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Research Article

Hypergeometric approach to Apéry-like series

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ABSTRACT
By reformulating four hypergeometric series formulae, we derive 36 Apéry-like series expressions for the Riemann zeta function, including a couple of identities conjectured by Sun [New series for some special values of L-functions. Nanjing Univ J Math. 2015;32(2): 189–218].

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1. Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers. The shifted factorials are defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for } n \in \mathbb{N},$$

which can be expressed as the following quotient of the Γ-function

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{where } \Gamma(x) = \int_0^\infty \tau^{x-1}\,\mathrm{e}^{-\tau}\,\mathrm{d}\tau \quad \text{for } \Re(x) > 0.$$

Their multiparameter forms will be abbreviated, respectively, as

$$\Gamma\begin{bmatrix} a, & b, & \ldots, & c \\ A, & B, & \ldots, & C \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)},$$
$$\begin{bmatrix} a, & b, & \ldots, & c \\ A, & B, & \ldots, & C \end{bmatrix}_n = \frac{(a)_n(b)_n\cdots(c)_n}{(A)_n(B)_n\cdots(C)_n}.$$

According to Bailey [1, §2.1], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix} a_0, & a_1, & a_2, & \ldots, & a_p \\ & b_1, & b_2, & \ldots, & b_p \end{matrix}\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k(a_2)_k\cdots(a_p)_k}{k!(b_1)_k(b_2)_k\cdots(b_p)_k} z^k.$$

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Recently, Sun [2] claimed to have discovered an infinite series identity which can be reproduced, with a slight modification, as follows

$$\frac{\pi^3}{48}=\sum_{n=1}^{\infty}\frac{2^nH_n^{\langle 2\rangle}}{\binom{2n}{n}(2n+1)}\quad\text{where } H_n^{\langle m\rangle}=\sum_{k=1}^{n}\frac{1}{k^m}. \tag{1.1}$$

However, Chu and Zheng [3, Equation 3b̃] found earlier a more general formula

$$\frac{(\pi/2)^{2m+1}}{(2m+1)!}=\sum_{n=1}^{\infty}\frac{2^n\sigma_m(n)}{\binom{2n}{n}(2n+1)}, \tag{1.2}$$

where $\sigma_m(n)$ denotes the mth elementary function of degree m in $\{1/k^2\}_{1\le k\le n}$. This formula recovers (1.1) with the very initial value $m=1$.

As a warm up, we illustrate a hypergeometric series proof for the last identity. Recall Gauss' second formula (cf. Bailey [1, §2.4])

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}a, c\\ \frac{1+a+c}{2}\end{matrix}\,\middle|\,\frac{1}{2}\right]=\Gamma\left[\begin{matrix}\frac{1}{2}, \frac{1+a+c}{2}\\ \frac{1+a}{2}, \frac{1+c}{2}\end{matrix}\right]\\
\text{and}\quad &{}_2F_1\left[\begin{matrix}1+x, & 1-x\\ & \frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{2}\right]=\frac{\sin\frac{\pi x}{2}}{x},
\end{aligned}
$$

where the latter follows from the former under the replacements $a\to 1+x$ and $c\to 1-x$. Observing that the second series can also be reformulated as

$$1+\sum_{n=1}^{\infty}\frac{2^n}{\binom{2n}{n}(2n+1)}\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)=\frac{\sin\frac{\pi x}{2}}{x},$$

we confirm the general infinite series identity (1.2) by extracting the coefficient of x^{2m} across the last equation. This elegant proof has best shown the power of the hypergeometric series approach devised by the author in [4].

For infinite series related to the Riemann Zeta function, De Doelder [5] established numerous interesting identities through evaluating improper integrals. Some of them are rederived by Borwein [6] by means of the Parseval identity on Fourier series. A couple of beautiful examples may be reproduced, with $H_n:=H_n^{\langle 1\rangle}$, as

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{\pi^4}{72}\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{17\pi^4}{360},$$

where the first one is due to Euler (1775). Other formulas and the related references may be found in Berndt [7, Chapter 9] and Chu [4].

The objective of the present paper is to explore further applications of the hypergeometric approach to Apéry-like series involving generalized harmonic numbers. By examining four non-terminating hypergeometric series, we shall derive 36 infinite series identities.

Some of them are remarkable, for instance, the Apéry-like series for $\zeta(4)$ (see Examples 3.11 and 3.12 with the first one due to Comtet [8, p.89])

$$\frac{17\pi^4}{3240} = \sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}} \quad \text{and} \quad \frac{7\pi^4}{1215} = \sum_{n=1}^{\infty} \frac{H_n^{\langle 2\rangle}}{n^2\binom{2n}{n}}$$

and two infinite series expressions for $\zeta(3)$ (see Examples 3.5 and 3.6) conjectured by Sun [9, Equations 3.1 & 3.2]

$$3\zeta(3) = \sum_{n=1}^{\infty} \frac{2+3nH_{2n}}{n^3\binom{2n}{n}} \quad \text{and} \quad \frac{5}{3}\zeta(3) = \sum_{n=1}^{\infty} \frac{2H_n + H_{2n}}{n^2\binom{2n}{n}}.$$

In order to understand how to extract the coefficient of x^m in a combinatorial expression, we record below some basic facts about the Bell polynomials and generalized harmonic numbers as well as power series expansions of the Γ-function. However for understanding the paper, it is not essential for the reader to study this part because some symbolic software like *Mathematica* or *Maple* can do it automatically.

For the indeterminates $y = \{y_k\}_{k\in\mathbb{N}}$, the Bell polynomials $\Omega_m(y)$ (or the cyclic indicators of symmetric groups [8, §3.3]) are defined by the generating function

$$\sum_{m\geq 0} \Omega_m(y)x^m = \exp\left\{\sum_{k\geq 1} \frac{x^k}{k} y_k\right\}. \tag{1.3}$$

There is the following explicit expression

$$\Omega_m(y) = \Omega_m(y_1, y_2, \ldots, y_m) = \sum_{\omega(m)} \prod_{k=1}^{m} \frac{y_k^{\ell_k}}{\ell_k! k^{\ell_k}}, \tag{1.4}$$

where the multiple sum runs over $\omega(m)$, the set of m-partitions represented by m-tuples of $(\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^{m} k\ell_k = m$.

Let $[x^m]f(x)$ stand for the coefficient of x^m in a formal power series $f(x)$. By means of the generating function method, it is not hard to show that (cf. Chen–Chu [10]) there hold the relations:

$$[x^m]\binom{n-\lambda x}{n} = \Omega_m(u), \quad u_k := -\left(\frac{\lambda}{n}\right)^k \mathbf{H}_k(n); \tag{1.5}$$

$$[x^m]\binom{n-\lambda x}{n}^{-1} = \Omega_m(v), \quad v_k := \left(\frac{\lambda}{n}\right)^k \mathbf{H}_k(n); \tag{1.6}$$

$$[x^m]\binom{n-1-\lambda x}{n-1} = \Omega_m(w), \quad w_k := \left(\frac{\lambda}{n}\right)^k \{1-\mathbf{H}_k(n)\}; \tag{1.7}$$

where the modified harmonic numbers of higher order, slightly different from the usual one, is defined by

$$\mathbf{H}_m(n)=\sum_{k=1}^{n}\left(\frac{n}{k}\right)^m=n^mH_n^{\langle m\rangle}\quad\text{for } m,\quad n\in\mathbb{N}.$$

Then the Bell polynomials corresponding to (1.5), (1.6) and (1.7) can be expressed as

$$\Omega_m(u)=\frac{\lambda^m}{n^m}\sum_{\omega(m)}\prod_{k=1}^{m}\frac{\{-\mathbf{H}_k(n)\}^{\ell_k}}{\ell_k!k^{\ell_k}},\tag{1.8}$$

$$\Omega_m(v)=\frac{\lambda^m}{n^m}\sum_{\omega(m)}\prod_{k=1}^{m}\frac{\mathbf{H}_k^{\ell_k}(n)}{\ell_k!k^{\ell_k}},\tag{1.9}$$

$$\Omega_m(w)=\frac{\lambda^m}{n^m}\sum_{\omega(m)}\prod_{k=1}^{m}\frac{\{1-\mathbf{H}_k(n)\}^{\ell_k}}{\ell_k!k^{\ell_k}}.\tag{1.10}$$

Furthermore, we shall utilize the power series expansions of the Γ-function [4]

$$\Gamma(1-x)=\exp\left\{\sum_{k\geq1}\frac{\sigma_k}{k}x^k\right\},\tag{1.11}$$

$$\Gamma\left(\frac{1}{2}-x\right)=\sqrt{\pi}\exp\left\{\sum_{k\geq1}\frac{\tau_k}{k}x^k\right\};\tag{1.12}$$

where σ_k and τ_k are defined, respectively, by

$$\sigma_1=\gamma\quad\text{and}\quad\sigma_m=\zeta(m)\quad\text{for } m\geq2;\tag{1.13}$$

$$\tau_1=\gamma+2\ln2\quad\text{and}\quad\tau_m=(2^m-1)\zeta(m)\quad\text{for } m\geq2;\tag{1.14}$$

with γ being the Euler–Mascheroni constant and $\zeta(x)$ the usual Riemann zeta function given, respectively, by

$$\gamma=\lim_{n\to\infty}(H_n-\ln n)\quad\text{and}\quad\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}.\tag{1.15}$$

2. Four hypergeometric series with argument $\frac{1}{4}$

As preliminaries, four hypergeometric series with argument $\frac{1}{4}$ will be reformulated in this section so that they will be utilized, in the next section, to derive Apéry-like series identities involving the generalized harmonic numbers by extracting coefficients from formal power series.

2.1. The first hypergeometric series

Recall the following hypergeometric series (cf. Chu–Zheng [3])

$$ {}_2F_1\left[\begin{matrix} 1+x, & 1-x \\ & 3/2 \end{matrix} \middle| y^2\right] = \frac{\sin(2x\arcsin y)}{2xy\sqrt{1-y^2}}. $$

Writing it, under the replacements $x \to 3x$ and $y \to 1/2$, explicitly

$$ \sum_{n=1}^{\infty} \frac{(1+3x)_{n-1}(1-3x)_{n-1}}{(2n-1)!} = \frac{2\sin \pi x}{3\sqrt{3}x}. $$

We can state it equivalently in the theorem below.

Theorem 2.1 (Hypergeometric summation formulae):

$$ \sum_{n=1}^{\infty} \frac{\binom{n-1+3x}{n-1}\binom{n-1-3x}{n-1}}{n\binom{2n}{n}} = \frac{\sin \pi x}{3\sqrt{3}x}. $$

2.2. The second hypergeometric series

There exists also the following counterpart series (cf. Slater [11, §1.5])

$$ {}_2F_1\left[\begin{matrix} x, & -x \\ & 1/2 \end{matrix} \middle| y^2\right] = \cos(2x\arcsin y), \quad \text{where } |y| < 1. \tag{2.1} $$

Its application to evaluation of trigonometric sums and mutlifold Euler sums can be found respectively in Chu [12] and Leshchiner [13]. Letting $x \to 6x$ and $y \to 1/2$, we can express the resulting series as

$$ \sum_{n=1}^{\infty} \frac{(6x)_n(-6x)_n}{(2n)!} = \cos(2\pi x) - 1 = -2\sin^2(\pi x). $$

This can be reformulated equivalently in the following theorem.

Theorem 2.2 (Hypergeometric summation formula):

$$ \sum_{n=1}^{\infty} \frac{\binom{n-1+6x}{n-1}\binom{n-1-6x}{n-1}}{n^2\binom{2n}{n}} = \frac{\sin^2 \pi x}{18x^2}. $$

2.3. The third hypergeometric series

In 1977, Gosper discovered a couple of non-terminating ${}_5F_4(\frac{1}{4})$-series identities. One of them can be derived as a limiting case $n \to \infty$ of a terminating ${}_7F_6(1)$-identity due to

Gessel–Stanton [14, Equation 1.7] (see also Chu [15, Equation 4.1d]):

$$\Gamma\left[\begin{matrix}1+d,\frac{1}{2}+d,1+a-b,\frac{1}{2}+a+b\\1+a,\frac{1}{2}+a,1-b+d,\frac{1}{2}+b+d\end{matrix}\right] = {}_5F_4\left[\left.\begin{matrix}2a,1+\frac{2a}{3},2b,1-2b,a-d\\\frac{2a}{3},1+a-b,\frac{1}{2}+a+b,1+2d\end{matrix}\right|\frac{1}{4}\right].$$

Under the parameter replacements

$$a \to ax, \quad b \to bx, \quad d \to dx,$$

we can express the last identity equivalently as the following theorem involving the variable x and three parameters $\{a, b, d\}$.

Theorem 2.3 (Summation formula):

$$\Gamma\left[\begin{matrix}1+dx,\frac{1}{2}+dx,1+ax-bx,\frac{1}{2}+ax+bx\\1+ax,\frac{1}{2}+ax,1-bx+dx,\frac{1}{2}+bx+dx\end{matrix}\right] = 1+\sum_{n=1}^{\infty}\frac{2b(a-d)x^2(2ax+3n)}{n^3\binom{2n}{n}}$$
$$\times\frac{\binom{n-1+2ax}{n-1}\binom{n-1+2bx}{n-1}\binom{n-1+ax-dx}{n-1}\binom{n+ax+bx}{n}\binom{n-2bx}{n}}{\binom{n+ax-bx}{n}\binom{n+2dx}{n}\binom{2n+2ax+2bx}{2n}}.$$

2.4. The fourth hypergeometric series

Another non-terminating ${}_5F_4(\frac{1}{4})$-series identity of Gosper can be obtained as a limiting case $n \to \infty$ of a terminating ${}_7F_6(1)$-identity due to Chu [15, Equation 5.1e]:

$$\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+b+d,1+a-b,1+a-d\\\frac{1}{2}+b,\frac{1}{2}+d,1+a,1+a-b-d\end{matrix}\right] = {}_5F_4\left[\left.\begin{matrix}a,1+\frac{2a}{3},2b,2d,1+2a-2b-2d\\\frac{2a}{3},1+a-b,1+a-d,\frac{1}{2}+b+d\end{matrix}\right|\frac{1}{4}\right].$$

Under the parameter replacements

$$a \to ax, \quad b \to bx, \quad d \to dx,$$

we can reformulate the last identity equivalently in the following theorem involving the variable x and three parameters $\{a, b, d\}$.

Theorem 2.4 (Summation formula):

$$\Gamma\begin{bmatrix}\frac{1}{2}, \frac{1}{2}+bx+dx, 1+ax-bx, 1+ax-dx\\ \frac{1}{2}+bx, \frac{1}{2}+dx, 1+ax, 1+ax-bx-dx\end{bmatrix} = 1+\sum_{n=1}^{\infty}\frac{2bdx^2(2ax+3n)}{n^3\binom{2n}{n}}$$
$$\times\frac{\binom{n-1+ax}{n-1}\binom{n-1+2bx}{n-1}\binom{n-1+2dx}{n-1}\binom{n+bx+dx}{n}\binom{n+2ax-2bx-2dx}{n}}{\binom{n+ax-bx}{n}\binom{n+ax-dx}{n}\binom{2n+2bx+2dx}{2n}}.$$

3. Apéry-like series for Riemann zeta function

For each theorem stated in the last section, we shall derive infinite series identities involving generalized harmonic numbers by following the procedure:

- For a fixed natural number m, extracting the coefficient of x^m from the binomial quotient by employing (1.5), (1.6) and (1.7).
- Extracting the coefficient of the same power x^m from the Γ-function quotient by utilizing (1.11) and (1.12).
- Specifying the parameters a,b,d to particular numerical values and simplifying the equality related to the above coefficients.

We do not need to write procedures to teach *Mathematica* or *Maple* how to extract the coefficient of x^m from combinatorial expressions inside summations, because a method for doing it is already implemented in the software. The two simplest cases follow immediately without making manipulations.

Example 3.1 (Theorem 2.1: $x = 0$; Elsner [16] and Lehmer [17]):

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}.$$

Example 3.2 (Theorem 2.2: $x = 0$; Euler: See [16,18,19]):

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

The last formula can also be obtained by letting $x = 0$ in Theorems 2.3 and 2.4.

Further identities are classified according to the integer variable of the zeta function $\zeta(m)$. For all the examples, the parameter settings for the theorem to be utilized will be highlighted in the headers. Except for the few examples, most formulae displayed below do not seem to have appeared previously. For the sake of brevity, we shall use '$[x^m]$ in Theorem n' to indicate the extraction of the coefficient $[x^m]$ from the equation appearing in Theorem n.

3.1. *Infinite Series for* $\zeta(3)$

Example 3.3 ($[x^2]$ in Theorem 2.1):

$$\frac{\pi^3}{162\sqrt{3}} = \sum_{n=1}^{\infty} \frac{\mathbf{H}_2(n) - 1}{n^3\binom{2n}{n}}.$$

Example 3.4 ($[x^3]$ in Theorem 2.4: $b \to 1$ and $d \to -1$):

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{3\mathbf{H}_1(n) - 1}{n^3\binom{2n}{n}}.$$

Example 3.5 ($[x^3]$ in Theorem 2.4: $a \to -2b - 2d$):

$$6\zeta(3) = \sum_{n=1}^{\infty} \frac{4 + 3\mathbf{H}_1(2n)}{n^3\binom{2n}{n}}.$$

Example 3.6 ($[x^3]$ in Theorem 2.4: $a \to -6b - 6d$):

$$\frac{10}{3}\zeta(3) = \sum_{n=1}^{\infty} \frac{4\mathbf{H}_1(n) + \mathbf{H}_1(2n)}{n^3\binom{2n}{n}}.$$

Example 3.7 ($[x^3]$ in Theorem 2.4: $a \to 4b + 4d$):

$$0 = \sum_{n=1}^{\infty} \frac{10 - 18\mathbf{H}_1(n) + 3\mathbf{H}_1(2n)}{n^3\binom{2n}{n}}.$$

We remark that the two formulae displayed in Examples 3.5 and 3.6 confirm the two conjectured series for $\zeta(3)$ made by Sun [9, Equations 3.1 & 3.2].

3.2. *Infinite Series for* $\zeta(4)$

Example 3.8 ($[x^2]$ in Theorem 2.2):

$$\frac{\pi^4}{1944} = \sum_{n=1}^{\infty} \frac{\mathbf{H}_2(n) - 1}{n^4\binom{2n}{n}}.$$

Example 3.9 ($[x^4]$ in Theorem 2.4: $a \to 0$ and $b \to -d$; Bailey et al. [20, Equation 2–6]):

$$\frac{\pi^4}{270} = \sum_{n=1}^{\infty} \frac{4 - 3\mathbf{H}_2(n)}{n^4\binom{2n}{n}}.$$

Example 3.10 ($[x^4]$ Theorem 2.4: $a \to d\sqrt{-2}$ and $b \to -d$):

$$\frac{2\pi^4}{45} = \sum_{n=1}^{\infty} \frac{10 + 2\mathbf{H}_1(n) - 3\mathbf{H}_1^2(n)}{n^4\binom{2n}{n}}.$$

By linear combinations of the equations displayed in Examples 3.8–3.10, we may derive easily the following three beautiful infinite series expressions for π^4 as well as two further annihilated series.

Example 3.11 (Comtet [8, p.89]: See also Lehmer [17]):

$$\frac{17\pi^4}{3240} = \sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}},$$

Example 3.12 (Ablinger [21, Equation 83]):

$$\frac{7\pi^4}{1215} = \sum_{n=1}^{\infty} \frac{\mathbf{H}_2(n)}{n^4\binom{2n}{n}}.$$

Example 3.13:

$$\frac{13\pi^4}{1620} = \sum_{n=1}^{\infty} \frac{3\mathbf{H}_1^2(n) - 2\mathbf{H}_1(n)}{n^4\binom{2n}{n}}.$$

Example 3.14:

$$0 = \sum_{n=1}^{\infty} \frac{51\mathbf{H}_2(n) - 56}{n^4\binom{2n}{n}}.$$

Example 3.15:

$$0 = \sum_{n=1}^{\infty} \frac{3\mathbf{H}_1^2(n) - 2\mathbf{H}_1(n) - 26/17}{n^4\binom{2n}{n}}.$$

In addition, by letting $a \to 15d/17, b \to 3d/17$ in Theorem 2.3 and $a \to 3d, b \to -3d/2$ in Theorem 2.4, we get, respectively, the following two equalities

$$\frac{6583\pi^4}{1620} = \sum_{n=1}^{\infty} \frac{118 + 72\mathbf{H}_1(2n) + 27\mathbf{H}_1^2(2n) + 27\mathbf{H}_2(2n)}{n^4\binom{2n}{n}}, \tag{3.1}$$

$$\frac{10\pi^4}{27} = \sum_{n=1}^{\infty} \frac{188\mathbf{H}_2(n) - 16\mathbf{H}_1^2(n) - \mathbf{H}_2(2n) - \mathbf{H}_1^2(2n) - 8\mathbf{H}_1(n)\mathbf{H}_1(2n) - 72}{n^4\binom{2n}{n}}. \tag{3.2}$$

In view of Examples 3.11 and 3.12, they can further be simplified to the following infinite series identities.

Example 3.16 (Reduction of (3.1)):

$$\frac{31\pi^4}{81} = \sum_{n=1}^{\infty} \frac{8\mathbf{H}_1(2n) + 3\mathbf{H}_1^2(2n) + 3\mathbf{H}_2(2n)}{n^4\binom{2n}{n}}.$$

Example 3.17 (Reduction of (3.2)):

$$\frac{7\pi^4}{90} = \sum_{n=1}^{\infty} \frac{6\mathbf{H}_1^2(n) - \mathbf{H}_1(2n) + 3\mathbf{H}_1(n)\mathbf{H}_1(2n)}{n^4\binom{2n}{n}}.$$

3.3. Infinite Series for $\zeta(5)$

Example 3.18 ($[x^4]$ Theorem 2.1):

$$\frac{\pi^5}{14580\sqrt{3}} = \sum_{n=1}^{\infty} \frac{\mathbf{H}_2^2(n) - \mathbf{H}_4(n) - 2\mathbf{H}_2(n) + 2}{n^5\binom{2n}{n}}.$$

Example 3.19 ($[x^5]$ in Theorem 2.4: $b \to a/\sqrt{-2}$ and $d \to -a/\sqrt{-2}$):

$$3\zeta(5) + \frac{\pi^2}{3}\zeta(3) = \sum_{n=1}^{\infty} \frac{2 - 10\mathbf{H}_1(n) - \mathbf{H}_1^2(n) + \mathbf{H}_1^3(n) + 20\mathbf{H}_3(n)}{n^5\binom{2n}{n}}.$$

Example 3.20 ($[x^5]$ in Theorem 2.4: First $b = -d = 1$ and then derivative with respect to a at $a = 0$):

$$\frac{\pi^2}{3}\zeta(3) - \zeta(5) = \sum_{n=1}^{\infty} \frac{4 - 12\mathbf{H}_1(n) - 3\mathbf{H}_2(n) + 6\mathbf{H}_3(n) + 9\mathbf{H}_1(n)\mathbf{H}_2(n)}{n^5\binom{2n}{n}}.$$

Example 3.21 ($[x^5]$ in Theorem 2.4: First $a = -2b-2d$ and then derivative with respect to b at $b \to -d$):

$$18\zeta(5) - 2\pi^2\zeta(3) = \sum_{n=1}^{\infty} \frac{40 - 12\mathbf{H}_2(n) - 39\mathbf{H}_3(n) + 12\mathbf{H}_1(2n) - 9\mathbf{H}_2(n)\mathbf{H}_1(2n)}{n^5\binom{2n}{n}}.$$

The next formula is derived from Theorem 2.3 by evaluating the limit $b \to -3d$ after letting $a = 3d$ and then dividing by $b+3d$ across the equation.

Example 3.22 ($[x^5]$ in Theorem 2.3):

$$\frac{32\zeta(5) - 2\pi^2\zeta(3)}{27} = \sum_{n=1}^{\infty} \frac{1}{n^5\binom{2n}{n}} \left\{ \begin{matrix} 4 - 2\mathbf{H}_1(n) - 2\mathbf{H}_2(n) + 2\mathbf{H}_1(n)\mathbf{H}_2(n) \\ +\mathbf{H}_1(2n) - \mathbf{H}_3(n) - \mathbf{H}_2(n)\mathbf{H}_1(2n) \end{matrix} \right\}.$$

By examining the linear combination

$$2 \text{ Example 3.20} + \text{Example 3.21} - 9 \text{ Example 3.22}$$

we get the following reduced series expression.

Example 3.23 (Reduced series):

$$\frac{16\zeta(5) - 2\pi^2\zeta(3)}{9} = \sum_{n=1}^{\infty} \frac{4 - 2\mathbf{H}_1(n) - 6\mathbf{H}_3(n) + \mathbf{H}_1(2n)}{n^5\binom{2n}{n}}.$$

By making use of the *Mathematica* package *HarmonicSums*, Ablinger [21] evaluated numerous infinite sums containing the central binomial coefficient and a single generalized harmonic number (instead of their combinations). It is possible to employ his results to simplify some formulae obtained in this paper. For instance, it is almost trivial to reduce the formula in the last example to

$$\frac{22}{9}\zeta(5)=\sum_{n=1}^{\infty}\frac{4-2\mathbf{H}_1(n)+\mathbf{H}_1(2n)}{n^5\binom{2n}{n}} \tag{3.3}$$

by invoking Ablinger's following remarkable one [21, Equation 108]

$$\frac{\zeta(5)}{9}+\frac{\pi^2\zeta(3)}{27}=\sum_{n=1}^{\infty}\frac{\mathbf{H}_3(n)}{n^5\binom{2n}{n}}. \tag{3.4}$$

It can also be utilized to reduce slightly those displayed in Examples 3.19–3.22.

3.4. *Infinite series for $\zeta(6)$*

Example 3.24 ($[x^4]$ in Theorem 2.2):

$$\frac{\pi^6}{262440}=\sum_{n=1}^{\infty}\frac{2-2\mathbf{H}_2(n)+\mathbf{H}_2^2(n)-\mathbf{H}_4(n)}{n^6\binom{2n}{n}}.$$

Example 3.25 ($[x^6]$ in Theorem 2.4: $a\to 0, b\to -d$; Bailey et al. [20, Equation 2–7]):

$$\frac{2\pi^6}{2835}=\sum_{n=1}^{\infty}\frac{32-24\mathbf{H}_2(n)+9\mathbf{H}_2^2(n)-15\mathbf{H}_4(n)}{n^6\binom{2n}{n}}.$$

Combinations of the last two formulae yield the following three reduced series.

Example 3.26 (15×Example 3.24−Example 3.25):

$$\frac{397\pi^6}{1224720}=\sum_{n=1}^{\infty}\frac{1+3\mathbf{H}_2(n)-3\mathbf{H}_2^2(n)}{n^6\binom{2n}{n}}.$$

Example 3.27 (9×Example 3.24−Example 3.25):

$$\frac{137\pi^6}{408240}=\sum_{n=1}^{\infty}\frac{7-3\mathbf{H}_2(n)-3\mathbf{H}_4(n)}{n^6\binom{2n}{n}}.$$

Example 3.28 (12×Example 3.24−Example 3.25):

$$\frac{101\pi^6}{153090}=\sum_{n=1}^{\infty}\frac{8-3\mathbf{H}_2^2(n)-3\mathbf{H}_4(n)}{n^6\binom{2n}{n}}.$$

3.5. *Infinite Series for* $\zeta(7)$

Example 3.29 ($[x^6]$ in Theorem 2.1):

$$\frac{\pi^7}{1837080\sqrt{3}} = \sum_{n=1}^{\infty} \frac{1}{n^7\binom{2n}{n}} \left\{ \begin{matrix} 6\mathbf{H}_2(n) + 3\mathbf{H}_4(n) - 3\mathbf{H}_2(n)\mathbf{H}_4(n) \\ -6 - 3\mathbf{H}_2^2(n) + \mathbf{H}_2^3(n) + 2\mathbf{H}_6(n) \end{matrix} \right\}.$$

The next identity is obtained by combining Theorem 2.3 with Theorem 2.4. First dividing the equation in Theorem 2.3 by $a-3d$ and letting $b \to -a$, $d \to a/3$ successively, we get an equality '**A**'. Then dividing the equation in Theorem 2.4 by a and letting $b \to -d$, $a \to 0$ successively, we get another equality '**B**'. Simplifying the difference '**A** − 9**B**', we find the following identity.

Example 3.30 (Combination of Theorem 2.3 with Theorem 2.4):

$$\frac{7}{120}\{1440\zeta(7) + 29\pi^4\zeta(3) + 60\pi^2\zeta(5)\}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^7\binom{2n}{n}} \left\{ \begin{matrix} 11 + 63[1 - 3\mathbf{H}_1(n)][7 - 3\mathbf{H}_2(n) - 3\mathbf{H}_4(n)] \\ +399\mathbf{H}_5(n) - 9[9 - 7\mathbf{H}_2(n)][1 - 21\mathbf{H}_3(n)] \end{matrix} \right\}.$$

We can derive another identity analogously by combining Theorem 2.3 with Theorem 2.4. Dividing the equation in Theorem 2.3 by $a+b$ and letting $d \to (3a + 2b)/3$, $a \to -b$ successively, we get an equality '**C**'. Then dividing the equation in Theorem 2.4 by $b+d$ and then letting $a \to -2b - 2d$, $b \to -d$ successively, we get another equality '**D**'. Simplifying the difference '**C** − 9**D**', we find the identity below.

Example 3.31 (Combination of Theorem 2.3 with Theorem 2.4):

$$\frac{2}{5}\{5760\zeta(7) - 29\pi^4\zeta(3) - 660\pi^2\zeta(5)\}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^7\binom{2n}{n}} \left\{ \begin{matrix} 4183 + 72[4 + 3\mathbf{H}_1(2n)][7 - 3\mathbf{H}_2(n) - 3\mathbf{H}_4(n)] \\ -6096\mathbf{H}_5(n) + 9[45 - 32\mathbf{H}_2(n)][5 - 24\mathbf{H}_3(n)] \end{matrix} \right\}.$$

3.6. *Infinite series for* $\zeta(8)$

Example 3.32 ($[x^6]$ in Theorem 2.2):

$$\frac{\pi^8}{44089920} = \sum_{n=1}^{\infty} \frac{1}{n^8\binom{2n}{n}} \left\{ \begin{matrix} 3\mathbf{H}_4(n) + 2\mathbf{H}_6(n) - 3\mathbf{H}_2(n)\mathbf{H}_4(n) \\ -6 + 6\mathbf{H}_2(n) - 3\mathbf{H}_2^2(n) + \mathbf{H}_2^3(n) \end{matrix} \right\}.$$

Example 3.33 ($[x^8]$ in Theorem 2.4: $a \to 0$ and $b \to -d$; Bailey et al. [20, Equation 2–8]):

$$\frac{\pi^8}{14175} = \sum_{n=1}^{\infty} \frac{1}{n^8\binom{2n}{n}} \left\{ \begin{matrix} 128 - 96\mathbf{H}_2(n) + 36\mathbf{H}_2^2(n) - 9\mathbf{H}_2^3(n) \\ -60\mathbf{H}_4(n) - 42\mathbf{H}_6(n) + 45\mathbf{H}_2(n)\mathbf{H}_4(n) \end{matrix} \right\}.$$

From the last two examples, we can get, by eliminating the mixed product term $\mathbf{H}_2(n)\mathbf{H}_4(n)$, the following slightly reduced series

Example 3.34 (15×Example 3.32+Example 3.33):

$$\frac{5209\pi^8}{73483200}=\sum_{n=1}^{\infty}\frac{38-6\mathbf{H}_2(n)-9\mathbf{H}_2^2(n)+6\mathbf{H}_2^3(n)-15\mathbf{H}_4(n)-12\mathbf{H}_6(n)}{n^8\binom{2n}{n}}.$$

According to Theorem 2.1 and Theorem 2.2, we can derive, by extracting the coefficient of x^{2m}, the following two general infinite series identities:

$$\frac{(\pi/3)^{1+2m}}{(2m+1)!\sqrt{3}}=\sum_{n=1}^{\infty}\frac{\sigma_m(n-1)}{n\binom{2n}{n}}\quad\text{and}\quad\frac{(\pi/3)^{2+2m}}{(2m+2)!}=\sum_{n=1}^{\infty}\frac{\sigma_m(n-1)}{n^2\binom{2n}{n}}.$$

For $m=4$, they can be reformulated in terms of the generalized harmonic numbers as the following two examples.

Example 3.35 ($[x^8]$ in Theorem 2.1):

$$\frac{\pi^9}{297606960\sqrt{3}}=\sum_{n=1}^{\infty}\frac{1}{n^9\binom{2n}{n}}\left\{\begin{array}{l}12\mathbf{H}_2^2(n)-4\mathbf{H}_2^3(n)+\mathbf{H}_2^4(n)+12\mathbf{H}_2(n)\mathbf{H}_4(n)\\ \quad-12\mathbf{H}_4(n)-8\mathbf{H}_6(n)-6\mathbf{H}_8(n)-6\mathbf{H}_2^2(n)\mathbf{H}_4(n)\\ \quad+24-24\mathbf{H}_2(n)+3\mathbf{H}_4^2(n)+8\mathbf{H}_2(n)\mathbf{H}_6(n)\end{array}\right\}.$$

Example 3.36 ($[x^8]$ in Theorem 2.2):

$$\frac{\pi^{10}}{8928208800}=\sum_{n=1}^{\infty}\frac{1}{n^{10}\binom{2n}{n}}\left\{\begin{array}{l}12\mathbf{H}_2^2(n)-4\mathbf{H}_2^3(n)+\mathbf{H}_2^4(n)+12\mathbf{H}_2(n)\mathbf{H}_4(n)\\ \quad-12\mathbf{H}_4(n)-8\mathbf{H}_6(n)-6\mathbf{H}_8(n)-6\mathbf{H}_2^2(n)\mathbf{H}_4(n)\\ \quad+24-24\mathbf{H}_2(n)+3\mathbf{H}_4^2(n)+8\mathbf{H}_2(n)\mathbf{H}_6(n)\end{array}\right\}.$$

It is curious that the summands in the last two examples differ only by factor n.

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