Algebraic $\mathcal{A}$-hypergeometric functions and their monodromy

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The polytopes on the cover are the convex hulls of the sets $\mathcal{A}$ for the Appell F_1 and F_4 and Horn H_1 functions.

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Algebraic $\mathcal{A}$-hypergeometric functions and their monodromy

Algebraïsche $\mathcal{A}$-hypergeometrische functies en hun monodromie

(met een samenvatting in het Nederlands)

Proefschrift

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Contents

Introduction

In this thesis, we study algebraic $\mathcal{A}$-hypergeometric functions and their monodromy groups. The systematic study of hypergeometric functions started with Gauss' paper [Gau63] on the function

$$_2F_1(a,b,c|z) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n n!} z^n$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. This function was generalized to functions with more parameters [Cla28, Tho70] and more variables [App82, App80, Lau93, Hor89, Hor31].

At the end of the 1980's, I.M. Gelfand, S.I. Gelfand, Graev and Zelevinsky published a series of papers about generalized hypergeometric functions [Gel86, GG86, GGZ87]. They defined a hypergeometric function to be a function satisfying certain homogeneity conditions under the action on a torus, as well as certain additional structure equations. These conditions are encoded in a system of linear partial differential equations, depending on a set $\mathcal{A}$ and a parameter vector $\boldsymbol{\beta}$. The system is denoted $H_{\mathcal{A}}(\boldsymbol{\beta})$ and the functions are called $\mathcal{A}$-hypergeometric functions. The function $_2F_1$ can be viewed as a dehomogenized $\mathcal{A}$-hypergeometric function.

The first important results were obtained by Gelfand, Zelevinsky and Kapranov. They showed that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is a holonomic system of which the rank around a generic point can easily be computed, provided $\mathcal{A}$ satisfies some conditions. They also showed how to construct analytic solutions of this system [GZK88, GKZ89]. In their honor, $\mathcal{A}$-hypergeometric functions are sometimes called GKZ-hypergeometric functions.

It is a classical question which hypergeometric functions are algebraic, i.e., satisfy a polynomial equation over $\mathbb{C}[z_1,\ldots,z_n]$, where n is the number of variables of the function. If the monodromy group acts irreducibly on the solution space of a system of differential equations, then either all solutions are transcendental, or they are all algebraic, with the latter case happening if and only if the monodromy group is finite. In this thesis we will restrict ourselves to such irreducible systems. This makes the classification of algebraic functions equivalent to the classification of systems $H_{\mathcal{A}}(\boldsymbol{\beta})$ with a finite monodromy group.

In 1873 Schwarz gave a list of all parameters (a,b,c) such that the Gauss function $_2F_1(a,b,c|z)$ is irreducible and algebraic [Sch73]. He obtained this list by computing the monodromy groups. Later on, this list was extended to some other classical hypergeometric functions, including the generalized hypergeometric functions ${}_pF_{p-1}$ [BH89] and the Appell functions F_1, F_2 and F_4 [CW92, Sas77, Kat00, Kat97].

In general, computing monodromy groups and finiteness conditions is hard. Therefore we will use another approach in this thesis. Some years ago Beukers proved a criterion for algebraicity of $\mathcal{A}$-hypergeometric functions, based on the combinatorics of their defining set $\mathcal{A}$ [Beu10]. This allows us to compute the parameters of algebraic $\mathcal{A}$-hypergeometric functions without first computing the monodromy groups. Knowing for which parameters the monodromy groups are finite, we are sometimes able to compute these groups afterwards. The main results are a classification of the irreducible algebraic Appell, Lauricella and Horn functions and partial results on the monodromy groups of the corresponding systems of differential equations and the equations these functions satisfy, as well as a classification of the irreducible algebraic $\mathcal{A}$-hypergeometric functions with $\mathcal{A} \subseteq \mathbb{Z}^2$ or $\mathcal{A} \subseteq \mathbb{Z}^3$.

This thesis is organized as follows.

In Chapter 1, we discuss classical hypergeometric functions and define $\mathcal{A}$-hypergeometric systems of differential equations. We collect some basic results about these systems and explain Beukers' combinatorial criterion for algebraicity.

Chapter 2 contains a classification of the irreducible algebraic Appell, Lauricella and Horn functions. We reproduce the known results for the Appell functions, greatly simplifying the existing proofs by using Beukers' combinatorial criterion, and extend them to the Lauricella and Horn functions.

Chapter 3 deals with $\mathcal{A}$-hypergeometric systems for which the defining set $\mathcal{A}$ is a subset of $\mathbb{Z}^2$ or $\mathbb{Z}^3$. By definition, $\mathcal{A}$ lies in a hyperplane, so the convex hull of $\mathcal{A}$ is a line segment or a polygon. We give a list of all polygons for which there exist parameters such that the system has irreducible algebraic solutions.

In Chapter 4, we study Γ-series and show how to construct a basis of the solution space. We compute the local monodromy group, generated by the monodromy matrices corresponding to loops of the variables around the origin, on bases induced by triangulations of $\mathcal{A}$ and explain the condition under which these local groups can be glued to give the local monodromy group on a basis of Mellin-Barnes integrals. We perform these calculations for the Appell and Horn functions.

Finally, in Chapter 5 we use the methods of Chapter 4 to compute the projective monodromy groups for most of the irreducible algebraic Appell, Lauricella and Horn functions. We end with explicit equations for the parametrized families of algebraic functions.

Notations

Throughout this thesis, we will use the following notations. Vectors will be printed in boldface. For vectors $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we write $|\boldsymbol{m}| = m_1 + \ldots + m_n$, $\boldsymbol{m}! = m_1! \cdots m_n!$ and $(\boldsymbol{x})_{\boldsymbol{m}} = (x_1)_{m_1} \cdots (x_n)_{m_n}$. For $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we write $\boldsymbol{z}^{\boldsymbol{m}} = z_1^{m_1} \cdots z_n^{m_n}$.

$\mathcal{A}$ and $\boldsymbol{\beta}$ will always be as in Definition 1.2.1, with $\mathcal{A} \subseteq \mathbb{Z}^r$, $|\mathcal{A}| = N$ and $\boldsymbol{\beta} \in \mathbb{C}^r$. Unless stated otherwise, $\boldsymbol{\beta}$ will be an element of $\mathbb{Q}^r$. We write $d = N - r$. The standard basis vectors will be denoted $\boldsymbol{e}_i$.

1

Hypergeometric functions

In this chapter we introduce hypergeometric functions and discuss their basic properties. The first section contains an overview of the relevant parts of the classical theory of hypergeometric functions. We introduce the Gauss hypergeometric function ${}_2F_1$ and study its integral representations, the monodromy group and the values of the parameters for which it is algebraic. Then we consider the 1-variable generalization ${}_pF_q$ with more parameters. We end this section with a discussion about hypergeometric series in more variables, in particular the Appell, Lauricella and Horn functions.

In the second section, we introduce $\mathcal{A}$-hypergeometric functions and the systems of partial linear differential equations they satisfy. We give some basic results about irreducibility and the solution space of these systems. We also show how the classical functions fit into the $\mathcal{A}$-hypergeometric framework.

The third section is devoted to algebraic $\mathcal{A}$-hypergeometric functions. We first show that the monodromy group of systems with algebraic solutions equals the Galois group of the extension generated by the solutions. Then we state the combinatorial criterion that will be our main tool for classifying the algebraic functions, and explain the algorithm we will use to determine these functions in practice.

In the final section we collect some additional properties of $\mathcal{A}$-hypergeometric functions that will help us to classify the algebraic functions. Among others, they concern triangulations of $\mathcal{A}$ and reductions to functions with less variables.

1.1 Classical hypergeometric series

In this section, we study the classical hypergeometric functions in some detail. Apart from their historical importance, there are two reasons to do this. First, large parts of this thesis are devoted to (the $\mathcal{A}$-hypergeometric counterpart of) the Appell, Lauricella and Horn functions: we classify the algebraic functions in Chapter 2 and compute (most of) their monodromy groups in Chapter 5.

Second, the Gauss function not only serves as a motivation but also as an illustration of the theory of $\mathcal{A}$-hypergeometric functions. Both for the Gauss function and $\mathcal{A}$-hypergeometric functions, the monodromy group (and hence algebraicity) only depends on the fractional parts of the parameters, and the parameters for which the function is algebraic form orbits under conjugation. Some results for the Gauss function will later be generalized to $\mathcal{A}$-hypergeometric functions: at the end of Section 1.3

the interlacing condition (1.8) will be generalized to arbitrary $\mathcal{A}$-hypergeometric functions. In Section 4.1 we will show how triangulations of $\mathcal{A}$ lead to local bases of solutions, similar to the solutions (1.5). The Barnes integral (1.4) will be generalized in Section 4.3. Finally, the formulas (1.6) will be generalized to the Lauricella functions in Section 5.2.

The Gauss function ${}_2F_1$

The systematic study of hypergeometric functions started in 1813 with a paper by Gauss, reprinted in [Gau63], in which he introduced a function that in nowadays called the Gauss hypergeometric function. It is denoted ${}_2F_1$ and is given by the power series

$$ {}_2F_1(a,b,c|z) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{1.1} $$

where a, b and c are complex parameters with $c \notin \mathbb{Z}_{\leq 0}$. Here $(a)_n$ is the Pochhammer symbol, defined by

$$ (a)_n = a \cdot (a+1) \cdots (a+n-1). $$

The series converges for all $z \in \mathbb{C}$ with $|z| < 1$. Many well-known functions can be expressed in terms of the Gauss function, including

$$ \begin{aligned} (1-z)^a &= {}_2F_1(-a,b,b|z) \\ \log(1+z) &= z \cdot {}_2F_1(1,1,2|-z) \\ \arcsin(z) &= z \cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2},\frac{3}{2}\middle| z^2\right) \\ P_n(z) &= 2^n \cdot {}_2F_1\left(-n,n+1,1\middle|\frac{z+1}{2}\right) \\ T_n(z) &= (-1)^n \cdot {}_2F_1\left(-n,n,\frac{1}{2}\middle|\frac{z+1}{2}\right) \end{aligned} $$

where P_n and T_n are the Legendre and Chebyshev polynomials, respectively.

${}_2F_1(a,b,c|z)$ satisfies the second order linear differential equation

$$ z(z-1)F''(z) + ((a+b+1)z-c)F'(z) + abF(z) = 0. \tag{1.2} $$

The power series (1.1) converges if $|z| < 1$. By expanding the integrand of

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{1.3} $$

in powers of z and using the Euler integral for the Beta function, one easily sees that this integral is equal to (1.1). It converges if $\Re(c) > \Re(b) > 0$, and can be evaluated if

$z \notin [1,\infty)$. Hence it allows us to define an analytic continuation of ${}_2F_1$. This can also be done by the Barnes integral [Bar08]:

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds. \tag{1.4} $$

where $z \notin [0,\infty)$. The contour is taken so that the poles $0,1,2,\ldots$ are separated from the poles $-a,-a-1,\ldots,-b,-b-1,\ldots$.

The equation (1.2) has regular singularities in 0, 1 and ∞. Around each of these, there are two independent solutions. The Euler integral can be used to find these solutions [Pro94]. The integral

$$ u(z) = \int_C \frac{s^{a-c}(s-1)^{c-b-1}}{(s-z)^a} ds $$

is a solution of (1.2) for each simple path C joining two of the singularities of the integrand, 0, 1, ∞ and z. The integrand converges if $0 < \Re(b) < \Re(c) < \Re(a+1) < 2$. For each of the six pairs of singularities, we obtain an integral that, after a suitable variable substitution, reduces to the Euler integral. In this way we find 6 solutions of (1.2), consisting of pairs of independent solutions near a singularity. Up to constant factors, these solutions are:

$$ \begin{aligned} u_1(a,b,c|z) &= {}_2F_1(a,b,c|z) \\ u_2(a,b,c|z) &= z^{1-c}\cdot {}_2F_1(b-c+1,a-c+1,2-c|z) \\ u_3(a,b,c|z) &= {}_2F_1(a,b,a+b-c+1|1-z) \\ u_4(a,b,c|z) &= (1-z)^{c-a-b}\cdot {}_2F_1(c-a,c-b,c-a-b+1|1-z) \\ u_5(a,b,c|z) &= z^{-a}\cdot {}_2F_1(a,a-c+1,a-b+1|z^{-1}) \\ u_6(a,b,c|z) &= z^{-b}\cdot {}_2F_1(b-c+1,b,b-a+1|z^{-1}) \end{aligned} \tag{1.5} $$

The substitutions $t = 1-s$, $t = \frac{s}{1-z+sz}$ and $t = \frac{1-s}{1-sz}$ in (1.3) yield the identities

$$ \begin{aligned} {}_2F_1(a,b,c|z) &= (1-z)^{c-a-b}\cdot {}_2F_1(c-a,c-b,c|z) = \\ &(1-z)^{-a}\cdot {}_2F_1\left(a,c-b,c\,\middle|\,\frac{z}{z-1}\right) = (1-z)^{-b}\cdot {}_2F_1\left(c-a,b,c\,\middle|\,\frac{z}{z-1}\right). \end{aligned} $$

Applying this to each of the 6 solutions in (1.5), we find 24 solutions of (1.2). A list of these solutions was given by Kummer in 1836 [Kum36].

For some special values of the parameters, the solutions u_i coincide. For example, if $c=1$, then $u_2(a,b,c|z) = {}_2F_1(a,b,c|z)$. In this case, we obtain a second solution by taking the difference of the two solutions for $c \neq 1$, dividing by $c-1$ and taking the limit $c \to 1$. This gives the second solution

$$ \log(z)\cdot {}_2F_1(a,b,1|z) + \sum_{n\geq 1} \frac{(a)_n(b)_n}{(n!)^2} z^n \left(\sum_{k=1}^{n} \frac{1}{a+k+1} + \frac{1}{b+k+1} - \frac{2}{k} \right). $$

It is a classical question for which parameters (a, b, c) the Gauss function is algebraic. Note that the parameters need to be rational, as they determine the local exponents of the solutions of the differential equation (1.2). This question was answered by Schwarz in 1873, by computing the projective monodromy groups of equation (1.2). In 1857, Riemann had introduced the monodromy group of the differential equation defining the Gauss function [Rie53]. Schwarz worked out a geometrical description of this group using the Schwarz map. We give a sketch of his main ideas; more information can be found in the original paper by Schwarz [Sch73] and the book by Klein [Kle33] or in the modern book about monodromy by Żołądek [Żoł06].

Given two independent solutions f and g of (1.2), the Schwarz map is defined by $\eta(z) = \frac{f(z)}{g(z)}$. It can be shown that η maps the upper half plane bijectively to the interior of a curvilinear triangle (i.e., a triangle whose edges are curved line segments) and can be continued to the closure of the upper half plane by mapping the real axis to the edges of this triangle. The angles of the triangle are determined by the differences of the local exponents in each of the singularities 0, 1 and ∞. The local exponents are 0 and $1-c$; 0 and $c-a-b$; $-a$ and $-b$, respectively (cf. Kummer's solutions (1.5)), so the angles are $\lambda\pi$, $\mu\pi$ and $\nu\pi$ with $\lambda = |1-c|$, $\mu = |c-a-b|$ and $\nu = |a-b| \pmod{\mathbb{Z}}$. The monodromy group is irreducible if and only if a, b, $c-a$ and $c-b$ are non-integral. Up to conjugation, in this case the monodromy group only depends on the fractional parts of a, b and c [Żoł06, Proposition 12.7]. Hence we can shift (a, b, c) over $\mathbb{Z}^3$ to have $0 \leq \lambda, \mu, \nu < 1$.

An important property of η is that it is algebraic if and only if f and g are algebraic. Indeed, the quotient of two algebraic functions is algebraic. On the other hand, if η is algebraic, then so is its derivative $\eta' = \frac{f'g-fg'}{g^2}$. The numerator is the Wronskian of (1.2) and equals $z^{-c}(1-z)^{c-a-b-1}$. This is an algebraic function, so g^2 is algebraic, implying that both f and g are algebraic. Hence we can study the monodromy group of (1.2) by studying the analytic continuation of η along loops.

If γ is a loop crossing the real axis in one of the intervals $(-\infty, 0)$, $(0, 1)$ or $(1, \infty)$, its image under η will cross the corresponding edge of the curvilinear triangle. Hence the projective monodromy group depends on the group of reflection in the edges of the triangle. As loops have an even number of crossings with the real axis, the projective monodromy group is the subgroup consisting of products of an even number of reflections. It follows that the projective monodromy group is finite if and only if the reflection group is finite. This happens only if the triangles are spherical, i.e., $\lambda + \mu + \nu > 1$, and the reflections of the triangles give a (finite) tessellation of the sphere. This tessellation is given by the projection of a double pyramid or one of the Platonic solids on the sphere. The triangle with angles (λ, μ, ν) induces such a tessellation if and only if (λ, μ, ν), $(\lambda, 1-\mu, 1-\nu)$, $(1-\lambda, \mu, 1-\nu)$ or $(1-\lambda, 1-\mu, \nu)$ is a permutation of one of the tuples in Table 1.1.

For the triple $(\frac{1}{2}, \frac{1}{2}, r)$, the tessellation is given by the projection of a double pyramid on the sphere. The projective monodromy group is the dihedral group with $2q$ elements, where q is the denominator of r. The triples $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ correspond to the projection of the tetrahedron on the sphere. The projective monodromy group is the alternating group A_4. The tessellation for $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ and $(\frac{2}{3}, \frac{1}{4}, \frac{1}{4})$ is given

Table 1.1: The tuples (λ, μ, ν) such that ${}_2F_1(a, b, c|z)$ is irreducible and algebraic

$(\frac{1}{2}, \frac{1}{2}, r)$ with $r \in \mathbb{Q} \setminus \mathbb{Z}$				
$(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, \frac{2}{5}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{5})$	$(\frac{2}{5}, \frac{1}{3}, \frac{1}{3})$	$(\frac{3}{5}, \frac{2}{5}, \frac{1}{3})$
$(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{2}, \frac{2}{5}, \frac{1}{5})$	$(\frac{2}{3}, \frac{1}{4}, \frac{1}{4})$	$(\frac{2}{5}, \frac{2}{5}, \frac{2}{5})$	$(\frac{4}{5}, \frac{1}{5}, \frac{1}{5})$
$(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5}, \frac{1}{5})$	$(\frac{3}{5}, \frac{1}{3}, \frac{1}{5})$	

by the cube and the projective monodromy group is the symmetric group S_4. For the remaining triples in Table 1.1, the tessellation is given by the projection of the dodecahedron and the monodromy group is the alternating group A_5.

To compute all triples (a, b, c) such that ${}_2F_1(a, b, c|z)$ is irreducible and algebraic, note that each triple (λ, μ, ν) in Table 1.1 in fact denotes 24 triples: the permutations of (λ, μ, ν), $(\lambda, 1-\mu, 1-\nu)$, $(1-\lambda, \mu, 1-\nu)$ and $(1-\lambda, 1-\mu, \nu)$. For each of these, we can shift the coordinates by integers, such that the sum is even (this corresponds to shifting a, b and c over the integers). Furthermore, since $\lambda = |1-c|$, $\mu = |c-a-b|$ and $\nu = |a-b| \pmod{\mathbb{Z}}$, we should also take into account the triples obtained by changing the sign of one or more of the coordinates. It follows that ${}_2F_1(a, b, c|z)$ is irreducible and algebraic if and only if $(a, b, c) = (\frac{1-\lambda'-\mu'-\nu'}{2}, \frac{1-\lambda'-\mu'+\nu'}{2}, 1-\lambda')$, where (λ', μ', ν') is obtained from a triple in the table by permutations, sign changes and addition of (l, m, n) with $l+m+n$ even. Addition of (l, m, n) corresponds to shifts of a, b and c by integers. We mentioned already that this does not influence the monodromy group. In what follows, we will take $0 \leq a, b, c < 1$.

For the tuple $(\lambda, \mu, \nu) = (\frac{1}{2}, \frac{1}{2}, r)$, there are 6 possibilities for (a, b, c): we have $(a, b, c) = (-\frac{r}{2}, \frac{r}{2}, \frac{1}{2})$, $(-\frac{r}{2}, \frac{1-r}{2}, \frac{1}{2})$, $(-\frac{r}{2}, \frac{1-r}{2}, 1-r)$, $(\frac{r-1}{2}, \frac{1-r}{2}, \frac{1}{2})$, $(\frac{r-1}{2}, \frac{r}{2}, \frac{1}{2})$ or $(\frac{r-1}{2}, \frac{r}{2}, \frac{1}{2}) \pmod{\mathbb{Z}}$. Writing $\tilde{r} = \{a\}$ gives $(a, b, c) = (\tilde{r}, 1-\tilde{r}, \frac{1}{2}), (\tilde{r}, \tilde{r}+\frac{1}{2}, \frac{1}{2})$ or $(\tilde{r}, \tilde{r}+\frac{1}{2}, 2\tilde{r}) \pmod{\mathbb{Z}}$. Since $r \in \mathbb{Q} \setminus \mathbb{Z}$, we have $2\tilde{r} \in \mathbb{Q} \setminus \mathbb{Z}$. We will call these tuples *Gauss triples of type 1.* This terminology is not standard, but will be used troughout this thesis.

For the other 14 tuples (λ, μ, ν), we compute (a, b, c) for all 48 tuples obtained from (λ, μ, ν) by sign changes and permutations. Up to equivalence modulo $\mathbb{Z}$, this gives 408 triples (a, b, c), which we call *Gauss tuples of type 2.* By the symmetry of the Gauss function, it is clear that the tuples come in pairs (a, b, c) and (b, a, c). Furthermore, if (a, b, c) is a Gauss triple and k is coprime with the greatest common denominator D of a, b and c, then $k(a, b, c) = (ka, kb, kc)$ turns out to be a Gauss triple as well. We call this action of $(\mathbb{Z}/D\mathbb{Z})^*$ on the tuples *conjugation.* In Table 1.2, the smallest element of each pair of orbits under conjugation is given (where $\frac{p}{q}$ is considered to be smaller than $\frac{u}{v}$ if either $q < v$, or $q = v$ and $p \leq u$. Tuples of fractions are ordered lexicographically). For later use we note that the denominators of a, b and c of a Gauss tuple of type 2 are at most 60, 60 and 5, respectively.

There are three families of irreducible algebraic Gauss functions: ${}_2F_1(r, -r, \frac{1}{2}|z)$, ${}_2F_1(r, r+\frac{1}{2}, \frac{1}{2}|z)$ and ${}_2F_1(r, r+\frac{1}{2}, 2r|z)$. The following formulas are well-known; the

Table 1.2: The tuples (a,b,c) such that ${}_2F_1(a,b,c|z)$ is irreducible and algebraic

$(r,-r,\frac{1}{2})$, $(r,r+\frac{1}{2},\frac{1}{2})$ and $(r,r+\frac{1}{2},2r)$ with $2r \notin \mathbb{Z}$				
$(\frac{1}{2},\frac{1}{6},\frac{1}{3})$	$(\frac{1}{6},\frac{5}{12},\frac{1}{4})$	$(\frac{1}{12},\frac{5}{12},\frac{1}{4})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2})$	$(\frac{1}{30},\frac{11}{30},\frac{1}{5})$
$(\frac{1}{4},\frac{3}{4},\frac{1}{3})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{3})$	$(\frac{1}{12},\frac{7}{12},\frac{1}{3})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{5})$	$(\frac{1}{30},\frac{19}{30},\frac{1}{3})$
$(\frac{1}{4},\frac{7}{12},\frac{1}{2})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{5})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{3})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{3})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{3})$
$(\frac{1}{4},\frac{7}{12},\frac{1}{3})$	$(\frac{1}{10},\frac{3}{10},\frac{1}{5})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{5})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{4})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{5})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{3})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{5})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{4})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{3}{5})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{4})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{5})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{5})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{3})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{5})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2})$
$(\frac{1}{6},\frac{5}{12},\frac{1}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{2}{5})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{3})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{3})$

second one can already be found in [Sch73]:

$$\begin{aligned}
{}_2F_1\left(r,-r,\frac{1}{2}|z\right) &= \frac{1}{2}\left((\sqrt{1-z}+i\sqrt{z})^{2r}+(\sqrt{1-z}-i\sqrt{z})^{2r}\right)\\
{}_2F_1\left(r,r+\frac{1}{2},\frac{1}{2}|z\right) &= \frac{1}{2}\left((1+\sqrt{z})^{-2r}+(1-\sqrt{z})^{-2r}\right)\\
{}_2F_1\left(r,r+\frac{1}{2},2r|z\right) &= \frac{(1+\sqrt{1-z})^{1-2r}}{2^{1-2r}\sqrt{1-z}}.
\end{aligned} \tag{1.6}$$

From these formulas we again see that the monodromy group is the dihedral group with $2q$ elements, where q is the denominator of $2r$. In Section 5.2 we will give similar formulas that can be used to compute the monodromy groups of families of algebraic Appell, Lauricella and Horn functions.

Generalized hypergeometric functions

A generalized hypergeometric function is a function in one complex variable, given by a series $F(z)=\sum_{n\geq 0} c_n z^n$, such that the quotient $\frac{c_{n+1}}{c_n}$ is a rational function $\frac{A(n)}{B(n)}$ of n, with polynomials A and B. We can factor A and B into linear factors a_i+n and b_j+n, respectively. It is assumed that $1+n$ is a factor of B. This can be done without loss of generality, as we can multiply both A and B by $1+n$. By this assumption, F satisfies an irreducible differential equation of degree $\max(\deg(A),\deg(B))$. It follows easily from

$$\frac{c_{n+1}}{c_n}=\frac{A(n)}{B(n)}=\frac{(a_1+n)\cdots(a_p+n)}{(b_1+n)\cdots(b_q+n)(1+n)}$$

that $c_n = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!}$. The power series with these coefficiens is denoted ${}_pF_q$, i.e.,

$$ {}_pF_q(a_1, \dots, a_p, b_1, \dots, b_q|z) = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} z^n. \tag{1.7} $$

For $p = 2$ and $q = 1$, we get the Gauss function ${}_2F_1$.

The functions with coefficients of degree 0, i.e., with $p = q + 1$, were introduced first by Clausen for $p = 3$ [Cla28] and Thomae for general p [Tho70]. These functions are also the most-studied case. Beukers and Heckman studied the algebraicity and monodromy groups [BH89] and showed that they have properties similar to the Gauss function. The monodromy group is reducible if there exists i, j such that $a_i \in \mathbb{Z}$ or $a_i - b_j \in \mathbb{Z}$. If it is irreducible, it is invariant under shifts of the parameters by integers. Furthermore, we have the following *interlacing condition*:

Theorem 1.1.1 ([BH89, Theorem 4.8]). *Suppose that the monodromy group of ${}_pF_{p-1}$ is irreducible and $a_i, b_j \in \mathbb{Q}$ for all i, j. Then the monodromy group is finite if and only if for all k coprime with the greatest common denominator of all parameters,*

$$ 0 < \{ka_{i_1}\} < \{kb_{j_1}\} < \ldots < \{ka_{i_{p-1}}\} < \{kb_{j_{p-1}}\} < \{ka_p\} < 1 \tag{1.8} $$

for some permutations $(i_1, \dots, i_p)$ of $(1, \dots, p)$ and $(j_1, \dots, j_{p-1})$ of $(1, \dots, p-1)$. Here $\{\cdot\}$ denotes the fractional part.

The parameters for which the interlacing condition is satisfied, and hence the function is algebraic, are given in Theorems 5.8 and 7.1 of [BH89]. The interlacing condition for the Gauss function was already given in [Kat72].

The Appell, Lauricella and Horn functions

Apart from the functions ${}_pF_q$ with $p + q$ parameters and one variable, some other generalizations of the ${}_2F_1$ function have been studied. Of the classical hypergeometric series in multiple variables, the series defined by Appell, Lauricella and Horn are probably the most famous examples.

In 1880, Appell started the study of multivariable hypergeometric series by introducing the functions F_1, F_2, F_3 and F_4 [App80]. In [App82] he explains that he arrived at these series by multiplying two Gauss functions ${}_2F_1(a_1, b_1, c_1|x)$ and ${}_2F_1(a_2, b_2, c_2|y)$ and replacing some of the products of Pochhammer symbols by single ones, like replacing $(a_1)_m(a_2)_n$ by $(a)_{m+n}$. This gives five distinct series in two variables, including ${}_2F_1(a, b, c|x + y)$. The four other series are

$$ F_1(a, b_1, b_2, c|x, y) = \sum_{m,n \geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m!n!} x^m y^n, $$

$$ F_2(a, b_1, b_2, c_1, c_2|x, y) = \sum_{m,n \geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!n!} x^m y^n, $$

$$ F_3(a_1, a_2, b_1, b_2, c|x, y) = \sum_{m,n \geq 0} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n} m!n!} x^m y^n \qquad \text{and} $$

$$F_4(a, b, c_1, c_2|x, y) = \sum_{m,n \geq 0} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n m!n!} x^m y^n.$$

In [App80] Appell studies the systems of partial differential equations these series satisfy and the number and form of the other independent solutions. The regions of convergence are computed in [App82]. The functions F_1 and F_3 converge if $|x|, |y| < 1$; the series F_2 converges if $|x| + |y| < 1$ and F_4 converges if $\sqrt{|x|} + \sqrt{|y|} < 1$. Appell also gives integral representations such as

$$F_1(a, b_1, b_2, c|x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b_1}(1-ty)^{-b_2} dt$$

if $\Re(a), \Re(c-a) > 0$ and

$$F_2(a, b_1, b_2, c_1, c_2|x, y) = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1)\Gamma(c_2-b_2)} \cdot \int_0^1 \int_0^1 s^{b_1-1}t^{b_2-1}(1-s)^{c_1-b_1-1}(1-t)^{c_2-b_2-1}(1-sx-ty)^{-a} ds dt$$

if $\Re(b_1), \Re(b_2), \Re(c-b_1), \Re(c-b_2) > 0$ (cf. (1.3)).

The obvious generalizations of the Appell series to series in n variables were introduced by Lauricella in 1893 [Lau93]. They are called F_D, F_A, F_B and F_C (note the order of the indices). Lauricella showed that $F_D(\boldsymbol{z})$ and $F_B(\boldsymbol{z})$ converge if $|z_1| + \ldots + |z_n| < 1$, the series $F_A(\boldsymbol{z})$ converges if $|z_1| + \ldots + |z_n| < 1$ and $F_C(\boldsymbol{z})$ converges if $\sqrt{|z_1|} + \ldots + \sqrt{|z_n|} < 1$. He gives the corresponding system of differential equations and shows that the dimension of the solution space is 2^n for F_A, F_B and F_C, and $n+1$ for F_D. He computes these solutions explicitly for F_A, F_B and F_C. Furthermore, he gives integral representations for F_A, F_B and F_D.

An overview of the theory on Appell and Lauricella series can be found in the classical book by Appell and Kampé de Fériet [AF26].

Meanwhile, in 1889, Horn gave a more general definition of hypergeometric functions in 2 and 3 variables [Hor89]. He defined a hypergeometric series in 2 variables to be a series

$$F(x, y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$$

such that

$$f(m, n) = \frac{a_{m+1,n}}{a_{m,n}} \quad \text{and} \quad g(m, n) = \frac{a_{m,n+1}}{a_{m,n}}$$

are rational functions of m and n. This definition can easily be extended to n-variable functions. The question arose what the general form of a hypergeometric series in 2 variables is. Of course, f and g have to satisfy

$$f(m, n)g(m+1, n) = \frac{a_{m+1,n+1}}{a_{m,n}} = f(m, n+1)g(m, n). \tag{1.9}$$

The rational solutions of this equation were found by Ore. He proved the following theorem:

Theorem 1.1.2 ([Ore30, Theorem 8]). *For integers $A_1, \ldots, A_t$ and $B_1, \ldots, B_t$, we define $a_i = A_1 + \ldots + A_{i-1}$, $b_i = B_2 + \ldots + B_i$, $A = A_1 + \ldots + A_t$ and $B = B_1 + \ldots + B_t$. Suppose that f and g are rational functions satisfying the compatibility condition* (1.9)*. Then f is a product of a rational function of m, factors of the form*

$$\prod_{i=1}^{t} \prod_{j=0}^{A_i - 1} A(m - b_i) + B(n + a_i + j) + C$$

with $A, B \neq 0$ and $C \in \mathbb{C}$, and factors of the form

$$\prod_{i=1}^{t} \prod_{j=0}^{A_i - 1} P(m - b_i, n + a_i + j)$$

with $A = B = 0$ and $P \in \mathbb{C}[x, y]$. Up to a rational function of n, g is a product of the factors

$$\prod_{i=1}^{t} \prod_{j=0}^{B_i - 1} A(m - b_i + j) + B(n + a_i) + C \qquad \text{and} \qquad \prod_{i=1}^{t} \prod_{j=0}^{B_i - 1} P(m - b_i + j, n + a_i),$$

with the same parameters as in f.

This theorem was generalized to hypergeometric series in any number of variables by Sato [Sat90, Appendix]. Using Ore's theorem, one can compute all hypergeometric series in two variables, by recursively computing the coefficients. This has been done by Erdélyi in the case $a_{m,n} \neq 0$ for all $m, n \geq 0$, although he did not state this condition on the coefficients.

Theorem 1.1.3 ([Erd51, Section 1]). *Let $F(x, y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$ be a hypergeometric series with $a_{m,n} \neq 0$ for all $m, n \geq 0$. Then there exists constants $a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k, d, e$ and a rational function $R(m, n)$ such that*

$$a_{m,n} = d^m e^n R(m, n) \prod_{i=1}^{k} \Gamma(a_i m + b_i n + c_i). \tag{1.10}$$

It was noted in 2008 by Abramov and Petkovšek that Theorem 1.1.3 need not hold if $a_{m,n} = 0$ for some m, n. They deduced from Sato's generalization of Theorem 1.1.2 that the n-variable analogue of Theorem 1.1.3 holds if $\operatorname{supp}(A) = \{\boldsymbol{m} \in \mathbb{Z}^m \mid a_{\boldsymbol{m}} \neq 0\}$ is Zariski-dense in $\mathbb{C}^n$ and is connected, i.e., there is a path in $\operatorname{supp}(A)$ consisting of unit steps $\pm \boldsymbol{e}_i$ between each two points of $\operatorname{supp}(A)$ [AP08, Corollary 2].

We return to bivariate functions and compare the formulas (1.7) and (1.10). The factor $d^m e^n$ in (1.10) can be removed by applying a transformation of the form $(x, y) \mapsto (dx, ey)$. The rational function $R(m, n)$ was not present in the one-variable function ${}_pF_q$, since the numerator and denominator of a one-variable rational function can be factored into linear factors, which can be written as Gamma functions. In two variables, this factorization need not exist.

The most-studied functions are those with $R(m,n) = 1$. Based on the result of Ore, Horn had already listed these series in the case where the numerator and denominator of f and g both have degree at most 2 in m and n, and the denominator of $a_{m,n}$ contains a factor $m!n!$. Excluding series derived from one-variable series or products of those, there are 34 of those. They can be found in [Hor31]; H_{11} is missing and can be found in [SK85]. Of particular interest are the 14 *complete* or *Horn* series, where all degrees are equal to 2. These consist of the 4 Appell series, 3 Horn G series and 7 Horn H series. The other 20 functions are the so-called *confluent* functions, where not all degrees are 2, but the degree of the denominators is at least equal to the degree of the numerator (otherwise the series does not converge). Horn analysed the corresponding systems of differential equations and computed the number of independent solutions and the local exponents.

We will compute the irreducible algebraic Appell, Lauricella and Horn functions in the next chapter, and compute some of the corresponding monodromy groups in Chapter 5.

1.2 $\mathcal{A}$-hypergeometric functions

At the end of the 1980's, I.M. Gelfand defined general hypergeometric functions by an integral on a certain line bundle over a Grassmannian manifold [Gel86]. Together with S.I. Gelfand, he showed that these general hypergeometric functions satisfy a system of linear partial differential equations [GG86]. These differential equations consist of first order equations, encoding homogeneity with respect to an action of the diagonal matrices and invariance with respect to an action of the invertible matrices, and homogeneous second order equations. They showed that this sytem is holonomic and has a finite dimensional solution space consisting of analytic functions.

A further generalization was given by I.M Gelfand, Graev and Zelevinsky [GGZ87]. Let $V \cong \mathbb{C}^N$ and $H \cong (\mathbb{C}^*)^r \subseteq \mathrm{GL}(V)$ such that $\mathbb{C}^*\mathrm{Id} \subseteq H$. Choose a basis $\{z_1, \ldots, z_N\}$ of V on which H consists of diagonal matrices. Then there exist $a_{ij} \in \mathbb{Z}$ such that

$$H \cong \{\mathrm{diag}(t_1^{a_{11}} \cdots t_r^{a_{1r}}, \ldots, t_1^{a_{N1}} \cdots t_r^{a_{Nr}}) \mid t_j \in \mathbb{C}^*\}.$$

The rank of the vectors $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{ir})$ with $1 \leq i \leq N$ is r and H acts on V by

$$X : (\mathbb{C}^*)^r \to \mathrm{GL}(V) : \boldsymbol{t} \mapsto (\boldsymbol{z} \mapsto (\boldsymbol{t}^{\boldsymbol{a}_1} z_1, \ldots, \boldsymbol{t}^{\boldsymbol{a}_N} z_N)).$$

Gelfand, Graev and Zelevinsky defined a hypergeometric function to be a function Φ that is homogeneous with respect to this torus action, i.e.,

$$\Phi(X(\boldsymbol{t})(\boldsymbol{z})) = \boldsymbol{t}^{\boldsymbol{\beta}}\Phi(\boldsymbol{z}) \tag{1.11}$$

for all $\boldsymbol{t} \in (\mathbb{C}^*)^r$ and $\boldsymbol{z} \in V$ and some $\boldsymbol{\beta} \in \mathbb{C}^r$, and satisfies certain equations of higher order induced by the relations between the vectors $\boldsymbol{a}_i$.

If V is a vector space of complex matrices and H is the group of linear transformations of V generated by the dilations of the rows and columns (i.e., multiplication of a row or column by a scalar), this definition is equivalent to the definition by Gelfand and Gelfand.

The usual definition of an $\mathcal{A}$-hypergeometric system starts with the vectors $\boldsymbol{a}_i$ but is equivalent to the definition by Gelfand, Graev and Zelevinsky. We recall this definition in Definition 1.2.1, but first make some remarks on the structure of the set $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N\} \subseteq \mathbb{Z}^r$.

As noted before, $\mathcal{A}$ has rank r. We even have $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$: by writing the matrix with columns $\boldsymbol{a}_i$ in Smith Normal Form, it can easily be seen that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$ if and only if all $r \times r$-minors have greatest common divisor 1. Let d be this gcd and let $I \subseteq \{1, \ldots, N\}$ have r elements. Then one can easily show that $t_j = e^{\frac{2\pi i}{d} \cdot (-1)^{j-1} \cdot \det(a_{ik})_{i \in I, k \neq j}}$ gives the identity matrix in H, and hence $t_j = 1$ for all j. It follows that d divides all $(r-1) \times (r-1)$-minors, so $d = 1$ and $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$.

Furthermore, the condition $\mathbb{C}^*\mathrm{Id} \subseteq H$ implies that the vectors $\boldsymbol{a}_i$ lie in an affine hyperplane given by $c_1 x_1 + \ldots + c_r x_r = 1$ with coefficients $c_j \in \mathbb{Z}$. This can be seen as follows: there exist $t_1, \ldots, t_r \in \mathbb{C}^*$ such that $\boldsymbol{t}^{\boldsymbol{a}_i} = e$ for all $1 \leq i \leq N$, where e is the base of the natural logarithm. Taking the logarithm of both sides shows that the vectors lie in an affine hyperplane. Each standard basis vector $\boldsymbol{e}_j$ can be written as $\boldsymbol{e}_j = \sum_i \lambda_{ij} \boldsymbol{a}_i$ with $\lambda_{ij} \in \mathbb{Z}$. Hence the coefficients satisfy $c_j = \sum_i \lambda_{ij}$ and are integral.

The above discussion serves as a motivation for the definition of $\mathcal{A}$-hypergeometric functions below. Instead of starting with the torus action, this definition starts with the set $\mathcal{A}$. To get a set coming from a torus action, we impose the conditions that $\mathcal{A}$ spans $\mathbb{Z}^r$ over $\mathbb{Z}$ and lies in an affine hyperplane (of which the defining equation can be taken to be $c_1 x_1 + \ldots + c_r x_r = 1$ with $c_j \in \mathbb{Z}$). Then we can define

$$H = \{\mathrm{diag}(\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_N} \mid \boldsymbol{t} \in (\mathbb{C}^*)^r\}.$$

Then we have $H \cong (\mathbb{C}^*)^r$ because $\mathcal{A}$ spans $\mathbb{Z}^r$. For $\lambda \in \mathbb{C}^*$, taking $t_j = \lambda^{c_j}$ gives $\lambda\mathrm{Id} \in H$, so H contains $\mathbb{C}^*\mathrm{Id}$ as a subgroup.

Definition 1.2.1. Let $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N\}$ be a finite subset of $\mathbb{Z}^r$ such that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$ and there exists a linear form h on $\mathbb{R}^r$ such that $h(\boldsymbol{a}_i) = 1$ for all i. The *lattice of relations* of $\mathcal{A}$ is $\mathbb{L} = \{(l_1, \ldots, l_N) \in \mathbb{Z}^N \mid l_1 \boldsymbol{a}_1 + \ldots + l_N \boldsymbol{a}_N = 0\}$. Let $\boldsymbol{\beta} \in \mathbb{C}^r$ and denote by ∂_i the differential operator $\frac{\partial}{\partial z_i}$. The *$\mathcal{A}$-hypergeometric system associated to $\mathcal{A}$ and $\boldsymbol{\beta}$*, denoted $H_{\mathcal{A}}(\boldsymbol{\beta})$, consists of two sets of differential equations:

- the *structure equations*: for all $\boldsymbol{l} = (l_1, \ldots, l_N) \in \mathbb{L}$

$$\Box_{\boldsymbol{l}} \Phi = \left(\prod_{l_i > 0} \partial_i^{l_i} \right) \Phi - \left(\prod_{l_i < 0} \partial_i^{-l_i} \right) \Phi = 0.$$

- the *homogeneity* or *Euler equations*: for $1 \leq i \leq r$

$$a_{1i} z_1 \partial_1 \Phi + \ldots + a_{Ni} z_N \partial_N \Phi = \beta_i \Phi.$$

The solutions $\Phi(z_1,\ldots,z_N)$ of this system are called $\mathcal{A}$-*hypergeometric functions.*

We make some comments on this definition. Since $\mathcal{A}$ lies in the affine hyperplane given by $h(\boldsymbol{a}_i)=1$, the structure equations are homogeneous: for all $\boldsymbol{l}\in\mathbb{L}$, we have $\sum_i l_i=\sum_i l_i h(\boldsymbol{a}_i)=h(\sum_i l_i\boldsymbol{a}_i)=0$ and hence $\sum_{l_i>0} l_i=\sum_{l_i<0} -l_i$. As $\mathbb{L}$ is infinite, $H_{\mathcal{A}}(\boldsymbol{\beta})$ contains an infinite number of structure equations. However, by Hilbert's basis theorem, the ring $\mathbb{Z}[\partial_1,\ldots,\partial_N]$ is Noetherian and hence the ideal $\langle\Box_{\boldsymbol{l}}\rangle_{\boldsymbol{l}\in\mathbb{L}}$ is finitely generated. Hence all structure equations follow from a finite number of them. Furthermore, the Euler equations are the infinitesimal analogue of the homogeneity condition (1.11):

Lemma 1.2.2. $\Phi(\boldsymbol{z})$ *satisfies the Euler equations if and only if* $\Phi(X(\boldsymbol{t})(\boldsymbol{z}))=\boldsymbol{t}^{\boldsymbol{\beta}}\Phi(\boldsymbol{z})$ *for all* $\boldsymbol{t}\in(\mathbb{C}^*)^r$.

Proof. Suppose that $\Phi(X(\boldsymbol{t})(\boldsymbol{z}))=\boldsymbol{t}^{\boldsymbol{\beta}}\Phi(\boldsymbol{z})$ for all $\boldsymbol{t}\in(\mathbb{C}^*)^r$. Let $1\le i\le r$ and choose $t_i=t$ and $t_j=1$ for all $j\neq i$. Differentiating the above expression with respect to t gives

$$\sum_{j=1}^{N} a_{ji}t^{a_{ji}-1}z_j\partial_j\Phi(t^{a_{1i}}z_1,\ldots,t^{a_{Ni}}z_N)=\beta_i\Phi(t^{a_{1i}}z_1,\ldots,t^{a_{Ni}}z_N).$$

Substitute $t=1$ to obtain the i^{th} Euler equation.

Conversely, suppose that Φ satisfies the Euler equations. Let $1\le i\le r$ and define $f_i(t,\boldsymbol{z})=\Phi(t^{a_{1i}}z_1,\ldots,t^{a_{Ni}}z_N)$. Substituting $t^{a_{ji}}z_j$ for z_j in the i^{th} Euler equation shows that f_i satisfies $(t\frac{\partial}{\partial t}-\beta_i)f_i(t,\boldsymbol{z})=0$. It is clear that $t^{\beta_i}\Phi(\boldsymbol{z})$ also satisfies this equation, so there is a constant λ such that $f_i(t,\boldsymbol{z})=\lambda t^{\beta_i}\Phi(\boldsymbol{z})$. Substituting $t=1$ shows that $\lambda=1$. It now follows easily that

$$\begin{aligned}\Phi(X(\boldsymbol{t})(\boldsymbol{z}))&=f_1(t_1,t_2^{a_{12}}\cdots t_r^{a_{1r}}z_1,\ldots,t_2^{a_{N2}}\cdots t_r^{a_{Nr}}z_N)=\\&t_1^{\beta_1}\Phi(t_2^{a_{12}}\cdots t_r^{a_{1r}}z_1,\ldots,t_2^{a_{N2}}\cdots t_r^{a_{Nr}}z_N)=\ldots=t_1^{\beta_1}\cdots t_r^{\beta_r}\Phi(z_1,\ldots,z_N).\end{aligned}$$ $\Box$

We introduce two more notations:

Definition 1.2.3. We denote the convex hull of $\mathcal{A}$ by $Q(\mathcal{A})$, and the real non-negative cone generated by $\mathcal{A}$ by $C(\mathcal{A})$, i.e., $C(\mathcal{A})=\mathbb{R}_{\geq 0}\mathcal{A}$.

In many interesting cases, the dimension of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$ can be computed from the combinatorics of $\mathcal{A}$.

Definition 1.2.4. $\mathcal{A}$ is called *normal* if the integral points in the non-negative cone spanned by $\mathcal{A}$ are integral non-negative combinations of the vectors $\boldsymbol{a}_i$, i.e., $C(\mathcal{A})\cap\mathbb{Z}^r=\mathbb{Z}_{\geq 0}\mathcal{A}$.

Theorem 1.2.5 (Gelfand, Kapranov, Zelevinsky, Adolphson). *Let* $\mathcal{A}$ *be as in Definition 1.2.1. Then* $H_{\mathcal{A}}(\boldsymbol{\beta})$ *is holonomic. If* $\mathcal{A}$ *is normal, then the dimension of the solution space is equal to Vol*$(Q(\mathcal{A}))$.

This theorem was first proven by stated by Gelfand, Zelevinsky and Kapranov in [GKZ89], without the normality condition. Adolphson [Ado94] pointed out that this condition is needed, after which Gelfand, Kapranov and Zelevinsky corrected the proof [GKZ93]. In this thesis, we will only consider normal sets $\mathcal{A}$, since this condition is also needed in the proof of the criterion we will use for determining algebraicity of hypergeometric functions (Theorem 1.3.7). If $\mathcal{A}$ is not normal, the dimension of the solution space can be larger than $\mathrm{Vol}(Q(\mathcal{A}))$. More information can be found in [MMW05, Ber11].

In Theorem 1.2.5 and everywhere else in this thesis, by 'volume' we mean the *simplex volume*. It is a normalization of the Euclidean volume, such that the simplex spanned by the standard basis has volume 1. Since the determinant is the only linear function that gives 0 if two of the arguments are equal and maps the standard basis to 1, the simplex volume of the simplex spanned by the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ equals $|\det(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)|$.

An important aspect of $\mathcal{A}$-hypergeometric systems of equations is their monodromy. Let $\Phi_1, \dots, \Phi_n$ be a local basis of solutions to $H_{\mathcal{A}}(\boldsymbol{\beta})$ around a point non-singular point z_0 and let c be a closed path with starting and end point z_0, avoiding all singularities. The functions Φ_i can be analytically continued along c, thereby changing into other solutions $\tilde{\Phi}$ of $H_{\mathcal{A}}(\boldsymbol{\beta})$ around z_0. These functions will again be a basis of the solution space around z_0. Hence the loop c induces a basis transformation, with can be represented by an invertible $n \times n$-matrix. It is clear that this transformation and matrix only depend on the class of c in the fundamental group. The *monodromy group* of $H_{\mathcal{A}}(\boldsymbol{\beta})$ is the group of all matrices corresponding to loops c.

The monodromy group depends on both the choice of the basis and the base point z_0. For any other basis, there is an invertible matrix giving the basis transformation. Conjugating the monodromy group with this matrix will give the monodromy group with respect to the new basis. Changing the base point also induces a conjugation of the monodromy group. Throughout this thesis, it will implicitly be understood that the monodromy group is defined up to conjugation.

The monodromy group acts on the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$. The group and the system $H_{\mathcal{A}}(\boldsymbol{\beta})$ are called *irreducible* if this action is irreducible, i.e., if there are no non-trivial subspaces of the solution space that are fixed under the action of the monodromy group. In this thesis, we will restrict ourselves to irreducible systems. The reasons for this will become clear in the next section. To be able to check that the monodromy group is irreducible without computing it, we use the almost equivalent concept of non-resonance.

Definition 1.2.6. $H_{\mathcal{A}}(\boldsymbol{\beta})$ is called *resonant* if $\boldsymbol{\beta} + \mathbb{Z}^r$ contains a point in a face of $C(\mathcal{A})$. Otherwise $H_{\mathcal{A}}(\boldsymbol{\beta})$ is *non-resonant.*

Theorem 1.2.7 ([GKZ90, Theorem 2.11]). *If $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant, then it is irreducible.*

We call $\mathcal{A}$ a *pyramid* if $N-1$ points of $\mathcal{A}$ lie in an $r-2$ dimensional affine plane, and the remaining point lies outside this plane. The converse of the above theorem is given by:

Theorem 1.2.8 ([SW12, Theorem 4.1]). *Let $H_{\mathcal{A}}(\boldsymbol{\beta})$ be resonant and suppose that $\mathcal{A}$ is not a pyramid. Then $H_{\mathcal{A}}(\boldsymbol{\beta})$ is reducible.*

We will sometimes need the stronger condition of total non-resonance.

Definition 1.2.9. $H_{\mathcal{A}}(\boldsymbol{\beta})$ is called *totally non-resonant* if $\boldsymbol{\beta}+\mathbb{Z}^r$ contains no point in any hyperplane spanned by $r-1$ independent elements of $\mathcal{A}$.

Note that total non-resonance implies non-resonance and hence irreducibility.

The connection between classical hypergeometric functions and systems of $\mathcal{A}$-hypergeometric differential equations is given by Γ-series [GKZ89]. Let $\boldsymbol{\gamma}\in\mathbb{C}^N$ such that $\gamma_1\boldsymbol{a}_1+\ldots+\gamma_N\boldsymbol{a}_N=\boldsymbol{\beta}$. Then it can easily be checked that

$$\Phi_{\boldsymbol{\gamma}}(z_1,\ldots,z_N)=\sum_{(l_1,\ldots,l_N)\in\mathbb{L}}\frac{z_1^{l_1+\gamma_1}\cdot\ldots\cdot z_n^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1)\cdot\ldots\cdot\Gamma(l_N+\gamma_N+1)} \tag{1.12}$$

is a formal solution of $H_{\mathcal{A}}(\boldsymbol{\beta})$. This function has N variables, but up to the monomial factor $z_1^{\gamma_1}\cdots z_N^{\gamma_N}$ it can be viewed as a function in $d=N-r$ variables: the lattice $\mathbb{L}$ has rank d and hence has a basis $\{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_d\}\subseteq\mathbb{Z}^N$. Now $\boldsymbol{z}^{\boldsymbol{b}_1},\ldots,\boldsymbol{z}^{\boldsymbol{b}_d}$ can be used as new variables. We will call this function in d variables the *dehomogenization* of $\Phi_{\boldsymbol{\gamma}}$.

For general choices of $\boldsymbol{\gamma}\in\mathbb{C}^N$ the series $\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$ does not converge. However, if I is an r-element subset of $\{1,\ldots,N\}$ such that $\{\boldsymbol{a}_i\mid i\in I\}$ is a linearly independent set and $\gamma_j\in\mathbb{Z}$ for all $j\notin I$, then the series has a positive radius of convergence [Sti07, Section 3]. We will consider the convergence of these series in more detail in Section 4.2.

We now show how the classical hypergeometric functions, in particular the Gauss function, fit into the framework of $\mathcal{A}$-hypergeometric functions. Recall that the Gauss function is given by

$$_2F_1(a,b,c|z)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n n!}z^n.$$

By Euler's reflection formula $\Gamma(x+n)=\frac{(-1)^n\pi}{\sin(x)\Gamma(1-x-n)}$, we have

$$_2F_1(a,b,c|z)=\frac{\pi^2}{\sin(a)\sin(b)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n\geq 0}\frac{z^n}{\Gamma(1-a-n)\Gamma(1-b-n)\Gamma(c+n)\Gamma(n+1)}.$$

If we substitute $\frac{z_3z_4}{z_1z_2}$, this function is up to a constant factor and a factor $z_1^{-a}z_2^{-b}z_3^{c-1}$ equal to the Γ-series with $\mathbb{L}=\mathbb{Z}(-1,-1,1,1)$ and $\boldsymbol{\gamma}=(-a,-b,c-1,0)$. This lattice is the lattice of relations of the set $\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3\}\subseteq\mathbb{Z}^3$. We have chosen $\mathcal{A}$ such that it spans $\mathbb{Z}^3$ over $\mathbb{Z}$ and its elements satisfy $x_1+x_2+x_3=1$. Hence $\mathcal{A}$ satisfies the conditions of Definition 1.2.1. Furthermore, $\mathcal{A}$ is normal, since both

$\mathbb{Z}_{\geq 0}\mathcal{A}$ and $\mathbb{R}_{\geq 0}\mathcal{A} \cap \mathbb{Z}^3$ equal $\{\boldsymbol{x} \in \mathbb{Z}^3 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_3 \geq 0, x_1 + x_4 \geq 0\}$. It follows that $z_1^{-a} z_2^{-b} z_3^{c-1} {}_2F_1(a, b, c | \frac{z_3 z_4}{z_1 z_2})$ is an $\mathcal{A}$-hypergeometric function.

For our purposes, we can identify ${}_2F_1(a, b, c|z)$ with $\Phi_{(-a,-b,c-1,0)}(z_1, z_2, z_3, z_4)$. It is easy to show that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant (and hence irreducible) if and only if $a, b, c - a, c - b \notin \mathbb{Z}$, which are the usual irreducibility conditions for the Gauss function. Furthermore, if $\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$ is algebraic, then substituting $z_1 = z_2 = z_3 = 1$ and $z_4 = z$ in its minimal polynomial gives a minimal polynomial for ${}_2F_1(z)$. On the other hand, if ${}_2F_1(z)$ is algebraic, then substituting $z = \frac{z_3 z_4}{z_1 z_2}$ shows that $\boldsymbol{z}^{-\boldsymbol{\gamma}} \Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$ is algebraic. The monomial factor $\boldsymbol{z}^{\boldsymbol{\gamma}}$ does not influence algebraicity, so $\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$ is algebraic if and only if ${}_2F_1(z)$ is algebraic.

The computation of the monodromy groups of the systems $H_{\mathcal{A}}(\boldsymbol{\beta})$ will be discussed in Chapter 4. We will compute the local monodromy groups, generated by the monodromy matrices corresponding to loops of the variables around the origin. At the end of Section 4.1, we will show that, up to scalars, these groups, are already generated by the matrices corresponding to the loops of $d = N - r$ variables, provided that $\mathcal{A}$ has a maximal independent subset with determinant 1. The scalars arise from the monomial factors $\boldsymbol{z}^{\boldsymbol{\gamma}}$. In case of the Gauss function, this means that the local monodromy group is determined by the monodromy matrices corresponding to loops of z_4. Hence the substitution $z_1 = z_2 = z_3 = 1$ and $z_4 = z$ induces an isomorphism between the projective local monodromy groups of ${}_2F_1(z)$ and $\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$.

Similarly, the other classical functions can be viewed as solutions of $\mathcal{A}$-hypergeometric systems of differential equations. The Pochhammer symbols determine the lattice and $\boldsymbol{\gamma}$. Then a suitable set $\mathcal{A}$ can be found, and taking $\boldsymbol{\beta} = \gamma_1 \boldsymbol{a}_1 + \ldots + \gamma_N \boldsymbol{a}_N$, the classical function can be identified with a Γ-series solution of $H_{\mathcal{A}}(\boldsymbol{\beta})$, at least for the purpose of classifying irreducible algebraic functions and computing projective monodromy groups.

In this process of homogenization, the number of variables and differential equations increases, compared to the classical systems. However, the torus action causes the function to be homogeneous, making the system more elegant. In many cases, all solutions of the $\mathcal{A}$-hypergeometric system correspond to solutions of the classical system. However, in some cases including the Horn G_3 function, $H_{\mathcal{A}}(\boldsymbol{\beta})$ contains more equations than the homogenization of the classical differential system. In this case, $H_{\mathcal{A}}(\boldsymbol{\beta})$ will have fewer solutions than the classical system. We will consider this situation for G_3 in more detail in Section 5.2.

In this thesis, we will mostly identify the classical hypergeometric functions with their $\mathcal{A}$-hypergeometric counterparts. This means that we start with the lattice $\mathbb{L}$ and choose a suitable set $\mathcal{A}$. This raises the question whether such a set $\mathcal{A}$ exists for all lattices, and up to what extent $\mathcal{A}$ is determined by $\mathbb{L}$. It is most convenient to state the answer to these questions in terms of the Gale dual of $\mathcal{A}$. The Gale dual will again become important in Chapter 4.

Definition 1.2.10. Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^r$ such that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$. Write $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N\}$ and let A be the $r \times N$-matrix with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$. Let

$d = N - r$. A *Gale dual* of $\mathcal{A}$ is an $N \times d$-matrix B such that $\ker_{\mathbb{Z}}(A) = B\mathbb{Z}^d$.

If B is a Gale dual of $\mathcal{A}$, then the lattice $\mathbb{L}$ is spanned by the columns of B. By the next lemma, given $\mathbb{L}$ we can easily check whether there exists a set $\mathcal{A}$.

Lemma 1.2.11. *Let B be an $N \times d$-matrix over $\mathbb{Z}$. Then there exists an $r \times N$-matrix A over $\mathbb{Z}$ such that $A\mathbb{Z}^N = \mathbb{Z}^r$ and $\ker_{\mathbb{Z}}(A) = B\mathbb{Z}^d$ if and only if the greatest common divisor of the $d \times d$-minors of B equals 1.*

Proof. Let $P \in \mathrm{GL}_N(\mathbb{Z})$ and $Q \in \mathrm{GL}_d(\mathbb{Z})$ such that $R = PBQ$ is the Smith Normal Form of B, with diagonal elements $b_1, \ldots, b_d$. Then

$$B\mathbb{Z}^d = P^{-1}(R\mathbb{Z}^d) = P^{-1}(b_1\mathbb{Z} \times \ldots \times b_d\mathbb{Z} \times \{0\}^r).$$

Suppose that $A\mathbb{Z}^N = \mathbb{Z}^r$ and $\ker_{\mathbb{Z}}(A) = B\mathbb{Z}^d$. Let $\boldsymbol{w} = P^{-1}(b_d, \ldots, b_d, 0, \ldots, 0)$. Then $\boldsymbol{w} \in B\mathbb{Z}^d$, so $A\boldsymbol{w} = 0$. Hence $\frac{1}{b_d}\boldsymbol{w} \in \ker_{\mathbb{Z}}(A)$, so $P^{-1}(1, \ldots, 1, 0, \ldots, 0) \in B\mathbb{Z}^d$. It follows that $b_d | 1$, so the greatest common divisor of the $d \times d$-minors of B equals the greatest common divisor of the 1×1-minors, which is 1.

Now suppose that the greatest common divisor of the $d \times d$-minors of B equals 1. Then all b_i are equal to 1 and hence $B\mathbb{Z}^d = P^{-1}(\mathbb{Z}^d \times \{0\}^r)$. Let the rows of P be $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$ and define A to be the matrix with rows $\boldsymbol{p}_{d+1}, \ldots, \boldsymbol{p}_N$. Then $AP^{-1} = (\boldsymbol{0}\ \mathrm{Id}_r)$ and $A\mathbb{Z}^N = AP^{-1}\mathbb{Z}^N = \mathbb{Z}^r$. One easily shows that $\ker_{\mathbb{Z}}(A) = B\mathbb{Z}^d$. $\square$

It is clear that the action of $\mathrm{GL}_N(\mathbb{Z})$ on $\mathcal{A}$ gives a set with the same lattice of relations, and hence the same Gale dual. Similarly, the action of $\mathrm{GL}_d(\mathbb{Z})$ on B does not change the lattice spanned by its columns. It can easily be seen that up to these actions, $\mathcal{A}$ and its Gale dual are uniquely determined by each other.

We will only consider normal sets $\mathcal{A}$ lying in an affine hyperplane giving by $h(\boldsymbol{x}) = 1$ for some linear form $h : \mathbb{R}^r \to \mathbb{R}$. The next lemma and propositions translate properties of $\mathcal{A}$ into properties of the Gale dual B.

Lemma 1.2.12. *Let $J \subseteq \{1, \ldots, N\}$ with $|J| = r$. Then $\det(\boldsymbol{a}_j)_{j \in J} \neq 0$ if and only if $\det(\boldsymbol{b}_i)_{i \notin J} \neq 0$.*

Proof. To simplify notation, assume that $J = \{1, \ldots, r\}$. Suppose that $\det(\boldsymbol{a}_j)_{j \in J} \neq 0$. Then there exist $\lambda_{ji} \in \mathbb{R}$ such that $\boldsymbol{a}_i = \lambda_{1i}\boldsymbol{a}_1 + \ldots + \lambda_{ri}\boldsymbol{a}_r$ for all $r + 1 \leq i \leq N$. For $1 \leq k \leq d$ we have $\sum_{i=1}^N b_{ik}\boldsymbol{a}_i = 0$ and hence $\sum_{j=1}^r (b_{jk} + \sum_{i=r+1}^N \lambda_{ji} b_{ik})\boldsymbol{a}_j = 0$. This implies that $b_{jk} + \sum_{i=r+1}^N \lambda_{ji} b_{ik} = 0$ for all j and k, so b_{jk} $(1 \leq j \leq r)$ is determined by b_{ik} $(r + 1 \leq k \leq N)$, i.e., by the last d rows of B. These rows must then be independent since B has rank d, so $\det(\boldsymbol{b}_i)_{i \notin J} \neq 0$.

The proof of the other implication is similar, using $\sum_{i=1}^N a_{il}\boldsymbol{b}_i = 0$ for all l. $\square$

Proposition 1.2.13. *The elements of $\mathcal{A}$ lie in an affine hyperplane if and only if the sum of each column of B equals 0.*

Proof. Write $B = (b_{ij})_{ij}$. If the elements of $\mathcal{A}$ lie in an affine hyperplane given by $h(\boldsymbol{a}_i) = 1$ for some linear form $h : \mathbb{R}^r \to \mathbb{R}$, then for all $1 \leq j \leq d$ we have $\sum_i b_{ij} = \sum_i b_{ij} h(\boldsymbol{a}_i) = h(\sum_i b_{ij}\boldsymbol{a}_i) = 0$.

On the other hand, suppose that the sum of each column of B is 0. Choose A such that $\ker_{\mathbb{Z}}(A) = B\mathbb{Z}^N$ and $A\mathbb{Z}^N = \mathbb{Z}^r$. Then A has r independent columns, say $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$. By Lemma 1.2.12, the last d rows of B are independent. Write $\tilde{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ and define the linear form $h : \mathbb{R}^r \to \mathbb{R}$ by $h(\boldsymbol{x}) = (1, \ldots, 1) \cdot \tilde{A}^{-1} \cdot \boldsymbol{x}$. Then for $1 \leq i \leq r$ we have $h(\boldsymbol{a}_i) = \sum_{j=1}^r (\tilde{A}^{-1} \cdot \boldsymbol{a}_i)_j = \sum_{j=1}^r \delta_{ij} = 1$. It remains to show that $h(\boldsymbol{a}_j) = 0$ for $r+1 \leq j \leq N$. Note that $\sum_{j=r+1}^N b_{jk} h(\boldsymbol{a}_j) = -\sum_{i=1}^r b_{i,k}$ for all k. This is a system of d equations in the unknowns $h(\boldsymbol{a}_j)$. The matrix $(b_{j,k})_{j,k}$ is invertible, so the system has a unique solution. Since the sum of each column of B is 0, this solution is $h(\boldsymbol{a}_{r+1}) = \ldots = h(\boldsymbol{a}_N) = 1$. □

Proposition 1.2.14 ([DHS09, Theorem 9]). *$\mathcal{A}$ is normal if and only if $\{\boldsymbol{x} \in \mathbb{R}^d \mid \forall 1 \leq i \leq N : (B\boldsymbol{x})_i > -1\} + \mathbb{Z}^d = \mathbb{R}^d$.*

For later use, we state the following lemma.

Lemma 1.2.15. *We have*

$$\gcd(\det(\boldsymbol{a}_j)_{j \in J}) = \gcd(\det(\boldsymbol{b}_j)_{j \notin J}) = 1$$

where the gcd*'s are taken over all $J \subseteq \{1, \ldots, N\}$ with $|J| = r$.*

Proof. Writing A in Smith Normal Form, the statement follows immediately from the assumption that $A\mathbb{Z}^N = \mathbb{Z}^r$. For the Gale dual B it follows from Lemma 1.2.11. □

1.3 Algebraic $\mathcal{A}$-hypergeometric functions

The goal of this thesis is to classify algebraic $\mathcal{A}$-hypergeometric functions and compute their (projective) monodromy groups. We will restrict ourselves to normal sets $\mathcal{A}$, since the normality condition is needed in several important theorems, such as Theorems 1.2.5 and 1.3.7. Furthermore, we will only consider irreducible systems, i.e., systems for which the monodromy group acts irreducibly on the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$. This condition is necessary to have a basis of Γ-series solutions of the form (1.12). However, the most important reason to impose irreducibility are given in the following proposition:

Proposition 1.3.1. *Let $\mathcal{A}$ be as in Definition 1.2.1 and suppose that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible. Then either all solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are algebraic, or they are all transcendental.*

Proof. Suppose that f is an algebraic solution. Denote by *Mon* the monodromy group, and let V the orbit of f under the action of the monodromy group. As V is a *Mon*-invariant subspace of the solution space and *Mon* acts irreducibly, all solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are contained in the space spanned by V. Note that f is a root of a polynomial whose coefficients are rational functions. The analytic continuation of f along a loop is again a root of this polynomial. Hence V consists of algebraic functions, and all solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are algebraic. □

It is well-known that the solutions of a regular system of differential equations are algebraic if and only if the monodromy group is finite. Related to this is the fact that in case of algebraicity, the monodromy group is equal to the Galois group. To prove this, we need a lemma:

Lemma 1.3.2. *Suppose that g is an algebraic function over $\mathbb{C}(z_1, \ldots, z_n)$, with monic minimal polynomial $P \in \mathbb{C}(z_1, \ldots, z_n)[X]$. Let $S \subseteq \mathbb{C}^n$ be the union of the zero-loci of the discriminant of P and the smallest common denominator of the coefficients of P. Let $\boldsymbol{z} \in \mathbb{C} \setminus S$ and choose a branch of g in a neighbourhood of $\boldsymbol{z}$. If the analytic continuation of g along every closed loop in $\mathbb{C} \setminus S$ is equal to g, then g is a rational function.*

Proof. First suppose that $n = 1$. Since g has trivial monodromy on $\mathbb{C} \setminus S$, it can be extended to a one-valued function on $\mathbb{P}^1 \setminus S$. Since g is algebraic, it grows at most polynomially in a neighbourhood of each of the singularities. Hence all singularities are regular points or poles of g. It follows that g is a meromorphic function on $\mathbb{P}^1$, and hence rational.

Now let $n \geq 2$. Since P is irreducible, Hilbert's irreducibility theorem for function fields [Lan83, Theorem 4.2] implies that there exist $\eta_2, \ldots, \eta_n \in \mathbb{C}$ such that $P(z_1, \eta_2, \ldots, \eta_n, X)$ is irreducible in $\mathbb{C}(z_1)[X]$. These η_i can be chosen so that the line $\{(z_1, \eta_2, \ldots, \eta_n) \mid z_1 \in \mathbb{C}\}$ is not entirely contained in S. The function $g(z_1, \eta_2, \ldots, \eta_n)$ is a one-variable function that is invariant under the monodromy restricted to this line. Hence it is a rational function, with a minimal polynomial of degree 1. Since $P(z_1, \eta_2, \ldots, \eta_n, X)$ is irreducible and $g(z_1, \eta_2, \ldots, \eta_n)$ is a root, it is the minimal polynomial of $g(z_1, \eta_2, \ldots, \eta_n)$. It follows that $P(z_1, \eta_2, \ldots, \eta_n, X)$ has degree 1, so $P(z_1, \ldots, z_n, X)$ also has degree 1 in X (because P is monic). This implies that $g(z_1, \ldots, z_n)$ is a rational function. □

Theorem 1.3.3. *Let f_1 be a solution of an irreducible system of differential equations in the variables $z_1, \ldots, z_n$. Suppose that f_1 is algebraic. Denote the other roots of its minimal polynomial by $f_2, \ldots, f_m$. Then the monodromy group equals $Gal(\mathbb{C}(z_1, \ldots, z_n, f_1, \ldots, f_m)/\mathbb{C}(z_1, \ldots, z_n))$.*

Proof. Elements of the monodromy group map roots of the minimal polynomial to other roots, and hence are elements of the Galois group. To show that the Galois group equals the monodromy group, let $g \in \mathbb{C}(z_1, \ldots, z_n, f_1, \ldots, f_m)$ and suppose that g is invariant under the action of the monodromy group. Multiplying g with a rational function, we can assume that $g \in \mathbb{C}[z_1, \ldots, z_n](f_1, \ldots, f_m)$. Hence the singularities of g are also singularities of the f_i. Now it follows from Lemma 1.3.2 that g is a rational function. It follows that the monodromy group and the Galois group fix the same functions, so these groups are equal. □

Our main tool for classifying irreducible algebraic functions is a combinatorial criterion found by Beukers [Beu10]. To state this criterion, we need the notion of apex points.

Definition 1.3.4. Let $K_{\mathcal{A}}(\boldsymbol{\beta}) = (\boldsymbol{\beta} + \mathbb{Z}^r) \cap C(\mathcal{A})$. A point $\boldsymbol{p} \in K_{\mathcal{A}}(\boldsymbol{\beta})$ is called an *apex point* if for every $\boldsymbol{q} \in K_{\mathcal{A}}(\boldsymbol{\beta})$ such that $\boldsymbol{p} \neq \boldsymbol{q}$, it holds that $\boldsymbol{p} - \boldsymbol{q} \notin C(\mathcal{A})$. The number of apex points is called the *signature* of $\mathcal{A}$ and $\boldsymbol{\beta}$ and is denoted $\sigma_{\mathcal{A}}(\boldsymbol{\beta})$.

Note that $\sigma_{\mathcal{A}}(\boldsymbol{\beta})$ only depends on the fractional part $\{\boldsymbol{\beta}\}$ of $\boldsymbol{\beta}$ (where $\{\boldsymbol{\beta}\}_i = \{\beta_i\} = \beta_i - \lfloor \beta_i \rfloor$). Under the assumption that $\mathcal{A}$ is normal, there is an easier way to tell whether a point is an apex point:

Lemma 1.3.5. *Suppose that $\mathcal{A}$ is normal and $\boldsymbol{p} \in K_{\mathcal{A}}(\boldsymbol{\beta})$. Then $\boldsymbol{p}$ is an apex point if and only if $\boldsymbol{p} - \boldsymbol{a}_i \notin C(\mathcal{A})$ for all $\boldsymbol{a}_i \in \mathcal{A}$.*

Proof. If there exists $\boldsymbol{a}_i \in \mathcal{A}$ such that $\boldsymbol{p} - \boldsymbol{a}_i \in C(\mathcal{A})$, then we can take $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{a}_i \in K_{\mathcal{A}}(\boldsymbol{\beta})$. Then $\boldsymbol{p} \neq \boldsymbol{q}$ and $\boldsymbol{p} - \boldsymbol{q} = \boldsymbol{a}_i \in C(\mathcal{A})$, so $\boldsymbol{p}$ is not an apex point.

Suppose that $\boldsymbol{p} \in K_{\mathcal{A}}(\boldsymbol{\beta})$ is not an apex point. Let $\boldsymbol{q} \in K_{\mathcal{A}}(\boldsymbol{\beta})$ such that $\boldsymbol{p} \neq \boldsymbol{q}$ and $\boldsymbol{p} - \boldsymbol{q} \in C(\mathcal{A})$. Since $\boldsymbol{q} \in C(\mathcal{A})$, there exists $\lambda_1, \ldots, \lambda_N \geq 0$ such that $\boldsymbol{q} = \lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_N \boldsymbol{a}_N$. Define $\boldsymbol{v} = \boldsymbol{p} - \boldsymbol{q}$. Then $\boldsymbol{v} \in C(\mathcal{A}) \cap \mathbb{Z}^r$. As $\mathcal{A}$ is normal, there exist $\mu_1, \ldots, \mu_N \in \mathbb{Z}_{\geq 0}$ such that $\boldsymbol{v} = \mu_1 \boldsymbol{a}_1 + \ldots + \mu_N \boldsymbol{a}_N$. It follows that $\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{v} = \sum_{i=1}^{N} (\lambda_i + \mu_i) \boldsymbol{a}_i$. Since $\boldsymbol{v} \neq 0$, there is some i such that $\mu_i \geq 1$. Now $\boldsymbol{p} - \boldsymbol{a}_i \in C(\mathcal{A})$. $\square$

Lemma 1.3.6 ([Beu10, Proposition 1.9]). *Let $\mathcal{A}$ as in Definition 1.2.1 and $\boldsymbol{\beta} \in \mathbb{R}^r$. Then $\sigma_{\mathcal{A}}(\boldsymbol{\beta}) \leq Vol(Q(\mathcal{A}))$.*

Now we can state the combinatorial criterion for algebraicity. As the Γ-series (1.12) is a Laurent series multiplied by $\boldsymbol{z}^{\boldsymbol{\gamma}}$, it can only be algebraic if $\boldsymbol{\gamma}$ has rational coordinates. Hence we must have $\boldsymbol{\beta} \in \mathbb{Q}^r$. In this thesis, $\boldsymbol{\beta}$ will always be assumed to be rational.

Theorem 1.3.7 ([Beu10, Theorem 1.10]). *Suppose that $\mathcal{A}$ is normal and $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant. Let D be the smallest common denominator of the coordinates of $\boldsymbol{\beta} \in \mathbb{Q}^r$. Then the solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are algebraic over $\mathbb{C}(\boldsymbol{z})$ if and only if $\sigma_{\mathcal{A}}(k\boldsymbol{\beta}) = Vol(Q(\mathcal{A}))$ for all $k \in \mathbb{Z}$ with $1 \leq k < D$ and $\gcd(k, D) = 1$.*

The proof of this theorem is based on a conjecture of Grothendieck, proven by Katz for Picard-Fuchs systems, on the existence of polynomial equations modulo p for almost each prime p.

Note that the above theorem implies that algebraicity of the solutions of the $\mathcal{A}$-hypergeometric system only depends on $\mathcal{A}$ and $\{\boldsymbol{\beta}\}$, but not on $\boldsymbol{\beta}$ itself, as we noticed already for ${}_2F_1$. This also follows from the fact that, up to isomorphism, the solution space only depends on the fractional part of $\boldsymbol{\beta}$:

Theorem 1.3.8. *Suppose that $\mathcal{A}$ is normal and $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant. If $\boldsymbol{\beta} - \boldsymbol{\beta}' \in \mathbb{Z}^d$, then $H_{\mathcal{A}}(\boldsymbol{\beta})$ and $H_{\mathcal{A}}(\boldsymbol{\beta}')$ are isomorphic.*

This theorem was first proven in a totally different framework by Dwork [Dwo90, Theorem 6.9.1]. A proof using the language of $\mathcal{A}$-hypergeometric functions can be found in [Sai01, Theorem 5.2].

The above theorem implies that algebraicity of the solutions and the isomorphism class of the monodromy group only depends on $\{\boldsymbol{\beta}\}$. Unless stated otherwise, we will assume that $0 \le \beta_i < 1$. Another consequence of Theorem 1.3.7 is the fact that either the solution set of $H_{\mathcal{A}}(k\boldsymbol{\beta})$ consists of algebraic functions for all k coprime to the smallest common denominator of the coordinates of $\boldsymbol{\beta}$, or there are transcendental solutions for all k. Note that we saw this behaviour already in the previous section, where we noted that ${}_2F_1(ka, kb, kc|z)$ is irreducible and algebraic if ${}_2F_1(a, b, c|z)$ is irreducible and algebraic. We will call the parameters $\boldsymbol{\beta}$ and $k\boldsymbol{\beta}$ *conjugated*, as we did for ${}_2F_1$.

Recall that the generalized hypergeometric function ${}_pF_q$ is irreducible and algebraic if and only if the interlacing condition (1.8) is satisfied. We can use Theorem 1.3.7 to find similar interlacing conditions for other hypergeometric functions. We will use the fact that the number of apex points is constant on certain parts of $[0, 1)^r$, when we let $\boldsymbol{\beta}$ vary:

Lemma 1.3.9. *Let $H_{\mathcal{A}}(\boldsymbol{\beta})$ be non-resonant and $C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^r \mid m_1(\boldsymbol{x}) \ge 0, \dots, m_n(\boldsymbol{x}) \ge 0\}$ where $m_1, \dots, m_n$ are linear forms with integral coefficients with greatest common divisor 1. Then $\sigma_{\mathcal{A}}(\boldsymbol{\beta})$ depends on $(\lfloor m_1(\boldsymbol{\beta})\rfloor, \dots, \lfloor m_n(\boldsymbol{\beta})\rfloor)$, but not on $\boldsymbol{\beta}$ itself.*

Proof. Let $\boldsymbol{x} \in \mathbb{Z}^r$. Then $\boldsymbol{x} + \boldsymbol{\beta}$ is an apex point if and only if $\boldsymbol{x} + \boldsymbol{\beta} \in C(\mathcal{A})$ and for all i, $\boldsymbol{x} - \boldsymbol{a}_i + \boldsymbol{\beta} \notin C(\mathcal{A})$. Equivalently, $m_j(\boldsymbol{x}) \ge -m_j(\boldsymbol{\beta})$ for all j, and for all i there exists j such that $m_j(\boldsymbol{x}) < m_j(\boldsymbol{a}_i) - m_j(\boldsymbol{\beta})$. Since $m_j(\boldsymbol{x})$ and $m_j(\boldsymbol{a}_i)$ are integral, whereas $m_j(\boldsymbol{\beta})$ is non-integral (because $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant), the apex points are those $\boldsymbol{x} + \boldsymbol{\beta}$ such that $m_j(\boldsymbol{x}) \ge -\lfloor m_j(\boldsymbol{\beta})\rfloor$ for all j, and for all i there exists j such that $m_j(\boldsymbol{x}) \le m_j(\boldsymbol{a}_i) - \lfloor m_j(\boldsymbol{\beta})\rfloor - 1$. Hence the conditions on $\boldsymbol{x} + \boldsymbol{\beta}$ to be an apex point only depend on $\lfloor m_j(\boldsymbol{\beta})\rfloor$. ☐

We now describe how to find a condition on $\boldsymbol{\beta}$ to have maximal signature, similar to the interlacing condition for ${}_2F_1$. As input we need the linear forms m_i that determine the faces of the cone $C(\mathcal{A})$. Write $m_i(\boldsymbol{x}) = \sum_j m_{ij} x_j$. Since we only consider $\boldsymbol{\beta}$ such that $\beta_i \in [0, 1)$ for all i, $\lfloor m_i(\boldsymbol{\beta})\rfloor$ can only take integral values between $\sum_j \min(m_{ij}, 0)$ and $\sum_j \max(m_{ij}, 0)$ (both boundaries are excluded, unless they are zero). Hence $(\lfloor m_1(\boldsymbol{\beta})\rfloor, \dots, \lfloor m_n(\boldsymbol{\beta})\rfloor)$ takes only finitely many values. For each of those, it suffices to find one corresponding $\boldsymbol{\beta}$ and compute the number of apex points. Finding such $\boldsymbol{\beta}$ boils down to solving a linear system of inequalities. This can easily be done by hand or using a computer algebra system, which will also detect the values of $(\lfloor m_1(\boldsymbol{\beta})\rfloor, \dots, \lfloor m_n(\boldsymbol{\beta})\rfloor)$ for which no $\boldsymbol{\beta}$ exists. Having found $\boldsymbol{\beta}$, finding apex points can again be done by solving a system of linear inequalities, in this case over the integers.

Using this algorithm, finding the interlacing condition can entirely be done by computer. However, this algorithm is very slow, although it can be used for most systems we consider in this thesis.

1.4 Triangulations and reductions

Theorem 1.3.7 will allow us to classify the algebraic Appell, Lauricella and Horn functions, as well as the functions corresponding to sets of dimension at most 2. In this section we collect some properties of the defining point configurations and functions that will simplify this classification. We will show that we can choose $\mathcal{A}$ in the hyperplane $z_r = 1$. Then we will consider triangulated sets $\mathcal{A}$ and show how triangulations help us to compute the maximal number of apexpoints, to show that $\mathcal{A}$ is normal and to show that resonant systems often have transcendental solutions. This last property makes it possible to reduce large sets $\mathcal{A}$ to subsets. Finally, we give some lemmas that simplify the application of Theorem 1.3.7.

Our first remark concerns isomorphic sets. Let $\mathcal{A}, \mathcal{A}' \subseteq \mathbb{Z}^r$ be as in Definition 1.2.1 with $|\mathcal{A}| = |\mathcal{A}'|$, and let $f : \mathbb{Z}^r \to \mathbb{Z}^r$ be an isomorphism such that $f(\mathcal{A}) = \mathcal{A}'$ (as sets, so the order of the vectors can be changed). Write $\boldsymbol{\beta}' = f(\boldsymbol{\beta})$. Then $C(\mathcal{A}') = f(C(\mathcal{A}))$ and $\sigma_{\mathcal{A}'}(\boldsymbol{\beta}') = \sigma_{\mathcal{A}}(\boldsymbol{\beta})$, so $H_{\mathcal{A}'}(\boldsymbol{\beta}')$ is non-resonant and has algebraic solutions if and only if $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant and has algebraic solutions. If the faces of $C(\mathcal{A})$ are given by linear forms m_i as in Lemma 1.3.9, then the faces of $C(\mathcal{A}')$ are given by $m_i \circ f^{-1}$ and $H_{\mathcal{A}'}(\boldsymbol{\beta}')$ has algebraic solutions if $\boldsymbol{\beta}' = f(\boldsymbol{\beta})$, with $\boldsymbol{\beta}$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ has algebraic solutions.

By choosing a suitable isomorphism, we can arrange that $\mathcal{A}$ lies in the hyperplane on height 1:

Proposition 1.4.1. *Let $\mathcal{A}$ be a subset of $\mathbb{Z}^r$ satisfying $\mathbb{Z}\mathcal{A} = \mathbb{Z}^r$ and let $h : \mathbb{R}^r \to \mathbb{R}$ be a linear form with $h(\boldsymbol{a}) = 1$ for all $\boldsymbol{a} \in \mathcal{A}$. Then there is an isomorphism f of $\mathbb{Z}^r$ such that $f(\mathcal{A}) \subseteq \mathbb{Z}^{r-1} \times \{1\}$.*

Proof. Write $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N\}$ and $h(\boldsymbol{x}) = h_1x_1 + \ldots + h_rx_r$. For each i there exist $\lambda_1, \ldots, \lambda_N \in \mathbb{Z}$ such that $\boldsymbol{e}_i = \lambda_1\boldsymbol{a}_1 + \ldots + \lambda_N\boldsymbol{a}_N$. It follows from $h(\boldsymbol{a}_j) = 1$ that $h_i = h(\boldsymbol{e}_i) = \lambda_1 + \ldots + \lambda_N \in \mathbb{Z}$. Furthermore, $\gcd(h_1, \ldots, h_r) = 1$ because $h(\boldsymbol{a}_1) = 1$. There exists a basis of $\mathbb{Z}^r$ containing the vector $(h_1, \ldots, h_r)$. The matrix whose rows are the elements of this basis, with r^{th} row $(h_1, \ldots, h_r)$, gives the desired isomorphism. $\square$

In most interesting cases, it is possible to choose $\mathcal{A}$ so that the standard basis vectors of $\mathbb{R}^r$ are elements of $\mathcal{A}$. Sometimes we will prefer this over choosing $\mathcal{A}$ in the hyperplane on height 1.

Next, we consider triangulations of $\mathcal{A}$.

Definition 1.4.2. Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^r$ in an affine hyperplane given by $h(\boldsymbol{x}) = 1$ for some linear form $h : \mathbb{R}^r \to \mathbb{R}$. A *triangulation* of $Q(\mathcal{A})$ is a finite set $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ such that each V_i is a subset of $\mathcal{A}$ consisting of r linearly independent elements, $Q(V_i) \cap Q(V_j) = Q(V_i \cap V_j)$ for all i and j and $Q(\mathcal{A}) = \cup_{i=1}^{l} Q(V_i)$. If all $Q(V_i)$ have simplex volume 1, then the triangulation is called *unimodular*. We will call $\{V_1, \ldots, V_l\}$ a (unimodular) triangulation of $\mathcal{A}$.

Remark 1.4.3. If $Q(\mathcal{A})$ has a unimodular triangulation, then its volume equals the number of simplices in the triangulation.

The following lemma will be helpful to find triangulations:

Lemma 1.4.4. *Suppose that $V_1, \ldots, V_l$ are subsets of $\mathcal{A}$ consisting of r vectors with determinant ± 1, such that $\mathcal{A} = \cup_{i=1}^{l} V_i$, $C(V_i) \cap C(V_j) \subseteq C(V_i \cap V_j)$ for all i and j and $\cup_{i=1}^{l} C(V_i)$ is convex. Then $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ is a unimodular triangulation of $Q(\mathcal{A})$.*

Proof. Since the determinant of the vectors in V_i is ± 1, the vectors are linearly independent and $Q(V_i)$ has volume 1. It is clear that $C(V_i \cap V_j) \subseteq C(V_i) \cap C(V_j)$, so $C(V_i) \cap C(V_j) = C(V_i \cap V_j)$. By taking the intersection with the hyperplane $h(\boldsymbol{x}) = 1$, we see that $Q(V_i) \cap Q(V_j) = Q(V_i \cap V_j)$. It is also clear that $\cup_{i=1}^{l} Q(V_i) \subseteq Q(\mathcal{A})$. Note that $\cup_{i=1}^{l} Q(V_i) = \cup_{i=1}^{l} C(V_i) \cap h^{-1}(\{1\})$ is a convex set. It contains $\mathcal{A} = \cup_{i=1}^{l} V_i$, so it also contains the convex hull of $\mathcal{A}$. This implies that $Q(\mathcal{A}) \subseteq \cup_{i=1}^{l} Q(V_i)$. $\square$

Proposition 1.4.5. *If $\mathcal{A}$ has a unimodular triangulation, then $\mathcal{A}$ is normal.*

Proof. Let $\{Q(V_1), \ldots, Q(V_l)\}$ be a unimodular triangulation of $Q(\mathcal{A})$ and let $\boldsymbol{x} \in C(\mathcal{A}) \cap \mathbb{Z}^r$. There exists V_i such that $\boldsymbol{x} \in C(V_i)$. Viewing the elements of V_i as the columns of an $r \times r$-matrix, there exists an vector $\boldsymbol{\lambda}$ such that $\boldsymbol{x} = V_i \cdot \boldsymbol{\lambda}$ and $\lambda_j \geq 0$ for all j. Since the triangulation is unimodular, the matrix V_i is invertible over the integers. Hence $\boldsymbol{\lambda} = V_i^{-1} \boldsymbol{x}$ is an integral vector, and $\boldsymbol{x} \in \mathbb{Z}_{\geq 0} \mathcal{A}$. $\square$

Lemma 1.4.6. *Let $\{V_1, \ldots, V_l\}$ be a unimodular triangulation of $\mathcal{A}$. Then each cone $C(V_i)$ contains at most one apex point.*

Proof. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be apex points in $C(V_i)$. By reordering the vectors in $\mathcal{A}$ if necessary, we can assume that $V_i = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$. There exist $\lambda_j, \mu_j \geq 0$ such that $\boldsymbol{p} = \lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_r \boldsymbol{a}_r$ and $\boldsymbol{q} = \mu_1 \boldsymbol{a}_1 + \ldots + \mu_r \boldsymbol{a}_r$. Since $\boldsymbol{p}$ is an apex point, we have $\boldsymbol{p} - \boldsymbol{a} \notin C(\mathcal{A})$ for all $\boldsymbol{a} \in \mathcal{A}$. In particular, $\boldsymbol{p} - \boldsymbol{a}_j \notin C(V_i)$, so $\lambda_j < 1$ for $j = 1, \ldots, r$. Similarly, $\mu_j < 1$.

Note that $\boldsymbol{p} - \boldsymbol{q} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r) \cdot (\boldsymbol{\lambda} - \boldsymbol{\mu})$, so $\boldsymbol{\lambda} - \boldsymbol{\mu} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)^{-1}(\boldsymbol{p} - \boldsymbol{q})$, where $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ is viewed as an invertible $r \times r$-matrix. As $\boldsymbol{p}$ and $\boldsymbol{q}$ are both apex points, we have $\{\boldsymbol{p}\} = \{\boldsymbol{q}\} = \{\boldsymbol{\beta}\}$, so $\boldsymbol{p} - \boldsymbol{q} \in \mathbb{Z}^r$. The matrix $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ has determinant ± 1 because the triangulation is unimodular. It follows that $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{Z}^r$. This implies that $\boldsymbol{\lambda} = \boldsymbol{\mu}$, and hence $\boldsymbol{p} = \boldsymbol{q}$. $\square$

The Appell and Lauricella functions are generalizations of the Gauss function to several variables. If such a function is algebraic, then all specializations to Gauss functions must be algebraic as well. In general, substituting one or more values in a multivariable algebraic function gives an algebraic function in fewer variables (substituting 0 for one of the variables of a Lauricella function just reduces the number of variables). However, note that substituting values for the variables can alter irreducibility.

The $\mathcal{A}$-hypergeometric analogue of substituting 0 for a variable is deleting a point of $\mathcal{A}$. For $\mathcal{A}$-hypergeometric functions, the number of variables equals the dimension of the lattice. In general, by deleting a point of $\mathcal{A}$ the number of independent relations between the points of $\mathcal{A}$ decreases by one. The following proposition states the conditions on $\mathcal{A}$ under which this does not affect the irreducibility or algebraicity of the solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$.

Proposition 1.4.7. *Let $\mathcal{A}' \subseteq \mathcal{A} \subseteq \mathbb{Z}^r$ be as in Definition 1.2.1. Suppose that $\mathcal{A}'$ has a unimodular triangulation that can be extended to a unimodular triangulation of $\mathcal{A}$. If $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant and has algebraic solutions, then $H_{\mathcal{A}'}(\boldsymbol{\beta})$ is also non-resonant with algebraic solutions.*

We will mostly use this proposition in cases where there is a subset $\mathcal{A}'$ such that there is no $\boldsymbol{\beta}$ for which the system is irreducible and has algebraic solutions. Then there are also no irreducible algebraic functions associated to $\mathcal{A}$.

To prove Proposition 1.4.7, we need the following lemma:

Lemma 1.4.8. *Suppose that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant but not totally non-resonant. Then there exist $r-1$ independent elements of $\mathcal{A}$ such that $\boldsymbol{\beta} + \mathbb{Z}^r$ contains a point of the hyperplane $\mathcal{F}$ through these elements of $\mathcal{A}$. Suppose that $\mathcal{A}$ has a unimodular triangulation $\{V_1, \ldots, V_l\}$ such that $\mathcal{F}$ is a face of one of the cones $C(V_i)$. Then $H_{\mathcal{A}}(\boldsymbol{\beta})$ has transcendental solutions.*

Proof. As the algebraicity of solutions only depends on $\{\boldsymbol{\beta}\}$, we can assume that $\boldsymbol{\beta}$ itself lies on $\mathcal{F}$. Since $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant, $\mathcal{F}$ is not a face of $C(\mathcal{A})$ and hence there is a $V_j \neq V_i$ such that $\mathcal{F}$ is also a face of $C(V_j)$. The sets V_i and V_j have $r-1$ points in common, say $V_i = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ and $V_j = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{r-1}, \boldsymbol{a}_{r+1}\}$. There exists a vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{r-1}, 0)$ such that $\boldsymbol{\beta} = V_i\boldsymbol{\lambda} = V_j\boldsymbol{\lambda}$. By again translating $\boldsymbol{\beta}$ if necessary, we can assume that $0 \leq \lambda_i < 1$ for all i. Suppose that $\boldsymbol{x} + \boldsymbol{\beta} \in C(V_i)$ is an apex point, with $\boldsymbol{x} \in \mathbb{Z}$. Then we can write $\boldsymbol{x} + \boldsymbol{\beta} = V_i\boldsymbol{\mu}$ with $0 \leq \mu_i < 1$. But then $\boldsymbol{\mu} - \boldsymbol{\lambda} = V_i^{-1}\boldsymbol{x} \in \mathbb{Z}^r$ with $-1 < \mu_i - \lambda_i < 1$, so $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\boldsymbol{x} = \boldsymbol{0}$. Hence if there is an apex point in $C(V_i)$, then it must be $\boldsymbol{\beta}$. Similarly, the only possible apex point in $C(V_j)$ is $\boldsymbol{\beta}$. By Lemma 1.4.6, the other $C(V_k)$ contain at most $l-2$ apexpoints, so $\sigma_{\mathcal{A}}(\boldsymbol{\beta}) \leq l-1$. Theorem 1.3.7 now implies that the solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are transcendental. $\square$

Remark 1.4.9. One can show that systems that are not totally non-resonant have solutions involving logarithms. Hence the above lemma also holds if $\mathcal{A}$ has no unimodular triangulation, but the proof is more involved.

Proof of Proposition 1.4.7. Suppose that $H_{\mathcal{A}'}(\boldsymbol{\beta})$ is resonant. Then $H_{\mathcal{A}}(\boldsymbol{\beta})$ is not totally non-resonant. Extending the triangulation of $\mathcal{A}'$ to $\mathcal{A}$ gives a triangulation satisfying the hypothesis of Lemma 1.4.8. Hence $H_{\mathcal{A}}(\boldsymbol{\beta})$ has transcendental solutions. It follows that $H_{\mathcal{A}'}(\boldsymbol{\beta})$ is non-resonant.

Let $k \in \mathbb{Z}$ be coprime with the smallest common denominator of the coordinates of $\boldsymbol{\beta}$. By Theorem 1.3.7, $\sigma_{\mathcal{A}}(k\boldsymbol{\beta}) = \mathrm{Vol}(Q(\mathcal{A}))$. Lemma 1.4.6 shows that each cone in the triangulation of $\mathcal{A}$ contains exactly one apex point. In particular, this holds for the cones in the triangulation of $\mathcal{A}'$. Hence $\sigma_{\mathcal{A}'}(k\boldsymbol{\beta}) = \mathrm{Vol}(Q(\mathcal{A}'))$ and $H_{\mathcal{A}'}(\boldsymbol{\beta})$ has algebraic solutions. □

We end with two easy lemmas that will make it easier to prove that certain functions are not algebraic. To find all parameters $\boldsymbol{\beta}$ satisfying a certain interlacing condition, we will have to find k such that $\gcd(k, D) = 1$ and $\sigma_{\mathcal{A}}(k\boldsymbol{\beta})$ is not maximal. By the following lemma, it suffices to find k coprime with the denominator of several, but not all, coefficients. Sometimes we can show that $\sigma_{\mathcal{A}}(k\boldsymbol{\beta})$ is not maximal if some parameter is close to $1/2$. The second lemma handles this case.

Lemma 1.4.10. *Let k, D and $\tilde{D}$ be positive integers such that $D|\tilde{D}$ and $\gcd(k, D) = 1$. Then there exists an integer l such that $l \equiv k \pmod{D}$ and $\gcd(l, \tilde{D}) = 1$.*

Proof. Let E be maximal such that $E|\tilde{D}$ and $\gcd(D, E) = 1$. By the Chinese remainder theorem, there exists l such that $l \equiv k \pmod{D}$ and $l \equiv 1 \pmod{E}$. This l satisfies $\gcd(l, \tilde{D}) = 1$. □

Lemma 1.4.11. *Let $r = p/q$ with $\gcd(p, q) = 1$ and $q \geq 3$. Define $d = 1$ if q is odd, $d = 2$ if 4 divides q and $d = 4$ if $q \equiv 2 \pmod{4}$. Let $t \in (0, 1/2)$. If $q \geq d/(1-2t)$, then there exists $k \in \mathbb{Z}$ with $\gcd(k, q) = 1$ such that $\{kr\} \in [t, 1/2)$.*

Proof. Choose $p' \in \mathbb{Z}$ such that $pp' \equiv 1 \pmod{q}$ and take $k = (q-d)p'/2$. Then k is an integer with $\gcd(k, q) = 1$. Since $q \geq d/(1-2t)$, we have $(q-d)/(2q) \geq t$. Hence $\{kr\} = \{(q-d)/2 \cdot (pp')/q\} = \{(q-d)/(2q)\} \in [t, 1/2)$. □

2

The Appell, Lauricella and Horn functions

In this chapter we use the apex point criterion 1.3.7 to find all irreducible algebraic Appell, Lauricella and Horn functions. Some of the results are known: the algebraic Appell functions F_1 and Lauricella functions F_D have been computed by Cohen and Wolfart [CW92] and Sasaki [Sas77]; the Appell functions F_2 and F_4 by Kato [Kat00, Kat97] and the Horn G_3 function by Schipper [Sch09]. In [CW92], Cohen and Wolfart also give some results on reducible algebraic F_2, F_3 and F_4 functions. In most of these papers, the algebraic functions were determined by computing the monodromy group and determining the parameters for which it is finite. With our approach we can reproduce these results and extend them to the Lauricella and Horn functions without computing the monodromy groups.

This chapter is based on [Bod12].

2.1 The Appell and Lauricella functions

The Appell F_1 and Lauricella F_D functions

The Lauricella F_D function is defined by

$$F_D(a, \boldsymbol{b}, c|\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\boldsymbol{m}|}(\boldsymbol{b})_{\boldsymbol{m}}}{(c)_{|\boldsymbol{m}|}\boldsymbol{m}!}\boldsymbol{z}^{\boldsymbol{m}}.$$

Up to a constant factor, this equals

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}} \frac{z_1^{m_1}\cdot\ldots\cdot z_n^{m_n}}{\Gamma(1-m_1-\ldots-m_n-a)\Gamma(\boldsymbol{1}-\boldsymbol{m}-\boldsymbol{b})\Gamma(m_1+\ldots+m_n+c)\Gamma(\boldsymbol{1}+\boldsymbol{m})}.$$

Hence the lattice is

$$\mathbb{L} = \bigoplus_{i=1}^{n} \mathbb{Z}(-\boldsymbol{e}_1 - \boldsymbol{e}_{i+1} + \boldsymbol{e}_{n+2} + \boldsymbol{e}_{n+i+2}) \subseteq \mathbb{Z}^{2n+2}$$

and $\boldsymbol{\gamma} = (-a, -\boldsymbol{b}, c-1, \boldsymbol{0}) \in \mathbb{R}^{2n+2}$. We can take

$$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_{n+2}, \boldsymbol{e}_1+\boldsymbol{e}_3-\boldsymbol{e}_{n+2}, \ldots, \boldsymbol{e}_1+\boldsymbol{e}_{n+1}-\boldsymbol{e}_{n+2}\} \subseteq \mathbb{Z}^{n+2}$$

and $\boldsymbol{\beta} = \sum_{i=1}^{2n+2} \gamma_i \boldsymbol{a}_i = (-a, -\boldsymbol{b}, c-1) \in \mathbb{Q}^{n+2}$.

Lemma 2.1.1. *$\mathcal{A}$ is normal.*

Proof. Suppose that $\boldsymbol{x} \in \mathbb{Z}^{n+2}$ lies in $\mathbb{R}_{\geq 0}\mathcal{A}$. Then there exist $\mu_1, \ldots, \mu_{2n+2} \in \mathbb{R}_{\geq 0}$ such that $\boldsymbol{x} = \sum_{i=1}^{2n+2} \mu_i \boldsymbol{a}_i$. Define $\lambda_{n+3} = \min(x_1, x_2)$, and for $n+4 \leq i \leq 2n+2$, define recursively $\lambda_i = \min(x_1 - \lambda_{n+3} - \ldots - \lambda_{i-1}, x_{i-n-1})$. Furthermore, for $2 \leq i \leq n+1$, let $\lambda_i = x_i - \lambda_{n+i+1}$. Finally, let $\lambda_1 = x_1 - \lambda_{n+3} - \ldots - \lambda_{2n+2}$ and $\lambda_{n+2} = x_{n+2} + \lambda_{n+3} + \ldots + \lambda_{2n+2}$. Then it is clear that $\lambda_i \in \mathbb{Z}$ for all i, and $\sum_{i=1}^{2n+2} \lambda_i \boldsymbol{a}_i = \boldsymbol{x}$. It remains to show that $\lambda_i \geq 0$ for all i. This is clear for all i except for $i = n+2$. If $\lambda_{n+i+1} = x_i$ for all $2 \leq i \leq n+1$, then $\lambda_{n+2} = x_2 + \ldots + x_{n+1} + x_{n+2} \geq 0$. If there exists $2 \leq i \leq n+1$ such that $\lambda_{n+i+1} = x_1 - \lambda_{n+3} - \ldots - \lambda_{n+i}$, then $\lambda_{n+j} = 0$ for all $i+2 \leq j \leq 2n+2$, so $\lambda_{n+2} = \lambda_{n+3} + \ldots + \lambda_{n+i} + x_1 - \lambda_{n+3} - \ldots - \lambda_{n+i} + x_{n+2} = x_1 + x_{n+2} \geq 0$. It follows that $\boldsymbol{x}$ lies in $\mathbb{Z}_{\geq 0}\mathcal{A}$. □

Lemma 2.1.2. *The positive real cone spanned by $\mathcal{A}$ is*

$$C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid x_1, \ldots, x_{n+1} \geq 0, x_1 + x_{n+2} \geq 0, x_2 + \ldots + x_{n+1} + x_{n+2} \geq 0\}.$$

Proof. It is clear that $C(\mathcal{A})$ is included in this set. Suppose that $x_1, \ldots, x_{n+1} \geq 0$, $x_1 + x_{n+2} \geq 0$ and $x_2 + \ldots + x_{n+1} + x_{n+2} \geq 0$. Define λ_i as in the proof of Lemma 2.1.1. Then $\boldsymbol{x} = \sum_{i=1}^{2n+2} \lambda_i \boldsymbol{a}_i$, and by an argument very similar to the one given in the proof of Lemma 2.1.1, it follows that all λ_i are non-negative. Hence $\boldsymbol{x} \in C(\mathcal{A})$. □

Corollary 2.1.3. *$F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ is irreducible if and only if a, $b_1, \ldots, b_n$, $c - a$, and $c - b_1 - \ldots - b_n$ are non-integral.*

Proof. By Theorems 1.2.7 and 1.2.8, $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ is irreducible if and only if $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant, i.e., there is no face of $C(\mathcal{A})$ containing a point in $\boldsymbol{\beta} + \mathbb{Z}^{n+2}$. Since $\boldsymbol{\beta} = (-a, -b_1, \ldots, -b_n, c-1)$, the statement follows from Lemma 2.1.2. □

To compute the simplex volume of the convex hull of $\mathcal{A}$, we map $\mathcal{A}$ to the hyperplane $v_{n+2} = 1$ by the invertible transformation $\boldsymbol{v} \mapsto (v_1, \ldots, v_{n+1}, v_1 + \ldots + v_{n+2})$. Now we can omit the last coordinate. This gives the set $\tilde{\mathcal{A}} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}, \boldsymbol{0}, \boldsymbol{e}_1 + \boldsymbol{e}_2, \ldots, \boldsymbol{e}_1 + \boldsymbol{e}_{n+1}\} \subseteq \mathbb{Z}^{n+1}$.

Lemma 2.1.4. *The convex hull of $\tilde{\mathcal{A}}$ is $\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid 0 \leq x_1, \ldots, x_{n+1} \leq 1, 0 \leq x_2 + \ldots + x_{n+1} \leq 1\}$.*

Proof. Denote the above set by V. It is clear that V is a convex set containing $\tilde{\mathcal{A}}$, so $Q(\tilde{\mathcal{A}})$ is contained in V. Let $\boldsymbol{x} \in V$. For $i \neq n+2$, define λ_i as in the proof of Lemma 2.1.1. Similar to Lemma 2.1.1, we have $\lambda_i \geq 0$ for all $i \neq n+2$. Note that $\boldsymbol{x} = \sum_{i=1}^{2n+2} \lambda_i \boldsymbol{a}_i$ for every value of λ_{n+2}. By an argument similar to the proof of Lemma 2.1.1, one can show that $\lambda_1 + \ldots + \lambda_{n+1} + \lambda_{n+3} + \ldots + \lambda_{2n+2}$ is equal to either x_1 or $x_2 + \ldots + x_{n+1}$. In both cases it is smaller than 1, so we can define $\lambda_{n+2} = 1 - (\lambda_1 + \ldots + \lambda_{n+1} + \lambda_{n+3} + \ldots + \lambda_{2n+2}) \geq 0$. □

The volume of $Q(\mathcal{A})$ can now be obtained by an $(n+1)$-fold integration.

Corollary 2.1.5. *The simplex-volume of $Q(\mathcal{A})$ is $n+1$, so there are at most $n+1$ apex points.*

Lemma 2.1.6. *Suppose that $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ is irreducible. Then there are $n+1$ apex points if and only if $(\lfloor\{c\}-\{a\}\rfloor, \lfloor\{c\}-\{b_1\}-\ldots-\{b_n\}\rfloor) \in \{(0,-n),(-1,0)\}$.*

Proof. Let $\boldsymbol{p} \in \mathbb{R}^{n+2}$. Since $\boldsymbol{p}$ is an apex point if and only if $\boldsymbol{p} \in K_{\mathcal{A}}(\boldsymbol{\beta})$ and $\boldsymbol{p} - \boldsymbol{a}_i \notin C(\mathcal{A})$ for all $\boldsymbol{a}_i \in \mathcal{A}$, the apex points are precisely the points $\boldsymbol{p} = \boldsymbol{x}+\boldsymbol{\beta}$, with $\boldsymbol{x} \in \mathbb{Z}^{n+2}$, satisfying the following conditions: $x_1, \ldots, x_{n+1} \geq 0$, $x_1 + x_{n+2} + \beta_1 + \beta_{n+2} \geq 0$ and $x_2 + \ldots + x_{n+2} + \beta_2 + \ldots + \beta_{n+2} \geq 0$; $x_1 = 0$ or $x_1 + x_{n+2} + \beta_1 + \beta_{n+2} < 1$; $x_2 + \ldots + x_{n+2} + \beta_2 + \ldots + \beta_{n+2} < 1$ or $x_i = 0$ for all $i \in \{2, \ldots, n+1\}$; $x_1 + x_{n+2} + \beta_1 + \beta_{n+2} < 1$ or $x_2 + \ldots + x_{n+2} + \beta_2 + \ldots + \beta_{n+2} < 1$; and $x_1 = 0$ or $x_i = 0$ for all $i \in \{2, \ldots, n+1\}$.

If $x_1 = \ldots = x_{n+1} = 0$, then $x_{n+2} + \beta_1 + \beta_{n+2} \geq 0$ so $x_{n+2} \geq -1$. Since we either have $x_{n+2} + \beta_1 + \beta_{n+2} < 1$ or $x_{n+2} + \beta_2 + \ldots + \beta_{n+2} < 1$, we also have $x_{n+2} \leq 0$. $x_{n+2} = -1$ gives a apex point if and only if $\beta_1 + \beta_{n+2} \geq 1$ and $\beta_2 + \ldots + \beta_{n+2} \geq 1$, and $x_{n+2} = 0$ gives an apex point in all other cases. Hence there is always exactly one apex point with $x_1 = \ldots = x_{n+1} = 0$.

If $x_1 = 0$ and there exists $2 \leq i \leq n+1$ with $x_i > 0$, then $x_2 + \ldots + x_{n+2} + \beta_2 + \ldots + \beta_{n+2} < 1$. From $x_{n+2} + \beta_1 + \beta_{n+2} \geq 0$ it follows that $x_{n+2} \geq -1$. Now $x_2, \ldots, x_{n+1} \geq 0$ and $x_2 + \ldots + x_{n+2} \leq 0$ imply that $x_i = 1$, $x_{n+2} = -1$ and $x_j = 0$ for all $j \neq i, n+2$. Hence there is at most one apex point, and this is indeed an apex point if and only if $\beta_1 + \beta_{n+2} \geq 1$ and $\beta_2 + \ldots + \beta_{n+2} < 1$. Since this condition is independent of i, there are either 0 or n apex points of this form.

Finally, if $x_1 > 0$, then we have $0 \leq x_1 + x_{n+2} + \beta_1 + \beta_{n+2} < 1$, $x_2 = \ldots = x_{n+1} = 0$ and $x_{n+2} + \beta_2 + \ldots + \beta_{n+2} \geq 0$. It follows that $-n-1 < -(\beta_2 + \ldots + \beta_{n+2}) \leq x_{n+2} < -(\beta_1 + \beta_{n+2}) \leq 0$, so $-n \leq x_{n+2} \leq -1$. For every such x_{n+2}, there is exactly one x_1 which satisfies the conditions, namely $x_1 = -x_{n+2} - \lfloor \beta_1 + \beta_{n+2} \rfloor$. This gives at most n apex points, and there are n apex points if and only if $\beta_1 + \beta_{n+2} < 1$ and $\beta_2 + \ldots + \beta_{n+2} \geq n$.

Hence there are $n+1$ apex points if and only if either $\beta_1 + \beta_{n+2} \geq 1$ and $\beta_2 + \ldots + \beta_{n+2} < 1$, or $\beta_1 + \beta_{n+2} < 1$ and $\beta_2 + \ldots + \beta_{n+2} \geq n$. Since $\boldsymbol{\beta} = (1-\{a\}, 1-\{b_1\}, \ldots, 1-\{b_n\}, \{c-1\})$, this is equivalent to the condition that $(\lfloor\{c\}-\{a\}\rfloor, \lfloor\{c\}-\{b_1\}-\ldots-\{b_n\}\rfloor) \in \{(0,-n),(-1,0)\}$. □

Now we have found an interlacing condition, we can easily check whether a parameter vector $(a, b_1, \ldots, b_n, c)$ gives rise to an irreducible algebraic function. To find all irreducible algebraic functions, we use a reduction to Lauricella functions with less variables:

Lemma 2.1.7. *If $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ is an irreducible algebraic function, then for every $i \in \{1, \ldots, n\}$, $F_D(a, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, c|z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ is also irreducible and algebraic.*

Proof. We can assume that $i = n$. Write $F_{D,n-1} = F_D(a, b_1, \ldots, b_{n-1}, c|z_1, \ldots, z_{n-1})$. From the irreducibility conditions for $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$, it follows that $F_{D,n-1}$ is irreducible unless $c - b_1 - \ldots - b_{n-1}$ is an integer. However, if this is an integer, then $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ doesn't satisfy the interlacing condition. Hence $F_{D,n-1}$ is also irreducible. Algebraicity of $F_{D,n-1}$ follows from the fact that $F_{D,n-1}$ can be obtained from $F_{D,n}$ by substituting $z_n = 0$. □

Lemma 2.1.8. *If $F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic, then ${}_2F_1(a, b_1 + b_2, c|z)$ is also irreducible and algebraic.*

Proof. Since $F_1(a, b_1, b_2, c|x, y)$ is irreducible, $a, c - a$ and $c - b_1 - b_2$ are non-integral. It follows from the interlacing condition for $F_1(a, b_1, b_2, c|x, y)$ that either $0 < \{b_1\} + \{b_2\} \leq \{c\} < 1$ or $1 < \{c\} + 1 < \{b_1\} + \{b_2\} < 2$, so $b_1 + b_2$ cannot be an integer. Hence ${}_2F_1(a, b_1 + b_2, c|z)$ is irreducible.

To prove that ${}_2F_1(a, b_1 + b_2, c|z)$ is algebraic, let D be the smallest common denominator of $a, b_1 + b_2$ and c and let $\tilde{D}$ be the smallest common denominator of a, b_1, b_2 and c. Then clearly $D|\tilde{D}$. Let $k \in \mathbb{Z}$ such that $1 \leq k < D$ and $\gcd(k, D) = 1$. By Lemma 1.4.10, there exists $l \in \mathbb{Z}$ such that $l \equiv k \pmod{D}$ and $\gcd(l, \tilde{D}) = 1$. Since $F_1(a, b_1, b_2, c|x, y)$ is algebraic, either $\lfloor \{lc\} - \{la\} \rfloor = -1$ and $\lfloor \{lc\} - \{lb_1\} - \{lb_2\} \rfloor = 0$, or $\lfloor \{lc\} - \{la\} \rfloor = 0$ and $\lfloor \{lc\} - \{lb_1\} - \{lb_2\} \rfloor = -2$,. In the first case $\{lb_1\} + \{lb_2\} \leq \{lc\} < 1$, so $\{l(b_1 + b_2)\} = \{lb_1\} + \{lb_2\}$ and $\lfloor \{c\} - \{b_1 + b_2\} \rfloor = \lfloor \{c\} - \{b_1\} - \{b_2\} \rfloor = 0$. In the second case $\{lb_1\} + \{lb_2\} > 1$, so $\{l(b_1 + b_2)\} = \{lb_1\} + \{lb_2\} - 1$ and $\lfloor \{c\} - \{b_1 + b_2\} \rfloor = \lfloor \{c\} - \{b_1\} - \{b_2\} \rfloor + 1 = -1$. In both cases, the interlacing condition for the Gauss functon is satisfied, so ${}_2F_1(a, b_1 + b_2, c|z)$ is algebraic. □

Remark 2.1.9. One can show that ${}_2F_1(a, b_1 + b_2, c|z) = F_1(a, b_1, b_2, c|z, z)$. This also implies that ${}_2F_1(a, b_1 + b_2, c|z)$ is algebraic if $F_1(a, b_1, b_2, c|x, y)$ is algebraic.

Theorem 2.1.10. *$F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic if and only if the tuple $(a, b_1, b_2, c) \pmod{\mathbb{Z}}$ is equal to one of the tuples $\pm(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3})$, $\pm(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}, \frac{1}{3})$, $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ or $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$.*

Proof. If $F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic, then by Lemma 2.1.7 and Lemma 2.1.8, $(a, b_1, c), (a, b_2, c)$ and $(a, b_1 + b_2, c)$ are Gauss triples.

First suppose that (a, b_1, c) and (a, b_2, c) are both Gauss triples of type 1. Then there exist $r, s \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that we have $(a, b_1, c) \in \{(r, -r, \frac{1}{2}), (r, r + \frac{1}{2}, \frac{1}{2}), (r, r + \frac{1}{2}, 2r)\}$ and $(a, b_2, c) \in \{(s, -s, \frac{1}{2}), (s, s + \frac{1}{2}, \frac{1}{2}), (s, s + \frac{1}{2}, 2s)\}$ (up to congruence modulo $\mathbb{Z}$). This implies that $r = s$. There are five possibilities for (a, b_1, b_2, c) and we obtain the four combinations

$$(a, b_1 + b_2, c) \equiv (r, -2r, \frac{1}{2}), (r, \frac{1}{2}, \frac{1}{2}), (r, 2r, \frac{1}{2}), (r, 2r, 2r) \pmod{\mathbb{Z}}$$

$(r, -2r, \frac{1}{2})$ is of type 1 if and only if $r = \pm\frac{1}{6}$. This gives $(a, b_1, b_2, c) = (\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$. The triple $(r, 2r, \frac{1}{2})$ is of type 1 if and only if $r = \pm\frac{1}{3}$, which gives the tuples $(a, b_1, b_2, c) = (\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$. In all other cases, $(a, b_1 + b_2, c)$ is of type 2, and hence the denominator of r is at most 60. There are finitely many possibilities and for each of these possibilities, we let the computer check whether all conjugates $k(a, b_1, b_2, c)$ (with $\gcd(k, D) = 1$, where D is the smallest common denominator of a, b_1, b_2 and c) satisfy the interlacing condition. It turns out that this gives the same four solutions as we found above.

If (a, b_1, c) is a Gauss triple of type 1 and (a, b_2, c) is of type 2, then the denominator of a is at most 60. This gives finitely many possibilities for the parameter r in (a, b_1, c). Again we check these by computer, and we get the solutions $\pm(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3})$. By symmetry, if (a, b_1, c) is of type 2 and (a, b_2, c) is of type 1, the solutions are $\pm(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}, \frac{1}{3})$.

Finally, if both (a, b_1, c) and (a, b_2, c) are of type 2, we have a finite list to check. This gives two more solutions: $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$. □

Theorem 2.1.11. *For $n = 3$, $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ is irreducible and algebraic if and only if $(a, \boldsymbol{b}, c) = \pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3}) \pmod{\mathbb{Z}}$. For $n \geq 4$, there are no irreducible algebraic Lauricella F_D functions.*

Proof. Let $n = 3$ and let $F_D(a, \boldsymbol{b}, c|\boldsymbol{z})$ be irreducible and algebraic. Then the functions $F_1(a, b_1, b_2, c|x, y)$, $F_1(a, b_1, b_3, c|x, y)$ and $F_1(a, b_2, b_3, c|x, y)$ are also irreducible and algebraic. Using Theorem 2.1.10, one easily computes that the only irreducible possibilities for (a, b_1, b_2, b_3, c) are $(a, b_1, b_2, b_3, c) = \pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$. They form an orbit and satisfy the interlacing condition.

For $n = 4$, the only possibilities for $(a, b_1, b_2, b_3, b_4, c)$ are $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$. However, both functions are reducible. Hence there are no irreducible algebraic functions in 4 variables. Lemma 2.1.7 implies that there are also no irreducible algebraic F_D functions in 5 or more variables. □

The Appell F_2 and Lauricella F_A functions

The Lauricella F_A function is defined by

$$F_A(a, \boldsymbol{b}, \boldsymbol{c}|\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\boldsymbol{m}|}(\boldsymbol{b})_{\boldsymbol{m}}}{(\boldsymbol{c})_{\boldsymbol{m}} \boldsymbol{m}!} \boldsymbol{z}^{\boldsymbol{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^{n} \mathbb{Z}(-\boldsymbol{e}_1 - \boldsymbol{e}_{i+1} + \boldsymbol{e}_{n+i+1} + \boldsymbol{e}_{2n+i+1}) \subseteq \mathbb{Z}^{3n+1}$ and we can take

$$\begin{aligned}
\boldsymbol{\gamma} &= (-a, -\boldsymbol{b}, \boldsymbol{c} - \boldsymbol{1}, \boldsymbol{0}) \in \mathbb{R}^{3n+1}, \\
\mathcal{A} &= \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{2n+1}, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_{n+2}, \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_{n+3}, \ldots, \boldsymbol{e}_1 + \boldsymbol{e}_{n+1} - \boldsymbol{e}_{2n+1}\} \subseteq \mathbb{Z}^{2n+1}, \\
\boldsymbol{\beta} &= \sum_{i=1}^{3n+1} \gamma_i \boldsymbol{a}_i = (-a, -\boldsymbol{b}, \boldsymbol{c} - \boldsymbol{1}) \in \mathbb{Q}^{2n+1}.
\end{aligned}$$

where $\boldsymbol{c} - \boldsymbol{1} = (c_1 - 1, \ldots, c_n - 1)$. For each $I \subseteq \{n+2, \ldots, 2n+1\}$, define $\tilde{I} = \{n+2, \ldots, 2n+1\} \setminus I$ and $V_I = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\} \cup \{\boldsymbol{e}_1 + \boldsymbol{e}_{i-n} - \boldsymbol{e}_i | i \in I\} \cup \{\boldsymbol{e}_i | i \in \tilde{I}\}$.

Lemma 2.1.12. $\{V_I \mid I \subseteq \{n+2, \ldots, 2n+1\}\}$ *is a triangulation of $\mathcal{A}$.*

Proof. Let $I \subseteq \{n+2, \dots, 2n+1\}$. Then the determinant of the vectors in V_I is ± 1, so the vectors are the vertices of a $2n$-dimensional simplex with volume 1. Furthermore,

$$\begin{aligned} C(V_I) = \{\boldsymbol{x} \in \mathbb{R}^{2n+1} \mid\ & \exists \lambda_1, \dots, \lambda_{2n+1} \geq 0 : x_1 = \lambda_1 + \sum_{i \in I} \lambda_i; \\ & \forall i \in \tilde{I} : x_{i-n} = \lambda_{i-n} \text{ and } x_i = \lambda_i; \\ & \forall i \in I : x_{i-n} = \lambda_{i-n} + \lambda_i \text{ and } x_i = -\lambda_i\} \\ = \{\boldsymbol{x} \in \mathbb{R}^{2n+1} \mid\ & x_1, \dots, x_{n+1} \geq 0; x_1 + \sum_{i \in I} x_i \geq 0; \\ & \forall i \in \tilde{I} : x_i \geq 0; \forall i \in I : x_i \leq 0 \text{ and } x_{i-n} + x_i \geq 0\}. \end{aligned}$$

and hence

$$\begin{aligned} \bigcup_I C(V_I) = \{\boldsymbol{x} \in \mathbb{R}^{2n+1} \mid\ & x_1, \dots, x_{n+1} \geq 0; x_2 + x_{n+2}, \dots, x_{n+1} + x_{2n+1} \geq 0; \\ & \text{for all } I \subseteq \{n+2, \dots, 2n+1\} : x_1 + \sum_{i \in I} x_i \geq 0\} \end{aligned}$$

so $\cup_I C(V_I)$ is convex. Since for all $I, J \subseteq \{n+2, \dots, 2n+1\}$ both $C(V_I) \cap C(V_J)$ and $C(V_I \cap V_J)$ equal

$$\begin{aligned} \{\boldsymbol{x} \in \mathbb{R}^{2n+1} \mid\ & x_1, \dots, x_{n+1} \geq 0; x_1 + \sum_{i \in I \cap J} x_i \geq 0; \forall i \in \tilde{I} \cap \tilde{J} : x_i \geq 0; \\ & \forall i \in I \cap J : x_i \leq 0 \text{ and } x_{i-n} + x_i \geq 0; \\ & \forall i \in (I \cap \tilde{J}) \cup (\tilde{I} \cap J) : x_i = 0\}, \end{aligned}$$

we have $C(V_I) \cap C(V_J) = C(V_I \cap V_J)$. It follows from Lemma 1.4.4 that $\{V_I \mid I \subseteq \{n+2, \dots 2n+1\}$ is a triangulation of $\mathcal{A}$. $\square$

Corollary 2.1.13. *$\mathcal{A}$ is normal, the volume of $Q(\mathcal{A})$ is 2^n and*

$$\begin{aligned} \bigcup_I C(V_I) = \{\boldsymbol{x} \in \mathbb{R}^{2n+1} \mid\ & x_1, \dots, x_{n+1} \geq 0; x_2 + x_{n+2}, \dots, x_{n+1} + x_{2n+1} \geq 0; \\ & \textit{for all } I \subseteq \{n+2, \dots, 2n+1\} : x_1 + \sum_{i \in I} x_i \geq 0\} \end{aligned}$$

$F_A(a, \boldsymbol{b}, \boldsymbol{c}|\boldsymbol{z})$ is irreducible if and only if $b_1, \dots, b_n$, $c_1 - b_1, \dots, c_n - b_n$ and $-a + \sum_{j \in J} c_j$ are non-integral for all $J \subseteq \{1, \dots, n\}$.

Corollary 2.1.14. *If $F_A(a, \boldsymbol{b}, \boldsymbol{c}|\boldsymbol{z})$ is irreducible and algebraic, then the function $F_A(a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n \mid z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$ is also irreducible and algebraic for every $i \in \{1, \dots, n\}$.*

Finding an interlacing condition isn't as easy as for F_D. Therefore, we will first find an interlacing condition for the Appell function and compute all irreducible algebraic functions for $n = 2$. Using this, we can prove that there are no irreducible algebraic functions for $n \geq 3$.

Lemma 2.1.15. *If $F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible, then there are 4 apex points if and only if* $(\lfloor\{c_1\} - \{a\}\rfloor, \lfloor\{c_2\} - \{a\}\rfloor, \lfloor\{c_1\} + \{c_2\} - \{a\}\rfloor, \lfloor\{c_1\} - \{b_1\}\rfloor, \lfloor\{c_2\} - \{b_2\}\rfloor) \in \{(-1, -1, -1, 0, 0), (-1, 0, 0, 0, -1), (0, -1, 0, -1, 0), (0, 0, 1, -1, -1)\}$.

Proof. Using the algorithm described at the end of Section 1.3 and the assumption $\beta_i \in [0, 1)$, one easily computes that there are 4 apex points if and only if $(\lfloor\beta_1 + \beta_4\rfloor, \lfloor\beta_1 + \beta_5\rfloor, \lfloor\beta_1 + \beta_4 + \beta_5\rfloor, \lfloor\beta_2 + \beta_4\rfloor, \lfloor\beta_3 + \beta_5\rfloor) \in \{(1, 1, 2, 0, 0), (0, 1, 1, 1, 0), (1, 0, 1, 0, 1), (0, 0, 0, 1, 1)\}$ Since $\boldsymbol{\beta} = (1 - \{a\}, 1 - \{b_1\}, 1 - \{b_2\}, \{c_1\}, \{c_2\})$, this is equivalent to the conditions given above. □

Lemma 2.1.16. *If $F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible and algebraic, then ${}_2F_1(a - c_2, b_1, c_1|x)$ is also irreducible and algebraic.*

Proof. The proof is similar to the proof of Lemma 2.1.8 and uses the interlacing condition from Lemma 2.1.15. □

Since $F_2(a, b_1, b_2, c_1, c_2|x, y) = F_2(a, b_2, b_1, c_2, c_1|y, x)$, the algebraic functions come in pairs.

Theorem 2.1.17. *$F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible and algebraic if and only if (a, b_1, b_2, c_1, c_2) or (a, b_2, b_1, c_2, c_1) (mod $\mathbb{Z}$) is conjugate to one of* $(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3})$, $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{2}{3})$, $(\frac{1}{10}, \frac{7}{10}, \frac{9}{10}, \frac{2}{5}, \frac{4}{5})$, $(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$, $(\frac{1}{12}, \frac{5}{6}, \frac{11}{12}, \frac{2}{3}, \frac{1}{2})$, $(\frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2})$ *or* $(\frac{1}{30}, \frac{5}{6}, \frac{7}{10}, \frac{2}{3}, \frac{2}{5})$.

Proof. If $F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible and algebraic, then Corollary 2.1.14 and Lemma 2.1.16 imply that (a, b_1, c_1), (a, b_2, c_2) and $(a - c_2, b_1, c_1)$ are Gauss triples.

First suppose that (a, b_1, c_1) and (a, b_2, c_2) are both Gauss triples of type 1. Then there exist $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that $(a, b_1, c_1), (a, b_2, c_2) \in \{(r, -r, \frac{1}{2}), (r, r + \frac{1}{2}, \frac{1}{2}), (r, r + \frac{1}{2}, 2r)\}$ (up to congruence modulo $\mathbb{Z}$). Hence $a - c_2, b_1 \in \{r + \frac{1}{2}, -r\}$ (mod $\mathbb{Z}$). If $(a - c_2, b_1, c_1)$ is a Gauss triple of type 1, then $a - c_2 \equiv -b_1$ or $a - c_2 \equiv b_1 + \frac{1}{2}$. However, this doesn't hold for $r \neq \frac{1}{2}$. Hence $(a - c_2, b_1, c_1)$ must be of type 2, so the denominator of $a - c_2$ is at most 60. This implies that the denominator of r is at most 60, or 2 (mod 4) and at most 120. This gives finitely many possibilities for r and using a computer it turns out that there are no solutions.

If (a, b_1, c_1) is a Gauss triple of type 1 and (a, b_2, c_2) is of type 2, then the denominator of a is at most 60 and there are again finitely many possibilities. The solutions are the 8 conjugates of $(\frac{1}{12}, \frac{11}{12}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$ and $(\frac{1}{12}, \frac{7}{12}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$. By symmetry, if (a, b_1, c_1) is of type 2 and (a, b_2, c_2) is of type 1, the solutions are the conjugates of $(\frac{1}{12}, \frac{5}{6}, \frac{11}{12}, \frac{2}{3}, \frac{1}{2})$ and $(\frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2})$.

Finally, when both (a, b_1, c_1) and (a, b_2, c_2) are of type 2, there are only finitely many possibilities. This gives the other 36 solutions listed above. □

Theorem 2.1.18. *For $n \geq 3$, there are no irreducible algebraic Lauricella F_A functions.*

Proof. First let $n = 3$. If $F_A(a, \boldsymbol{b}, \boldsymbol{c}|\boldsymbol{z})$ is irreducible and algebraic, then each of the three tuples (a, b_1, b_2, c_1, c_2), (a, b_1, b_3, c_1, c_3) and (a, b_2, b_3, c_2, c_3) must give an irreducible algebraic F_2 function. From Theorem 2.1.17, it easily follows that $(a, \boldsymbol{b}, \boldsymbol{c}) = \pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Hence $\boldsymbol{\beta}$ is equal to $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the corresponding functions are irreducible. There are 5 and 7 apex points, respectively. Since the volume of $Q(\mathcal{A})$ is 8, the functions are not algebraic. By Corollary 2.1.14, this implies that there are no irreducible algebraic functions for $n \geq 4$. $\square$

The Appell F_3 and Lauricella F_B functions

The Lauricella F_B function is defined by

$$F_B(\boldsymbol{a}, \boldsymbol{b}, c|\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}} \frac{(\boldsymbol{a})_{\boldsymbol{m}} (\boldsymbol{b})_{\boldsymbol{m}}}{(c)_{|\boldsymbol{m}|} \boldsymbol{m}!} \boldsymbol{z}^{\boldsymbol{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\boldsymbol{e}_i - \boldsymbol{e}_{n+i} + \boldsymbol{e}_{2n+1} + \boldsymbol{e}_{2n+i+1}) \subseteq \mathbb{Z}^{3n+1}$. We can take

$$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{2n+1}, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_{n+2}, \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_{n+3}, \ldots, \boldsymbol{e}_1 + \boldsymbol{e}_{n+1} - \boldsymbol{e}_{2n+1}\} \subseteq \mathbb{Z}^{2n+1}$$

and $\boldsymbol{\gamma} = (-\boldsymbol{a}, -\boldsymbol{b}, c-1, \boldsymbol{0})$. Then $\boldsymbol{\beta} = (-\boldsymbol{a}, -\boldsymbol{b}, c-1)$. Consider the map $f : \mathbb{Z}^{2n+1} \to \mathbb{Z}^{2n+1} : \boldsymbol{x} \mapsto (x_2 + x_{n+2}, \ldots, x_{n+1} + x_{2n+1}, x_2, \ldots, x_{n+1}, x_1 - x_2 - \ldots - x_{n+1}$. Its inverse is $f^{-1} : \boldsymbol{x} \mapsto (x_{n+1} + \ldots + x_{2n} + x_{2n+1}, x_{n+1}, \ldots, x_{2n}, x_1 - x_{n+1}, \ldots, x_n + x_{2n})$. It maps the set $\mathcal{A}$ of F_A to the set $\mathcal{A}$ of F_B. Hence these sets are isomorphic, and we can translate the results for F_A to F_B.

Lemma 2.1.19. *$F_B(\boldsymbol{a}, \boldsymbol{b}, c|\boldsymbol{z})$ is irreducible if and only if $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and $c - d_1 - \ldots - d_n$ with $d_i \in \{a_i, b_i\}$ are non-integral.*

Theorem 2.1.20. *$F_3(a_1, a_2, b_1, b_2, c|x, y)$ is irreducible and algebraic if and only if at least one of the tuples (a_1, a_2, b_1, b_2, c), (a_2, a_1, b_2, b_1, c), (b_1, b_2, a_1, a_2, c) and (b_2, b_1, a_2, a_1, c) (mod $\mathbb{Z}$) is conjugate to $(\frac{1}{4}, \frac{1}{6}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{10}, \frac{5}{6}, \frac{9}{10}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{1}{3})$ or $(\frac{1}{10}, \frac{3}{10}, \frac{9}{10}, \frac{7}{10}, \frac{1}{2})$. There are no irreducible algebraic Lauricella F_B functions for $n \geq 3$.*

The Appell F_4 and Lauricella F_C functions

The Lauricella F_C function is defined by

$$F_C(a, b, \boldsymbol{c}|\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\boldsymbol{m}|} (b)_{|\boldsymbol{m}|}}{(\boldsymbol{c})_{\boldsymbol{m}} \boldsymbol{m}!} \boldsymbol{z}^{\boldsymbol{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_{i+2} + \boldsymbol{e}_{n+i+2}) \subseteq \mathbb{Z}^{2n+2}$ and we can choose

$$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4, \ldots, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_{n+2}\} \subseteq \mathbb{Z}^{n+2}.$$

We have $\boldsymbol{\gamma} = (-a, -b, \boldsymbol{c} - \boldsymbol{1}, \boldsymbol{0})$, so $\boldsymbol{\beta} = (-a, -b, \boldsymbol{c} - \boldsymbol{1}) \in \mathbb{Q}^{n+2}$.
For $I \subseteq \{3, \ldots, n+2\}$, let $\tilde{I} = \{3, \ldots, n+2\} \setminus I$ and $V_I = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} \cup \{\boldsymbol{e}_i | i \in \tilde{I}\} \cup \{\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_i | i \in I\}$.

Lemma 2.1.21. $\{V_I \mid I \subseteq \{3,\ldots,n+2\}\}$ *is a triangulation of* $\mathcal{A}$.

Proof. Let $I, J \subseteq \{3,\ldots,n+2\}$. The determinant of the vectors in V_I equals ± 1, so the vectors are the vertices of an $(n+1)$-dimensional simplex. We have

$$C(V_I) = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid x_1, x_2 \geq 0; \forall i \in I : x_i \leq 0; \forall i \in \tilde{I} : x_i \geq 0; \\ x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i \in I} x_i \geq 0\}$$

so

$$\bigcup_I C(V_I) = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid \forall I \subseteq \{3,\ldots,n+2\} : x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i\in I} x_i \geq 0\}.$$

Furthermore, both $C(V_I) \cap C(V_J)$ and $C(V_I \cap V_J)$ equal

$$\{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid x_1, x_2 \geq 0; \forall i \in I \cap J : x_i \leq 0; \forall i \in \tilde{I} \cap \tilde{J} : x_i \geq 0; \\ \forall i \in (I \cap \tilde{J}) \cup (\tilde{I} \cap J) : x_i = 0; x_1 + \sum_{i \in I\cap J} x_i \geq 0; x_2 + \sum_{i\in I \cap J} x_i \geq 0\}.$$

Hence $C(V_I) \cap C(V_J) = C(V_I \cap V_J)$ so $\{V_I \mid I \subseteq \{3,\ldots,n_2\}\}$ is a triangulation by Lemma 1.4.4. ☐

Corollary 2.1.22. $\mathcal{A}$ *is normal, the volume of* $Q(\mathcal{A})$ *is* 2^n *and*

$$C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid \forall I \subseteq \{3,\ldots,n+2\} : x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i\in I} x_i \geq 0\}.$$

$F_C(a, b, \mathbf{c}|\boldsymbol{z})$ *is irreducible if and only if* $-a+\sum_{i\in I} c_i$ *and* $-b+\sum_{i\in I} c_i$ *are non-integral for all* $I \subseteq \{1,\ldots,n\}$.

Corollary 2.1.23. *If* $F_C(a, b, \mathbf{c}|\boldsymbol{z})$ *is irreducible and algebraic, then the function* $F_C(a, b, c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n|z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ *is also irreducible and algebraic for every* $i \in \{1,\ldots,n\}$.

Lemma 2.1.24. *Suppose that* $\{a\} \leq \{b\}$ *and for all* $I, J \subseteq \{1,\ldots,n\}$ *with* $|I|$ *even and* $|J|$ *odd,*

$$\lfloor -\{a\} + \sum_{i\in I} \{c_i\} \rfloor = \lfloor -\{b\} + \sum_{i\in I} \{c_i\} \rfloor = \frac{|I|}{2} - 1$$

and

$$\lfloor -\{a\} + \sum_{j\in J} \{c_j\} \rfloor = \frac{|J|-1}{2} \text{ and } \lfloor -\{b\} + \sum_{j\in J} \{c_j\} \rfloor = \frac{|J|-3}{2}.$$

Then there are 2^n *apex points. In particular, if* $\{c_3\} = \ldots = \{c_n\} = \frac{1}{2}$, *then there are* 2^n *apex points if*

$$\{a\} \leq \{c_1\}, \{c_2\}, \frac{1}{2} < \{b\} \leq \{c_1\} + \frac{1}{2}, \{c_2\} + \frac{1}{2}, \{c_1\} + \{c_2\} < \\ \{a\} + 1 \leq \{c_1\} + \{c_2\} + \frac{1}{2} < \{b\} + 1.$$

For $n = 2$ and $n = 3$, the number of apex points equals 2^n if and only if the above condition holds.

Proof. The interlacing condition for $n = 2$ and $n = 3$ can easily computed using the algorithm at the end of Section 1.3. For $F_4(a, b, c_1, c_2|x, y)$, we find that there are 4 apex points if and only if $(\lfloor\beta_1+\beta_3\rfloor, \lfloor\beta_1+\beta_4\rfloor, \lfloor\beta_1+\beta_3+\beta_4\rfloor, \lfloor\beta_2+\beta_3\rfloor, \lfloor\beta_2+\beta_4\rfloor, \lfloor\beta_2+\beta_3+\beta_4\rfloor)$ equals $(1, 1, 1, 0, 0, 1)$ or $(0, 0, 1, 1, 1, 1)$. Note that $F_C(a, b, c_1, c_2, c_3|z_1, z_2, z_3)$ can only be algebraic if all induced F_4 functions are also algebraic. Hence we only have to find the values of $(\lfloor\beta_1 + \beta_3 + \beta_4 + \beta_5\rfloor, \lfloor\beta_2 + \beta_3 + \beta_4 + \beta_5\rfloor)$, given the values of the other linear forms induced by the F_4 functions. This gives the conditions stated in the lemma.

For the first statement, we claim that all points $\boldsymbol{x} + \boldsymbol{\beta}$ with $x_3, \ldots, x_{n+2} \in \{-1, 0\}$ and $x_k = |I| - \lfloor\beta_k + \sum_{i\in I} \beta_i\rfloor$ are apexpoints, where $k = 1, 2$ and $I = \{i \in \{3, \ldots, n + 2\} \mid x_i = -1\}$. Let $I' = \{i - 2 \mid i \in I\}$. Note that $x_k + \beta_k + \sum_{i\in I}(x_i + \beta_i) = \{\beta_k + \sum_{i\in I} \beta_i\}$. This equals $1 - a + \sum_{i\in I'} c_i - \frac{|I|}{2}$ or $1 - b + \sum_{i\in I'} c_i - \frac{|I|}{2}$ if $|I|$ is even, and $1 - a + \sum_{i\in I'} c_i - \frac{|I|+1}{2}$ or $1 - b + \sum_{i\in I'} c_i - \frac{|I|-1}{2}$ if $|I|$ is odd.

To show that $\boldsymbol{x} + \boldsymbol{\beta} \in C(\mathcal{A})$, we have to show that $x_k + \beta_k + \sum_{j\in J}(x_j + \beta_j) \geq 0$ for $k = 1, 2$ and $J \subseteq \{3, \ldots, n + 2\}$. Since $x_j + \beta_j \geq 0$ if and only if $j \notin I$, it suffices to take $J = I$. But then $x_k + \beta_k + \sum_{j\in J}(x_j + \beta_j) = \{\beta_k + \sum_{i\in I} \beta_i\}$, which is clearly non-negative. Since it is smaller than 1, we also have $\boldsymbol{x} + \boldsymbol{\beta} - \boldsymbol{e}_1, \boldsymbol{x} + \boldsymbol{\beta} - \boldsymbol{e}_2 \notin C(\mathcal{A})$.

Now we show that $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\beta} - \boldsymbol{e}_l \notin C(\mathcal{A})$ for $3 \leq l \leq n + 2$, so we have to find J such that $y_k + \beta_k + \sum_{j\in J}(y_j + \beta_j) < 0$. Take $J = I \cup \{l\}$. If $l \in I$, then $y_k + \beta_k + \sum_{j\in J}(y_j + \beta_j) = \{\beta_k + \sum_{i\in I} \beta_i\} - 1$, which is negative. Hence we can assume that $l \notin I$. Then $y_k + \beta_k + \sum_{j\in J}(y_j + \beta_j) = \{\beta_k + \sum_{i\in I} \beta_i\} + \beta_l - 1$. If $|I|$ is odd, take $k = 1$. Then $\{\beta_k + \sum_{i\in I} \beta_i\} = 1 - a + \sum_{i\in I'} c_i - \frac{|I|+1}{2}$, so $y_k + \beta_k + \sum_{j\in J}(y_j + \beta_j) = -a + \sum_{i\in I'} c_i - \frac{|I|+1}{2} + c_{l-2} = -a + \sum_{j\in J'} c_j - \frac{|J|}{2} < 0$. Similarly, if $|I|$ is even, take $k = 2$ to get $\{\beta_k + \sum_{i\in I} \beta_i\} = 1 - b + \sum_{i\in I'} c_i - \frac{|I|}{2}$ and $y_k + \beta_k + \sum_{j\in J}(y_j + \beta_j) = -b + \sum_{j\in J'} c_j - (|J| - 1)/2 < 0$.

Finally, we show that $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{\beta} - (\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_l) \notin C(\mathcal{A})$ for $3 \leq l \leq n + 2$. Take $J = I \setminus \{l\}$. If $l \notin I$, then $z_k + \beta_k + \sum_{j\in J}(z_j + \beta_j) = \{\beta_k + \sum_{i\in I} \beta_i\} - 1$, which is negative. Let $l \in I$. Then $z_k + \beta_k + \sum_{j\in J}(z_j + \beta_j) = \{\beta_k + \sum_{i\in I} \beta_i\} - \beta_1$. If $|I|$ is even, take $k = 2$. This gives $z_k + \beta_k + \sum_{j\in J}(z_j + \beta_j) = 1 - b + \sum_{i\in I'} c_i - \frac{|I|}{2} - c_{l-2} = -b + \sum_{i\in I'\setminus\{l\}} c_i - \frac{|I'\setminus\{l\}|-1}{2} < 0$. If $|I|$ is odd, we take $k = 1$ to get $\{\beta_k + \sum_{i\in I} \beta_i\} - \beta_1 = 1 - a + \sum_{i\in I'} c_i - \frac{|I|+1}{2} - c_{1-2} = -a + \sum_{i\in I'\setminus\{l\}} c_i - \frac{|I'\setminus\{l\}|}{2} < 0$.

The second statement follows easily from this. □

Lemma 2.1.25. *Suppose that ${}_2F_1(a, b, c_1|x)$ and ${}_2F_1(a, b, c_2|x)$ are irreducible and algebraic, and either $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Then $F_4(a, b, c_1, c_2|x, y)$ is also irreducible and algebraic.*

Proof. Note that $F_4(a, b, c_1, c_2|x, y)$ is irreducible if $c_1 + c_2 - a$ and $c_1 + c_2 - b$ are non-integral, and that $\{a\} < \frac{1}{2}$ or $\{b\} < \frac{1}{2}$ for all Gauss triples (a, b, c).

Suppose that $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$. Then $c_1+c_2-a \equiv b$ is non-integral, and the same holds for $c_1+c_2-b \equiv a$. Let k be coprime with the denominators of a, b, c_1 and c_2. Then we can assume that $\{ka\} \leq \{kc_1\}, \{kc_2\}, \frac{1}{2} < \{kb\}$. Then $\{kc_1\}+\{kc_2\} = \{ka\}+\{kb\}$, so the interlacing condition is satisfied.

Now suppose that at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. We can assume that $c_1 \equiv \frac{1}{2}$ and $\{ka\} \leq \frac{1}{2}, \{kc_2\} < \{kb\}$. If $c_2 \equiv \frac{1}{2}$, then $c_1+c_2-a \equiv -a$ and $c_1+c_2-b \equiv -b$ are non-integral and the interlacing condition is satisfied. If $b-a \equiv \frac{1}{2}$, then $c_1+c_2-a \equiv c_2-b$ and $c_1+c_2-b \equiv c_2-a$ are non-integral. Since $\{ka\} \leq \frac{1}{2}, \{kc_2\} < \{ka\}+\frac{1}{2} \leq 1+\left(\{kc_2\}-\frac{1}{2}\right) < 1+\{ka\}$, the interlacing condition is again satisfied. $\square$

Theorem 2.1.26. *$F_4(a, b, c_1, c_2|x, y)$ is irreducible and algebraic if and only if (a, b, c_1) and (a, b, c_2) are Gauss triples and either $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Up to conjugation and permutations of $\{a, b\}$ and of $\{c_1, c_2\}$, the parameters of the irreducible algebraic functions are the tuples in Table 2.1.*

Proof. It suffices to find all tuples satisfying the interlacing condition and prove that they satisfy $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$ or at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$.

First suppose that (a, b, c_1) and (a, b, c_2) are both Gauss triples of type 1. Then we have $(a, b, c_1, c_2) \in \{(r, -r, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, 2r), (r, r+\frac{1}{2}, 2r, \frac{1}{2}), (r, r+\frac{1}{2}, 2r, 2r)\}$ for some $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ (up to equivalence modulo $\mathbb{Z}$). By Lemma 2.1.25, all these tuples give algebraic functions, possibly except for $(r, r+\frac{1}{2}, 2r, 2r)$, so suppose that $(a, b, c_1, c_2) = (r, r+\frac{1}{2}, 2r, 2r)$. Write $r = \frac{p}{q}$ with $\gcd(p, q) = 1$. Then for every k coprime with $2q$ such that $\{kr\} > 1/2$, the interlacing condition implies that $\{kr\} - \frac{1}{2} \leq 2\{kr\} - 1 < \{kr\} \leq 4\{kr\} - 2 < \{kr\} + \frac{1}{2}$, so $\{kr\} \in [\frac{2}{3}, \frac{5}{6})$. Hence for every k with $\gcd(k, 2q) = 1$, it must hold that $\{kr\} < \frac{5}{6}$. There exists k such that $kp \equiv -1 \pmod{q}$. Choose l such that $\gcd(l, 2q) = 1$ and $lp \equiv -1 \pmod{q}$. If $q \geq 6$, then $\{lr\} = \frac{q-1}{q} \geq \frac{5}{6}$. Contradiction, so for all algebraic solutions, it holds that $q < 6$. By symmetry, $r+\frac{1}{2}$ also has a denominator smaller than 6, so $r = \frac{1}{4}$ or $r = \frac{3}{4}$. However, this only gives solutions of the form $(r, -r, \frac{1}{2}, \frac{1}{2})$.

If (a, b, c_1) is a Gauss triple of type 1, and (a, b, c_2) is a Gauss triple of type 2, then the denominator of a is at most 60. This gives 72 solutions, which all turn out to satisfy $c_1 = \frac{1}{2}$ and $b-a \equiv \frac{1}{2} \pmod{\mathbb{Z}}$. By symmetry, if (a, b, c_1) is of type 2 and (a, b, c_2) is of type 1, we get 72 solutions which all satisfy $c_2 = \frac{1}{2}$ and $b-a \equiv \frac{1}{2} \pmod{\mathbb{Z}}$.

Finally, if (a, b, c_1) and (a, b, c_2) are both of type 1, then there are 480 irreducible algebraic functions, and all the tuples (a, b, c_1, c_2) either satisfy $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. $\square$

We proved Theorem 2.1.26 by computing all tuples for which $F_4(a, b, c_1, c_2|x, y)$ is irreducible and algebraic, and checking whether for each of the tuples either $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Unfortunately, this doesn't give much insight. In [Kat97], Kato proves the

Table 2.1: The tuples (a, b, c_1, c_2) such that $F_4(a, b, c_1, c_2|x, y)$ is irreducible and algebraic

$(r, -r, \frac{1}{2}, \frac{1}{2})$, $(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(r, r+\frac{1}{2}, \frac{1}{2}, 2r)$ with $2r \notin \mathbb{Z}$			
$(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{10}, \frac{7}{10}, \frac{2}{5}, \frac{2}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{10}, \frac{9}{10}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{5}, \frac{2}{5})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{3})$
$(\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{10}, \frac{9}{10}, \frac{1}{5}, \frac{4}{5})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{30}, \frac{11}{30}, \frac{1}{5}, \frac{1}{5})$
$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{10}, \frac{13}{30}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{30}, \frac{19}{30}, \frac{1}{3}, \frac{1}{3})$
$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{12}, \frac{5}{12}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{3})$
$(\frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{5})$
$(\frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{3}, \frac{1}{5})$
$(\frac{1}{6}, \frac{5}{6}, \frac{1}{5}, \frac{4}{5})$	$(\frac{1}{15}, \frac{7}{15}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{6}, \frac{5}{12}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{15}, \frac{11}{15}, \frac{1}{5}, \frac{3}{5})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{5})$
$(\frac{1}{6}, \frac{11}{30}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{15}, \frac{13}{15}, \frac{1}{3}, \frac{3}{5})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{10}, \frac{3}{10}, \frac{1}{5}, \frac{1}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{3})$

same theorem using monodromy groups, without computing all solutions explicitly. However, to find all non-resonant algebraic F_C functions in more than 2 variables, we do need to know the solutions for $n = 2$ explicitly.

For the Lauricella F_D, F_A and F_B functions, from a certain number of parameters on, there are no irreducible algebraic functions. However, for the Lauricella F_C function, the situation is entirely different: for every number of parameters there are three infinite families of irreducible algebraic functions.

Theorem 2.1.27. *For $n \geq 3$, $F_C(a, b, \boldsymbol{c}|\boldsymbol{z})$ is a irreducible algebraic function if and only if up to permutations of $\{a, b\}$ and permutations of $\{c_1, \ldots, c_n\}$, we have $\{c_3\} = \ldots = \{c_n\} = \frac{1}{2}$ and the tuple $(a, b, c_1, c_2) \pmod{\mathbb{Z}}$ is conjugate to one of the tuples in Table 2.2.*

Proof. First we show that all tuples in Table 2.2 indeed give irreducible algebraic functions. For irreducibility, it suffices to prove that $-a, -a+c_1, -a+c_2, -a+c_1+c_2, -b, -b+c_1, -b+c_2, -b+c_1+c_2$ are not half-integral. This can easily be checked for all tuples. To prove that the functions are algebraic, we use the second statement of Lemma 2.1.24 to show that all conjugates have 2^n apexpoints. Again, this is an easy check.

Now we show that all irreducible algebraic functions have parameters from Table 2.2. For $n = 3$, we use the interlacing condition from Lemma 2.1.24 and the fact that both (a, b, c_1, c_2) and (a, b, c_1, c_3) must be F_4 tuples. If both (a, b, c_1, c_2) and (a, b, c_1, c_3) are of type 1, then we have, up to permutations of $\{c_1, c_2, c_3\}$, $(a, b, c_1, c_2, c_3) \in$

Table 2.2: The tuples (a, b, c_1, c_2) such that $F_C(a, b, c_1, c_2, \frac{1}{2}, \dots, \frac{1}{2}|\boldsymbol{z})$ is irreducible and algebraic

$(r, -r, \frac{1}{2}, \frac{1}{2})$, $(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(r, r+\frac{1}{2}, \frac{1}{2}, 2r)$ with $2r \notin \mathbb{Z}$			
$(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{5}, \frac{2}{5})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{5})$
$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{3})$
$(\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{3})$	
$(\frac{1}{12}, \frac{7}{12}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{5})$	
$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{3}, \frac{1}{5})$	

$\{(r, -r, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2r), (r, r+\frac{1}{2}, \frac{1}{2}, 2r, 2r)\}$ (mod $\mathbb{Z}$). The first three give irreducible algebraic functions. Let $(a, b, c_1, c_2, c_3) = (r, r+\frac{1}{2}, \frac{1}{2}, 2r, 2r)$. Then (a, b, c_2, c_3) must also be an F_4 tuple, but it is not of type 1 (unless $r = \pm\frac{1}{4}$, in which case it equals $(r, -r, \frac{1}{2}, \frac{1}{2})$). Hence it is of type 2, so the denominator of $2r$ is at most 5. If the denominator of r is 4, then the tuple is $(r, -r, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and if the denominator of r equals 3 or 6, then the function will be reducible. Therefore, we can assume that r has denominator 5, 8 or 10. Using the interlacing condition, we easily compute all algebraic functions.

If one of the tuples (a, b, c_1, c_2) and (a, b, c_1, c_3) is of type 1, then the parameter has denominator at most 60, so there are finitely many possibilities. The same holds if both (a, b, c_1, c_2) and (a, b, c_1, c_3) are of type 2. This gives the 720 conjugates of the tuples in Table 2.2.

Finally, let $n \geq 4$, and suppose that all irreducible algebraic functions in $n-1$ variables are given in Table 2.2. Let $(a, b, c_1, \dots, c_n)$ correspond to a irreducible algebraic function. Of each $n-1$ c_i's, at least $n-3$ have to be equal to $\frac{1}{2}$. Hence at least $n-2$ of $c_1, \dots, c_n$ are equal to $\frac{1}{2}$, so we can assume that $c_3 = \dots = c_n = \frac{1}{2}$. Since $(a, b, c_1, \dots, c_{n-1})$ must also give a irreducible algebraic function, (a, b, c_1, c_2) must be one of the tuples in Table 2.2. $\square$

2.2 The Horn G functions

The Horn G_1 function

The G_1 function is defined by

$$G_1(a, b_1, b_2|x, y) = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}}{m!n!} x^m y^n.$$

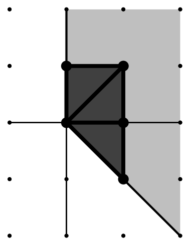


Figure 2.1: The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ for G_1

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1,1,-1,1,0) \oplus \mathbb{Z}(-1,-1,1,0,1)$. We choose $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}$ and $\boldsymbol{\gamma} = (-a, -b_1, -b_2, 0, 0)$. Then $\boldsymbol{\beta} = (-a, -b_1, -b_2)$.

$\mathcal{A}$ lies in the hyperplane $x_1 + x_2 + x_3 = 1$. By projecting $\mathcal{A}$ onto the (x_1, x_2)-plane, we get the set shown in Figure 2.1. The thick dots represent $\mathcal{A}$, the dark gray region is the set $Q(\mathcal{A})$ and light gray region is a part of the set $C(\mathcal{A})$. From this figure, it is clear that $Q(\mathcal{A})$ has volume 3 and has a unimodular triangulation. Hence $\mathcal{A}$ is normal. It is clear that

$$C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 \geq 0, x_1 + x_2 \geq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0\}.$$

Hence $G_1(a, b_1, b_2|x, y)$ is irreducible if and only if a, $a + b_1$, $a + b_2$ and $b_1 + b_2$ are non-integral.

Lemma 2.2.1. *If $G_1(a, b_1, b_2|x, y)$ is irreducible, then there are 3 apex points if and only if* $(\lfloor\{a\} + \{b_1\}\rfloor, \lfloor\{a\} + \{b_2\}\rfloor, \lfloor\{b_1\} + \{b_2\}\rfloor) \in \{(0,0,1), (1,1,0)\}$.

Proof. It is easy to show that there are 3 apex points if and only if $(\lfloor\beta_1 + \beta_2\rfloor, \lfloor\beta_1 + \beta_3\rfloor, \lfloor\beta_2 + \beta_3\rfloor) \in \{(0,0,1), (1,1,0)\}$. Since $a \notin \mathbb{Z}$, we have $\beta_1 = 1 - \{a\}$. We have either $\beta_1 + \beta_2 \geq 1$ or $\beta_2 + \beta_3 \geq 1$, so $\beta_2 > 0$. Similarly, $\beta_3 > 0$ and hence $\boldsymbol{\beta} = (1 - \{a\}, 1 - \{b_1\}, 1 - \{b_2\})$. ☐

Lemma 2.2.2. *If $G_1(a, b_1, b_2|x, y)$ is irreducible and algebraic, then ${}_2F_1(a, b_1, a + b_1 + b_2|z)$ is also irreducible and algebraic.*

Proof. Irreducibility follows from the interlacing condition for $G_1(a, b_1, b_2|x, y)$. We show that the interlacing condition for $G_1(a, b_1, b_2|x, y)$ implies the interlacing condition for ${}_2F_1(a, b_1, a + b_1 + b_2|z)$. Suppose that $(\lfloor\{a\} + \{b_1\}\rfloor, \lfloor\{a\} + \{b_2\}\rfloor, \lfloor\{b_1\} + \{b_2\}\rfloor) = (0,0,1)$. Then $1 = \lfloor b_1 + b_2\rfloor \leq \lfloor a + b_1 + b_2\rfloor \leq \lfloor a + b_1\rfloor + 1 = 1$ so $\{a + b_1 + b_2\} = \{a\} + \{b_1\} + \{b_2\} - 1$. Hence $\lfloor\{a + b_1 + b_2\} - \{a\}\rfloor = \lfloor\{b_1\} + \{b_2\}\rfloor - 1 = 0$ and $\lfloor\{a + b_1 + b_2\} - \{b_1\}\rfloor = \lfloor\{a\} + \{b_2\}\rfloor - 1 = -1$. The other case is similar. ☐

Theorem 2.2.3. *$G_1(a, b_1, b_2|x, y)$ is irreducible and algebraic if and only if (a, b_1, b_2) (mod $\mathbb{Z}$) is one of the following: $\pm(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$ and $\pm(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.*

Proof. We only have to consider (a, b_1, b_2) such that $(a, b_1, a + b_1 + b_2)$ is a Gauss triple. Suppose that $(a, b_1, a + b_1 + b_2)$ is a Gauss triple of type 1. Then $(a, b_1, b_2) \in \{(r, -r, \frac{1}{2}), (r, r+\frac{1}{2}, -2r), (r, r+\frac{1}{2}, \frac{1}{2})\ (\mathrm{mod}\ \mathbb{Z})$. If $(a, b_1, b_2) = (r, -r, \frac{1}{2})$, then $a+b_1 \in \mathbb{Z}$, so $G_1(a, b_1, b_2|x, y)$ is reducible. Suppose that $(a, b_1, b_2) = (r, r + \frac{1}{2}, -2r)$. Then the function is irreducible if $r \neq \frac{1}{4}$. The interlacing condition is satisfied if and only if $r \leq \frac{1}{4}$ or $r > \frac{3}{4}$. Hence we must have $\{kr\} \leq \frac{1}{4}$ or $\{kr\} > \frac{3}{4}$ for all k coprime with the denominator of r. By Lemma 1.4.11, this happens only if the denominator of r is 6, so $r = \frac{1}{6}$ or $r = \frac{5}{6}$. This gives the solutions $(a, b_1, b_2) = \pm(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$. If $(a, b_1, b_2) = (r, r + \frac{1}{2}, \frac{1}{2})$, then the interlacing condition again reduces to $\{kr\} \leq \frac{1}{4}$ or $\{kr\} > \frac{3}{4}$, so the solutions are $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$. If (a, b_1, b_2) is a Gauss triple of type 2, then there are only finitely many possibilities. This gives two more irreducible algebraic functions, with $(a, b_1, b_2) = \pm(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$. $\square$

The Horn G_2 function

The G_2 function is defined by

$$G_2(a_1, a_2, b_1, b_2|x, y) = \sum_{m,n\geq 0} \frac{(a_1)_m (a_2)_n (b_1)_{n-m} (b_2)_{m-n}}{m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1, 0, 1, -1, 1, 0) \oplus \mathbb{Z}(0, -1, -1, 1, 0, 1)$. We choose $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_4, \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_4\}$. With $\boldsymbol{\gamma} = (-a_1, -a_2, -b_1, -b_2, 0, 0)$, we get $\boldsymbol{\beta} = (-a_1, -a_2, -b_1, -b_2)$. The function $f : \mathbb{Z}^4 \to \mathbb{Z}^4 : x \mapsto (x_2, x_3, x_3 + x_4, x_1 - x_3)$ is an isomorphism of $\mathbb{Z}^4$, with inverse $f^{-1} : \mathbb{Z}^4 \to \mathbb{Z}^4 : \boldsymbol{x} \mapsto (x_2 + x_4, x_1, x_2, -x_2 + x_3)$. It maps F_1 to G_2, so we get:

Lemma 2.2.4. *$G_2(a_1, a_2, b_1, b_2|x, y)$ is irreducible if and only if a_1, a_2, a_1+b_1, a_2+b_2 and $b_1 + b_2$ are non-integral.*

Theorem 2.2.5. *$G_2(a_1, a_2, b_1, b_2|x, y)$ is irreducible and algebraic if and only if (a_1, a_2, b_1, b_2) or (a_2, a_1, b_2, b_1) $(\mathrm{mod}\ \mathbb{Z})$ is equal to one of the tuples $\pm(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $\pm(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3})$ and $\pm(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.*

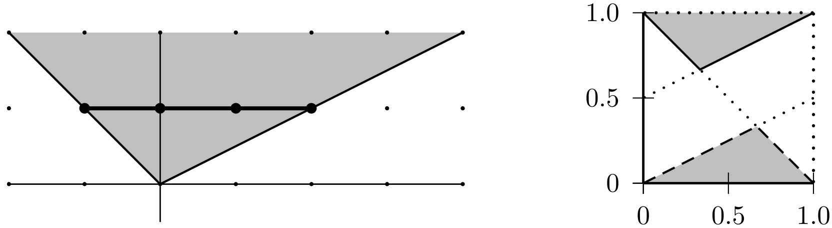
The Horn G_3 function

The G_3 function is defined by

$$G_3(a_1, a_2|x, y) = \sum_{m,n\geq 0} \frac{(a_1)_{2n-m} (a_2)_{2m-n}}{m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(1, -2, 1, 0) \oplus \mathbb{Z}(-2, 1, 0, 1)$. We choose $\mathcal{A} = \{\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_2, -\boldsymbol{e}_1 + \boldsymbol{e}_2, 2\boldsymbol{e}_1 + \boldsymbol{e}_2\}$ and $\boldsymbol{\gamma} = (-a_1, -a_2, 0, 0)$. This gives $\boldsymbol{\beta} = (-a_1, -a_1 - a_2)$.

In Figure 2.2(a), the thick dots represent the set $\mathcal{A}$; the thick line is $Q(\mathcal{A})$ and the gray region is a part of the cone $C(\mathcal{A})$. It is clear that $Q(\mathcal{A})$ has volume 3 and has a unimodular triangulation, so $\mathcal{A}$ is normal.



(a) The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ (b) The interlacing condition

Figure 2.2: The Horn G_3 function

Since

$$C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0, -x_1 + 2x_2 \geq 0\},$$

the function $G_3(a_1, a_2|x, y)$ is irreducible if and only if $2a_1 + a_2$ and $a_1 + 2a_2$ are non-integral.

One easily computes that there are 3 apex points if and only if either $(\lfloor -\beta_1 + 2\beta_2 \rfloor, \lfloor \beta_1 + \beta_2 \rfloor) \in \{(-1, 0), (1, 1)\}$. Figure 2.2(b) gives a graphical interpretation of the interlacing condition. There are 3 apex points if and only if $\boldsymbol{\beta}$ lies in the gray region. This figure also gives a idea how to find all algebraic functions: if some multiple of β_2 is close enough to $\frac{1}{2}$, then the function will not be algebraic. Hence the denominator of β_2 must be smal.. Furthermore, if the numerator of a multiple of β_1 equals 1, then β_2 must be sufficiently small. Hence the denominator of β_1 can not be too large.

Theorem 2.2.6. *$G_3(a_1, a_2|x, y)$ is irreducible and algebraic if and only if $a_1 + a_2 \in \mathbb{Z}$ or, up to equivalence modulo $\mathbb{Z}$, $(a_1, a_2) \in \{\pm(\frac{1}{2}, \frac{1}{3}), \pm(\frac{1}{3}, \frac{1}{2})\}$.*

Proof. Write $\beta_1 = \frac{p}{q}$ and $\beta_2 = \frac{u}{v}$ with $\gcd(p, q) = \gcd(u, v) = 1$, $0 \leq p < q$ and $0 \leq u < v$. It follows immediately from the interlacing condition that $\beta_1 \neq 0$ and $\beta_2 \neq \frac{1}{2}$. Furthermore, if $\beta_2 = 0$ (i.e., $a_1 + a_2 \in \mathbb{Z}$), then the interlacing condition is satisfied for all β_1. Therefore, we will assume that $q \geq 2$, $v \geq 3$ and $p, u \neq 0$.

The interlacing condition is not satisfied if there exists k such that $\{k\beta_2\} \in [\frac{1}{3}, \frac{1}{2})$ and $\gcd(k, qv) = 1$. It follows from Lemma 1.4.11 that $v \in \{4, 6, 10\}$. Choose l such that $lp \equiv 1 \pmod q$ and $\gcd(l, qv) = 1$. Then $\{l\beta_1\} = \frac{1}{q}$. Write $\{l\beta_2\} = \frac{t}{v}$ with $0 \leq t < v$. If $\frac{t}{v} < \frac{1}{3}$, then it must hold that $2\{l\beta_2\} < \{l\beta_1\}$, i.e., $2tq < v$. This gives $(\{l\beta_1\}, \{l\beta_2\}) \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{10}), (\frac{1}{3}, \frac{1}{10}), (\frac{1}{4}, \frac{1}{10})\}$. If $\frac{t}{v} \geq \frac{2}{3}$, then we must have $\{l\beta_1\} + \{l\beta_2\} \geq 1$, so $q \leq 4$ if $v = 2$, $q \leq 6$ if $v = 6$, $q \leq 3$ if $\frac{t}{v} = \frac{7}{10}$ and $q \leq 10$ if $\frac{t}{v} = \frac{9}{10}$. Now one easily checks that all conjugates satisfy the interlacing condition if and only if $(\{l\beta_1\}, \{l\beta_2\}) \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{5}{6}), (\frac{1}{3}, \frac{5}{6})\}$.

It is easy to check that these parameters, as well as $\beta_2 = 0$, give irreducible functions. Hence the irreducible algebraic functions are given by the orbits of these parameters. $\square$

2.3 The Horn H functions

The Horn H_1 function

The H_1 function is defined by

$$H_1(a,b,c,d|x,y) = \sum_{m,n\geq 0} \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1,-1,0,1,1,0) \oplus \mathbb{Z}(1,-1,-1,0,0,1)$. Take $\mathcal{A} = \{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3,\ \boldsymbol{e}_4,\ \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4,\ -\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3\}$ and $\boldsymbol{\beta} = (-a,-b,-c,d-1)$. Using Lemma 1.4.4, it is easily checked that $V_1 = \{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\}$, $V_2 = \{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4\}$, $V_3 = \{-\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\}$ and $V_4 = \{-\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4\}$ are a unimodular triangulation of $\mathcal{A}$. This implies the following lemma:

Lemma 2.3.1. *$\mathcal{A}$ is normal, the volume of $Q(\mathcal{A})$ is 4 and*

$$\begin{aligned} C(\mathcal{A}) = \{\boldsymbol{x}\in\mathbb{R}^4 \mid{} & x_2\geq 0, x_3\geq 0, x_1+x_2\geq 0, x_1+x_3\geq 0,\\ & x_2+x_4\geq 0, x_1+x_2+2x_4\geq 0, x_1+x_3+x_4\geq 0\}. \end{aligned}$$

$H_1(a,b,c,d|x,y)$ is irreducible if and only if b, c, $a+b$, $a+c$, $d-b$, $2d-a-b$ and $d-a-c$ are non-integral.

Lemma 2.3.2. *Suppose that $H_1(a,b,c,d|x,y)$ is irreducible. Then there are 4 apex points if and only if $(\lfloor\{a\}+\{b\}\rfloor, \lfloor\{a\}+\{c\}\rfloor, \lfloor\{d\}-\{b\}\rfloor, \lfloor 2\{d\}-\{a\}-\{b\}\rfloor, \lfloor\{d\}-\{a\}-\{c\}\rfloor) \in \{(0,1,-1,0,-1),(1,0,-1,0,-1),(1,0,-1,-1,0),(0,1,0,0,-2),$ $(0,1,0,-1,-1),\ (1,0,0,-1,-1)\}$.*

Proof. One easily computes that there are 4 apex points if and only if $(\lfloor\beta_1+\beta_2\rfloor$, $\lfloor\beta_1+\beta_3\rfloor$, $\lfloor\beta_2+\beta_4\rfloor$, $\lfloor\beta_1+\beta_2+2\beta_4\rfloor$, $\lfloor\beta_1+\beta_3+\beta_4\rfloor) \in \{(1,0,0,2,1),\ (0,1,0,2,1),$ $(0,1,0,1,2),\ (1,0,1,2,0),\ (1,0,1,1,1),\ (0,1,1,1,1)\}$. Since either $\beta_1+\beta_2\geq 1$ or $\beta_1+\beta_3\geq 0$, we have $a\notin\mathbb{Z}$ and $\boldsymbol{\beta} = (1-\{a\},1-\{b\},1-\{c\},\{d\})$. ☐

Lemma 2.3.3. *Suppose that $H_1(a,b,c,d|x,y)$ is irreducible and algebraic. Then ${}_2F_1(a,b,d|z)$ and ${}_2F_1(b-d,c,d-a|z)$ are also irreducible and algebraic.*

Proof. Irreducibility follows from the irreducibility and the interlacing condition of $H_1(a,b,c,d|x,y)$. Note that ${}_2F_1(a,b,d|z) = H_1(a,b,c,d|z,0)$. Hence it suffices to show that the interlacing condition for $H_1(a,b,c,d|x,y)$ implies the interlacing condition for ${}_2F_1(b-d,c,d-a|z)$.

Suppose that $(\lfloor\{a\}+\{b\}\rfloor, \lfloor\{a\}+\{c\}\rfloor, \lfloor\{d\}-\{b\}\rfloor, \lfloor 2\{d\}-\{a\}-\{b\}\rfloor, \lfloor\{d\}-\{a\}-\{c\}\rfloor) \in \{(0,1,-1,0,-1),(1,0,-1,0,-1),(1,0,-1,-1,0)\}$. Then $\{b-d\} = \{b\}-\{d\}+\lfloor\{d\}-\{b\}\rfloor+1 = \{b\}-\{d\}$. In the first and second case, $\lfloor 2\{d\}-\{a\}-\{b\}\rfloor > \lfloor\{d\}-\{b\}\rfloor$, so $\{d\}-\{a\}>0$ and hence $\{d-a\} = \{d\}-\{a\}$. In the third case, *entier*$\{d\}-\{a\}-\{c\} = 0$ so $\{d-a\}>0$ and again $\{d-a\} = \{d\}-\{a\}$. Hence in all cases $(\lfloor\{d-a\}-\{b-d\}\rfloor, \lfloor\{d-a\}-\{c\}\rfloor) = (\lfloor 2\{d\}-\{a\}-\{b\}\rfloor, \lfloor\{d\}-\{a\}-\{c\}\rfloor) \in \{(0,-1),(-1,0)\}$. The three other cases follow by symmetry of the interlacing condition with respect to multiplication of $\boldsymbol{\beta}$ by -1. ☐

Theorem 2.3.4. *$H_1(a,b,c,d|x,y)$ is irreducible and algebraic if and only if, up to equivalence modulo $\mathbb{Z}$, (a,b,c,d) is one of the following: $\pm(\frac{1}{3},\frac{5}{6},\frac{1}{2},\frac{2}{3})$, $\pm(\frac{1}{4},\frac{7}{12},\frac{5}{6},\frac{1}{2})$ and $\pm(\frac{1}{4},\frac{11}{12},\frac{1}{6},\frac{1}{2})$.*

Proof. If $H_1(a,b,c,d|x,y)$ is irreducible and algebraic, then (a,b,d) and $(b-d,c,d-a)$ are Gauss triples. Suppose that (a,b,d) is of type 1. Then there exists $r\in(0,1)\cap\mathbb{Q}\backslash\{\frac{1}{2}\}$ such that $(b-d,d-a)\in\{(-r+\frac{1}{2},-r+\frac{1}{2}),(r,-r+\frac{1}{2}),(-r+\frac{1}{2},r)\} \pmod{\mathbb{Z}}$. Note that $d-a\neq\frac{1}{2}$. Hence if $(b-d,c,d-a)$ is also of type 1, then there exists $s\in(0,1)\cap\mathbb{Q}\backslash\{\frac{1}{2}\}$ such that $(b-d,c,d-a)=(s,s+\frac{1}{2},2s)$. Modulo $\mathbb{Z}$, we have $d-a\equiv 2(b-d)$. This implies that $(b-d,d-a)=(r,-r+\frac{1}{2})$ with $r=\pm\frac{1}{6}$ or $(b-d,d-a)=(-r+\frac{1}{2},r)$ with $r=\pm\frac{1}{3}$. Hence the possibilities for (a,b,c,d) are $\pm(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{2})$ and $\pm(\frac{1}{3},\frac{5}{6},\frac{2}{3},\frac{2}{3})$. However, they don't satisfy the interlacing condition. If $(b-d,c,d-a)$ is of type 2, then the denominator of $d-a$ is at most 5. Hence the denominator of r is at most 10. We check all possibilities, and find as solutions the tuples $\pm(\frac{1}{3},\frac{5}{6},\frac{1}{2},\frac{2}{3})$.

Now suppose that (a,b,d) is of type 2. Then the denominator of $b-d$ is at most 60. If $(b-d,c,d-a)$ is of type 1, then it is of the form $(s,-s,\frac{1}{2}),(s,s+\frac{1}{2},\frac{1}{2})$ or $(s,s+\frac{1}{2},2s)$ where the denominator of s is at most 60. This gives finitely many possibilities, and there are no algebraic functions.

Finally, if both (a,b,d) and $(b-d,c,d-a)$ are of type 2, then the solutions are the conjugates of $(\frac{1}{4},\frac{7}{12},\frac{5}{6},\frac{1}{2})$. □

The Horn H_2 function

The H_2 function is defined by

$$H_2(a,b,c,d,e|x,y)=\sum_{m,n\geq 0}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!n!}x^m y^n.$$

The lattice is $\mathbb{L}=\mathbb{Z}(-1,-1,0,0,1,1,0)\oplus\mathbb{Z}(1,0,-1,-1,0,0,1)$. We can take $\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4,\boldsymbol{e}_5,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_5,-\boldsymbol{e}_1+\boldsymbol{e}_3+\boldsymbol{e}_4\}$ and $\boldsymbol{\gamma}=(-a,-b,-c,-d,e-1,0,0)$. Then $\boldsymbol{\beta}=(-a,-b,-c,-d,e-1)$. The function $f:\boldsymbol{x}\mapsto(x_1+x_3,x_2,x_3+x_5,x_3,x_4)$ maps F_2 to H_2.

Lemma 2.3.5. *$H_2(a,b,c,d,e|x,y)$ is irreducible if and only if b,c,d, $a+c,a+d$, $e-b$, $e-a-c$ and $e-a-d$ are non-integral.*

Theorem 2.3.6. *$H_2(a,b,c,d,e|x,y)$ is irreducible and algebraic if and only if, up to equivalence modulo $\mathbb{Z}$ and permutations of $\{c,d\}$, (a,b,c,d,e) is conjugate to one of the following tuples: $(\frac{1}{2},\frac{1}{6},\frac{5}{12},\frac{11}{12},\frac{1}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{4},\frac{3}{4},\frac{2}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{6},\frac{5}{6},\frac{2}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{10},\frac{9}{10},\frac{2}{3})$, $(\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},\frac{1}{2})$, $(\frac{1}{4},\frac{7}{12},\frac{1}{6},\frac{5}{6},\frac{1}{2})$, $(\frac{1}{5},\frac{7}{10},\frac{1}{6},\frac{5}{6},\frac{2}{5})$, $(\frac{1}{5},\frac{7}{10},\frac{1}{10},\frac{9}{10},\frac{2}{5})$ and $(\frac{1}{6},\frac{5}{6},\frac{5}{12},\frac{11}{12},\frac{2}{3})$.*

The Horn H_3 function

The H_3 function is defined by

$$H_3(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}m!n!}x^m y^n.$$

Hence the lattice is equals $\mathbb{L} = \mathbb{Z}(-2,0,1,1,0) \oplus \mathbb{Z}(-1,-1,1,0,1)$. Choose $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 2\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}$ and $\boldsymbol{\gamma} = (-a, -b, c-1, 0, 0)$. Then $\boldsymbol{\beta} = (-a, -b, c-1)$. Consider $f : \boldsymbol{x} \mapsto (x_1 + x_3, x_2 + x_3, -x_3)$. This function maps the set $\mathcal{A}$ of G_1 to the set $\mathcal{A}$ of H_3. Hence:

Lemma 2.3.7. *$H_3(a, b, c|x, y)$ is irreducible if and only if a, b, $c - a$ and $2c - a - b$ are non-integral.*

Theorem 2.3.8. *$H_3(a, b, c|x, y)$ is irreducible and algebraic if and only if (a, b, c) is one of the following: $\pm(\frac{1}{3}, \frac{5}{6}, \frac{1}{2})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{3})$ and $\pm(\frac{1}{6}, \frac{5}{6}, \frac{1}{3})$.*

The Horn H_4 function

The H_4 function is defined by

$$H_4(a, b, c, d|x, y) = \sum_{m,n \geq 0} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-2,0,1,0,1,0) \oplus \mathbb{Z}(-1,-1,0,1,0,1)$. Take $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, 2\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4\}$. With $\boldsymbol{\gamma} = (-a, -b, c-1, d-1, 0, 0)$ we get $\boldsymbol{\beta} = (-a, -b, c-1, d-1)$. The sets $V_1 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$, $V_2 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4\}$, $V_3 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_4, 2\boldsymbol{e}_1 - \boldsymbol{e}_3\}$ and $V_4 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, 2\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4\}$ form a unimodular triangulation of $\mathcal{A}$. Hence:

Lemma 2.3.9. *$\mathcal{A}$ is normal, the volume of $Q(\mathcal{A})$ is 4 and*

$$C(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1, x_2 \geq 0, x_1 + 2x_3 \geq 0, x_1 + x_4 \geq 0, x_2 + x_4 \geq 0, x_1 + 2x_3 + x_4 \geq 0\}.$$

$H_4(a, b, c, d|x, y)$ is irreducible if and only if a, b, $2c - a$, $d - a$, $d - b$ and $2c + d - a$ are non-integral.

Lemma 2.3.10. *Suppose that $H_4(a, b, c, d|x, y)$ is irreducible. Then there are 4 apex points if and only if $(\lfloor 2\{c\} - \{a\} \rfloor, \lfloor \{d\} - \{a\} \rfloor, \lfloor \{d\} - \{b\} \rfloor, \lfloor 2\{c\} + \{d\} - \{a\} \rfloor) \in \{(0, 0, -1, 1), (0, -1, 0, 0)\}$.*

Proof. There are 4 apex points if and only if $(\lfloor \beta_1 + 2\beta_3 \rfloor, \lfloor \beta_1 + \beta_4 \rfloor, \lfloor \beta_2 + \beta_4 \rfloor, \lfloor \beta_1 + 2\beta_3 + \beta_4 \rfloor)$ equals $(1, 1, 0, 2)$ or $(1, 0, 1, 1)$. □

Lemma 2.3.11. *If $H_4(a, b, c, d|x, y)$ is irreducible and algebraic, then the functions ${}_2F_1(\frac{a}{2}, \frac{a+1}{2}, c|z)$, ${}_2F_1(a, b, d|z)$ and ${}_2F_1(b, a - 2c, d|z)$ are also irreducible and algebraic.*

Proof. Irreducibility is clear. Since we have ${}_2F_1(\frac{a}{2}, \frac{a+1}{2}, c|z) = H_4(a, b, c, d|\frac{z}{4}, 0)$ and ${}_2F_1(a, b, d|z) = H_4(a, b, c, d|0, z)$, these functions are algebraic.

If $(\lfloor 2\{c\} - \{a\} \rfloor, \lfloor \{d\} - \{a\} \rfloor, \lfloor \{d\} - \{b\} \rfloor, \lfloor 2\{c\} + \{d\} - \{a\} \rfloor) \in \{(0, 0, -1, 1), (0, -1, 0, 0)\}$, then $0 \leq 2\{c\} - \{a\} < 1$ so $\{a - 2c\} = 1 - \{2c - a\} = 1 - 2\{c\} - \{a\}$. It follows that $(\lfloor \{d\} - \{b\} \rfloor, \lfloor \{d\} - \{a - 2c\} \rfloor) = (\lfloor \{d\} - \{b\} \rfloor, \lfloor 2\{c\} + \{d\} - \{a\} \rfloor - 1) \in \{(-1, 0), (0, -1)\}$. Hence the interlacing condition for $H_4(a, b, c, d|x, y)$ implies the interlacing condition for ${}_2F_1(b, a - 2c, d|z)$. □

Theorem 2.3.12. *$H_4(a,b,c,d|x,y)$ is irreducible and algebraic if and only if (a,b,c,d) is conjugate to one of the tuples in Table 2.3.*

Proof. We only have to consider (a,b,c,d) such that $(\frac{a}{2},\frac{a+1}{2},c)$, (a,b,d) and $(b,a-2c,d)$ are Gauss triples. Suppose that $(\frac{a}{2},\frac{a+1}{2},c)$ is of type 1. Then either $c=\frac{1}{2}$ or $c=a$. If (a,b,d) is also of type 1, then there exists $r\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ such that $(a,b,c,d) \pmod{\mathbb{Z}} \in \{(r,-r,\frac{1}{2},\frac{1}{2}),\ (r,-r,r,\frac{1}{2}),\ (r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2}),\ (r,r+\frac{1}{2},r,\frac{1}{2}),\ (r,r+\frac{1}{2},\frac{1}{2},2r),\ (r,r+\frac{1}{2},r,2r)\}$. This gives $(b,a-2c,d) \pmod{\mathbb{Z}} \in \{(-r,\pm r,\frac{1}{2}),\ (r+\frac{1}{2},\pm r,\frac{1}{2}),\ (r+\frac{1}{2},\pm r,\frac{1}{2})$. One easily checkes that precisely the triples with $a-2c$ are indeed Gauss triples. In those cases, all conjugates of (a,b,c,d) satisfy the interlacing condition. If (a,b,d) is of type 2, we can just check the interlacing condition for all tuples $(a,b,\frac{1}{2},d)$ and (a,b,a,d). This gives 408 solutions.

If $(\frac{a}{2},\frac{a+1}{2},c)$ is of type 2 and (a,b,d) is of type 1, then the denominator of a is at most 30. We check the interlacing condition for all possibilities and find 8 solutions.

Finally, if both $(\frac{a}{2},\frac{a+1}{2},c)$ and (a,b,d) are of type 2, then there are finitely many possibilities. This gives another 36 solutions. Of all 452 solutions, the smallest conjugate is given in Table 2.3. □

The Horn H_5 function

The H_5 function is defined by

$$H_5(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_n m!n!}x^m y^n.$$

Hence the lattice is $\mathbb{L}=\mathbb{Z}(-2,1,0,1,0)\oplus\mathbb{Z}(-1,-1,1,0,1)$. We can take $\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,2\boldsymbol{e}_1-\boldsymbol{e}_2,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3\}$ and $\boldsymbol{\gamma}=(-a,-b,c-1,0,0)$, so $\boldsymbol{\beta}=(-a,-b,c-1)$.

The projection of $\mathcal{A}$ onto the (x_1,x_2)-plane is shown in Figure 2.3. The thick dots represent $\mathcal{A}$, the dark gray region is the set $Q(\mathcal{A})$ and light gray region is a part of the set $C(\mathcal{A})$. It is clear that $Q(\mathcal{A})$ has a unimodular triangulation and has volume 4, so $\mathcal{A}$ is normal. Furthermore,

$$C(\mathcal{A})=\{\boldsymbol{x}\in\mathbb{R}^3 \mid x_1\geq 0, x_1+2x_2\geq 0, x_1+x_3\geq 0, x_1+2x_2+3x_3\geq 0\}.$$

Hence $H_5(a,b,c|x,y)$ is irreducible if and only if a, $a+2b$, $c-a$ and $3c-a-2b$ are non-integral.

Lemma 2.3.13. *If $H_5(a,b,c|x,y)$ is irreducible, then there are 4 apex points if and only if $(\lfloor\{a\}+2\{b\}\rfloor,\lfloor\{c\}-\{a\}\rfloor,\lfloor 3\{c\}-\{a\}-2\{b\}\rfloor)\in\{(1,-1,0),(1,0,-1)\}$.*

Proof. There are 4 apex points if and only if $(\lfloor\beta_1+2\beta_2\rfloor,\lfloor\beta_1+\beta_3\rfloor,\lfloor\beta_1+2\beta_2+3\beta_3\rfloor)$ equals $(1,0,3)$ or $(1,1,2)$. In those cases, β_2 is non-integral, so $\boldsymbol{\beta}=(1-\{a\},1-\{b\},\{c\})$. □

Theorem 2.3.14. *$H_5(a,b,c|x,y)$ is irreducible and algebraic if and only if (a,b,c) is, up to equivalence modulo $\mathbb{Z}$, conjugate to one of the following: $(r,-r,\frac{1}{2})$ for some $r\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$, $(\frac{1}{6},\frac{1}{2},\frac{1}{3})$, $(\frac{1}{6},\frac{2}{3},\frac{1}{3})$, $(\frac{1}{6},\frac{5}{6},\frac{1}{3})$, $(\frac{1}{10},\frac{3}{5},\frac{1}{5})$ and $(\frac{1}{12},\frac{3}{4},\frac{1}{2})$.*

Table 2.3: The tuples (a,b,c,d) such that $H_4(a,b,c,d|x,y)$ is irreducible and algebraic

$(r,-r,\frac{1}{2},\frac{1}{2})$, $(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(r,r+\frac{1}{2},\frac{1}{2},2r)$ with $2r \notin \mathbb{Z}$

$(\frac{1}{2},\frac{1}{6},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{7}{10},\frac{2}{5},\frac{2}{5})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{2},\frac{1}{6},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{2},\frac{2}{3})$
$(\frac{1}{4},\frac{3}{4},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{4},\frac{7}{12},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{4}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{2},\frac{3}{5})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{5},\frac{2}{3})$
$(\frac{1}{4},\frac{7}{12},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{5},\frac{4}{5})$	$(\frac{1}{15},\frac{13}{15},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{30},\frac{7}{10},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{15},\frac{13}{15},\frac{1}{2},\frac{3}{5})$	$(\frac{1}{30},\frac{7}{10},\frac{1}{2},\frac{2}{5})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{30},\frac{7}{10},\frac{1}{3},\frac{2}{5})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{2}{3})$	$(\frac{1}{12},\frac{3}{4},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{30},\frac{11}{30},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{12},\frac{3}{4},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{30},\frac{19}{30},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{12},\frac{3}{4},\frac{1}{3},\frac{1}{2})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{2}{3})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{2},\frac{2}{3})$	$(\frac{1}{20},\frac{17}{20},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{20},\frac{17}{20},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2},\frac{1}{2})$
$(\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{4},\frac{2}{3})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{6},\frac{11}{30},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2},\frac{1}{2})$
$(\frac{1}{6},\frac{11}{30},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{12},\frac{7}{12},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{12},\frac{7}{12},\frac{1}{3},\frac{1}{2})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2},\frac{1}{4})$	
$(\frac{1}{10},\frac{7}{10},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{12},\frac{11}{12},\frac{1}{3},\frac{1}{2})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2},\frac{1}{2})$	

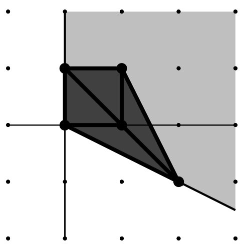


Figure 2.3: The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ for H_5

Proof. Suppose that $H_5(a,b,c|x,y)$ is non-resonant and algebraic. Substituting $y=0$ shows that ${}_2F_1(a,b,c|x)$ is also algebraic and irreducible, so (a,b,c) is a Gauss triple.

First suppose that (a,b,c) is a Gauss triple of type 1. If $(a,b,c)=(r,-r,\frac{1}{2})$, then the function is non-resonant and $(\lfloor\{a\}+2\{b\}\rfloor,\lfloor\{c\}-\{a\}\rfloor,\lfloor 3\{c\}-\{a\}-2\{b\}\rfloor)=(\lfloor 2-r\rfloor,\lfloor\frac{1}{2}-r\rfloor,\lfloor r-\frac{1}{2}\rfloor)$. This equals $(1,0,-1)$ if $r<\frac{1}{2}$ and $(1,-1,0)$ if $r>\frac{1}{2}$, so the interlacing condition is satisfied. Since all conjugates are of the same form, the function is algebraic.

Suppose that $(a,b,c)=(r,r+\frac{1}{2},\frac{1}{2})$. The function is irreducible if r is not equal to $\pm\frac{1}{3}$, or $\pm\frac{1}{6}$. Then the interlacing condition is satisfied if and only if $r\in(\frac{1}{6},\frac{1}{3}]\cup(\frac{2}{3},\frac{5}{6}]$. All conjugates of r also have to be in this set. By choosing a conjugate with numerator 1, we see that the denominator of r can at most be 5. $r=\pm\frac{1}{4}$ gives the same tuple as for $(r,-r,\frac{1}{2})$. If r has denominator 5, then $\frac{2}{5}$ is a conjugate that doesn't satisfy the condition. Hence this gives no more algebraic functions.

Finally, suppose that $(a,b,c)=(r,r+1/2,2r)$. Then the function is irreducible if r is not equal to $\pm\frac{1}{3}$. The interlacing condition is satisfied if and only if $r\in(0,\frac{1}{3})\cup(\frac{2}{3},1)$. Lemma 1.4.11 implies that the denominator of r is 4, 6 or 10. If the denominator is 4, then we again get the tuple $(r,-r,\frac{1}{2})$. There are two solutions with denominator 6: $(\frac{1}{6},\frac{2}{3},\frac{1}{3})$ and $(\frac{5}{6},\frac{1}{3},\frac{2}{3})$. With denominator 10, we find the solutions $(\frac{1}{10},\frac{3}{5},\frac{1}{5})$, $(\frac{3}{10},\frac{4}{5},\frac{3}{5})$, $(\frac{7}{10},\frac{1}{5},\frac{2}{5})$ and $(\frac{9}{10},\frac{2}{5},\frac{4}{5})$. For all these tuples, the interlacing condition is indeed satisfied.

If (a,b,c) is a Gauss triple of type 2, then there are only finitely many possibilities. There are 8 solutions: $\pm(\frac{1}{6},\frac{1}{2},\frac{1}{3})$, $\pm(\frac{1}{6},\frac{5}{6},\frac{1}{3})$, $\pm(\frac{1}{12},\frac{3}{4},\frac{1}{2})$ and $\pm(\frac{5}{12},\frac{3}{4},\frac{1}{2})$. ☐

The Horn H_6 function

The H_6 function is defined by

$$H_6(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m-n}(b)_{n-m}(c)_n}{m!n!}x^m y^n.$$

Hence the lattice is $\mathbb{L}=\mathbb{Z}(-2,1,0,1,0)\oplus\mathbb{Z}(1,-1,-1,0,1)$. We choose $\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,2\boldsymbol{e}_1-\boldsymbol{e}_2,-\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3\}$ and $\boldsymbol{\gamma}=(-a,-b,-c,0,0)$. Then $\boldsymbol{\beta}=(-a,-b,-c)$. The function $f:\boldsymbol{x}\mapsto(x_1-x_2,x_2,x_2+x_3)$ maps G_1 to H_6. Hence:

Lemma 2.3.15. *$H_6(a,b,c|x,y)$ is irreducible if and only if $a+b$, $a+2b$, c and $a+c$ are non-integral.*

Theorem 2.3.16. *$H_6(a,b,c|x,y)$ is irreducible and algebraic if and only if up to equivalence modulo $\mathbb{Z}$, (a,b,c) equals $\pm(\frac{1}{2},\frac{1}{3},\frac{2}{3})$, $\pm(\frac{1}{2},\frac{1}{3},\frac{5}{6})$ or $\pm(\frac{1}{3},\frac{1}{2},\frac{5}{6})$.*

The Horn H_7 function

The H_7 function is defined by

$$H_7(a,b,c,d|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m!n!}x^m y^n.$$

Table 2.4: The tuples (a,b,c,d) such that $H_7(a,b,c,d|x,y)$ is irreducible and algebraic

$(\frac{1}{2},r,-r,\frac{1}{2})$, $(\frac{1}{2},r,r+\frac{1}{2},\frac{1}{2})$ and $(-2r,r,r+\frac{1}{2},\frac{1}{2})$ with $2r\notin\mathbb{Z}$			
$(\frac{1}{2},\frac{1}{4},\frac{7}{12},\frac{1}{2})$	$(\frac{1}{3},\frac{1}{6},\frac{5}{6},\frac{1}{3})$	$(\frac{1}{3},\frac{11}{60},\frac{59}{60},\frac{1}{2})$	$(\frac{1}{5},\frac{7}{10},\frac{9}{10},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{12},\frac{7}{12},\frac{1}{3})$	$(\frac{1}{3},\frac{5}{6},\frac{1}{12},\frac{1}{2})$	$(\frac{1}{4},\frac{1}{6},\frac{5}{6},\frac{1}{2})$	$(\frac{1}{5},\frac{9}{10},\frac{7}{30},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{20},\frac{13}{20},\frac{1}{2})$	$(\frac{1}{3},\frac{5}{6},\frac{1}{30},\frac{1}{2})$	$(\frac{1}{4},\frac{1}{6},\frac{5}{6},\frac{1}{4})$	$(\frac{1}{5},\frac{4}{15},\frac{13}{15},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{24},\frac{17}{24},\frac{1}{2})$	$(\frac{1}{3},\frac{1}{10},\frac{9}{10},\frac{1}{2})$	$(\frac{1}{4},\frac{1}{6},\frac{11}{12},\frac{1}{2})$	$(\frac{1}{5},\frac{4}{15},\frac{14}{15},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{24},\frac{19}{24},\frac{1}{2})$	$(\frac{1}{3},\frac{1}{10},\frac{9}{10},\frac{1}{3})$	$(\frac{1}{4},\frac{7}{12},\frac{11}{12},\frac{1}{2})$	$(\frac{1}{5},\frac{8}{15},\frac{13}{15},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{60},\frac{41}{60},\frac{1}{2})$	$(\frac{1}{3},\frac{1}{10},\frac{23}{30},\frac{1}{2})$	$(\frac{1}{4},\frac{7}{12},\frac{19}{24},\frac{1}{2})$	$(\frac{1}{5},\frac{7}{20},\frac{17}{20},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{60},\frac{49}{60},\frac{1}{2})$	$(\frac{1}{3},\frac{5}{12},\frac{1}{12},\frac{1}{2})$	$(\frac{1}{4},\frac{7}{24},\frac{23}{24},\frac{1}{2})$	$(\frac{1}{5},\frac{7}{20},\frac{19}{20},\frac{1}{2})$
$(\frac{1}{3},\frac{1}{2},\frac{5}{6},\frac{1}{2})$	$(\frac{1}{3},\frac{2}{15},\frac{11}{15},\frac{1}{2})$	$(\frac{1}{5},\frac{1}{6},\frac{5}{6},\frac{1}{2})$	$(\frac{1}{5},\frac{9}{20},\frac{19}{20},\frac{1}{2})$
$(\frac{1}{3},\frac{1}{4},\frac{3}{4},\frac{1}{2})$	$(\frac{1}{3},\frac{5}{24},\frac{17}{24},\frac{1}{2})$	$(\frac{1}{5},\frac{1}{6},\frac{5}{6},\frac{1}{5})$	$(\frac{1}{5},\frac{19}{30},\frac{29}{30},\frac{1}{2})$
$(\frac{1}{3},\frac{1}{4},\frac{3}{4},\frac{1}{3})$	$(\frac{1}{3},\frac{5}{24},\frac{23}{24},\frac{1}{2})$	$(\frac{1}{5},\frac{1}{6},\frac{29}{30},\frac{1}{2})$	$(\frac{1}{5},\frac{19}{60},\frac{49}{60},\frac{1}{2})$
$(\frac{1}{3},\frac{1}{4},\frac{11}{12},\frac{1}{2})$	$(\frac{1}{3},\frac{11}{30},\frac{29}{30},\frac{1}{2})$	$(\frac{1}{5},\frac{1}{10},\frac{9}{10},\frac{1}{2})$	$(\frac{1}{5},\frac{19}{60},\frac{59}{60},\frac{1}{2})$
$(\frac{1}{3},\frac{1}{6},\frac{5}{6},\frac{1}{2})$	$(\frac{1}{3},\frac{11}{60},\frac{41}{60},\frac{1}{2})$	$(\frac{1}{5},\frac{1}{10},\frac{9}{10},\frac{1}{5})$	$(\frac{1}{6},\frac{5}{12},\frac{11}{12},\frac{1}{3})$

The lattice is $\mathbb{L}=\mathbb{Z}(-2,0,0,1,1,0)\oplus\mathbb{Z}(1,-1,-1,0,0,1)$. We can take $\mathcal{A}=\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, 2\boldsymbol{e}_1-\boldsymbol{e}_4, -\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3\}$ and $\boldsymbol{\gamma}=(-a,-b,-c,d-1,0,0)$. Then $\boldsymbol{\beta}=(-a,-b,-c,d-1)$.

The function $f:\boldsymbol{x}\mapsto(x_1-x_2,x_2,x_2+x_4,x_3)$ maps H_4 to H_7. Hence:

Lemma 2.3.17. *$H_7(a,b,c,d|x,y)$ is irreducible if and only if b, c, $a+b$, $a+c$, $2d-a-b$ and $2d-a-c$ are non-integral.*

Theorem 2.3.18. *$H_7(a,b,c,d|x,y)$ is irreducible and algebraic if and only if (a,b,c,d) or (a,c,b,d) is conjugate to one of the tuples in Table 2.4.*

3

Planar point configurations

In this chapter, we will consider planar point configurations, i.e., subsets of $\mathbb{Z}^2$ and $\mathbb{Z}^3$. After a coordinate transformation, we can assume that the elements of $\mathcal{A}$ lie on the line $x_2 = 1$ (for $\mathcal{A} \subseteq \mathbb{Z}^2$) or in the plane $x_3 = 1$ (for $\mathcal{A} \subseteq \mathbb{Z}^3$). Then the convex hull $Q(\mathcal{A})$ is a line segment or a polygon on height 1. To sets $\mathcal{A} \subseteq \mathbb{Z}^3$ we associate the polygon $P(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (\boldsymbol{x}, 1) \in Q(\mathcal{A})\}$.

The main result of this chapter is the following:

Theorem 3.1. *Let $\mathcal{A}$ be a finite normal subset of $\mathbb{Z}^2$, lying on an affine line. Then there exists $\boldsymbol{\beta} \in \mathbb{Q}^2$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant and has algebraic solutions.*

Let $\mathcal{A}$ be a finite normal subset of $\mathbb{Z}^3$, lying in an affine plane. If $\mathcal{A}$ is a pyramid, then there exists $\boldsymbol{\beta} \in \mathbb{Q}^3$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant and has algebraic solutions. If $\mathcal{A}$ is not a pyramid, then such $\boldsymbol{\beta}$ exists if and only if $P(\mathcal{A})$ is isomorphic to one of the polygons in Figure 3.1.

We start by analyzing one-dimensional sets in the first section. For the two-dimensional sets, we will mostly use the polygons $P(\mathcal{A})$ instead of the sets $\mathcal{A}$. In Section 3.2, we consider these polygons in more detail and show that they have unimodular triangulations. This allows us to use a reduction of $\mathcal{A}$ to subsets, as stated in Proposition 1.4.7. In Section 3.3 we will compute all polygons with at most 2 interior lattice points and determine the irreducible algebraic functions. Then we show in Section 3.4 that all polygons with sufficiently many interior and boundary lattice points contain smaller subpolygons, to which we can reduce the problem of finding irreducible algebraic functions. Finally, the polygons with many interior points and few boundary points will be treated in Section 3.5.

The results of this chapter can also be found in the preprint [Bod13a].

3.1 Collinear point configurations

Suppose that $\mathcal{A}$ is a finite normal subset of $\mathbb{Z}^2$. By applying a coordinate transformation, we can assume that $\mathcal{A}$ lies on the line $x_2 = 1$. Furthermore, we can shift $\mathcal{A}$ so that $\mathcal{A} = \{(k, 1) | -1 \leq k \leq N-2\}$. Since $\mathcal{A}$ spans $\mathbb{Z}^2$, we have $N \geq 2$. We compute the irreducible algebraic functions for each N. $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible if and only if $\beta_1 + \beta_2, -\beta_1 + (N-2)\beta_2 \notin \mathbb{Z}$, as can be seen in Figure 3.2(a). The interlacing condition is a generalization of the interlacing condition for the Horn G_3 function:

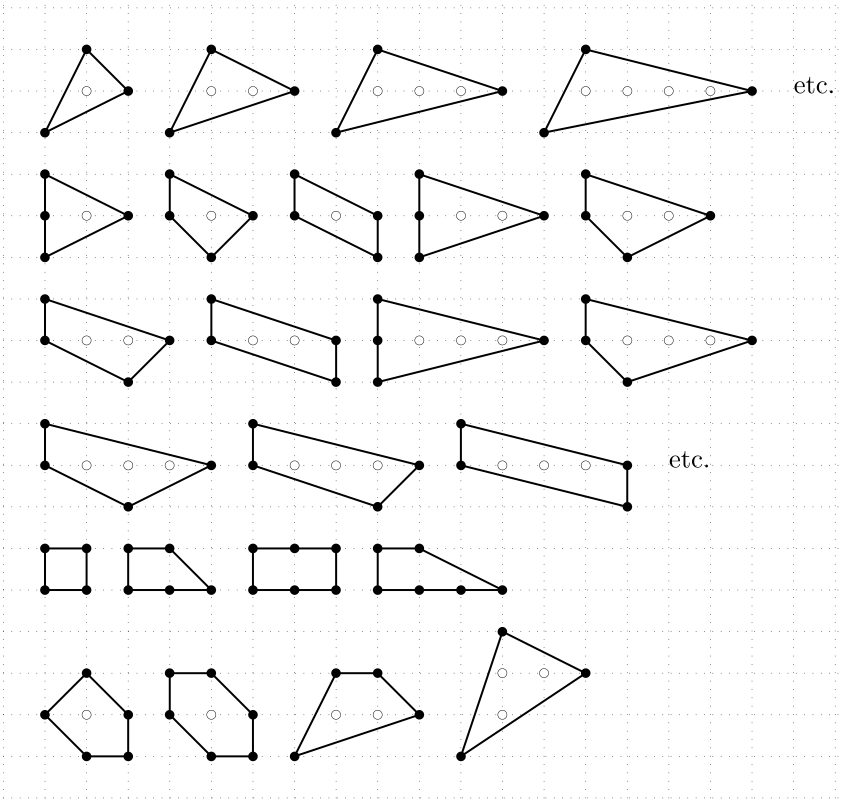


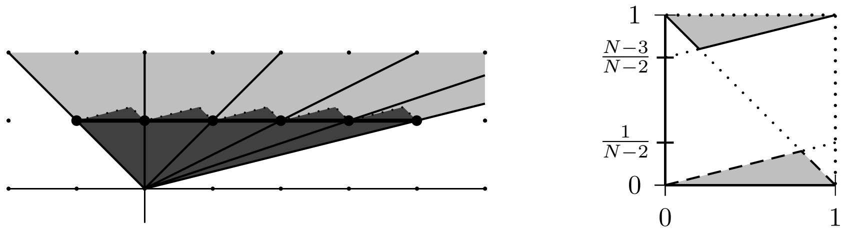
Figure 3.1: The polygons $P(\mathcal{A})$ such that there exists $\boldsymbol{\beta}$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions

Lemma 3.1.1. *For* $0 \le i \le N-2$*, let* $V_i = \{(i-1,1),(i,1)\}$ *and let* $(x,y) \in C(V_i)$*. Then* (x,y) *is an apex point if and only if* $x+y < i+1$ *and* $x > (N-2)y-N+i+1$*. There are* $N-1$ *apex points if and only if* $(\lfloor \beta_1+\beta_2 \rfloor, \lfloor -\beta_1+(N-2)\beta_2 \rfloor) \in \{(-1,0),(N-3,1)\}$*.*

Proof. Suppose that $(x,y) \in C(V_i)$ is an apex point. Since $(x,y) \in C(V_i)$, we have $(i-1)y \le x \le iy$. Then $(x,y) - \boldsymbol{a}_j \notin C(\mathcal{A})$ for all $\boldsymbol{a}_j$, so in particular $(x-i,y-1),(x-i+1,y-1) \notin C(\mathcal{A})$, i.e., $x+y < i+1$ or $x-i > (N-2)(y-1)$, and $x+y < i$ or $x > (N-2)y-N+i+1$. We consider two cases: $y < 1$ and $y \ge 1$. If $y < 1$, then $x+y \le (i+1)y < i+1$. Furthermore, if $x+y < i$, then $(N-2)y-N+i+1 = (N-2)(y-1)+i-1 < (i-1)(y-1)+i-1 = (i-1)y \le x$. If $y \ge 1$, then $x-i \le i(y-1) \le (N-2)(y-1)$, so we again have $x+y < i+1$. It also holds that $x+y \ge iy \ge i$, so $x > (N-2)y-N+i+1$.

On the other hand, for points $(x,y) \in C(V_i)$ satisfying $x+y < i+1$ and $x > (N-2)y-N+i+1$, it is easily checked that $(x,y)-(k,1) \notin C(\mathcal{A})$ for all $-1 \le k \le N-2$.

It follows that the apex points are the points with fractional part $\boldsymbol{\beta}$ lying in the



(a) The convex hull (thick line), positive cone (light gray) and apex points (dark gray)

(b) The interlacing condition

Figure 3.2: A collinear point configuration ($N = 6$)

Table 3.1: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible and has algebraic solutions if $\mathcal{A} = \{(k, 1) | -1 \leq k \leq N-2\}$

N	$\boldsymbol{\beta}$			
All $N \geq 2$	$(r, 0)$ with $r \notin \mathbb{Z}$			
2	All irreducible functions are algebraic			
3	$(\frac{1}{3}, \frac{1}{4})$	$(\frac{1}{3}, \frac{5}{6})$	$(\frac{1}{4}, \frac{1}{6})$	$(\frac{1}{5}, \frac{1}{10})$
	$(\frac{1}{3}, \frac{1}{6})$	$(\frac{1}{3}, \frac{1}{10})$	$(\frac{1}{5}, \frac{1}{6})$	$(\frac{1}{5}, \frac{9}{10})$
4	$(\frac{1}{2}, \frac{1}{6})$	$(\frac{1}{3}, \frac{5}{6})$		
5	$(\frac{1}{3}, \frac{5}{6})$			

dark gray area in Figure 3.2(a). Hence there are $N-1$ apex points if and only if $-\beta_1 + (N-2)\beta_2 < 0$ and $\beta_1 + \beta_2 < 1$, or $-\beta_1 + (N-2)\beta_2 \geq N-3$ and $\beta_1 + \beta_2 \geq 1$. $\square$

A graphical interpretation of the interlacing condition is given in Figure 3.2(b).

Now we have found the interlacing condition, we can determine the irreducible algebraic functions:

Theorem 3.1.2. *Let $N \geq 2$ and $\mathcal{A} = \{(k, 1) | -1 \leq k \leq N-2\}$. Then $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible and has algebraic solutions if and only if $\boldsymbol{\beta} \pmod{\mathbb{Z}}$ is a conjugate of one of the tuples in Table 3.1.*

Except for $N = 3$, these results can also be found in [Sch09].

Proof. If $N = 2$, then the lattice $\mathbb{L}$ is trivial, so the Γ-series solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are monomials $z_1^{-\beta_1} z_2^{\beta_1+\beta_2}$, and hence algebraic.

If $N = 3$, the Gauss function ${}_2F_1(\frac{-\beta_1-\beta_2}{2}, \frac{-\beta_1-\beta_2-1}{2}, -\beta_1 + 1 | 4z)$ is a solution. It is irreducible if and only if $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible. Checking the interlacing condition for all parameters such that the Gauss function is algebraic shows that $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $\boldsymbol{\beta}$ is of the form $(\frac{1}{2}, r)$ or $(r, 0)$ with $r \notin \mathbb{Z}$ or is one of the 36 conjugates of $(\frac{1}{3}, \frac{1}{4})$, $(\frac{1}{3}, \frac{1}{6})$, $(\frac{1}{3}, \frac{5}{6})$, $(\frac{1}{3}, \frac{1}{10})$, $(\frac{1}{4}, \frac{1}{6})$, $(\frac{1}{5}, \frac{1}{6})$, $(\frac{1}{5}, \frac{1}{10})$ and $(\frac{1}{5}, \frac{9}{10})$.

For $N = 4$, we have the Horn G_3 function, which is irreducible and algebraic if and only if $\boldsymbol{\beta} \in \{(r, 0), \pm(\frac{1}{2}, \frac{1}{6}), \pm(\frac{1}{3}, \frac{5}{6})\}$ with $r \notin \mathbb{Z}$.

Now let $N = 5$. By Proposition 1.4.7, if $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions, then it is also irreducible for $N = 4$ and has algebraic solutions. Hence we only have to check $(r, 0), \pm(\frac{1}{2}, \frac{1}{6})$ and $\pm(\frac{1}{3}, \frac{5}{6})$. It turns out that $(r, 0)$ is a solution for all $r \notin \mathbb{Z}$, as well as $\pm(\frac{1}{3}, \frac{5}{6})$, but $\pm(\frac{1}{2}, \frac{1}{6})$ is resonant.

For $N = 6$, we only have to check the solutions $(r, 0)$ and $\pm(\frac{1}{3}, \frac{5}{6})$ of $N = 5$. Now $\pm(\frac{1}{3}, \frac{5}{6})$ is resonant, but $(r, 0)$ is a solution.

Finally, for $N \geq 6$, the tuple $(r, 0)$ gives irreducible functions if $r \notin \mathbb{Z}$ and it satisfies the interlacing condition. $\square$

3.2 Unimodular triangulations of planar polygons

In this section, we will show that planar sets $\mathcal{A}$ have unimodular triangulations that are compatible with the triangulations of subsets $\mathcal{A}'$ as in Proposition 1.4.7. To this end we study triangulations of the associated polygons $P(\mathcal{A}) = \{\boldsymbol{x} \in \mathbb{R}^{r-1} \mid (\boldsymbol{x}, 1) \in Q(\mathcal{A})\}$. They are convex lattice polygons, in the sense of the following definition:

Definition 3.2.1. A *simple polygon* P is a subset of $\mathbb{R}^2$ bounded by a closed chain of line segments, that does not intersect itself. If the points in which the line segments meet are integral, then P is called a *lattice polygon.* Two lattice polygons P and P' are *isomorphic* if P' is the image of P under a linear isomorphism of $\mathbb{Z}^2$. A *triangulation* of a convex lattice polygon P is a dissection of P into triangles, of which the vertices are lattice points in P. It is *unimodular* if all triangles have simplex area 1.

Each convex lattice polygon P is associated to a finite set $\mathcal{A} \subseteq \mathbb{Z}^3$: by defining $\mathcal{A} = (P \cap \mathbb{Z}^2) \times \{1\}$, we have $P = P(\mathcal{A})$. It follows from Corollary 3.2.5 that P has a unimodular triangulation, inducing a unimodular triangulation of $\mathcal{A}$. This implies that $\mathcal{A}$ spans $\mathbb{Z}^2$ over $\mathbb{Z}$ and is normal. Note that isomorphisms of polygons correspond to isomorphisms of the associated point configurations.

To check that a triangulation is unimodular, we use Pick's formula:

Lemma 3.2.2 (Pick's formula). *Let P be a convex lattice polygon with i interior lattice points and b lattice points on the boundary. Then its simplex area is equal to $2i + b - 2$.*

It follows that a triangle with integral vertices has area 1 if and only if the only integral points are the vertices.

The next lemma and corollary allow us to use Proposition 1.4.7.

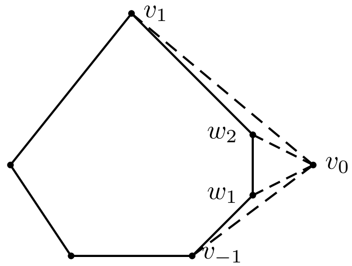


Figure 3.3: Adding a vertex to a triangulated polygon

Lemma 3.2.3. *Let P, P' be convex lattice polygons with $P' \subseteq P$. Then every unimodular triangulation of P' can be extended to a unimodular triangulation of P.*

Proof. Let V be the set of lattice points in P that are not in P'. It is clear that it suffices to show that the lemma holds for $|V| = 1$, since we can then add the points in V one by one to P' while preserving the triangulation. So suppose that $V = \{v_0\}$. Then v_0 is a vertex of P, since otherwise it would be contained in P', which is the convex hull of all other lattice points in P. Let v_{-1} and v_1 be the previous and next lattice point on the boundary of P (in counterclockwise order; see Figure 3.3). There can be lattice points on the boundary of P' in between v_{-1} and v_1; call them $w_1, \ldots, w_k$. By connecting v_0 to $v_{-1}, v_1, w_1, \ldots, w_k$, we clearly get a triangulation of P that extends the triangulation of P'. Since the only lattice points in each of the triangles are the vertices, all triangles have area 1 and the triangulation is unimodular. $\square$

Lemma 3.2.4. *Every lattice triangle has a unimodular triangulation.*

Proof. Let P be a lattice triangle. We show that P can be divided into smaller triangles. Since the area of such triangles is positive and integral, after finitely many steps we will find a unimodular triangulation of P. If P doesn't have interior lattice points or lattice points on the boundary except for the vertices, then the area is 1 and P is triangulated already. If P has an interior lattice point, then connecting this interior point with the three vertices divides the triangle into three smaller triangles. If there is a lattice point on an edge of P, which is not a vertex, then connecting this with the opposite vertex will divide P into two smaller triangles. $\square$

Corollary 3.2.5. *Every convex lattice polygon has a unimodular triangulation.*

Proof. We prove this by induction on the number of vertices. Let P be a lattice polygon. If P is a triangle, then the statement follows from Lemma 3.2.4. Suppose that P has at least 4 vertices. Then there is a diagonal dividing P in two smaller polygons. By induction, we can find a unimodular triangulation of one of these. Lemma 3.2.3 shows that we can extend this to a unimodular triangulation of P. $\square$

Corollary 3.2.6. *Let $P' \subseteq P$ be convex lattice polygons, with associated sets $\mathcal{A}' \subseteq \mathcal{A} \subseteq \mathbb{Z}^3$. If $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions, then $H_{\mathcal{A}'}(\boldsymbol{\beta})$ is irreducible and its solutions are algebraic. If there are no $\boldsymbol{\beta}$ such that $H_{\mathcal{A}'}(\boldsymbol{\beta})$ has irreducible algebraic solutions, then such $\boldsymbol{\beta}$ also don't exist for $H_{\mathcal{A}}(\boldsymbol{\beta})$.*

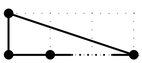
Proof. This follows from Lemma 3.2.3, Corollary 3.2.5, and Proposition 1.4.7. □

3.3 Polygons with at most 2 interior points

In this section, we will determine the algebraic functions corresponding to polygons with at most 2 lattice interior points. In the remainder of this chapter, we will abbreviate 'interior lattice point' and 'lattice point on the boundary' to 'interior point' and 'boundary point', respectively.

Polygons without interior points

Rabinowitz has given a classification of the polygons without interior points [Rab89]. Up to isomorphism, there are three types: triangles with vertices $(0,0), (p,0)$ and $(0,1)$, the triangle with vertices $(0,0), (2,0)$ and $(0,2)$, and trapezoids with vertices $(0,0), (p,0), (q,1)$ and $(0,1)$. We denote the corresponding sets by $\mathcal{A}_p^{(1)}$, $\mathcal{A}_1$ and $\mathcal{A}_{p,q}^{(2)}$, respectively.



Lemma 3.3.1. *Let* $\mathcal{A}_p^{(1)} = \begin{pmatrix} 0 & 1 & 2 & \dots & p & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$. *Then $H_{\mathcal{A}_p^{(1)}}(\boldsymbol{\beta})$ is non-resonant and has algebraic solutions if and only if, up to conjugation and equivalence modulo $\mathbb{Z}$, $\boldsymbol{\beta}$ is one of the tuples in Table 3.2.*

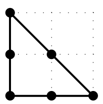
Proof. Let $\tilde{\mathcal{A}}_p^{(1)} = \mathcal{A}_p^{(1)} \setminus \{\boldsymbol{a}_{p+2}\}$ and $\mathcal{A}'_p = \begin{pmatrix} 0 & 1 & 2 & \dots & p \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}$. Note that $\mathbb{L}(\mathcal{A}_p^{(1)}) = \mathbb{L}(\tilde{\mathcal{A}}_p^{(1)}) \times \{0\} = \mathbb{L}(\mathcal{A}'_p) \times \{0\}$. The Γ-series

$$\Phi_{\mathcal{A}_p^{(1)},\boldsymbol{\gamma}}(z_1, \dots, z_{p+2}) = \sum_{\boldsymbol{l} \in \mathbb{L}(\mathcal{A}_p^{(1)})} \frac{z_1^{l_1+\gamma_1} \cdots z_{p+2}^{l_{p+2}+\gamma_{p+2}}}{\Gamma(1+l_1+\gamma_1) \cdots \Gamma(1+l_{p+2}+\gamma_{p+2})}$$

is a solution of $H_{\mathcal{A}_p^{(1)}}(\boldsymbol{\beta})$ for all $\boldsymbol{\gamma} \in \mathbb{C}^{p+2}$ such that $\mathcal{A}_p^{(1)}\boldsymbol{\gamma} = \boldsymbol{\beta}$. This series is equal to $\frac{z_{p+2}^{\gamma_{p+2}}}{\Gamma(1+\gamma_{p+2})} \Phi_{\tilde{\mathcal{A}}_p^{(1)},(\gamma_1,\dots,\gamma_{p+1})}(z_1, \dots, z_{p+1})$. Hence $H_{\mathcal{A}_p^{(1)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $H_{\tilde{\mathcal{A}}_p^{(1)}}(\beta_1, 0, \beta_3 - \beta_2)$ has irreducible algebraic solutions. This is equivalent to the condition that $H_{\mathcal{A}'_p}(\beta_1, \beta_3 - \beta_2)$ has irreducible algebraic solutions. It is clear that $\mathcal{A}'_p$ is isomorphic to the set $\mathcal{A}$ of Section 3.1, with a isomorphism given by $f(x,y) = (x+y, y)$. Hence $H_{\mathcal{A}_p^{(1)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $\boldsymbol{\beta} = (\beta'_1 + \beta'_2, s, \beta'_2 + s)$ and $\boldsymbol{\beta}'$ gives irreducible algebraic solutions for the collinear point configuration. The results of Section 3.1 now give the parameters of Table 3.2. □

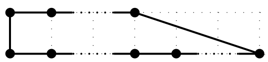
Table 3.2: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_p^{(1)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions

p	$\boldsymbol{\beta}$		
1	All non-resonant functions are algebraic		
2	(r,s,s)	$(r,s,r+s+\frac{1}{2})$	
	$(\frac{1}{2},r,r+\frac{1}{6})$	$(\frac{1}{10},r,r+\frac{9}{10})$	$(\frac{1}{30},r,r+\frac{5}{6})$
	$(\frac{1}{6},r,r+\frac{5}{6})$	$(\frac{1}{12},r,r+\frac{3}{4})$	$(\frac{1}{30},r,r+\frac{7}{10})$
	$(\frac{1}{10},r,r+\frac{7}{10})$	$(\frac{1}{12},r,r+\frac{5}{6})$	with $r,s\notin\mathbb{Z}$
3	(r,s,s)		
	$(\frac{1}{3},r,r+\frac{5}{6})$	$(\frac{1}{6},r,r+\frac{5}{6})$	with $r,s\notin\mathbb{Z}$
4	(r,s,s)	$(\frac{1}{6},r,r+\frac{5}{6})$	with $r,s\notin\mathbb{Z}$
≥ 5	(r,s,s)	with $r,s\notin\mathbb{Z}$	



Lemma 3.3.2. *Let* $\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then there are no* $\boldsymbol{\beta}$ *such that* $H_{\mathcal{A}_1}(\boldsymbol{\beta})$ *has irreducible algebraic solutions.*

Proof. One can easily show that there are never four apexpoints. □



Lemma 3.3.3. *Let*

$$\mathcal{A}_{p,q}^{(2)} = \begin{pmatrix} 0 & 1 & 2 & \dots & p & 0 & 1 & 2 & \dots & q \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

with $p \geq q$. *Then* $H_{\mathcal{A}_{p,q}^{(2)}}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to conjugation and equivalence modulo* $\mathbb{Z}$, $\boldsymbol{\beta}$ *is one of the tuples in Table 3.3.*

Proof. Note that $\mathcal{A}_{1,1}^{(2)}$ is isomorphic to the set $\mathcal{A}$ for the Gauss function ${}_2F_1$. The parameters such that ${}_2F_1$ is irreducible and algebraic are well-known and can for example be found in [Sch73] or Table 1.2. In Section 2.2, the irreducible algebraic Horn G_1 functions are determined. For this function, the set $\mathcal{A}$ is isomorphic to $\mathcal{A}_{2,1}^{(2)}$. This gives the first two cases in Table 3.3.

Let $(p,q) = (2,2)$. The interlacing condition is $(\lfloor -\beta_1+2\beta_3 \rfloor, \lfloor -\beta_2+\beta_3 \rfloor) \in \{(-1,0), (1,-1)\}$. It is clear that $\mathcal{A}_{2,1}^{(2)}$ is included in $\mathcal{A}_{2,2}^{(2)}$. Hence we only have to check the

Table 3.3: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}^{(2)}_{p,q}}(\boldsymbol{\beta})$ has irreducible algebraic solutions

(p,q)	$\boldsymbol{\beta}$				
$(1,1)$	Up to permutations of $\{\beta_1,\beta_2\}$:				
	$(r,-r,\frac{1}{2})$, $(r,r+\frac{1}{2},\frac{1}{2})$ and $(r,r+\frac{1}{2},-2r)$ with $2r\notin\mathbb{Z}$				
	$(\frac{1}{2},\frac{1}{6},\frac{2}{3})$	$(\frac{1}{6},\frac{5}{12},\frac{3}{4})$	$(\frac{1}{12},\frac{5}{12},\frac{3}{4})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2})$	$(\frac{1}{30},\frac{11}{30},\frac{4}{5})$
	$(\frac{1}{4},\frac{3}{4},\frac{1}{3})$	$(\frac{1}{6},\frac{11}{30},\frac{2}{3})$	$(\frac{1}{12},\frac{7}{12},\frac{2}{3})$	$(\frac{1}{20},\frac{13}{20},\frac{4}{5})$	$(\frac{1}{30},\frac{19}{30},\frac{2}{3})$
	$(\frac{1}{4},\frac{7}{12},\frac{1}{2})$	$(\frac{1}{6},\frac{11}{30},\frac{4}{5})$	$(\frac{1}{15},\frac{7}{15},\frac{2}{3})$	$(\frac{1}{24},\frac{13}{24},\frac{2}{3})$	$(\frac{1}{60},\frac{31}{60},\frac{2}{3})$
	$(\frac{1}{4},\frac{7}{12},\frac{2}{3})$	$(\frac{1}{10},\frac{3}{10},\frac{4}{5})$	$(\frac{1}{15},\frac{7}{15},\frac{4}{5})$	$(\frac{1}{24},\frac{13}{24},\frac{3}{4})$	$(\frac{1}{60},\frac{31}{60},\frac{4}{5})$
	$(\frac{1}{6},\frac{5}{6},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{3})$	$(\frac{1}{15},\frac{11}{15},\frac{2}{5})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2})$
	$(\frac{1}{6},\frac{5}{6},\frac{1}{4})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{4}{5})$	$(\frac{1}{24},\frac{17}{24},\frac{3}{4})$	$(\frac{1}{60},\frac{41}{60},\frac{4}{5})$
	$(\frac{1}{6},\frac{5}{6},\frac{1}{5})$	$(\frac{1}{10},\frac{13}{30},\frac{2}{3})$	$(\frac{1}{20},\frac{11}{20},\frac{3}{5})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2})$
	$(\frac{1}{6},\frac{5}{12},\frac{2}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{4}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{4}{5})$	$(\frac{1}{24},\frac{19}{24},\frac{2}{3})$	$(\frac{1}{60},\frac{49}{60},\frac{2}{3})$
$(2,1)$	$(\frac{1}{3},\frac{5}{6},\frac{2}{3})$	$(\frac{1}{6},\frac{2}{3},\frac{1}{2})$	$(\frac{1}{6},\frac{5}{6},\frac{2}{3})$		
$(2,2)$	$(\frac{1}{6},\frac{5}{6},\frac{2}{3})$				
$(3,1)$	$(\frac{1}{6},\frac{5}{6},\frac{2}{3})$				

interlacing condition for $\pm(\frac{1}{3},\frac{5}{6},\frac{2}{3})$, $\pm(\frac{1}{6},\frac{2}{3},\frac{1}{2})$ and $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$. It turns out that this condition is satisfied only for $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$.

For $p=3$ and $q=1$, we again have the inclusion $\mathcal{A}^{(2)}_{2,1}\subseteq\mathcal{A}^{(2)}_{3,1}$. The interlacing condition is $(\lfloor-\beta_1-2\beta_2+3\beta_3\rfloor,\lfloor-\beta_2+\beta_3\rfloor)\in\{(-1,0),(0,-1)\}$. Checking $\pm(\frac{1}{3},\frac{5}{6},\frac{2}{3})$, $\pm(\frac{1}{6},\frac{2}{3},\frac{1}{2})$ and $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$ shows that only $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$ gives an irreducible algebraic function.

For $p=3$ and $q=2$, the interlacing condition is $(\lfloor-\beta_1-\beta_2+3\beta_3\rfloor,\lfloor-\beta_2+\beta_3\rfloor)\in\{(-1,0),(1,-1)\}$. Due to the inclusion $\mathcal{A}^{(2)}_{3,1}\subseteq\mathcal{A}^{(2)}_{3,2}$, we only have to check this condition for $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$. This tuple doesn't satisfy the condition, so there are no irreducible algebraic functions. It follows immediately from this that there are also no irreducible algebraic functions for any $\mathcal{A}^{(2)}_{p,q}$ with $p\geq 3$ and $q\geq 2$.

This leaves us with the case $p\geq 4$ and $q=1$. Let $p=4$. The interlacing condition is $(\lfloor-\beta_1-3\beta_2+4\beta_3\rfloor,\lfloor-\beta_2+\beta_3\rfloor)\in\{(-1,0),(0,-1)\}$ and we only have to check $\pm(\frac{1}{6},\frac{5}{6},\frac{2}{3})$. Again there are no irreducible algebraic functions. Hence there are also no irreducible algebraic functions with $p>4$ and $q=1$. $\square$

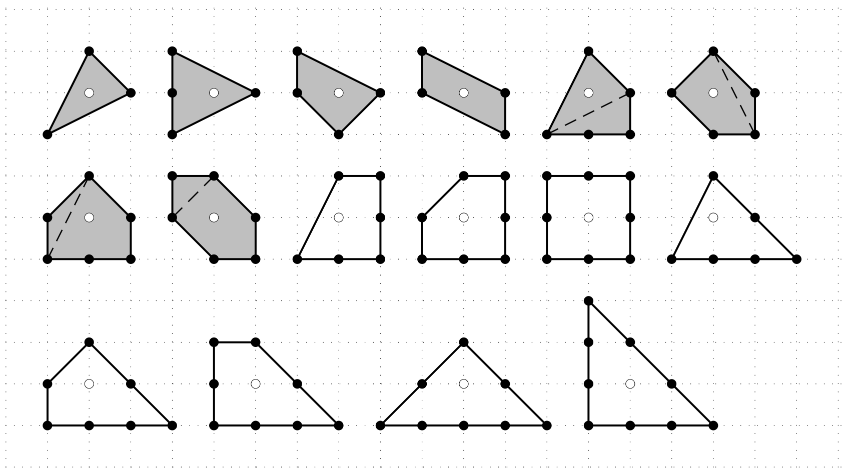
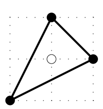


Figure 3.4: The polygons with exactly one interior point

Polygons with exactly one interior point

In [Sco76] Scott proves that there exists a lattice polygon with i interior points and b boundary points if and only if either $i = 0$, or $i = 1$ and $3 \leq b \leq 9$, or $i \geq 2$ and $3 \leq b \leq 2i + 6$. Furthermore, Theorem 2 in [LZ91] states that a polygon of normalized area V is, up to isomorphism, contained in a square of side length $2V$. Since the number of interior and boundary points determine the area by Pick's formula, this implies that there are only finitely many non-isomorphic polygons with a given number of interior and boundary points. Polygons with exactly one interior point have 3 to 9 boundary points. A classification of these can be found in both [Rab89] and [PRV00]. There are 16 isomorphism classes. They are shown in Figure 3.4.

For each of these, we can compute the interlacing condition. Then we compute the irreducible algebraic functions by using a reduction to a polygon for which we know the algebraic functions already, as in Corollary 3.2.6. For the non-shaded polygons in Figure 3.4, one easily computes that there exists no $\boldsymbol{\beta} \in \mathbb{Q}^3$ such that $\sigma_{\mathcal{A}}(\boldsymbol{\beta}) = \mathrm{Vol}(Q(\mathcal{A}))$. Hence $H_{\mathcal{A}}(\boldsymbol{\beta})$ never has irreducible algebraic solutions. We now consider the shaded polygons one by one. We denote the corresponding sets $\mathcal{A}$ by $\mathcal{A}_2$ up to $\mathcal{A}_9$.



Lemma 3.3.4. *Let* $\mathcal{A}_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_2}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to conjugation and equivalence modulo* $\mathbb{Z}$, $\boldsymbol{\beta}$ *is one of the tuples in Table 3.4.*

Table 3.4: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_2}(\boldsymbol{\beta})$ has irreducible algebraic solutions

$(\frac{1}{3}, \frac{2}{3}, r)$ with $r \notin \mathbb{Z}$				
$(r, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, r, \frac{1}{2})$ and $(r, r+\frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$				
$(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{4}, \frac{3}{4}, \frac{1}{3})$	$(\frac{1}{5}, \frac{4}{5}, \frac{1}{2})$	$(\frac{1}{7}, \frac{3}{7}, \frac{1}{2})$
$(\frac{1}{2}, \frac{1}{4}, \frac{1}{3})$	$(\frac{1}{3}, \frac{5}{6}, \frac{1}{3})$	$(\frac{1}{5}, \frac{2}{5}, \frac{1}{2})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{7}, \frac{5}{7}, \frac{1}{2})$
$(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{2})$	$(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$	

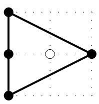
Table 3.5: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_3}(\boldsymbol{\beta})$ has irreducible algebraic solutions

$(0, \frac{1}{2}, r)$ with $r \notin \mathbb{Z}$					
$(r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin 2\mathbb{Z}+1$					
$(0, \frac{1}{3}, \frac{1}{2})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{10}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{10}, \frac{1}{2}, \frac{4}{5})$
$(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$	$(\frac{1}{10}, \frac{1}{2}, \frac{1}{5})$	

Proof. The lattice is given by $\mathbb{L} = \mathbb{Z}(1, -3, 1, 1)$. The Γ-series

$$\Phi(z_1, z_2, z_3, z_4) = \sum_{n \in \mathbb{Z}} \frac{z_1^{n+\gamma_1} z_2^{-3n+\gamma_2} z_3^{n+\gamma_3} z_4^{n+\gamma_4}}{\Gamma(1+n+\gamma_1)\Gamma(1-3n+\gamma_2)\Gamma(1+n+\gamma_3)\Gamma(1+n+\gamma_4)}$$

is a formal solution of $H_{\mathcal{A}_2}(\boldsymbol{\beta})$ whenever $\mathcal{A}_2\boldsymbol{\gamma} = \boldsymbol{\beta}$. Choosing $\boldsymbol{\gamma} = (-\beta_2, -\beta_1 + 2\beta_2 + \beta_3, \beta_1 - \beta_2, 0)$ gives a convergent solution. Note that Φ is irreducible and algebraic if and only if the higher hypergeometric function ${}_3F_2(\frac{-\gamma_2}{3}, \frac{-\gamma_2+1}{3}, \frac{-\gamma_2+2}{3}; \gamma_1+1, \gamma_3+1|z)$ is irreducible and algebraic. The algebraic higher hypergeometric functions have been determined by Beukers and Heckman [BH89]. From the irreducible algebraic ${}_3F_2$ functions in [BH89], we select the functions whose first 3 parameters differ by $\frac{1}{3}$ and compute $\boldsymbol{\beta} = (-\gamma_1 + \gamma_3, -\gamma_1, \gamma_1 + \gamma_2 + \gamma_3)$. This gives the tuples in Table 3.4. $\square$



Lemma 3.3.5. *Let* $\mathcal{A}_3 = \begin{pmatrix} -1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_3}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to conjugation and equivalence modulo* $\mathbb{Z}$, $\boldsymbol{\beta}$ *is one of the tuples in Table 3.5.*

Proof. With $\mathbb{L} = \mathbb{Z}(0, 1, -2, 1, 0) \oplus \mathbb{Z}(1, -2, 0, 0, 1)$ and $\boldsymbol{\gamma} = (-\beta_2, -\beta_1 + \beta_2, \beta_1 + \beta_3, 0, 0)$, we get the Γ-series

$$\Phi(\boldsymbol{z}) = \sum_{m,n \geq 0} \frac{z_1^{n-\beta_2} z_2^{m-2n-\beta_1+\beta_2} z_3^{-2m+\beta_1+\beta_3} z_4^m z_5^n}{\Gamma(1+n-\beta_2)\Gamma(1+m-2n-\beta_1+\beta_2)\Gamma(1-2m+\beta_1+\beta_3)m!n!},$$

which is irreducible and algebraic if and only if

$$\Psi(x,y)=\sum_{m,n\geq 0}\frac{(\beta_1-\beta_2)_{-m+2n}(-\beta_1-\beta_3)_{2m}}{(-\beta_2+1)_n m!n!}x^m y^n$$

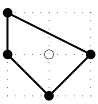
is irreducible and algebraic. If this happens, then $\Psi_x(x) := \Psi(x,0)$ and $\Phi_y(y) := \Psi(0,y)$ are also algebraic. Note that $\Psi_x(x) = {}_2F_1(\frac{-\beta_1-\beta_3}{2}, \frac{-\beta_1-\beta_3+1}{2}, -\beta_1+\beta_2+1|4x)$ and $\Psi_y(y) = {}_2F_1(\frac{\beta_1-\beta_2}{2}, \frac{\beta_1-\beta_2+1}{2}, -\beta_2+1|4y)$.

$H_{\mathcal{A}_3}(\boldsymbol{\beta})$ is irreducible if and only if $-\beta_1-2\beta_2+\beta_3, -\beta_1+2\beta_2+\beta_3, \beta_1+\beta_3 \notin \mathbb{Z}$. As ${}_2F_1(\alpha_1,\alpha_2,\alpha_3|z)$ is irreducible if and only if $\alpha_1,\alpha_2,\alpha_1-\alpha_3,\alpha_2-\alpha_3 \notin \mathbb{Z}$, irreducibility of $H_{\mathcal{A}_2}(\boldsymbol{\beta})$ implies irreducibility of Φ_x. However, Φ_y is reducible if $\beta_1 \pm \beta_2 \in \mathbb{Z}$. But in this case, $H_{\mathcal{A}_3}(\boldsymbol{\beta})$ is not totally non-resonant, so this doesn't give any algebraic functions. Hence if $H_{\mathcal{A}_3}(\boldsymbol{\beta})$ has irreducible and algebraic solutions, then both Ψ_x and Ψ_y are irreducible and algebraic.

The tuples $(\alpha_1,\alpha_2,\alpha_3)$ such that ${}_2F_1(\alpha_1,\alpha_2,\alpha_3|z)$ is irreducible and algebraic can be computed from Table 3.3, $p=q=1$, or can be found in [Sch73] or Table 1.2. We select the pairs $(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})$ of tuples satisfying $\alpha_2^{(i)} \equiv \alpha_1^{(i)} + \frac{1}{2} \pmod{\mathbb{Z}}$ and $\alpha_3^{(1)} + 2\alpha_1^{(2)} \in \mathbb{Z}$, and compute the corresponding $\boldsymbol{\beta}$. Then we check the interlacing condition for $\mathcal{A}_3$. This is given by $(\lfloor -2\beta_1-\beta_2+\beta_3\rfloor, \lfloor \beta_2+\beta_3\rfloor, \lfloor 2\beta_1-\beta_2+\beta_3\rfloor) \in \{(-2,1,0),(-1,0,1)\}$.

The triples $\boldsymbol{\alpha}^{(i)}$ can be either of the form $(r, r+\frac{1}{2}, \frac{1}{2})$ or $(r, r+\frac{1}{2}, 2r)$, or can be one of the other 408 triples for which the Gauss function is irreducible and algebraic. Hence there are several cases to check for $(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})$. In most cases, one can easily show that there are only finitely many possibilities for the parameter(s), using the fact that $\alpha_3^{(1)} + 2\alpha_1^{(2)} \in \mathbb{Z}$. We only discuss the case in which $\boldsymbol{\alpha}^{(1)} = (r, r+\frac{1}{2}, \frac{1}{2})$ and $\boldsymbol{\alpha}^{(2)}$ is one of the other 408 triples. Since $\alpha_3^{(1)} + 2\alpha_1^{(2)} \in \mathbb{Z}$, we have $\alpha_1^{(2)} = \pm\frac{1}{4}$ and hence $\alpha_2^{(2)} = \mp\frac{1}{4}$. Checking all 408 triples, we get $\alpha_3^{(2)} = \pm\frac{1}{3}$. Hence we find $\boldsymbol{\beta} = \pm(\frac{1}{6}, \frac{2}{3}, s)$ for some parameter s. This satisfies the interlacing condition if and only if $\frac{1}{2} \leq s < \frac{5}{6}$ (for $\boldsymbol{\beta} = (\frac{1}{6}, \frac{2}{3}, s)$) or $\frac{1}{6} \leq s < \frac{1}{2}$ (for $\boldsymbol{\beta} = (\frac{5}{6}, \frac{1}{3}, s)$). By Lemma 1.4.10, it follows that the denominator of s must divide 6, and we easily find $\boldsymbol{\beta} = (\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.

Checking all cases, we find the 2 families and 42 other tuples in Table 3.5. ☐



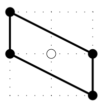
Lemma 3.3.6. *Let* $\mathcal{A}_4 = \begin{pmatrix} 0 & -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_4}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if* $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$ *with* $2r \notin \mathbb{Z}$ *or, up to conjugation and equivalence modulo* $\mathbb{Z}$, $\boldsymbol{\beta}$ *equals one of the tuples* $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$, $(\frac{1}{5}, \frac{3}{5}, \frac{1}{2})$ *and* $(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$.

Proof. The parameters such that the Horn H_5 function is irreducible and algebraic are determined in Section 2.3. H_5 is given by $\mathcal{A}_{H_5} = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$. The map $f(x,y,z) = (-y,-z,x+y+z)$ is an isomorphism of $\mathbb{Z}^3$, mapping $\mathcal{A}_{H_5}$ to $\mathcal{A}_4$.

Table 3.6: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_5}(\boldsymbol{\beta})$ has irreducible algebraic solutions

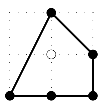
$(0, \frac{1}{2}, r)$, $(r, \frac{1}{2} - r, \frac{1}{2})$, with $r \notin \mathbb{Z}$					
$(r, \frac{1}{2}, \frac{1}{2})$, with $2r \notin 2\mathbb{Z}+1$					
$(0, \frac{1}{3}, \frac{1}{2})$	$(\frac{1}{5}, \frac{1}{5}, \frac{1}{2})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{6}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{10}, \frac{1}{2}, \frac{4}{5})$	$(\frac{1}{12}, \frac{2}{3}, \frac{1}{2})$
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{2})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{6}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{10}, \frac{2}{5}, \frac{1}{3})$	$(\frac{1}{12}, \frac{1}{4}, \frac{1}{2})$
$(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{10}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{10}, \frac{2}{5}, \frac{1}{5})$	$(\frac{1}{15}, \frac{1}{3}, \frac{1}{2})$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{3})$	$(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$	$(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{10}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{10}, \frac{2}{5}, \frac{4}{5})$	$(\frac{1}{15}, \frac{3}{5}, \frac{1}{2})$

Hence $H_{\mathcal{A}_4}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $H_{\mathcal{A}_{H_5}}(f^{-1}(\boldsymbol{\beta}))$ has irreducible algebraic solutions. ☐



Lemma 3.3.7. *Let* $\mathcal{A}_5 = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_5}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to conjugation and equivalence modulo* $\mathbb{Z}$, $\boldsymbol{\beta}$ *is one of the tuples in Table 3.6.*

Proof. We have $\mathbb{L} = \mathbb{Z}(0,1,-2,1,0) \oplus \mathbb{Z}(1,0,-2,0,1)$. Similar to the proof of Lemma 3.3.4, one checks that the Γ-series with $\boldsymbol{\gamma} = (-\beta_2, -\beta_1-\beta_2, \beta_1+2\beta_2+\beta_3, 0, 0)$ is irreducible and algebraic if and only if the Appell F_4 function with parameters $\frac{-\beta_1-2\beta_2-\beta_3}{2}, \frac{-\beta_1-2\beta_2-\beta_3+1}{2}, 1-\beta_1-\beta_2, 1-\beta_2|x,y)$ is irreducible and algebraic. The tuples (a,b,c_1,c_2) such that $F_4(a,b,c_1,c_2|x,y)$ is irreducible and algebraic can be found in Section 2.1. We select the tuples satisfying $a-b \equiv \frac{1}{2} \pmod{\mathbb{Z}}$ and compute the corresponding $\boldsymbol{\beta} = (c_2-c_1, -c_2, -2a+c_1+c_2)$. ☐



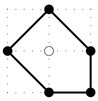
Lemma 3.3.8. *Let* $\mathcal{A}_6 = \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then there are no* $\boldsymbol{\beta}$ *such that* $H_{\mathcal{A}_6}(\boldsymbol{\beta})$ *has irreducible algebraic solutions.*

Proof. Note that $\mathcal{A}_2$ is a subset of $\mathcal{A}_6$, as is also shown in Figure 3.4. It follows from Corollary 3.2.6 that all $\boldsymbol{\beta}$ for which the number of apexpoints is maximal for all conjugates must also be listed in Table 3.4. The interlacing condition for $\mathcal{A}_6$ is given by

$$(\lfloor -\beta_1-\beta_2+\beta_3 \rfloor, \lfloor -\beta_1+\beta_3 \rfloor, \lfloor \beta_2+\beta_3 \rfloor, \lfloor 2\beta_1-\beta_2+\beta_3 \rfloor) \in \{(-1,0,0,1),(-1,-1,1,0)\}.$$

The families for $\mathcal{A}_2$ never satisfy the interlacing condition, except for $(\frac{1}{2}, r, \frac{1}{2})$, which satisfies this condition if $0 < r < \frac{1}{2}$. However, not all conjugates satisfy this condition.

Furthermore, one easily checks that none of the other 48 tuples for $\mathcal{A}_2$ gives $\boldsymbol{\beta}$ such that all conjugates satisfy the interlacing condition for $\mathcal{A}_6$. $\square$

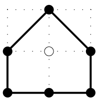


Lemma 3.3.9. *Let* $\mathcal{A}_7 = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_7}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to equivalence modulo* $\mathbb{Z}$*, we have* $\boldsymbol{\beta} = \pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$.

Proof. The proof is similar to the proof of Lemma 3.3.8. In this case, $f(\mathcal{A}_4)$ is a subset of $\mathcal{A}_7$, with $f(x, y, z) = (y, x, z)$. Hence we only have to check the interlacing condition for $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_4}(f^{-1}(\boldsymbol{\beta}))$ is irreducible and has algebraic solutions. The interlacing condition is

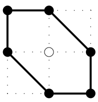
$$(\lfloor -\beta_1 - \beta_2 + \beta_3 \rfloor, \lfloor -\beta_1 + \beta_3 \rfloor, \lfloor \beta_2 + \beta_3 \rfloor, \lfloor \beta_1 - \beta_2 + \beta_3 \rfloor, \lfloor \beta_1 + \beta_2 + \beta_3 \rfloor) \in \\ \{(-1, -1, 1, 0, 1), (-1, 0, 0, 0, 1)\}.$$

One easily checks that $(\frac{1}{2}, r, \frac{1}{2})$ only satisfies this condition for $0 < r < \frac{1}{2}$, so there are no r such that all conjugates satisfy the condition. Furthermore, the only $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_4}(\boldsymbol{\beta})$ is irreducible and has algebraic solutions that satisfies the interlacing condition for $\mathcal{A}_7$ is $\boldsymbol{\beta} = \pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$. $\square$



Lemma 3.3.10. *Let* $\mathcal{A}_8 = \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then there are no* $\boldsymbol{\beta}$ *such that* $H_{\mathcal{A}_8}(\boldsymbol{\beta})$ *has irreducible algebraic solutions.*

Proof. This follows immediately from Corollary 3.2.6, using the inclusion $\mathcal{A}_6 \subseteq \mathcal{A}_8$. $\square$



Lemma 3.3.11. *Let* $\mathcal{A}_9 = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_9}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to conjugation and equivalence modulo* $\mathbb{Z}$*, we have* $\boldsymbol{\beta} = \pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$.

Proof. Since $\mathcal{A}_7 \subseteq \mathcal{A}_9$, it suffices to check the interlacing condition for all $\boldsymbol{\beta}$ such that $\sigma_{A_7}(k\boldsymbol{\beta})$ is maximal for all k, i.e., $\boldsymbol{\beta} = \pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$. The interlacing condition is

$$(\lfloor -\beta_1 - \beta_2 + \beta_3 \rfloor, \lfloor -\beta_1 + \beta_3 \rfloor, \lfloor -\beta_2 + \beta_3 \rfloor, \lfloor \beta_2 + \beta_3 \rfloor, \lfloor \beta_1 + \beta_3 \rfloor, \lfloor \beta_1 + \beta_2 + \beta_3 \rfloor) \in \\ \{(-1, -1, -1, 1, 1, 1), (-1, 0, 0, 0, 0, 1)\}.$$

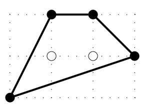
It is easy to see that $\pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ satisfies this condition and is non-resonant for $\mathcal{A}_9$. $\square$

Polygons with exactly two interior points

Polygons with two interior points have 3 up to 10 boundary points. This implies that their area lies between 5 and 12. Furthermore, up to isomorphism each polygon lies in a square whose sides have length 24. By shifting the polygon, we can assume that the lower left corner of the square is the origin. By applying a translation, we can assume that each polygon has a vertex on each of the lower and left sides of the square. This makes it feasible to compute all these polygons.

After computing all polygons in this square, we compute the different isomorphism classes. Note that the number of vertices, the number of interior points and the discrete lengths of the sides are invariant under isomorphisms. By the *discrete length* of an edge we mean the number of lattice points on this edge minus 1. Hence for each pair of polygons, we can first check whether these invariants are the same. If they do, we check whether there exists an isomorphism. This can be done by computing the functions that map sets of three lattice points of the first polygon to a fixed set of three lattice points of the second polygon. It turns out that there are 45 isomorphism classes of polygons with two interior points. They are shown in Figure 3.5.

As in the previous section, we will compute all parameters such that the corresponding functions are algebraic. For the 38 non-shaded polygons, a set isomorphic to $\mathcal{A}_6$ is included in $\mathcal{A}$. These inclusions are also drawn in Figure 3.5. In all but three cases, this subset of $\mathcal{A}$ is a translation or reflection of $\mathcal{A}_6$. For the remaining three polygons, it might not be immediately clear that the indicated subset is indeed isomorphic to $\mathcal{A}_6$. In these cases, note that $\mathcal{A}_6$ is the only convex lattice polygon with one interior point, area 5 and an edge of length 2. This leaves us with the seven shaded polygons. Five of these are elements of one of the two families of polygons with algebraic functions, which will be discussed in Section 3.5. The second and the seventh shaded polygon aren't elements of such families. For the second polygon, the inclusion of a set isomorphic to $\mathcal{A}_4$ gives us a proof of the following lemma, similar to the proof of Lemma 3.3.8:



Lemma 3.3.12. *Let* $\mathcal{A}_{10} = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_{10}}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if, up to equivalence modulo* $\mathbb{Z}$, *we have* $\boldsymbol{\beta} = \pm(\frac{1}{3}, \frac{5}{6}, \frac{1}{2})$.

Similarly, again using a inclusion of $\mathcal{A}_4$, one can show that there are no irreducible algebraic functions for the seventh shaded polygon.

3.4 Polygons with at least 3 interior points and at least 5 boundary points

In this section, we will show that the hypergeometric functions associated to polygons with at least 3 interior points and at least 5 boundary points are never irreducible and algebraic. We do this by proving that such polygons contain subpolygons with 1

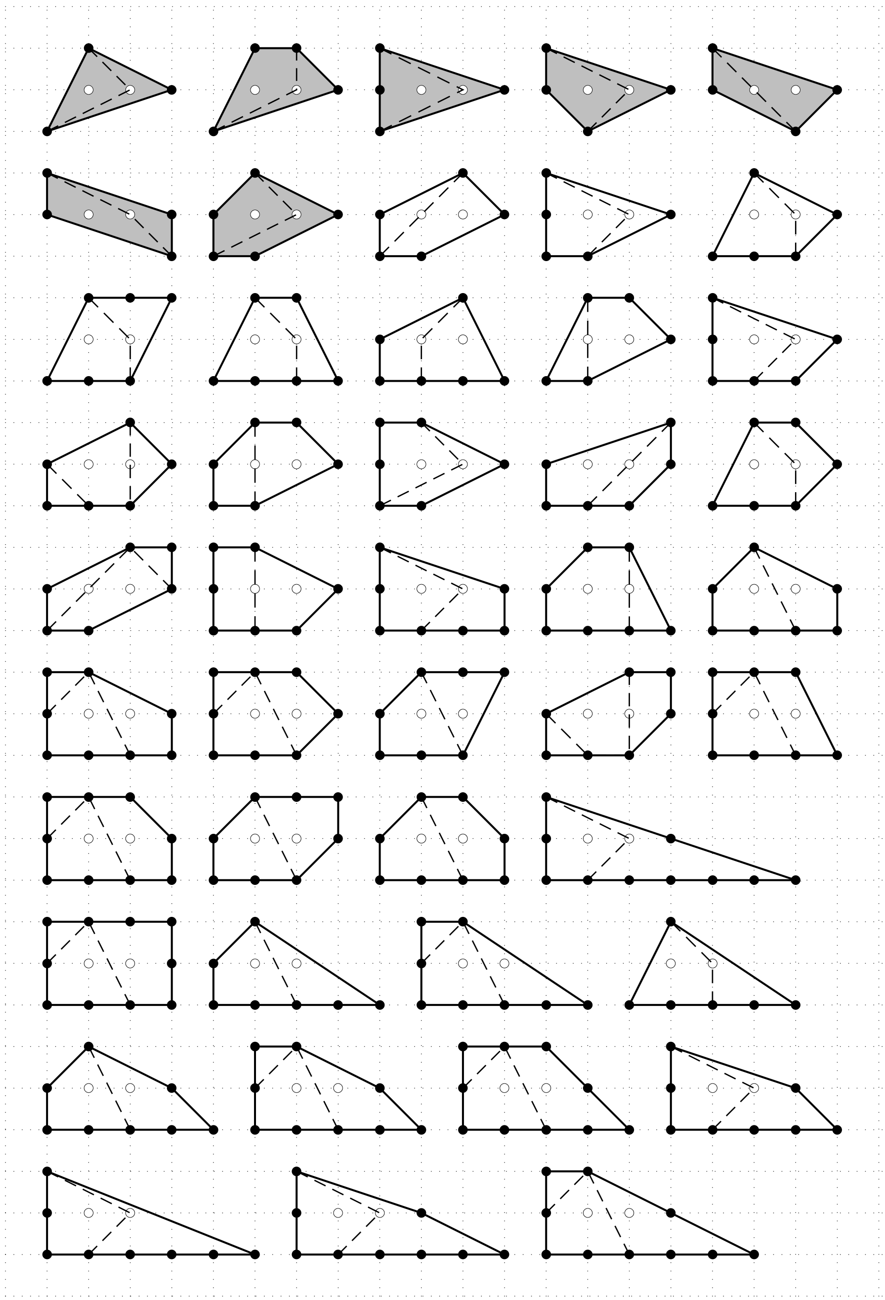


Figure 3.5: The polygons with exactly 2 interior points

or 2 boundary points without irreducible algebraic functions. Polygons with exactly 3 or 4 boundary points will be treated in the next section.

Definition 3.4.1. We say that P has *type* (i, b) if P has i interior points and b boundary points. We write $t(P) = (i, b)$. If $P = P(\mathcal{A})$, then we also call $\mathcal{A}$ of *type* (i, b) and write $t(\mathcal{A}) = (i, b)$.

Definition 3.4.2. Let $T = \{(i, b) \in \mathbb{N}^2 \mid i = 1, b \geq 7 \text{ or } i \geq 2, b \geq 5\}$ and $S = \{(i, b) \in T \mid i = 1 \text{ or } i = 2\}$. Let $\prec$ be the lexicographical ordering on T and S.

Remark 3.4.3. Note that $t(P) \in T$ if and only if P has at least one interior point, at least 5 boundary points and normalized area at least 7. If P and P' are convex lattice polygons with $P' \subsetneq P$, then $t(P') \prec t(P)$.

Lemma 3.4.4. *Let P be a convex lattice polygon with $t(P) \in T \setminus S$. Then there exists a convex lattice polygon $P' \subsetneq P$ of type $t(P') \in T$.*

Corollary 3.4.5. *Let P be a convex lattice polygon with $t(P) \in T$. Then there exists a convex lattice polygon $P' \subseteq P$ of type $t(P') \in S$.*

Proof. If $t(P) \in S$, then we can take $P' = P$. Otherwise, we repeatedly apply Lemma 3.4.4 to find a series of subpolygons of decreasing type, eventually leading to a subpolygon with type in S. $\square$

To prove Lemma 3.4.4, we consider 6 cases: polygons of type (3,5); triangles, quadrilaterals and pentagons with at least 4 interior points and exactly 5 boundary points; triangles with at least 3 interior points and at least 6 boundary points and 2 edges of length 1; and other polygons with at least 3 interior points and at least 6 boundary points.

Lemma 3.4.6. *If P is a lattice polygon with exactly 3 interior points and 5 boundary points, then there is a subpolygon $P' \subseteq P$ with 2 interior points and 5 boundary points.*

Proof. There are 12 convex lattice polygons with 3 interior lattice points and 5 lattice points on the boundary. They are shown in Figure 3.4. For each of these polygons, a subpolygon with 2 interior points and 5 boundary points is indicated. $\square$

Lemma 3.4.7. *Suppose that P is a lattice triangle of type $(i, 5)$ with $i \geq 4$. Then there exists a lattice polygon $P' \subsetneq P$ with $t(P') \in T$.*

Proof. Suppose that P doesn't have an edge of discrete length 3. Then the lengths of the edges are 1, 2 and 2. However, if the lattice points on the boundary are $(0, 0)$, v_1, $v_2 = 2v_1$, v_3 and $v_4 = 2v_3$, then $v_1 + v_3$ is also a lattice point on the boundary. Hence P has an edge of discrete length 3, and the other two edges have length 1. After a suitable transformation, the lattice points on the boundary are $(0, 0)$, $(1, 0)$, $(2, 0)$, $(3, 0)$ and (c, d) for some $c, d \in \mathbb{Z}$ with $\gcd(c, d) = \gcd(c - 3, d) = 1$. We can assume

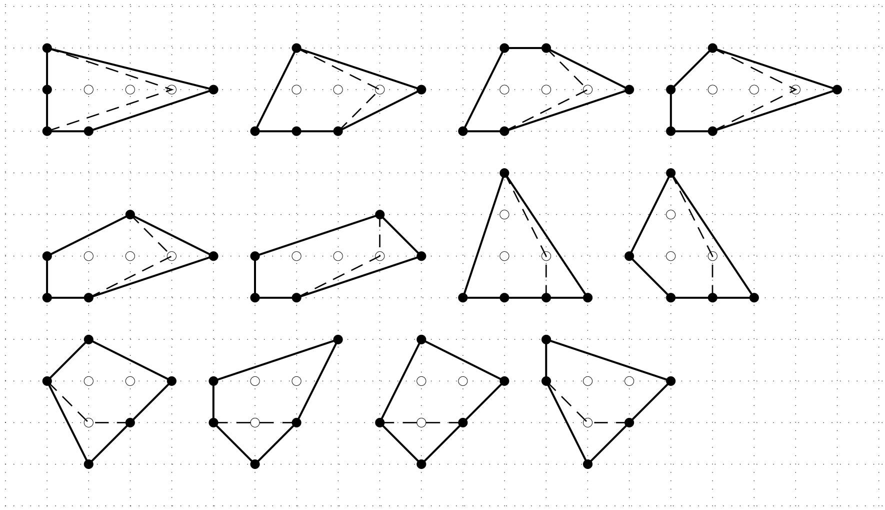


Figure 3.6: The lattice polygons with 3 interior points and 5 boundary points

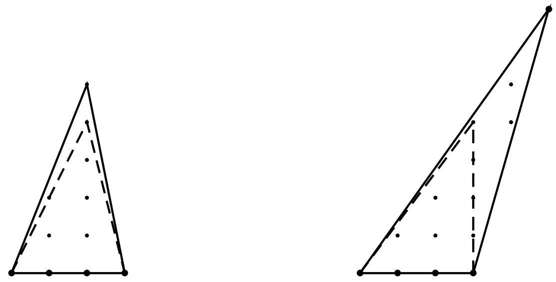
that $d > 0$ and $c \geq 2$. Furthermore, we can apply an isomorphism $(x, y) \mapsto (x + ny, y)$ with $n \in \mathbb{Z}$. This maps the basis $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$ to itself, and it maps (c, d) to $(c + nd, d)$. It is possible to choose n so that $0 \leq c + nd < d$. Hence we can assume that $0 \leq c < d$.

Suppose that $c = 2$. Then d is odd and the interior points are the points $(1, k)$ wih $1 \leq k \leq \frac{d-1}{2}$ and $(2, k)$ with $1 \leq k \leq d - 1$ (see Figure 3.7(a)). Let P' be the triangle with vertices $(0,0)$, $(3,0)$ and $(2, d-1)$. Then P' has 6 boundary points and $i - 2$ interior points, so $t(P') \in T$.

Note that $c \neq 3$, because $\gcd(c-3, d) = 1$. Suppose that $4 \leq c < d$. Then P is given by the inequalities $x_2 \geq 0$, $cx_2 \leq dx_1$ and $(c-3)x_2 \geq d(x_1 - 3)$, so $(3, \lfloor \frac{3d}{c} \rfloor)$ is an interior point (note that $c \nmid 3d$) (see Figure 3.7(b)). Let P' be the triangle with vertices $(0,0)$, $(3,0)$ and $(3, \lfloor \frac{3d}{c} \rfloor)$. Since $\frac{3d}{c} > 3$, the point $(2,1)$ is an interior point and the area of P' is at least 9. P' has $4 + \lfloor \frac{3d}{c} \rfloor \geq 5$ boundary points, so $t(P') \in T$ by Remark 3.4.3. $\square$

Lemma 3.4.8. *Suppose that P is a convex lattice quadrilateral of type $(i, 5)$ with $i \geq 4$. Then there exists a convex lattice polygon $P' \subsetneq P$ with $t(P') \in T$.*

Proof. A quadrilateral with exactly 5 lattice points on the boundary must have edges of discrete length 1, 1, 1 and 2. Let the vertices, in counterclockwise order, be $v_0, \ldots, v_4$. After a suitable transformation, we have $v_0 = (-1, 0)$, $v_1 = (0,0)$ and $v_2 = (1,0)$ and $v_{32}, v_{42} > 0$. Suppose that the triangles with vertices $(\pm 1, 0)$, v_3 and v_4 both have area 1. Then the triangles with vertices $(-1,0)$, $(1,0)$ and v_i with $i = 3, 4$ both have area $\mathrm{Vol}(P) - 1$ (see Figure 3.8(a)). This implies that $v_{32} = v_{42} = \frac{\mathrm{Vol}(P)-1}{2}$. The area of P is equal to $v_{32} + v_{42} + (v_{31}v_{42} - v_{41}v_{32})$ (by dissecting P with the dotted lines as in



(a) $c = 2$

(b) $4 \leq c < d$

Figure 3.7: Subpolygons of triangles

Figure 3.8(a)). Hence $(v_{31} - v_{41})v_{32} = 1$, so $v_{32} = 1$. But then $\mathrm{Vol}(P) = 2v_{32} + 1 = 3$, contradicting the assumption that P has at least 4 interior points.

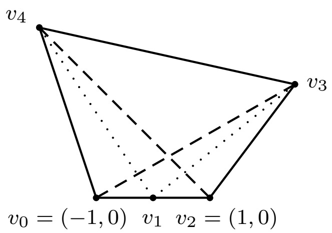
Hence we can assume that the triangle with vertices $(-1, 0)$, v_3 and v_4 has area at least 2. This implies that there is an interior point of P on or above the line from $(-1, 0)$ to v_3. Let P' be the convex hull of all lattice points in or on the boundary of P, except for v_4.

If P' has at least one interior point, then P' satisfies the conditions of Remark 3.4.3: P' contains $i + 4$ lattice points, so $\mathrm{Vol}(P) \geq 2 \cdot 1 + (i + 3) - 2 = i + 3 \geq 7$, and P' has at least 5 boundary points: v_0, v_1, v_2, v_3 and the interior point of P on or above the line from $(-1, 0)$ to v_3.

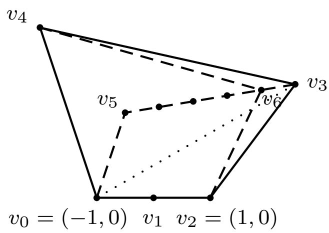
Suppose that P' has no interior lattice points. By [Rab89], the only polygons without interior points are (up to isomorphism) triangles with vertices $(0, 0), (p, 0)$ and $(0, 1)$, the triangles with vertices $(0, 0), (2, 0)$ and $(0, 2)$, and trapezoids with vertices $(0, 0), (p, 0), (q, 1)$ and $(0, 1)$. If P' is a triangle, then all interior points of P lie on the line from $(-1, 0)$ to v_3. Hence P' has edges of discrete length 1, 2 and $i + 1$. It follows that P' is a quadrilateral, and hence has two opposite edges of discrete length 1. Let v_5 be the fourth vertex. Then all interior points of P must be on the line from v_3 to v_5, as in Figure 3.8(b). Let v_6 be the vertex closest to v_3 and let P'' be the convex hull of v_0, v_2, v_6 and v_4. Then P'' has $i - 1$ interior points and 5 boundary points, so $t(P'') \in T$. □

Lemma 3.4.9. *Suppose that P is a convex lattice pentagon of type $(i, 5)$ with $i \geq 4$. Then there exists a convex lattice polygon $P' \subsetneq P$ with $t(P') \in T$.*

Proof. The proof is similar to the proof of Lemma 3.4.8. Let the vertices be $v_0, \ldots, v_4$ (in counterclockwise order). By applying a suitable transformation, we can assume that $v_0 = (0, 0)$ and $v_1 = (1, 0)$, and $v_{i2} \geq 1$ for $i = 2, 3, 4$. We claim that there exist vertices v_i and v_{i+2} such that an interior point of P lies on or at the side of v_{i+1} of the line from v_i to v_{i+2} (indices modulo 5). Suppose that such vertices do not exist. Then every triangle with vertices v_i, v_{i+1} and v_{i+2} contains no lattice points,



(a) Quadrilateral



(b) Quadrilateral with interior points on a line

Figure 3.8: A quadrilateral with five boundary points

except for these vertices, and hence has normalized area 1 (see Figure 3.9(a)). Hence $v_{22} = v_{42} = 1$, and

$$(v_{21} - 1)v_{32} - (v_{31} - 1)v_{22} = 1$$
$$(v_{31} - v_{21})(v_{42} - v_{22}) - (v_{41} - v_{21})(v_{32} - v_{22}) = 1$$
$$v_{31}v_{42} - v_{41}v_{32} = 1.$$

It follows that $(v_{41} - v_{21})(v_{32} - v_{22}) = -1$ and hence $v_{32} - 1 = \pm 1$. This implies $v_{32} = 2$, so the equations reduce to $2v_{21} - v_{31} = 2$, $v_{21} - v_{41} = 1$ and $v_{31} - 2v_{41} = 1$. These equations have no solution.

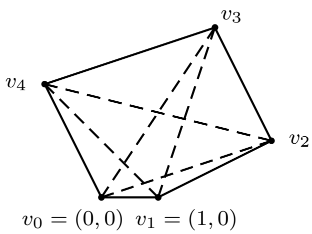
We can assume that there is an interior point of P on or above the line from v_2 to v_4. Let P' be the convex hull of all lattice points in P except for v_3. Then it is clear that $P' \subsetneq P$, so it remains to show that $t(P') \in T$.

Suppose that P' has an interior point. Since P' contains $i + 4$ lattice points, the area is at least $2 \cdot 1 + (i + 3) - 2 = i + 3 \geq 7$. Furthermore, P' has 5 boundary points: v_0, v_1, v_2, v_4 and the interior point of P on or above the line from v_2 to v_4. Hence $t(P') \in T$ by Remark 3.4.3.

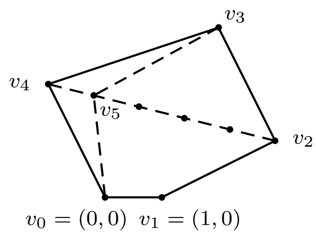
If P' doesn't have an interior point, then all interior points of P lie on the edges of P'. It is shown in [Rab89] that the only polygons without interior points are (up to isomorphism) triangles with vertices $(0, 0), (p, 0)$ and $(0, 1)$, the triangle with vertices $(0, 0), (2, 0)$ and $(0, 2)$, and trapezoids with vertices $(0, 0), (p, 0), (q, 1)$ and $(0, 1)$. Since P' has at least 4 edges, it must be a trapezoid and all interior points of P are on the edge from v_2 to v_4 (see Figure 3.9(b)). Let v_5 be the interior point that is closest to v_4 and let P'' be the convex hull of v_0, v_1, v_2, v_3 and v_5. Then P'' is of type $(i - 1, 5) \in T$. □

Lemma 3.4.10. *Suppose that P is a lattice triangle of type (i, b) with $i \geq 3$ and $b \geq 6$, such that two edges have discrete length 1. Then there exists a convex lattice polygon $P' \subsetneq P$ with $t(P') \in T$.*

Proof. By applying a suitable transformation, we can assume that the boundary points are $v_0 = (0, 0), v_1 = (1, 0), \ldots, v_{b-2} = (b - 2, 0)$ and $v_{b-1} = (c, d)$ with $d > 0$. Let



(a) Pentagon with lines from v_i to v_{i+2}



(b) Pentagon with interior points on a line

Figure 3.9: A pentagon with five boundary points

P' be the triangle with vertices $v_0 = (0, 0)$, $v_{b-3} = (b-3, 0)$ and $v_{b-1} = (c, d)$, and let P'' be the triangle with vertices $v_1 = (1, 0)$, $v_{b-2} = (b-2, 0)$ and $v_{b-1} = (c, d)$ (see Figure 3.10). Note that $(b-2)d = \mathrm{Vol}(P) = 2i + b - 2$, so $d = \frac{2i}{b-2} + 1$. Since $d \in \mathbb{N}$, we have $d \geq 2$. Hence $\mathrm{Vol}(P') = \mathrm{Vol}(P'') = (b-3)d$, which is at least 7, unless $d = 2$ and $b = 6$. But in this case $\mathrm{Vol}(P) = 8$ and hence $i = 2$, which contradicts the assumptions on P. As both P' and P'' have $b - 1 \geq 5$ boundary points, Remark 3.4.3 implies that it suffices to show that at least one of P' and P'' has an interior lattice point.

Suppose that P' has no interior lattice points. Then all interior lattice points of P lie on or to the right of the line through v_{b-3} and v_{b-1}. However, in this case, all interior lattice points of P are also interior points of P''. □

Lemma 3.4.11. *Let P is a convex lattice polygon of type (i, b) with $i \geq 3$ and $b \geq 6$. Suppose that P is not a triangle with two edges of discrete length 1. Then there exists a convex lattice polygon $P' \subsetneq P$ with $t(P') \in T$.*

Proof. Choose a vertex v_0 of P and let P' be the convex hull of all lattice points inside or on the boundary of P, except for v_0. Note that P' contains $i + b - 1$ lattice points. Hence if P' has an interior lattice point, then $\mathrm{Vol}(P') \geq 2 \cdot 1 + (i+b-2) - 2 = i+b-2 \geq 7$ and P' satisfies the conditions. So suppose P' contains no interior lattice point. Let v_{-1} and v_1 be the previous and next lattice point on the boundary of P (in counterclockwise order; see Figure 3.11). Then all interior lattice points of P lie on the line from v_{-1} to v_1, or at the side of v_0. Now consider a vertex $v_i \neq v_{-1}, v_0, v_1$. This clearly exists if P is not a triangle. If P is not a triangle, at least one of v_{-1} and v_1 is not a vertex (since otherwise there are two sides of discrete length 1), and hence P has a third vertex unequal to v_{-1}, v_0 and v_1. Define $v_{i\pm1}$ similar to $v_{\pm1}$ and let P'' be the convex hull of all lattice points in P except for v_i. Similar to P', it suffices to show that P'' has an interior lattice point. The line from v_{i-1} to v_{i+1} has at most one point in common with the line from v_{-1} to v_i (because P has more than 4 boundary points), so all interior lattice points of P lie in the interior of P''. It is clear that P' and P'' have at least $b - 1 \geq 5$ boundary points, so $t(P') \in T$ or $t(P'') \in T$. □

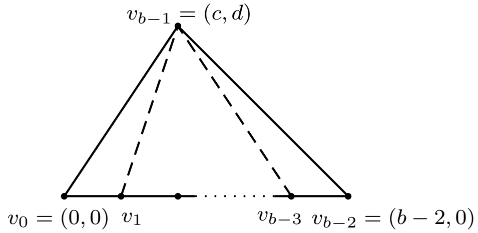


Figure 3.10: The triangle of Lemma 3.4.10

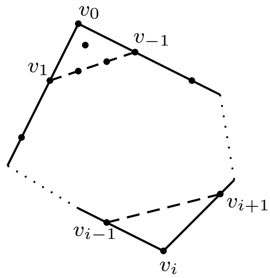


Figure 3.11: The polygon of Lemma 3.4.11

Proof of Lemma 3.4.4. This follows immediately from Lemmas 3.4.6 up to 3.4.11. □

Theorem 3.4.12. *If $P(\mathcal{A})$ has one interior lattice point and at least 7 boundary points, or at least two interior points and at least 5 boundary points, then there are no $\boldsymbol{\beta} \in \mathbb{Q}^3$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is irreducible and has algebraic solutions.*

Proof. By Corollary 3.4.5, there exists a convex lattice polygon $P' \subseteq P(\mathcal{A})$ of type $t(P') \in S$. Let $\mathcal{A}'$ be the set of lattice points in P' (including the boundary). Then $P' = P(\mathcal{A}')$ and $t(\mathcal{A}') \in S$. Hence by Section 3.3, the statement of the theorem holds for $\mathcal{A}'$. Now the statement follows for $\mathcal{A}$ by Corollary 3.2.6. □

3.5 Polygons with at least 3 interior points and 3 or 4 boundary points

In the previous section, we have seen that polygons with at least three interior points and at least five boundary points do not admit algebraic hypergeometric functions. The final section of this chapter will be devoted to polygons with three or four boundary points. We will show that there are families of polygons and choices of $\boldsymbol{\beta}$ for which the associated functions are algebraic.

Theorem 3.5.1. *Suppose that $t(\mathcal{A}) = (i, b)$ with $i \geq 3$ and there exists $\boldsymbol{\beta}$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions. Then $b \in \{3, 4\}$ and $\mathcal{A}$ is one of the following:*

$$\mathcal{A}_{11} = \begin{pmatrix} 0 & 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \mathcal{A}_i^{(3)} = \begin{pmatrix} -1 & 0 & 1 & 2 & \dots & i & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

$$\textit{or} \qquad \mathcal{A}_{i,k}^{(4)} = \begin{pmatrix} k & -1 & 0 & 1 & \dots & i & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \qquad (-1 \leq k \leq i).$$

Proof. The first statement follows immediately from Theorem 3.4.12. For the second statement, we use induction on i. For $3 \leq i \leq 7$, we check this theorem by computing all polygons with 3 or 4 boundary points. We find the 5 polygons $P(\mathcal{A}_i^{(3)})$, the 35 polygons $P(\mathcal{A}_{i,k}^{(4)})$ and 40 polygons shown in Figure 3.12.

The families $\mathcal{A}_i^{(3)}$ and $\mathcal{A}_{i,k}^{(4)}$ will be treated in Lemmas 3.5.4 and 3.5.5. For all non-shaded polygons in Figure 3.12, we indicated a subset that is isomorphic to either $\mathcal{A}_1$ or $\mathcal{A}_6$. Hence in these cases, there are no irreducible algebraic functions. This leaves us with the family on the first line of Figure 3.12 and the first polygon on the second line, which is $P(\mathcal{A}_{18})$. We will determine the irreducible algebraic function for $\mathcal{A}_{18}$ in Lemma 3.5.3. For the remaining family, note that all polygons include the smallest one, with $i = 3$. Hence it suffices to show that there are no irreducible algebraic functions for $\mathcal{A} = \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Note that $\mathcal{A}_{3,1}^{(2)}$ is included in this set, so by Lemma 3.3.3 it suffices to check that $\boldsymbol{\beta} = \pm(\frac{1}{6}, \frac{5}{6}, \frac{2}{3})$ doesn't give algebraic functions. This can easily be done by computing the interlacing condition for $\mathcal{A}$, which is

$$(\lfloor -\beta_1 - 2\beta_2 + 3\beta_3 \rfloor, \lfloor -\beta_1 + 4\beta_2 + 3\beta_3 \rfloor, \lfloor -\beta_2 + \beta_3 \rfloor, \lfloor 2\beta_1 - \beta_2 + \beta_3 \rfloor) \in \{(-1, 2, 0, 1), (0, 3, -1, 0)\}.$$

Now we assume that $i \geq 8$. To simplify notation, we will omit the third coordinate of points in $\mathcal{A}$, which always equals 1. Let $\tilde{\mathcal{A}}$ be the set of interior points of $\mathcal{A}$. Then $P(\tilde{\mathcal{A}})$ is either a line segment or a polygon with at least 8 points and fewer points than $\mathcal{A}$, such that $H_{\tilde{\mathcal{A}}}(\boldsymbol{\beta})$ has irreducible algebraic solutions. If $P(\tilde{\mathcal{A}})$ is not a line segment, the induction hypothesis and the results of the previous sections imply that $\tilde{\mathcal{A}}$ is of the form $\mathcal{A}_{i-1}^{(1)}$, $A_{i'}^{(3)}$ or $A_{i',k'}^{(4)}$ with $i' \in \{i-3, i-4\}$. In all cases, at least $i-2$ points in $\mathcal{A}$ lie on a line. By applying a suitable isomorphism of $\mathbb{Z}^2$, this line be can chosen to be $x_2 = 0$.

Now consider the polygon corresponding to the points of $\mathcal{A}$ with $x_2 \geq 0$. Note that this includes at least the $i-2$ points of $\tilde{\mathcal{A}}$ satisfying $x_2 = 0$, as well as at least one point with $x_2 > 0$, since otherwise the points with $x_2 = 0$ cannot be interior points. This polygon contains fewer points than $P(\mathcal{A})$ (as $\mathcal{A}$ also has a point with $x_2 < 0$), and an edge of length at least $i - 2 \geq 6$. The only such polygon admitting irreducible algebraic functions is a triangle $\mathcal{A}_p^{(1)}$. Hence $P(\tilde{\mathcal{A}})$ is a line segment and $\mathcal{A}$ has exactly one point with $x_2 > 0$. Similarly, there is exactly one point with $x_2 < 0$. As $P(\mathcal{A})$ has exactly 3 or 4 boundary points, there must also be 1 or 2 boundary points with $x_2 = 0$. We can assume that $\tilde{\mathcal{A}} = \{(0,0), (1,0), \ldots, (i-1,0)\}$. Then the only possible boundary points with $x_2 = 0$ are $(-1, 0)$ and $(i, 0)$.

Suppose that $\mathcal{A}$ has exactly 3 boundary points. By symmetry, we can assume that $(i, 0)$ is a boundary point, but $(-1, 0)$ is not. After applying a coordinate transformation, the unique point with $x_2 > 0$ is $(0, 1)$. Let the remaining boundary point be (c, d) with $d < 0$. Since $(0, 0)$ is not a boundary point, we have $c < 0$ (see Figure 3.13(a)). The area of $P(\mathcal{A})$ equals $-c - id + i$. On the other hand, a

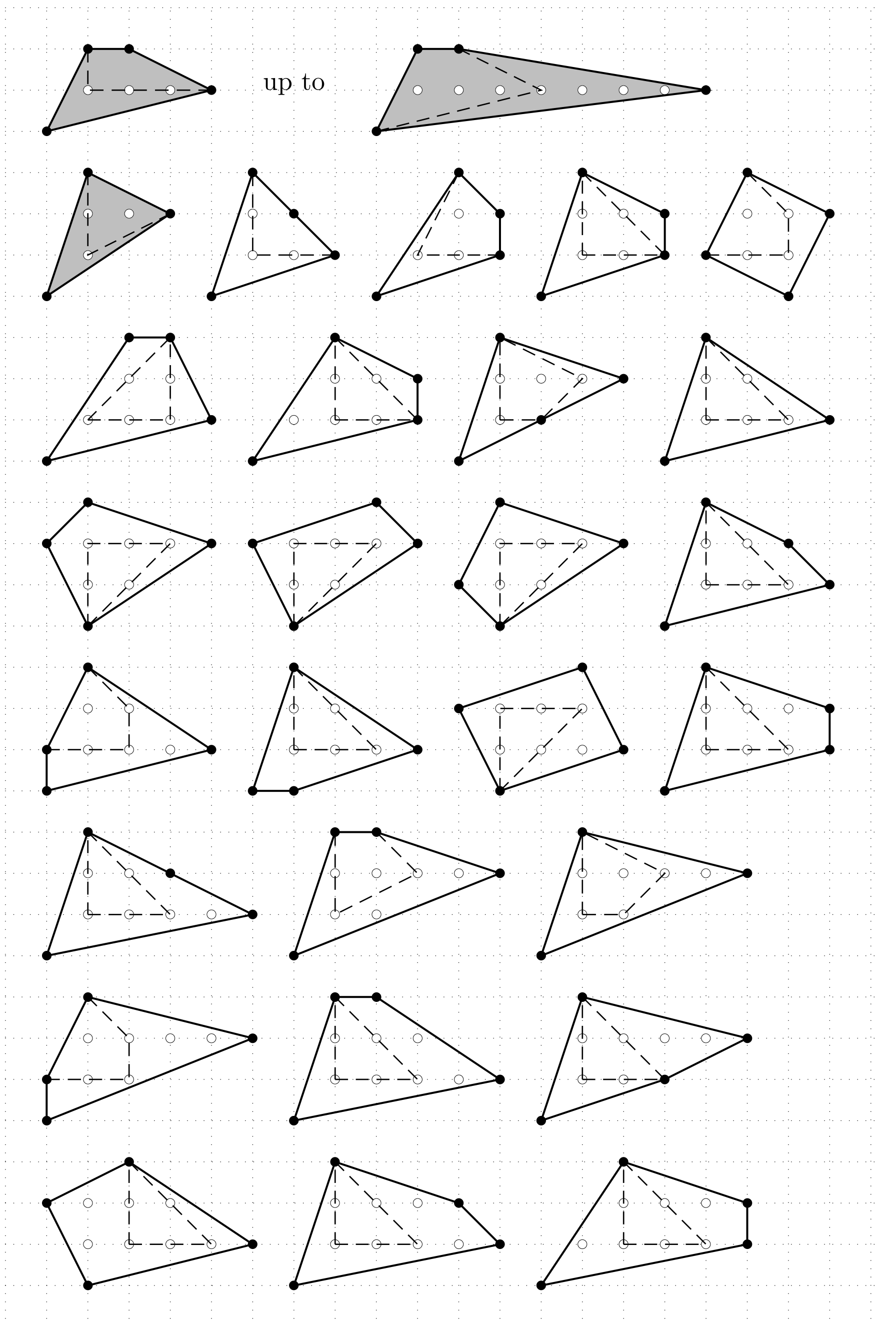


Figure 3.12: The polygons with 3 to 7 interior points and 3 or 4 boundary points (continued on next page)

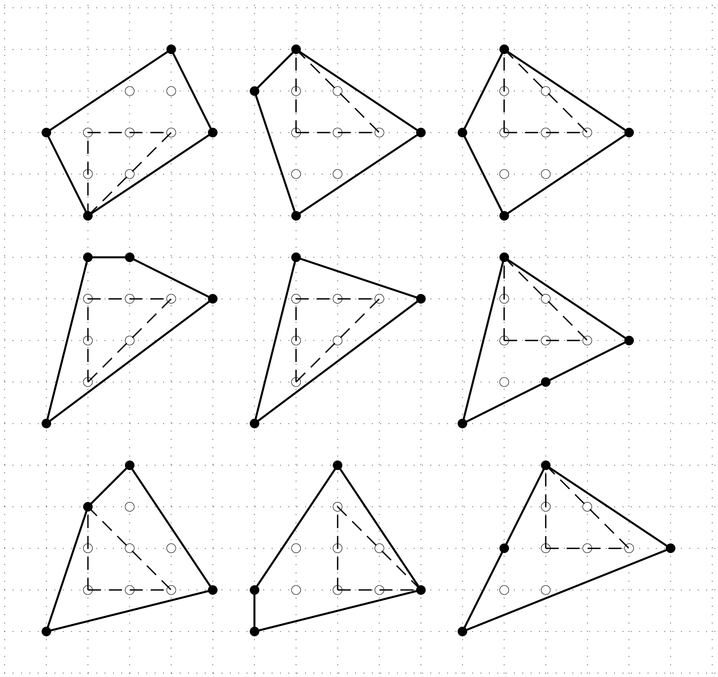
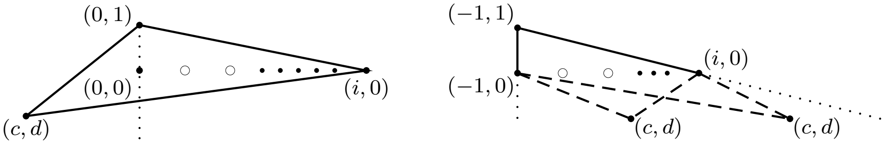


Figure 3.12: The polygons with 3 to 7 interior points and 3 or 4 boundary points (continued)

polygon with 3 boundary points and i interior points has area $2i+1$. This implies that $-(c+id) = i+1$. Now it follows from $c, d < 0$ that $c = d = -1$, and we have $\mathcal{A} = \mathcal{A}_i^{(3)}$.

Now suppose that $\mathcal{A}$ has exactly 4 boundary points. Then both $(-1,0)$ and $(i,0)$ are boundary points. Again we can apply a coordinate transformation, so that $(-1,1)$ is a vertex of $P(\mathcal{A})$. Let the remaining boundary point again be (c,d) with $d < 0$. Since $(-1,0)$ and $(i,0)$ are boundary points, we have $-1 \leq c \leq 2i+1$ (see Figure 3.13(b)). Furthermore, the area of $P(\mathcal{A})$ is $(i+1)(1-d)$. It equals $2i+2$, so $d = -1$. If $-1 \leq c \leq i$, then we have $\mathcal{A} = \mathcal{A}_{i,k}^{(4)}$ with $k = c$. Otherwise, apply the transformation $f(x,y,z) = (-x + (i-c)y + (i-1)z, -y, z)$. This maps $\mathcal{A}$ to $\mathcal{A}_{i,k}^{(4)}$ with $k = 2i - c$. $\square$

Remark 3.5.2. One can show that there are exactly 3 families of polygons with the interior points on a line: $P(\mathcal{A}_i^{(3)})$, $P(\mathcal{A}_{i,k}^{(4)})$ and the family shown in Figure 3.12. Hence if $\mathcal{A}$ has at least 3 interior points and is not isomorphic to $\mathcal{A}_{11}$, then there exists $\boldsymbol{\beta}$ such that $H_{\mathcal{A}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if the interior points lie on a line, and there are 2 boundary points not on this line.



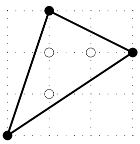
(a) Polygon with 3 boundary points (b) Polygon with 4 boundary points

Figure 3.13: Polygons with 3 or 4 boundary points of Theorem 3.5.1

Table 3.7: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_i^{(3)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions

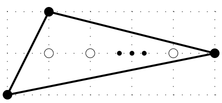
i	$\boldsymbol{\beta}$		
All i	$(r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$		
0	All non-resonant functions are algebraic		
1	$\boldsymbol{\beta}$ in Table 3.4		
2	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$	$(\frac{1}{3}, \frac{2}{3}, \frac{1}{2})$	$(\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$
3	$(\frac{1}{3}, \frac{2}{3}, \frac{1}{2})$		

We now consider the two families and the other polygon we found in Theorem 3.5.1.



Lemma 3.5.3. *Let* $\mathcal{A}_{11} = \begin{pmatrix} 0 & 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_{11}}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if* $\boldsymbol{\beta} = (0, \frac{1}{2}, \frac{1}{2}) \pmod{\mathbb{Z}}$.

Proof. Similar to the proof of Lemma 3.3.8, using the inclusion of $\mathcal{A}_3$ as indicated in Figure 3.12. □



Lemma 3.5.4. *Let* $\mathcal{A}_i^{(3)} = \begin{pmatrix} -1 & 0 & 1 & 2 & \dots & i & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_i^{(3)}}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if* $\boldsymbol{\beta}$ *is, up to conjugation and equivalence modulo* $\mathbb{Z}$, *one of the tuples in Table 3.7.*

Proof. $H_{\mathcal{A}_i^{(3)}}(\boldsymbol{\beta})$ is irreducible if and only if $-\beta_1 - i\beta_2 + i\beta_3 \notin \mathbb{Z}$, $2\beta_1 - \beta_2 + \beta_3 \notin \mathbb{Z}$ and $-\beta_1 + (i+1)\beta_2 + i\beta_3 \notin \mathbb{Z}$.

For $i = 0, 1$, the $\boldsymbol{\beta}$ giving irreducible algebraic functions are (up to an isomorphism) given in Lemmas 3.3.1 and 3.3.4. For $i = 2, 3, 4$, one easily computes that the interlacing condition is given by

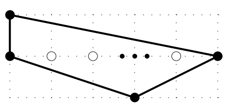
$$(\lfloor -\beta_1 - i\beta_2 + i\beta_3 \rfloor, \lfloor 2\beta_1 - \beta_2 + \beta_3 \rfloor, \lfloor -\beta_1 + (i+1)\beta_2 + i\beta_3 \rfloor) \in \{(-1, 1, i-1), (-1, 0, i)\}.$$

Using the fact that $\mathcal{A}_{i-1}^{(3)} \subseteq \mathcal{A}_i^{(3)}$ for all i, it follows easily that the solutions for $i \leq 4$ are irreducible and algebraic if and only if $\boldsymbol{\beta}$ is in Table 3.7. It also follows immediately from this that the only possibility for $i > 4$ is $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$.

We now show that $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$ always gives irreducible algebraic solutions. $H_{\mathcal{A}_i^{(3)}}(\boldsymbol{\beta})$ is clearly irreducible and $\mathrm{Vol}(Q(A)) = 2i + 1$, so it suffices to give $2i + 1$ apex points for all r. We claim that $(k, 0, 1) + \boldsymbol{\beta}$ with $0 \leq k \leq i - 1$ if $0 < r < \frac{1}{2}$ and $-1 \leq k \leq i - 1$ if $\frac{1}{2} < r < 1$ are apex points, as well as $(l, -1, 1) + \boldsymbol{\beta}$ with $-1 \leq l \leq i - 1$ if $0 < r < \frac{1}{2}$ and $-1 \leq l \leq i - 2$ if $\frac{1}{2} < r < 1$. One can easily check this using the characterisation of apex points of Lemma 1.3.5 and the fact that

$$C(\mathcal{A}_i^{(3)}) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid -x_1 - ix_2 + ix_3 \geq 0, 2x_1 - x_2 + x_3 \geq 0, -x_1 + (i+1)x_2 + ix_3 \geq 0\}.$$

□



Lemma 3.5.5. *Let* $\mathcal{A}_{i,k}^{(4)} = \begin{pmatrix} k & -1 & 0 & 1 & \dots & i & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$. *Then* $H_{\mathcal{A}_{i,k}^{(4)}}(\boldsymbol{\beta})$ *has irreducible algebraic solutions if and only if* $\boldsymbol{\beta}$ *is, up to conjugation and equivalence modulo* $\mathbb{Z}$, *one of the tuples in Table 3.8.*

Proof. $A_{0,-1}^{(4)}$, $A_{0,0}^{(4)}$, $A_{1,-1}^{(4)}$, $A_{1,0}^{(4)}$ and $A_{1,1}^{(4)}$ are isomorphic to $\mathcal{A}_2^{(1)}$, $\mathcal{A}_{1,1}^{(2)}$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively (see Lemmas 3.3.1, 3.3.3, 3.3.5, 3.3.6 and 3.3.7). For $i = 2$, the proofs are similar to the proof of Lemma 3.3.8, by computing the interlacing conditions and using the inclusions as shown in Figure 3.5.

For $i \geq 3$ and $k \neq i$, note that $\mathcal{A}_{i-1,k}^{(4)} \subseteq \mathcal{A}_{i,k}^{(4)}$. Hence for $i = 3$ we can compute all $\boldsymbol{\beta}$ such that $H_{\mathcal{A}_{i,k}^{(4)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions by computing the number of apex points for all $\boldsymbol{\beta}$ we have found for $i = 2$. For $(i, k) = (3, 3)$, note that $f(x, y, z) = (-x + y + z, -y, z)$ maps $\mathcal{A}_{2,1}^{(4)}$ to $\mathcal{A}_{3,3}^{(4)}$. Hence we only have to compute the number of apex points for all $\boldsymbol{\beta}$ coming from $\mathcal{A}_{2,1}^{(4)}$. Similarly, for $i = 4$ we use the inclusions $\mathcal{A}_{3,k}^{(4)} \subseteq \mathcal{A}_{4,k}^{(4)}$ for $k \neq 4$. For $(i, k) = (4, 4)$, the map $f(x, y, z) = (-x + y + 2z, -y, z)$ maps $\mathcal{A}_{3,2}^{(4)}$ to a subset of $\mathcal{A}_{4,4}^{(4)}$. In all cases, we find that $H_{\mathcal{A}_{i,k}^{(4)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$ if k is even and $2r \notin 2\mathbb{Z} + 1$ if k is odd.

Let $i \geq 5$. We claim that $H_{\mathcal{A}_{i,k}^{(4)}}(\boldsymbol{\beta})$ has irreducible algebraic solutions if and only if $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$ if k is even and $2r \notin 2\mathbb{Z} + 1$ if k is odd. It is easy to show that $H_{\mathcal{A}_{i,k}^{(4)}}(\boldsymbol{\beta})$ is irreducible if and only if $\beta_1 + \beta_3, \beta_1 + (k+1)\beta_2 + \beta_3, -\beta_1 - (i+1)\beta_2 + i\beta_3, -\beta_1 + (i - k)\beta_2 + i\beta_3 \notin \mathbb{Z}$. For $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$, this holds exactly under the condition stated above. We use induction on i to show that $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$

Table 3.8: The parameters $\boldsymbol{\beta}$ such that $H_{\mathcal{A}^{(4)}_{i,k}}(\boldsymbol{\beta})$ has irreducible algebraic solutions

(i,k)	$\boldsymbol{\beta}$
All (i,k)	$(r, \frac{1}{2}, \frac{1}{2})$ with $2r \notin \mathbb{Z}$ if k is even, $2r \notin 2\mathbb{Z}+1$ if k is odd
$(0,-1)$	$(\beta_2 - \beta_3, \beta_1 + \beta_2 - \beta_3, \beta_3)$ with $\boldsymbol{\beta}$ in Table 3.2, $p = 1$
$(0,0)$	$(-\beta_1, \beta_1 - \beta_2, \beta_3)$ with $\boldsymbol{\beta}$ in Table 3.3, $(p,q) = (1,1)$
$(1,-1)$	$\boldsymbol{\beta}$ in Table 3.5
$(1,0)$	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ $(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$ $(\frac{1}{5}, \frac{3}{5}, \frac{1}{2})$ $(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$
$(1,1)$	$\boldsymbol{\beta}$ in Table 3.6
$(2,-1)$	$(0, \frac{1}{2}, \frac{1}{3})$ $(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$
$(2,0), (3,0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$
$(2,1)$	$(0, \frac{1}{2}, \frac{1}{3})$ $(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$ $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$
$(3,\pm 1)$	$(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$
$(3,3)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ $(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$

is the only possibility. It suffices to find a subpolygon for which $\boldsymbol{\beta}$ can only be $(r, \frac{1}{2}, \frac{1}{2})$. If $k \neq i$, we can use the inclusion $\mathcal{A}^{(4)}_{i-1,k} \subseteq \mathcal{A}^{(4)}_{i,k}$. For $k = i$, note that $f(x,y,z) = (-x + y + (i-2)z, -y, z)$ maps $\mathcal{A}^{(4)}_{i-1,i-2}$ to a subset of $\mathcal{A}^{(4)}_{i,i}$. Under this isomorphism, we have $f(r, \frac{1}{2}, \frac{1}{2}) = (-r + \frac{i-1}{2}, -\frac{1}{2}, \frac{1}{2})$, which is equivalent modulo $\mathbb{Z}$ to $(s, \frac{1}{2}, \frac{1}{2})$ for $s = -r + \frac{i-1}{2}$. This proves the claim.

It remains to show that $\boldsymbol{\beta} = (r, \frac{1}{2}, \frac{1}{2})$ indeed gives irreducible algebraic solutions. As in the proof of Lemma 3.5.4, we can do this by showing that there are $\mathrm{Vol}(Q(\mathcal{A}^{(4)}_{i,k}) = 2i + 2$ apex points. They are given by $(l, 0, 1) + \boldsymbol{\beta}$ for $\lfloor -2r \rfloor \leq l \leq \lfloor i - 2r \rfloor$, and $(m, -1, 1) + \boldsymbol{\beta}$ for $\lfloor \frac{k}{2} - r \rfloor \leq m \leq \lfloor i + \frac{k}{2} - r \rfloor$. $\square$

4

Computation of the local monodromy matrices

In the previous chapters, we determined the irreducible algebraic Appell, Lauricella and Horn functions and $\mathcal{A}$-hypergeometric functions with planar sets $\mathcal{A}$. The monodromy groups of these functions are finite. A second goal of this thesis is to compute these monodromy groups. In this chapter we will explain how to compute the local monodromy group. This is the group generated by the monodromy matrices corresponding to loops of the variables z_i around the origin, and it might be smaller than the full monodromy group. This group will be computed on a basis of Mellin-Barnes integrals, provided such a basis exists. In the next chapter we will use this to compute the monodromy group for most of the irreducible algebraic Appell, Lauricella and Horn functions.

The computation of the local monodromy group is based on correspondences between regular triangulations of $\mathcal{A}$, local bases of power series solutions, the directions in which these series converge and the secondary fan. In Section 4.1 we show how triangulations of $\mathcal{A}$ lead to local bases of Γ-series solutions and compute the triangulations of the Appell and Horn function. To see which triangulations give a basis of series with a common domain of convergence, we introduce the concept of convergence direction in Section 4.2. We study the convergence directions of the Γ-series, show how they are related to the secondary fan and explicitly give the above correspondences, showing that we can compute the regular triangulations of $\mathcal{A}$ by computing the secondary fan. This allows us to compute the local monodromy matrices on local bases of Γ-series. In Section 4.3, Mellin-Barnes integrals are used to glue together these local groups. Throughout this chapter, we will perform the calculations for all Appell and Horn functions.

Except for these calculations, most results in this chapter are not new. The correspondence between regular triangulations and bases of local series solutions in Section 4.1 goes back to Gelfand, Zelevinsky and Kapranov [GZK88, GKZ89]. Much more about triangulations can be found in their book [GKZ94]. In [Beu11], Beukers also explains this correspondence, as well as the computation of the triangulations using convergence directions. The correspondence between the convergence directions and the secondary fan can be found in [Sti07] The computation of the local monodromy on a basis of Mellin-Barnes integrals is described in [Beu]. Large parts of this chapter are mostly a more detailed exposition of these papers.

4.1 Summation sectors, triangulations and local bases of Γ-series solutions

Let $\mathcal{A}$ be as in Definition 1.2.1. We write $d = N - r$ and denote the matrix with columns $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N\}$ by A and its Gale dual by B. If $A\boldsymbol{\gamma} = \boldsymbol{\beta}$, then the Γ-series

$$\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z}) = \sum_{\boldsymbol{l} \in \mathbb{L}} \prod_{j=1}^{N} \frac{z_j^{l_j + \gamma_j}}{\Gamma(l_j + \gamma_j + 1)} \tag{4.1}$$

is a formal solution of $H_{\mathcal{A}}(\boldsymbol{\beta})$. Recall from Section 1.2 that the series converges if sufficiently many γ_j are integral. Since the Gamma-function has poles on the non-positive integers, the sum will then only be taken over a cone. In this section, we will show how different choices for $\boldsymbol{\gamma}$ give rise to a basis of solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$.

Definition 4.1.1. Let $\mathcal{S}(\mathcal{A})$ be the set of vertices of subsimplices of $Q(\mathcal{A})$, i.e., $\mathcal{S}(\mathcal{A}) = \{J \subseteq \{1, \ldots, N\} \mid |J| = r \text{ and } \det(\boldsymbol{a}_j)_{j \in J} \neq 0\}$. For $J \in \mathcal{S}(\mathcal{A})$, we denote by $Q(J)$ and $C(J)$ the convex hull (simplex) and the positive cone spanned by $\{\boldsymbol{a}_j \mid j \in J\}$, respectively. For $\boldsymbol{x} \in Q(\mathcal{A})$, let $\mathcal{S}(\boldsymbol{x}) = \{J \in \mathcal{S}(\mathcal{A}) \mid \boldsymbol{x} \in Q(J)\}$, the set of vertices of simplices containing $\boldsymbol{x}$.

Definition 4.1.2. Let $I \subseteq \{1, \ldots, N\}$ with $I^c \in \mathcal{S}(\mathcal{A})$, where I^c denotes the complement of I. We write $\Delta_I = |\det(\boldsymbol{a}_j)_{j \in I^c}|$. The *summation sector with index I* is the cone $V(I) = \{\boldsymbol{l} \in \mathbb{L} \mid l_i \geq 0 \text{ for all } i \in I\}$.

Since $\mathbb{L}$ is d-dimensional, all l_j with $j \in I^c$ are completely determined by the l_i with $i \in I$. Hence the summation sector is indeed a cone. Choose $\gamma_i \in \mathbb{Z}$ for all $i \in I$. Then $\frac{1}{\Gamma(l_i + \gamma_i + 1)}$ equals 0 if $l_i + \gamma_i < 0$, so the series $\Phi_{\boldsymbol{\gamma}}$ contains only terms with $l_i + \gamma_i \geq 0$ for all $i \in I$. Note that we sum over a translate of the summation sector with index I. This explains the name 'summation sector'.

To ensure that the power series is non-trivial, we need the assumption that $\gamma_j \notin \mathbb{Z}$ for all $j \in I^c$. This also makes the summation sector unique. To assure this, we impose the condition that the system is totally non-resonant.

Lemma 4.1.3. *Suppose that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is totally non-resonant. Let $I^c \in \mathcal{S}(\mathcal{A})$ and $\boldsymbol{\gamma} \in \mathbb{R}^r$ such that $A\boldsymbol{\gamma} = \boldsymbol{\beta}$ and $\gamma_i \in \mathbb{Z}$ for all $i \in I$. Then $\gamma_j \notin \mathbb{Z}$ for all $j \in I^c$.*

Proof. Suppose that there exists $j \in I^c$ such that $\gamma_j \in \mathbb{Z}$. Define $\boldsymbol{p} = \sum_{i \in I} \gamma_i \boldsymbol{a}_i + \gamma_j \boldsymbol{a}_j$. Then $\boldsymbol{p} \in \mathbb{Z}^r$, and $\boldsymbol{\beta} - \boldsymbol{p} = \sum_{k \in I^c, k \neq j} \gamma_k \boldsymbol{a}_k$. Hence the hyperplane spanned by $\{\boldsymbol{a}_k\}_{k \in I^c \setminus \{j\}}$ contains a point in $\boldsymbol{\beta} + \mathbb{Z}^r$, contradicting the total non-resonance. □

In the remainder of this thesis, we will only consider totally non-resonant functions.

For given $\boldsymbol{\beta}$ and I, we have some freedom in the choice of $\boldsymbol{\gamma}$. This gives us independent Γ-series solutions:

Proposition 4.1.4. *Let $H_{\mathcal{A}}(\boldsymbol{\beta})$ be totally non-resonant. Let I be the index of a summation sector, i.e., $I^c \in \mathcal{S}(\mathcal{A})$. Modulo $\mathbb{L}$, there are Δ_I choices for $\boldsymbol{\gamma}$ with $\gamma_i \in \mathbb{Z}$ for all $i \in I$, leading to Δ_I series of the form* (4.1)*. These series are linearly independent solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$.*

Proof. Since $\boldsymbol{\gamma}$ is completely determined by $(\gamma_i)_{i\in I}$ and the equation $A\boldsymbol{\gamma} = \boldsymbol{\beta}$, the number of choices for $\boldsymbol{\gamma}$ equals the number of choices for $(\gamma_i)_{i\in I}$. It easily follows from formula (4.1) that $\boldsymbol{\gamma}$'s that differ by a vector in $\mathbb{L}$ will give the same power series. Hence the number of choices for $(\gamma_i)_{i\in I}$ equals $[\mathbb{M} : \mathbb{L}]$, where $\mathbb{M} = \{\boldsymbol{l} \in \mathbb{L}\otimes\mathbb{R} \mid \forall i \in I : l_i \in \mathbb{Z}\}$. Define

$$\phi : \mathbb{M} \to \operatorname{Span}_{\mathbb{Z}}((\boldsymbol{a}_i)_{i\in I})/(\operatorname{Span}_{\mathbb{Z}}((\boldsymbol{a}_i)_{i\in I}) \cap \operatorname{Span}_{\mathbb{Z}}((\boldsymbol{a}_j)_{j\in I^c})) : \boldsymbol{l} \mapsto [\sum_{i\in I} l_i\boldsymbol{a}_i].$$

We will abbreviate $\operatorname{Span}_{\mathbb{Z}}((\boldsymbol{a}_i)_{i\in I})$ to $\operatorname{Span}(I)$. It is clear that ϕ is surjective and the kernel contains $\mathbb{L}$. Conversely, suppose that $\boldsymbol{l} \in \ker(\phi)$, i.e., $\sum_{i\in I} l_i\boldsymbol{a}_i \in \operatorname{Span}(I) \cap \operatorname{Span}(I^c)$. Then there exist $m_j \in \mathbb{Z}$ such that $\sum_{i\in I} l_i\boldsymbol{a}_i = \sum_{j\in I^c} m_j\boldsymbol{a}_j$. The vector $\tilde{\boldsymbol{l}}$ with $\tilde{l}_i = l_i$ if $i \in I$ and $\tilde{l}_j = -m_j$ if $j \in I^c$ lies in $\mathbb{L}$. Since $\tilde{l}_i = l_i$ for all $i \in I$ and vectors in $\mathbb{M}$ are uniquely determined by their i^{th} components ($i \in I$), we must have $\tilde{\boldsymbol{l}} = \boldsymbol{l}$ and hence $\boldsymbol{l} \in \mathbb{L}$. It follows that $\ker(\phi) = \mathbb{L}$, and $[\mathbb{M} : \mathbb{L}] = |\operatorname{Span}(I)/(\operatorname{Span}(I)\cap\operatorname{Span}(I^c))|$.

Define

$$\psi : \operatorname{Span}(I) \to \mathbb{Z}^r/\operatorname{Span}(I^c) : \boldsymbol{x} \mapsto [\boldsymbol{x}].$$

This is well-defined because $\operatorname{Span}(I^c)$ is a sublattice of $\mathbb{Z}^r$. Note that $\ker(\psi) = \operatorname{Span}(I) \cap \operatorname{Span}(I^c)$. Furthermore, ψ is surjective because $\mathbb{Z}^r = \operatorname{Span}(I \cup I^c)$. Hence

$$|\operatorname{Span}(I)/(\operatorname{Span}(I) \cap \operatorname{Span}(I^c))| = |\operatorname{Im}(\psi)| = |\mathbb{Z}^r/\operatorname{Span}(I^c)| = \Delta_I$$

since the index of a sublattice of $\mathbb{Z}^r$ is given by the determinant of its generators. This shows that there are Δ_I choices for $\boldsymbol{\gamma}$ with $\gamma_i \in \mathbb{Z}$ for all $i \in I$.

Consider a linear combination $\lambda_1\Phi_{\boldsymbol{\gamma}^{(1)}}(\boldsymbol{z})+\ldots+\lambda_{\Delta_I}\Phi_{\boldsymbol{\gamma}^{(\Delta_I)}}(\boldsymbol{z})$ with all $\boldsymbol{\gamma}^{(i)}$ different modulo $\mathbb{L}$. The exponents of $\boldsymbol{z}$ in $\Phi_{\boldsymbol{\gamma}^{(i)}}(\boldsymbol{z})$ are modulo $\mathbb{L}$ equal to $\boldsymbol{\gamma}^{(i)}$. Hence such a linear combination can only be zero if all λ_i are zero, so the series are linearly independent. □

Theorem 4.1.5. *Let $H_{\mathcal{A}}(\boldsymbol{\beta})$ be totally non-resonant and let $\mathcal{T}$ be a triangulation of $\mathcal{A}$. For each $I \subseteq \{1, \ldots, N\}$ such that $I^c \in \mathcal{T}$ there are Δ_I choices for $\boldsymbol{\gamma}$ so that the resulting power series $\Phi_{I,\boldsymbol{\gamma}}$ are distinct. The union of these sets of power series is a basis of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$.*

Proof. The first statement follows immediately from Proposition 4.1.4. For each $I^c \in \mathcal{T}$, we have Δ_I power series. Note that

$$\sum_{I^c\in\mathcal{T}} \Delta_I = \sum_{I^c\in\mathcal{T}} |\det(\boldsymbol{a}_j)_{j\in I^c}| = \sum_{J\in\mathcal{T}} \operatorname{Vol}(Q(J)) = \operatorname{Vol}(Q(\mathcal{A})).$$

In the last step we use that the sets $Q(J)$ intersect in faces with volume 0. Hence the power series form a basis, provided that they are linearly independent. Suppose that there exist a linear combination $\sum_{I^c\in\mathcal{T}}\sum_{i=1}^{\Delta_I} \lambda_{I,i}\Phi_{I,i} = 0$. By total non-resonance, a series $\Phi_{I,i}$ contains integral powers of z_i if and only if $i \in I$. Hence the above relation implies that $\sum_{i=1}^{\Delta_I} \lambda_{I,i}\Phi_{I,i} = 0$ for each $I^c \in \mathcal{T}$. However, for each I the series $\Phi_{I,i}$ are independent by Proposition 4.1.4, so $\lambda_{I,i} = 0$ for all I and i, and the series form a basis. □

Example 4.1.6. We apply Theorem 4.1.5 to the Gauss function. The set $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}$ has two triangulations: $\mathcal{T}_1 = \{\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}\}$ and $\mathcal{T}_2 = \{\{\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}, \{\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}\}$. The first triangulation gives $\boldsymbol{\gamma}^{(1)} = (-a, -b, c-1, 0)$ and $\boldsymbol{\gamma}^{(2)} = (-a+c-1, -b+c-1, 0, 1-c)$. This gives the solutions u_1 and u_2 from (1.5). For the second triangulation, we get $\boldsymbol{\gamma}^{(1)} = (-a+b, 0, c-b-1, -b)$ and $\boldsymbol{\gamma}^{(2)} = (0, a-b, -c-a-1, -a)$. This corresponds to the solutions u_6 and u_5, respectively. Note that the solutions u_3 and u_4 are not Γ-series in z, and hence do not correspond to a triangulation of $\mathcal{A}$. $\diamondsuit$

If $\mathcal{T}$ is a triangulation and Φ_k is a Γ-series solution corresponding to this triangulation, then there is $I^c \in \mathcal{S}(\mathcal{A})$ such that $\Phi_k(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\gamma}} \tilde{\Phi}_k(\boldsymbol{z})$ where $\boldsymbol{\gamma}_i \in \mathbb{Z}$ for all $i \in I$ and $\tilde{\Phi}_k$ is a holomorphic function. The local monodromy is determined by the analytic continuation along loops of z_j around the origin. Analytic continuation along of a loop of z_j around the origin will change the factor $z_j^{\gamma_j}$ into $e^{2\pi i \gamma_j} z_j^{\gamma_j}$. Hence the function $\tilde{\Phi}_k$ will be multiplied by $e^{2\pi i \gamma_j}$ and the local monodromy matrix is a diagonal matrix. Note that the monodromy is completely determined by the local exponents.

The above discussion shows that each triangulation induces basis of Γ-series solutions and that it is easy to compute the local monodromy matrices on this basis. To use this to compute the monodromy groups of the algebraic Appell and Horn functions, we need to compute the triangulations of $\mathcal{A}$. A naive way to compute all triangulations is to compute the set $\mathcal{S}(\mathcal{A})$, by computing the determinant of each subset of $\mathcal{A}$ with r elements. Then we compute all pairs $(I, J) \in \mathcal{S}(\mathcal{A})^2$ such that $Q(I) \cap Q(J) = Q(I \cap J)$. As we have $Q(I \cap J) \subseteq Q(I) \cap Q(J)$ for all I and J, it suffices to let the computer determine whether there is a point in $Q(I \cap J) \setminus (Q(I) \cap Q(J))$. Now we compute all sets $\mathcal{T}$ consisting of subsets of $\mathcal{A}$, such that for all $\mathcal{T}$ each pair (I, J) of subsets of $\mathcal{A}$ satisfies $Q(I) \cap Q(J) = Q(I \cap J)$. Finally, for each $\mathcal{T}$ we check whether $C(\mathcal{A}) = \cup_{I \in \mathcal{T}} C(I)$, by determining whether there are points in $C(\mathcal{A}) \setminus (\cup_{I \in \mathcal{T}} C(I))$. This will give us all triangulations of $\mathcal{A}$. This method is very slow, but it works for all Appell and Horn functions.

We used this algorithm to compute the triangulations for the sets $\mathcal{A}$ of all (non-isomorphic) Appell and Horn functions, as well as the Lauricella F_D function with $n = 3$. The results can be found in Table 4.1. For the Lauricella F_C function, there seem to be $n + 1$ triangulations, which can be computed for small n, and are generalizations of the triangulations for F_4. We do not list them in Table 4.1, because we don't have a proof for this. In Section 4.3, we will see that we are not able to compute the monodromy group because there is no basis of Mellin-Barnes integrals, so computing the local monodromy matrices is not very useful anyway.

We will only be able to compute the monodromy groups up to roots of unity. Therefore, to make computations easier, we can now already consider the local monodromy group up to roots of unity. In almost all interesting situations (e.g., in case of the Appell and Horn functions), there exists $I^c \in \mathcal{S}(\mathcal{A})$ such that $\Delta_I = 1$. For

Table 4.1: Triangulations for the Appell and Horn functions

Function	$\mathcal{A}$	Triangulations
F_1	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,3,5,6\}\}$ $\mathcal{T}_2 = \{\{1,2,3,4\},\{1,2,3,6\},\{1,2,5,6\}\}$ $\mathcal{T}_3 = \{\{1,2,4,6\},\{1,2,5,6\},\{2,3,4,6\}\}$ $\mathcal{T}_4 = \{\{1,3,4,5\},\{1,3,5,6\},\{2,3,4,5\}\}$ $\mathcal{T}_5 = \{\{1,4,5,6\},\{2,3,4,5\},\{3,4,5,6\}\}$ $\mathcal{T}_6 = \{\{1,4,5,6\},\{2,3,4,6\},\{2,4,5,6\}\}$
F_D $(n=3)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{1,2,3,4,5\},\{1,2,3,4,6\},\{1,3,4,6,7\},\{1,4,6,7,8\}\}$ $\mathcal{T}_2 = \{\{1,2,3,5,8\},\{1,2,3,6,8\},\{1,3,6,7,8\},\{2,3,4,5,8\}\}$ $\mathcal{T}_3 = \{\{1,2,5,7,8\},\{1,2,6,7,8\},\{2,3,4,5,7\},\{2,4,5,7,8\}\}$ $\mathcal{T}_4 = \{\{1,5,6,7,8\},\{2,3,4,5,6\},\{3,4,5,6,7\},\{4,5,6,7,8\}\}$ and 20 other triangulations, obtained by permuting $\{\{2,6\},\{3,7\},\{4,8\}\}$.
F_2	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{1,2,3,4,5\},\{1,2,3,4,7\},\{1,2,3,5,6\},\{1,2,3,6,7\}\}$ $\mathcal{T}_2 = \{\{1,2,4,5,7\},\{1,2,5,6,7\},\{2,3,4,5,7\},\{2,3,5,6,7\}\}$ $\mathcal{T}_3 = \{\{1,3,4,5,6\},\{1,3,4,6,7\},\{2,3,4,5,6\},\{2,3,4,6,7\}\}$ $\mathcal{T}_4 = \{\{1,4,5,6,7\},\{2,3,4,5,6\},\{2,3,4,6,7\},\{3,4,5,6,7\}\}$ $\mathcal{T}_5 = \{\{1,4,5,6,7\},\{2,3,4,5,7\},\{2,3,5,6,7\},\{2,4,5,6,7\}\}$
F_4	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$ $\mathcal{T}_2 = \{\{1,3,4,5\},\{1,3,5,6\},\{2,3,4,5\},\{2,3,5,6\}\}$ $\mathcal{T}_3 = \{\{1,3,4,6\},\{1,4,5,6\},\{2,3,4,6\},\{2,4,5,6\}\}$

Table 4.1: Triangulations for the Appell and Horn functions

Function	$\mathcal{A}$	Triangulations
G_1	$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{2,3,4\},\{2,4,5\}\}$ $\mathcal{T}_2 = \{\{2,3,5\},\{3,4,5\}\}$ $\mathcal{T}_3 = \{\{1,2,3\},\{1,2,5\},\{1,3,4\}\}$ $\mathcal{T}_4 = \{\{1,2,4\},\{1,2,5\},\{2,3,4\}\}$ $\mathcal{T}_5 = \{\{1,3,4\},\{1,3,5\},\{2,3,5\}\}$
G_3	$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{3,4\}\}$ $\mathcal{T}_2 = \{\{1,3\},\{1,4\}\}$ $\mathcal{T}_3 = \{\{2,3\},\{2,4\}\}$ $\mathcal{T}_4 = \{\{1,2\},\{1,4\},\{2,3\}\}$
H_1	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\mathcal{T}_1 = \{\{1,3,4,5\},\{2,4,5,6\},\{3,4,5,6\}\}$ $\mathcal{T}_2 = \{\{1,2,3,4\},\{1,2,3,5\},\{2,3,4,6\},\{2,3,5,6\}\}$ $\mathcal{T}_3 = \{\{1,2,4,6\},\{1,2,5,6\},\{1,3,4,6\},\{1,3,5,6\}\}$ $\mathcal{T}_4 = \{\{1,3,4,5\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\}\}$ $\mathcal{T}_5 = \{\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\},\{2,4,5,6\}\}$
H_4	$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{2,3,4,5\},\{2,3,5,6\}\}$ $\mathcal{T}_2 = \{\{2,3,4,6\},\{2,4,5,6\},\{3,4,5,6\}\}$ $\mathcal{T}_3 = \{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$ $\mathcal{T}_4 = \{\{1,3,4,6\},\{1,4,5,6\},\{2,3,4,6\},\{2,4,5,6\}\}$
H_5	$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$\mathcal{T}_1 = \{\{2,3,4\},\{2,4,5\}\}$ $\mathcal{T}_2 = \{\{2,3,5\},\{3,4,5\}\}$ $\mathcal{T}_3 = \{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,5\}\}$ $\mathcal{T}_4 = \{\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,5\}\}$

each $\boldsymbol{\gamma} \in \mathbb{R}^N$ such that $A\boldsymbol{\gamma} = \boldsymbol{\beta}$, we have

$$(\gamma_j)_{j\in I^c} = (\boldsymbol{a}_j)^{-1}_{j\in I^c} \cdot (\boldsymbol{\beta} - (\boldsymbol{a}_i)_{i\in I} \cdot (\gamma_i)_{i\in I}).$$

Since $(\boldsymbol{a}_j)_{j\notin I}$ has determinant ± 1, each γ_j (with $j \in I^c$) is an integral linear combination of the coordinates of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Hence up to a root of unity (corresponding to the terms coming from $\boldsymbol{\beta}$), M_j is a product of integral powers of the M_i with $i \in I$. This reduces the number of monodromy matrices to $|I| = d$ for each triangulation.

Suppose that the solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ are irreducible and algebraic. By Theorem 1.3.8, we can change the coordinates of $\boldsymbol{\beta}$ by integers without changing the monodromy group. Therefore we will compute the monodromy group only for parameters with coordinates in $[0,1)$. Furthermore, if k is coprime with the least common denominator of $\boldsymbol{\beta}$, then the solutions of $H_{\mathcal{A}}(k\boldsymbol{\beta})$ are also algebraic. From the computation of the local monodromy matrices, we easily see that multiplying $\boldsymbol{\beta}$ by k corresponds to taking the k^{th} power of the monodromy matrices. The entries of the monodromy matrices are roots of unity, whose order equals the common denominator of $\boldsymbol{\beta}$. If k is coprime with all denominators, then the k^{th} power of the monodromy matrix has the same order. Hence the k^{th} powers of the monodromy matrices generate the same group. It follows that we only have to compute the monodromy matrix for one element in each orbit of the $\boldsymbol{\beta}$'s.

We illustrate this with two examples. We will return to these examples throughout this chapter and the next one.

Example 4.1.7. The algebraic Horn function $G_3(\frac{1}{2}, \frac{1}{3}|x,y)$ is given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\beta} = \left(-\tfrac{1}{2}, -\tfrac{5}{6}\right)^t.$$

It is clear that the triangulations of $\mathcal{A}$ are $\mathcal{T}_1 = \{\{3,4\}\}$, $\mathcal{T}_2 = \{\{1,3\},\{1,4\}\}$, $\mathcal{T}_3 = \{\{2,3\},\{2,4\}\}$ and $\mathcal{T}_4 = \{\{1,2\},\{1,4\},\{2,3\}\}$. For the first triangulation, we have to find three vectors $\boldsymbol{\gamma}$ with $\gamma_1, \gamma_2 \in \mathbb{Z}$ that are distinct modulo $\mathbb{L}$. A possible choice is $\boldsymbol{\gamma}^{(1)} = (0,0,-\frac{7}{18},-\frac{4}{9})$, $\boldsymbol{\gamma}^{(2)} = (0,1,-\frac{19}{18},-\frac{7}{9})$ and $\boldsymbol{\gamma}^{(3)} = (1,0,-\frac{13}{18},-\frac{10}{9})$. For the second triangulation, we choose $\boldsymbol{\gamma}^{(1)} = (-\frac{2}{3},0,-\frac{1}{6},0)$ and $\boldsymbol{\gamma}^{(2)} = (-\frac{7}{6},1,-\frac{2}{3},0)$ with $\gamma_2^{(i)}, \gamma_4^{(i)} \in \mathbb{Z}$ and $\boldsymbol{\gamma}^{(3)} = (-\frac{7}{6},0,0,\frac{1}{3})$ with $\gamma_2^{(3)}, \gamma_3^{(3)} \in \mathbb{Z}$. Similarly, $\{\{2,3\},\{2,4\}\}$ and $\{\{1,2\},\{1,4\},\{2,3\}\}$ give $\boldsymbol{\gamma}^{(1)} = (0,-\frac{4}{3},\frac{1}{2},0)$, $\boldsymbol{\gamma}^{(2)} = (0,-\frac{7}{12},0,-\frac{1}{4})$, $\boldsymbol{\gamma}^{(3)} = (1,-\frac{13}{12},0,-\frac{3}{4})$ and $\boldsymbol{\gamma}^{(1)} = (-\frac{1}{2},-\frac{1}{3},0,0)$, $\boldsymbol{\gamma}^{(2)} = (-\frac{7}{6},0,0,\frac{1}{3})$, $\boldsymbol{\gamma}^{(3)} = (0,-\frac{4}{3},\frac{1}{2},0)$, respectively.

As noted above, up to roots of unity the monodromy matrices are determined by the matrices corresponding to loops of variables z_i ($i \in I$) with $I^c \in \mathcal{S}(\mathcal{A})$ and $\Delta_I = 1$. We choose $I = \{3,4\}$ and find $(\gamma_1, \gamma_2) = (-\frac{1}{2} + \gamma_3 - 2\gamma_4, -\frac{1}{3} - 2\gamma_3 + \gamma_4)$ and hence $P_{j1} = -P_{j3}P_{j4}^{-2}$ and $P_{j2} = -\zeta P_{j3}^{-2}P_{j4}$, where $\zeta = e^{\frac{\pi i}{3}}$ and P_{jk} denotes the monodromy matrix corresponding to the j^{th} triangulation and a loop of z_k around the origin. The matrices P_{j3} and P_{j4} can be found in Table 4.2. Note that the monodromy matrices for different triangulations are written on different bases. ♢

Table 4.2: The local monodromy matrices for $G_3(\frac{1}{2}, \frac{1}{3}|x, y)$. Notation: $\xi = e^{\frac{\pi i}{9}}$ and $\zeta = e^{\frac{\pi i}{3}}$.

Triang.	I and $\boldsymbol{\gamma}$	Monodromy matrices
$\mathcal{T}_1$	$\{1,2\}$: $(0, 0, -\frac{7}{18}, -\frac{4}{9})$	$P_{13} = \mathrm{diag}(-\xi^2, -\xi^8, \xi^5)$
	$\{1,2\}$: $(0, 1, -\frac{19}{18}, -\frac{7}{9})$	$P_{14} = \mathrm{diag}(-\xi, \xi^4, -\xi^7)$
	$\{1,2\}$: $(1, 0, -\frac{13}{18}, -\frac{10}{9})$	
$\mathcal{T}_2$	$\{2,3\}$: $(-\frac{7}{6}, 0, 0, \frac{1}{3})$	$P_{23} = \mathrm{diag}(1, -\zeta^2, \zeta^2)$
	$\{2,4\}$: $(-\frac{2}{3}, 0, -\frac{1}{6}, 0)$	$P_{24} = \mathrm{diag}(\zeta^2, 1, 1)$
	$\{2,4\}$: $(-\frac{7}{6}, 1, -\frac{2}{3}, 0)$	
$\mathcal{T}_3$	$\{1,3\}$: $(0, -\frac{7}{12}, 0, -\frac{1}{4})$	$P_{33} = \mathrm{diag}(1, 1, -1)$
	$\{1,3\}$: $(1, -\frac{13}{12}, 0, -\frac{3}{4})$	$P_{34} = \mathrm{diag}(-i, i, 1)$
	$\{1,4\}$: $(0, -\frac{4}{3}, \frac{1}{2}, 0)$	
$\mathcal{T}_4$	$\{1,4\}$: $(0, -\frac{4}{3}, \frac{1}{2}, 0)$	$P_{43} = \mathrm{diag}(-1, 1, 1)$
	$\{2,3\}$: $(-\frac{7}{6}, 0, 0, \frac{1}{3})$	$P_{44} = \mathrm{diag}(1, \zeta^2, 1)$
	$\{3,4\}$: $(-\frac{1}{2}, -\frac{1}{3}, 0, 0)$	

Example 4.1.8. Our other example is the function $F_2(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$. Here we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\beta} = \left(-\tfrac{1}{12}, -\tfrac{3}{4}, -\tfrac{5}{6}, -\tfrac{1}{2}, -\tfrac{1}{3}\right)^t .$$

$\mathcal{S}(\mathcal{A})$ consists of the 15 subsets of $\mathcal{A}$ with 5 elements and non-zero determinant. We find 5 sets of subsets of $\mathcal{A}$, in which each pair (I, J) satisfies $Q(I) \cap Q(J) = Q(I \cap J)$. It turns out that these 5 sets all satisfy $C(\mathcal{A}) = \cup_{I \in \mathcal{T}} C(I)$, and hence are triangulations. The triangulations are: $\mathcal{T}_1 = \{\{1,2,3,4,5\}, \{1,2,3,4,7\}, \{1,2,3,5,6\}, \{1,2,3,6,7\}\}$, $\mathcal{T}_2 = \{\{1,2,4,5,7\}, \{1,2,5,6,7\}, \{2,3,4,5,7\}, \{2,3,5,6,7\}\}$, $\mathcal{T}_3 = \{\{1,3,4,5,6\}, \{1,3,4,6,7\}, \{2,3,4,5,6\}, \{2,3,4,6,7\}\}$, $\mathcal{T}_4 = \{\{1,4,5,6,7\}, \{2,3,4,5,6\}, \{2,3,4,6,7\}, \{3,4,5,6,7\}\}$ and $\mathcal{T}_5 = \{\{1,4,5,6,7\}, \{2,3,4,5,7\}, \{2,3,5,6,7\}, \{2,4,5,6,7\}\}$. For each of these we compute the vectors $\boldsymbol{\gamma}$ and the local monodromy matrices. We only consider loops of z_6 and z_7 around the origin, because $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (-\frac{1}{12} - \gamma_6 - \gamma_7, -\frac{3}{4} - \gamma_6, \frac{5}{6} - \gamma_7, -\frac{1}{2} + \gamma_6, \frac{1}{3} + \gamma_7)$ and hence $P_{j1} = -i\omega^2 P_{j6}^{-1} P_{j7}^{-1}$, $P_{j2} = iP_{j6}^{-1}$, $P_{j3} = \omega^2 P_{j7}^{-1}$, $P_{j4} = -P_{j6}$ and $P_{j5} = -\omega^2 P_{j7}$, where $\omega = e^{\frac{\pi i}{6}}$. The results can be found in Table 4.3. $\diamondsuit$

Table 4.3: The local monodromy matrices for $F_2(\frac{1}{12},\frac{3}{4},\frac{5}{6},\frac{1}{2},\frac{2}{3}|x,y)$. Notation: $\omega = e^{\frac{\pi i}{6}}$.

Triang.	I and γ	Monodromy matrices
$\mathcal{T}_1$	$\{4,5\}$: $(-\frac{11}{12},-\frac{5}{4},-\frac{7}{6},0,0,\frac{1}{2},\frac{1}{3})$	$P_{16}=\operatorname{diag}(-1,-1,1,1)$
	$\{4,7\}$: $(-\frac{7}{12},-\frac{5}{4},-\frac{5}{6},0,-\frac{1}{3},\frac{1}{2},0)$	$P_{17}=\operatorname{diag}(i\omega,1,i\omega,1)$
	$\{5,6\}$: $(-\frac{5}{12},-\frac{3}{4},-\frac{7}{6},-\frac{1}{2},0,0,\frac{1}{3})$	
	$\{6,7\}$: $(-\frac{1}{12},-\frac{3}{4},-\frac{5}{6},-\frac{1}{2},-\frac{1}{3},0,0)$	
$\mathcal{T}_2$	$\{1,4\}$: $(0,-\frac{5}{4},-\frac{1}{4},0,-\frac{11}{12},\frac{1}{2},-\frac{7}{12})$	$P_{26}=\operatorname{diag}(-1,1,-1,1)$
	$\{1,6\}$: $(0,-\frac{3}{4},-\frac{3}{4},-\frac{1}{2},-\frac{5}{12},0,-\frac{1}{12})$	$P_{27}=\operatorname{diag}(i\omega^2,-i\omega^2,\omega^2,\omega^2)$
	$\{3,4\}$: $(-\frac{1}{4},-\frac{5}{4},0,0,-\frac{7}{6},\frac{1}{2},-\frac{5}{6})$	
	$\{3,6\}$: $(\frac{3}{4},-\frac{3}{4},0,-\frac{1}{2},-\frac{7}{6},0,-\frac{5}{6})$	
$\mathcal{T}_3$	$\{1,5\}$: $(0,-\frac{1}{3},-\frac{7}{6},-\frac{11}{12},0,-\frac{5}{12},-\frac{1}{3})$	$P_{36}=\operatorname{diag}(-\omega,-i\omega^2,i,i)$
	$\{1,7\}$: $(0,-\frac{2}{3},-\frac{5}{6},-\frac{7}{12},-\frac{1}{3},-\frac{1}{12},0)$	$P_{37}=\operatorname{diag}(i\omega,1,i\omega,1)$
	$\{2,5\}$: $(\frac{1}{3},0,-\frac{7}{6},-\frac{5}{4},0,-\frac{3}{4},\frac{1}{3})$	
	$\{2,7\}$: $(\frac{2}{3},0,-\frac{5}{6},-\frac{5}{4},-\frac{1}{3},-\frac{3}{4},0)$	
$\mathcal{T}_4$	$\{1,2\}$: $(0,0,-\frac{3}{2},-\frac{5}{4},\frac{1}{3},-\frac{3}{4},\frac{2}{3})$	$P_{46}=\operatorname{diag}(i,-\omega,-i\omega^2,i)$
	$\{1,5\}$: $(0,-\frac{1}{3},-\frac{7}{6},-\frac{11}{12},0,-\frac{5}{12},-\frac{1}{3})$	$P_{47}=\operatorname{diag}(-\omega^2,i\omega,1,\omega^2)$
	$\{1,7\}$: $(0,-\frac{2}{3},-\frac{5}{6},-\frac{7}{12},-\frac{1}{3},-\frac{1}{12},0)$	
	$\{2,3\}$: $(\frac{3}{2},0,0,-\frac{5}{4},-\frac{7}{6},-\frac{3}{4},-\frac{5}{6})$	
$\mathcal{T}_5$	$\{1,3\}$: $(0,-\frac{3}{2},0,\frac{1}{4},-\frac{7}{6},\frac{3}{4},-\frac{5}{6})$	$P_{56}=\operatorname{diag}(-i,-1,1,i)$
	$\{1,4\}$: $(0,-\frac{5}{4},-\frac{1}{4},0,-\frac{11}{12},\frac{1}{2},-\frac{7}{12})$	$P_{57}=\operatorname{diag}(\omega^2,i\omega^2,-i\omega^2,\omega^2)$
	$\{1,6\}$: $(0,-\frac{3}{4},-\frac{3}{4},-\frac{1}{2},-\frac{5}{12},0,-\frac{1}{12})$	
	$\{2,3\}$: $(\frac{3}{2},0,0,-\frac{5}{4},-\frac{7}{6},-\frac{3}{4},-\frac{5}{6})$	

4.2 Convergence directions and the secondary fan

Using triangulations of $\mathcal{A}$, we constructed a basis of Γ-series solutions of $H_{\mathcal{A}}(\beta)$. On this basis, it is easy to compute the local monodromy matrices. The question arises which triangulations lead to a basis of series with a common domain of convergence. In this section we study the convergence of Γ-series and show that the convergence directions of $\Phi_{I,\gamma}$ only depend on I. We explain how the convergence directions are related to the secondary fan, and why the maximal cones of the secondary fan give triangulations of $\mathcal{A}$. We then show that this gives a bijection between the bases of Γ-series with a common convergence direction, equivalence classes of convergence directions (or maximal cones of the secondary fan) and regular triangulations. Hence we can compute the bases of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$ by computing the triangulations corresponding to the maximal cones of the secondary fan.

We start by studying the convergence directions of Γ-series.

Definition 4.2.1. A vector $\boldsymbol{\rho} \in \mathbb{R}^n$ is called a *convergence direction* of a Γ-series Φ if there exists $\varepsilon > 0$ such that $\Phi(\boldsymbol{z})$ converges for all $\boldsymbol{z} \in \mathbb{C}^n$ with $|z_j| = t^{\sigma_j}$ and $t > 0$ sufficiently small, for all $\boldsymbol{\sigma} \in \mathbb{R}^N$ such that $\max(|\sigma_i - \rho_i|) < \varepsilon$.

Lemma 4.2.2. *Let $\Phi_{\boldsymbol{\gamma}}$ have summation sector with index I. Then $\boldsymbol{\rho} \in \mathbb{R}^N$ is a convergence direction of $\Phi_{\boldsymbol{\gamma}}$ if and only if $\boldsymbol{\rho} \cdot \boldsymbol{l} > 0$ for all $\boldsymbol{l} \neq 0$ in the summation sector $V(I)$.*

To prove this lemma, we need two estimates on the growth of the coefficients of Γ-series.

Lemma 4.2.3. *Let $s \in \mathbb{C}$. Asymptotically for $n \to \infty$, we have*

$$\frac{1}{\Gamma(s+n)} \sim \frac{n^{1-s}}{n!} \qquad \text{and} \qquad \frac{1}{\Gamma(s-n)} \sim (-1)^n \frac{\sin(\pi s)}{\pi} \frac{n!}{n^s}.$$

Proof. It follows from Euler's identity $\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}$ that

$$\lim_{n\to\infty} \frac{n!}{n^{1-s}\Gamma(s+n)} = \lim_{n\to\infty} \frac{n!}{n^{1-s}s(s+1)\cdots(s+n-1)\Gamma(s)} = \lim_{n\to\infty} \frac{s+n}{n} = 1.$$

For the second statement, we apply the reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$. □

Lemma 4.2.4. *Let $\boldsymbol{l} = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that $l_1 + \ldots + l_N = 0$. Write $|\boldsymbol{l}| = |l_1| + \ldots + |l_N|$. Then*

$$\frac{1}{N^{\frac{|\boldsymbol{l}|}{2}}} \leq \frac{\prod_{l_j<0}(|l_j|!)}{\prod_{l_j>0}(l_j!)} \leq N^{\frac{|\boldsymbol{l}|}{2}}.$$

Proof. Note that $\sum_{l_j<0} |l_j| = \sum_{l_j>0} l_j = \frac{|\boldsymbol{l}|}{2}$. This implies that $\prod_{l_j<0}(|l_j|!) \leq (\sum_{l_j<0} |l_j|)! = \left(\frac{|\boldsymbol{l}|}{2}\right)!$. Furthermore,

$$\sum_{k_1+\ldots+k_N=\frac{|\boldsymbol{l}|}{2}} \frac{1}{k_1! \cdot \ldots \cdot k_N!} \geq \frac{1}{\max(l_1,0)! \cdot \ldots \cdot \max(l_N,0)!} = \frac{1}{\prod_{l_i>0} l_i!}.$$

Putting this together, we get

$$\frac{\prod_{l_j<0}(|l_j|!)}{\prod_{l_j>0}(l_j!)} \leq \frac{\left(\frac{|\boldsymbol{l}|}{2}\right)!}{\prod_{l_j>0} l_j!} \leq \sum_{k_1+\ldots+k_N=\frac{|\boldsymbol{l}|}{2}} \frac{\left(\frac{|\boldsymbol{l}|}{2}\right)!}{k_1! \cdot \ldots \cdot k_N!} = (1+\ldots+1)^{\frac{|\boldsymbol{l}|}{2}} = N^{\frac{|\boldsymbol{l}|}{2}}.$$

For the other inequality, the proof is similar. □

Proof of Lemma 4.2.2. First suppose that $\boldsymbol{\rho} \cdot \boldsymbol{l} > 0$ for all $\boldsymbol{l} \neq 0$ in the summation sector. Let $\boldsymbol{z} \in \mathbb{C}^n$ such that $|z_j| = t^{\sigma_j}$. We estimate each of the terms in the series $\Phi_{\boldsymbol{\gamma}}$ and show that there exists ε such that the series converges if $t > 0$ is sufficiently small and $\max(|\sigma_j - \rho_j|) < \varepsilon$. By choosing ε small, we can ensure that $\boldsymbol{\sigma} \cdot \boldsymbol{l} > 0$ for all $\boldsymbol{l} \neq \mathbf{0}$ in the summation sector. For $i \in I$, let $\boldsymbol{l}^{(i)}$ be the vector in $\mathbb{L} \otimes \mathbb{R}$ with $l_i = 1$ and $l_k = 0$ for all $k \in I \setminus \{i\}$. Define $\tau = \min_{i \in I} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{l}^{(i)}}{2|\boldsymbol{l}^{(i)}|}$. Then $\tau > 0$ and for all $i \in I$ we have $\boldsymbol{\sigma} \cdot \boldsymbol{l}^{(i)} > \tau |\boldsymbol{l}^{(i)}|$. Let $\boldsymbol{l} \neq 0$ in the summation sector. Then we can write $\boldsymbol{l} = \sum_{i \in I} \lambda_i \boldsymbol{l}^{(i)}$ with $\lambda_i \geq 0$. Hence

$$\boldsymbol{\sigma} \cdot \boldsymbol{l} = \sum_{i \in I} \lambda_i \boldsymbol{\sigma} \cdot \boldsymbol{l}^{(i)} > \sum_{i \in I} \lambda_i \tau |\boldsymbol{l}^{(i)}| \geq \tau \left| \sum_{i \in I} \lambda_i \boldsymbol{l}^{(i)} \right| = \tau |\boldsymbol{l}|.$$

Take $t < 1$. Then

$$\left| \prod_{j=1}^{N} z_j^{l_j + \gamma_j} \right| = \prod_{j=1}^{N} t^{\sigma_j (l_j + \gamma_j)} = t^{\boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}} < t^{\tau |\boldsymbol{l}| + \sigma \cdot \boldsymbol{\gamma}}.$$

By Lemma 4.2.3 there exist constants c_j such that for all $\boldsymbol{l} \in \mathbb{L}$ we have $|\frac{1}{\Gamma(l_j + \gamma_j + 1)}| \leq c_j \frac{l_j^{-\gamma_j}}{l_j!}$ if $l_j > 0$ and $|\frac{1}{\Gamma(l_j + \gamma_j + 1)}| \leq c_j \frac{|l_j|!}{|l_j|^{\gamma_j + 1}}$ if $l_j < 0$. Hence there is a constant C such that for all $\boldsymbol{l} \in \mathbb{L}$

$$\left| \prod_{j=1}^{N} \frac{1}{\Gamma(l_j + \gamma_j + 1)} \right| \quad \leq \quad C \quad \cdot \quad \frac{\prod_{l_j < 0} |l_j|!}{\prod_{l_j > 0} l_j!} \quad \cdot \quad \frac{1}{\prod_{l_j > 0} l_j^{\gamma_j} \cdot \prod_{l_j < 0} |l_j|^{\gamma_j + 1}}. \tag{4.2}$$

We apply Lemma 4.2.4 to the first term. For the second term, there exists a constant D, dependent on $\boldsymbol{\gamma}$, such that this term is smaller than $|\boldsymbol{l}|^{c_2}$. Hence we find that

$$\left| \prod_{j=1}^{N} \frac{1}{\Gamma(l_j + \gamma_j + 1)} \right| \leq C \cdot N^{\frac{|\boldsymbol{l}|}{2}} \cdot |\boldsymbol{l}|^D.$$

It follows that

$$|\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})| = \left| \sum_{\boldsymbol{l} \in \mathbb{L}} \prod_{j=1}^{N} \frac{z_j^{l_j + \gamma_j}}{\Gamma(l_j + \gamma_j + 1)} \right| \leq \sum_{\boldsymbol{l} \in \mathbb{L}} C t^{\tau |\boldsymbol{l}| + \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}} \cdot N^{\frac{|\boldsymbol{l}|}{2}} \cdot |\boldsymbol{l}|^D.$$

The convergence of $\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})$ depends on the growth of the exponential factors $t^{\tau |\boldsymbol{l}|}$ and $N^{\frac{|\boldsymbol{l}|}{2}}$. By choosing t sufficiently small, compared to N, we can arrange that the product of these factors decreases exponentially. Hence $\Phi_{\boldsymbol{\gamma}}$ converges.

Now suppose that $\boldsymbol{\rho} \cdot \boldsymbol{l}_0 \leq 0$ for some $\boldsymbol{l}_0$ in the summation sector. Again we estimate the terms of the sequence. We apply the lower bound from Lemma 4.2.4 to formula (4.2) to obtain

$$|\Phi_{\boldsymbol{\gamma}}(\boldsymbol{z})| \geq C \sum_{\boldsymbol{l} \in \mathbb{L}} t^{\boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}} \cdot \frac{1}{N^{\frac{|\boldsymbol{l}|}{2}}} \cdot |\boldsymbol{l}|^D$$

for some constants C and D. If this series converges, then the sum over the multiples of $\boldsymbol{l}_0$ also converges. However, by choosing $\boldsymbol{\sigma}$ close enough to $\boldsymbol{\rho}$, we can arrange that $\boldsymbol{\sigma} \cdot \boldsymbol{l}_0 < 0$. Now $t^{\boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{\sigma} \cdot \boldsymbol{\gamma}}$ decreases exponentially, so if t if sufficiently small compared to N, then the series diverges. $\square$

Definition 4.2.5. Fix a choice for the Gale dual B. We denote the rows of B by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$ and the columns by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$ of $\mathbb{L}$. For $\boldsymbol{\rho} \in \mathbb{R}^N$, we define $\tilde{\boldsymbol{\rho}} = (\boldsymbol{\rho} \cdot \boldsymbol{w}_1, \ldots, \boldsymbol{\rho} \cdot \boldsymbol{w}_d) \in \mathbb{R}^d$.

Note that the map $\mathbb{R}^N \to \mathbb{R}^d : \boldsymbol{\rho} \mapsto \tilde{\boldsymbol{\rho}}$ is surjective: the kernel is spanned by the rows of the matrix A with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ and hence is r-dimensional.

By Lemma 4.2.2, the convergence directions of $\Phi_{I,\boldsymbol{\gamma}}$ only depend on I. Hence we can speak of the convergence directions of (the index of) a summation sector instead of the convergence directions of a Γ-series. We have the following alternative description of the convergence directions of I:

Lemma 4.2.6. *Let I be the index of a summation sector. Then $\boldsymbol{\rho} \in \mathbb{R}^N$ is a convergence direction of I if and only if there exist $\tau_i > 0$ such that $\tilde{\boldsymbol{\rho}} = \sum_{i \in I} \tau_i \boldsymbol{b}_i$.*

Proof. Let $\boldsymbol{\rho}$ be a convergence direction of I. Since the $r \times r$-matrix with columns $\boldsymbol{a}_j$ ($j \in I^c$) is invertible, for each $i \in I$, there exists a $\boldsymbol{l}^{(i)} \in \mathbb{L} \otimes \mathbb{R}$ such that $l_i^{(i)} = 1$ and $l_k^{(i)} = 0$ for all $k \in I \setminus \{i\}$. The vector $\boldsymbol{l}^{(i)}$ is an element of $V(I) \otimes \mathbb{R}$, and hence satisfies $\boldsymbol{\rho} \cdot \boldsymbol{l}^{(i)} > 0$. Define $\boldsymbol{\tau} = \boldsymbol{\rho} \cdot \boldsymbol{l}^{(i)}$. For all $1 \leq k \leq d$, the k^{th} component of $\sum_{i \in I} \tau_i \boldsymbol{b}_i$ is $\sum_{i \in I} (\boldsymbol{\rho} \cdot \boldsymbol{l}^{(i)}) b_{ik} = \boldsymbol{\rho} \cdot (\sum_{i \in I} w_{ki} \boldsymbol{l}_{(i)})$. Note that vectors in the subspace of $\mathbb{R}^N$ spanned by $\boldsymbol{l}_{(i)}$ with $i \in I$ are completely determined by their i^{th} components. Hence $\sum_{i \in I} w_{ki} \boldsymbol{l}_{(i)} = \boldsymbol{w}_k$ and $\sum_{i \in I} \tau_i \boldsymbol{b}_i = \tilde{\boldsymbol{\rho}}$.

Now suppose that $\tilde{\boldsymbol{\rho}} = \sum_{i \in I} \tau_i \boldsymbol{b}_i$ with $\tau_i > 0$. Let $\boldsymbol{l} \in V(I)$ with $\boldsymbol{l} \neq \boldsymbol{0}$. Since $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$ is a $\mathbb{Z}$-basis of $\mathbb{L}$, there exist $n_1, \ldots, n_d$ such that $\boldsymbol{l} = n_1 \boldsymbol{w}_1 + \ldots + n_d \boldsymbol{w}_d$. Hence

$$\begin{aligned} \boldsymbol{\rho} \cdot \boldsymbol{l} = n_1 \boldsymbol{\rho}_1 \cdot \boldsymbol{w}_1 + \ldots + n_d \boldsymbol{\rho}_d \cdot \boldsymbol{w}_d = n_1 \tilde{\rho}_1 + \ldots + n_d \tilde{\rho}_d = \\ \sum_{i \in I} n_1 \tau_i b_{i1} + \ldots + n_d \tau_i b_{id} = \sum_{i \in I} \tau_i (n_1 w_{1i} + \ldots + n_d w_{di}) = \sum_{i \in I} \tau_i l_i. \end{aligned}$$

Now it follows from $\boldsymbol{l} \in V(I)$, $\boldsymbol{l} \neq \boldsymbol{0}$ and $\tau_i > 0$ that $\boldsymbol{\rho} \cdot \boldsymbol{l} > 0$. Hence $\boldsymbol{\rho}$ is a convergence direction of I. $\square$

By the above lemma, we can relate the convergence directions to the maximal cones in the secondary fan.

Definition 4.2.7. For $\boldsymbol{t} \in \mathbb{R}^d$ we define

$$T_{\boldsymbol{t}} = \{J \subseteq \{1, \dots, N\} \mid (\boldsymbol{b}_i)_{i \in J^c} \text{ are linearly independent; } \exists \tau_i > 0 : \boldsymbol{t} = \sum_{i \in J^c} \tau_i \boldsymbol{b}_i\}.$$

Let $\sim$ be the equivalence relation on $\mathbb{R}^d$ defined by $\boldsymbol{t} \sim \boldsymbol{t}'$ if and only if $T_{\boldsymbol{t}} = T_{\boldsymbol{t}'}$. The *secondary fan of* $\mathcal{A}$ is the set of equivalence classes.

One easily sees that the equivalence class of $\boldsymbol{t}$ is

$$[\boldsymbol{t}] = \bigcap_{J \in T_{\boldsymbol{t}}} \{\sum_{i \in J^c} \tau_i \boldsymbol{b}_i \mid \tau_i > 0\}.$$

Hence the secondary fan consists of open cones. The equivalence class of $\boldsymbol{t}$ is an open cone of maximal dimension if and only if all $J \in T_{\boldsymbol{t}}$ have r elements. In this case, Lemma 1.2.12 implies that

$$T_{\boldsymbol{t}} = \{J \in \mathcal{S}(\mathcal{A}) \mid \exists \tau_i > 0 : \boldsymbol{t} = \sum_{i \in J^c} \tau_i \boldsymbol{b}_i\}.$$

The following is immediate from Lemma 4.2.6:

Corollary 4.2.8. *Let I be the index of a summation sector. Then the convergence directions of I are $\{\boldsymbol{\rho} \in \mathbb{R}^N \mid I^c \in T_{\tilde{\boldsymbol{\rho}}}\}$.*

Definition 4.2.9. We write $\mathcal{C} = \{\boldsymbol{\rho} \in \mathbb{R}^N \mid [\tilde{\boldsymbol{\rho}}] \text{ is a cone of maximal dimension}\}$. Furthermore, if I is the index of a summation sector, let $CDir(I) = \{\boldsymbol{\rho} \in \mathcal{C} \mid I^c \in T_{\tilde{\boldsymbol{\rho}}}\}$. Define an equivalence relation on $\mathcal{C}$ by $\boldsymbol{\rho} \sim \boldsymbol{\rho}'$ if and only if $\tilde{\boldsymbol{\rho}} \sim \tilde{\boldsymbol{\rho}}'$.

Remark 4.2.10. Let $\boldsymbol{\rho} \in \mathcal{C}$. Then it follows from Corollary 4.2.8 that

$$T_{\tilde{\boldsymbol{\rho}}} = \{J \in \mathcal{S}(\mathcal{A}) \mid \boldsymbol{\rho} \in CDir(J^c)\}.$$

We summarize the results we obtained so far. In Section 4.1, we associated a basis of formal Γ-series to each triangulation of $\mathcal{A}$. We showed that the convergence of such series only depends on the summation sector, and explained the connected between convergence directions and cones in the secondary fan. Next, we will show that such cones give triangulations, the so-called regular triangulations. This will allow us to show that there are bijections between the set of regular triangulations, the bases of Γ-series with a common convergence direction and the maximal cones in the secondary fan.

We first need a lemma:

Lemma 4.2.11. *Let $\boldsymbol{\lambda} \in \mathbb{R}^N$ with $\lambda_j > 0$ for all j. Define $M_{\boldsymbol{\lambda}} = \{f : \mathbb{R}^r \to \mathbb{R} \text{ homomorphism} \mid \forall j : f(\boldsymbol{a}_j) + \lambda_j \geq 0\}$. Let $\pi : \mathbb{R}^N \to \mathbb{R}^d : \boldsymbol{y} \mapsto (\boldsymbol{y} \cdot \boldsymbol{w}_1, \dots, \boldsymbol{y} \cdot \boldsymbol{w}_d)$ and let $\hat{\pi}$ be the restriction of π to $(\mathbb{R}_{\geq 0})^N$. Then the following holds:*
(i) For all $\boldsymbol{y} \in \mathbb{R}^N$, we have $\pi(\boldsymbol{y}) = y_1 \boldsymbol{b}_1 + \ldots + y_N \boldsymbol{b}_N$.
(ii) If $\boldsymbol{t} = \lambda_1 \boldsymbol{b}_1 + \ldots + \lambda_N \boldsymbol{b}_N$, then $\hat{\pi}^{-1}(\boldsymbol{t}) = \boldsymbol{\lambda} + M_{\boldsymbol{\lambda}}$, where $M_{\boldsymbol{\lambda}}$ is viewed as a subset of $\mathbb{R}^N$ by $f \mapsto (f(\boldsymbol{a}_1), \dots, f(\boldsymbol{a}_N))$.
(iii) Let $\boldsymbol{x} \in \mathbb{R}^N$ with $x_j \geq 0$ for all j, and let $\boldsymbol{t} \in \mathbb{R}^d$. Then $\boldsymbol{x}$ is a vertex of $\hat{\pi}^{-1}(\boldsymbol{t})$ if and only if $\boldsymbol{t} = x_1 \boldsymbol{b}_1 + \ldots + x_N \boldsymbol{b}_N$ and $\{j \mid x_j = 0\} \in T_{\boldsymbol{t}}$.

Proof. *(i)* This follows from an easy calculation.
(ii) It follows from *(i)* that $\pi(\boldsymbol{\lambda}) = \boldsymbol{t}$, so $\boldsymbol{\lambda} \in \hat{\pi}^{-1}(\boldsymbol{t})$. Since we have the exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^N \longrightarrow \mathbb{Z}^r \longrightarrow 0 \qquad \text{with dual} \qquad 0 \longrightarrow \mathbb{R}^r \longrightarrow \mathbb{R}^N \xrightarrow{\pi} \mathbb{R}^d \longrightarrow 0,$$

the kernel of π is the image of $\mathbb{R}^r$ in $\mathbb{R}^N$ under the map $\boldsymbol{v} \mapsto (\boldsymbol{v}\cdot\boldsymbol{a}_1, \ldots, \boldsymbol{v}\cdot\boldsymbol{a}_N)$. Hence

$$\begin{aligned}
&\boldsymbol{x} \in \hat{\pi}^{-1}(\boldsymbol{t}) \Leftrightarrow \boldsymbol{x} - \boldsymbol{\lambda} \in \ker\pi \text{ and } \forall j : x_j \geq 0 \Leftrightarrow \\
&\qquad \exists \boldsymbol{v} \in \mathbb{R}^r : \boldsymbol{x} - \boldsymbol{\lambda} = (\boldsymbol{v}\cdot\boldsymbol{a}_1, \ldots, \boldsymbol{v}\cdot\boldsymbol{a}_N) \text{ and } (\boldsymbol{x} - \boldsymbol{\lambda})_j + \lambda_j \geq 0 \Leftrightarrow \\
&\qquad\qquad \boldsymbol{x} - \boldsymbol{\lambda} \in M_{\boldsymbol{\lambda}} \Leftrightarrow \boldsymbol{x} \in \boldsymbol{\lambda} + M_{\boldsymbol{\lambda}}.
\end{aligned}$$

(iii) Suppose that $\boldsymbol{x}$ is a vertex of $\hat{\pi}^{-1}(\boldsymbol{t})$. Then $\pi(\boldsymbol{x}) = \boldsymbol{t}$, so $\boldsymbol{t} = x_1\boldsymbol{b}_1 + \ldots + x_N\boldsymbol{b}_N$. Hence $\{j \mid x_j = 0\} \in T_{\boldsymbol{t}}$ if and only if $\{\boldsymbol{b}_i \mid x_i \neq 0\}$ is a linearly independent set. So suppose that there is a non-trivial relation $\sum_i v_i\boldsymbol{b}_i = 0$ with $v_i = 0$ if $x_i = 0$. We can choose $|v_i| \leq x_i$ for all i. Then the set $\{\boldsymbol{x} + \mu\boldsymbol{v} \mid -1 \leq \mu \leq 1\}$ lies in $\hat{\pi}^{-1}(\boldsymbol{t})$. Hence $\boldsymbol{x}$ is the midpoint of an interval in $\hat{\pi}^{-1}(\boldsymbol{t})$, so it is not a vertex of $\hat{\pi}^{-1}(\boldsymbol{t})$.

Conversely, suppose that $\boldsymbol{t} = x_1\boldsymbol{b}_1 + \ldots + x_N\boldsymbol{b}_N$ but $\boldsymbol{x}$ is not a vertex of $\hat{\pi}^{-1}(\boldsymbol{t})$. Then there is an interval of the form $\{\boldsymbol{x} + \mu\boldsymbol{v} \mid -1 \leq \mu \leq 1\}$ contained in $\hat{\pi}^{-1}(\boldsymbol{t})$. It follows that $|v_i| \leq x_i$ for all i, so in particular, if $x_i = 0$ then $v_i = 0$. Hence there is a non-trivial relation $\sum_{x_i \neq 0} v_i\boldsymbol{b}_i = 0$ and the vectors $\{\boldsymbol{b}_i \mid x_i \neq 0\}$ are linearly dependent. $\square$

Lemma 4.2.12. *Let $\boldsymbol{t} \in \mathbb{R}^d$ such that the equivalence class of $\boldsymbol{t}$ in the secondary fan is an open cone of maximal dimension. Then $T_{\boldsymbol{t}}$ is a triangulation of $\mathcal{A}$.*

Proof. By definition, $T_{\boldsymbol{t}} \subseteq \mathcal{S}(\mathcal{A})$. It suffices to show that $Q(\mathcal{A}) \subseteq \cup_{J \in T_{\boldsymbol{t}}} Q(J)$ and that $Q(I) \cap Q(J) \subseteq Q(I \cap J)$ for all $I, J \in T_{\boldsymbol{t}}$.

To show that $Q(\mathcal{A}) \subseteq \cup_{J \in T_{\boldsymbol{t}}} Q(J)$, let $\boldsymbol{x} \in Q(\mathcal{A})$. Choose $\lambda_i > 0$ such that $\boldsymbol{t} = \sum_{i \in I} \lambda_i\boldsymbol{b}_i$ for some $I \in T_{\boldsymbol{t}}$. Since the equivalence class of $\boldsymbol{t}$ is an open cone, we can shift $\boldsymbol{t}$ a little bit in the direction of $\boldsymbol{b}_j$ with $j \notin I$, while staying in the same equivalence class. Hence we can assume that $\boldsymbol{t} = \lambda_1\boldsymbol{b}_1 + \ldots + \lambda_N\boldsymbol{b}_N$ with $\lambda_i > 0$ for all i. There is a multiple $\mu\boldsymbol{x}$ of $\boldsymbol{x}$ on a codimension 1 face of $Q(\boldsymbol{0}, \frac{1}{\lambda_1}\boldsymbol{a}_1, \ldots, \frac{1}{\lambda_N}\boldsymbol{a}_N)$. This face is spanned by $\{\frac{1}{\lambda_j}\boldsymbol{a}_j \mid j \in J\}$ for some $J \in \mathcal{S}(\mathcal{A})$. Since both $\boldsymbol{x}$ and all $\boldsymbol{a}_j$ satisfy $h(\boldsymbol{x}) = 1$ for some linear form (as in Definition 1.2.1), we have $\boldsymbol{x} \in Q(J)$.

It remains to show that $J \in T_{\boldsymbol{t}}$. There is a linear map $f : \mathbb{R}^r \to \mathbb{R}$ such that $f(\frac{1}{\lambda_j}\boldsymbol{a}_j) = -1$ for all $j \in J$ and $f(\frac{1}{\lambda_i}\boldsymbol{a}_i) > -1$ for all $i \in J^c$. Now f is an element of $M_{\boldsymbol{\lambda}}$, and it is even a vertex of $M_{\boldsymbol{\lambda}}$ since $f(\boldsymbol{a}_j) + \lambda_j = 0$ for r independent vectors $\boldsymbol{a}_j$. Lemma 4.2.11*(ii)* implies that $M_{\boldsymbol{\lambda}} = \hat{\pi}^{-1}(\boldsymbol{t}) - \boldsymbol{\lambda}$. Hence f is of the form $\boldsymbol{x} - \boldsymbol{\lambda}$ with $\boldsymbol{x}$ a vertex of $\hat{\pi}^{-1}(\boldsymbol{t})$. It follows for Lemma 4.2.11*(iii)* that $\{j \mid x_j = 0\} \in T_{\boldsymbol{t}}$, i.e., $\{j \mid f(\boldsymbol{a}_j) = \lambda_j\} \in T_{\boldsymbol{t}}$. This last set equals J, so $J \in T_{\boldsymbol{t}}$.

Now we show that $Q(I) \cap Q(J) \subseteq Q(I \cap J)$ for all $I, J \in T_{\boldsymbol{t}}$. Let $I, J \in T_{\boldsymbol{t}}$ and $\boldsymbol{p} \in Q(I) \cap Q(J)$. There exist $\lambda_i, \mu_j \geq 0$ such that $\boldsymbol{p} = \sum_{i \in I} \lambda_i\boldsymbol{a}_i = \sum_{j \in J} \mu_j\boldsymbol{a}_j$. Define $\boldsymbol{l} \in \mathbb{R}^N$ by $l_i = \lambda_i$ if $i \in I \setminus J$, $l_i = -\mu_i$ if $i \in J \setminus I$, $l_i = \lambda_i - \mu_i$ if $i \in I \cap J$ and $l_i = 0$ if $i \notin I \cup J$. Then $A\boldsymbol{l} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$, so $\boldsymbol{l} \in \mathbb{L} \otimes \mathbb{R}$. There exists $\boldsymbol{\rho} \in \mathbb{R}^N$ such that $\tilde{\boldsymbol{\rho}} = \boldsymbol{t}$. By Lemma 4.2.10, we have $\boldsymbol{\rho} \in CDir(I^c) \cap CDir(J^c)$. If $j \in I^c$, then l_j

equals 0 or $-\mu_j$, so in either case, $-l_i \geq 0$. Hence $-\boldsymbol{l} \in V(I^c) \otimes \mathbb{R}$, and hence $\boldsymbol{l} = 0$ or $\boldsymbol{\rho} \cdot \boldsymbol{l} < 0$. Similarly, $\boldsymbol{l} \in V(J^c) \otimes \mathbb{R}$, and $\boldsymbol{l} = 0$ or $\boldsymbol{\rho} \cdot \boldsymbol{l} > 0$. It follows that $\boldsymbol{l} = 0$ and hence $\boldsymbol{p} \in Q(I \cap J)$. □

Definition 4.2.13. A triangulation is called *regular* if it is of the form $T_{\tilde{\boldsymbol{\rho}}}$ with $\boldsymbol{\rho} \in \mathcal{C}$.

Lemma 4.2.14. *Let $H_{\mathcal{A}}(\boldsymbol{\beta})$ be totally non-resonant and let $\{\Phi_{I_1,\boldsymbol{\gamma}_1}, \ldots, \Phi_{I_n,\boldsymbol{\gamma}_n}\}$ a basis of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$ with $\Phi_{I_i,\boldsymbol{\gamma}_i}$ of the form* (4.1) *with summation sector I_i. Suppose that $I_1, \ldots, I_n$ have a common direction of convergence $\boldsymbol{\rho}$. Then for all i we have $|\{j \mid I_j = I_i\}| = \Delta_{I_i}$ and $T_{\tilde{\boldsymbol{\rho}}} = \{I_1^c, \ldots, I_n^c\}$.*

Proof. By Proposition 4.1.4, for each I_i there are at most Δ_{I_i} series with summation sector with index I_i. Note that $I_i^c \in T_{\tilde{\boldsymbol{\rho}}}$ by Remark 4.2.10. Since $T_{\tilde{\boldsymbol{\rho}}}$ is a triangulation, we have

$$n \leq \sum_{I_i} \Delta_{I_i} = \sum_{I_i} \mathrm{Vol}(Q(I_i^c)) \leq \sum_{J \in T_{\tilde{\boldsymbol{\rho}}}} \mathrm{Vol}(Q(J)) = \mathrm{Vol}(Q(\mathcal{A})) = n.$$

Hence the inequalities must be equalities, so each I_i occurs Δ_{I_i} times and $T_{\tilde{\boldsymbol{\rho}}} = \{I_1^c, \ldots, I_n^c\}$. □

Theorem 4.2.15. *Suppose that $H_{\mathcal{A}}(\boldsymbol{\beta})$ is totally non-resonant. Let $RTr(\mathcal{A})$ be the set of regular triangulations of $\mathcal{A}$, and let Sol be the set of all bases of power series solutions of $H_{\mathcal{A}}(\boldsymbol{\beta})$ consisting of Γ-series $\Phi_{I,\boldsymbol{\gamma}}$ of the form* (4.1) *with summation sector I such that all I have a common convergence direction in $\mathcal{C}$. Define*

$$\begin{aligned}
&\Phi : RTr(\mathcal{A}) \to Sol : \mathcal{T} \mapsto \{\Phi_{I,\boldsymbol{\gamma}} \mid I^c \in \mathcal{T}, \gamma_j \in \mathbb{Z} \text{ for all } j \in I^c\},\\
&\Psi : Sol \to \mathcal{C}/\sim : \{\Phi_{I_1,\boldsymbol{\gamma}_1}, \ldots, \Phi_{I_n,\boldsymbol{\gamma}_n}\} \mapsto [\boldsymbol{\rho}] \text{ with } \boldsymbol{\rho} \in \cap_{i=1}^n CDir(I_i) \qquad \text{and}\\
&\Xi : \mathcal{C}/\sim \to RTr(\mathcal{A}) : [\boldsymbol{\rho}] \mapsto T_{\tilde{\boldsymbol{\rho}}}.
\end{aligned}$$

Then the maps Φ, Ψ and Ξ are well-defined bijections and $\Xi \circ \Psi \circ \Phi$, $\Phi \circ \Xi \circ \Psi$ and $\Psi \circ \Phi \circ \Xi$ are the identity maps on $RTr(\mathcal{A})$, Sol and $\mathcal{C}/\sim$, respectively.

Proof. By definition of regular triangulations, Ξ is well-defined and surjective. It is injective by definition of $\sim$.

To show that Φ is well-defined, we have to show that the basis corresponding to a regular triangulation has a common convergence direction. Let $\mathcal{T}$ be a regular triangulation. Then there exists $\boldsymbol{\rho} \in \mathcal{C}$ such that $\mathcal{T} = T_{\tilde{\boldsymbol{\rho}}}$. By Corollary 4.2.8, $\boldsymbol{\rho}$ is a convergence direction for all I^c with $I \in \mathcal{T}$. Hence the Γ-series have a common convergence direction and Φ is well-defined. As the summation sector is uniquely determined by the series, the set $\{\Phi_{I,\boldsymbol{\gamma}} \mid I^c \in \mathcal{T}, \gamma_j \in \mathbb{Z} \text{ for all } j \in I^c\}$ determines the indices I of the summation sector, and hence $\mathcal{T}$. It follows that Φ is injective. For surjectivity, let $\{\Phi_{I_1,\boldsymbol{\gamma}_1}, \ldots, \Phi_{I_n,\boldsymbol{\gamma}_n}\}$ be a basis with a common convergence direction $\boldsymbol{\rho}$. Then it follows from Lemma 4.2.14 that $T_{\tilde{\boldsymbol{\rho}}} = \{I_1^c, \ldots, I_n^c\}$ and hence $\{\Phi_{I_1,\boldsymbol{\gamma}_1}, \ldots, \Phi_{I_n,\boldsymbol{\gamma}_n}\} = \Phi(T_{\tilde{\boldsymbol{\rho}}})$.

Let $\{\Phi_{I_1,\gamma_1},\dots,\Phi_{I_n,\gamma_n}\}\in Sol$ and suppose that $\boldsymbol{\rho},\boldsymbol{\sigma}\in\cap_{i=1}^n CDir(I_i)$. Then it follows from Lemma 4.2.14 that $T_{\tilde{\boldsymbol{\rho}}}=\{I_1^c,\dots,I_n^c\}=T_{\tilde{\boldsymbol{\sigma}}}$, so $\boldsymbol{\rho}\sim\boldsymbol{\sigma}$. Hence Ψ is well-defined.

Let $\boldsymbol{\rho}\in\mathcal{C}$. Then

$$\Psi(\Phi(\Xi([\boldsymbol{\rho}])))=\Psi(\Phi(T_{\tilde{\boldsymbol{\rho}}}))=\Psi(\{\Phi_{I,\gamma}\mid I^c\in T_{\tilde{\boldsymbol{\rho}}}\})=\Psi(\{\Phi_{I,\gamma}\mid\boldsymbol{\rho}\in CDir(I)\})=[\boldsymbol{\rho}]$$

where we use Remark 4.2.10 in the third equality. Hence $\Psi\circ\Phi\circ\Xi$ is the identity map on $V/\sim$. Since $\Phi\circ\Xi$ is a bijection, this implies that Ψ is also a bijection and $\Xi\circ\Psi\circ\Phi$ and $\Phi\circ\Xi\circ\Psi$ are the identity maps on $RTr(\mathcal{A})$ and Sol, respectively. ☐

Example 4.2.16. We continue Example 4.1.7, where we computed the basis corresponding to each of the triangulations of $\mathcal{A}$. These triangulations are $\mathcal{T}_1=\{\{3,4\}\}$, $\mathcal{T}_2=\{\{1,3\},\{1,4\}\}$, $\mathcal{T}_3=\{\{2,3\},\{2,4\}\}$ and $\mathcal{T}_4=\{\{1,2\},\{1,4\},\{2,3\}\}$. Now we compute the secondary fan to show that all triangulations are regular. We choose

$$B=\begin{pmatrix}1&-2&1&0\\-2&1&0&1\end{pmatrix}^t$$

as a Gale dual, so we have $\boldsymbol{b}_1=(1,-2)$, $\boldsymbol{b}_2=(-2,1)$, $\boldsymbol{b}_3=(1,0)$ and $\boldsymbol{b}_4=(0,1)$. Any two vectors $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$ are independent, so $\mathcal{S}(\mathcal{A})$ consists of all subsets of $\mathcal{A}$ with 2 elements. The vectors $\boldsymbol{b}_i$ have been drawn in Figure 4.1. The equivalence classes in the secondary fan are the intersections of the cones spanned by two vectors $\boldsymbol{b}_i$. One easily sees that there are four 2-dimensional open cones in the secondary fan, given by the four sets the plane is divided into by the vectors $\boldsymbol{b}_i$. For each cone, the sets $J\in\mathcal{S}(\mathcal{A})$ such that the cone is included in the positive span of $\{\boldsymbol{b}_i\mid i\in J^c\}$ are also given in Figure 4.1. In each cone these sets form a triangulation of $\mathcal{A}$, confirming Lemma 4.2.12. We find the same triangulations as we computed in Example 4.1.7, so all triangulations are regular. ♢

Example 4.2.17. For F_2, we computed the triangulations in Example 4.1.8. There we found that there are 5 triangulations. Now we check that they are regular. We fix the Gale dual

$$B=\begin{pmatrix}-1&-1&0&1&0&1&0\\-1&0&-1&0&1&0&1\end{pmatrix}^t.$$

The vectors $\boldsymbol{b}_i$ and the maximal open cones of the secondary fan have been drawn in Figure 4.2. As for G_3, for each cone one can compute the sets $J\in\mathcal{S}(\mathcal{A})$ such that the cone is included in the positive span of $\{\boldsymbol{b}_i\mid i\in J^c\}$. Again each cone corresponds to a triangulation, as indicated in Figure 4.2. We find the same 5 triangulations as in Example 4.1.8, so all triangulations are regular. ♢

Example 4.2.18. We give an example of a non-regular triangulation (reproduced from [BFS90]). The set

$$\begin{pmatrix}4&0&0&2&1&1\\0&4&0&1&2&1\\0&0&4&1&1&2\end{pmatrix}$$

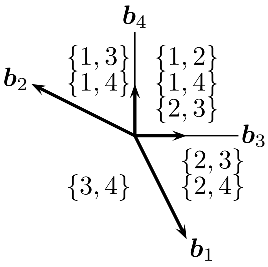


Figure 4.1: The secondary fan for G_3

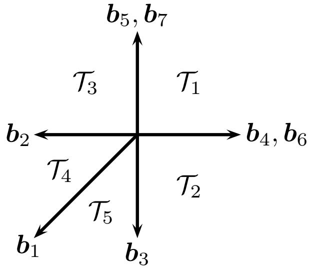


Figure 4.2: The secondary fan for F_2

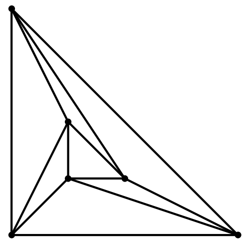


Figure 4.3: A non-regular triangulation

has 18 triangulations, including the one in Figure 4.3. This triangulation is given by $\mathcal{T} = \{\{1,2,4\},\{1,3,6\},\{1,4,6\},\{2,3,5\},\{2,4,5\},\{3,5,6\},\{4,5,6\}\}$. If it were a regular triangulation, there would be $\boldsymbol{t} \in \mathbb{R}^3$ such that $\mathcal{T} = T_{\boldsymbol{t}}$. Then $\boldsymbol{t} \in \bigcap_{J\in\mathcal{T}}\{\sum_{i\in J^c} \tau_i \boldsymbol{b}_i \mid \tau_i > 0\}$, where we can choose the vectors $\boldsymbol{b}_i$ equal to the rows of the Gale dual

$$B = \begin{pmatrix} 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{pmatrix}^t .$$

For $J = \{1,4,6\} \in \mathcal{T}$, we have $\{\sum_{i\in J^c} \tau_i \boldsymbol{b}_i \mid \tau_i > 0\} = \{\boldsymbol{s} \in \mathbb{R}^3 \mid s_3 > s_1 > 0, 3s_1 + s_2 > 0\}$. Similarly, $\{2,4,5\}$ and $\{3,5,6\}$ give $s_1 > s_2 > 0, 3s_2 + s_3 > 0$ and $s_2 > s_3 > 0, s_1 + 3s_3 > 0$, respectively. It is clear that they have empty intersection, contradicting the existence of $\boldsymbol{t}$. $\Diamond$

Using the bijections Ψ, Φ and Ξ we can compute the local monodromy matrices on local bases. The first step is to compute the set $\mathcal{S}(\mathcal{A})$. If $\mathcal{A}$ is sufficiently small, this can be done by computing the determinant of each subset with r elements. Then we compute the positive span of $\{\boldsymbol{b}_i \mid i \in J^c\}$ for all $J \in \mathcal{S}(\mathcal{A})$. We take the intersections of these spans to find the open cones of maximal dimension in the secondary fan. The corresponding sets J then give a triangulation of $\mathcal{A}$. Then we proceed as in Section 4.1 to compute local bases of Γ-series solutions and the local monodromy matrices.

For the Appell, Lauricella and Horn functions, we computed the triangulations in the previous section. There we did not check for regularity, so it is possible that we found bases of Γ-series without a common region of convergence. Therefore, we compute the secondary fans. For the Appell and Horn functions, the vectors $\boldsymbol{b}_i$ lie in $\mathbb{Z}^2$ and we can draw the secondary fan as we did in Examples 4.2.16 and 4.2.17 for G_3 and F_2. The results can be found in Figure 4.4. We see that all triangulations are regular.

The Lauricella F_D function with $n = 3$ has a 3-dimensional secondary fan. Performing the calculations outlined above, we find that it has 24 3-dimensional cones. Four of them are given by $0 < t_1 < t_2 < t_3$, $t_3 < 0 < t_1 < t_2$, $t_2 < t_3 < 0 < t_1$ and

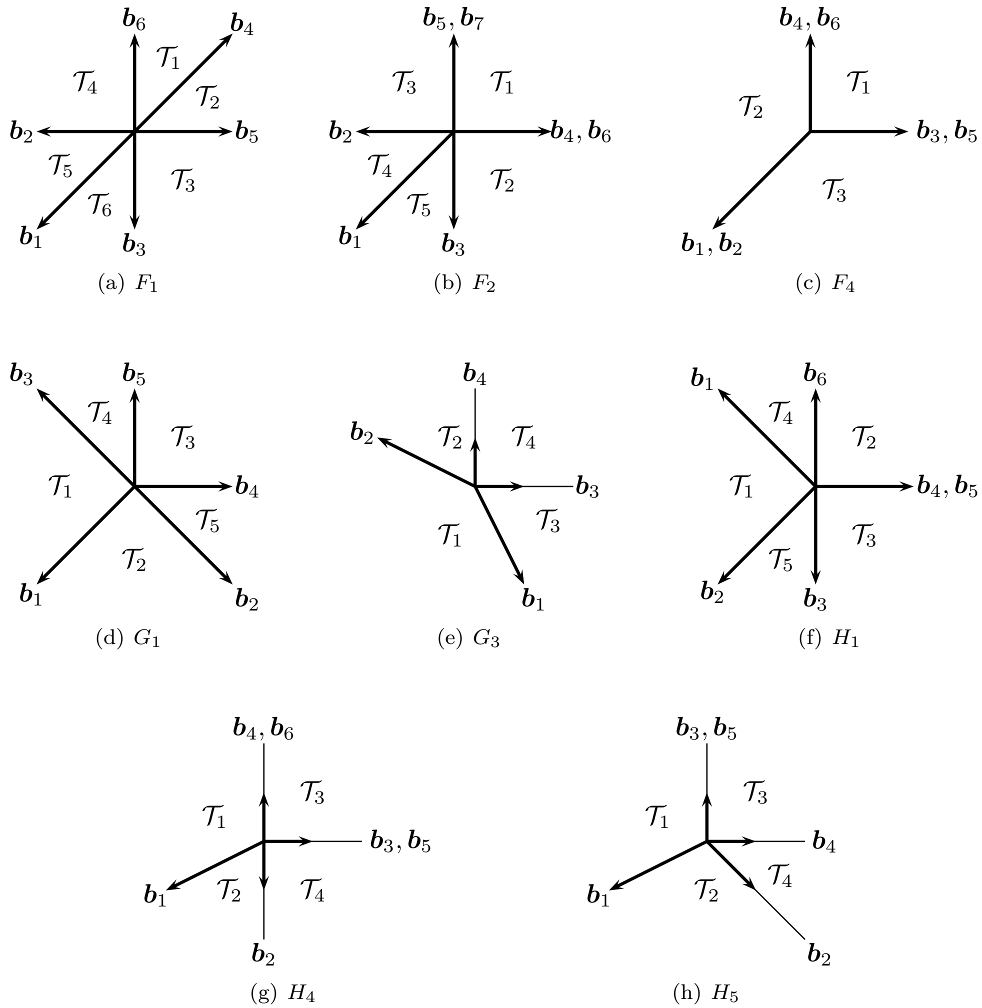


Figure 4.4: Secondary fans for the Appell and Horn functions

$t_1 < t_2 < t_3 < 0$, respectively. The other cones are obtained by permuting t_1, t_2 and t_3 in the above inequalities. The four cones given above give the triangulations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ as in Table 4.1. We see that all triangulations are regular. Again we don't give results for the F_C function, but the triangulations seem to be regular as well.

4.3 The local monodromy group on a basis of Mellin-Barnes integrals

In Section 4.1 we computed the local monodromy matrices on a basis of Γ-series. Each triangulation gives such a basis, leading to (in most cases) d monodromy matrices. However, the monodromy matrices corresponding to distinct triangulations are written on distinct bases. In this section we introduce Mellin-Barnes integrals and show how the local monodromy matrices can be written on a basis of Mellin-Barnes integrals, independent of the triangulation. Unfortunately, such a basis need not exist. However, for most Appell and Horn functions, there is indeed a basis of Mellin-Barnes integrals.

In 1908, Barnes [Bar08] showed that the Gauss function satisfies

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds. \tag{4.3} $$

The contour is taken so that the poles $0, 1, 2, \ldots$ are separated from the poles $-a, -a-1, \ldots, -b, -b-1, \ldots$. Recall that a Gale dual B of $\mathcal{A}$ is a $N \times d$ matrix, whose columns form a $\mathbb{Z}$-basis of the lattice $\mathbb{L}$. We denote the rows of B by $\boldsymbol{b_j}$. As a vector space, $\mathbb{L}(\mathbb{R}) = \mathbb{L} \otimes \mathbb{R}$ is isomorphic to $\mathbb{R}^d$. This isomorphism is explicitly given by B: the map $\boldsymbol{s} \mapsto (\boldsymbol{b}_1 \cdot \boldsymbol{s}, \ldots, \boldsymbol{b}_N \cdot \boldsymbol{s})$ maps $\mathbb{R}^d$ to $\mathbb{L}(\mathbb{R})$. Fix $\boldsymbol{\gamma} \in \mathbb{R}^N$ such that $\sum_{j=1}^N \gamma_j \boldsymbol{a}_j = \boldsymbol{\beta}$. The *Mellin-Barnes integral* is defined by

$$ M(\boldsymbol{z}) = \int_{i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s}) z_j^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s}} ds_1 \ldots ds_d. $$

For the Gauss function, $M(1,1,1,z)$ is up to a constant equal to the Barnes integral. In computing the Mellin-Barnes integrals, we have some freedom to choose the arguments of z_j. By making a suitable choice, the integral will converge and will be a solution of $H_{\mathcal{A}}(\boldsymbol{\beta})$. Moreover, if there are sufficiently many possibilities to choose the arguments, we will get a basis of Mellin-Barnes integrals. We can then write all monodromy matrices on this basis. We will now consider these steps in more detail.

We start by showing that the integrals satisfy the hypergeometric equations. We need the assumption $\gamma_j < 0$ for all $1 \leq j \leq N$. This cannot be satisfied for all $\boldsymbol{\beta}$. However, if $H_{\mathcal{A}}(\boldsymbol{\beta})$ is non-resonant, shifting $\boldsymbol{\beta}$ by an integral vector does not change the monodromy group (Theorem 1.3.8. By substracting a_j from $\boldsymbol{\beta}$ if $\gamma_j \geq 0$, we can arrange that $\gamma_j < 0$.

Lemma 4.3.1. *Suppose that the Mellin-Barnes integral $M(\boldsymbol{z})$ converges and assume that $\gamma_j < 0$ for all $1 \leq j \leq N$. Then $M(\boldsymbol{z})$ is a solution of $H_{\mathcal{A}}(\boldsymbol{\beta})$.*

Proof. We have to show that for all $1 \leq k \leq r$

$$(a_{1k}z_1\partial_1 + \ldots + a_{Nk}z_n\partial_N)M(z_1,\ldots,z_N) = \beta_k M(z_1,\ldots,z_N)$$

and

$$\prod_{l_j>0} \partial_j^{l_j} M(z_1,\ldots,z_N) = \prod_{l_j<0} \partial_j^{-l_j} M(z_1,\ldots,z_N)$$

for all $\boldsymbol{l} \in \mathbb{L}$. For the first set of equations, note that

$$(\sum_{l=1}^{N} a_{lk}z_l\partial_l)M(\boldsymbol{z}) = \int_{i\mathbb{R}^d} \sum_{l=1}^{N} a_{lk}(\gamma_l + \boldsymbol{b}_l \cdot \boldsymbol{s}) \cdot \prod_{j=1}^{N} \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s}) z_j^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s}} ds_1 \ldots ds_d.$$

$\sum_{l=1}^{N} a_{lk}\boldsymbol{b}_l$ is the k^{th} row of AB and hence equal to zero. It follows that $\sum_{l=1}^{N} a_{lk}(\gamma_l + \boldsymbol{b}_l \cdot \boldsymbol{s}) = \sum_{l=1}^{N} a_{lk}\gamma_l = \beta_k$ so the first set of equations is satisfied.

For the second set of equations, let $\boldsymbol{l} \in \mathbb{L}$ and write $\boldsymbol{l} = \boldsymbol{l}^+ - \boldsymbol{l}^-$, with $l_i^+ = \max(l_i, 0)$ and $l_i^- = \max(-l_i, 0)$. Let $|\boldsymbol{l}^+| = l_1^+ + \ldots + l_N^+$. Note that this equals $|l^-|$. Hence

$$\prod_{l_j>0} \partial_j^{l_j} M(\boldsymbol{z}) = \prod_{j=1}^{N} \partial_j^{l_j^+} M(\boldsymbol{z}) =$$

$$(-1)^{|l^+|} \int_{i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s} + l_j^+) z_j^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s} - l_j^+} ds_1 \ldots ds_d.$$

Choose $\boldsymbol{s_0} \in \mathbb{R}^d$ such that $(\boldsymbol{b}_1 \cdot \boldsymbol{s_0}, \ldots, \boldsymbol{b}_N \cdot \boldsymbol{s_0}) = \boldsymbol{l}$. Then

$$\prod_{l_j>0} \partial_j^{l_j} M(\boldsymbol{z}) = (-1)^{|l^-|} \int_{-\boldsymbol{s_0}+i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s} + l_j^-) z_j^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s} - l_j^-} ds_1 \ldots ds_d.$$

To show that this integral equals $\prod_{l_j<0} \partial_j^{-l_j} M(\boldsymbol{z})$, it suffices to show that the domain of integration can be changed from $-\boldsymbol{s_0} + i\mathbb{R}^d$ to $i\mathbb{R}^d$. Let $\boldsymbol{s} = \lambda\boldsymbol{s_0} + \boldsymbol{t}$ with $\lambda \in [-1, 0]$ and $\boldsymbol{t} \in i\mathbb{R}^d$. Then $\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s} - l_j^- = \gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{t} + \lambda l_j - l_j^-$, with real part in $[\gamma_j - l_j^+, \gamma_j]$ if $\boldsymbol{l}_j \geq 0$ and in $[\gamma_j - l_j^-, \gamma_j]$ if $\boldsymbol{l}_j \leq 0$. In both cases, the real part is negative because $\gamma_j < 0$. Hence none of the Γ-functions has a pole in $-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s} + l_j^-$ for $\boldsymbol{s} \in \{\lambda\boldsymbol{s_0} + \boldsymbol{t} \mid \lambda \in [-1, 0], \boldsymbol{t} \in i\mathbb{R}^d\}$ and the domain of integration can be changed from $-\boldsymbol{s_0} + i\mathbb{R}^d$ to $i\mathbb{R}^d$. $\square$

Next, we explain the conditions under which the integral converges. As noted before, the integral depends on the choice for the arguments of $\boldsymbol{z}$.

Definition 4.3.2. The *zonotope* of B is $Z_B = \{\frac{1}{4}\sum_{j=1}^{N} \lambda_j \boldsymbol{b}_j \mid \lambda_j \in (-1, 1)\}$.

Lemma 4.3.3. *Let $\boldsymbol{z} \in \mathbb{R}^N$ be given and fix a choice for the arguments $\theta_j = \arg(z_j)$. The Mellin-Barnes integral $M(\boldsymbol{z})$ converges absolutely if and only if $\sum_{j=1}^{N} \frac{\theta_j}{2\pi} \boldsymbol{b}_j \in Z_B$.*

Proof. Let $\boldsymbol{s} \in i\mathbb{R}^d$ and write $\boldsymbol{s} = i\boldsymbol{t}$ with $\boldsymbol{t} \in \mathbb{R}^d$. Note that for all $a, b \in \mathbb{R}$ we have $|z_j^{a+bi}| = |z_j|^a e^{-b\theta_j}$ and hence

$$\prod_{j=1}^{N} |z_j^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s}}| = \prod_{j=1}^{N} |z_j^{\gamma_j + i\boldsymbol{b}_j \cdot \boldsymbol{t}}| = \prod_{j=1}^{N} |z_j|^{\gamma_j} e^{-\boldsymbol{b}_j \cdot \boldsymbol{t}\theta_j} = c_1 \exp(\sum_{j=1}^{N} -(\boldsymbol{b}_j \cdot \boldsymbol{t})\theta_j)$$

where c_1 is a constant depending on $\boldsymbol{z}$ and $\boldsymbol{\gamma}$. Furthermore, we have the following estimate [AAR99, Corollary 1.4.4]: if $a \in (a_1, a_2)$ and $|b| \to \infty$, then

$$|\Gamma(a + bi)| = \sqrt{2\pi}|b|^{a-\frac{1}{2}} e^{-\frac{\pi|b|}{2}} \left(1 + O\left(\frac{1}{|b|}\right)\right)$$

with the constant of the O-term only depending on a_1 and a_2. The growth of the integrand of $M(\boldsymbol{z})$ is determined by the exponential factors

$$\exp\left(-\sum_{j=1}^{N}\left((\boldsymbol{b}_j \cdot \boldsymbol{t})\theta_j + \frac{\pi|\boldsymbol{b}_j \cdot \boldsymbol{t}|}{2}\right)\right).$$

It follows that the integral converges if and only if $\sum_{j=1}^{N}((\boldsymbol{b}_j \cdot \boldsymbol{t})\theta_j + \frac{\pi|\boldsymbol{b}_j \cdot \boldsymbol{t}|}{2}) > 0$ if a coordinate of $\boldsymbol{t}$ goes to $\pm\infty$. This happens if and only if $\sum_{j=1}^{N}((\boldsymbol{b}_j \cdot \boldsymbol{t})\theta_j + \frac{\pi|\boldsymbol{b}_j \cdot \boldsymbol{t}|}{2}) > 0$ for all $\boldsymbol{t} \in \mathbb{R}^d$. Substituting $-\boldsymbol{t}$ for $\boldsymbol{t}$, we find that the condition is equivalent to $\sum_{j=1}^{N} \frac{\pi}{2}|\boldsymbol{b}_j \cdot \boldsymbol{t}| > |(\sum_{j=1}^{N} \theta_j \boldsymbol{b}_j) \cdot \boldsymbol{t}|$.

Suppose that $\sum_{j=1}^{N} \frac{\theta_j}{2\pi}\boldsymbol{b}_j \in Z_B$. Then there exist $-1 < \lambda_j < 1$ such that $\sum_j \frac{\theta_j}{2\pi}\boldsymbol{b}_j = \frac{1}{4}\sum_j \lambda_j \boldsymbol{b}_j$. Hence

$$\left|\left(\sum_{j=1}^{N} \theta_j \boldsymbol{b}_j\right) \cdot \boldsymbol{t}\right| = \frac{\pi}{2}\left|\sum_{j=1}^{N} \lambda_j (\boldsymbol{b}_j \cdot \boldsymbol{t})\right| \leq \frac{\pi}{2}\sum_{j=1}^{N} |\lambda_j||\boldsymbol{b}_j \cdot \boldsymbol{t}| < \frac{\pi}{2}\sum_{j=1}^{N} |\boldsymbol{b}_j \cdot \boldsymbol{t}|$$

where the strict inequality follows from the fact that $|\lambda_j| < 1$ for all j and there exists j such that $\boldsymbol{t}$ is not orthogonal to $\boldsymbol{b}_j$, because $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$ span $\mathbb{R}^d$.

Conversely, suppose that $\sum_{j=1}^{N} \frac{\theta_j}{2\pi}\boldsymbol{b}_j \notin Z_B$ Since Z_B is a convex set, it is bounded by hyperplanes that are given by linear forms. There is a linear form, say $\boldsymbol{x} \mapsto \boldsymbol{x} \cdot \boldsymbol{t}$ for some $\boldsymbol{t} \in \mathbb{R}^d$, which is negative on Z_B but positive on $\sum_{j=1}^{N} \frac{\theta_j}{2\pi}\boldsymbol{b}_j$.

$$|(\sum_{j=1}^{N} \theta_j \boldsymbol{b}_j) \cdot \boldsymbol{t}| > 0 > \frac{\pi}{2}\sum_{j=1}^{N} \lambda_j \boldsymbol{b}_j \cdot \boldsymbol{t}$$

for all $\lambda_j \in (-1, 1)$. Now let λ_j approach ± 1 to find that $\sum_{j=1}^{N} \frac{\pi}{2}|\boldsymbol{b}_j \cdot \boldsymbol{t}| \leq |(\sum_{j=1}^{N} \theta_j \boldsymbol{b}_j) \cdot \boldsymbol{t}|$. □

To prove that different choices for $\arg(\boldsymbol{z})$ give independent power series, we use the Mellin-inverse theorem:

Theorem 4.3.4 ([McL53, Theorem 1 in Appendix 4]). *Let $\phi : \mathbb{C} \to \mathbb{C}$ be analytic on a strip $a < \Re(z) < b$. Suppose that $\int_{-\infty}^{\infty} |\phi(x+iy)|dy$ converges for all $a < x < b$ and that for all $\varepsilon > 0$ we have $\phi(z) \to 0$ for $|y| \to \infty$ uniformly in $x \in (a+\varepsilon, b-\varepsilon)$. Let $a < c < b$ and define $f(\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \theta^{-z}\phi(z)dz$ for all $\theta \in \mathbb{R}_{>0}$. Then $\phi(z) = \int_0^\infty \theta^{z-1}f(\theta)d\theta$.*

Theorem 4.3.5. *Let $\boldsymbol{z} \in (\mathbb{C}^*)^N$ and let $\Theta \subseteq \mathbb{R}^N$ be a finite set of choices for* $\arg(\boldsymbol{z})$*, such that the corresponding points $\frac{1}{2\pi}\sum_{j=1}^d \theta_i \boldsymbol{b}_i$ are distinct for all $\boldsymbol{\theta} \in \Theta$. Then the corresponding Mellin-Barnes integrals are linearly independent over $\mathbb{C}$.*

Proof. Denote the Mellin-Barnes integral corresponding to $\boldsymbol{\theta} \in \Theta$ by $M_{\boldsymbol{\theta}}$. Writing z_j in polar form gives

$$M_{\boldsymbol{\theta}}(\boldsymbol{z}) = \int_{i\mathbb{R}^d} \prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s})|z_j|^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s}} \exp(i\theta_j \boldsymbol{b}_j \cdot \boldsymbol{s} + i\gamma_j\theta_j)d\boldsymbol{s}.$$

Suppose that $\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}} = 0$ with $\lambda_{\boldsymbol{\theta}} \in \mathbb{C}$. Then

$$\begin{aligned} 0 = \sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}}(\boldsymbol{z}) &= \int_{i\mathbb{R}^d} \prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j \cdot \boldsymbol{s})|z_j|^{\gamma_j + \boldsymbol{b}_j \cdot \boldsymbol{s}} \sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\theta_j \boldsymbol{b}_j\cdot\boldsymbol{s} + i\gamma_j\theta_j} d\boldsymbol{s} \\ &= \left(\sum_{\boldsymbol{\theta}\in\Theta} e^{i\boldsymbol{\gamma}\cdot\boldsymbol{\theta}}\right) \prod_{j=1}^N |z_j|^{\gamma_j} \int_{i\mathbb{R}^d} \left(\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}\right) \prod_{j=1}^N |z_j|^{\boldsymbol{b}_j\cdot\boldsymbol{s}} \Gamma(-\gamma_j - \boldsymbol{b}_j\cdot\boldsymbol{s})d\boldsymbol{s}. \end{aligned}$$

Write $x_j = |z_1|^{b_{1j}} \cdots |z_N|^{b_{Nj}}$. Then

$$\int_{i\mathbb{R}^d} \left(\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}\right) x_1^{s_1}\cdots x_d^{s_d} \prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j\cdot\boldsymbol{s})d\boldsymbol{s} = 0.$$

Define

$$\phi(s_d) = \int_{i\mathbb{R}^{d-1}} \left(\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}\right) x_1^{s_1}\cdots x_{d-1}^{s_{d-1}} \prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j\cdot\boldsymbol{s})ds_1\cdots ds_{d-1}$$

and $f(x_d) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x_d^{-s_d}\phi(s_d)ds_d$. Then $f(\frac{1}{x_d}) = 0$ and ϕ satisfies the conditions of the Mellin-inverse theorem. Hence

$$\phi(s_d) = \int_0^\infty x_d^{s_d-1} f(x_d)dx_d = \int_0^\infty x_d^{-s_d-1} f\left(\frac{1}{x_d}\right)dx_d = 0$$

so

$$\int_{i\mathbb{R}^{d-1}} \left(\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}\right) x_1^{s_1}\cdots x_{d-1}^{s_{d-1}} \prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j\cdot\boldsymbol{s})ds_1\cdots ds_{d-1} = 0.$$

Repeated application of the Mellin-inverse theorem shows that

$$\left(\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}\right)\prod_{j=1}^N \Gamma(-\gamma_j - \boldsymbol{b}_j\cdot\boldsymbol{s}) = 0.$$

Since the Gamma function has no zeros, this implies that $\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}} = 0$. All points $\frac{1}{2\pi}\sum_{j=1}^N \theta_j \boldsymbol{b}_j$ are distinct, so $\sum_{\boldsymbol{\theta}\in\Theta} \lambda_{\boldsymbol{\theta}} e^{i\sum_{j=1}^N \theta_j \boldsymbol{b}_j\cdot\boldsymbol{s}}$ is a sum of different exponential functions, that can only be identically zero if the coefficients $\lambda_{\boldsymbol{\theta}}$ are all zero. Hence the functions $M_{\boldsymbol{\theta}}$ are linearly independent. □

The above discussion shows that there exists a basis of Mellin-Barnes integrals around a point $\boldsymbol{z}\in\mathbb{C}^N$ if and only if there are $\mathrm{Vol}(Q(\mathcal{A}))$ choices of the arguments corresponding to different points in the zonotope. These points will differ by integral points. On the other hand, if there are $\mathrm{Vol}(Q(\mathcal{A}))$ points in the zonotope with integral differences, then there are $\mathrm{Vol}(Q(\mathcal{A}))$ choices for the arguments of $\boldsymbol{z}$ leading to these points: for the points $\boldsymbol{p_1},\ldots,\boldsymbol{p}_{\mathrm{Vol}(\boldsymbol{Q}(\boldsymbol{\mathcal{A}}))}$ in the zonotope with $\boldsymbol{p_i}-\boldsymbol{p_1}\in\mathbb{Z}^d$, we can write $\boldsymbol{p_1}=\frac{1}{2\pi}\sum_{j=1}^N \theta_j^{(1)}\boldsymbol{b}_j$ with $\theta_j^{(1)}\in(-\frac{\pi}{2},\frac{\pi}{2})$. By solving $\sum_{j=1}^N k_j^{(i)}\boldsymbol{b}_j=\boldsymbol{p_i}-\boldsymbol{p_1}$ for $k_j^{(i)}\in\mathbb{Z}$ and defining $\theta_j^{(i)}=\theta_j^{(1)}+k_j^{(i)}$, we find choices $\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(\mathrm{Vol}(Q(\mathcal{A})))}$ for the arguments of a single point, corresponding to the points $\boldsymbol{p_i}$ in the zonotope. It is possible to take $k_j^{(i)}\in\mathbb{Z}$ because the rows of B span $\mathbb{Z}$, as the greatest common divisor of the $d\times d$-minors of B is 1 (Lemma 1.2.15).

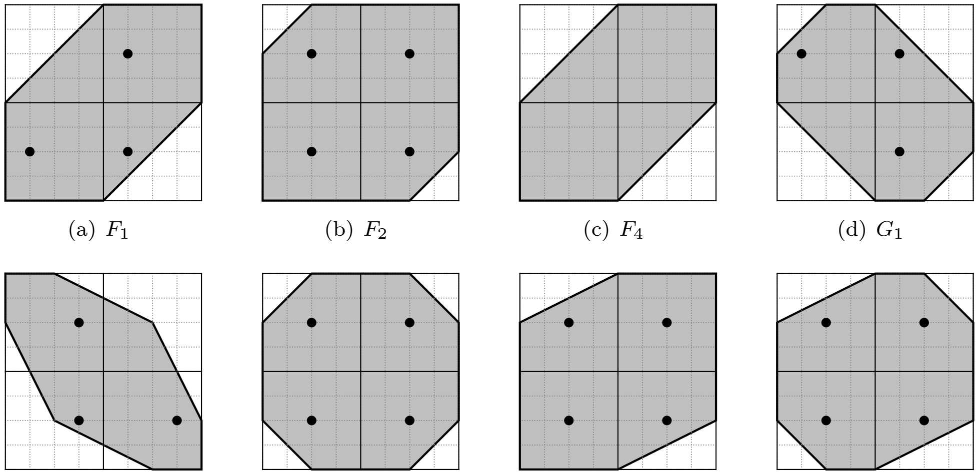
In general, it is not known under which conditions for $\mathcal{A}$ there are $\mathrm{Vol}(Q(\mathcal{A}))$ points in the zonotope with integral differences. For $d=2$, Nilsson and Passare showed that the zonotope and the so-called coamoeba of the $\mathcal{A}$-discriminant together cover the torus $\mathrm{Vol}(Q(\mathcal{A}))$ times [NP10, Theorems 2 and 3]. Hence there exists a basis of Mellin-Barnes integrals if and only if the coamoeba does not completely cover the torus. However, it is not known for which sets $\mathcal{A}$ this holds.

We are particularly interested in the algebraic Appell, Lauricella and Horn functions. For the sets $\mathcal{A}$ of the Appell and Horn functions, the zonotope is two-dimensional and can be drawn easily. The zonotopes are shown in Figure 4.5. For all functions except for F_4, there exists a point around which there is a basis of Mellin-Barnes integrals. The choices for the arguments are also shown in Figure 4.5. The zonotope of the Lauricella F_D and F_C functions are equal. They are n-dimensional, and are given by

$$Z_B=\{\boldsymbol{x}\in\mathbb{R}^n \mid \forall i: |x_i|<1, \forall i,j: |x_i-x_j|<1\}.$$

It is easy to show that there are $n+1$ points with integral differences, but not 2^n points. Hence there is a basis of converging Mellin-Barnes integrals for the F_D function, but not for the F_C function.

In the remainder of this chapter and the next one, we will assume that there exists a basis of Mellin-Barnes integrals. In particular, we will only be concerned with Appell, Lauricella and Horn functions other than F_4 and F_C. It would also be interesting to



(a) F_1 (b) F_2 (c) F_4 (d) G_1

(e) G_3 (f) H_1 (g) H_4 (h) H_5

Figure 4.5: Zonotopes for the Appell and Horn functions

compute the monodromy groups for the algebraic functions with planar sets $\mathcal{A}$, as we found in Chapter 3. However, in these cases there often does not exist a basis of Mellin-Barnes integrals.

Given the local monodromy matrices on a basis of triangulations, we use the basis of Mellin-Barnes integrals to write the monodromy matrices on a fixed basis. We will denote the (normalized) volume of $Q(\mathcal{A})$ by n and assume that $\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(n)}$ satisfy $\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{(1)} \in (2\pi\mathbb{Z})^N$ and $\frac{1}{2\pi}\sum_{j=1}^N \theta_j^{(i)} \boldsymbol{b}_j \in Z_B$. Then there is a basis $\{M_1, \dots, M_n\}$ of Mellin-Barnes integrals around each $\boldsymbol{z} \in (\mathbb{C}^*)^N$ with $\arg(z_j) \equiv \theta_j^{(1)} \pmod{2\pi\mathbb{Z}}$. Let $\{\Phi_1, \dots, \Phi_n\}$ be a basis of Γ-series solutions, obtained from a triangulation as described in Section 4.1, with a common convergence direction $\boldsymbol{\rho}$. Define $\boldsymbol{z} \in (\mathbb{C}^*)^N$ by $z_j = t^{\rho^j} \cdot e^{2\pi i \theta_j^{(1)}}$ with $t > 0$ sufficiently small. Then both the Γ-series and the Mellin-Barnes integrals converge on a neighbourhood of $\boldsymbol{z}$.

For $\boldsymbol{n} \in \mathbb{Z}^N$, define the loop $c(\boldsymbol{n}) : [0,1] \to \mathbb{C}^N$ by $t \mapsto (e^{2\pi i n_1 t} z_1, \dots, e^{2\pi i n_N t} z_N)$. This is a loop around the origin, through $\boldsymbol{z}$, that winds n_k times in the k^{th} coordinate direction. We compute the analytic continuations of M_1 and Φ_k $(1 \le k \le n)$ along the curves $c(\boldsymbol{n}_j)$, where $\boldsymbol{n}_j = \frac{\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^{(1)}}{2\pi}$.

If we continue M_1 along a curve that winds n_k times in the k^{th} coordinate direction, then the argument of z_j will increase by $2\pi i n_k$. Hence analytic continuation along $c(\boldsymbol{n}_j)$ will change M_1 into M_j.

For the analytic continuation of Φ_k, write $\Phi_k(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\gamma}^{(k)}} \Psi_k(\boldsymbol{z})$ where Ψ_k is an analytic function. For any $\boldsymbol{n} \in \mathbb{Z}^N$, the loop $c(\boldsymbol{n})$ will change $z_k^{\gamma_k}$ into $e^{2\pi i \gamma_k n_k} z_k^{\gamma_k}$. Hence under $c(\boldsymbol{n}_j)$, the factor $\boldsymbol{z}^{\boldsymbol{\gamma}^{(k)}}$ will be changed into $e^{2\pi i \boldsymbol{\gamma}^{(k)} \cdot \boldsymbol{n}_j} \boldsymbol{z}^{\boldsymbol{\gamma}^{(k)}}$, which equals

$e^{i(\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^{(1)})\cdot\boldsymbol{\gamma}^{(k)}}\boldsymbol{z}^{\boldsymbol{\gamma}^{(k)}}$. It follows that $\Phi_j(\boldsymbol{z})$ will become $e^{i(\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^{(1)})\cdot\boldsymbol{\gamma}^{(k)}}\Psi_j(\boldsymbol{z})$. Writing $\chi_{j,k} = e^{i(\boldsymbol{\theta}^{(j)}-\boldsymbol{\theta}^{(1)})\cdot\boldsymbol{\gamma}^{(k)}}$, analytic continuation along $c(\boldsymbol{n}_j)$ multiplies Φ_k by a factor $\chi_{j,k}$.

Since the power series form a basis of the solution space, we can write $M_1 = \lambda_1\Phi_1 + \ldots + \lambda_n\Phi_n$ with $\lambda_j \in \mathbb{C}$. Analytic continuation along $c(\boldsymbol{n}_j)$ shows that $M_j = \lambda_1\chi_{j,1}\Phi_1 + \ldots + \lambda_n\chi_{j,n}\Phi_n$. This implies that all M_j are contained in the space spanned by $\lambda_1\Phi_1, \ldots, \lambda_n\Phi_n$. Since the M_j span the solution space, so do $\lambda_1\Phi_1, \ldots, \lambda_n\Phi_n$. It follows that $\lambda_j \neq 0$ for all j. By renormalizing Φ_j, we can assume that $\lambda_j = 1$ for all j. Now we have $M_j = \chi_{j,1}\Phi_1 + \ldots + \chi_{j,n}\Phi_n$.

Let X be the matrix with entries $\chi_{j,k}$. Then $(M_1, \ldots, M_m)^t = X(\Phi_1, \ldots, \Phi_m)^t$. Hence if P is a local monodromy matrix on the basis of Γ-series solutions, then the corresponding matrix on the basis of Melling-Barnes integrals is given by XPX^{-1}.

Example 4.3.6. We return to the example $G_3(\frac{1}{2}, \frac{1}{3}|x, y)$. The zonotope for G_3 is shown in Figure 4.5(e). One sees that there indeed exists a basis of solutions of Mellin-Barnes integrals. For example, choose arguments $\boldsymbol{\theta}^{(1)}$ of $\boldsymbol{z}$ such that $\frac{1}{2\pi}\sum_{i=1}^{4}\theta_i^{(1)}\boldsymbol{b}_i = (-0.25, -0.25)$. The two choices $\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(1)} + (0, 0, 2\pi, 0)$ and $\boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(1)} + (0, 0, 0, 2\pi)$ give independent converging integrals, so these three integrals form a basis. For each triangulation $\mathcal{T}_i$, we have the matrix $X_i = (\chi_{j,k})_{j,k}$ with $\chi_{1,k} = 1$, $\chi_{2,k} = e^{2\pi i\gamma_3^{(k)}}$ and $\chi_{3,k} = e^{2\pi i\gamma_4^{(k)}}$. This gives the matrices

$$X_1 = \begin{pmatrix} 1 & 1 & 1 \\ -\xi^2 & -\xi^8 & \xi^5 \\ -\xi & \xi^4 & -\xi^7 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\zeta^2 & \zeta^2 \\ \zeta^2 & 1 & 1 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -i & i & 1 \end{pmatrix} \quad \text{and} \quad X_4 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & \zeta^2 & 1 \end{pmatrix}$$

with $\xi = e^{\frac{\pi i}{9}}$ and $\zeta = e^{\frac{\pi i}{3}}$. For each of the matrices P_{i3} and P_{i4} computed in Example 4.1.7, the local monodromy matrix on the basis of Mellin-Barnes integrals is given by $N_{ij} = X_i P_{ij} X_i^{-1}$. Computing these 8 products show that there are 6 local monodromy matrices:

$$N_{13} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\zeta \\ \zeta & 0 & 0 \end{pmatrix}, \qquad N_{14} = \begin{pmatrix} 0 & 0 & 1 \\ \zeta & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$N_{23} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\zeta \\ -1 & 1 & 1 \end{pmatrix}, \qquad N_{24} = N_{44} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -\zeta^2 & 0 & \zeta \end{pmatrix},$$

$$N_{33} = N_{43} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{34} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

◊

Example 4.3.7. Consider $F_2(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$. From Figure 4.5(b) we see that there exists a basis of Mellin-Barnes integrals. Choose $\boldsymbol{\theta}^{(1)}$ such that $\frac{1}{2\pi}\sum_{i=1}^{7}\theta_i \boldsymbol{b}_i = (-0.5, -0.5)$ and $\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(1)} + (0,0,0,0,0,2\pi,0)$, $\boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(1)} + (0,0,0,0,0,0,2\pi)$ and $\boldsymbol{\theta}^{(4)} = \boldsymbol{\theta}^{(1)} + (0,0,0,0,0,2\pi,2\pi)$. For the triangulation $\mathcal{T}_i$ we have the corresponding transition matrix $X_i = (\chi_{j,k})_{j,k}$ with $\chi_{1,k} = 1$, $\chi_{2,k} = e^{2\pi i \gamma_6^{(k)}}$, $\chi_{3,k} = e^{2\pi i \gamma_7^{(k)}}$ and $\chi_{4,k} = e^{2\pi i(\gamma_6^{(k)} + \gamma_7^{(k)})}$. The local monodromy matrices are then given by $N_{i6} = X_i P_{i6} X_i^{-1}$ and $N_{17} = X_i P_{i7} X_i^{-1}$. The transition matrices are

$$X_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ i\omega & 1 & i\omega & 1 \\ -i\omega & -1 & i\omega & 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ i\omega^2 & -i\omega^2 & \omega^2 & \omega^2 \\ -i\omega^2 & -i\omega^2 & -\omega^2 & \omega^2 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\omega & -i\omega^2 & i & i \\ i\omega & 1 & i\omega & 1 \\ -i\omega^2 & -i\omega^2 & -\omega & i \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -\omega & -i\omega^2 & i \\ -\omega^2 & i\omega & 1 & \omega^2 \\ -i\omega^2 & -i\omega^2 & -i\omega^2 & i\omega^2 \end{pmatrix} \quad \text{and}$$

$$X_5 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & -1 & 1 & i \\ \omega^2 & i\omega^2 & -i\omega^2 & \omega^2 \\ -i\omega^2 & -i\omega^2 & -i\omega^2 & i\omega^2 \end{pmatrix}$$

with $\omega = e^{\frac{\pi i}{6}}$. It turns out that there are four distinct monodromy matrices:

$$N_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad N_{17} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -i\omega & 0 & \omega^2 & 0 \\ 0 & -i\omega & 0 & \omega^2 \end{pmatrix},$$

$$N_{27} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\omega & \omega^2 & -i\omega^2 \\ -\omega & 0 & -i\omega^2 & \omega^2 \end{pmatrix}, \qquad N_{36} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -i\omega & \omega \\ 0 & 0 & 0 & 1 \\ -\omega^2 & -i\omega^2 & 0 & i \end{pmatrix},$$

$$N_{47} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \omega & 0 & 0 \\ -\omega & 0 & -i\omega^2 & \omega^2 \end{pmatrix} \quad \text{and} \quad N_{56} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -i & i\omega & \omega \\ 0 & 0 & 0 & 1 \\ -\omega^2 & -i\omega^2 & 0 & i \end{pmatrix}.$$

For the other matrices we have $N_{26} = N_{16}$, $N_{37} = N_{17}$, $N_{57} = N_{27}$ and $N_{36} = N_{46}$.$\Diamond$

5

Monodromy groups and formulas for the algebraic Appell, Lauricella and Horn functions

In this chapter, we compute the projective monodromy groups for most of the irreducible algebraic Appell, Lauricella and Horn functions. Using the method in the previous chapter, we can compute bases of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$ consisting of Γ-series. In the first section, we compute the monodromy groups of most of the irreducible algebraic Appell and Horn functions that do not belong to a parametrized family of algebraic functions. This will be based on the classification of the finite unitary reflection groups by Shephard and Todd. For the Appell functions, similar results can be found in [Sas77, Kat00, Kat97]. In the second section, we use the bases of Γ-series solutions to find equations for the families of irreducible algebraic Appell, Lauricella and Horn functions. The monodromy group then equals the Galois group of these equations (Theorem 1.3.3). The results of the second section can also be found in the preprint [Bod13b].

5.1 Monodromy groups of the Appell, Lauricella and Horn functions

For the hypergeometric systems for which there exists a basis of Mellin-Barnes integrals, the methods in the previous chapter allow us to compute the local monodromy matrices (the group generated by the monodromy matrices corresponding to loops of the variables around the origin) on this basis. In particular, we can compute these matrices for all irreducible algebraic Appell, Lauricella and Horn functions except for F_4 and F_C. In this section we try to determine the monodromy group, which is finite in the case of algebraic functions. Throughout this chapter, we will denote the monodromy group by *Mon*. Note that this group can be larger than the group generated by the local monodromy matrices.

Our computation of the monodromy group is based on the classification of the finite unitary reflection groups by Shephard and Todd [ST54]. The relevant parts of this classification can be found in Table 5.1. The groups are called ST1 up to ST32, following the numbering Shephard and Todd used. For each group, the dimension of the space it acts on is given, as well as the order of the group. The groups can be identified by the number of reflections of order 2 and 3 and the degrees of the invariant

Table 5.1: Finite irreducible unitary reflection groups. For group ST2, p is a divisor of m and $q = \frac{m}{p}$.

Group	Dim.	Order	Degrees of basis of inv. polynomials	Number of reflections order 2	order 3
ST1	n	$(n+1)!$	$2, 3, \ldots, n+1$	$\frac{n(n+1)}{2}$	0
ST2	n	$qm^{n-1}n!$	$m, 2m, \ldots,$ $(n-1)m, qn$	$\frac{mn(n-1)}{2}$	$n\phi(d)$ of order d with $d\|q$ and $d > 1$
ST23	3	120	2, 6, 10	15	0
ST24	3	336	4, 6, 14	21	0
ST25	3	648	6, 9, 12	0	24
ST26	3	1296	6, 12, 18	9	24
ST27	3	2160	6, 12, 30	45	0
ST28	4	1152	2, 6, 8, 12	24	0
ST29	4	7680	4, 8, 12, 20	40	0
ST30	4	14400	2, 12, 20, 30	60	0
ST31	4	$64 \cdot 6!$	8, 12, 20, 24	60	0
ST32	4	$216 \cdot 6!$	12, 18, 24, 30	0	80

polynomials. These polynomials form a polynomial ring with finite basis. The degrees of the basis elements are independent of the choice of the basis and are also given in Table 5.1.

To compute the projective monodromy group, we first compute a reflection subgroup H of the local monodromy group. Then we use Shephard and Todd's classification of the finite irreducible unitary reflection groups to determine to which group H is conjugate. Let G be the maximal reflection subgroup of *Mon*. Using Shephard and Todd's classification and the fact that $H \leq G$, we determine candidates for G. Very often G is uniquely determined. By Theorem 5.1.5 by Cohen, *Mon* is determined by G up to scalars. Note that we can use the classification of unitary groups since it is well-known that each finite complex group is unitary with respect to a suitable inner product.

There are some problems with this approach. First, we need a non-trivial reflection subgroup H. We do not know under which conditions this exists, but this turns out not to be a problem in case of the Appell and Horn functions. Second, H has to be irreducible to be able to use Shephard and Todd's classification. We can test this by computing the invariant Hermitian forms: there is a unique invariant form if and only if the group is irreducible (Proposition 5.1.3). Third, Cohen's theorem can only be applied if G is primitive, i.e., does not permute non-trivial subspaces of $\mathbb{C}^n$. The only imprimitive groups are the groups ST2, also denoted $G(m, p, n)$. We will come back to these issues in the course of this exposition.

To compute the monodromy group, we need a non-trivial reflection subgroup of the monodromy group. A *reflection group* is a group that is generated by reflections. By a *reflection* we mean a non-trivial element of $GL(\mathbb{C}, n)$ which leaves a hyperplane pointwise fixed. To check whether a matrix is a reflection, one can just compute the eigenvalues, or use the fact that a matrix R is a reflection if and only if $R - \mathrm{Id}$ has rank 1. We assume that the local monodromy matrices are known, and will denote a set of generators of these by $N_1, \ldots, N_k$. In many cases it is possible to find such a subgroup by checking whether some matrices of the form $N_i N_j^{-1}$, or just N_i itself, are reflections. Note that, if $N_i N_j^{-1}$ is a reflection fixing a hyperplane V, then $N_j N_i^{-1}$ is also a reflection fixing V, and $N_i^{-1} N_j$ and $N_j^{-1} N_i$ are reflections fixing $N_i^{-1}(V)$.

Example 5.1.1. We computed the local monodromy matrices of $G_3(\frac{1}{2}, \frac{1}{3}|x, y)$ in Example 4.3.6. By computing the eigenvalues, one sees that N_{24} and N_{33} are reflections. Two of their eigenvalues equal 1 and the remaining eigenvalues are ζ^2 and -1, respectively. Furthermore, all products of the form $N_{ij} N_{kl}^{-1}$ or $N_{ij}^{-1} N_{kl}$ with $\{(i,j),(k,l)\} \in \{\{(1,3),(2,3)\}, \{(1,4),(3,4)\}, \{(2,3),(3,3)\}, \{(2,4),(3,4)\}\}$ give reflections of order 3. Some of these reflections are equal, so in total we find 12 reflection matrices, including

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \zeta & -1 & -\zeta \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -\zeta^2 & 0 & \zeta \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 1 & 0 & 0 \\ \zeta & -\zeta & -\zeta \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_5 = \begin{pmatrix} \zeta^2 & -\zeta^2 & -\zeta^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_6 = \begin{pmatrix} \zeta & 0 & -1 \\ 0 & 1 & 0 \\ \zeta^2 & 0 & 0 \end{pmatrix} \text{ and}$$

$$R_7 = \begin{pmatrix} 0 & -\zeta & -\zeta^2 \\ -1 & -\zeta^2 & -\zeta^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The other 5 reflection matrices are the square of one of the matrices $R_1, \ldots, R_7$. The group generated by these reflections contains reflections of both order 2 and order 3.$\diamondsuit$

Example 5.1.2. The local monodromy matrices for $F_2(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$ are given in Example 4.3.7. By computing the products of the form $N_{ij} N_{kl}^{-1}$ or $N_{ij}^{-1} N_{kl}$ with $\{(i,j),(k,l)\} \in \{\{(1,6),(5,6)\}, \{(1,7),(4,7)\}, \{(2,7),(4,7)\}, \{(3,6),(5,6)\}\}$, we find 6 reflections of order 2:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \omega^2 & -i\omega^2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & \omega & -i\omega \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -i\omega^2 & 0 & i \\ 0 & \omega^2 & -i & 0 \end{pmatrix}, \qquad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i & 0 & \omega & i\omega \\ 0 & 0 & 1 & 0 \\ -i\omega^2 & -\omega^2 & i & 0 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} 0 & -i & i\omega & \omega \\ 0 & 1 & 0 & 0 \\ -\omega^2 & -i\omega^2 & 0 & i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R_6 = \begin{pmatrix} 0 & i & -i\omega & 0 \\ -i & 0 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \diamondsuit$$

Similar to these examples we compute reflections R_i for all irreducible algebraic Appell and Horn functions, except for the F_4 and F_C functions, for which there is no basis of converging Mellin-Barnes integrals. We denote the group generated by these reflections by H. In general we don't know whether there are any reflections at all of the form N_i or $N_i N_j^{-1}$, but it turns out that H is non-empty in all cases of interest.

Now we would like to use Shephard and Todd's classification to determine H. However, this is only possible if H is irreducible. Since we only consider irreducible algebraic functions, *Mon* is irreducible, but as H is only a subgroup of *Mon*, it can be reducible. However, we can easily test whether a finite group is irreducible:

Proposition 5.1.3. *Let H be a finite matrix group acting on $\mathbb{C}^n$. Then H acts irreducibly if and only if it has a unique invariant Hermitian form (up to scalars).*

Proof. Suppose that H is irreducible and suppose that h_1 and h_2 are invariant Hermitian forms. Choose $\boldsymbol{v} \in \mathbb{C}^n$ with $\boldsymbol{v} \neq \boldsymbol{0}$. Then $h_i(\boldsymbol{v}, \boldsymbol{v}) \neq 0$, so we can normalize h_i so that $h_i(\boldsymbol{v}, \boldsymbol{v}) = 1$. Now $\ker(h_1 - h_2)$ is non-trivial (as it contains $(\boldsymbol{v}, \boldsymbol{v})$ and it is an invariant subspace of $\mathbb{C}^n$, so it equals $\mathbb{C}^n$. Hence up to normalization, $h_1 = h_2$.

Conversely, suppose that H is reducible and let V be an invariant subspace. Each finite group has an invariant hermitian form; choose such a form and denote it by h. Then the complement $V^\perp$ with respect to this inner product is also invariant and $\mathbb{C}^n = V \oplus V^\perp$. For each $\lambda \in \mathbb{C}$, we can define an hermitian form h_λ which equals h on V, and equals λh on $V^\perp$. This is an invariant form for H, but it is not a scalar multiple of h. $\square$

Given the generators of H, the invariant hermitian forms can easily be computed. An hermitian form is given by an hermitian matrix h, with the form defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\dagger h \boldsymbol{y}$. It is invariant for H if and only if $R^\dagger h R = h$ for all generators R of H. This gives a system of linear equations for the entries of h, which can easily be solved. Computing the invariant hermitian forms for each of the reflection subgroups of the algebraic Appell F_1 and F_2, Lauricella F_D and Horn G_1, G_3, H_1, H_4 and H_5 functions shows that the groups H are irreducible, except for some choices for the parameters of H_4. These can be found in Table 5.2.

To determine which group H is, we consider invariant polynomials.

Definition 5.1.4. Let H be a matrix group acting on $\mathbb{C}^n$. An *H-invariant polynomial* is a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(R(\boldsymbol{x})) = f(\boldsymbol{x})$ for all $R \in H$ and all $\boldsymbol{x} \in \mathbb{C}^n$.

The invariant polynomials form a polynomial ring with a finite basis, and the degrees of the polynomials in this basis are independent of the choice of the basis. The degrees in Table 5.1 are the degrees of the polynomials in such a basis. To find all invariant polynomials of a given degree, one can consider the coefficients of the polynomial as unknowns. By comparing the coefficients of the different powers of the x_k, the conditions $f(R_i(\boldsymbol{x})) = f(\boldsymbol{x})$ for the generators R_i of H give a system of linear equations in the coefficients of f. As we are only interested in the number of invariant polynomials of a given degree, we don't actually have to solve the system; it suffices to compute the dimension of the solution space. In theory, this can be done for any group and any degree, but in practice the calculation time increases very fast. Therefore we only compute whether there are invariant polynomials of degrees 2, 4, 6, 8 and 9. Note that if there is an invariant polynomial of degree 2, then there will also be an invariant polynomial of degree 4, namely the square of the polynomial of degree 2. Since we are interested in polynomials which form a basis, we only count polynomials that are independent of the polynomials of lower degrees. The results can be found in Table 5.2.

We use this information to determine which group in Table 5.1 H equals. For each group H, we compute the orders of the generators and the degrees of some invariant polynomials. The groups H for F_1, G_1 and G_3 act on $\mathbb{C}^3$. For $F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$, all generators have order 3 and there exist invariant polynomials of order 6 and 9. Since the product of reflections of order 3 can never have order 2, H cannot be imprimitive and must be equal to the group ST25. In all other cases, there are generators of order 2 and 3 and there is an invariant polynomial of degree 6, but not of degree 9. As imprimitive groups $G(m, p, 3)$ have either 0 or 6 reflections of order 3, we can show that H is primitive by finding at least 7 reflections of order 3. For $F_1(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$, $F_1(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3}|x, y)$, $G_1(\frac{1}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$ and $G_3(\frac{1}{2}, \frac{1}{3}|x, y)$, there are 7, 11, 8 and 8 generators of order 3, respectively, so H is primitive. In the other two cases, note that the square of a reflection of order 3 is again a reflection of order 3. Squaring the generators gives in total 10 reflections of order 3 for both $F_1(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3}|x, y)$ and $G_1(\frac{1}{6}, \frac{2}{3}, \frac{2}{3}|x, y)$. Hence H is primitive and equals the group ST26 in all cases.

Now we consider the monodromy groups of F_D, F_2, H_1, H_4 and H_5, acting on $\mathbb{C}^4$. As can be seen from Table 5.2, there are 4 cases. Suppose that the generators all have order 2. If there are invariant polynomials of degrees 2, 4, 4 and 6, then H must be imprimitive and is equal to $G(2, 2, 4)$. If there are invariant polynomials of degree 2, 6 and 8 but not of degree 4, then H is equal to the group ST28. There are also groups for which the only invariant polynomial (of degree at most 9) has degree 2. In this case, H must be equal to ST30. In the remaining case, the generators have order 3 and there are no invariant polynomials of degree 2, 4, 6, 8 or 9. As H contains no elements of order 2, it must be primitive and hence be ST32.

The next step is to determine the maximal reflection subgroup G of *Mon*. We will use the fact that the order of H divides the order of G because $H \leq G$. Note that any subgroup of an imprimitive group is again imprimitive. Hence if H is primitive, then so is G.

For the groups acting on $\mathbb{C}^3$, we again have to make a distinction between $F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$ and the other groups. For $F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$, the reflections have order 3 and H is the group ST25. Since H is primitive, G must be primitive as well and hence equals ST25 or ST26. For the other groups acting on $\mathbb{C}^3$, G is primitive because G contains at least as many reflections of order 3 as H. It follows that G is ST26 in those cases.

For the groups acting on $\mathbb{C}^4$, we again consider 4 possibilities. In case $H = G(2, 2, 4)$, we find only reflections of order 2. This suggest that G is an imprimitive group with $m = p$, but we don't have a proof of this. For some functions, we found that H is ST30 or ST32. These groups are maximal in the sense that no other primitive group contains then, as can easily be seen from the orders. As G must be primitive, we have $G = H$. This leaves us with the functions for which H is the group ST28. Then G is again primitive, and we see from the orders of the groups and the number of reflections that G is either ST28 or ST31. The group ST28 is the Weyl group of the root system F_4; ST31 is the group EW(N_4). In [Coh76, Theorem on page 409] we find that ST31 is given by the roots $\boldsymbol{\alpha}_0 = \frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4)$, $\boldsymbol{\alpha}_1 = \frac{1}{2}(1 + i)(\boldsymbol{e}_2 + \boldsymbol{e}_4)$, $\boldsymbol{\alpha}_2 = \frac{1}{2}(1 + i)(-\boldsymbol{e}_2 + \boldsymbol{e}_3)$, $\boldsymbol{\alpha}_3 = \frac{1}{2}(-\boldsymbol{e}_1 + i\boldsymbol{e}_2 - \boldsymbol{e}_3 + i\boldsymbol{e}_4)$ and $\boldsymbol{\alpha}_4 = \boldsymbol{e}_4$. The roots $\boldsymbol{\alpha}_4$, $-\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ give the root system F_4 and hence the group ST28. It follows that ST28 is indeed a subgroup of ST31. Hence in the cases where $H =$ ST28, we have two possibilities for G: ST28 and ST31. These results can also be found in Table 5.2.

The final step is the determination of the monodromy group from these reflection subgroups. This is based on the following theorem:

Theorem 5.1.5 ([Coh76, Proposition 5.14]; erratum in [Coh78]). *Let μ_∞ be the set of complex roots of unity. Let G be a complex primitive n-dimensional reflection group ($n \geq 3$), and let $M \subseteq U_n(\mathbb{C})$ be a finite group such that $G \trianglelefteq M$. Then $M \subseteq \mu_\infty \cdot G$ except when G is the group $ST25$ and $ST26 \subseteq M \subseteq \mu_\infty \cdot ST26$.*

Lemma 5.1.6. *Let G be the maximal reflection subgroup of Mon. Then G is normal in Mon.*

Proof. As G is finite, it is generated by a finite number of reflections $g_1, \ldots, g_l$. It suffices to show that for all $m \in$ *Mon*, we have $mg_jm^{-1} \in G$. This follows immediately from the fact that mg_jm^{-1} is again a reflection and G is the maximal reflection subgroup of *Mon*. □

For most functions, G is equal to one of the groups ST26, ST30 or ST32. In these cases, Theorem 5.1.5 teaches us that *Mon* is included in $\mu_\infty G$. In some cases, we could not determine G, but expected it to be imprimitive. As Cohen's theorem cannot be applied to imprimitive groups, we cannot say anything about *Mon*. This leaves us with $F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$, $F_2(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$, $H_1(\frac{1}{3}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3}|x, y)$, $H_4(\frac{1}{12}, \frac{3}{4}, \frac{1}{3}, \frac{1}{2}|x, y)$

and $H_4(\frac{1}{12}, \frac{5}{6}, \frac{1}{4}, \frac{2}{3}|x, y)$. Here we had two possibilities for G. For $F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}|x, y)$, we found that H equals ST25 and G is ST25 or ST26, and the only thing Cohen's theorem tells us is ST25 $\leq$ *Mon* $\leq \mu_\infty$ST25 or ST26 $\leq$ *Mon* $\leq \mu_\infty$ST26. For the other four cases, we know that H is ST28 and G is either ST28 or ST31. Hence it follows from Cohen's theorem that ST28 $\leq$ *Mon* $\leq \mu_\infty$ST28 or ST31 $\leq$ *Mon* $\leq \mu_\infty$ST31.

The results can be found in Table 5.2.

5.2 Explicit formulas for parametrized families of irreducible algebraic Appell, Lauricella and Horn functions

In this section we give explicit formulas for all one-parameter families of algebraic Appell, Lauricella and Horn functions. Such families exist for the Appell F_4, Lauricella F_C and Horn G_3, H_4 and H_5 functions. The results can be found in Tables 5.3 up to 5.10. The formulas for the Appell and Lauricella functions can be viewed as generalizations of the well-known formulas (1.6) for the Gauss functions ${}_2F_1$.

Our results are inspired by an observation by Beukers. In [Beu10] he notes that the function $G_3(r, 1-r|x, y)$ is of a very special form:

$$G_3(r, 1-r|x, y) = f(x, y)^r \cdot \sqrt{\frac{g(x, y)}{\Delta(x, y)}} \tag{5.1}$$

where f and g generate the same cubic extension of $\mathbb{C}[x, y]$ and Δ is the discriminant of the minimal polynomial of f. More explicitly:

$$\begin{aligned} yf(x,y)^3 + f(x,y)^2 - f(x,y) - x &= 0, \\ g(x,y) &= -3y^2 f(x,y)^2 - 2yf(x,y) + 4y + 1 \qquad \text{and} \\ \Delta(x,y) &= 1 + 4x + 4y + 18xy - 27x^2y^2. \end{aligned} \tag{5.2}$$

In particular, $G_3(r, 1-r|x, y)$ only depends on r in the exponent of f.

By computing $\frac{\Phi(r|\boldsymbol{z})^2}{\Phi(2r|\boldsymbol{z})}$ for several values of r, we see that the other algebraic Appell, Lauricella and Horn functions are not of the form

$$\Phi(r|\boldsymbol{z}) = f(\boldsymbol{z})^r g(\boldsymbol{z}). \tag{5.3}$$

However, it is possible to find other solutions of the same hypergeometric system that indeed are of this form, with f and g generating the same extension of $\mathbb{C}(\boldsymbol{z})$, with degree equal to the rank of the $\mathcal{A}$-hypergeometric system. In Tables 5.3 up to 5.10 we give such Φ and the formulas for f and g for all irreducible algebraic one-parameter families of Appell, Lauricella and Horn functions. For the Appell and Horn functions we also give a triangulation and the corresponding basis, and indicate how Φ can be written on this basis. As the proofs of these results are all similar, we only give proofs for $F_4(r, -r, \frac{1}{2}, \frac{1}{2}|x, y)$, $F_C(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}|\boldsymbol{z})$ and $G_3(r, 1-r|x, y)$ (Lemmas 5.2.1, 5.2.2 and 5.2.3).

Table 5.2: Monodromy groups for the algebraic Appell, Lauricella and Horn functions

Function	Orders and number of reflections	Degrees invariant polynomials	H	G	Mon/μ_∞
$F_1(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2} \vert x, y)$	3 (20)	6, 9	ST25	ST25 or ST26	ST25 or ST26
$F_1(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3} \vert x, y)$	2 (7), 3 (9)	6	ST26	ST26	ST26
$F_1(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2} \vert x, y)$	2 (6), 3 (8)	6	ST26	ST26	ST26
$F_1(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3} \vert x, y)$	2 (2), 3 (18)	6	ST26	ST26	ST26
$F_D(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3} \vert x, y, z)$	3 (59)	-	ST32	ST32	ST32
$F_2(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3} \vert x, y)$	2 (6)	2, 4, 4, 6	$G(2,2,4)$	?	?
$F_2(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{2}{3} \vert x, y)$	2 (6)	2, 4, 4, 6	$G(2,2,4)$	?	?
$F_2(\frac{1}{10}, \frac{7}{10}, \frac{9}{10}, \frac{2}{5}, \frac{4}{5} \vert x, y)$	2 (7)	2	ST30	ST30	ST30
$F_2(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3} \vert x, y)$	2 (6)	2, 6, 8	ST28	ST28 or ST31	ST28 or ST31
$F_2(\frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2} \vert x, y)$	3 (12)	-	ST32	ST32	ST32
$F_2(\frac{1}{12}, \frac{5}{6}, \frac{11}{12}, \frac{2}{3}, \frac{1}{2} \vert x, y)$	3 (12)	-	ST32	ST32	ST32
$F_2(\frac{1}{30}, \frac{5}{6}, \frac{7}{10}, \frac{2}{3}, \frac{2}{5} \vert x, y)$	2 (8)	2	ST30	ST30	ST30
$F_4(a, b, c_1, c_2 \vert x, y)$ with (a, b, c_1, c_2) in Table 2.1	No basis of converging Mellin-Barnes integrals			?	?

Table 5.2: Monodromy groups for the algebraic Appell, Lauricella and Horn functions

Function	Orders and number of reflections	Degrees invariant polynomials	H	G	Mon/μ_∞
$F_C(a,b,\boldsymbol{c}\|\boldsymbol{z})$ with $(a,b,\boldsymbol{c})$ in Table 2.2	No basis of converging Mellin-Barnes integrals			?	?
$G_1(\frac{1}{6},\frac{1}{2},\frac{2}{3}\|x,y)$	2 (1), 3 (15)	6	ST26	ST26	ST26
$G_1(\frac{1}{6},\frac{2}{3},\frac{2}{3}\|x,y)$	2 (4), 3 (8)	6	ST26	ST26	ST26
$G_3(\frac{1}{2},\frac{1}{3}\|x,y)$	2 (1), 3 (11)	6	ST26	ST26	ST26
$H_1(\frac{1}{3},\frac{5}{6},\frac{1}{2},\frac{2}{3}\|x,y)$	2 (8)	2, 6, 8	ST28	ST28 or ST31	ST28 or ST31
$H_1(\frac{1}{4},\frac{7}{12},\frac{5}{6},\frac{1}{2}\|x,y)$	3 (16)	-	ST32	ST32	ST32
$H_4(\frac{1}{2},\frac{1}{6},\frac{1}{3},\frac{1}{3}\|x,y)$	2 (4)	2, 4, 4, 6	$G(2,2,4)$	?	?
$H_4(\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{2}{3}\|x,y)$	2 (4)	2, 4, 4, 6	$G(2,2,4)$	?	?
$H_4(\frac{1}{10},\frac{7}{10},\frac{2}{5},\frac{2}{5}\|x,y)$	2 (4)	2	ST30	ST30	ST30
$H_4(\frac{1}{10},\frac{9}{10},\frac{1}{5},\frac{4}{5}\|x,y)$	2 (4)	2	ST30	ST30	ST30
$H_4(\frac{1}{12},\frac{3}{4},\frac{1}{3},\frac{1}{2}\|x,y)$	2 (4)	2, 6, 8	ST28	ST28 or ST31	ST28 or ST31
$H_4(\frac{1}{12},\frac{5}{6},\frac{1}{4},\frac{2}{3}\|x,y)$	2 (5)	2, 6, 8	ST28	ST28 or ST31	ST28 or ST31
$H_4(\frac{1}{12},\frac{7}{12},\frac{1}{3},\frac{1}{2}\|x,y)$	3 (8)	-	ST32	ST32	ST32
$H_4(\frac{1}{12},\frac{11}{12},\frac{1}{3},\frac{1}{2}\|x,y)$	3 (8)	-	ST32	ST32	ST32

Table 5.2: Monodromy groups for the algebraic Appell, Lauricella and Horn functions

Function	Orders and number of reflections	Degrees invariant polynomials	H	G	Mon/μ_∞
$H_4(\frac{1}{30}, \frac{5}{6}, \frac{1}{5}, \frac{2}{3} \| x, y)$	2 (4)	2	ST30	ST30	ST30
$H_4(\frac{1}{30}, \frac{7}{10}, \frac{1}{3}, \frac{2}{5} \| x, y)$	2 (4)	2	ST30	ST30	ST30
$H_4(a, b, c, d \| x, y)$ with (a, b, c, d) in Table 2.3 and not conjugated to the above tuples	Reducible reflection subgroups		-	?	?
$H_5(\frac{1}{6}, \frac{1}{2}, \frac{1}{3} \| x, y)$	3 (12)	-	ST32	ST32	ST32
$H_5(\frac{1}{6}, \frac{2}{3}, \frac{1}{3} \| x, y)$	2 (6)	2, 4, 4, 6	$G(2, 2, 4)$	?	?
$H_5(\frac{1}{6}, \frac{5}{6}, \frac{1}{3} \| x, y)$	3 (12)	-	ST32	ST32	ST32
$H_5(\frac{1}{10}, \frac{3}{5}, \frac{1}{5} \| x, y)$	2 (6)	2	ST30	ST30	ST30
$H_5(\frac{1}{12}, \frac{3}{4}, \frac{1}{2} \| x, y)$	3 (12)	-	ST32	ST32	ST32

Table 5.3: Formulas for $F_C\left(r,-r,\frac{1}{2},\ldots,\frac{1}{2}|z_1,\ldots,z_n\right)$

Function:	$F_C\left(r,-r,\frac{1}{2},\ldots,\frac{1}{2}\vert z_1,\ldots,z_n\right)$
Horn eqs. ($n=2$):	$(\theta_x(\theta_x-\frac{1}{2})-x(\theta_x+\theta_y+r)(\theta_x+\theta_y-r))F=0$
	$(\theta_y(\theta_y-\frac{1}{2})-y(\theta_x+\theta_y+r)(\theta_x+\theta_y-r))F=0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_{n+2},\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3,$
	$\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4,\ldots,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_{n+2}\}$
	$\boldsymbol{\beta}=(-r,r,-\frac{1}{2},\ldots,-\frac{1}{2})$
Rank:	2^n
Triang. ($n=2$):	$\{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$
Basis ($n=2$):	$\Phi_1(r\vert x,y)=F_4\left(r,-r,\frac{1}{2},\frac{1}{2}\vert x,y\right)$
	$\Phi_2(r\vert x,y)=\sqrt{y}F_4\left(r+\frac{1}{2},-r+\frac{1}{2},\frac{1}{2},\frac{3}{2}\vert x,y\right)$
	$\Phi_3(r\vert x,y)=\sqrt{x}F_4\left(r+\frac{1}{2},-r+\frac{1}{2},\frac{3}{2},\frac{1}{2}\vert x,y\right)$
	$\Phi_4(r\vert x,y)=\sqrt{xy}F_4\left(r+1,-r+1,\frac{3}{2},\frac{3}{2}\vert x,y\right)$
Sol. of (5.3) ($n=2$):	$\Phi=\Phi_1+2ir\Phi_2-2ir\Phi_3+4r^2\Phi_4$
Formulas:	$\Phi(r\vert\boldsymbol{z})=f(\boldsymbol{z})^r$
	$f(\boldsymbol{z})=h(\boldsymbol{z})+\sqrt{h(\boldsymbol{z})^2-1}$
	$h(\boldsymbol{z})=1-2(\sqrt{z_1}+\ldots+\sqrt{z_n})^2$

The results for the Appell and Horn functions are stated in terms of the classical Horn systems. These systems are described in [Dic08] and consist of two second order partial differential equations. These equations can also be found in the tables, using the notation $\theta_x=x\frac{\partial}{\partial x}$ and $\theta_y=y\frac{\partial}{\partial y}$. The Horn systems can have more independent solutions than the $\mathcal{A}$-hypergeometric systems: apart from the Γ-series solutions, there can be monomial solutions. By [DMS05, Theorem 2.5], the rank of all Horn systems we consider is 4. For the F_4, H_4 and H_5 functions, this equals the rank of the $\mathcal{A}$-hypergeometric system. However, for $G_3(r,1-r|x,y)$ the Horn system has a so called Puiseux monomial solution $x^{\frac{r-2}{3}}y^{\frac{-r-1}{3}}$ that is not a solution of the $\mathcal{A}$-hypergeometric system.

We now prove the formulas for $F_4(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2}|x,y)$, $F_C(r,r+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}|\boldsymbol{z})$ and $G_3(r,1-r|x,y)$. The proofs for the other functions are similar.

Lemma 5.2.1. *Let $\Phi(r|x,y)=(\sqrt{x}+\sqrt{y}-1)^{-2r}$. Then $\Phi(r|x,y)$ is a solution of the system of Horn equations for $F_4(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2}|x,y)$. Furthermore, on the basis $\{\Phi_1,\Phi_2,\Phi_3,\Phi_4\}$ of the solution space from Table 5.4, we have $\Phi=\Phi_1+2r\Phi_2-2r\Phi_3+$*

Table 5.4: Formulas for $F_C\left(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} | z_1, \ldots, z_n\right)$

Function:	$F_C\left(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \| z_1, \ldots, z_n\right)$
Horn eqs. $(n=2)$:	$(\theta_x(\theta_x - \frac{1}{2}) - x(\theta_x + \theta_y + r)(\theta_x + \theta_y + r + \frac{1}{2}))F = 0$
	$(\theta_y(\theta_y - \frac{1}{2}) - y(\theta_x + \theta_y + r)(\theta_x + \theta_y + r + \frac{1}{2}))F = 0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3,$
	$\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4, \ldots, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_{n+2}\}$
	$\boldsymbol{\beta} = (-r, -r - \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$
Rank:	2^n
Triang. $(n=2)$:	$\{\{1,2,3,4\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,5,6\}\}$
Basis $(n=2)$:	$\Phi_1(r\|x,y) = F_4\left(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \| x, y\right)$
	$\Phi_2(r\|x,y) = \sqrt{y} F_4\left(r+\frac{1}{2}, r+1, \frac{1}{2}, \frac{3}{2} \| x, y\right)$
	$\Phi_3(r\|x,y) = \sqrt{x} F_4\left(r+\frac{1}{2}, r+1, \frac{3}{2}, \frac{1}{2} \| x, y\right)$
	$\Phi_4(r\|x,y) = \sqrt{xy} F_4\left(r+1, r+\frac{3}{2}, \frac{3}{2}, \frac{3}{2} \| x, y\right)$
Sol. of (5.3) $(n=2)$:	$\Phi = \Phi_1 + 2r\Phi_2 + 2r\Phi_3 + 2r(2r+1)\Phi_4$
Formulas:	$\Phi(r\|\boldsymbol{z}) = f(\boldsymbol{z})^r$
	$f(\boldsymbol{z}) = (\sqrt{z_1} + \ldots + \sqrt{z_n} - 1)^{-2}$

$2r(2r+1)\Phi_4$.

Proof. It can easily be checked that $\Phi(r|x,y)$ is a solution of the Horn system of differential equations from Table 5.4.

The Gale dual

$$B = \begin{pmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}^t$$

has a minor equal to 1 and, in the notation of [DMS05, Theorem 2.5], all ν_{ij} are equal to 0. Hence the rank of the Horn system is the rank $\mathrm{Vol}(\mathcal{A}) = 4$ of the $\mathcal{A}$-hypergeometric system. The triangulation $\{\{1,2,3,4\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,5,6\}\}$ of $\mathcal{A}$ gives the basis from Table 5.4 of the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta})$. Since $\Phi(r|x,y)$ is a solution, there exist functions $c_i(r)$, independent of x and y, such that

$$\Phi(r|x,y) = c_1(r)\Phi_1(r|x,y) + c_2(r)\Phi_2(r|x,y) + c_3(r)\Phi_3(r|x,y) + c_4(r)\Phi_4(r|x,y).$$

The lowest order terms in the Taylor series of $\Phi(r|x,y)$ are

$$(\sqrt{x} + \sqrt{y} - 1)^{-2r} = 1 + 2r\sqrt{x} + 2r\sqrt{y} + (r + 2r^2)x + 2r(2r+1)\sqrt{xy} + (r + 2r^2)y + \ldots$$

Hence $a_1 = 1$, $a_2 = a_3 = 2r$ and $a_4 = 2r(2r+1)$, so indeed $\Phi(r|x,y) = (\sqrt{x} + \sqrt{y} - 1)^{-2r}$. □

Table 5.5: Formulas for $F_C\left(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 2r|z_1, \ldots, z_n\right)$

Function:	$F_C\left(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 2r\middle|z_1, \ldots, z_n\right)$
Horn eqs. ($n=2$):	$(\theta_x(\theta_x - \frac{1}{2}) - x(\theta_x+\theta_y+r)(\theta_x+\theta_y+r+\frac{1}{2}))F = 0$
	$(\theta_y(\theta_y + 2r - 1) - y(\theta_x+\theta_y+r)(\theta_x+\theta_y+r+\frac{1}{2}))F = 0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3,$
	$\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4, \ldots, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_{n+2}\}$
	$\boldsymbol{\beta} = (-r, -r-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, 2r-1)$
Rank:	2^n
Triang. ($n=2$):	$\{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$
Basis ($n=2$):	$\Phi_1(r\|x,y) = F_4\left(r, r+\frac{1}{2}, \frac{1}{2}, 2r\middle|x,y\right)$
	$\Phi_2(r\|x,y) = y^{1-2r}F_4\left(-r+1, -r+\frac{3}{2}, \frac{1}{2}, -2r+r\middle|x,y\right)$
	$\Phi_3(r\|x,y) = \sqrt{x}F_4\left(r+\frac{1}{2}, r+1, \frac{3}{2}, 2r\middle|x,y\right)$
	$\Phi_4(r\|x,y) = \sqrt{x}y^{1-2r}F_4\left(-r+\frac{3}{2}, -r+2, \frac{3}{2}, -2r+2\middle|x,y\right)$
Sol. of (5.3) ($n=2$):	$\Phi = \Phi_1 + 2r\Phi_3$
Formulas:	$\Phi(r\|\boldsymbol{z}) = f(\boldsymbol{z})^r g(\boldsymbol{x})$
	$f(\boldsymbol{z}) = \frac{8h(\boldsymbol{z})^2 - 4z_n + 8h(\boldsymbol{z})\sqrt{h(\boldsymbol{z})^2 - z_n}}{z_n^2}$
	$g(\boldsymbol{z}) = \frac{1}{2} - \frac{h(\boldsymbol{z})}{2\sqrt{h(\boldsymbol{z})^2 - z_n}}$
	$h(\boldsymbol{z}) = \sqrt{z_1} + \ldots + \sqrt{z_{n-1}} - 1$

Lemma 5.2.2. $\Phi(r|\boldsymbol{z}) = (\sqrt{z_1} + \ldots + \sqrt{z_n} - 1)^{-2r}$ *is a dehomogenized solution of the $\mathcal{A}$-hypergeometric system for $F_C(r, r+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}|z_1, \ldots, z_n)$.*

Proof. The homogeneous function corresponding to Φ is

$$\tilde{\Phi}(r|\boldsymbol{z}) = z_1^{-r} z_2^{-r-\frac{1}{2}} z_3^{-\frac{1}{2}} \cdots z_{n+2}^{-\frac{1}{2}} \left(\sqrt{\frac{z_3 z_{n+3}}{z_1 z_2}} + \ldots + \sqrt{\frac{z_{n+2} z_{2n+2}}{z_1 z_2}} - 1\right)^{-2r}$$

with z_i in Φ equal to $\frac{z_{i+2}z_{n+i+2}}{z_1 z_2}$ in $\tilde{\Phi}$. The $\mathcal{A}$-hypergeometric system is given by $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4, \ldots, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_{n+2}\}$ and $\boldsymbol{\beta} = (-r, -r-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$. Hence the lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_{i+2} + \boldsymbol{e}_{n+i+2}) \subseteq \mathbb{Z}^{2n+2}$. One can easily show that the structure equations all follow from

$$\partial_1\partial_2\tilde{\Phi} = \partial_3\partial_{n+3}\tilde{\Phi} = \ldots = \partial_{n+2}\partial_{2n+2}\tilde{\Phi}.$$

Table 5.6: Formulas for $G_3(r, 1-r|x,y)$

Function:	$G_3(r,1-r\|x,y)$
Horn eqs.:	$(\theta_x(-\theta_x+2\theta_y+r)-x(2\theta_x-\theta_y+1-r)(2\theta_x-\theta_y+2-r))F=0$
	$(\theta_y(2\theta_x-\theta_y+1-r)-y(-\theta_x+2\theta_y+r)(-\theta_x+2\theta_y+r+1))F=0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A}=\{\boldsymbol{e}_1+\boldsymbol{e}_2,\boldsymbol{e}_2,-\boldsymbol{e}_1+\boldsymbol{e}_2,2\boldsymbol{e}_1+\boldsymbol{e}_2\}$
	$\boldsymbol{\beta}=(-r,-1)$
Rank:	Horn system: 4; $\mathcal{A}$-hypergeometric system: 3
Triang.:	$\{\{1,2\},\{2,3\},\{1,4\}\}$
Basis:	$\Phi_1(r\|x,y)=G_3(r,1-r\|x,y)$
	$\Phi_2(r\|x,y)=x^r\sum\frac{(r+1)_{2m+3n}}{(r+1)_{m+2n}m!n!}(-x)^m(x^2y)^n$
	$\Phi_3(r\|x,y)=y^{1-r}\sum\frac{(2-r)_{3m+2n}}{(2-r)_{2m+n}m!n!}(xy^2)^m y^n$
	Horn system: $\Phi_4(r\|x,y)=x^{\frac{r-2}{3}}y^{\frac{-r-1}{3}}$
Sol. of (5.3):	$\Phi=\Phi_1$
Formulas:	$\Phi(r\|x,y)=f(x,y)^r\sqrt{\frac{g(x,y)}{\Delta(x,y)}}$
	$yf(x,y)^3+f(x,y)^2-f(x,y)-x=0$
	$g(x,y)=-3y^2f(x,y)^2-2yf(x,y)+4y+1$
	$\Delta(x,y)=1+4x+4y+18xy-27x^2y^2$

An easy computation shows that both $\partial_1\partial_2\tilde{\Phi}$ and $\partial_k\partial_{n+k}\tilde{\Phi}$ equal

$$z_1^{-r-1}z_2^{-r-\frac{3}{2}}z_3^{-\frac{1}{2}}\cdots z_{n+2}^{-\frac{1}{2}}\left(\sqrt{\frac{z_3z_{n+3}}{z_1z_2}}+\ldots+\sqrt{\frac{z_{n+2}z_{2n+2}}{z_1z_2}}-1\right)^{-2r-2}$$

for all $3\le k\le n+2$. Hence the structure equations are satisfied. The Euler equations

$$(z_1\partial_1+z_{n+3}\partial_{n+3}+\ldots+z_{2n+2}\partial_{2n+2})\tilde{\Phi}(r|\boldsymbol{z})=-r\tilde{\Phi}(r|\boldsymbol{z})$$
$$(z_2\partial_2+z_{n+3}\partial_{n+3}+\ldots+z_{2n+2}\partial_{2n+2})\tilde{\Phi}(r|\boldsymbol{z})=-(r+\frac{1}{2})\tilde{\Phi}(r|\boldsymbol{z})$$
$$(z_k\partial_k-z_{n+k}\partial_{n+k})\tilde{\Phi}(r|\boldsymbol{z})=-\frac{1}{2}\tilde{\Phi}(r|\boldsymbol{z})\quad\text{with } 3\le k\le n+2$$

are easily checked. □

Lemma 5.2.3. *$G_3(r,1-r|x,y)$ satisfies the formulas (5.1) and (5.2).*

Table 5.7: Formulas for $H_4\left(r, -r, \frac{1}{2}, \frac{1}{2}|x, y\right)$

Function:	$H_4\left(r, -r, \frac{1}{2}, \frac{1}{2}\|x, y\right)$
Horn eqs.:	$(\theta_x(\theta_x - \frac{1}{2}) - x(2\theta_x + \theta_y + r)(2\theta_x + \theta_y + r + 1))F = 0$
	$(\theta_y(\theta_y - \frac{1}{2}) - y(2\theta_x + \theta_y + r)(\theta_y - r))F = 0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, 2\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4\}$
	$\boldsymbol{\beta} = (-r, r, -\frac{1}{2}, -\frac{1}{2})$
Rank:	4
Triang.:	$\{\{1,2,3,4\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,5,6\}\}$
Basis:	$\Phi_1(r\|x, y) = H_4\left(r, -r, \frac{1}{2}, \frac{1}{2}\|x, y\right)$
	$\Phi_2(r\|x, y) = \sqrt{y} H_4\left(r + \frac{1}{2}, -r + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\|x, y\right)$
	$\Phi_3(r\|x, y) = \sqrt{x} H_4\left(r + 1, -r, \frac{3}{2}, \frac{1}{2}\|x, y\right)$
	$\Phi_4(r\|x, y) = \sqrt{xy} H_4\left(r + \frac{3}{2}, -r + \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\|x, y\right)$
Sol. of (5.3):	$\Phi = \Phi_1 - 2ir\Phi_2 + 2r\Phi_3 - 2ir(2r+1)\Phi_4$
Formulas:	$\Phi(r\|x, y) = f(x, y)^r$
	$f(x, y) = \frac{1 - 2\sqrt{x} + 2y + 2\sqrt{y(-1 + 2\sqrt{x} + y)}}{(1 - 2\sqrt{x})^2}$

Proof. For $G_3(r, 1-r|x, y)$, the Horn system of differential equations is

$$\begin{aligned} (x(2\theta_x - \theta_y + 1 - r)(2\theta_x - \theta_y + 2 - r) - (-\theta_x + 2\theta_y + r)\theta_x)\Phi &= 0 \\ (y(-\theta_x + 2\theta_y + r)(-\theta_x + 2\theta_y + 1 - r) - (2\theta_x - 2\theta_y + 1 - r)\theta_y)\Phi &= 0 \end{aligned} \tag{5.4}$$

where $\theta_x = x\frac{\partial}{\partial x}$ and $\theta_y = y\frac{\partial}{\partial y}$. Define $\Phi(r|x, y) = f(x, y)^r \sqrt{\frac{g(x,y)}{\Delta(x,y)}}$ where f, g and Δ are as in (5.2). We will first show that Φ is a solution of the Horn system and then show that $\Phi(r|x, y) = G_3(r, 1-r|x, y)$.

To show that Φ is a solution of the Horn system, we compute the partial derivatives of f and g using implicit differentiation. For example,

$$\frac{\partial f}{\partial x} = \frac{1}{-1 + 2f(x, y) + 3yf(x, y)^2}.$$

Substituting this in (5.4) and dividing by $\Phi(r|x, y)$ gives two expressions containing integral powers of f and g. Using $g(x, y) = -3y^2 f(x, y)^2 - 2yf(x, y) + 4y + 1$, we obtain expressions containing integral powers of f only. As f satisfies an equation of degree 3, these expressions can be reduced to contain only the powers 1, f and f^2. It turns out that these expressions are 0, so $\Phi(r|x, y)$ is a solution of (5.4).

Table 5.8: Formulas for $H_4\left(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x,y\right)$

Function:	$H_4\left(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x,y\right)$
Horn eqs.:	$(\theta_x(\theta_x-\frac{1}{2}) - x(2\theta_x+\theta_y+r)(2\theta_x+\theta_y+r+1))F = 0$
	$(\theta_y(\theta_y-\frac{1}{2}) - y(2\theta_x+\theta_y+r)(\theta_y+r+\frac{1}{2}))F = 0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, 2\boldsymbol{e}_1-\boldsymbol{e}_3, \boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4\}$
	$\boldsymbol{\beta} = (-r, -r-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
Rank:	4
Triang.:	$\{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$
Basis:	$\Phi_1(r|x,y) = H_4\left(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x,y\right)$
	$\Phi_2(r|x,y) = \sqrt{y}H_4\left(r+\frac{1}{2}, r+1, \frac{1}{2}, \frac{3}{2}|x,y\right)$
	$\Phi_3(r|x,y) = \sqrt{x}H_4\left(r+1, r+\frac{1}{2}, \frac{3}{2}, \frac{1}{2}|x,y\right)$
	$\Phi_4(r|x,y) = \sqrt{xy}H_4\left(r+\frac{3}{2}, r+1, \frac{3}{2}, \frac{3}{2}|x,y\right)$
Sol. of (5.3):	$\Phi = \Phi_1 - 2r\Phi_2 + 2r\Phi_3 - 2r(2r+1)\Phi_4$
Formulas:	$\Phi(r|x,y) = f(x,y)^r$
	$f(x,y) = (\sqrt{1-2\sqrt{x}}+\sqrt{y})^{-2}$

To show that $\Phi(r|x,y) = G_3(r, 1-r|x,y)$, we write Φ on a basis of the solution space of the Horn system. However, the Horn system is not equivalent to the $\mathcal{A}$-hypergeometric system. The Gale dual of the $\mathcal{A}$-hypergeometric system is

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}^t.$$

In the notation of [DMS05, Theorem 2.5], we have $\nu_{12} = 1$ and all other ν_{12} are 0, so the rank of (5.4) is $\mathrm{Vol}(\mathcal{A}) + 1 = 4$. Apart from the three independent solutions of the $\mathcal{A}$-hypergeometric system, there is also a Puiseux monomial solution. With the triangulation $\{\{1,2\},\{2,3\},\{1,4\}\}$ of $\mathcal{A}$, we get the three solutions

$$\begin{aligned} \Phi_1(r|x,y) &= G_3(r, 1-r|x,y), \\ \Phi_2(r|x,y) &= x^r \sum_{m,n\geq 0} \frac{(r+1)_{2m+3n}}{(r+1)_{m+2n}m!n!}(-x)^m(x^2y)^n \qquad \text{and} \\ \Phi_3(r|x,y) &= y^{1-r} \sum_{m,n\geq 0} \frac{(2-r)_{3m+2n}}{(2-r)_{2m+n}m!n!}(xy^2)^m y^n. \end{aligned}$$

The Puiseux monomial solution is $\Phi_4(r|x,y) = x^{\frac{r-2}{3}}y^{\frac{-r-1}{3}}$. Note that $\Phi(r|x,y)$ is a holomorphic solution of the system (5.4), so it can be expanded in a series that contains

Table 5.9: Formulas for $H_4\left(r, r+\frac{1}{2}, \frac{1}{2}, 2r|x,y\right)$

Function:	$H_4\left(r, r+\frac{1}{2}, \frac{1}{2}, 2r\|x,y\right)$
Horn eqs.:	$(\theta_x(\theta_x-\frac{1}{2})-x(2\theta_x+\theta_y+r)(2\theta_x+\theta_y+r+1))F=0$
	$(\theta_y(\theta_y+2r-1)-y(2\theta_x+\theta_y+r)(\theta_y+r+\frac{1}{2}))F=0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4,2\boldsymbol{e}_1-\boldsymbol{e}_3,\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_4\}$
	$\boldsymbol{\beta}=(-r,-r-\frac{1}{2},-\frac{1}{2},2r-1)$
Rank:	4
Triang.:	$\{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$
Basis:	$\Phi_1(r\|x,y)=H_4\left(r,r+\frac{1}{2},\frac{1}{2},2r\|x,y\right)$
	$\Phi_2(r\|x,y)=y^{1-2r}H_4\left(1-r,\frac{3}{2}-r,\frac{1}{2},2-2r\|x,y\right)$
	$\Phi_3(r\|x,y)=\sqrt{x}H_4\left(r+1,r+\frac{1}{2},\frac{3}{2},2r\|x,y\right)$
	$\Phi_4(r\|x,y)=\sqrt{x}y^{1-2r}H_4\left(-r+2,-r+\frac{3}{2},\frac{3}{2},2-2r\|x,y\right)$
Sol. of (5.3):	$\Phi=\Phi_1+2r\Phi_3$
Formulas:	$\Phi(r\|x,y)=f(x,y)^r g(x,y)$
	$f(x,y)=\frac{-16\sqrt{x}-4y+8-4\sqrt{(2\sqrt{x}-1)(2\sqrt{x}+y-1)}}{y^2}$
	$g(x,y)=\frac{1}{2}+\frac{1-2\sqrt{x}}{2\sqrt{(2\sqrt{x}-1)(2\sqrt{x}+y-1)}}$
	$h(x,y)=\sqrt{(2\sqrt{x}-1)(2\sqrt{x}+y-1)}$

only integral powers of x and y. Since it can be written on the basis mentioned above, it is a linear combination of Φ_1, Φ_2, Φ_3 and Φ_4. By comparing the local exponents, we see that $\Phi(r|x,y)=\Phi_1(r|x,y)=G_3(r|x,y)$ for all $r\notin\mathbb{Z}$. By continuity, this also holds for $r\in\mathbb{Z}$. □

Although the proofs of the above results are elementary, it might not be clear how to find these functions Φ and the formulas. We end with some remarks on this and illustrate it with $F_4(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2},x,y)$.

Let $H_{\mathcal{A}}(\boldsymbol{\beta}(r))$ by a family of $\mathcal{A}$-hypergeometric systems depending on a parameter r and suppose that there exists a family $\Phi(r|\boldsymbol{z})$ of algebraic solutions of the form (5.3). We compute the regular triangulations and the corresponding bases for the solution space of $H_{\mathcal{A}}(\boldsymbol{\beta}(r))$ and choose a suitable basis $\{\Phi_1(r|\boldsymbol{z}),\ldots,\Phi_k(r|\boldsymbol{z})\}$ (in practice, we choose a basis containing the Appell or Horn function). Then $\Phi(r|\boldsymbol{z})$ is a linear combination of the basis elements and hence can be written as

$$\Phi(r|\boldsymbol{z})=c_1(r)\Phi_1(r|\boldsymbol{z})+\ldots+c_k(r)\Phi_k(r|\boldsymbol{z})$$

Table 5.10: Formulas for $H_5(r, -r, \frac{1}{2}|x, y)$

Function:	$H_5(r, -r, \frac{1}{2}\|x, y)$
Horn eqs.:	$(\theta_x(-\theta_x + \theta_y - r) - x(2\theta_x + \theta_y + r)(2\theta_x + \theta_y + r + 1))F = 0$
	$(\theta_y(\theta_y - \frac{1}{2}) - y(2\theta_x + \theta_y + r)(-\theta_x + \theta_y - r))F = 0$
$H_{\mathcal{A}}(\boldsymbol{\beta})$:	$\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 2\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}$
	$\boldsymbol{\beta} = (-r, r, -\frac{1}{2})$
Rank:	4
Triang.:	$\{\{1,2,3,4\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,5,6\}\}$
Basis:	$\Phi_1(r\|x, y) = H_5(r, -r, \frac{1}{2}\|x, y)$
	$\Phi_2(r\|x, y) = \sqrt{y}H_5(r + \frac{1}{2}, -r + \frac{1}{2}, \frac{3}{2}\|x, y)$
	$\Phi_3(r\|x, y) = x^{-r} \sum \frac{(-r)_{2m+3n}}{(\frac{1}{2})_n(1-r)_{m+n}m!n!}(-x)^m(xy)^n$
	$\Phi_4(r\|x, y) = x^{-r+\frac{1}{2}}\sqrt{y} \sum \frac{(\frac{3}{2}-r)_{2m+3n}}{(\frac{3}{2}-r)_{m+n}(\frac{3}{2})_n m!n!}(-x)^m(xy)^n$
Sol. of (5.3):	$\Phi = \Phi_1 + 2ir\Phi_2$
Formulas:	$\Phi(r\|x, y) = f(x, y)^r$
	$x^2f^4 + 2xf^3 + (1 - 2x)f^2 + (4y - 2)f + 1 = 0$

with unknown coefficients $c_i(r)$ depending on r. Note that for all r, s,

$$\Phi(r|\boldsymbol{z})\Phi(s|\boldsymbol{z}) = f(\boldsymbol{z})^{r+s}g(\boldsymbol{z})^2 = \Phi\left(\frac{r+s}{2}|\boldsymbol{z}\right)^2. \tag{5.5}$$

Substituting values for r and s (for example, $r = 0$ and $s = 2$) and comparing the first few terms of these power series of the left and right hand side gives a system of quadratic equations for $c_i(r)$, $c_i(s)$ and $c_i(\frac{r+s}{2})$. We solve this system and guess that $c_i(r)$ is an 'easy' function of r, for example linear or quadratic. This gives us a guess for Φ. As a further check, we compute

$$\left(\frac{\Phi(r|x,y)}{\Phi(s|x,y)}\right)^{\frac{1}{r-s}} = \left(\frac{f(x,y)^r g(x,y)}{f(x,y)^s g(x,y)}\right)^{\frac{1}{r-s}} = f$$

for several values of r and s and check that the first terms of the power series coincide for all choices of r and s.

Having convinced ourselves that Φ is of the desired form, we try to find the formulas for f and g. First we compute the first terms of the power series expansions of f and g. This can easily be done, for example by using $g(x, y) = \Phi(0|x, y)$ and $f(x, y) = \frac{\Phi(1|x,y)}{\Phi(0|x,y)}$. Finding the formulas for f and g is the hardest part and requires some luck and good guesses. For some functions, we find $g = 1$. For the functions that contain square roots of x and y, substitutions like $u = \sqrt{x}$ and $v = \sqrt{y}$ can be useful. The same holds for functions with Pochhammer symbols of $\frac{1}{2}$: the coefficients of $x^m y^n$ contains factors 2^m, which can be removed by substituting $u = \frac{x}{2}$. In this way, we try to get a power series of which we recognize the coefficients. The Online Database of Integer Sequences [Onl] can be helpful in this. In many cases, substituting $x = 0$ or $y = 0$ gives a well-known function such as the Gauss function ${}_2F_1$. This implies that substituting $x = 0$ or $y = 0$ in the equation for Φ must give an equation the Gauss function satisfies.

We illustrate this with an example.

Example 5.2.4. For $F_4(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x, y)$, we have $\mathcal{A} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_4\}$ and $\boldsymbol{\beta} = (-r, -r-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Suppose that $\Phi(r|z)$ is a solution of the form (5.3). There are coefficients $c_1(r), \ldots, c_4(r)$ such that after dehomogenization

$$\Phi(r|x,y) = c_1(r)\Phi_1(r|x,y) + c_2(r)\Phi_2(r|x,y) + c_3(r)\Phi_3(r|x,y) + c_4(r)\Phi_4(r|x,y).$$

As it suffices to determine Φ up to a scalar, we simplify by assuming that $c_1(r) = 1$ (this need not be possible, if all solutions have $c_1(r) = 0$ for some r, but we can try it as a first guess). By taking $r = 0$ and $s = 2$, applying relation (5.5) and comparing the coefficients of the first few powers of x and y, we get a system of quadratic equations for the coefficients $c_i(0)$, $c_i(1)$ and $c_i(2)$. Solving this, we obtain $c_2(0) = c_3(0) = 0$, $c_2(1) = c_3(1) = 2$, $c_2(2) = c_3(2) = 4$, $c_4(0) = 0$, $c_4(1) = 6$ and $c_4(2) = 20$. Assuming that that coefficients are easy functions of r, we guess that $c_1(r) = 1$, $c_2(r) = c_3(r) = 2r$ and $c_4(r) = 2r(2r+1)$. Hence we get

$$\begin{aligned}\Phi(r|x,y) = {} & F_4\left(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x,y\right) + 2r\sqrt{y}F_4\left(r+\frac{1}{2}, r+1, \frac{1}{2}, \frac{3}{2}|x,y\right) + \\ & 2r\sqrt{x}F_4\left(r+\frac{1}{2}, r+1, \frac{3}{2}, \frac{1}{2}|x,y\right) + 2r(2r+1)\sqrt{xy}F_4\left(r+1, r+\frac{3}{2}, \frac{3}{2}, \frac{3}{2}|x,y\right).\end{aligned}$$

Substituting $r = 0$, we see that $g(x,y) = F_4(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}|x,y) = 1$, so $f(x,y) = \Phi(1|x,y)$. Note that f is symmetric in x and y. Define $h(x,y) = f(x^2, y^2)$. One easily computes that $h(x,0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + 10x^9 + \ldots$. These are the first terms of the Tayler series of $\frac{1}{(1-x)^2}$, so we expect $h(x,0)$ and $h(0,y)$ to be equal to $\frac{1}{(1-x)^2}$ and $\frac{1}{(1-y)^2}$, respectively. The 'easiest' function satisfying this is $h(x,y) = \frac{1}{(1-x-y)^2}$. Computing the power series of both functions, we see that this indeed holds up to degree 10. Hence we guess that $\Phi(r|x,y) = \frac{1}{(1-\sqrt{x}-\sqrt{y})^2}$. ◊

To find formulas for the algebraic Lauricella F_C functions, we simply guess the most obvious generalizations of the formulas for the Appell F_4 functions. The proofs are similar to the proof of 5.2.2.

Samenvatting

Dit proefschrift gaat over algebraïsche $\mathcal{A}$-hypergeometrische functies en hun monodromiegroepen. In deze samenvatting zal ik eerst uitleggen wat een hypergeometrische functie is. De definitie van $\mathcal{A}$-hypergeometrische functies is vrij gecompliceerd; daarom blijft deze samenvatting grotendeels beperkt tot de klassieke hypergeometrische functies. $\mathcal{A}$-hypergeometrische functies zijn hiervan een generalisatie. Vervolgens bespreek ik algebraïsche functies. Om uit te leggen wat monodromiegroepen zijn, wordt eerst analytische voortzetting van complexe functies behandeld. Daarna bespreek ik monodromiegroepen van differentiaalvergelijkingen en het verband met algebraïciteit. Tot slot geef ik een overzicht van de inhoud van dit proefschrift.

De Gauss-hypergeometrische functie

De systematische studie van hypergeometrische functies is begonnen met het verschijnen van een artikel van Gauss in 1813. De bekendste hypergeometrische functie is dan ook naar hem vernoemd. Hij definieerde de *Gaussfunctie* ${}_2F_1(a,b,c|x)$, afhankelijk van drie parameters a, b en c en één variabele x, door de oneindige som

$$
\begin{aligned}
{}_2F_1(a,b,c|x) = 1 + \frac{a\cdot b}{c\cdot 1}x + \frac{a(a+1)\cdot b(b+1)}{c(c+1)\cdot 1\cdot 2}x^2 + {}& \\
\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{c(c+1)(c+2)\cdot 1\cdot 2\cdot 3}&x^3 + \ldots.
\end{aligned}
$$

Voor de meeste waarden van a, b en c moet x hierbij tussen -1 en 1 liggen; anders wordt de rechterkant van de formule oneindig groot. Men noteert deze functie ook wel als ${}_2F_1(a,b,c|x) = \sum_n \frac{(a)_n(b)_n}{(c)_n n!}x^n$, waarbij $(a)_0 = 1$ en $(a)_n = a\cdot(a+1)\cdot\ldots\cdot(a+n-1)$ voor $n \geq 1$, en $\sum_n$ aangeeft dat we sommeren over alle niet-negatieve gehele getallen n. Deze functie wordt ${}_2F_1$ genoemd omdat er twee parameters in de teller staan en één parameter in de noemer.

Voor bepaalde waardes van de parameters is de Gaussfunctie een bekende functie, zoals

$$
\begin{aligned}
x^a &= {}_2F_1(-a,b,b|1-x) && \text{voor } 0 < x < 2 \qquad \text{en} \\
\ln(x) &= (x-1)\cdot {}_2F_1(1,1,2|1-x) && \text{voor } 0 < x < 2.
\end{aligned}
$$

Het blijkt dat de Gaussfunctie een oplossing is van een differentiaalvergelijking, een vergelijking die een verband geeft tussen de functie en zijn afgeleiden. De functie ${}_2F_1(a,b,c|x)$ voldoet aan

$$x(x-1)F''(x) + ((a+b+1)x - c)F'(x) + abF(x) = 0. \tag{1}$$

Deze vergelijking heeft nog meer oplossingen: behalve $F(x) = {}_2F_1(a,b,c|x)$ is ook $G(x) = x^{1-c} \cdot {}_2F_1(b-c+1, a-c+1, 2-c|x)$ een oplossing; alle oplossingen blijken van de vorm $\alpha F(x) + \beta G(x)$ te zijn, waarbij α en β constanten zijn.

Gegeneraliseerde hypergeometrische functies

De Gaussfunctie kan op een aantal manieren gegeneraliseerd worden. We kunnen het aantal parameters veranderen, zodat we reeksen van de vorm

$$
{}_pF_q(a_1,\ldots,a_p,b_1,\ldots,b_q|x) = \sum_n \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} x^n =
$$
$$
1 + \frac{a_1 \cdots a_p}{b_1 \cdots b_q \cdot 1} x + \frac{a_1(a_1+1) \cdots a_p(a_p+1)}{b_1(b_1+1) \cdots b_q(b_q+1) \cdot 1 \cdot 2} x^2 + \ldots
$$

krijgen. Voor $p = 2$ en $q = 1$ geeft dit weer ${}_2F_1$.

Een andere mogelijkheid is om het aantal variabelen te verhogen, bijvoorbeeld naar 2 variabelen x en y. De meest directe generalisaties die we zo krijgen zijn de vier *Appellfuncties*. Een voorbeeld hiervan is

$$F_4(a,b,c_1,c_2|x,y) = \sum_{m,n} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m (c_2)_n m! n!} x^m y^n.$$

De bijbehorende functies in meer dan 2 variabelen zijn de *Lauricellafuncties*.

Een algemenere definitie van hypergeometrische functies in meer variabelen werd aan het einde van de negentiende eeuw gegeven door Horn. Als we de coëfficiënten van de Gaussfunctie nog eens bekijken zien we het volgende patroon: als we van x^n naar x^{n+1} gaan, dan wordt de coëfficiënt vermenigvuldigd met $\frac{(a+n)(b+n)}{(c+n)(n+1)} = \frac{n^2+(a+b)n+ab}{n^2+(c+1)n+c}$. Gebaseerd op deze observatie definieerde Horn een hypergeometrische functie in twee variabelen als volgt: als we van $x^m y^n$ naar $x^{m+1}y^n$ of $x^m y^{n+1}$ gaan, dan moet de coëfficiënt met een rationale functie van m en n vermenigvuldigd worden, d.w.z. met een functie waarvan de teller en noemer alleen gehele machten van m en n bevatten. Tegenwoordig wordt een dergelijke functie een *Hornfunctie* genoemd als hij aan een aantal extra voorwaarden voldoet, waaronder de eis dat de graden van m en n in de teller en noemer van de rationale functie hoogstens 2 zijn.

De Appell-, Lauricella- en Hornfuncties voldoen aan stelsels differentiaalvergelijkingen die lijken op (1).

De $\mathcal{A}$-hypergeometrische functies uit de titel van dit proefschrift zijn een generalisatie van de klassieke hypergeometrische functies zoals die door Appell, Lauricella en

Horn bestudeerd werden. Deze zijn aan het einde van de jaren negentig van de vorige eeuw geïntroduceerd door Gelfand, Graev, Kapranov en Zelevinsky. Hiervoor beginnen we met een verzameling punten $\mathcal{A}$, bestaande uit roosterpunten in een r-dimensionale ruimte. We noteren de punten in $\mathcal{A}$ meestal als $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$. Nu associëren we met $\mathcal{A}$ een bepaald stelsel differentiaalvergelijkingen; $\mathcal{A}$-hypergeometrische functies zijn functies in N variabelen $z_1, \ldots, z_N$ die oplossingen van dergelijke stelsels zijn. De differentiaalvergelijkingen hangen onder andere af van de relaties tussen de punten in $\mathcal{A}$; als bijvoorbeeld de punten $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ en $\boldsymbol{a}_3$ op een lijn liggen, dan moet er een bepaalde relatie tussen de afgeleiden naar z_1, z_2 en z_3 gelden[1]. De klassieke hypergeometrische functies van Gauss, Appell, Lauricella en Horn zijn in zekere zin equivalent aan $\mathcal{A}$-hypergeometrische functies en kunnen daarom bestudeerd worden met behulp van de theorie van $\mathcal{A}$-hypergeometrische functies. Een groot deel van dit proefschrift gaat over de Appell-, Lauricella- en Hornfuncties. Ik heb voornamelijk deze functies bestudeerd omdat ze historisch belangrijk zijn geweest in de ontwikkeling van de theorie van hypergeometrische functies, terwijl er nog open vragen over de algebraïciteit en de monodromiegroepen waren. In Hoofdstuk 3 bekijk ik echter $\mathcal{A}$-hypergeometrische functies waarbij de punten van $\mathcal{A}$ roosterpunten in een vlak zijn. Deze functies hebben over het algemeen geen klassieke tegenhanger.

Algebraïsche functies

We hebben hierboven opgemerkt dat hypergeometrische functies oplossingen zijn van differentiaalvergelijkingen, zoals (1). Voor sommige waarden van de parameters voldoen de functies ook aan een 'gewone' vergelijking, die een relatie geeft tussen de machten van de functie. Zo is voor $-1 < x < 1$ bijvoorbeeld

$$ {}_2F_1\left(-\frac{1}{2}, 1, 1 \middle| x\right) = \sqrt{1-x} $$

en dus

$$ {}_2F_1\left(-\frac{1}{2}, 1, 1 \middle| x\right)^2 + x - 1 = 0. \tag{2} $$

Een ander voorbeeld van een Gauss-hypergeometrische functie die voldoet aan een vergelijking is

$$ {}_2F_1\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2} \middle| x\right) = \frac{1}{2}\left(\frac{1}{\sqrt{1-\sqrt{x}}} + \frac{1}{\sqrt{1+\sqrt{x}}}\right). \tag{3} $$

waarbij $0 < x < 1$. Voor deze functie kan men aantonen dat

$$ 4(1-x)^2 \cdot {}_2F_1\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2} \middle| x\right)^4 - 4(1-x) \cdot {}_2F_1\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2} \middle| x\right)^2 + x = 0. \tag{4} $$

[1] Voor de wiskundigen onder de lezers: naast deze zogeheten structuurvergelijkingen bevat het stelsel ook vergelijkingen waardoor de oplossingen homogeen zijn ten opzichte van een torusactie. Het aantal vergelijkingen is groter dan in het klassieke geval, maar het $\mathcal{A}$-hypergeometrische stelsel is eleganter dan het stelsel differentiaalvergelijkingen waaraan de klassieke functies voldoen.

In deze vergelijkingen zijn de coëfficiënten voor de machten van ${}_2F_1$ functies die zijn opgebouwd uit machten van x en getallen. In vergelijking (2) is de coëfficiënt van ${}_2F_1\left(-\frac{1}{2}, 1, 1\middle|x\right)^2$ gelijk aan 1 en de coëfficiënt van ${}_2F_1\left(-\frac{1}{2}, 1, 1\middle|x\right)^0 = 1$ gelijk aan $x - 1$. Voor ${}_2F_1\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}\middle|x\right)$ zijn de coëfficiënten van ${}_2F_1{}^4$, ${}_2F_1{}^3$, ${}_2F_1{}^2$, ${}_2F_1$ en 1 respectievelijk gelijk aan $4(1-x)^2$, 0, $-4(1-x)$, 0 en x. Een functie die aan zo'n vergelijking voldoet, waarbij de coëfficiënten van de machten van de functie opgebouwd zijn uit machten van x, heet *algebraïsch*.

Een voorbeeld van een algebraïsche functie in twee variabelen is

$$\Phi(x, y) = \sqrt{x} + \sqrt{y} - 1$$

met $x, y > 0$. Dit is een van de oplossingen van het stelsel differentiaalvergelijkingen voor de Appellfunctie $F_4(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}|x, y)$ en voldoet aan

$$\Phi^4 + 4\Phi^3 + 2(3 - x - y)\Phi^2 + 4(1 - x - y)\Phi + (x^2 + y^2 - 2xy - 2x - 2y + 1) = 0.$$

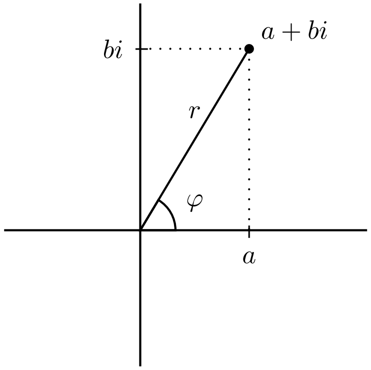
Φ is een algebraïsche functie omdat de coëfficiënten van de machten van Φ functies zijn die opgebouwd zijn uit machten van x en y.

Algebraïciteit is een bijzondere eigenschap, die hypergeometrische functies voor verreweg de meeste waarden van de parameters niet hebben. In 1873 heeft Schwarz de waarden van de parameters bepaald waarvoor ${}_2F_1(a, b, c|x)$ algebraïsch is. Ook voor bepaalde generalisaties, zoals de functies ${}_pF_{p-1}$ en de Appellfuncties, waren de waarden van de parameters waarvoor de functies algebraïsch zijn al langer bekend. Het belangrijkste doel van dit proefschrift is om deze classificaties van algebraïsche functies uit te breiden naar een grotere klasse van $\mathcal{A}$-hypergeometrische functies. Men kan aantonen dat, onder bepaalde voorwaarden, de oplossingen van een stelsel differentiaalvergelijkingen allemaal algebraïsch zijn, of allemaal niet-algebraïsch. Omdat $\mathcal{A}$-hypergeometrische functies gedefinieerd zijn als oplossingen van een stelsel differentiaalvergelijkingen, is het doel van dit proefschrift dus om de stelsels te bepalen waarvan alle oplossingen algebraïsch zijn. Dit heb ik niet gedaan door formules zoals (3) te bepalen. Beukers heeft een aantal jaar geleden een combinatorisch criterium gevonden waarmee, gegeven een verzameling $\mathcal{A}$ en waarden voor de parameters, eenvoudig gecontroleerd kan worden of het $\mathcal{A}$-hypergeometrische stelsel differentiaalvergelijkingen algebraïsche oplossingen heeft. In dit proefschrift gebruik ik dit criterium om voor bepaalde types van verzamelingen $\mathcal{A}$ alle stelsels te bepalen waarvan de oplossingen algebraïsch zijn.

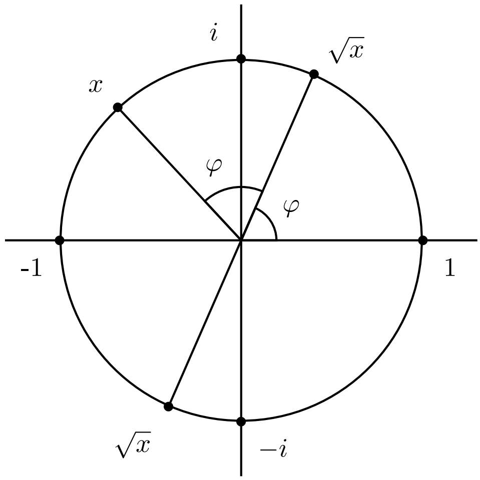
Analytische voortzetting

Het tweede doel van dit proefschrift is het berekenen van de monodromiegroepen van de differentiaalvergelijkingen die algebraïsche oplossingen hebben. Enkele pagina's verderop zal ik de monodromiegroep van een differentiaalvergelijking definiëren. Hiervoor moet ik eerst uitleggen wat de analytische voortzetting van een functie is.

In alle voorgaande formules is x een variabele waarvoor je willekeurige getal kunt invullen (die soms moeten voldoen aan een voorwaarde zoals $-1 < x < 1$). De meeste



Figuur 1: Twee karakterisaties van complexe getallen: als $a + bi$, en door de afstand tot de oorsprong en de hoek met de positieve reële as



Figuur 2: De wortel uit een complex getal op de eenheidscirkel

lezers van deze samenvatting zullen hierbij aan de reële getallen denken: breuken en getallen met oneindig veel decimalen achter de komma. Wiskundigen werken echter meestal met de complexe getallen. Hiervoor voeren we een nieuw 'imaginair' getal i in, dat voldoet aan $i^2 = -1$. De complexe getallen zijn dan de getallen die je kunt schrijven als som van een reëel en een imaginair getal; het zijn dus getallen van de vorm $a + bi$, waarbij a en b 'gewone' reële getallen zijn. Je kunt met complexe getallen rekenen zoals je gewend bent: je kunt ze optellen, aftrekken, vermenigvuldigen en delen. Bij het vermenigvuldigen gebruiken we de regel $i^2 = -1$; zo is bijvoorbeeld

$$(1 + 2i) \cdot (3 + 4i) = 1 \cdot 3 + 1 \cdot 4i + 2i \cdot 3 + 2i \cdot 4i = 3 + 4i + 6i - 8 = -5 + 10i.$$

Een van de voordelen van de complexe getallen is dat alle getallen nu een wortel hebben: voor alle $a + bi$ bestaat er een $c + di$ zodat $(c + di)^2 = a + bi$.

Zoals de reële getallen op een getallenlijn liggen, vormen de complexe getallen een vlak: de reële getallen liggen op de horizontale as, de imaginaire getallen op de verticale as, en het punt (a, b) in het vlak representeert het getal $a + bi$. Hiermee kunnen we ieder complex getal ook op een andere manier karakteriseren: met de afstand tot de oorsprong, en de hoek die de lijn van de oorsprong naar het getal maakt met de positieve reële as (zie Figuur 1). Vermenigvuldigen blijkt nu een meetkundige betekenis te hebben: als we 2 complexe getallen vermenigvuldigen, worden de afstanden tot de oorsprong vermenigvuldigd, en de hoeken met de positieve reële as opgeteld. Lezers die bekend zijn met poolcoördinaten kunnen dit eenvoudig inzien door x in de vorm $x = re^{i\varphi}$ te schrijven, zodat $\sqrt{x} = \sqrt{r}e^{\frac{i\varphi}{2}}$.

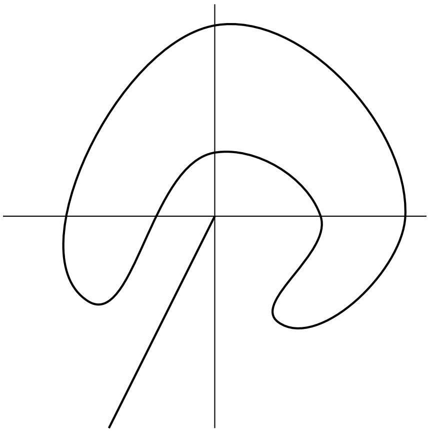
Nu kunnen we in de hypergeometrische functies zoals ${}_2F_1(a, b, c|x)$ voor x ook complexe getallen invullen. Ook nu moeten we x weer zo kiezen dat de som niet oneindig groot is; men kan laten zien dat we voor x alle getallen kunnen invullen waarvan de afstand tot de oorsprong kleiner dan 1 is. Voor sommige waarden van de parameters kunnen we voor x ook andere getallen invullen. De functie is dus in elk geval gedefinieerd binnen de eenheidscirkel, de cirkel met middelpunt de oorsprong en straal 1, en soms op een groter domein. Als we de andere oplossingen van de differentiaalvergelijking (1) bekijken, blijkt echter dat we deze niet binnen zo'n cirkel kunnen definiëren, ook niet met een kleinere straal. Ik zal dit gedrag verduidelijken aan de hand van een eenvoudiger voorbeeld: de functie $f(x) = \sqrt{x}$.

Zoals we hierboven al opgemerkt hebben, hebben alle complexe getallen een wortel. Ieder getal heeft zelfs 2 wortels: als $(c + di)^2 = a + bi$, dan is ook $(-c - di)^2 = a + bi$. Dit geldt ook voor de reële getallen: zowel 3 als -3 zijn wortels van 9. Bij de reële getallen hebben we afgesproken om alleen 3 de wortel van 9 te noemen, maar eigenlijk is -3 evengoed een wortel. Voor de complexe getallen blijkt het niet goed mogelijk te zijn om zo'n afspraak te maken. Laten we bijvoorbeeld proberen om de wortel te definiëren van alle getallen op de eenheidscirkel, ervan uitgaande dat we de afspraak voor de reële getallen aanhouden. We hebben gezien dat als we 2 complexe getallen vermenigvuldigen, de afstanden tot de oorsprong worden vermenigvuldigd en de hoeken met de positieve reële as worden opgeteld. Beide wortels van een getal x op de eenheidscirkel liggen dus ook op de eenheidscirkel, en een van beide heeft een half zo grote hoek met de positieve reële as als x. De andere wortel ligt hier recht tegenover (zie Figuur 2).

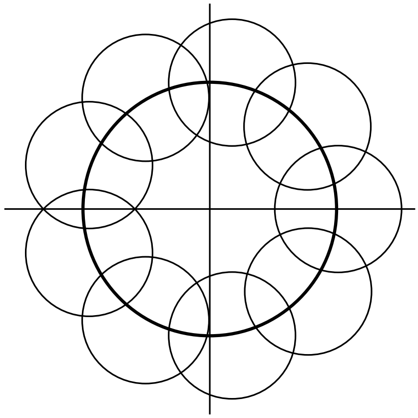
Als we nu x over de eenheidscirkel laten lopen, te beginnen in 1, dan loopt $\sqrt{x}$ half zo snel over de eenheidscirkel. Zo vinden we $\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$, en $\sqrt{-1} = i$, wat klopt met de definitie van i. Echter, als we via de onderste helft van de cirkel weer teruglopen naar 1, dan krijgen we $\sqrt{1} = -1$. Nu is dit in zoverre correct dat inderdaad geldt dat $(-1)^2 = 1$, maar we hadden al afgesproken dat $\sqrt{1} = 1$. Doordat $\sqrt{x}$ half zo snel over de eenheidscirkel loopt als x, krijgen we dus een rare sprong in de wortelfunctie. Dergelijke functies met sprongen heten *discontinu*, en missen veel mooie eigenschappen die continue functies wel hebben. Daarom bestuderen wiskundigen over het algemeen alleen continue functies.

Het is dus niet mogelijk om de wortelfunctie op het hele complexe vlak op een goede, continue manier te definiëren. Het beste wat we kunnen doen is deze functie op een deel van het complexe vlak, dat geen cirkel rond de oorsprong bevat, definiëren. Men kan aantonen dat de wortelfunctie continu gedefinieerd kan worden op het complexe vlak, waaruit een halve lijn door de oorsprong is weggelaten. Als we bijvoorbeeld $\sqrt{x}$ willen definiëren voor alle complexe getallen x op de lus in Figuur 3, dan kunnen we de aangegeven halve lijn weglaten. Voor alle getallen buiten deze lijn, en dus ook voor alle getallen op de lus, kunnen we de wortelfunctie dan continu definiëren. Hierbij maken we dan een keuze voor een van beide mogelijkheden van de wortel van een complex getal. Als we de variabele nu over de lus laten lopen, verandert de keuze van de wortelfunctie niet.

Het blijkt vaak ook nuttig te zijn om te bestuderen hoe een functie zich gedraagt als we de variabele over een willekeurige lus in het complexe vlak laten lopen, waarop



Figuur 3: Een lus en een halve lijn die de lus niet snijdt



Figuur 4: Een overdekking van de eenheidscirkel met cirkeltjes waarop de wortelfunctie gedefinieerd kan worden

we de wortelfunctie niet in één keer kunnen definiëren. Als we x bijvoorbeeld over de eenheidscirkel laten lopen, dan is $\sqrt{x}$ niet gedefinieerd bij de weggelaten halve lijn. Het blijkt echter wel mogelijk te zijn om elkaar gedeeltelijk overlappende cirkels in het complexe vlak te vinden, zodat de lus binnen deze cirkels ligt en we de wortelfunctie op iedere cirkel kunnen definiëren, waarbij deze definities netjes op elkaar aansluiten als de cirkels overlappen. Voor iedere cirkel kiezen we dan een halve lijn die niet door de cirkel loopt, die we weglaten uit het definitiegebied van de wortelfunctie. De keuze van deze halve lijn zal over het algemeen per cirkel verschillend zijn. Als we x nu over de lus laten lopen gebruiken we in iedere cirkel de definitie van $\sqrt{x}$ die we voor die cirkel gekozen hebben. Dan maakt $\sqrt{x}$ geen rare sprongen, maar nadat we de hele lus doorlopen hebben en terug zijn in het beginpunt kan de keuze voor de wortel wel veranderd zijn. Deze methode om een functie op een lus te bestuderen door hem op overlappende cirkels te definiëren, heet het *analytisch voortzetten* van de functie langs de lus.

Als we als lus bijvoorbeeld de eenheidscirkel nemen, dan kunnen we de cirkels kiezen zoals in Figuur 4. Op ieder van deze cirkels kunnen we de wortelfunctie definiëren en deze definities sluiten netjes op elkaar aan, maar als we de hele cirkel doorlopen hebben is $\sqrt{x}$ veranderd in $-\sqrt{x}$.

Voor lussen die niet rond de oorsprong lopen blijkt de keuze van de wortel niet te veranderen bij analytische voortzetting. Bij analytische voortzetting langs lussen die wel rond de oorsprong lopen blijkt het voor de keuze van de wortelfunctie in het eindpunt niet uit te maken hoe de lus precies loopt. Het enige wat van belang is, is hoe vaak de lus rond de oorsprong loopt: na het doorlopen van een lus die een oneven aantal keer rond de oorsprong loopt, is $\sqrt{x}$ veranderd in $-\sqrt{x}$, bij een lus die een even aantal keer rond de oorsprong loopt verandert er niets.

Monodromiegroepen en algebraïciteit

Met behulp van het bovenstaande kan de monodromiegroep van een differentiaalvergelijking gedefinieerd worden. Een differentiaalvergelijking heeft een *basis van oplossingen*: een eindig aantal oplossingen zodat we iedere oplossing op een unieke manier als som hiervan kunnen schrijven. De functies $y(x) = \sin(x)$ en $y(x) = \cos(x)$ zijn bijvoorbeeld oplossingen van de differentiaalvergelijking $y'' = -y$. Alle oplossingen van deze vergelijking worden gegeven door $y(x) = \alpha \sin(x) + \beta \cos(x)$, waarbij α en β constanten zijn. Een basis van de oplossingsruimte bestaat dus uit $\sin(x)$ en $\cos(x)$. Voor de vergelijking (1) zijn $F(x) = {}_2F_1(a, b, c|x)$ en $G(x) = x^{1-c} \cdot {}_2F_1(b-c+1, a-c+1, 2-c|x)$ een basis: alle oplossingen van (1) zijn van de vorm $\alpha F(x) + \beta G(x)$, met α en β constanten. Dergelijke oplossingen van een differentiaalvergelijking kunnen over het algemeen niet op het hele complexe vlak gedefinieerd worden: er zijn punten (zoals $0 = 0 + 0i$ voor de wortelfunctie) zodat het doorlopen van een lus rond dat punt een verandering van de functiewaarde geeft. We bekijken nu alle lussen die niet door dergelijke punten gaan en bekijken wat er met de basis gebeurt als we deze analytisch voortzetten langs zo'n lus. Neem aan dat de basis uit 2 functies F en G bestaat, zoals voor (1). Men kan aantonen dat de analytische voortzetting langs een lus weer een oplossing van de differentiaalvergelijking geeft, die dus geschreven kan worden als lineaire combinatie van de oplossingen F en G. Stel dat F na analytische voortzetting overgaat in $aF + bG$, en G in $cF + dG$. De werking van de lus op de oplossingen van de differentiaalvergelijking is dan volledig vastgelegd door de getallen a, b, c en d. Deze worden meestal genoteerd als een matrix, een rechthoekig getallenschema $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. De *monodromiegroep* van de differentiaalvergelijking is nu gedefinieerd als de verzameling van alle matrices $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ die bij zo'n lus horen[2]. Als de basis meer dan 2 elementen heeft krijgen we een grotere matrix, en als de vergelijking meer variabelen heeft dan alleen x kunnen we een soortgelijke definitie voor lussen in de ruimte geven.

Laten we als voorbeeld de monodromiegroep van de vergelijking (1) met $(a, b, c) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$ berekenen. Dan vormen $F(x) = {}_2F_1(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}|x)$ en $G(x) = \sqrt{x} \cdot {}_2F_1(\frac{5}{4}, \frac{3}{4}, \frac{3}{2}|x)$ een basis. Er geldt, net als in (3), dat

$$F(x) = \frac{1}{2}\left(\frac{1}{\sqrt{1-\sqrt{x}}} + \frac{1}{\sqrt{1+\sqrt{x}}}\right).$$

Men kan laten zien dat voor de andere basisoplossing G geldt dat

$$G(x) = \frac{1}{\sqrt{1-\sqrt{x}}} - \frac{1}{\sqrt{1+\sqrt{x}}}.$$

Omdat de keuze van de wortel alleen verandert bij analytische voortzetting als de variabele over een lus rond de oorsprong loopt en in deze formules zowel $\sqrt{x}$ als

[2] Het is de representatie van de fundamentaalgroep op de oplossingsruimte, gegeven door de basistransformaties geïnduceerd door analytische voortzetting langs lussen die de singulariteiten vermijden.

$\sqrt{1-\sqrt{x}}$ voorkomt, moeten we lussen bekijken die rond 0 of 1 lopen, maar niet door 0 of 1 gaan.

Bij een lus die een oneven aantal keer rond 0 gaat, verandert $\sqrt{x}$ in $-\sqrt{x}$. Na analytische voortzetting van F krijgen we dus weer F, maar analytische voortzetting van G geeft $-G$. Bij een lus die een oneven aantal keer rond 0 gaat hoort dus de matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$; bij lussen die een even aantal keer rond 0 gaan krijgen we $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Nu bekijken we de lussen rond 1. Dan verandert $\sqrt{1+\sqrt{x}}$ niet, maar omdat $1-\sqrt{x}$ rond 0 loopt als x rond 1 loopt verandert $\sqrt{1-\sqrt{x}}$ in $-\sqrt{1-\sqrt{x}}$. Dit betekent dat F verandert in $-\frac{1}{2}G$, en G in $-2F$, zodat we de matrix $\begin{pmatrix}0 & -2\\ -\frac{1}{2} & 0\end{pmatrix}$ krijgen. Als we deze lus nogmaals doorlopen verandert $-\frac{1}{2}G$ weer in F en $-2F$ in G, en vinden we $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Men kan aantonen dat de lussen die zowel rond 0 als rond 1 lopen ook nog de matrices $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\begin{pmatrix}0 & 2\\ \frac{1}{2} & 0\end{pmatrix}$, $\begin{pmatrix}0 & -2\\ \frac{1}{2} & 0\end{pmatrix}$ en $\begin{pmatrix}0 & 2\\ -\frac{1}{2} & 0\end{pmatrix}$ opleveren. In totaal heeft de monodromiegroep dus acht elementen.

De monodromiegroep wordt bestudeerd omdat deze veel informatie geeft over de structuur van de oplossingen van de differentiaalvergelijking. Punten zoals 0 en 1 in bovenstaand voorbeeld zijn speciale punten van de differentiaalvergelijking, en het is van belang om te weten hoe de oplossingen zich rond deze punten gedragen. Als alle oplossingen van de differentiaalvergelijking gedefinieerd kunnen worden op het hele complexe vlak, dan heeft de monodromiegroep maar 1 element: alleen $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. De monodromiegroep geeft dus aan in hoeverre het niet mogelijk is om de oplossingen op het hele complexe vlak te definiëren. In dit proefschrift bestudeer ik monodromiegroepen voornamelijk omdat er een verband is tussen algebraïciteit van de oplossingen en de monodromiegroep: onder bepaalde voorwaarden is de monodromiegroep eindig dan en slechts dan als de oplossingen van de differentiaalvergelijking algebraïsche functies zijn. In het bovenstaande voorbeeld zagen we dat de monodromiegroep acht elementen heeft en dus eindig is. Hieruit blijkt opnieuw dat ${}_2F_1(\frac{1}{4},\frac{3}{4},\frac{1}{2}|x)$ algebraïsch is, zoals we ook al hadden gezien aan vergelijking (4)[3]. Het bepalen van de parameters van algebraïsche hypergeometrische functies is dus equivalent aan het bepalen van de parameters waarvoor de monodromiegroep eindig is. Nadat deze parameters berekend zijn, is het interessant om deze groepen ook daadwerkelijk te bepalen.

Inhoud van dit proefschrift

In het verleden heeft men algebraïsche functies bepaald door de monodromiegroep te berekenen en te bepalen wanneer deze eindig is. In dit proefschrift pak ik het omgekeerd aan: ik bepaal eerst de algebraïsche functies, en bereken daarna de monodromiegroepen, gebruikmakend van het feit dat ze eindig zijn.

Hoofdstuk 1 is een inleiding in de theorie van hypergeometrische functies en het bepalen van de algebraïsche functies. Vervolgens bepaal ik in Hoofdstuk 2 de algebraïsche Appell-, Lauricella- en Hornfuncties. Hoewel dit klassieke hypergeometrische

[3]De monodromiegroep is in dit geval de dihedrale groep met 8 elementen. Het is geen toeval dat deze gelijk is aan de Galoisgroep van $\mathbb{C}(x, {}_2F_1(\frac{1}{4},\frac{3}{4},\frac{1}{2}|x))/\mathbb{C}(x)$.

functies zijn, bekijk ik hier hun $\mathcal{A}$-hypergeometrische tegenhanger, zodat het combinatorische criterium van Beukers kan worden toegepast. In Hoofdstuk 3 bekijk ik $\mathcal{A}$-hypergeometrische functies waarvan de verzameling $\mathcal{A}$ hoogstens 2-dimensionaal is. Ook hier bepaal ik alle algebraïsche functies. Vervolgens worden de monodromiegroepen berekend. Hoofdstuk 4 bevat algemene theorie over de berekening van monodromiematrices en de toepassing op de Appell- en Hornfuncties. In Hoofdstuk 5 gebruik ik dit en het gegeven dat de monodromiegroep van een algebraïsche functie eindig is om de monodromiegroepen van de Appell- en Hornfuncties te berekenen. Dit proefschrift wordt afgesloten met formules zoals (3) voor de 1-parameterfamilies van algebraïsche Appell-, Lauricella- en Hornfuncties.

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Curriculum vitae

Esther Bod werd op 3 maart 1987 in Nijmegen geboren. Daar heeft ze in 2003 haar VWO-diploma behaald aan het Stedelijk Gymnasium Nijmegen. Vervolgens is ze wiskunde gaan studeren aan de Katholieke Universiteit Nijmegen (tegenwoordig Radboud Universiteit), alwaar ze in 2004 cum laude haar propedeuse haalde. Vervolgens heeft ze haar bacheloropleiding wiskunde voortgezet aan de Universiteit Utrecht en gelijktijdig ook de bacheloropleiding Cognitieve Kunstmatige Intelligentie gevolgd. In 2007 heeft ze beide opleidingen cum laude afgerond. Daarna heeft ze de masteropleiding Mathematical Sciences gevolgd en deze in 2009 cum laude afgerond met een scriptie over generalisaties van het tiende probleem van Hilbert.

In deze jaren heeft ze deelgenomen aan diverse nationale en internationale competities: de International Mathematical Olympiad (IMO) in 2002 en 2003, de Landelijke Interuniversitaire Mathematische Olympiade (LIMO) in 2005, 2006 en 2008 (respectievelijk 3^{e}, 1^{e} en 3^{e} plaats; lid van de organisatie in 2007) en de International Mathematics Competition for University Students (IMC) in 2006 en 2007 (beide keren 3^{e} prijs).

In 2009 is ze begonnen aan haar promotieonderzoek onder begeleiding van Frits Beukers. De resultaten van dit onderzoek staan in dit proefschrift beschreven. Daarnaast heeft ze werkcolleges gegeven bij diverse bachelorvakken en een aantal internationale congressen en summer schools bezocht. Daarnaast heeft ze de Utrechtse afvaardiging naar de IMC in 2010 begeleid en is ze coördinator geweest bij de Benelux Mathematical Olympiad in 2009 en de IMO in 2011.

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