutions of L defined over $H(D(0, 1^-))$, the ring of analytic elements on $D(0, 1^-)$, which agrees with the transformation $x \to x^{q^b}$ of independent variable, the integer b depending possibly upon p. More explicitly, there exists (in the case under consideration) a non-degenerate $n \times n$ matrix $\mathcal{A}$ with coefficients in $H(D(0, 1^-))$ such that for each solution (0.1) of L, there exists a solution, $z \in \mathbf{Q}_p[\![x]\!]$ such that

2.1.1 $$\mathcal{A}\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}(x^{p^b}) = \begin{pmatrix} z \\ z' \\ \vdots \\ z^{(n-1)} \end{pmatrix}(x).$$

Thus if V is the $\mathbf{Q}_p$ space spanned by all n-tuples

$$v = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

with y a solution of L in $\mathbf{Q}[\![x]\!]$, then

2.1.2 $$v \mapsto \mathcal{A}v(x^{q^b})$$

is an endomorphism of V. This property gives the analytic continuation of the elements of V to functions meromorphic on $D(0, 1^-)$, but indeed we may replace «meromorphic» by «analytic» if L has no singularity in $D(0, 1^-)$.

2.2 The problem of the p-adic nature of solutions near singular points is particularly troublesome. If the singularity (say at the origin) of L is regular with rational exponents then we may write a basis for local formal solutions in the form

2.2.1 $$(y_1, \ldots, y_n) = (F_1, \ldots, F_n) \exp(\theta \log x)$$

where $F_1, \ldots, F_n \in \mathbf{Q}[\![x]\!]$, θ is an $n \times n$ matrix with coefficients in $\mathbf{Q}$.

We have used the property of « over convergence » of the matrix $\mathcal{A}$ to show that in certain cases [9, Theorem 6] the series $F_1, \ldots, F_n$ converge in $D(0, 1^-)$.

2.3 A second consequence of (2.1.2) is the « law of logarithmic growth » of solution of L, namely that there exists v real, $v > 0$ such that as $j \to \infty$

2.3.1 $$|b_j|_p = O(j^v).$$

This follows most easily by extending the field of coefficients of V to a sufficiently large p-adic field, extending (2.1.2) to a semi linear map of V onto itself. Then V is spanned by « eigenvectors » v such that

2.3.2 $$\mathcal{A} v^{\sigma^b}(x^{p^b}) = \lambda v(x)$$

where σ is a suitable lifting of Frobenius to the extension field. From equation 2.3.2 we may bound the growth of $v(x)$ as $|x| \to 1$. From this estimate we may obtain (2.3.1) with estimates for v in terms of $\mathrm{ord}_p\, \lambda$.

In a series of articles culminating in our joint article [17] with Robba, we have largely freed the question of logarithmic growth from dependence upon the mapping (2.1.2). We state our main result.

2.3.3. Let K be an extension of $\mathbf{Q}_p$ with valuation extending that of $\mathbf{Q}_p$. Let $u_1, \ldots, u_n$ be elements of $K[\![x]\!]$ which converge in $D(0, 1^-)$ and whose wronskian has no zeros in that disk. If u is any element in the K span of $u_1, \ldots, u_n$ then

$$u = \sum_{s=0} a_s x^s$$

with

2.3.3.1 $$|a_s| \leqslant \{s, n-1\}_p \sup_{0 \leqslant k \leqslant n-1} |a_k|$$

where

$$1/\{s, n-1\}_p = \operatorname{Inf} |\lambda_1 \ldots \lambda_{n-1}|_p,$$

the infimum being over all sets of $n-1$ *distinct* natural numbers bounded by s.

2.3.4 The corresponding result for the situation (2.2.1) is not well understood except for the case in which θ has one Jordan block. In this case estimates for the series F_j have been obtained by Adolphson and Sperber.

We complete the discussion of orders of growth with two remarks.

2.3.5 The estimates of (2.3.3) are effective but in some cases better non-effective estimates for order of growth can be obtained by means of (2.3.2) using estimates for the p-adic magnitude of roots of the zeta function. Thus in the cases involving the Fuchs-Picard differential equation, equation (2.3.3.1) would give (2.3.1) with $v+1$ equal to the order of the differential equation while using Deligne's estimates, v may be taken to be the dimension of the variety.

2.3.6 The method of proof of 2.3.3.1 involves the systematic use of the formulation of Frobenius in which a differential equation of order n is reduced to one of order $n-1$ after one solution is given. We point out that the same procedure had been used by T. Honda [19] in the article appearing posthumously in this volume. We regret that his work had not been made available at an earlier time.

2.4 To make our discussion of 2.1.2 more concrete we give some examples. Our first two examples are based on the fact that for $\pi = (-p)^{1/p-1}$, the formal series

$$\exp \pi(x - x^p) = 1 + \pi x + \pi^2 x^2/2! + \dots$$

converges in the p-adic disk, $D(0, 1+\varepsilon)$ with some $\varepsilon > 0$.

2.4.1 Let l, m be natural numbers and let $\lambda, v_1, \dots, v_m, t_1, \dots, t_l$ denote $l+m+1$ variables. Let $f_1, \dots, f_l$ be elements of $\mathbb{Z}[\lambda, v_1, \dots, v_m]$ (which need not be homogeneous as polynomials in $v_1, \dots, v_m$). Let

$$F(v, t, \lambda) = \sum_{i=1}^{l} t_i f_i$$

which we view as a polynomial in v, t parametrized by λ. We put

$$\mathfrak{L} = \mathbf{Q}(\lambda)[v, t],$$

a $\mathbb{Q}(\lambda)$ module stable under the $l+m$ commuting differential mappings

$$D_i = t_i \frac{\partial}{\partial t_i} + t_i f_i \qquad i \leqslant i \leqslant l,$$

$$C_j = v_j \frac{\partial}{\partial v_j} + v_j \frac{\partial F}{\partial v_j} \qquad 1 \leqslant j \leqslant m .$$

We know that these operators give rise to a Koszul complex with finite factor spaces but we restrict our attention to related spaces. For each natural number s, let $\mathfrak{L}^{(s)}$ be the sum of the images of $\mathfrak{L}$ under each form in $C_1, \ldots, C_m, D_1, \ldots, D_l$ of degree s. Then $W_s = \mathfrak{L}/\mathfrak{L}^{(s)}$ is a finite dimensional $\mathbb{Q}(\lambda)$ space. The derivation $d/d\lambda$ of $\mathbb{Q}(\lambda)$ is extended to W_s by means of

$$\sigma = \frac{\partial}{\partial \lambda} + \frac{\partial F}{\partial \lambda}$$

which commutes with the C_j and D_i.

These differential modules are associated with the geometry of the variety of common zeros of $f_1, \ldots, f_l$. For fixed s, the module is known to have an action of Frobenius which induces an action of type 2.1.2 on the solutions of the differential equations corresponding to the «horizontal elements» of W_s. These differential equations are independent of p and hence by Katz's theory of global nilpotence (§ 5.6.2 below) these equations have regular singular points and rational exponents. Furthermore these actions of Frobenius are equipped with a structure of over convergence.

2.4.2 Let $\lambda, v_1, \ldots, v_m$ be as in 2.4.1. Let $f \in \mathbb{Z}[\lambda, v_1, \ldots, v_m]$ and to fix ideas we suppose f not homogeneous as polynomial in $v_1, \ldots, v_m$. Let $\pi^{p-1} = -p$, $K = \mathbb{Q}(\pi)$ (so K depends upon p). Let

$$\mathfrak{L} = K(\lambda)[v_1, \ldots, v_m]$$

$$D_i = v_i \frac{\partial}{\partial v_i} + \pi v_i \frac{\partial f}{\partial v_i} \qquad i \leqslant i \leqslant m .$$

Let $\mathfrak{L}^{(s)}$ be the sum of the images of $\mathfrak{L}$ under forms of degree s in $D_1, \ldots, D_m$. Let

$$W_s = \mathfrak{L}/\mathfrak{L}^{(s)} .$$

Then W_s is a finite $Q(\lambda)$ module with differential mapping induced by

$$\sigma = \frac{\partial}{\partial \lambda} + \pi \frac{\partial f}{\partial \lambda}$$

which is a mapping of $\mathcal{L}$ which commutes with each D_i. In this case the corresponding differential equation may depend non-trivially upon p. For almost all p there is again an action of Frobenius (with over convergence). In this case the associated differential equations need not have regular singularities.

2.4.3 Sperber [28, 29] and the author [14] have studied examples similar to (2.4.2) but with $\mathcal{L}$ replaced by the ring of polynomials in $v_1, \ldots, v_m, 1/v_1, \ldots, 1/v_m$. This formulation has simplified explicit calculations and has played a critical role in Sperber's theory of hyper Kloosterman sums but it seems possible that these examples may be subsumed by 2.4.2.

Berthelot's work on cristalline cohomology [20] has shown that an action of Frobenius (2.1.2) may be associated with the Fuchs-Picard differential equations associated with smooth varieties. The relation between example 2.4.1 and de Rahm cohomology was explained long ago by Katz [21] in the case of smooth hypersurfaces. The theory of zeta functions suggests that the two theories must be closely connected. We emphasize that we are concerned here with an « almost all p » theory.

We recall that example 2.4.2 is associated with the theory of exponential sums involving the reduction of $f(x)$. This is again related to the Fuchs-Picard equations associated with the cohomology of the covering

$$z^p - z = f(v)$$

of $(v_1, \ldots, v_m)$ space. The equations in this case need not have regular singular points since the basis of cohomology is non-rational.

This completes our discussion of how geometrical considerations lead to information concerning question (0.4).

We ask whether the examples deduced from (2.4.1) lead to collections of series, (0.1) which are closed under multiplication and integration.

3. **Direct method.**

For algebraic functions we have Eisentein's theorem and then the response to (0.4) is readily deduced for the indefinite integrals of algebraic functions.

For many classical differential equations we may explicitly compute the solutions at some special points, generally at a singular point. For example if we are in the situation (2.2.1) and if the $F_1, \ldots, F_n$

converge in $D(0, 1^-)$ and if the eigenvalues of θ lie in $Q \cap \mathbb{Z}_p$ then by transfer theorems [12 § 1, 15 § 4] we may conclude that for each residue class not containing a singular point, there is a basis of solutions converging on the residue class.

This method of determining the radius of convergence works particularly well in the case of second order differential equations. Here an examination of the wronskian and of one solution is often sufficient. Thus in particular we mention the Gauss hyper geometric function and the Bessel function J_α with $\alpha \in Q \cap \mathbb{Z}_p$.

However these classical examples do not seem to go beyond the method of § 2 since in general they have integral representations which reveal their cohomological interpretation [14, 28]. The theory is not completely developed. Thus integral representations of generalized hyper geometric functions have been given (for example) by Erdelyi [18] but the details of the corresponding crystal are not readily available.

4. Exceptional cases.

Before the work of Apery we were under the impression that the discussion of § 2 covered all examples of equations (0.2) whose solutions converged in $D(0, 1^-)$ for almost all primes. We are indebted to Apery for the following three examples which do not seem to come from geometry but which (after a shift or origin) have solutions converging in $D(0, 1^-)$ for almost all primes.

4.1 $$L_1 = (x - 11x^2 - x^3)D^2 + (1 - 22x - 3x^2)D - (3 + x).$$

The unique solution regular at the origin is given by (0.1) with

$$b_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j} \in \mathbb{Z}.$$

4.2 $$L_2 = x(8x - 1)(x + 1)D^2 + (24x^2 + 14x - 1)D + (8x + 2).$$

The unique solution regular at the origin is given by (0.1) with

$$b_n = \sum_{j=0}^{n} \binom{n}{j}^3.$$

4.3 $$L_3 = (1 - 34x + x^2)x^2D^3 + (3 - 153x + 6x^2)xD^2 + \\ + (1 - 112x + 7x^2)D + (-5 + x).$$

The unique solution regular at the origin is given by (0.1) with

$$b_n = \sum \binom{n}{j}^2 \binom{n+j}{j}^2.$$

We recall that any second order differential equation with four regular singular points and given exponents may be reduced to the form

4.4 $$D^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a}\right) D + \frac{\alpha\beta x - B}{x(x-1)(x-a)}$$

known as Heun's equation [30, *p.* 576]. The associated Riemann data (*i.e.* table of singular points with corresponding exponents) is

4.4.1 $$\begin{pmatrix} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta \end{pmatrix}$$

and the sum of the exponents is equal to 2 (in general a second order differential equation over a Riemann surface of genus g with m singular points has $2g-2+m$ as the sum of exponents). We note that the Riemann data determines (4.4) except for the accessory parameter B. No integral formula is known for the solutions of (4.4) except in degenerate cases which reduce to fewer than four singularities.

Specializing the exponents, we obtain the Lamé equation

4.5 $$f(x)D^2 + \tfrac{1}{2}f'(x)D - [m(m+1)x + B]$$

where

$$f(x) = 4(x-e_1)(x-e_2)(x-e_3)$$

with Riemann data

$$\begin{pmatrix} e_1 & e_2 & e_3 & 0 \\ 0 & 0 & 0 & \dfrac{m+1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -m/2 \end{pmatrix}.$$

With no loss in generality we may assume Re $m \geqslant -\frac{1}{2}$. The Halphen transform [26] is obtained by putting

$$x = \wp(u) \qquad \left(u = \int_x^\infty dt/f(t)\right),$$

letting $v = u/2$ and changing the dependent variable y by setting

$$Y = y(\wp(v))^m \tag{4.6.2}$$

and the independent variable by letting

$$X = \wp(v) \tag{4.6.3}$$

so that x is rational function of X. The transformed equation is

$$\left(f(X)\frac{d^2}{dX^2} + \left(\frac{1}{2} - m\right)f'(X)\frac{d}{dX} + 4\big(m(2m-1)X - B\big)\right)Y = 0 \tag{4.6.4}$$

with Riemann data

$$\begin{pmatrix} e_1 & e_2 & e_3 & \infty \\ 0 & 0 & 0 & -2m \\ m+\frac{1}{2} & m+\frac{1}{2} & m+\frac{1}{2} & \frac{1}{2}-m \end{pmatrix}.$$

For $m = -\frac{1}{2}$, the exponent differences of (4.6.4) are all zero and hence at each singular point there is precisely one analytic solution.

Examples (4.1), (4.2) of Apery, are both specializations of (4.6.4) with $m = -\frac{1}{2}$. They differ in location of singular points and in value of B. Example (4.3) is the symmetric square of a second order differential equation, L_4, with Riemann data

$$\begin{pmatrix} 0 & \theta_1 & \theta_2 & \infty \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

where θ_1, θ_2 are roots of $1 - 34\theta + \theta^2 = 0$. Specifically the unique solution analytic at zero of L_3 is the square of a solution of L_4. Equation L_4 is similar to a Lamé equation.

4.7 Note: I am indebted to R. Askey for explaining how these differential equations may be deduced from the contiguity relations for generalized hypergeometric functions. Thus for example in the case of (4.1) above

$$b_n = {}_3F_2\left(\begin{matrix} -n, & -n, & n+1; & 1 \\ 1, & 1 & & \end{matrix}\right).$$

In particular there are many examples of this kind. Using Pochhammer type of integral formulae for generalized hypergeometric functions the author has been led to the conclusion that the differential equations discussed in this section are the limits of differential equations which arise from geometry.

Note added in proof: C. Sevici and the author have verified that L_1, L_2, L_3 above all come from geometry. In particular L_1 is the pull-back of a hypergeometric differential equation by a rational map.

5. General theory.

Let us assume that the origin is not a singular point of (0.2). Associated with L there is infinite set of rational functions $H_{s,j}$ $(0 \leqslant s,$ $0 \leqslant j \leqslant m-1)$ defined by

$$D^s \equiv \sum_{j=0}^{n-1} H_{s,j} D^j \bmod Q(x)[D]L .$$

Each $H_{s,j}$ lies in the differential ring generated by the ratios $\{A_i/A_0\}_{i \leqslant i \leqslant n}$ of coefficients of L. It follows that for almost all p, the numbers $H_{s,j}(0)$ are all p-integral and the functions $H_{s,j}$ are all p-integral in the sense of Gauss norm.

At this point we introduce for each prime p the generic point t lying in some trascendental extension of $\mathbb{Q}_p$ such that $|t|_p = 1$ and such that the residue class of t is trascendental over $\mathbb{F}_p$, the prime field of characteristic p. For almost all p we have

5.2.1 $$|H_{s,j}(0)|_p \leqslant 1$$

5.2.2 $$|H_{s,j}(t)|_{\text{Gauss}} \leqslant 1$$

and hence the solutions of L at $x = 0$ converge for

5.3.1 $$\operatorname{ord}_p x > \frac{1}{p-1}$$

and those at t converge for

5.3.2 $$\operatorname{ord}_p (x-t) > \frac{1}{p-1} .$$

The excluded primes are those for which either A_0 has a zero in $D(0, 1^-)$ or those for which the Gauss norm of A_j/A_0 exceeds 1 for some j.

5.4.1 The transfer theorem referred to in § 3 implies that if L has n independent solutions converging in $D(0, 1^-)$ then L has n independent solutions converging in $D(t, 1^-)$. The converse holds provided A_0 has no zero in $D(0, 1^-)$.

5.4.2. It is also known [15] that the number of independent solutions converging in $D(t, 1^-)$ is not less than the number of independent solutions converging in $D(0, 1^-)$.

5.4.3 If the solutions of L at t all converge in $D(t, r^-)$ for some $r < 1$ then (supposing A_0 has no zero in $D(0, 1^-)$ we may conclude that all solutions of L at $x = 0$ converge in $D(0, r^-)$).

5.4.4 We do not believe the converse of 5.4.3 to be true. It seems likely that a counter example may be obtained by considering solutions of the modular equation at a modulus corresponding to an elliptic curve with supersingular reduction.

Hence when discussing radii of convergence lying between unity and that of (5.3) we shall restrict our attention to solutions at generic points.

5.5 Lemma (Katz): Suppose that $|A_j/A_0|_{\text{Gauss}} \leqslant 1$ for $1 \leqslant j \leqslant n$. If the solutions of L at the generic point t all converge in a disk

5.5.1 $$\operatorname{ord}_p (x - t) > \frac{1}{p-1} - \varepsilon$$

for some $\varepsilon > 0$ then they all converge for

5.5.2 $$\operatorname{ord}_p (x - t) > \frac{1}{p-1} - \frac{1}{pn}.$$

We sketch the proof. It follows from (5.5.1), using the p-adic Gauss norm, that

5.5.3 $$|H_{s,j}|_{\text{Gauss}} \to 0$$

as $s \to \infty$. Using the ideal theory of $R = \mathbb{F}_p((x))\,[D]$ we may conclude that there exists v such that

5.5.4 $$D^{pv} \equiv 0 \bmod R\bar{L}.$$

where $\bar{L}$ is the reduction of $L \bmod p$. This situation (nilpotent p-curvature) has been studied by Katz [22] (for an elementary account see the article of Honda [19] referred to previously; some elementary indications have been given in [15]). It follows from 5.5.4 using the ideal theory of R that

5.5.5 $$\bar{L} = L_1 L_2 \dots L_n$$

where each L_j is an element of R of order 1 such that

5.5.6 $$D^p \equiv 0 \bmod R\bar{L}.$$

It follows that

5.5.7 $$D^{pn} \equiv 0 \bmod R\bar{L}$$

and hence using the discreteness of the valuation

5.5.8 $$\operatorname{ord}_p H_{p^n,j} \geqslant 1 \qquad (0 \leqslant j \leqslant n-1).$$

From this it follows that for $s > 0$,

5.5.9 $$\operatorname{ord}_p H_{s,j} \geqslant [s/pn],$$

from which (5.5.2) may be deduced.

5.6 The differential equation with constant coefficients, $D-p$, has nilpotent, (indeed zero) p-curvature at p but not at other primes. The following conjecture has been proposed by E. Bombieri.

5.6.1 Conjecture: If (5.5.7) holds for almost all p then the solutions of L at t converge in $D(t, 1^-)$ for almost all p.

We do know:

5.6.2 Theorem (Katz) [19]: If (5.5.7) holds for an infinite set of primes then the singularities are regular. If (5.5.7) holds for a set of primes of density one then the exponents are all rational.

It is extremely difficult to determine whether L has nilpotent curvature. For this reason one should not consider 5.6.1 as an adeguate solution to (0.4).

5.6.3 Conjecture (Grothendieck): If equation (5.5.4) holds with $v=1$ for almost all p then the solutions of L are algebraic.

This has been verified by Katz [24] for the Fuchs-Picard differential equations and for equations associated with certain subspaces of cohomology. It is open for (4.4) and for (4.5).

5.6.4 E. Bombieri has deduced from (2.3.3) that if all solutions of L at t converge for

5.6.4.1 $$\operatorname{ord}_p (x-t) > \varepsilon_p$$

and if $\sum \varepsilon_p \log p$ converges then each solution of L at a non-singular algebraic point is *a* G-function.

Under the hypothesis of 5.6.3, we see that for almost all p, equation 5.6.4.1 holds with

5.6.4.2 $$\varepsilon_p = \frac{1}{(p-1)p}\,.$$

Thus under the hypothesis of 5.6.3 all solutions of L at non-singular algebraic points are G functions. This may be deduced without the use of (2.3.3) since under the hypothesis of 5.6.3, for almost all p, the solutions at t of L not only converge in the disk,

5.6.4.3 $$\operatorname{ord}_p(x-t) > \frac{1}{p(p-1)}$$

but in addition solutions normalized by the condition of having unit initial data (at t) are bounded on this disk independently of p.

We emphasize that we do not know how to show that the hypothesis of 5.6.3 implies that for almost all p the solutions of L at t all converge in $D(t, 1^-)$.

6. **Lamé again.**

A counter example of Deligne shows that the converse to 5.6.2 is false. In this section we show how the counter example is related to the theory of the Lamé equation. This explanation may serve to support our surmise that the examples of Apery (§ 4) do not « come from geometry ». (But see § 4.7.)

Let $\mathbf{Q}^{\mathrm{alg}}$ denote the algebraic closure of $\mathbf{Q}$, let

6.1.1 $$N = D^2 + PD + Q \in \mathbf{Q}^{\mathrm{alg}}(x)[D]$$

and let φ be a solution of the symmetric square of N, *i.e.*

6.1.2 $$\varphi = y_1 y_2$$

where y_1, y_2 are non zero (but possibly dependent) solutions of N. Let

6.1.3 $$\tau = y_2/y_1$$

6.1.4 $$\gamma_i = y_i'/y_i \qquad (i = 1, 2)\,.$$

Finally let

6.1.5 $$-h = (\varphi'/\varphi)^2 + 2(\varphi'/\varphi)' + 2(\varphi'/\varphi)P + 4Q\,.$$

6.2 Lemma (Fuchs):

6.2.0 $$(\tau'/\tau)^2 = h$$

i.e.

6.2.0.1 $$(2\gamma_1 - \varphi'/\varphi)^2 = h\,.$$

Proof: The functions γ_i satisfy the Riccati equation

6.2.1 $$\gamma^2 + \gamma' + P\gamma + Q = 0\,.$$

Furthermore by (6.1.2) and (6.1.3)

6.2.2 $$\tau'/\tau = \gamma_2 - \gamma_1$$

6.2.3 $$\varphi'/\varphi = \gamma_2 + \gamma_1\,.$$

The lemma follows by writing (6.2.1) with $\gamma = \gamma_1, \gamma_2$, adding the two equations and then eliminating γ_1 and γ_2 with the help of (6.2.2) and and (6.2.3).

We now assume

6.3.1 The function φ is the radical of an element of $\mathbb{Q}^{\mathrm{alg}}(x)$ (hence $h \in \mathbb{Q}^{\mathrm{alg}}(x)$).

There are two possibilities:

Lemma 6.4: If y_1 and y_2 are dependent then

6.4.1 $$y_1 \text{ is radical of element of } \mathbb{C}(x)$$

6.4.2 $$\gamma_1 \in \mathbb{Q}^{\mathrm{alg}}(x)$$

6.4.3 $$h = 0$$

6.4.4 $$N \text{ is reducible in } \mathbb{Q}^{\mathrm{alg}}(x)[D]\,.$$

Proof: Obvious. We note that conversely (6.4.3) implies the dependence of y_1 and y_2.

Lemma 6.5: If y_1 and y_2 are independent then

6.5.1 $$\tau'/\tau = w/\varphi$$

where w is « the » wronskian of N and

6.5.2 γ_1 and γ_2 lie either in $\mathbb{Q}^{\mathrm{alg}}(x)$ or

in a quadratic extension. In particular if N is irreducible over $\mathbb{Q}^{\mathrm{alg}}(x)[D]$ then γ_1 and γ_2 are conjugates over $\mathbb{Q}^{\mathrm{alg}}(x)$.

Proof: Equation 6.5.1 follows from (6.1.2) and (6.1.3). Statement 6.5.2 follows from (6.2.2), (6.2.3), (6.2) and (6.3.1).

Remark:

6.6.1 Under the hypothesis of Lemma 6.5, equation (6.5.1) is not well determined since the wronskian is only determined up to a constant. On the other hand Lemma 6.2 determines τ'/τ uniquely once the vector space $\mathbb{C}\cdot\varphi$ is fixed.

6.6.2 Under the hypothesis of (6.5) and (6.3.1) the ratio $(\varphi/w)^2$ is a rational function. In particular if $P = 0$ then w is a constant and φ^2 is rational.

Lemma 6.7: We assume 6.1.2 and 6.3.1.

6.7.1 Under the hypothesis 6.4 if w is algebraic over $\mathbb{Q}(x)$ then the solutions of N at t converge in $D(t, 1^-)$ for almost all p (and hence N is globally nilpotent).

6.7.2 Under hypothesis 6.5, if N is globally nilpotent then N has zero p-curvature for almost all p and in particular equation (6.2) has a solution mod p in $\mathbb{F}_p(x)^{\mathrm{alg}}$ for almost all p.

Proof: Under hypothesis 6.4, $y = \sqrt{\varphi}$ is a solution of N and a second solution may be obtained from

$$\varphi\int w/\varphi\,\mathrm{d}x$$

so that the assertion follows from § 1.

Under hypothesis 6.5, both φ and γ_1 are algebraic over $\mathbb{Q}^{\mathrm{alg}}(x)$ and hence have reductions mod p for almost all p. By hypothesis $h \neq 0$ and hence excluding a finite set of primes we have reductions

$\bar{N}$, $\bar{\gamma}_1$, $\bar{\varphi}$, $\bar{h} \neq 0$. Clearly

6.7.3 $$\bar{N} = (D + \bar{\gamma}_1 + \bar{P})(D - \bar{\gamma}_1)$$

a decomposition in $\mathbb{F}_p(x)^{\mathrm{alg}}$ and hence by Honda [19, Theorem 4] if $\bar{N}$ has nilpotent p-curvature, the same holds for each factor, *i.e.*

$$\bar{\gamma}_1 = y'/y$$

for some $y \in \mathbb{F}_p(x)^{\mathrm{alg}}$. Clearly y is a solution of $\bar{N}$. By reduction of 6.2.0, and 6.2.1

$$(\bar{\varphi}'/\bar{\varphi} - 2\bar{\gamma}_1)^2 = \bar{h}$$

$$\bar{\gamma}_1^2 + \bar{\gamma}_1' + \bar{P}\bar{\gamma}_1' + \bar{Q} = 0\,.$$

From these equations we deduce that $\bar{\varphi}/y$ is a solution, z, of $\bar{N}$ and that the ratio $\xi = z/y$ satisfies

$$\xi' = \xi\bar{h}\,.$$

This completes the proof of the lemma.

6.8 We apply the preceeding remarks to equation (4.5) with $f \in \mathbb{Q}(x)$ and $m \in N$. It is known in this case [30] that the symmetric square of (4.5) has a solution $\varphi \in \mathbb{Z}[x, B]$ of degree m which is unique up to a factor independent of x. For $2m+1$ special values of B, φ is the square of a solution of (4.5) and (6.7.1) applies. For all other values of B, we are in the situation (6.5).

In particular if $m = 0$ then excluding one special value of B, and excluding a finite set of primes, nilpotence of p-curvature of (4.5) implies by 6.7.2 that the reduction of

6.8.1 $$\tau' = \tau\,(4B/f(x))^{\frac{1}{2}}$$

has a solution in $\mathbb{F}_p(x)^{\mathrm{alg}}$. Since $\tau'\,dx/\tau$ would then be invariant under the Cartier operator, we may reduce to the condition that B^{1-p} bears a suitable realation to the Hasse invariant of the reduction of the elliptic curve

6.8.2 $$y^2 = f(x)\,.$$

Thus global nilpotence, regardless of B is impossible if 6.8.2 has complex multiplication.

6.8.3 We are thus tempted to conjecture that for $m \in \mathbb{Z}$ equation (4.5) with $f \in \mathbb{Q}(x)$ is globally nilpotent only for the $2m+1$ special value of B (for which h reduces to zero). For $m \in \frac{1}{2} + \mathbb{Z}$ we do not have 6.1.2 and there are $m + \frac{1}{2}$ values of B for which all solutions of (4.5) are algebraic [26]. The examples of Apery (§ 4) show that these do not exhaust the list with global nilpotence.

6.9 It has been shown by Robba [27] that if L has for fixed p no solution converging in $D(t, 1^-)$ the same holds for all differential equations of the same order which aproximate L p-adically.

We know of no global result of this type.

CONJECTURE: Let S be the set of all n^{th} order elements of $\mathbb{Q}^{\text{alg}}(x)[D]$ having given Riemann data. Thus S is parametrized by a certain number of accessory parameters. Let S_1 be the subset of S corresponding to equations where solutions converge in $D(t, 1^-)$ for almost all p. We conjecture that S_1 corresponds to an algebraic subset of S.

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