MATHEMATICS

HYPERGEOMETRIC FUNCTIONS. II

BY

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CHARACTERIZATION OF THE SOLUTION SPACE OF A DIFFERENTIAL EQUATION WITH A WEAK SINGULARITY

II. 1. *Introduction*

Hypergeometric functions are usually considered from the point of view of the hypergeometric differential equation (see e.g. [1], formula (1.1)). This is a differential equation of Fuchsian type ([2], p. 370–372, or [3], p. 129–130): In this field the key-notion is that of singularity of the first kind ([3], p. 111) or weak singularity ([4], II, p. 125). The solution space of a differential equation (or system of linear differential equations of the first order) in a neighbourhood of a weakly singular point has been investigated by many authors (for this theory and references see e.g. [3], ch. 4). The standard technique consists in substituting formal power series (whose coefficients are matrices in the case of a system of differential equations) multiplied by a power of the variable. Next, the recurrence relations for the coefficients which are obtained by equating the coefficients of equal powers of the variable have to be solved. A convergence proof has to be given afterwards. By this method existence and important properties of the solutions are translated into the solvability of recurrence relations which are difficult to survey. The theory becomes unpleasantly intricate when explicit formulae are required ([4]. II, Kap. XIV, § 10).

Our aim is a characterization of hypergeometric functions by properties in the large. To this end we have to characterize the solution space of an analytic differential equation with isolated singularities and to define the important notion of exponents without introducing the differential equation itself. In this chapter the solution spaces of such differential equations and of systems of linear differential equations are investigated in neighbourhoods of the singular points. A space of this kind can be characterized by the conditions (A) and (B) of section 2. Condition (A) concerns the transformation, induced by the analytic continuation of the elements of the space in question around the singular point, whereas condition (B) regards the orders of these elements at the singular point. In section 2 spaces satisfying the conditions (A) and (B) are analysed and exponents are defined. In section 3 we shall expound the connection with the theory of differential equations with weakly singular points and thus the most important theorems on these equations will be derived in a new way.

II. 2. *Analysis of singular points*

We start this section by making some arrangements concerning notations and definitions. Let Z be the complex number sphere, and let S be the interior of a circle on Z with centre at 0. By S' we mean $S-\{0\}$, and by $\pi_1(S')$ the fundamental group of S'. As is well-known, $\pi_1(S')$ is the infinite cyclic group, generated by the homotopy class δ of a positively oriented circular path on S'. When W denotes the universal covering surface of S' and p the projection, then p is a locally conformal mapping of W onto S'. Therefore, at each point of W the projection p can be used as a local parameter. A covering transformation α of W is a conformal mapping of W onto itself such that $p \circ \alpha = p$. $\pi_1(S')$ can be regarded in a natural way as the group of all covering transformations of W.

Furthermore, we denote by $M(S)$ and $M(W)$ the fields of meromorphic functions on S and W, respectively. When m and n are positive integers, $M^{m\times n}(S)$ and $M^{m\times n}(W)$ are the linear spaces (over the complex number field) of all $m\times n$ matrices over $M(S)$ and $M(W)$, respectively. We shall write $M^m(S)$ instead of $M^{m\times 1}(S)$ and $M^m(W)$ instead of $M^{m\times 1}(W)$. To each $\alpha \in \pi_1(S')$ there corresponds an automorphism α^* of $M^{m\times n}(W)$ defined by

$$(\alpha^* f)(w) = f(\alpha w).$$

Evidently we have $(\alpha\beta)^* = \beta^*\alpha^*$, and α^* is the identity mapping if α is the identity element of $\pi_1(S')$.

Definition 2.1. An element $f \in M^{m\times n}(W)$ is called of finite order at 0 if a real number λ can be found such that

$$\lim_{p(w)\to 0,\, w\in G} (p(w))^{-\lambda} f(w) = 0 \tag{2.1}$$

for every "sector" G of W (a "sector" is a subset of W that is conformally mapped by p onto a sector of S').

In this section we shall investigate n-dimensional subspaces V (over the complex number field) of $M^m(W)$ satisfying the following set of conditions:

(A) V is mapped onto itself by δ^* (the restriction of δ^* to V is also denoted by δ^*).

(B) The elements of V are of finite order at 0.

Let $\varrho_1, \ldots, \varrho_r$ be the different eigenvalues of the linear transformation δ^* of V (by virtue of condition (A) all ϱ's are different from 0), and let $\mu_1, \ldots, \mu_r$ be the corresponding multiplicities. Then a well-known theorem on linear transformations says ([5], Ch. X, theorem 22):

V is the direct sum of subspaces $V_1, \ldots, V_r$ which are invariant under δ^*. To each $i (i=1, \ldots, r)$ there exists a smallest positive integer s_i such that $(\delta^* - \varrho_i I)^{s_i} V_i = 0$ (I denotes the identity transformation. It will also be used for the identity matrices of all dimensions). Moreover, we have $\dim V_i = \mu_i$.

On account of this result we may restrict ourselves to the study of a μ-dimensional subspace V of $M^m(W)$ which satisfies the above conditions (A) and (B), whereas δ^* has the only eigenvalue ϱ and $(\delta^* - \varrho I)^s = 0$. From now on up to theorem 2.5 we assume that this is the case.

In V we can define a valuation φ in the following way: When $f \in V$, $f \neq 0$, $\varphi(f)$ is equal to the largest integer N such that (2.1) holds for all $\lambda < N$. Furthermore, we define $\varphi(0) = \infty$.

The proof of the following simple theorem is left to the reader.

Theorem 2.1. The valuation φ has the properties:
(1) $\varphi(f)$ is an integer for every $f \in V$, $f \neq 0$, and $\varphi(0) = \infty$.
(2) $\varphi(f+g) \geqslant \min(\varphi(f), \varphi(g))$ for all $f, g \in V$.
(3) $\varphi(\lambda f) = \varphi(f)$ for all $f \in V$ and all constants $\lambda \neq 0$.
(4) $\varphi(\delta^* f) = \varphi(f)$ for all $f \in V$.

The valuation φ assumes only a finite number of different values and defines, in a natural way, a chain of subspaces of V. These facts are expressed in the next theorem.

Theorem 2.2. The valuation φ assumes only a finite number of different values
$$\infty = \varphi_0 > \varphi_1 > \ldots > \varphi_t.$$
Let X_i $(i = 0, \ldots, t)$ be defined by
$$X_i = \{f \mid f \in V \quad \text{and} \quad \varphi(f) \geqslant \varphi_i\}, \tag{2.2}$$
then X_i is a linear subspace of V and
$$\{0\} = X_0 \subset X_1 \subset \ldots \subset X_t = V, \tag{2.3}$$
where all X_i's are different.

Finally, when the integer m_i is defined by
$$m_i = \dim X_i / X_{i-i} \qquad (i = 1, \ldots, t), \tag{2.4}$$
we have
$$m_1 + \ldots + m_t = \mu = \dim V.$$

Proof. Define $Y_j = \{f \mid f \in V \text{ and } \varphi(f) \geqslant j\}$. Then Y_j is a linear subspace of V. For, let $\varphi(f) \geqslant j$ and $\varphi(g) \geqslant j$, then $\varphi(f+g) \geqslant \min(\varphi(f), \varphi(g)) \geqslant j$. Further, when λ is a constant and $f \in Y_j$, we have $\varphi(\lambda f) \geqslant \varphi(f) \geqslant j$. From the definition it is evident that $Y_j \supset Y_{j+1}$ for every j. So we have an infinite chain of subspaces of the finite dimensional space V. Such a chain can only have a finite number of different members. From this fact the first assertion of our theorem follows at once, $\varphi_1, \ldots, \varphi_t$ being those values of j for which $Y_{j+1} \neq Y_j$. The remaining assertions of the theorem are trivial.

Definition 2.2. When α is the complex number such that $e^{2\pi i\alpha}=\varrho$ and $0 \leqslant \mathrm{Re}\,\alpha < 1$, the exponents of V (corresponding to the eigenvalue ϱ of δ^*) are the numbers $\alpha+\varphi_1, \alpha+\varphi_2, \ldots, \alpha+\varphi_t$. For each $j (1 \leqslant j \leqslant t)$, m_j defined by (2.4), is called the multiplicity of the exponent $\alpha+\varphi_j$.

Next we construct a special base in V. From theorem 2.1 (4) it is evident that δ^* maps X_i into itself $(i=0, 1, \ldots, t)$. Therefore δ^* induces a linear transformation into X_i/X_{i-1} $(i=1, \ldots, t)$.
Hence, there exists a base $\{x_{i1}, \ldots, x_{im_i}\}$ in X_i/X_{i-1} such that the matrix of δ^* with respect to this base is in Jordan canonical form. Let $\{v_{11}, \ldots, v_{1m_1}, \ldots, v_{t1}, \ldots, v_{tm_t}\}$ be a set of elements of V such that $v_{ij} \in x_{ij}$ $(i=1, \ldots, t;\ j=1, \ldots, m_i)$. Then $\{v_{11}, \ldots, v_{1m_1}, \ldots, v_{t1}, \ldots, v_{tm_t}\}$ is a base of V with the properties described in the subsequent theorem.

Theorem 2.3. There exists a base $\{v_{11}, \ldots, v_{tm_t}\}$ in V with the properties:

(1) $\{v_{i1}, \ldots, v_{im_i}\}$ is a base of X_i with respect to X_{i-1} $(i=1, \ldots, t)$,

(2) the matrix D of δ^* with respect to the base $\{v_{11}, \ldots, v_{tm_t}\}$ consists of blocks D_{jh} with m_j rows and m_h columns $(j, h=1, \ldots, t)$.
Moreover, $D_{jh}=0$ if $j>h$, and $D_{11}, D_{22}, \ldots, D_{tt}$ are in Jordan canonical form, all eigenvalues being equal to ϱ.

Remark. If we choose $\{v_{11}, \ldots, v_{tm_t}\}$ in a suitable way, we can also find a special form for the matrices D_{jh} with $j<h$. However, we will not do this, as it does not simplify our future exposition.

Our next aim is to obtain analytic expressions for the elements of V. For this purpose we need the logarithm of a square matrix.

Definition 2.3. Let T be a triangular matrix of complex numbers which has the only eigenvalue $\varrho \neq 0$. Then $\log T$, the principal value of the logarithm of T, is defined by

$$\log T=\gamma I+\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\left(\frac{1}{\varrho} T-I\right)^j, \tag{2.5}$$

where γ is the unique complex number with $e^{\gamma}=\varrho$ and $0 \leqslant \mathrm{Im}\,\gamma<2\pi$.

Furthermore, let T be an arbitrary non-singular matrix of complex numbers. T can be written in the form

$$T=S^{-1}\begin{pmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_r \end{pmatrix} S, \tag{2.6}$$

where $T_1, \ldots, T_r$ are triangular matrices, T_i having ϱ_i as its only eigenvalue. Then the principal value of the logarithm of T is defined by

$$\log T=S^{-1}\begin{pmatrix} \log T_1 & & 0 \\ & \ddots & \\ 0 & & \log T_r \end{pmatrix} S.$$

Remark 1. Since $\varrho^{-1}T-I$ in formula (2.5) is a nilpotent matrix, $\log T$ is a polynomial in T, where the coefficients are scalar functions of T.

Remark 2. If T is an arbitrary non-singular matrix, it can be proved that $\log T$ does not depend on the special representation (2.6) and that $\log T$ satisfies $\mathrm{e}^{\log T}=T$ (for matrix functions we refer to [4], Teil I, Kap. V und VIII).

Theorem 2.4. The space V is generated by the columns of a matrix $Y \in M^{m\times\mu}(W)$, where

$$Y(w)=U(z)\,w^{A}w^{F} \qquad (z=p(w)) \tag{2.7}$$

and

(1) U is a holomorphic $m\times\mu$ matrix at 0.
(2) A is a $\mu\times\mu$ diagonal matrix, the elements in the diagonal being the exponents $\alpha_1(m_1$ times), $\alpha_2(m_2$ times), ..., $\alpha_t(m_t$ times) in succession, where α_i denotes $\alpha+\varphi_i$.
(3) $F=-\alpha I+(1/2\pi i)\log D$ is a nilpotent $\mu\times\mu$ matrix.

Proof. Let Y be the matrix the columns of which are the elements $v_{11}, \ldots, v_{tm_t}$ of V (cf. theorem 2.3). Putting $E=(1/2\pi i)\log D$, we find

$$\delta^*(Y(w)\,w^{-E})=(\delta^*\,Y(w))\,(\delta^*\,\mathrm{e}^{-E\log w})=Y(w)\,D\,\mathrm{e}^{-E(\log w+2\pi i)}=$$
$$=Y(w)\,D\,\mathrm{e}^{-2\pi i E}\,\mathrm{e}^{-E\log w}=Y(w)\,w^{-E}.$$

This proves that $Y(w)\,w^{-E}$ is invariant under δ^*, and so an $m\times\mu$ matrix $L(z)$ exists such that $L(z)$ is meromorphic on S' and that $L(p(w))=Y(w)\,w^{-E}$. Since $Y(w)$ and w^{-E} are of finite order at 0, the same thing is true for $L(z)$. As a consequence $L(z)$ can be regarded as a meromorphic matrix on S. Let $\tilde{A}$ be equal to $A-\alpha I$. So $\tilde{A}$ is a diagonal matrix, and the elements in the diagonal are

$$\varphi_1=[\operatorname{Re}\alpha_1],\ \varphi_2=[\operatorname{Re}\alpha_2],\ \ldots,\ \varphi_t=[\operatorname{Re}\alpha_t].$$

($[\operatorname{Re}\alpha_i]$ denotes the largest integer not exceeding $\operatorname{Re}\alpha_i$). We finish the proof of the theorem by showing that $L(z)\,z^{-\tilde{A}}$ is holomorphic at 0.

If $\varepsilon>0$, $\lim\limits_{w\to 0} Y(w)\,w^{-\tilde{A}}w^{\varepsilon}=0$ (in the sense of definition 2.1). This follows from the facts that the columns of $Y(w)$ are $v_{11}, \ldots, v_{1m_1}, \ldots, v_{t1}, \ldots, v_{tm_t}$, that the corresponding elements in the diagonal of $z^{-\tilde{A}}$ are $z^{-\varphi_1}, \ldots, z^{-\varphi_1}, \ldots, z^{-\varphi_t}, \ldots, z^{-\varphi_t}$, and that $v_{i1}, \ldots, v_{im_i}$ is a base of X_i with respect to X_{i-1}. Further, F is a nilpotent upper triangular matrix. Hence, $w^{-F}=I-F\log w+(1/2!)\,F^2\log^2 w-\ldots+(-1)^{s-1}(1/(s-1)!))\,F^{s-1}\log^{s-1} w$. Now $w^{\tilde{A}}Fw^{-\tilde{A}}$ is a matrix of monomials in z, and so we have $\lim\limits_{w\to 0} w^{\varepsilon}w^{\tilde{A}}w^{-F}w^{-\tilde{A}}=0$. Combining these results we find $\lim\limits_{w\to 0} L(z)z^{-\tilde{A}}w^{\alpha+2\varepsilon}=$ $=\lim\limits_{w\to 0} Y(w)\,w^{-\tilde{A}}w^{\varepsilon}\cdot\lim\limits_{w\to 0} w^{\tilde{A}}w^{-F}w^{-\tilde{A}}w^{\varepsilon}=0$. As $\operatorname{Re}\alpha<1$, it follows that $L(z)z^{-\tilde{A}}$ cannot have a pole at 0.

The next theorem summarizes the results of this section. The notations of the theorem will be used in the sequel without reference.

Theorem 2.5. Let V be an n-dimensional subspace of $M^m(W)$ which satisfies the conditions:
(A) V is mapped onto itself by δ^*,
(B) The elements of V are of finite order at 0 (definition 2.1).

Furthermore, let $\varrho_1, \ldots, \varrho_r$ be the different eigenvalues of δ^* with multiplicities $\mu_1, \ldots, \mu_r$, respectively.

Then V is generated by the columns of a matrix $Y \in M^{m\times n}(W)$ such that

$$\delta^* Y = YD,\ Y(w) = U(z)\, w^A w^F, \tag{2.8}$$

and

(1) U is a holomorphic $m\times n$ matrix at 0.
(2) D is an $n\times n$ matrix of complex numbers consisting of blocks $D_1, \ldots, D_r$ along the diagonal.
(3) For every i $(1\leqslant i\leqslant r)$ D_i is a $\mu_i\times\mu_i$ matrix. When $m_{i1}, \ldots, m_{it_i}$ are the multiplicities of the exponents $\alpha_{i1}, \ldots, \alpha_{it_i}$ corresponding to ϱ_i, then D_i consists of blocks D_{ijh} with m_{ij} rows and m_{ih} columns $(1\leqslant j, h\leqslant t_i)$. Moreover, $D_{ijh}=0$ if $j>h$, $D_{i11}, D_{i22}, \ldots, D_{it_it_i}$ are in Jordan canonical form, all eigenvalues being equal to ϱ_i.
(4) A consists of blocks $A_1, \ldots, A_r$ along the diagonal, corresponding to $D_1, \ldots, D_r$. A_i is in diagonal form. The elements in the diagonal of A_i are $\alpha_{i1}(m_{i1}$ times), $\ldots, \alpha_{it_i}(m_{it_i}$ times) in succession, where $\operatorname{Re}\alpha_{i1}\geqslant\operatorname{Re}\alpha_{i2}\geqslant\ldots\geqslant\operatorname{Re}\alpha_{it_i}$.
(5) $F=\tilde{A}-A+(1/2\pi i)\log D$, where $\tilde{A}$ is obtained from A if every element α_{ij} in the diagonal is replaced by $[\operatorname{Re}\alpha_{ij}]$ (=the largest integer $\leqslant\operatorname{Re}\alpha_{ij}$). The matrix F consits of blocks $F_1, \ldots, F_r$ along the diagonal, corresponding to the blocks $D_1, \ldots, D_r$ of D.

Remark. In the next section the "Jordan canonical part" of the matrix D will play a rôle. By this we mean the matrix D_0 which results from D when all blocks D_{ijh} with $j\neq h$ are replaced by 0.

II. 3. *Linear differential equations*

In the first part of this section the special case $m=n$ of theorem 2.5 will be related to the theory of systems of differential equations with analytic coefficients. In the second part the case $m=1$ of theorem 2.5 will be our object. This will lead to ordinary differential equations with analytic coefficients.

Definition 2.4. The space V satisfying the conditions of theorem 2.5 (with $m=n$) is said to have a weak singularity (or singularity of the first kind) at the point 0, if V is generated by the columns of a matrix $Y(w)\in M^{n\times n}(W)$ with

$$\det\ Y(w) = w^\sigma\varphi(p(w)),$$

where σ is the sum of the exponents of V, $\varphi(z)$ is holomorphic at 0 and $\varphi(0)\neq 0$.

The reader should notice that on account of theorem 2.5 the requirement $\varphi(0)\neq 0$ is the only non-trivial condition of definition 2.4 (cf. formula (2.11) below). Obviously it is irrelevant which matrix $Y(w)$ is used to generate V.

Theorem 2.6. When V satisfies the conditions of theorem 2.5 (with $m=n$), and V has a weak singularity at 0, then V is the solution space of a system of linear differential equations

$$\vartheta v = P(z)v, \tag{2.9}$$

where

$$\vartheta = z\frac{d}{dz}, \quad z=p(w) \tag{2.10}$$

and $P(z)\in M^{n\times n}(S)$ is holomorphic at 0.

The matrix $P(0)$ is equivalent to $\tilde{A}+(1/2\pi i)\log D_0$, where D_0 is the Jordan canonical part of D (cf. remark to theorem 2.5).

Proof. Applying theorem 2.5 to V, we have a matrix $Y(w)=U(z)w^A w^F$ (see (2.9)) the columns of which generate V. Furthermore, the identity

$$\det Y(w) = \det U(z)\cdot \det w^A \cdot \det w^F = \det U(z)\cdot w^\sigma \cdot 1 \tag{2.11}$$

holds and, since V has a weak singularity at 0, we have $\det U(0)\neq 0$. A simple calculation shows

$$(\vartheta Y)\,Y^{-1} = (\vartheta U(z))\,U(z)^{-1} + U(z)\{A + w^A F w^{-A}\}\,U(z)^{-1}. \tag{2.12}$$

Next we consider the matrix $A+w^A F w^{-A}$. Since A and F consist of blocks $A_1, \ldots, A_r$ and $F_1, \ldots, F_r$ along the diagonal (cf. theorem 2.5) we can restrict ourselves to the i-th block of A and F, respectively. For the moment we adopt the notations of the theorems 2.2, 2.3 and 2.4. Accordingly A, F and D are $\mu\times\mu$-matrices corresponding to the eigenvalue ϱ of δ^*. The following identities are evident

$$\left\{\begin{aligned} A + w^A F w^{-A} &= A + w^A\left(\frac{1}{2\pi i}\log D - \alpha I\right) w^{-A} = \\ &= A - \alpha I + w^A\left(\frac{1}{2\pi i}\log D\right) w^{-A} = \\ &= \tilde{A} + w^A\left(\frac{1}{2\pi i}\log D\right) w^{-A} = \tilde{A} + \frac{1}{2\pi i}\log w^A D w^{-A}. \end{aligned}\right. \tag{2.13}$$

(The latter equality follows from the fact that by definition 2.3, remark 1 the matrix $\log D$ is equal to a polynomial in D).

According to theorem 2.3 the matrix $w^A D w^{-A}$ consists of blocks $D_{jh}w^{\alpha_j-\alpha_h}$ ($j, h=1, \ldots, t$).

Since $D_{jh}=0$ if $j>h$, and $\alpha_j-\alpha_h$ is a positive integer if $j<h$, we see that $w^A D w^{-A}$ is a triangular matrix of monomials in z.

When $z=0$ only the blocks $D_{11}, D_{22}, \ldots, D_{tt}$ remain. Let D_0 denote the (non-singular) matrix consisting of these blocks along the diagonal. Now $\log w^A D w^{-A}$ is a polynomial in $w^A D w^{-A}$ and hence, a polynomial in z which assumes the value $\log D_0$ if $z=0$. Thus we have

$$\log w^A D w^{-A} = \log D_0 + O(z),$$

and by (2.13)

$$w^A F w^{-A} = \tilde{A} - A + \frac{1}{2\pi i} \log D_0 + O(z). \tag{2.14}$$

Let us now return to the situation described at the beginning of the proof, where different ϱ's were considered. Then, as is obvious, formula (2.14) is still right, when A, $\tilde{A}$, F and D_0 are used in the sense of the theorems 2.5 and 2.6. Now the proof of our theorem can be completed. As $\det U(0) \neq 0$, $U(z)^{-1}$ is holomorphic at $z=0$. Let $P(z)$ denote the right-hand member of formula (2.12). Then $P(z)$ is holomorphic at $z=0$. Passage to the limit $z \to 0$ yields, by (2.14), $P(0) = U(0)\{\tilde{A} + (1/2\pi i) \log D_0\} U(0)^{-1}$. Since a base of V satisfies the equation (2.9), and V is an n-dimensional linear space, we see that V is in fact the solution space of (2.10).

Theorem 2.7. When V is the solution space of the system of differential equations (2.9) and $P(z) \in M^{n \times n}(S)$ is holomorphic at 0, then the solution space V is an n-dimensional subspace of $M^n(W)$ which satisfies the conditions (A) and (B) of theorem 2.5. Moreover, the point 0 is a weak singularity of V.

Proof. From the theory of linear differential equations we use the following well-known facts (cf. [3], ch. 4):

(1) The solution space of the system (2.9) is an n-dimensional subspace of $M^n(W)$.
(2) V is mapped onto itself by δ^*.
(3) The elements of V are of finite order at 0.

Hence, V satisfies the conditions (A) and (B).

Next we prove that 0 is a weak singularity of V. By theorem 2.5 there exists a matrix $Y(w) = U(z) w^A w^F$ the columns of which generate V (Y is called a fundamental matrix of the system (2.9)). It suffices to prove that $\det U(0) \neq 0$. Substituting $Y = U(z) w^A w^F$ in (2.9) we find

$$\vartheta U(z) + U(z)\{A + w^A F w^{-A}\} = P(z)\, U(z).$$

Applying (2.14) and taking $z=0$, we obtain

$$U(0)\left\{\tilde{A} + \frac{1}{2\pi i} \log D_0\right\} = P(0)\, U(0). \tag{2.15}$$

Assume that $U(0)$ is a singular matrix. Then the null-space K of $U(0)$ is different from $\{0\}$. By formula (2.15) we see that $\tilde{A} + (1/2\pi i) \log D_0$ maps K into itself. Hence K contains an eigenvector l (a column vector of n

complex numbers) of $\tilde{A}+(1/2\pi i)\log D_0$. Let α_i be the eigenvalue belonging to l.

Now $\tilde{A}+(1/2\pi i)\log D_0$ is a matrix which consists of blocks along the diagonal corresponding to the different eigenvalues of A. This means that the only coordinates of l which may differ from 0 are those corresponding to $v_{i1},\ldots,v_{im_i}$. (We use again the notation of theorem 2.3). So we have $w^{-A}l=lw^{-\alpha_i}$ and $(\tilde{A}-A+(1/2\pi i)\log D_0)\,l=0$. As $F^n=0$ we find, using formula (2.14),

$$w^{-\alpha_i}(l_{i1}v_{i1}(w)+\ldots+l_{im_i}v_{im_i}(w))=w^{-\alpha_i}Y(w)l=$$

$$=U(z)\,w^A w^F w^{-A}l=U(z)\left(I+\sum_{j=1}^{n-1}\frac{1}{j\,!}\log^j w(w^A Fw^{-A})^j\right)l=$$

$$=U(z)l+\sum_{j=1}^{n-1}\frac{1}{j\,!}\log^j w\left(\tilde{A}-A+\frac{1}{2\pi i}\log D_0+O(z)\right)^j l=$$

$$=U(z)l+\sum_{j=1}^{n-1}\frac{1}{j\,!}\log^j w\left\{\left(\tilde{A}-A+\frac{1}{2\pi i}\log D_0\right)^j+O(z)\right\}l=$$

$$=U(z)l+O(z)l\sum_{j=1}^{n-1}\frac{1}{j\,!}\log^j w=U(z)l+O(\sqrt{z})l.$$

Now $U(0)l=0$, and thus $U(z)l$ vanishes at least to the first order if $z\to 0$. It follows that $w^{-\alpha_i}(l_{i1}v_{i1}(w)+\ldots+l_{im_i}v_{im_i}(w))=O(\sqrt{z})$. This means that $\varphi(l_{i1}v_{i1}+\ldots+l_{im_i}v_{im_i})>\varphi_i$, and so $v_{i1},\ldots,v_{im_i}$ are not independent with respect to X_{i-1}. We have deduced a contradiction. So our assumption $\det U(0)=0$ is wrong.

Theorem 2.8. Let V be the solution space of the linear system (2.9), where $P(z)\in M^{n\times n}(S)$ is holomorphic at 0. Let r be an integer, $0\leqslant r\leqslant n-1$. Then the following conditions are equivalent:

(1) rank $P(0)\leqslant r$,

(2) V contains at least $n-r$ elements $v_1,\ldots,v_{n-r}$ such that $\lim_{w\to 0} v_i(w)=v_i(0)$ (in the sense of definition 2.1) exists for $i=1,\ldots,n-r$, and $v_1(0),\ldots,v_{n-r}(0)$ are linearly independent.

(If (1) or (2) are satisfied, then we shall say that the $(n-r)$-dimensional initial-value problem is solvable).

Proof. Let the second condition of the theorem be satisfied and let $v(w)$ be one of $v_1(w),\ldots,v_{n-r}(w)$. By the theorems 2.7 and 2.5 we have

$$v(w)=\sum_{i,j\geqslant 0}\varphi_{ij}(z)\,w^{\beta_i}\log^j w,$$

where the $\varphi_{ij}(z)$ are holomorphic vectors, $\varphi_{ij}(0)\neq 0$ and all β_i's are different. As $\lim_{w\to 0} v(w)$ exists we see that $\operatorname{Re}\beta_i\geqslant 0$, and $\operatorname{Re}\beta_i>0$ if non-zero terms $\varphi_{ij}(z)w^{\beta_i}\log^j w$ can be found with $j\neq 0$ or $\operatorname{Im}\beta_i\neq 0$. From this it easily follows that $\lim_{w\to 0}\vartheta v(w)=0$. Since $\vartheta v(w)=P(z)v(w)$, we find by taking the limit $w \to 0$ that $P(0)v(0)=0$. So $P(0)$ annihilates $n-r$ independent vectors, and rank $P(0) \leqslant r$ as a consequence.

Next we assume the first condition of the theorem, viz. rank $P(0) \leqslant r$. Then $\tilde{A}+(1/2\pi i)\log D_0$, which by (2.15) is equivalent to $P(0)$, annihilates at least $n-r$ independent constant vectors $l_1, \ldots, l_{n-r}$. By arguments analogous to those used in the proof of the preceding theorem, we see that, if $j=1, \ldots, n-r$, $l_j = w^{-A} l_j$ and that $Y(w)l_j = U(z)l_j + O(\sqrt{z})$. Hence, $\lim_{w\to 0} Y(w)l_j$ exists and is equal to $U(0)l_j$. Since $U(0)$ is non-singular, we see that $U(0)l_1, \ldots, U(0)l_{n-r}$ are linearly independent vectors.

The last part of this section is devoted to the case $m=1$ of the theorem 2.5. This special case can be connected with the foregoing theory by the next theorem.

Theorem 2.9. Let V satisfy the conditions (A) and (B) of theorem 2.5 with $m=1$. Let Θ be defined by

$$\Theta y = \begin{pmatrix} y \\ \vartheta y \\ \vdots \\ \vartheta^{n-1} y \end{pmatrix} \tag{2.16}$$

for all $y \in M(W)$. Then we have the following results:

(1) Θ is an isomorphism of V onto an n-dimensional subspace V_1 of $M^n(W)$, and Θ commutes with δ^*.
(2) V_1 satisfies the conditions (A) and (B) of theorem 2.5 with $m=n$.
(3) V_1 has a weak singularity at 0.
(4) V and V_1 have the same set of exponents.

Proof. We shall only prove the non-trivial part of the theorem, viz. the statements (3) and (4).

In virtue of theorem 2.5 we can find a base $\{y_1, \ldots, y_n\}$ in V of the form

$$(y_1, \ldots, y_n) = (u_{11}(z), \ldots, u_{1n}(z))\, w^A w^F$$

with the properties (1)–(5) of that theorem. Applying the operator ϑ we find

$$(\vartheta y_1, \ldots, \vartheta y_n) = (u_{21}(z), \ldots, u_{2n}(z))\, w^A w^F,$$

where

$$\begin{aligned} (u_{21}(z), \ldots, u_{2n}(z)) &= (\vartheta u_{11}(z), \ldots, \vartheta u_{1n}(z)) + \\ &\quad + (u_{11}(z), \ldots, u_{1n}(z))\, w^A (A+F)\, w^{-A}. \end{aligned} \tag{2.17}$$

Now $w^A(A+F)w^{-A}$ is a matrix of monomials in z (the proof of this statement is similar to that of the same statement about $w^A D w^{-A}$ in the proof of theorem 2.6). Furthermore, $u_{11}(z), \ldots, u_{1n}(z)$ are holomorphic at $z=0$. Hence $u_{21}(z), \ldots, u_{2n}(z)$ are holomorphic at $z=0$. From (2.17) we derive by formula (2.14) that

$$(u_{21}(0), \ldots, u_{2n}(0)) = (u_{11}(0), \ldots, u_{1n}(0)) \left(\tilde{A} + \frac{1}{2\pi i}\log D_0\right).$$

Proceeding in this way we find

$$(2.18)\qquad \Theta(y_1,\dots,y_n)\equiv\begin{pmatrix} y_1 & \cdots\cdots\cdots & y_n \\ \vartheta y_1 & & \vartheta y_n \\ \vdots & & \vdots \\ \vartheta^{n-1} y_1 & \dots & \vartheta^{n-1} y_n \end{pmatrix} = U(z)\, w^A w^F,$$

where $U \in M^{n\times n}(S)$ is holomorphic at 0 and

$$(u_{j1}(0),\ \dots,\ u_{jn}(0))=(u_{11}(0),\ \dots,\ u_{1n}(0))\left(\tilde{A}+\frac{1}{2\pi i}\log D_0\right)^{j-1}$$

for $1\leqslant j\leqslant n$.

Let K be the space of column vectors of n complex numbers such that $U(0)l=0$, or equivalently,

$$(2.19)\qquad (u_{11}(0),\ \dots,\ u_{1n}(0))\left(\tilde{A}+\frac{1}{2\pi i}\log D_0\right)^{j-1} l=0$$

for $1\leqslant j\leqslant n$. Since, by Cayley's theorem, $\tilde{A}+(1/2\pi i)\log D_0$ satisfies a polynomial equation of degree n, we see that complex numbers $\lambda_1, \dots, \lambda_n$ can be found such that

$$\left(\tilde{A}+\frac{1}{2\pi i}\log D_0\right)^{n}=\sum_{j=1}^{n}\lambda_j\left(\tilde{A}+\frac{1}{2\pi i}\log D_0\right)^{j-1}.$$

It is obvious from this relation that (2.19) is also satisfied for $j=n+1$. Hence, $l\in K$ implies $(\tilde{A}+(1/2\pi i)\log D_0)\, l\in K$. This means that K is mapped into itself by $\tilde{A}+(1/2\pi i)\log D_0$. Assume that $U(0)$ is a singular matrix. Then K is different from $\{0\}$, and so an eigenvector l of $\tilde{A}+(1/2\pi i)\log D_0$ can be found. Following the reasoning at the end of the proof of theorem 2.7, we derive a contradiction. Hence, $U(0)$ is non-singular.

Now the proof of theorem 2.9 can easily be completed. The crucial point in the proof of theorem 2.6 was the regularity of $U(0)$. Since in (2.18) $U(0)$ is non-singular we obtain the same results as in theorem 2.6, viz. V_1 is the solution space of a differential equation (2.9), where $P(z)$ is holomorphic at 0, and the eigenvalues of $P(0)$ are equal to the elements in the diagonal of A. In virtue of theorem 2.7 the space V_1 has a weak singularity at 0. The assertion (4) follows from the fact that the exponents of V_1 are equal to the eigenvalues of $P(0)$.

Theorem 2.10. Let V satisfy the conditions (A) and (B) of theorem 2.5 with $m=1$. Then V is the solution space of a differential equation

$$(2.20)\qquad \vartheta^n y+a_1(z)\,\vartheta^{n-1}y+\dots+a_n(z)y=0,$$

where $a_1, \dots, a_n$ are holomorphic functions at 0. The equation (2.20) can also be written as

$$(2.21)\qquad z^n\frac{d^n y}{dz^n}+t_1(z)\,z^{n-1}\frac{d^{n-1}y}{dz^{n-1}}+\dots+t_n(z)\,y=0,$$

where $t_1, \dots, t_n$ are holomorphic functions at 0.

Moreover, the exponents of V are the roots of the equation

$$x^n + a_1(0)\, x^{n-1} + \ldots + a_n(0) = 0, \tag{2.22}$$

which is equivalent to

$$x(x-1)\ldots(x-n+1) + t_1(0)\, x(x-1)\ldots(x-n+2) + \ldots + t_n(0) = 0. \tag{2.23}$$

Conversely, the solution space of the equation (2.20) (or (2.21)) is an n-dimensional subspace of $M(W)$ and satisfies the conditions (A) and (B).

Proof. Let V satisfy the conditions (A) and (B) of theorem 2.5 with $m=1$. Then, in virtue of theorem 2.9, $V_1 = \Theta V$ satisfies the conditions of theorem 2.6. Hence, we can find an $n \times n$ matrix $P(z)$ holomorphic at $z=0$ such that V_1 is the solution space of the equation $\vartheta v = P(z) v$. Since every $v \in V_1$ has the form Θy with $y \in V$, we find

$$\vartheta^i y = \sum_{j=1}^{n} p_{ij}(z)\, \vartheta^{j-1} y \tag{2.24}$$

for $1 \leqslant i \leqslant n$ and every $y \in V$. If $i < n$, the relation (2.24) is a differential equation of order $\leqslant n-1$ for the elements of V. However, dim $V = n$ and so we have, if $1 \leqslant i \leqslant n-1$, $p_{ij}(z) \equiv 0$ for $j \neq i+1$ and $p_{ij}(z) \equiv 1$ for $j = i+1$. Putting $p_{nj}(z) = -a_{n-j+1}(z)$ $(1 \leqslant j \leqslant n)$, the only equation resulting from (2.24) is (2.20). From the preceding theorem we know that V and V_1 have the same set of exponents. The exponents of V_1 are the eigenvalues of the matrix $P(0)$. It can easily be verified that (2.22) is the characteristic equation of $P(0)$. Hence, the exponents of V are the roots of the equation (2.22). The statements about the equations (2.21) and (2.23) are left to the reader.

Finally, when V is the solution space of the equation (2.20), we may consider the space $V_1 = \Theta V$. V_1 satisfies the conditions of the theorem 2.7, and the assertion concerning V can be derived from the properties of V_1 in an obvious way.

Theorem 2.11 [1]). If V is the solution space of the differential equation (2.21), where $t_1(z), \ldots, t_n(z)$ are holomorphic at 0, and if r is an integer with $0 \leqslant r \leqslant n-1$, then the following conditions are equivalent:

(1) $t_j(z) = z^{j-r}\, u_j(z)$ $(j = r+1, \ldots, n)$, and $u_j(z)$ is holomorphic at 0,

(2) V contains $n-r$ elements $y_1, \ldots, y_{n-r}$ such that the vectors

$$\lim_{w \to 0} \begin{pmatrix} y_i(w) \\ \dfrac{d}{dz}\, y_i(w) \\ \vdots \\ \dfrac{d^{n-r-1}}{dz^{n-r-1}}\, y_i(w) \end{pmatrix} \quad (i = 1, \ldots, n-r)$$

exist and are linearly independent.

[1]) This theorem, which generalizes a result of E. Goursat ([6], p. 265), was conjectured during a discussion with Prof. G. W. Veltkamp, who also proved the special case $r = 1$ by a different method.

(If (1) or (2) hold we say that the $(n-r)$-dimensional initial-value problem is solvable).

Proof. A proof will be sketched; the details are omitted. Let condition (2) be satisfied. Just as in the proof of theorem 2.8 we can show that $\lim_{w\to 0} z(d/dz)\,y(w)=0$, if $y\in V$ and $\lim_{w\to 0}(d/dz)\,y(w)$ exists. By induction it follows that $\lim_{w\to 0} z^j(d^j/dz^j)\,y(w)=0$. From condition (2) it is evident that elements $f_1, \ldots, f_{n-r}\in V$ can be found such that $\lim_{w\to 0}(d^h/dz^h)\,f_i(w)=\delta_{hi}$ $(i, h=1, \ldots, n-r)$. Taking $y=f_1, \ldots, f_{n-r}$ in (2.21) subsequently, and each time passing to the limit we see that condition (1) is satisfied. Next assume that condition (1) is fulfilled. Then it can be seen that (2.21) is equivalent to

$$\vartheta^{r+1}\frac{d^{n-r-1}y}{dz^{n-r-1}}+p_1(z)\,\vartheta^r\frac{d^{n-r-1}y}{dz^{n-r-1}}+\ldots+p_r(z)\,\vartheta\frac{d^{n-r-1}y}{dz^{n-r-1}}+$$

$$p_{r+1}(z)\frac{d^{n-r-1}y}{dz^{n-r-1}}+\ldots+p_n(z)\,y=0,$$

where $p_1(z), \ldots, p_{n-r}(z)$ are holomorphic at $z=0$. To the latter equation corresponds the system

$$\vartheta\begin{pmatrix} y\\ \cdots\\ y^{(n-r-1)}\\ \vartheta y^{(n-r-1)}\\ \vdots\\ \vartheta^r y^{(n-r-1)}\end{pmatrix}=\begin{pmatrix} 0 & z & 0 & \cdots\cdots & 0\\ \vdots & & \ddots & & \vdots\\ \vdots & & z & & \vdots\\ \vdots & & & 1 & \vdots\\ \vdots & & & \ddots & 0\\ 0 & \cdots\cdots & & 0 & 1\\ -p_n(z) & \cdots\cdots & & & -p_1(z)\end{pmatrix}\begin{pmatrix} y\\ \vdots\\ y^{(n-r-1)}\\ \vartheta y^{(n-r-1)}\\ \vdots\\ \vartheta^r y^{(n-r-1)}\end{pmatrix}.$$

By theorem 2.8 we see that the $(n-r)$-dimensional initial-value problem can be solved for this system. From this the solvability of the corresponding problem for the equation (2.21) can be inferred in an obvious way.

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(To be continued)