Arithmetic nature of some infinite series and integrals

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Dedicated to Professor M. Ram Murty on his 60th birthday.

Abstract. We give an overview of some results on the transcendence nature of the sums of some infinite series. We also give some new results on the transcendence nature of some series involving mildly non-periodic functions and some integrals.

1. Some results on the transcendence of infinite series

Let $P(X)$ be a polynomial with algebraic coefficients; $Q(X)$ be a polynomial with rational coefficients and having only simple rational zeros in the interval $[-1,0)$ with deg $P <$ deg Q; $g(X)=\sum_{\lambda=1}^{\ell} P_\lambda(X)\alpha_\lambda^X$ where $P_\lambda(X)$ has algebraic coefficients and α_λ are algebraic numbers and $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ be a periodic function mod q with $q\geq 2$ an integer. In [**1**], it was shown that if the infinite series

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)},\ T=\sum_{n=0}^{\infty}\frac{f(n)}{Q(n)},\ U=\sum_{n=0}^{\infty}\frac{g(n)}{Q(n)}$$

converge, then their sum is either 0 or transcendental. This was achieved by showing that these series can be expressed as linear forms in logarithms of algebraic numbers with algebraic coefficients and applying the well known theorem of Baker in transcendence theory.

Theorem B. *Let $\alpha_1,\cdots,\alpha_n$ be non-zero algebraic numbers and $\beta_1,\cdots,\beta_n$ be algebraic numbers, not all zero. Then the linear form*

$$\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n$$

is either 0 *or a transcendental number.*

1.1. The series $S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}$. Lehmer [**12**], while studying generalized Euler constants, showed

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)}=\frac{\pi}{3}\ ;$$

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$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} =$$

$$\frac{1}{4320}\left(192\log 2 - 81\log 3 - 7\pi\sqrt{3}\right).$$

While the first series is obviously transcendental, the second series requires the application of Theorem B for such an assertion. In [**1**], transcendence of the series S was studied using partial fractions. In [**25**], Saradha and Tijdeman gave explicit conditions under which series of the form S is transcendental whenever $P(n) = (-1)^n(\alpha n+\beta)$ and $Q(n) = (qn+s_1)(qn+s_2)$ or $P(n) = (\alpha n+\beta)$ and $Q(n) = (qn+s_1)(qn+s_2)(qn+s_3)$ with $\alpha, \beta \in \overline{\mathbb{Q}}$, and $s_1, s_2, s_3 \in \mathbb{Z}$. Recently Ram Murty and Weatherby (see [**23**],[**35**]) simplified arguments in [**1**] and also extended the study to the series

$$\sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)}$$

when $B(X)$ has only simple roots $-p_1/q_1, \cdots, -p_k/q_k \in \mathbb{Q}$ with $\gcd(p_i, q_i) = 1$ and there is a $q_j > 1$ which is coprime to each of the other q_i's. Here $'$ indicates that the sum is over integers where $B(n) \neq 0$. In the same spirit as in [**1**] the following exponential series

$$\sum_{n=0}^{\infty}{}' \frac{z^n A(n)}{B(n)}$$

was studied in [**35**] via Stirling numbers of the second kind and shown to be a computable algebraic number or transcendental whenever $z \neq 1$ is algebraic with $|z| \leq 1$. See also [**10**] for more results. For excluding the possibility that the series under consideration may vanish, the following theorem of Baker, Birch and Wirsing [**2**] is applied.

Theorem BBW. *Suppose that f is a periodic function with period q satisfying*

(i) $f(r) = 0$ *if* $1 < \gcd(r, q) < q$

(ii) *the qth cyclotomic polynomial is irreducible over* $\mathbb{Q}(f(1), \cdots, f(q))$,

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In particular, if q is a prime and f is rational valued, then any convergent series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

is non-zero and hence represents a transcendental number. Chowla, in 1969, raised the question about the non-vanishing of such series and Theorem BBW answers his question. By the well known result of Dirichlet that $L(1, \chi) \neq 0$ for any non-principal Dirichlet character mod q, we immediately get

$$L(1,\chi) \textit{ is transcendental.}$$

This is a well known fact for quadratic characters by the class number formula for quadratic fields. The above assertion was noted in [**1**].

1.2. The series $T = \sum_{n=0}^{\infty} \frac{f(n)}{Q(n)}$. In [**19**], Ram Murty and Saradha studied the arithmetic nature of these series via the theory of Hurwitz zeta function. It can be seen that

$$L(s,f) = q^{-s} \sum_{a=1}^{q} f(a)\zeta\left(s, \frac{a}{q}\right)$$

where $\zeta(s,x)$ is the classical Hurwitz zeta function. This led to showing

Theorem MS. *Let ψ be the digamma function obtained by taking the logarithmic derivative of the classical gamma function. Then at most one of $\psi(a/q), 1 \leq a \leq q,\ \gcd(a,q)=1$ is algebraic. Further all the numbers $\psi(a/q)+\gamma$ for $1 \leq a \leq q,\ \gcd(a,q)=1$ are transcendental, where γ is the Euler constant.*

The latter result was first proved by Bundschuh [**3**] by a different method. The $p-$adic analog was considered in [**20**]. They also showed that any linear combination

$$\sum_{\chi \neq \chi_0} a_\chi L(1,\chi)$$

with $a_\chi \in \mathbb{Q}(\xi)$, not all zero and ξ a primitive $\varphi(q)$-th root of unity with $\gcd(q,\varphi(q)) = 1$, is non-zero and hence transcendental. The corresponding result over $\mathbb{Q}$ is already shown in [**2**].

This can be extended to any linear combination of $L(1,\chi)$ as χ ranges over non-trivial primitive characters mod q with coefficients from a field

$$K_1 = K(\xi) \text{ with } K_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} \tag{1.1}$$

where K is any algebraic number field, and ζ_q is a primitive q-th root of unity. See [**19**]. Gun, Rath and Ram Murty [**6**] have given conditional result on the transcendence of $L(1,\chi)$ when χ is an Artin character.

1.3. Problem of Erdős on $L(1,f)$. Suppose f assumes only the values ± 1 and $f(q)=0$. Then Erdős conjectured that $L(1,f) \neq 0$. Convergence of such series imposes the condition that q is odd. Using Theorem BBW, it can be seen that the conjecture is true whenever q is prime. Okada [**17**], by refining the conditions in Theorem BBW showed that the conjecture is true whenever $2\varphi(q)+1 > q$. This covers the case when q is a prime power or a product of two odd primes. This condition has been refined by Saradha in [**26**] and [**27**]. In [**22**], Saradha and Ram Murty have shown that the conjecture is true whenever $q \equiv 3 \pmod 4$. The case of $q \equiv 1 \pmod 4$ still remains *open*. Using the criteria of Okada, Tijdeman [**31**] could show that if f is either completely multiplicative or multiplicative with $|f(p^k)| < p-1$ for every prime divisor p of q, then $L(1,f) \neq 0$.

1.4. From $L(1,\chi)$ to $L(k,\chi)$. Let χ be a Dirichlet character (mod q) and let k be any integer > 1 such that χ and k have the same parity. Then one can show fairly easily that $L(k,\chi)$ is a non-zero algebraic multiple of π^k. No result is known if k and χ are of opposite parity. In [**21**], the following linear independence result was shown.

> *Let K_1 be as in* (1.1). *For any fixed integer k, the set of numbers $L(k,\chi)$ as χ ranges over Dirichlet characters* $\pmod q$ *with the same parity as k are linearly independent over K_1.*

Let $q = 4$ and let χ be the odd nontrivial character mod 4. Then $L(2,\chi)$ is called the *Catalan constant*. Evaluation of $L(2,\chi)$ involves Barnes double gamma function $G(z)$. Rivoal and Zudilin [**24**] showed that at least one of the seven numbers $L(2k,\chi), 1 \le k \le 7$ is irrational. Gun, Rath and Ram Murty [**8**] have shown that at least one of

$$L(2,\chi)/\pi^2, G(1/4)G(3/4)$$

is transcendental. In this connection we recall a conjecture of P. and S. Chowla. We know that

$$\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\{\frac{1}{n+z} - \frac{1}{n}\}.$$

By taking derivatives, we get the polygamma functions

$$\psi_k(z) = (-1)^k k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}$$

and for any q-periodic function f we have

$$\begin{aligned} L(k,f) &= \frac{(-1)^k}{(k-1)!q^k} \sum_{a=1}^{q} f(a)\psi_{k-1}(a/q) \\ &= q^{-k} \sum_{a=1}^{q} f(a)\zeta(k,a/q). \end{aligned}$$

The polygamma functions are connected to cotangent functions as follows:

$$-\frac{d^k}{dz^k}(\pi\cot\pi z) = \psi_k(z) + (-1)^{k+1}\psi_k(1-z). \tag{1.2}$$

Suppose $f(a) = m$ for $1 \le a \le q-1$. Then

$$L(k,f) = \frac{m}{q^k}\sum_{a=1}^{q-1}\zeta(k,a/q) + \frac{f(q)\zeta(k)}{q^k}.$$

Since

$$(1-q^{-s})\zeta(s) = \frac{1}{q^s}\sum_{a=1}^{q-1}\zeta(s,a/q),$$

we get

$$L(k,f) = \zeta(k)\left(\frac{f(q) + m(q^k-1)}{q^k}\right).$$

Thus if $f(q) = -m(q^k-1)$, then $L(k,f) = 0$. P. and S. Chowla *conjectured* that this is the only example if $k = 2$ and q an odd prime. Under this conjecture it can be seen [**20**] that

$\psi_1(a/q)$, $1 \le a \le q-1$, q an odd prime, are linearly independent over $\mathbb{Q}$.

Note that when $q = 2, \psi_1(1/2) = \pi^2/2$. Also by (1.2) one can see that at least one of $\psi_k(a/q), \psi_k(1-a/q)$ is transcendental with $1 \le a < q, \gcd(a,q) = 1$. In fact it is an algebraic multiple of π^{k+1}. The fact that it is a non-zero multiple follows by a result of Okada [**16**] on the linear independence of derivatives of cotangent function. The conjecture of Chowlas was generalized by Milnor. In [**7**], the following conjecture is termed as Chowla-Milnor conjecture:

The numbers $\zeta(k, a/q)$, $1 \leq a \leq q$, $\gcd(a, q) = 1$, *are linearly independent over* $\mathbb{Q}$.

It was shown in [**7**] that Chowla-Milnor conjecture for $q = 4$ implies the irrationality of $\zeta(2k+1)/\pi^{2k+1}$ for $k \geq 1$. Further it was shown that the conjecture is true for k odd and $q = 3$ or $q = 4$. The arithmetic nature of the values of $L(1, f)$ when f is an algebraic valued function defined over the ideal class group of an imaginary quadratic field was also extensively studied by Ram Murty and Kumar Murty [**18**].

1.5. Series involving mildly non-periodic functions. In this section we modify the series $V = \sum_{n=1}^{\infty} \frac{f(n)}{n}$ where f is a periodic function with period q and then analyze the transcendence of the modified series. Towards this, let α be an irrational algebraic number of degree $D(\alpha)$ with minimal polynomial

$$a_0 \prod_{i=1}^{D(\alpha)} (X - \alpha_i).$$

Let

$$M(\alpha) = |a_0| \prod_{i=1}^{D(\alpha)} \max(1, |\alpha_i|)$$

be the Mahler measure of α. Define the absolute logarithmic height of α by

$$h(\alpha) = \frac{1}{D(\alpha)} \log M(\alpha).$$

It is known that

$$h(\alpha\beta) \leq h(\alpha) + h(\beta)$$

and

$$h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log 2.$$

(See [**33**]). Note that for any rational number r/s with $s > r$, we have

$$h\left(\alpha + \frac{r}{s}\right) \leq h(\alpha) + \log s + \log 2 \leq 2(h(\alpha) + \log s). \tag{1.3}$$

First, we prove a general result and then apply it to the series V.

Theorem DS. *Let θ be a transcendental number. Suppose that there exist positive absolute constants c_1 and c_2 such that*

$$|\theta - \alpha| > \exp(-c_1 \ D(\alpha)^{c_2} \ h(\alpha)) \tag{1.4}$$

for any algebraic number α. Let ξ be any real number. Suppose that there exists an infinite sequence of rationals $\{p_i/q_i\}$, $q_i \geq 2$, satisfying

$$\left|\xi - \frac{p_i}{q_i}\right| \leq \exp(-(2c_1 \ D(\alpha)^{c_2} + \delta_i) \log q_i) \tag{1.5}$$

where $\delta_i \to \infty$ as $i \to \infty$. Assume further that $\{\epsilon_i\}$ is a sequence of real numbers with $\epsilon_i \to 0$ as $i \to \infty$ such that $\epsilon_i \ \delta_i \to \ c_3$ as $i \to \infty$ for some absolute constant c_3 and

$$\delta_i \log q_i \geq \delta_{i-1} \log q_{i-1} \geq \epsilon_i \delta_i \log q_i.$$

Then $\theta + \xi$ is transcendental. Further, for any algebraic number α

$$|\theta + \xi - \alpha| > \exp(-c_4 \ D(\alpha)^{2c_2} h(\alpha))$$

for some absolute constant $c_4 > 0$.

PROOF. Consider

$$
\begin{aligned}
|\theta+\xi-\alpha| &= \left|\theta+\xi-\alpha+\frac{p_i}{q_i}-\frac{p_i}{q_i}\right| \\
&\geq \left|\theta-\alpha+\frac{p_i}{q_i}\right|-\left|\xi-\frac{p_i}{q_i}\right| \\
&\geq \exp(-2c_1\ D(\alpha)^{c_2}(h(\alpha)+\log q_i))-\exp(-(2c_1\ D(\alpha)^{c_2}+\delta_i)\log q_i)
\end{aligned}
$$

by (1.3), (1.4) and (1.5). Choose the least i, say i_0, such that

$$2c_1\ D(\alpha)^{c_2}\ h(\alpha)+\log 2<\delta_{i_0}\log q_{i_0}.$$

Then

$$|\theta+\xi-\alpha| \geq \frac{1}{2}\exp(-2c_1\ D(\alpha)^{c_2}(h(\alpha)+\log q_{i_0})). \tag{1.6}$$

By the choice of i_0,

$$2c_1\ D(\alpha)^{c_2}\ h(\alpha)+\log 2 \quad\geq\quad \delta_{i_0-1}\log q_{i_0-1}\geq \epsilon_{i_0}\delta_{i_0}\log q_{i_0}.$$

Hence

$$\log q_{i_0}\leq c_5(2c_1\ D(\alpha)^{c_2}\ h(\alpha)+\log 2)$$

since $\{\epsilon_i\delta_i\}$ is a bounded sequence. Substituting the estimate for $\log q_{i_0}$ in (1.6) we get the result. □

We shall now apply Theorem DS with $\theta=V$ provided $V\neq 0$, where $V=\sum_{n=1}^{\infty}\frac{f(n)}{n}$. In [**1**, Theorem 1], it was shown that if $V\neq 0$, then for any algebraic number α

$$|V-\alpha|\geq \exp\left(-c^q q^{3q}(D(\alpha)d_f)^{q+3}\max(h(\alpha),h_f)\right)$$

where

$$d_f=[\mathbb{Q}(f(0),\ldots,f(q-1)):\mathbb{Q}],\ \ h_f=\sum_{j=0}^{q-1}h(f(j))$$

and c is some computable absolute constant. Thus for any fixed periodic function f with $V\neq 0$, we have

$$|V-\alpha|\geq \exp(-c_6(f)\ D(\alpha)^{q+3}h(\alpha)).$$

Hence (1.4) is satisfied with $c_1=c_6(f)$ and $c_2=q+3$. Therefore by Theorem DS, we conclude that $V+\xi$ is transcendental for any ξ as in Theorem DS and

$$|V+\xi-\alpha|>\exp(-c_7(f)D(\alpha)^{q+3}h(\alpha))$$

for any algebraic number α.

EXAMPLE 1.1. Let $a>1$ be an integer and let $\{b_k\}$ be an increasing sequence of positive integers such that

$$\limsup\frac{b_{k+1}}{b_k}=\infty.$$

Then

$$\xi=\sum_{n=1}^{\infty}\frac{(-1)^n}{a^{b_n}}$$

is a Liouville number (See [**13**, p. 165]) and it can be checked that (1.5) is satisfied. Hence by Theorem DS,

$$\sum_{n=1}^{\infty} \frac{f(n) + (-1)^n/a^{b_n}}{n}$$

is transcendental for any periodic function f for which $V \neq 0$. In particular,

$$\sum_{n=1}^{\infty} \frac{f(n) + (1/10^{n!})}{n}$$

is transcendental for functions f as above.

2. Integrals

In this section we study the arithmetic nature of several integrals. In 1931, Siegel [**29**] computed that the perimeter of the Lemniscate

$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

is equal to

$$K_0 = \sqrt{2} \int_0^1 \frac{dt}{\sqrt{1 - t^4}} = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}. \tag{2.1}$$

In 1919, Hardy and Ramanujan [**9**] showed that

$$144 \left(\frac{K_0}{\pi} \right)^8 = 1 + 480 \sum_{n \geq 1} \frac{n^7}{e^{2n\pi} - 1}.$$

K_0 is also connected to the elliptic curve $y^2 = 4x^3 - 4x$. The field of CM for this curve is $\mathbb{Q}(i)$, $g_3 = 0$ and $j = 1728$. The fundamental periods of this curve are given by

$$\omega_1 = \int_1^{\infty} \frac{dt}{\sqrt{t^3 - t}} = \frac{\Gamma(1/4)^2}{2^{3/2}\sqrt{\pi}} \textit{ and } \omega_2 = i\omega_1. \tag{2.2}$$

Another example is the elliptic curve $y^2 = 4x^3 - 4$ with the field of CM as $\mathbb{Q}(\rho)$ where $\rho = e^{2\pi i/3}, g_2 = 0 = j$. Its periods are given by

$$\omega_1 = \int_1^{\infty} \frac{dt}{\sqrt{t^3 - 1}} = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \textit{ and } \omega_2 = \rho\omega_1. \tag{2.3}$$

In fact, in 1936, Schneider [**28**] showed that if the invariants g_2 and g_3 of the Weierstrass elliptic function $\wp$ are algebraic then any non-zero period ω of $\wp$ is transcendental.

2.1. Periods. In [**5**], Griffiths calls integrals of rational functions over a contour as "periods." In [**11**], Kontsevich and Zagier describe "periods" as complex numbers whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by some polynomial inequalities with rational coefficients. In fact, one can replace "rational functions" and "rational coefficients" by "algebraic function" and "algebraic coefficients" without changing the set of numbers one obtains. The irrational number $\sqrt{2}$ can be expressed as

$$\sqrt{2} = \int_{2x^2 \leq 1} dx.$$

In fact, every algebraic number is a period. The simplest non-algebraic number which is a period is π since

$$\pi = \int_{x^2+y^2\leq 1} dx\ dy = 2\int_{-1}^{1}\sqrt{1-x^2}dx = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

or by the contour integral

$$2\pi i = \oint \frac{dz}{z}$$

in $\mathbb{C}$ around 0. In [**11**] the following principle is proposed:

> **Principle 1.** *Whenever you meet a new number, and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period.*

Further, the set of periods form an algebra so we get new periods by taking sums and products of known ones. By the terminology used in [**19**], we call a number "Baker period" if it can be expressed as a linear form in logarithms of algebraic numbers with algebraic coefficients. The set of Baker periods form a sub algebra of the algebra of periods. Further in [**11**], "exponential periods" are defined to be those numbers which can be represented by absolutely convergent integrals of the product of an algebraic function with the exponent of an algebraic function over a semi algebraic set where all polynomials entering the definition have algebraic coefficients. Since

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}dx;\ \Gamma(p/q) = \int_{0}^{\infty} e^{-t}t^{p/q-1}dt,$$

the algebra of exponential periods include $\sqrt{\pi}$ and $\Gamma(p/q)$ with p, q positive integers satisfying $\gcd(p, q) = 1$. After including the Euler constant γ in this algebra, it is predicted in [**11**] that all known mathematical constants will belong to the algebra of exponential periods.

REMARK 2.1. By the result mentioned in Section 1.2, we know that $\gamma+\psi(p/q)$ is transcendental for $1 \leq p < q$, $\gcd(p, q) = 1$. In fact, it was shown that it is a Baker period and hence a period. This can also be seen from the Legendre's integral representation

$$\gamma + \psi(p/q) = \int_{0}^{1}\frac{x^{p/q}-1}{x-1}dx.$$

2.2. Integrals which are transcendental. In 1949, Siegel [**29**] observed that the theory of transcendental numbers at that time was as yet unable to determine the arithmetic nature of integrals like

$$\int_{0}^{1}\frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right).$$

This was more so for the integrals in (2.1), (2.2) and (2.3). By Theorem B it is now easy to see that the last integral is indeed a transcendental number and $\frac{1}{3}(\log 2 + \frac{\pi}{\sqrt{3}})$ is a period which can also be seen from the fact that both $\log 2$ and $\frac{\pi}{\sqrt{3}}$ are periods. What can we say about the integrals in (2.1), (2.2) and (2.3)? It took another three decades when Nesterenko [**15**] in 1996 showed the following theorem:

THEOREM N. *Let $\wp(z)$ be a Weierstrass $\wp$-function with algebraic invariants and with complex multiplication by an order of the imaginary quadratic field K. Let ω be a non-zero period and let $\tau \in K$, with $\Im(\tau) \neq 0$. Then, π, ω and $e^{2\pi i \tau}$ are algebrically independent.*

The above theorem has several interesting applications for infinite series and also on the transcendence of the continued fraction of Rogers-Ramanujan. See [**34**] for more details. As a consequence of Theorem N and equations (2.2) & (2.3), we get

COROLLARY 2.2.
- (i) *The numbers $\pi, \Gamma(1/4), e^{\pi}$ are algebraically independent.*
- (ii) *The numbers $\pi, \Gamma(1/3), e^{\pi\sqrt{3}}$ are algebraically independent.*
- (iii) *The numbers π and $e^{\pi\sqrt{d}}$ are algebraically independent for any natural number d.*

We shall denote by W the following set

$$W = \left\{ s, e^{\pi}, \Gamma(1/4), \Gamma(1/3), e^{\pi\sqrt{d}} \right\}$$

where s is any algebraic number. Note that every element of W is contained in the algebra of exponential periods. By Corollary 2.2, we see that π and any element $a \in W$ are algebraically independent.

2.3. Integrand as rational function. Using Theorem B, van der Poorten [**30**] gave a convenient criterion for determining the arithmetic nature of definite integrals of rational functions. It can be seen that the theory of residues plays an important role here.

THEOREM V1. *Let $P(z)$ and $Q(z)$ denote polynomials with algebraic coefficients and with no common factor. Denote by $\alpha_1, \alpha_2, \cdots, \alpha_n$ the distinct zeros of $Q(z)$ and by $r_1, r_2, \cdots, r_n$ the residues respectively at the poles of the rational function $P(z)/Q(z)$. Further let C be some contour in the complex plane for which the definite integral*

$$\int_C \frac{P(z)}{Q(z)} dz$$

exists, and suppose that C is either closed, or has endpoints which are algebraic or infinite. Then the definite integral is algebraic if and only if

$$\int_C \left(\sum_{k=1}^{n} \frac{r_k}{z - \alpha_k} \right) dz = 0.$$

As an immediate corollary, we find that

COROLLARY 2.3. *If $P(z)$ and $Q(z)$ are as above with deg $P(z) <$ deg $Q(z)$ and the zeros of $Q(z)$ are distinct, then*

$$\int_C \frac{P(z)}{Q(z)} dz$$

is either zero or transcendental.

We give the proofs of Theorem V1 and Corollary 2.3. Writing

$$Q(z) = q_0(z-\alpha_1)^{\rho_1}\cdots(z-\alpha_n)^{\rho_n},$$

we find that $P(z)/Q(z)$ has partial fraction expansion of the form

$$\frac{P(z)}{Q(z)} = p(z) + \sum_{k=1}^{n}\left\{\frac{r_k}{z-\alpha_k} + \frac{r_{k,2}}{(z-\alpha_k)^2} + \cdots + \frac{r_{k,\rho_k}}{(z-\alpha_k)^{\rho_k}}\right\} \tag{2.4}$$

where $p(z)$ has algebraic coefficients and the numbers $\alpha_k, r_k, r_{k,h}$ are all algebraic. We assume that

$$\int_C \frac{P(z)}{Q(z)}$$

exists. If C contains infinity, then necessarily, $p(z) = \sum r_k = 0$. Thus for all the contours C considered in the Theorem V1,

$$\int_C p(z)dz \text{ and } \int_C \frac{r_{k,h}}{(z-\alpha_k)^h}dz$$

for $2 \le h \le \rho_k, 1 \le k \le n$, are all algebraic. Thus the given definite integral is algebraic if and only if

$$\int_C \left(\sum_{k=1}^{n}\frac{r_k}{z-\alpha_k}\right)dz$$

is algebraic.

Let δ_1 and δ_2 be two finite points on C and let us denote by $C(\delta_1,\delta_2)$ the curve from δ_1 to δ_2 along C. Then it can be easily seen that

$$\int_{C(\delta_1,\delta_2)} \left(\sum_{k=1}^{n}\frac{r_k}{z-\alpha_k}\right)dz = \operatorname{Log}\left\{\prod_{k=1}^{n}\left(\frac{\delta_2-\alpha_k}{\delta_1-\alpha_k}\right)^{r_k}\right\} + 2\pi i\ m(\delta_1,\delta_2),$$

where Log denotes the principal branch of log and $m(\delta_1,\delta_2)$ is an integer valued function depending on $C(\delta_1,\delta_2)$. Using Theorem B, the assertion of the theorem follows if C does not contain infinity.

Suppose C is a curve from δ_1 to ∞. Then

$$\int_{C(\delta_1,\infty)} \left(\sum_{k=1}^{n}\frac{r_k}{z-\alpha_k}\right)dz = \operatorname{Log}\left\{\prod_{k=1}^{n}(\delta_1-\alpha_k)^{r_k}\right\} + 2\pi i \lim_{\delta_2\to\infty} m(\delta_1,\delta_2),$$

and since the integral on the left hand side exists, it follows that $\lim_{\delta_2\to\infty} m(\delta_1,\delta_2)$ exists and $m(\delta_1,\delta_2) = m(\delta_1)$, an integer for large δ_2. The assertion follows in this case also.

Suppose C is a curve in the complex plane from ∞ to ∞. Then by letting $\delta_1 \to \infty$, we get

$$\int_{C(\delta_1,\infty)} \left(\sum_{k=1}^{n}\frac{r_k}{z-\alpha_k}\right)dz = 2\pi i\ m$$

where $\lim_{\delta_1,\delta_2\to\infty} m(\delta_1,\delta_2)$ exists and equals m. This gives the assertion in this case as well. □

In particular, we get

Theorem V2. *If C lies along the real axis, then*

(i) $\int_{\delta_1}^{\delta_2} \frac{P(x)}{Q(x)}dx$ *is algebraic if and only if*

$$\sum_{k=1}^{n} r_k\{\mathrm{Log}(\delta_2-\alpha_k)-\mathrm{Log}(\delta_1-\alpha_k)\}=0;$$

(ii) $\int_{\delta}^{\infty} \frac{P(x)}{Q(x)}dx$ *is algebraic if and only if*

$$\sum_{k=1}^{n} r_k\mathrm{Log}(\delta-\alpha_k)=0;$$

(iii) $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx$ *is algebraic if and only if it is zero.*

Thus Theorems V1 and V2 cover the case when the integrand is a rational function with some reasonable conditions so that Theorem B can be applied.

Remark 2.4. By Theorem V1, we see that

$$\int_C \frac{P(z)}{Q(z)}dz$$

is a Baker period and hence or otherwise a period and when non-zero, it is transcendental.

We relax the conditions on the coefficients of $P(z)$ and $Q(z)$ in the following theorem.

Theorem 2.5. *Let $P(z)=az^m+a_1z^{m-1}+\cdots+a_m$ and $Q(z)=bz^n+b_1z^{n-1}+\cdots+b_n$ with $P(z)$ and $Q(z)$ having no common zeros. If C is a simple closed curve which surrounds all the zeros of Q, then*

$$\int_C \frac{P(z)}{Q(z)} \text{ is } 0 \text{ if } n-m\geq 2$$

and transcendental if

$$n-m=1 \text{ and } \frac{\pi a}{b} \text{ is transcendental .}$$

Proof. From the partial fraction expansion of $\frac{P(z)}{Q(z)}$ in (2.4) one can easily see by Cauchy Residue Theorem that

$$\int_C \frac{P(z)}{Q(z)}dz = 2\pi i\sum_{k=1}^{h} r_k = 2\pi i \lim_{|z|\to\infty}\frac{zP(z)}{Q(z)} = \begin{cases} \frac{2\pi i a}{b} \text{ if } n-m=1 \\ 0 \text{ if } n-m\geq 2 \end{cases}$$

from which the result follows. □

Remark 2.6.

(i) With P,Q as in Theorem 2.5 with $n-m=1$ and if $a/b\in W$ then the integral is transcendental.
(ii) For any algebraic number α we know that $\pi\alpha$ is a period. By Theorem 2.5, we can give integral representation for $\pi\alpha$ by taking $P,Q\in\mathbb{A}[X]$ with $n-m=1$ and $a/b=\alpha/(2i)$.

2.4. Integrand as $\frac{P(z)}{Q(z)}\exp(i\alpha z)$. We now consider another generalization of Theorem V2 (ii) and (iii). The following result is well known in Residue theory, see [**14**, p. 110].

THEOREM R. *Let P and Q be polynomials and let $\alpha \geq 0$. Assume that the polynomial Q has no real roots on the real line and*

$$\deg Q \geq 2 + \deg P \text{ if } \alpha = 0 \text{ and } \deg Q \geq 1 + \deg P \text{ otherwise }.$$

Let $a_1, \cdots, a_k$ be all the roots of Q in the upper half plane. Then

$$\int_{-\infty}^{\infty} f(t)dt = 2\pi i \sum_{\nu=1}^{k} Res(f; a_\nu)$$

where $f(z) = \frac{P(z)}{Q(z)}\exp(i\alpha z)$.

Many of the integrals under this category take the value as an algebraic multiple of π. We do not intend to give an exhaustive list. We shall present certain non-obvious cases.

THEOREM 2.7. *The following integrals represent transcendental numbers.*

(i) $\int_0^\infty \frac{cost}{t^2+\pi^2 a}dt$ *with a any integer,*
(ii) $\int_0^\infty \frac{tsint}{t^2+\pi^2 a}dt$ *with a any integer,*
(iii) $\int_0^\infty \frac{cos2\pi t}{t^4+t^2+1}dt$
(iv) $\int_{-\infty}^\infty \frac{cost}{e^t+e^{-t}}dt$.

PROOF. Applying Theorem R, the integrals in (i)-(iv) of the theorem can be evaluated as

$$(i) = e^{-\pi\sqrt{a}}/\sqrt{a};\ (ii) = \frac{\pi e^{-\pi\sqrt{a}}}{2};\ (iii) = -\frac{\pi}{2\sqrt{3}}e^{-\pi\sqrt{3}};\ (iv) = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}.$$

See [**4**, p. 129-132] for a proof of (i) and (ii). We prove (iii) and (iv) below. Let $I_1 = \int_0^\infty \frac{cos2\pi t}{t^4+t^2+1}dt$. Consider the contour integral

$$I = \int_C \frac{\exp(2\pi i z)}{z^4+z^2+1}dz,$$

where $C = C_R \cup L_R$ is the contour shown in Figure 1 and $R > \sqrt{3}/2$.

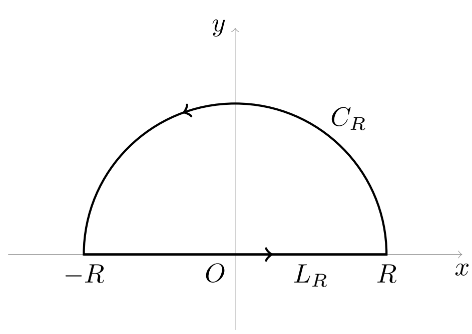

FIGURE 1

Let $w_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ and $w_2 = \frac{-1}{2} + \frac{\sqrt{3}i}{2}$. The function $f(z) = 1/(z^4 + z^2 + 1) = 1/((z - w_1)(z - w_2)(z - \bar{w_1})(z - \bar{w_2}))$ is analytic in the upper half plane except for simple poles at w_1 and w_2. Therefore

$$\begin{aligned} I &= 2\pi i \left(Res_{z=w_1}(\exp(2\pi i z) f(z)) + Res_{z=w_2}(\exp(2\pi i z) f(z)) \right) \\ &= \frac{\exp(-\pi\sqrt{3})}{\sqrt{3}(\sqrt{3} - i)} + \frac{-\exp(-\pi\sqrt{3})}{\sqrt{3}(\sqrt{3} + i)} \\ &= \frac{-\pi \exp(-\sqrt{3}\pi)}{\sqrt{3}}. \end{aligned}$$

Also

$$I = \int_{C_R} \frac{\exp(2\pi i z)}{z^4 + z^2 + 1} dz + \int_{-R}^{R} \frac{\exp(2\pi i t)}{t^4 + t^2 + 1} dt.$$

Now

$$\left| \int_{C_R} \frac{\exp(2\pi i z)}{z^4 + z^2 + 1} dz \right| \leq \int_{C_R} \frac{1}{|(z - w_1)(z - w_2)(z - \bar{w_1})(z - \bar{w_2})|} |dz| \leq \frac{\pi R}{R^4 - 1}$$

which $\to 0$ as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{\exp(2\pi i t)}{t^4 + t^2 + 1} dt = \frac{-\pi \exp(-\pi\sqrt{3})}{\sqrt{3}}.$$

Comparing the real parts, we get

$$\int_{-\infty}^{\infty} \frac{cos 2\pi t}{t^4 + t^2 + 1} dt = \frac{-\pi \exp(-\pi\sqrt{3})}{\sqrt{3}}.$$

Thus

$$I_1 = \frac{-\pi \exp(-\pi\sqrt{3})}{2\sqrt{3}}.$$

This proves (iii).

Let $I_1 = \int_{-\infty}^{\infty} \frac{cos t}{\exp(t) + \exp(-t)} dt$. Since sine function is odd,

$$I_1 = \int_{-\infty}^{\infty} \frac{\exp(it)}{\exp(t) + \exp(-t)} dt.$$

Consider the contour integral

$$I = \int_{C} \frac{\exp(iz)}{\exp(z) + \exp(-z)} dz,$$

where $C = L_1 \cup L_2 \cup L_3 \cup L_4$ is the contour shown in Figure 2 and $R > 0$.

The function $f(z) = \frac{\exp(it)}{\exp(t) + \exp(-t)}$ is analytic in the upper half plane except for simple poles at $\frac{\pi i}{2} + k\pi i$, $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} I &= 2\pi i \; Res_{z=\pi i/2} f(z) \\ &= \pi \exp(-\pi/2). \end{aligned}$$

Also

$$I = \int_{L_1} f(t) dt + \int_{L_2} f(z) dz + \int_{L_3} f(z) dz + \int_{L_4} f(z) dz.$$

Now

$$\left| \int_{L_2} f(z) dz \right| \leq \frac{\pi}{\exp(R)(1 - \exp(-2R))}$$

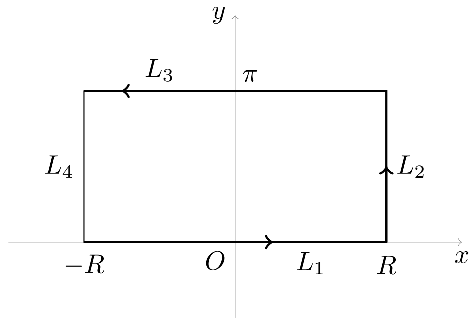

FIGURE 2

which $\to 0$ as $R \to \infty$. Similarly, $\int_{L_4} f(z)dz \to 0$ as $R \to \infty$. Further,

$$\begin{aligned} \int_{L_3} f(z)dz &= \int_R^{-R} \frac{\exp(i(t+\pi i))}{\exp(t+\pi i)+\exp(-t-\pi i)}dt \\ &= \exp(-\pi)\int_R^{-R} \frac{\exp(it)}{-\exp(t)-\exp(-t)} \\ &= \exp(-\pi)\int_{L_1} f(t)dt. \end{aligned}$$

Hence

$$I = I_1(1+\exp(-\pi)).$$

Thus

$$I_1 = \frac{\pi\exp(-\pi/2)}{1+\exp(-\pi)} = \frac{\pi}{\exp(\pi/2)+\exp(-\pi/2)}.$$

Now we apply Theorem N to conclude that the values of the integrals (i) to (iv) are transcendental. □

REMARK 2.8. We refer to the paper of Tijdeman [**32**] for more results on infinite series where Theorem N has been applied. Tijdeman writes on p. 286: " F. Beukers gave the following advice to me: check whether all the zeros of the denominator Q are located in some quadratic number field $\mathbb{Q}(\sqrt{-D})$ for $D \in \{1, 3\}$. If so, try to find an explicit expression for S in terms of π and $e^{\pi\sqrt{D}}$. If you suceed, apply the result of Nesterenko to conclude the transcendence of the number." From Theorem 2.7, we see that a similar advice is applicable in the case of integrals as well.

2.5. Application of Jensen's formula. The well known Jensen's formula is as follows.

Suppose f is analytic in $G = \{z : |z| < R\}$ except for a finite number of poles at non-zero points $p_1, \cdots, p_n$ and regular and non-zero on $\Gamma = \{z : |z| = R\}$. Let f have zeros at non-zero points $\nu_1, \cdots, \nu_m$ in G. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{\log f(z)}{z}dz = \log f(0) + \sum_{k=1}^m \log\frac{R}{\nu_k} - \sum_{k=1}^n \log\frac{R}{p_k}.$$

As a direct application of Jensen's theorem and Theorem B we get the following result.

THEOREM 2.9. *Suppose f is analytic in $G = \{z : |z| < R\}$ except for a finite number of poles at non-zero algebraic points $p_1, \cdots, p_n$ and regular and non-zero on $\Gamma = \{z : |z| = R\}$ with R algebraic. Let f have zeros at non-zero algebraic points $\nu_1, \cdots, \nu_m$ in G. Then the integral*

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log f(z)}{z}dz$$

is either 0 or transcendental.

EXAMPLE 2.10. Let $f(z) = cos\ \pi z$ and $R = m$ where $m > 0$ is an integer. Then $f(z)$ has only zeros in G at $\nu_k = \pm(k - \frac{1}{2})$ for $1 \leq k \leq m$. Further

$$\log f(0) + \sum_{k=1}^{m}\log\frac{R}{\nu_k} = 2m\log m - 2\sum_{k=1}^{m}\log\left(k - \frac{1}{2}\right) + K\pi i$$

for some integer K. We see that the real part on the right hand side never vanishes. Thus, by Theorem 2.9,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log cos\ \pi z}{z}dz$$

is transcendental.

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