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Computing the Lie algebra of the differential Galois group: The reducible case ☆

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In this paper, we explain how to compute the Lie algebra of the differential Galois group of a reducible linear differential system. We achieve this by showing how to transform a block-triangular linear differential system into a Kolchin-Kovacic reduced form. We combine this with other reduction results to propose a general algorithm for computing a reduced form of a general linear differential system. In particular, this provides directly the Lie algebra of the differential Galois group without an a priori computation of this Galois group.

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0. Introduction

Let $\mathcal{A}(x) \in \mathcal{M}_n(\mathbf{k})$ denote an $n \times n$ matrix with coefficients in a differential field $(\mathbf{k}, \partial)$ of characteristic zero, for instance $\mathbf{k} = \overline{\mathbb{Q}}(x)$. We consider the linear differential system $[\mathcal{A}]: Y'(x) = \mathcal{A}(x)Y(x)$. The *differential Galois group* G of $[\mathcal{A}]$ is an algebraic group which somehow measures the algebraic relations among the entries of a fundamental solution matrix of $[\mathcal{A}]$. The aim of this paper is to explain how to compute effectively the Lie algebra $\mathfrak{g}$ of the differential Galois group G, without computing G.

Goal of the paper. Given an invertible matrix $P(x) \in \mathrm{GL}_n(\mathbf{k})$, the change of variable ("gauge transformation") $Y(x) = P(x).Z(x)$ produces the linear differential system noted $Z'(x) = P(x)[\mathcal{A}(x)]\, Z(x)$, with $P(x)[\mathcal{A}(x)] := P^{-1}(x)\mathcal{A}(x)P(x) - P^{-1}(x)P'(x)$. The differential system $[\mathcal{A}]$ is called *reducible* if there exists a gauge transformation $P(x)$, such that $A(x) := P(x)[\mathcal{A}(x)]$ is of the form

$$A(x) = \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline S(x) & A_2(x) \end{array}\right).$$

There exist algorithms to test and realize this *factorization*; they may be found in Singer (1996); Barkatou (2007) for the completely reducible case, and in the appendix of Compoint and Weil (2004) (and references therein) for the general case.

Let G denote the differential Galois group of $[A]: Y'(x) = A(x)Y(x)$. Let $\mathfrak{g}$ be its Lie algebra. In this paper, we show how to compute $\mathfrak{g}$ by using the theory of reduced forms of linear differential systems. Finding a *reduced form* of $[A]$ amounts to finding a gauge transformation $P(x)$ (possibly over an algebraic extension $\mathbf{k_0}$ of $\mathbf{k}$) such that $P(x)[A(x)] \in \mathfrak{g}(\mathbf{k_0})$. This is similar to the Lie-Vessiot-Guldberg theories of reduction of connections in differential geometry (see Blázquez-Sanz and Morales-Ruiz, 2010, 2012 for the latter and their connections with the Kolchin-Kovacic theory of reduced forms). Our contribution is to provide an algorithm to compute such a reduction matrix $P(x)$ for a reducible system.

In Barkatou et al. (2016), it is explained how to put a completely reducible block-diagonal system into reduced form. We will show that, to reduce $[\mathcal{A}]$, it is thus sufficient to be able to reduce $[A]$ under the assumption that the block diagonal differential system

$$Y'(x) = A_{\mathrm{diag}}(x)Y(x), \quad \text{with} \quad A_{\mathrm{diag}}(x) = \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline 0 & A_2(x) \end{array}\right),$$

is in reduced form. We, with A. Aparicio-Monforte, had solved this problem in Aparicio-Monforte et al. (2016) in the special case when the Lie algebra of $A_{\mathrm{diag}}(x)$ is abelian. This was extended by Casale and the second author in Casale and Weil (2018) to families of SL_2-systems. In this paper, we treat the problem in the general case. A review of this work with an emphasis on down-to-earth exposition, relations to questions of theoretical physics and examples can be found in Dreyfus and Weil (2021).

General algorithms for computing differential Galois groups. Using the classification of the algebraic subgroups of SL_2, Kovacic gave an efficient algorithm for computing liouvillian solutions, which in turn allows to essentially obtain the differential Galois group when $n = 2$. This approach was systematized by Singer and Ulmer (1993b,a) and then Singer and Ulmer (1997), notably in the case $n = 3$.

Let us now describe general procedures that work for an arbitrary n. Compoint and Singer gave a decision procedure in Compoint and Singer (1999) to compute the differential Galois group in the case of completely reducible (direct sums of irreducible) systems. Berman and Singer gave an algorithm extending (Compoint and Singer, 1999) for a large class of reducible systems (Berman, 2002; Berman and Singer, 1999). Using model theory, Hrushovski gave in (Hrushovski, 2002) the first general decision procedure computing the Galois group. It was clarified and improved by Feng (2015), see also Sun (2019). More recently, the paper (Amzallag et al., 2021) introduces new ideas to further improve the bounds in Hrushovski's algorithm. A symbolic-numeric algorithm was proposed by van der Hoeven (2007), based on the Schlesinger-Ramis density theorems. None of these general algorithms is currently implemented, either because their complexity is prohibitive (especially Hrushovski's algorithm) or because it is not yet known how to implement some of the required building blocks.

General algorithms for computing reduced forms. In the last decade, a strategy has been developed to compute the Lie algebra, instead of the Galois group, by computing a reduced form of the differential system. The Kolchin-Kovacic reduction theorems appear in the Kovacic program on the inverse problem Kovacic (1969, 1971) and in works of Kolchin on the logarithmic derivative (Kolchin, 1973, 1999). Further studies of Lie-Kolchin reduction methods are carried out by Blázquez-Sanz and Morales-Ruiz (2010, 2012). As a computation strategy, reduced forms are used in Aparicio-Monforte and Weil (2011, 2012); Aparicio-Monforte et al. (2016) (the strategy in Nguyen and van der Put, 2010 is also related to this approach). A characterization of reduced forms in terms of invariants is proposed in Aparicio-Monforte et al. (2013); the latter paper also contains a decision procedure for putting the system into reduced form when the Galois group is reductive. A more elaborate, and much more efficient, algorithm is given in Barkatou et al. (2016) in the case of an absolutely irreducible system.

Some motivations for reductions of reducible systems. In papers on differential Galois theory, the case of a reducible system is sometimes brushed aside for two reasons. First, if one solves the irreducible diagonal blocks, then the full system can be solved by variation of constants. Second, a generic system is irreducible anyway so it may seem futile, at first glance, to spend energy on rare reducible systems.

Regarding the first objection, it would require to first solve irreducible systems, and then construct a big Picard-Vessiot extension; variation of constants would then require the computation of integrals of transcendental functions. Namely, in the above notations, a fundamental matrix is

$$U = \left(\begin{array}{c|c} U_1 & 0 \\ \hline U_2 V & U_2 \end{array}\right), \quad \text{with} \quad \begin{cases} U_i' &= A_i U_i, \\ V' &= U_2^{-1} S U_1, \end{cases}$$

where S denotes the lower triangular block in A. In contrast, the approach developed here uses essentially rational solutions of linear differential systems with coefficients in the base field; in return, it may actually be used to study properties of integrals of holonomic transcendental functions, see Bertrand (2001). Indeed, a reduced form gives us all algebraic relations between these integrals of holonomic functions; in particular, we will obtain a basis of transcendental integrals to express all the other ones.

Regarding the second objection, it turns out that, in many practical applications, the differential systems or operators that occur happen to be reducible. Indeed we next describe several examples of this.

The context which was our initial motivation is the Morales-Ramis-Simó theory: it studies integrability properties of dynamical systems by studying successive differential systems, the variational equations, which can be viewed as a cascade of reducible systems. Algorithms to obtain reduced forms, and hence integrability criteria, for such systems are elaborated in Aparicio-Monforte and Weil (2011, 2012); Aparicio-Monforte et al. (2016). For more general (non-integrable) non-linear differential systems, the Lie algebras of the differential Galois groups of variational equations give information on the Malgrange groupoïd of the system. This is shown by Casale (2009) and developed in Casale and Weil (2018) to compute the Malgrange groupoïd of (non-linear) second order differential equations.

Once again, the fact that the variational equations are reducible systems turns out to be an important ingredient.

Reducible operators also appear very naturally in the holonomic world of statistical mechanics or combinatorics, see Bostan et al. (2009, 2011) or the reference book (McCoy, 2010). In this context, objects (or generating series) appear as convergent holonomic power series with integer coefficients; they are solutions of linear differential operators and their minimal operator is often reducible, see e.g. Bostan et al. (2009, 2011).

Last, we may also mention prolongations of systems which appear in works on generic Galois groups (situations with mixed differential and q-difference structures), see Di Vizio and Hardouin (2010) and references therein. These are also (structured) reducible linear differential systems and tools from this work may hence be used for a better understanding of generic or particular parametrized differential Galois groups. Similar prolongations also appear when studying singularly perturbed linear differential systems and studying solutions as series in the perturbation parameter, see e.g. the PhD of Maddah (2015) and references therein. The methods that we elaborate here may lead to simplification methods for such systems.

Structure of the paper. The paper is organized as follows. In §1, we recall some basic facts of differential Galois theory. We present the theory of reduced forms, notably the Kolchin Kovacic reduction theorem, which is the heart of our paper. In §2 we prove that the reduction matrix may be chosen to have a particular shape: it is a unipotent triangular matrix. The action of such a gauge transformation on the matrix $A(x)$ of the system is governed by the adjoint action of the block-diagonal part of $A(x)$ on its off-diagonal parts. The results of this first part of §2 are generalizations of Aparicio-Monforte et al. (2016). Then, we recall the construction of an isotypical flag, which will be adapted to the adjoint action in the reduction process. In §3 we give examples of the reduction process of §4. We have chosen to take examples in increasing degrees of complexity in order to show step by step what the difficulties are. In §4, we present the main contribution of the paper. We explain how to put a block-triangular linear differential system into reduced form. Applying linear algebra and standard module-theoretic tools (isotypical decomposition, flags of indecomposable modules, etc.), we generalize the techniques of Aparicio-Monforte et al. (2016) to this *non-abelian* setting.[1] The gauge transformation which reduces the system is then derived from the computation of rational solutions of successive linear differential systems with parametrized right-hand-side, see Theorem 4.4. We believe that this part will generally be algorithmically efficient because it uses mostly linear algebra and rational solutions of linear differential systems of bounded size. We show this in several examples; see also the maple worksheet (Dreyfus and Weil, 2020). In §5, we present another contribution. We show how the results of §4 may be combined with other results in order to have a general algorithm for reducing a general linear differential system.

The last two short sections are mostly expository and included for self-containedness. In §6, we explain, given a system in reduced form, how to compute the Lie algebra $\mathfrak{g}$ of the differential Galois group. In §7, we describe how, having computed the Galois-Lie algebra $\mathfrak{g}$ of a reduced linear differential system, one can recover its differential Galois group G (using connectedness). The material in §6 and §7 is mostly known.

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[1] The main difference between this paper and Aparicio-Monforte et al. (2016) is that, in the previous paper, the Lie algebra of $A_{\mathrm{diag}}(x)$ was abelian. This had the consequence that the eigenvalues of the adjoint action belonged to $\mathbf{k}$ and a convenient Lie subalgebra of $A(x)$ admitted a basis of constant matrices in which the matrix associated to the adjoint action was in Jordan normal form. The reduction problem was then reduced to rational solutions of first order scalar linear differential equations. This is no longer true here.

1. Differential Galois theory and reduced forms

1.1. The base field

Let us consider a differential field of characteristic zero $(\mathbf{k}, \partial)$, i.e. a field equipped with a derivation. We will use the classical notation c', for the derivative of $c \in \mathbf{k}$. We assume that its constant field $\mathcal{C} := \{c \in \mathbf{k} \mid c' = 0\}$ is algebraically closed. We need to make assumptions about our base field $\mathbf{k}$ to elaborate our algorithms.

① First we assume that $\mathbf{k}$ is an effective field, i.e. that one can compute representatives of the four operations $+, -, \times, /$ and one can effectively test whether two elements of $\mathbf{k}$ are equal.
② We also assume that, given a homogeneous linear differential system $[A] : Y'(x) = A(x)Y(x)$ with $A(x) \in \mathcal{M}_n(\mathbf{k})$, we can effectively find a basis of its rational solutions, i.e. its solutions $Y(x) \in \mathbf{k}^n$.
③ Finally, we assume that, given a homogeneous linear differential system $[A] : Y'(x) = A(x)Y(x)$ with $A(x) \in \mathcal{M}_n(\mathbf{k})$, we can effectively find a basis of its exponential, also called hyperexponential, solutions (see Barkatou et al., 2012).

The standard example of such a field would be $\mathbf{k} = \mathcal{C}(x)$ with $\mathcal{C} = \overline{\mathbb{Q}}$. When $\mathbf{k} = \mathcal{C}(x)$, a fast algorithm for rational solutions of linear differential systems is given in Barkatou (1999). A Maple package IntegrableConnections, based on ISOLDE (Barkatou and Pfluegel, 2022), for this task is proposed in Barkatou et al. (2022). Algorithms for ② and ③ and generalizations appear in Barkatou et al. (2012) (and references therein).

Remark 1.1. Assumption ③ is used only in the factorization algorithm which is a preliminary step to our reduction method. The specific algorithm proposed in this paper only uses the rational algorithms of assumption ② and also ①.

Singer showed, in Singer (1991), Lemma 3.5 and Theorem 4.1, that if $\mathbf{k}$ is an elementary extension of $\mathcal{C}(x)$ or if $\mathbf{k}$ is an algebraic extension of a purely transcendental Liouvillian extension of $\mathcal{C}(x)$, then $\mathbf{k}$ satisfies the above conditions and hence suits our purposes.

To simplify the exposition, we will further assume that $\mathbf{k}$ is a $\mathcal{C}^1$-field.[2]

1.2. Differential Galois theory

We review classical elements of differential Galois theory. We refer to van der Put and Singer (2003) or Crespo and Hajto (2011); Singer (2009) for details and proofs. Let us consider a linear differential system of the form $[A] : Y'(x) = A(x)Y(x)$, with $A(x) \in \mathcal{M}_n(\mathbf{k})$. A *Picard-Vessiot extension* for $[A]$ is a differential field extension K of $\mathbf{k}$, generated over $\mathbf{k}$ by the entries of a fundamental solution matrix of $[A]$ and such that the field of constants of K is $\mathcal{C}$. The Picard-Vessiot extension K exists and is unique up to differential field isomorphism.

The *differential Galois group* G of the system $[A]$ is the group of field automorphisms of the Picard-Vessiot extension K which commute with the derivation and leave all elements of $\mathbf{k}$ invariant. Let $U(x) \in \mathrm{GL}_n(K)$ be a fundamental solution matrix of $Y'(x) = A(x)Y(x)$ with coefficients in K. For any $\varphi \in G$, $\varphi(U(x))$ is also a fundamental solution matrix, so there exists a constant matrix $C_\varphi \in \mathrm{GL}_n(\mathcal{C})$ such that $\varphi(U(x)) = U(x).C_\varphi$. The map $\rho_U : \varphi \longmapsto C_\varphi$ is an injective group morphism. The group G, identified with Im ρ_U, may be viewed as a linear algebraic subgroup of $\mathrm{GL}_n(\mathcal{C})$.

The *Lie algebra* $\mathfrak{g}$ of the linear algebraic group $G \subset \mathrm{GL}_n(\mathcal{C})$ is the tangent space to G at the identity. Equivalently, it is the set of matrices $N \in \mathcal{M}_n(\mathcal{C})$ such that $\mathrm{Id}_n + \varepsilon N$ satisfies the defining equations of the algebraic group G modulo ε^2. The Lie algebra $\mathfrak{g}$ of the differential Galois group is referred to as the

[2] A field $\mathbf{k}$ is a $\mathcal{C}^1$-field when every non-constant homogeneous polynomial P over $\mathbf{k}$ has a non-trivial zero provided that the number of its variables is more than its degree. For example, $\mathcal{C}(x)$ is a $\mathcal{C}^1$-field and any algebraic extension of a $\mathcal{C}^1$-field is a $\mathcal{C}^1$-field (Tsen's theorem).

Galois-Lie algebra of the differential system $[A]$. The dimension of the Lie algebra $\mathfrak{g}$, as a vector space, is the transcendence degree of a Picard-Vessiot extension. Consequently, if we are able to compute the dimension of the Galois-Lie algebra, it will help us to prove results of algebraic independence among solutions of a linear differential system. The following example illustrates this by showing how our techniques allow to prove or disprove algebraic dependence of integrals of D-finite functions.

Example 1.2. Let $A_1 := \begin{pmatrix} 0 & 1 \\ \frac{3x^2-6x+7}{144x(x-1)^2} & -\frac{2}{3x} - \frac{2}{3(x-1)} \end{pmatrix}$. A basis of solutions of the equation associated to $[A_1]$ is given by Heun functions $f_1(x), f_2(x)$. The Kovacic algorithm, see Kovacic (1986); van Hoeij and Weil (2005), shows that the differential Galois group is a finite extension of $\mathrm{SL}_2(\mathcal{C})$; so the Galois Lie algebra has dimension 3. The system $[A]$ given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3x^2-6x+7}{144(x-1)^3x^2} & -\frac{2}{3x} - \frac{2}{3(x-1)} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has fundamental solution matrix

$$\begin{pmatrix} f_1(x) & f_2(x) & 0 \\ f_1'(x) & f_2'(x) & 0 \\ \int^x f_1(t)\mathrm{d}t & \int^x f_2(t)\mathrm{d}t & 1 \end{pmatrix}.$$

One can show, for example with the techniques of this paper, that the Galois-Lie algebra of $[A]$ has dimension 5. It follows that the $\int f_i(t)dt$ are transcendental and algebraically independent over $\mathcal{C}(x)(f_1, f_2, f_1', f_2')$. However, suppose we had started from

$$A_1 = \begin{pmatrix} 0 & 1 \\ \frac{1}{36}\frac{1}{x(x-1)} & -\frac{7}{12x} - \frac{1}{6(x-1)} \end{pmatrix}.$$

Its differential Galois group is also a finite extension of the group $\mathrm{SL}_2(\mathcal{C})$. The maple implementation of van Hoeij and Weil (2005) gives us two hypergeometric solutions $f_1(x) = {}_2F_1([-1/3, 1/12], [7/12])(x)$ and $f_2(x) = x^{5/12}{}_2F_1([1/12, 1/2], [17/12])(x)$. The Galois-Lie algebra of $[A]$, with $A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{36}\frac{1}{x(x-1)} & -\frac{7}{12x} - \frac{1}{6(x-1)} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, turns out to have dimension 3 and reduction techniques applied to $[A]$ give us the relations

$$\int^x f_i(t)\,\mathrm{d}t = -\frac{9}{11}x(x-1)f_i'(x) + \frac{15}{44}(3x-1)f_i(x) + \frac{9}{11}c_i$$

satisfied by the f_i, for some constants c_i.

For a factorized reducible system $[A]$ of the form

$$A(x) = \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline S(x) & A_2(x) \end{array}\right) = A_{diag}(x) + A_{sub}(x),$$

we have a fundamental solution matrix of the form

$$U = \left(\begin{array}{c|c} U_1 & 0 \\ \hline U_2V & U_2 \end{array}\right) = \left(\begin{array}{c|c} U_1 & 0 \\ \hline 0 & U_2 \end{array}\right)\left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline V & \mathrm{Id}_{n_2} \end{array}\right).$$

Once U_1 and U_2 are known, V is given by integrals: $V' = U_2^{-1} S U_1$. Let $K_{diag} := \mathbf{k}(U_1, U_2)$ be a Picard-Vessiot extension of $\mathbf{k}$ for $[A_{diag}]$, with differential Galois group G_{diag}. Then $K := K_{diag}(V)$ is a Picard-Vessiot extension of $\mathbf{k}$ for $[A]$ with differential Galois group G. Note that K_{diag} has field of constants $\mathcal{C}$, so that we may consider the differential Galois group over K_{diag}. Letting $G_u := Gal(K/K_{diag})$ be the differential Galois group of $[A]$ over K_{diag}, we have by Galois correspondence, see Proposition 1.34 of van der Put and Singer (2003), that G_u is the set of elements of G of the form $\left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline G_{2,1} & \mathrm{Id}_{n_2} \end{array}\right)$ as the U_i are fixed. Then, $G_u \lhd G$ and

$$\overbrace{\mathbf{k} \subset K_{diag}}^{G_{diag} \simeq G/G_u} := \overbrace{\mathbf{k}(U_1, U_2) \subset K}^{G_u \lhd G} := K_{diag}(V).$$

Remark 1.3. Given $g \in G$, we have $g(U_i) = U_i.C_i$ for invertible constant matrices C_i. Then $G_{diag} \simeq \left\{ \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) \middle| \exists g \in G, g(U_i) = U_i.C_i \text{ for } i = 1, 2 \right\}$. In other words, for any matrix $\left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) \in G_{diag}$, there exists a matrix $M \in G$ with $M = \left(\begin{array}{c|c} C_1 & 0 \\ \hline C_{2,1} & C_2 \end{array}\right)$.

We define $C_g := C_2 g(V) - V C_1$. In virtue of

$$g(V') = g(U_2^{-1} S U_1) = C_2^{-1} U_2^{-1} S U_1 C_1 = C_2^{-1} V' C_1,$$

we find that C_g is a constant matrix. Since $g(V) = C_2^{-1} V C_1 + C_2^{-1} C_g$, we see that $g(U_2 V) = U_2 V C_1 + U_2 C_g$ and

$$g(U) = \left(\begin{array}{c|c} U_1 & 0 \\ \hline U_2 V & U_2 \end{array}\right) \cdot \left(\begin{array}{c|c} C_1 & 0 \\ \hline C_g & C_2 \end{array}\right) = U \cdot \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) \left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline C_2^{-1} \cdot C_g & \mathrm{Id}_{n_2} \end{array}\right).$$

Last, we recall a useful lemma to switch from group to Lie algebra in specific cases:

Lemma 1.4. *Let $\mathfrak{n} \subset \mathcal{M}_n(\mathcal{C})$ be a $\mathcal{C}$-vector space of lower triangular matrices with zero entries on the diagonal. Assume that, for all $N, N' \in \mathfrak{n}$, $N \cdot N' = (0)$. Then, $U := \left\{ \mathrm{Id}_n + N, N \in \mathfrak{n} \right\}$ is a connected algebraic group and $\mathfrak{n}$ is its Lie algebra.*

Furthermore, we have two bijective maps which are inverses of each other

$$\begin{array}{cccc} \exp: & \mathfrak{n} & \longrightarrow & U \\ & N & \mapsto & \mathrm{Id}_n + N \\ \log: & U & \longrightarrow & \mathfrak{n} \\ & \mathrm{Id}_n + N & \mapsto & N. \end{array}$$

Proof. The algebraic group U is abelian thanks to the assumption; the fact that $\mathfrak{n}$ is its Lie algebra is easily derived from the definition. Let $N \in \mathfrak{n}$. We have $\exp(N) = \mathrm{Id}_n + N$ because, by assumption, $N^2 = 0$. The same argument shows that $\log(\mathrm{Id}_n + N) = N$. It follows that exp and log are bijective on the required sets and are inverses of each other. This also proves the connectedness of U. □

1.3. Reduced forms of linear differential systems

Let $A(x) \in \mathcal{M}_n(\mathbf{k})$, G be the differential Galois group of $[A]: Y'(x) = A(x)Y(x)$ and $\mathfrak{g}$ its Lie algebra. As usual, the notation $\mathfrak{g}(\mathbf{k})$ stands for the extension of scalars $\mathfrak{g}(\mathbf{k}) = \mathfrak{g} \otimes_{\mathcal{C}} \mathbf{k}$. Let $\overline{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$.

Definition 1.5. Let us consider $A(x), B(x) \in \mathcal{M}_n(\mathbf{k})$. The two linear differential systems $[A]: Y'(x) = A(x)Y(x)$ and $[B]: Z'(x) = B(x)Z(x)$ are called *equivalent over* $\mathbf{k}$ (or *gauge equivalent* over $\mathbf{k}$) when there exists $P(x) \in \mathrm{GL}_n(\mathbf{k})$ such that

$$B(x) = P^{-1}(x)A(x)P(x) - P^{-1}(x)P'(x).$$

The notation is $B = P[A]$ and P is called a *gauge transformation matrix.*

Solutions of $[A]$ and $[B]$ are then linked by the relation $Y(x) = P(x)Z(x)$.

Definition 1.6. Let $A(x) \in \mathcal{M}_n(\mathbf{k})$. We say that the system $[A]: Y'(x) = A(x)Y(x)$ is *in reduced form* (or in *Kolchin-Kovacic reduced form*) when $A(x) \in \mathfrak{g}(\mathbf{k})$.

Otherwise, we say that a matrix $B(x) \in \mathcal{M}_n\left(\overline{\mathbf{k}}\right)$ (resp. a system $[B]$) is a *reduced form of* $[A]$ when there exists $P(x) \in \mathrm{GL}_n(\overline{\mathbf{k}})$ such that $B(x) = P(x)[A(x)]$ and $B(x)$ is in reduced form, i.e. $B(x) \in \mathfrak{g}(\overline{\mathbf{k}})$.

The existence and relevance of reduced forms are given by the following Kolchin-Kovacic reduction result. A proof can be found in van der Put and Singer (2003), Proposition 1.31 and Corollary 1.32. See also Blázquez-Sanz and Morales-Ruiz (2010), Theorem 5.8, and Aparicio-Monforte et al. (2013), § 5.3 after Remark 31.

Proposition 1.7 *(Kolchin-Kovacic reduction theorem). Let $A(x) \in \mathcal{M}_n(\mathbf{k})$. Let G be the differential Galois group of the differential system $[A]: Y'(x) = A(x)Y(x)$ and $\mathfrak{g}$ be the Lie algebra of G. Let $H \subset \mathrm{GL}_n(\mathcal{C})$ be a connected linear algebraic group, with Lie algebra $\mathfrak{h}$, such that $A(x) \in \mathfrak{h}(\mathbf{k})$.*

(1) *The Galois group G is contained in (a conjugate of) H.*
(2) *There exists a gauge transformation $P(x) \in H(\overline{\mathbf{k}})$ such that $P(x)[A(x)] \in \mathfrak{g}(\overline{\mathbf{k}})$. If we further assume that G is connected and that $\mathbf{k}$ is a $\mathcal{C}^1$-field, then there exists a gauge transformation $P(x) \in H(\mathbf{k})$ such that $P(x)[A(x)] \in \mathfrak{g}(\mathbf{k})$.*

We now construct a Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}$ is the Lie algebra of some algebraic group H, $A(x) \in \mathfrak{h}(\mathbf{k})$ and $\mathfrak{h}$ has minimal dimension for that property.

Following Wei and Norman (1963); Aparicio-Monforte et al. (2013), a *Wei-Norman decomposition* of $A(x)$ is a finite sum of the form

$$A(x) = \sum a_i(x)M_i,$$

where the matrices M_i have coefficients in $\mathcal{C}$ and the $a_i(x) \in \mathbf{k}$ form a basis of the $\mathcal{C}$-vector space spanned by the entries of $A(x)$. The M_i depend on the choice of $a_i(x)$ but the $\mathcal{C}$-vector space generated by the M_i is independent of the choice of the $a_i(x)$. This shows that the notation $\mathrm{Lie}(A)$ below does not depend upon the choice of the Wei-Norman decomposition and is well defined.

Recall that a Lie algebra is called an *algebraic Lie algebra* when it is the Lie algebra of an algebraic group.

Definition 1.8. For a matrix $A(x) \in \mathcal{M}_n(\mathbf{k})$, let constant matrices M_i denote the generators of a Wei-Norman decomposition of $A(x)$. We define $\mathrm{Lie}(A)$, called *the Lie algebra associated to* A, as the smallest algebraic Lie algebra which contains all matrices M_i, i.e. the algebraic envelope of the Lie algebra generated by the M_i.

The link between $\mathfrak{g}$ and $\mathrm{Lie}(A)$ is made in the following remark.

Remark 1.9. By Proposition 1.7, $\mathfrak{g} \subset \mathrm{Lie}(A)$. We see that the system $Y'(x) = A(x)Y(x)$ is in reduced form if and only if $\mathfrak{g} = \mathrm{Lie}(A)$.

The approach that we elaborate in this paper was initiated in Aparicio-Monforte and Weil (2011); Aparicio-Monforte et al. (2016) and Casale and Weil (2018). It is based on a criterion for reduced forms, which is given in the following lemma.

Lemma 1.10 *(Aparicio-Monforte et al. (2016), Lemma 1.3). Given $A(x) \in \mathcal{M}_n(\mathbf{k})$, let G be the differential Galois group of the system $[A] : Y'(x) = A(x)Y(x)$ and $\mathfrak{g}$ be its Lie algebra. Let H be the connected linear algebraic group whose Lie algebra is $\mathrm{Lie}(A)$. Assume that G is connected. The system $[A]$ is in reduced form, i.e. $G = H$ and $\mathfrak{g} = \mathrm{Lie}(A)$, if and only if, for all gauge transformation matrices $P(x)$ in $H(\mathbf{k})$, we have $\mathrm{Lie}(A) = \mathrm{Lie}(P[A])$.*

2. Decomposition and flags for the off-diagonal part

Let us consider a matrix

$$A(x) := \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline S(x) & A_2(x) \end{array}\right) \in \mathcal{M}_n(\mathbf{k})$$

where $A_i(x)$ are square matrices in $\mathcal{M}_{n_i}(\mathbf{k})$. We have $A(x) = A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x)$, where

$$A_{\mathrm{diag}}(x) := \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline 0 & A_2(x) \end{array}\right) \quad \text{and} \quad A_{\mathrm{sub}}(x) := \left(\begin{array}{c|c} 0 & 0 \\ \hline S(x) & 0 \end{array}\right).$$

Let us assume that $Y'(x) = A_{\mathrm{diag}}(x)Y(x)$ is in reduced form. The aim of the paper is to show how to then put the full system $Y'(x) = A(x)Y(x)$ in reduced form, see §4. We are going to see in §5 that solving this problem will give us a complete algorithm to put a general system into reduced form.

2.1. *The off-diagonal algebra $\mathfrak{gl}_{\mathrm{sub}}$*

Let $\mathfrak{g}_{\mathrm{diag}} := \mathrm{Lie}(A_{\mathrm{diag}})$ be the Lie algebra associated to $A_{\mathrm{diag}}(x)$. Let $\mathfrak{gl}_{\mathrm{sub}} := \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right), C \in \mathcal{M}_{n_2 \times n_1}(\mathcal{C}) \right\}$ denote the space of off-diagonal constant matrices.

We now list some useful simple properties of $\mathfrak{gl}_{\mathrm{sub}}$.

Lemma 2.1. *Let us consider a block-diagonal matrix $M = \left(\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array}\right) \in \mathfrak{g}_{\mathrm{diag}}$ and off-diagonal matrices $B_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline C_1 & 0 \end{array}\right)$ and $B_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline C_2 & 0 \end{array}\right) \in \mathfrak{gl}_{\mathrm{sub}}$.*

(1) *$B_1.B_2 = (0)$ and $\mathfrak{gl}_{\mathrm{sub}}$ is an abelian Lie algebra.*
(2) *$M.B_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline N_2C_1 & 0 \end{array}\right) \in \mathfrak{gl}_{\mathrm{sub}}$ and $B_1.M = \left(\begin{array}{c|c} 0 & 0 \\ \hline C_1N_1 & 0 \end{array}\right) \in \mathfrak{gl}_{\mathrm{sub}}$.*
(3) *$[M, B_1] = \left(\begin{array}{c|c} 0 & 0 \\ \hline N_2C_1 - C_1N_1 & 0 \end{array}\right) \in \mathfrak{gl}_{\mathrm{sub}}$.*

Proof. This is a simple calculation. □

As a consequence of the third point we obtain the following lemma.

Lemma 2.2. *The Lie algebra $\mathfrak{gl}_{\mathrm{sub}}$ is stable under $[\mathfrak{g}_{\mathrm{diag}}, \bullet]$.*

Remark 2.3. Lemma 1.4 applied to $\mathfrak{gl}_{\mathrm{sub}}$ gives that $\left\{\mathrm{Id}_n + B, B \in \mathfrak{gl}_{\mathrm{sub}}\right\}$ is a connected algebraic group with Lie algebra $\mathfrak{gl}_{\mathrm{sub}}$. More generally, given a vector subspace W of $\mathfrak{gl}_{\mathrm{sub}}$, for every $M, N \in W$, we have $MN \in (0)$. Then W is an algebraic abelian Lie algebra with additive abelian group $\{\mathrm{Id}_n + B, B \in W\}$.

2.2. The shape of the reduction matrix

The aim of this subsection is to generalize Theorem 3.3 of Aparicio-Monforte et al. (2016) to our context. As above, we consider a system $[A]$ given by

$$A(x) = A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x),$$

with

$$A_{\mathrm{diag}}(x) := \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline 0 & A_2(x) \end{array}\right) \text{ and } A_{\mathrm{sub}}(x) := \left(\begin{array}{c|c} 0 & 0 \\ \hline S(x) & 0 \end{array}\right).$$

We assume in the sequel that $[A_{\mathrm{diag}}]$ *is in reduced form.*

Theorem 2.4. *There exists a gauge transformation*

$$P(x) \in \left\{\mathrm{Id}_n + B(x), B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\},$$

such that $Y'(x) = P(x)[A(x)]Y(x)$ is in reduced form.

We first prove the following auxiliary lemma. Let G be the differential Galois group of $Y'(x) = A(x)Y(x)$ and $\mathfrak{g}$ be the Lie algebra of G. Let H be the connected algebraic group with Lie algebra $\mathrm{Lie}(A)$.

Lemma 2.5. *The differential Galois group G is connected.*

Proof of Lemma 2.5. The elements of G are of the form $\left(\begin{array}{c|c} G_1 & 0 \\ \hline G_{2,1} & G_2 \end{array}\right) \in \mathrm{GL}_n(\mathcal{C})$. Let G_u be the subgroup of elements of G of the form $\left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline G_{2,1} & \mathrm{Id}_{n_2} \end{array}\right)$. As we have seen in §1.2, G_u is a normal subgroup of G and $G \simeq G_u \rtimes G_{\mathrm{diag}}$, where G_{diag} denotes the differential Galois group of $Y'(x) = A_{\mathrm{diag}}(x)Y(x)$. Since the system $Y'(x) = A_{\mathrm{diag}}(x)Y(x)$ is in reduced form, we find that G_{diag} is connected, see Lemma 32 in Aparicio-Monforte et al. (2013). Now G_u is a vector group and hence it is a connected linear algebraic group as well. $\square$

Proof of Theorem 2.4. The differential Galois group G satisfies the inclusion $G \subset H$ (because of the first point of Proposition 1.7). Lemma 2.5 shows that G is connected. So we may use the second point of Proposition 1.7 to obtain the existence of $Q(x) := \left(\begin{array}{c|c} D_1(x) & 0 \\ \hline S_Q(x) & D_2(x) \end{array}\right) \in H(\mathbf{k})$ such that the linear differential system $Q(x)[A(x)]$ is in reduced form.

Since $[A_{\mathrm{diag}}]$ is in reduced form, Remark 1.9 implies that G_{diag}, the differential Galois group of $[A_{\mathrm{diag}}]$, admits $\mathfrak{g}_{\mathrm{diag}}$ as Lie algebra. By construction, $\mathrm{Lie}(A)$ is included in the smallest algebraic Lie algebra containing $\mathfrak{g}_{\mathrm{diag}} \oplus \mathfrak{gl}_{\mathrm{sub}}$. As a consequence of Lemma 2.1, we deduce that $\mathfrak{g}_{\mathrm{diag}} \oplus \mathfrak{gl}_{\mathrm{sub}}$ is a Lie algebra. It is even an algebraic Lie algebra whose algebraic group is $G_{\mathrm{diag}} \times \left\{\mathrm{Id}_n + B, B \in \mathfrak{gl}_{\mathrm{sub}}\right\}$, proving that $\mathrm{Lie}(A) \subset \mathfrak{g}_{\mathrm{diag}} \oplus \mathfrak{gl}_{\mathrm{sub}}$. As $Q(x) \in H(\mathbf{k})$, we have $\left(\begin{array}{c|c} D_1(x) & 0 \\ \hline 0 & D_2(x) \end{array}\right) \in G_{\mathrm{diag}}(\mathbf{k})$. Now, as $G_{diag} \simeq G/G_u$, Remark 1.3 shows that $G(\mathbf{k})$ contains a block-triangular matrix of the form $R(x) := \left(\begin{array}{c|c} D_1(x) & 0 \\ \hline T_{2,1}(x) & D_2(x) \end{array}\right)$. Then

$$R(x)^{-1} = \left(\begin{array}{c|c} D_1^{-1}(x) & 0 \\ \hline -D_2^{-1}T_{2,1}D_1^{-1}(x) & D_2^{-1}(x) \end{array}\right) \in G(\mathbf{k}).$$

By hypothesis, $Lie(Q[A]) = \mathfrak{g}$ (as $[Q[A]]$ is in reduced form). Now, a gauge transformation of an element $Q[A] \in \mathfrak{g}(\mathbf{k})$ by an element of $G(\mathbf{k})$ transforms $Q[A]$ into another element of $\mathfrak{g}(\mathbf{k})$, see (Mitschi and Singer, 2002, Proposition 5.1). Then, $[R^{-1}[Q[A]]]$ is in reduced form. A fundamental solution of $[Q[A]]$ is given by $Q^{-1}U$, where U is a fundamental solution of $[A]$. Therefore, $[R^{-1}[Q[A]]]$ has a fundamental solution $RQ^{-1}U = (QR^{-1})^{-1}U$, and we find that $R^{-1}[Q[A]] = (QR^{-1})[A]$. It follows that $[(QR^{-1})[A]]$ is in reduced form. We have

$$QR^{-1} = \left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline (S_Q - T_{2,1})D_1^{-1} & \mathrm{Id}_{n_2} \end{array}\right) \in \left\{\mathrm{Id}_n + \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\}.$$

Then, QR^{-1} is the expected gauge transformation. □

The following corollary will be a key ingredient for the reduction procedure of §4.

Corollary 2.6. *Assume that, for all gauge transformations of the form* $P(x) \in \left\{\mathrm{Id}_n + B(x), B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\}$, *we have* $\mathrm{Lie}(A) \subseteq \mathrm{Lie}(P[A])$. *Then, the linear differential system* $Y'(x) = A(x)Y(x)$ *is in reduced form.*

Proof. Theorem 2.4 provides $P(x) = \mathrm{Id}_n + B(x)$ with $B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$ such that the system $Y'(x) = P(x)[A(x)]Y(x)$ is in reduced form. By assumption, we have $\mathrm{Lie}(A) \subseteq \mathrm{Lie}(P[A])$. But since $Y'(x) = P(x)[A(x)]Y(x)$ is in reduced form, we have $\mathrm{Lie}(P[A]) = \mathfrak{g}$. This shows that $\mathrm{Lie}(A) \subseteq \mathfrak{g}$. By Remark 1.9, we had $\mathfrak{g} \subseteq \mathrm{Lie}(A)$ so $\mathrm{Lie}(A) = \mathfrak{g}$, which proves the result. □

2.3. The adjoint action $\Psi = [A_{\mathrm{diag}}(x), \bullet]$

We refer to §1 and §2.1 for the notations and definitions used in this subsection. The adjoint action is $[A_{\mathrm{diag}}(x), \bullet]$ for consistence of formulas. In §2.2, we have proved the existence of a gauge transformation matrix $P(x) \in \left\{\mathrm{Id}_n + B(x), B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\}$ such that the system $Y'(x) = P(x)[A(x)]Y(x)$ is in reduced form. Let $B_1, \dots, B_\sigma \in \mathcal{M}_n(\mathcal{C})$ be a basis of $\mathfrak{gl}_{\mathrm{sub}}$. The next proposition generalizes (Aparicio-Monforte et al., 2016, Proposition 3.6) in our context.

Proposition 2.7. *Let* $P(x) := \mathrm{Id}_n + \sum_{i=1}^{\sigma} f_i(x)B_i$, *with* $f_i(x) \in \mathbf{k}$. *The gauge transformation of [A] by* $P(x)$ *is*

$$P(x)[A(x)] = A_{\mathrm{diag}}(x) + \underbrace{A_{\mathrm{sub}}(x) + \sum_{i=1}^{\sigma} f_i(x)[A_{\mathrm{diag}}(x), B_i] - \sum_{i=1}^{\sigma} f_i'(x)B_i}_{\textit{off-diagonal part to be reduced}}.$$

Proof. The gauge transformation of $[A]$ by $P(x)$ is given by the formula $P(x)[A(x)]=P(x)^{-1}A(x)P(x) - P^{-1}(x)P'(x)$, see Definition 1.5. Computations are made simple by the fact that the product of two elements of $\mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$ is zero. We have the equalities $P^{-1}(x) = \mathrm{Id}_n - \sum_{i=1}^{\sigma} f_i(x)B_i$ and $P^{-1}(x)A(x) = A(x) - \sum_{i=1}^{\sigma} f_i(x)B_iA_{\mathrm{diag}}(x)$. As $A(x) = A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x)$, we find that

$$\begin{aligned} P(x)^{-1}A(x)P(x) &= \left(A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x) - \textstyle\sum_{i=1}^{\sigma} f_i(x)B_iA_{\mathrm{diag}}(x)\right)\left(\mathrm{Id}_n + \textstyle\sum_{i=1}^{\sigma} f_i(x)B_i\right) \\ &= A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x) + \textstyle\sum_{i=1}^{\sigma}\left(-f_i(x)B_iA_{\mathrm{diag}}(x) + f_i(x)A_{\mathrm{diag}}(x)B_i\right) \\ &= A_{\mathrm{diag}}(x) + A_{\mathrm{sub}}(x) + \textstyle\sum_{i=1}^{\sigma} f_i(x)[A_{\mathrm{diag}}(x), B_i]. \end{aligned}$$

Similarly, we have

$$P^{-1}(x)P'(x) = \left(\mathrm{Id}_n - \sum_{j=1}^{\sigma} f_j(x)B_j\right)\left(\sum_{i=1}^{\sigma} f_i'(x)B_i\right) = \sum_{i=1}^{\sigma} f_i'(x)B_i.$$

This yields the desired result. □

The Lie algebra $\mathfrak{gl}_{\text{sub}}$ is stable under the bracket $[\mathfrak{g}_{\text{diag}}, \bullet]$, see Lemma 2.2. This implies that the $\mathbf{k}$-linear map $\Psi := [A_{\text{diag}}(x), \bullet]$, which is the adjoint action of $\mathfrak{g}_{\text{diag}}(\mathbf{k})$ on $\mathfrak{gl}_{\text{sub}}(\mathbf{k})$, is well defined:

$$\begin{array}{cccc} \Psi: & \mathfrak{gl}_{\text{sub}}(\mathbf{k}) & \longrightarrow & \mathfrak{gl}_{\text{sub}}(\mathbf{k}) \\ & B(x) & \longmapsto & [A_{\text{diag}}(x), B(x)]. \end{array}$$

Writing

$$B(x) = \left(\begin{array}{c|c} 0 & 0 \\ \hline B_s(x) & 0 \end{array}\right),$$

the action of Ψ on B induces an action $\bar{\Psi}$ on B_s given by

$$\bar{\Psi}(B_s(x)) = A_2(x)B_s(x) - B_s(x)A_1(x).$$

Lemma 2.8. *With the above notations, the matrix Ψ of the adjoint action of $A_{\text{diag}}(x)$ on $\mathfrak{gl}_{\text{sub}}$ is*

$$\Psi = A_2 \otimes \mathrm{Id}_{\mathrm{n}_1} - \mathrm{Id}_{\mathrm{n}_2} \otimes A_1^T.$$

Proof. The row-$\mathtt{vec}$ operator transforms a matrix into a vector by transposing each row and stacking them into a vector. It is well known that, for general matrices, the row-$\mathtt{vec}$ operator satisfies $\mathtt{vec}(MX) = (M \otimes \mathrm{Id})\mathtt{vec}(X)$ and $\mathtt{vec}(XM) = (\mathrm{Id} \otimes M^T)\mathtt{vec}(X)$, see Petersen et al. (2008), Section 10.2. It follows that $\mathtt{vec}(\bar{\Psi}(B_s)) = \left(A_2 \otimes \mathrm{Id}_{\mathrm{n}_1} - \mathrm{Id}_{\mathrm{n}_2} \otimes A_1^T\right) \cdot \mathtt{vec}(B_s)$. □

Remark 2.9. In this remark, we use the language of differential modules as in van der Put and Singer (2003); Aparicio-Monforte et al. (2013). The differential system $[A_2 \otimes \mathrm{Id}_{\mathrm{n}_1} - \mathrm{Id}_{\mathrm{n}_2} \otimes A_1^T]$ corresponds to the differential module $\mathcal{M}_2 \otimes \mathcal{M}_1^{\star}$, where the connections on each $\mathcal{M}_i$ have matrices A_i respectively. Let us show that $[\Psi]$ is in reduced form. Indeed, Ψ is the matrix of the connection on $\mathcal{M}_2 \otimes \mathcal{M}_1^{\star}$ which is a submodule of a tensor construction on $\mathcal{M}_1 \oplus \mathcal{M}_2$. So any semi-invariant of $\mathcal{M}_2 \otimes \mathcal{M}_1^{\star}$ in a construction, i.e. an exponential solution of the corresponding differential system, can be extended, by adding zeroes, into a semi-invariant of $\mathcal{M}_1 \oplus \mathcal{M}_2$. Now, the matrix of the connection on $\mathcal{M}_1 \oplus \mathcal{M}_2$ is A_{diag} which is in reduced form. So, Theorem 1 of Aparicio-Monforte et al. (2013) shows that any semi-invariant of $\mathcal{M}_1 \oplus \mathcal{M}_2$ has constant coefficients. It follows that any semi-invariant of $\mathcal{M}_2 \otimes \mathcal{M}_1^{\star}$ will then have constant coefficients. So, Theorem 1 of Aparicio-Monforte et al. (2013) implies that $[\Psi]$ is in reduced form.

2.4. Decomposition of $\mathfrak{gl}_{\text{sub}}$ into Ψ-spaces

2.4.1. Isotypical decomposition of $\mathfrak{gl}_{\text{sub}}$ into Ψ-spaces

Proposition 2.7 shows that reduction will be essentially governed by the adjoint map Ψ. We had the Wei-Norman decomposition $A_{\text{diag}}(x) = \sum_{i=1}^{\delta} g_i(x)M_i$, where the $g_i(x) \in \mathbf{k}$ were $\mathcal{C}$-linearly independent. For each $i \in \{1, \dots, \delta\}$, we define the $\mathcal{C}$-linear map $\Psi_i : \begin{array}{ccc} \mathfrak{gl}_{\text{sub}} & \longrightarrow & \mathfrak{gl}_{\text{sub}} \\ B & \longmapsto & [M_i, B] \end{array}$ so that $\Psi = \sum_{i=1}^{\delta} g_i(x)\Psi_i$.

Definition 2.10. Consider a vector space $W \subset \mathfrak{gl}_{\text{sub}}$. We say that W is a Ψ*-space* when $\Psi(W) \subset W \otimes_{\mathcal{C}} \mathbf{k}$. Let $\mathcal{R} := \mathcal{C}[\Psi_1, \dots, \Psi_\delta]$ denote the $\mathcal{C}$-algebra generated by the Ψ_i. We say that W is an $\mathcal{R}$*-module* (or submodule of $\mathfrak{gl}_{\text{sub}}$) when, for all $i \in \{1, \dots, \delta\}$, $\Psi_i(W) \subset W$.

Note that, by Remark 2.3, any vector subspace of $\mathfrak{gl}_{\text{sub}}$ is an algebraic Lie algebra so any Ψ-space is an algebraic Lie algebra. The map Ψ acts naturally on the Lie algebra $\mathfrak{g}$; its action on the off-diagonal matrices of $\mathfrak{g}$ will govern our reduction strategy, as suggested by the following lemma.

Lemma 2.11. *Let $\mathfrak{g}_{sub} := \mathfrak{gl}_{\text{sub}} \bigcap \mathfrak{g}$ denote the subspace of off-diagonal matrices of $\mathfrak{g}$. Then $\mathfrak{g}_{sub}$ is a Ψ-space.*

Proof. Let $\left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) \in \mathfrak{g}_{\text{diag}}$; as noted in Remark 1.3, there exists a corresponding element $\left(\begin{array}{c|c} C_1 & 0 \\ \hline C & C_2 \end{array}\right) \in \mathfrak{g}$. Let $\left(\begin{array}{c|c} 0 & 0 \\ \hline C' & 0 \end{array}\right) \in \mathfrak{g}_{sub}$. We have

$$\left[\left(\begin{array}{c|c} C_1 & 0 \\ \hline C & C_2 \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ \hline C' & 0 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & 0 \\ \hline C_2C' - C'C_1 & 0 \end{array}\right) \in \mathfrak{g}_{sub}$$

and so $\mathfrak{g}_{\text{sub}}$ is stable under the bracket with $\mathfrak{g}_{\text{diag}}$. Since $[A_{\text{diag}}]$ is in reduced form, the M_i of its Wei-Norman decomposition are in $\mathfrak{g}_{\text{diag}}$ so that $\mathfrak{g}_{\text{sub}}$ is stable under the bracket with each M_i. The result follows with $\Psi = \sum_{i=1}^{\delta} g_i(x)\Psi_i$. $\square$

In this section, we describe the structure of the Ψ-subspaces of $\mathfrak{gl}_{\text{sub}}$ and how to compute them in order to be able to reduce the off-diagonal part of the Lie algebra of the system.

Lemma 2.12. *A vector subspace W of $\mathfrak{gl}_{\text{sub}}$ is a Ψ-space if and only if for all $i \in \{1, \dots, \delta\}$, $\Psi_i(W) \subset W$, i.e. if and only if W is an $\mathcal{R}$-module.*

Proof. By construction, we have $\Psi = \sum_{i=1}^{\delta} g_i(x)\Psi_i$. If for all $i \in \{1, \dots, \delta\}$, $\Psi_i(W) \subset W$, it is clear that $\Psi(W) \subset W \otimes_{\mathcal{C}} \mathbf{k}$. Conversely, let us assume that W is a Ψ-space. Let $E_1, \dots, E_\kappa$ be a basis of W; we complete it into a basis $E_1, \dots, E_\sigma$ of $\mathfrak{gl}_{\text{sub}}$. We recall that the Ψ_i are $\mathcal{C}$-linear maps. For $i \in \{1, \dots, \delta\}$ and $u \in W$, let $c_{i,j,u} \in \mathcal{C}$ such that $\Psi_i(u) = \sum_{j=1}^{\sigma} c_{i,j,u} E_j$. Since $\Psi = \sum_{i=1}^{\delta} g_i(x)\Psi_i$, we find that $\Psi(u) = \sum_{i=1}^{\delta} \sum_{j=1}^{\sigma} g_i(x) c_{i,j,u} E_j$. For $u \in W$, the fact that $\Psi(u) \in W \otimes_{\mathcal{C}} \mathbf{k}$ implies that we have, for all $j \in \{\kappa+1, \dots, \sigma\}$, $\sum_{i=1}^{\delta} g_i(x) c_{i,j,u} = 0$. But the $g_i(x) \in \mathbf{k}$ are $\mathcal{C}$-linearly independent so, for all $i \in \{1, \dots, \delta\}$ and all $j \in \{\kappa+1, \dots, \sigma\}$, we have $c_{i,j,u} = 0$. This proves that, for all $i \in \{1, \dots, \delta\}$, $\Psi_i(u) \in W$ and hence $\Psi_i(W) \subset W$. $\square$

Lemma 2.12, applied with $W = \mathfrak{gl}_{\text{sub}}$, tells us that for all $i \in \{1, \dots, \delta\}$, $\Psi_i(\mathfrak{gl}_{\text{sub}}) \subset \mathfrak{gl}_{\text{sub}}$. So the abelian Lie-algebra $\mathfrak{gl}_{\text{sub}}$ is endowed with a natural structure of $\mathcal{R}$-module.

An $\mathcal{R}$-module W is called *decomposable* if it admits two proper $\mathcal{R}$-submodules W_1 and W_2 such that $W = W_1 \oplus W_2$; it is *indecomposable* otherwise. A morphism $\phi : W_1 \to W_2$ is a morphism of $\mathcal{R}$-modules if each W_i is an $\mathcal{R}$-module and, for all $M \in \mathcal{R}$ and $N \in W_1$, $\phi(M.N) = M.\phi(N)$. We write $W_1 \simeq_{\mathcal{R}} W_2$ when the W_i are isomorphic $\mathcal{R}$-modules.

Proposition 2.13 *(Lam (2001), Corollary 19.22, page 288). The $\mathcal{R}$-module $\mathfrak{gl}_{\text{sub}}$ admits a decomposition $\mathfrak{gl}_{\text{sub}} = \bigoplus_{i=1}^{\kappa} W_i$, such that:*

- *Each W_i is an $\mathcal{R}$-module.*
- *Each W_i admits a decomposition $W_i = \bigoplus_{j=1}^{\nu_i} V_{i,j}$, where the $(V_{i,j})_{1 \le j \le \nu_i}$ are pairwise isomorphic indecomposable $\mathcal{R}$-modules.*
- *For $i_1 \neq i_2$, all non zero indecomposable $\mathcal{R}$-modules $V_{i_1,s} \subset W_{i_1}$ and $V_{i_2,t} \subset W_{i_2}$ are non-isomorphic $\mathcal{R}$-modules.*

Moreover, this decomposition is unique up to $\mathcal{R}$-module isomorphisms.

Remark 2.14. With a standard slight abuse of notations, we may write $W_i \simeq_{\mathcal{R}} \nu_i V_i$, for some indecomposable $\mathcal{R}$-module V_i. The numbers κ and ν_i, as well as the $\mathcal{R}$-module isomorphism class of V_i are uniquely determined in the isotypical decomposition.

Definition 2.15. The decomposition $\mathfrak{gl}_{\text{sub}} = \bigoplus_{i=1}^{\kappa} W_i$ in Proposition 2.13 is called the *isotypical decomposition* of the $\mathcal{R}$-module $\mathfrak{gl}_{\text{sub}}$.

A maximal direct sum of pairwise isomorphic indecomposable $\mathcal{R}$-modules, i.e. one of the spaces W_i, is called an *isotypical block* in the isotypical decomposition of $\mathfrak{gl}_{\text{sub}}$.

Computing an isotypical decomposition is classically achieved by studying the *eigenring*

$$\mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}}) := \{M \in \mathrm{End}_{\mathcal{C}}(\mathfrak{gl}_{\text{sub}}) \,|\, \forall i \in \{1, \dots, \delta\},\ M.\Psi_i = \Psi_i.M\},$$

where $\mathrm{End}_{\mathcal{C}}(\mathfrak{gl}_{\text{sub}})$ is the algebra of $\mathcal{C}$-linear endomorphisms of $\mathfrak{gl}_{\text{sub}}$. We review in the next paragraph how to use $\mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}})$ to compute a decomposition. This will follow, for example, from Fitting's Lemma in Lam (2001), Lemma 19.16, page 285, or Barkatou (2007), where the process is described in the context of Ore-modules and Barkatou et al. (2005), where the case of several matrices is addressed. The following known algorithm (Barkatou, 2007) computes the isotypical decomposition of $\mathfrak{gl}_{\text{sub}}$.

Input: the list of matrices Ψ_i.
Output: isotypical decomposition of $\mathfrak{gl}_{\text{sub}}$.

(1) Pick a matrix M with indeterminate coefficients. Solving the linear conditions $M.\Psi_i = \Psi_i.M$ gives a basis of the eigenring $\mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}})$.
(2) Pick a "sufficiently general" element $P \in \mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}})$ (see Barkatou, 2007).
(3) Factor its characteristic polynomial as $\chi_P(\lambda) = \prod_i \chi_i(\lambda)^{m_i}$.
(4) For each factor, compute a basis of the generalized eigenspaces $\ker\left(\chi_i(P)^{m_i}\right)$.

Return: A matrix T whose columns are bases of the $\ker\left(\chi_i(P)^{m_i}\right)$.

Note that the invariant subspaces of P are the $\mathcal{R}$-submodules of $\mathfrak{gl}_{\text{sub}}$ and $T^{-1}\Psi T$ is in block diagonal form, where each block is an indecomposable $\mathcal{R}$-module. This process is described in detail in Barkatou (2007), see also van der Put and Singer (2003), Proposition 2.40 or Barkatou et al. (2019). Indeed, as the differential system $[\Psi]$ is in reduced form, see Remark 2.9, Theorem 1 of Aparicio-Monforte et al. (2013) shows that $\mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}})$ is the eigenring of $[\Psi]$ (in the usual sense) and Barkatou (2007) applies mutatis mutandis. In Barkatou et al. (2019), refinements are given on how to use the eigenring structure to compute the isomorphism classes inside each isotypical block; this will be used in problem **P2** below.

Remark 2.16. It would be tempting to use the factors of Π_Ψ, the minimal polynomial of Ψ, to compute an isotypical decomposition. However, this would be the source of mistakes, as the characteristic spaces of Ψ are defined over $\mathbf{k}$, not over $\mathcal{C}$. There are examples, see §2.5, where $\mathfrak{gl}_{\text{sub}}$ is indecomposable while Π_Ψ is the product of several coprime polynomials.

2.4.2. The flag decomposition of an indecomposable Ψ-space

Let U be a Ψ-space. We say that U is an *irreducible* Ψ-space if its only Ψ-subspaces are $\{0\}$ and U. Schur's lemma shows that any automorphism of an irreducible Ψ-space is a scalar multiple of the identity. This is generalized in the following lemma, which is sometimes known as Goursat's lemma.

Lemma 2.17 *(Goursat's lemma, Compoint and Singer (1998), Lemma 2.2). Let $V := U_1 \oplus \cdots \oplus U_\nu$ where $U_1, \dots, U_\nu$ denote pairwise isomorphic irreducible Ψ-spaces. Let $\widetilde{\phi}_j : U_1 \to U_j$ be an isomorphism from U_1 to U_j. Let W be an irreducible Ψ-subspace of V. Then, there exist $c_1, \dots c_\nu \in \mathcal{C}$ such that $W = U_{\underline{c}}$, where*

$U_{\underline{c}} := \{\sum_{j=1}^{\nu} c_j \widetilde{\phi}_j(u), u \in U_1\}$. *Any* Ψ*-subspace of such a* V *is a direct sum of modules* $U_{\underline{c}}$ *as described in the lemma.*

We now construct a special flag for an isotypical bloc, called a Ψ-*isotypical flag*, adapted to our reduction process. We are going to proceed in two steps. First, we construct such a flag in the case of an indecomposable Ψ-space V. Let U_1 be an irreducible Ψ-subspace of V. There may be other subspaces which are Ψ-isomorphic to U_1; let n_1 be the maximal number of Ψ-subspaces of V whose direct sum is a subspace of V and which are all isomorphic to U_1. We define $V^{[1]}$ as this direct sum, so that $V^{[1]} \simeq n_1 U_1 \subset V$. If $V^{[1]} = V$ we are done. Otherwise, we look at the quotient $V/V^{[1]}$ and apply the same construction: we obtain a subspace $\tilde{W}_2 := n_2 U_2$ of $V/V^{[1]}$. Note that if $E_1, \dots, E_n$ denotes a basis of V such that $E_1, \dots, E_k$ is a basis of $V^{[1]}$, then $V/V^{[1]}$ is isomorphic, as a vector space, to $\mathrm{Vect}_{\mathcal{C}}(E_{k+1}, \dots, E_n) \subset V$. Then, we may lift $\tilde{W}_2$ to a Ψ-subspace $V^{[2]}$ of V which contains $V^{[1]}$ and such that $V^{[2]}/V^{[1]} = \tilde{W}_2$. We iterate and the result of this construction is what we call a Ψ-*isotypical flag*:

$$V = V^{[\mu]} \supsetneq V^{[\mu-1]} \supsetneq \cdots \supsetneq V^{[1]} \supsetneq V^{[0]} = \{0\},$$

such that each $V^{[j]}/V^{[j-1]}$ is a direct sum of pairwise isomorphic irreducible Ψ-spaces.

We now define a Ψ-isotypical flag for an isotypical block W among the W_i given by the isotypical decomposition $\mathfrak{gl}_{\mathrm{sub}} = \bigoplus_{i=1}^{\kappa} W_i$ of Proposition 2.13. Let W denote an isotypical block in the isotypical decomposition of $\mathfrak{gl}_{\mathrm{sub}}$. We have a decomposition $W = \bigoplus_{j=1}^{\nu} V_j$, where the $(V_j)_{1\le j\le \nu}$ are pairwise isomorphic indecomposable $\mathcal{R}$-modules. We first construct, as above, a Ψ-isotypical flag for V_1:

$$V_1 = V_1^{[\mu_1]} \supsetneq V_1^{[\mu_1-1]} \supsetneq \cdots \supsetneq V_1^{[1]} \supsetneq V_1^{[0]} = \{0\},$$

such that each $V_1^{[k]}/V_1^{[k-1]}$ is a direct sum of several pairwise isomorphic irreducible Ψ-spaces. Now let $\phi_j : V_1 \to V_j$ denote an $\mathcal{R}$-module isomorphism. We set $V_j^{[k]} := \phi_j(V_1^{[k]})$ for all k and this defines a Ψ-isotypical flag for V_j. Now, we define $W^{[k]}$ by $W^{[k]} := \bigoplus_{j=1}^{\nu} V_j^{[k]}$.

Definition 2.18. Let W denote an isotypical block in the isotypical decomposition of $\mathfrak{gl}_{\mathrm{sub}}$. The flag

$$W = W^{[\mu]} \supsetneq W^{[\mu-1]} \supsetneq \cdots \supsetneq W^{[1]} \supsetneq W^{[0]} = \{0\},$$

constructed above is called an *isotypical flag* (or Ψ-isotypical flag) for W.

We now discuss how to compute such a Ψ-isotypical flag. We have explained how to compute the isotypical decomposition so we start by computing the Ψ-isotypical flag of an indecomposable Ψ-space. We thus need to be able to solve the following two problems:

P1: Given a Ψ-space V, find an irreducible Ψ-subspace $U \subseteq V$.
P2: If $U \subset V$ is an irreducible Ψ-subspace, determine the maximal $n \in \mathbb{N}^*$ such that V contains a direct sum of n subspaces Ψ-isomorphic to U, i.e. $nU \subseteq V$.

In order to address Problem **P1**, we will use the following notion.

Definition 2.19. Given a subspace $\mathcal{W}$ of $\mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$ stable under Ψ, its *associated* Ψ*-space* is the smallest Ψ-subspace W of $\mathfrak{gl}_{\mathrm{sub}}$ such that $\mathcal{W} \subset W \otimes_{\mathcal{C}} \mathbf{k}$.

Computation of the Ψ-space W associated to $\mathcal{W}$ can be achieved as follows.

Input: $\mathcal{W}$, a subspace of $\mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$.
Output: the associated Ψ-space W.

(1) For each element of a basis of $\mathcal{W}$, compute its Wei-Norman decomposition.
(2) Compute a basis $\mathcal{B}$ of the orbits under $\mathcal{R}$ of all elements of these Wei-Norman decompositions.
(3) The Ψ-space W is the vector space generated by $\mathcal{B}$.

Let V denote a Ψ-subspace of $\mathfrak{gl}_{\text{sub}}$. We now show how to find an irreducible Ψ-subspace $U \subseteq V$.

We apply the eigenring method for the isotypical decomposition as above; we let U be an indecomposable subspace in the decomposition (or $U = V$ if V is indecomposable). We identify Ψ with its restriction to $U \otimes_{\mathcal{C}} \mathbf{k}$ in this paragraph. Let $\chi_\Psi(\lambda)$ denote its characteristic polynomial. If $\tilde{U}$ is an irreducible Ψ-subspace of U, then the characteristic polynomial of the restriction of Ψ to $\tilde{U} \otimes_{\mathcal{C}} \mathbf{k}$ divides $\chi_\Psi(\lambda)$. We compute a factorization $\chi_\Psi(\lambda) = f_1(\lambda)^{m_1} \cdots f_d(\lambda)^{m_d}$ where the f_i are pairwise coprime irreducible polynomials over $\mathbf{k}$. For each i, we compute $E_i := \ker(f_i(\Psi))$; then we compute the Ψ-space[3] $W_i \subset \mathfrak{gl}_{\text{sub}}$ associated to E_i. If all W_i are equal to U then U is an irreducible Ψ-space and we return U. Otherwise, pick a W_i of minimal dimension and repeat the above steps with W_i in place of U (eigenring, generalized eigenspaces, Ψ-space).

The dimension decreases strictly at each step so the process terminates and produces an irreducible Ψ-subspace $U \subseteq V$.

Input: V, a Ψ-subspace of $\mathfrak{gl}_{\text{sub}}$.
Output: an irreducible Ψ-space $U \subset V$.

(1) Compute an isotypical decomposition. Let U denote one of the indecomposable subspaces.
(2) Factor the characteristic polynomial of Ψ on $U \otimes_{\mathcal{C}} \mathbf{k}$: $\chi_\Psi(\lambda) = f_1(\lambda)^{m_1} \cdots f_d(\lambda)^{m_d}$ where the f_i are coprime irreducible polynomials.
(3) Compute the $E_i := \ker(f_i(\Psi))$ and the associated Ψ-space W_i.
(4) If all W_i are equal to U, then U is irreducible: return U. Otherwise, apply recursively this procedure to a W_i of minimal dimension.

Now let us consider Problem **P2**. Assume that we have found an irreducible Ψ-subspace U of V. Finding $\mathcal{R}$-submodules of V which are Ψ-isomorphic to U amounts to finding elements in $\mathrm{Hom}_{\mathcal{R}}(U, V)$. Let s be the dimension of U and d be the dimension of V. We have $s \leq d$. The matrix of Ψ restricted to V is of the form

$$\Psi|_V = \left(\begin{array}{c|c} \Phi & \star \\ \hline 0 & \star \end{array}\right),$$

where Φ is a square matrix of size s representing $\Psi|_U$. Let Ψ_j denote the constant matrices in a Wei-Norman decomposition of $\Psi|_V$. They induce matrices Φ_j that generate a Wei-Norman decomposition of Φ. Elements of $\mathrm{Hom}_{\mathcal{R}}(U, V)$ are represented by matrices $L \in \mathcal{M}_{d\times s}(\mathcal{C})$ such that, for all j, we have $\Psi_j \cdot L = L \cdot \Phi_j$. This gives a linear system of equations for the entries of L. Once a basis $L_1, \dots, L_m$ of these L is found, we let

$$P := \left(\begin{array}{c|c|c|c} L_1 & \dots & L_m & \begin{matrix} 0 \\ \mathrm{Id}_{d-sm} \end{matrix} \end{array}\right).$$

The conjugation given by $P^{-1}\Psi|_V P$ puts $\Psi|_V$ in a form where the north-west block is a direct sum of m copies of Φ.

[3] Note that E_i is a vector space over $\mathbf{k}$ whereas W_i is a vector space over $\mathcal{C}$; even though E_i is an irreducible subspace of $U \otimes_{\mathcal{C}} \mathbf{k}$, the space W_i may still be a reducible Ψ-space: see the $B3 \times B2$ Example in §2.5.3.

2.5. Examples of decomposition

In this subsection, we compute the isotypical decomposition and the flags with the desired properties in several examples. A Maple worksheet[4] with these examples may be found at Dreyfus and Weil (2020). In what follows, the $E_{i,j}$ are the elementary matrices forming the canonical basis of $\mathcal{M}_n$, i.e. $E_{i,j}$ has a 1 on the (i, j) entry and 0 elsewhere. We first focus on the isotypical decomposition; we will first expose our reduction technique on these examples as we believe that it may help the reader when we establish the theory in §4. In each of the five examples below, we compute the isotypical decomposition using only the block-diagonal part of systems which will be fully be written down in §3.

2.5.1. The "$SO_3 \times SL_2$" example

We consider a system whose diagonal part is given by

$$A_{diag}(x) := \left(\begin{array}{ccc|cc} 0 & 1 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -x & 0 \end{array}\right).$$

This matrix is the block-diagonal part of the system studied later in Section 3.1. The diagonal blocks are in the Lie algebras $\mathfrak{so}_3$ of the 3-dimensional special orthogonal group and $\mathfrak{sl}_2$ of the special linear group.

The matrix $A_{diag}(x)$ is in reduced form and its associated Lie algebra is of dimension 6, as we may see using Aparicio-Monforte et al. (2013); Barkatou et al. (2016). In this example and the following one, $(B_i)_{1 \le i \le n_1 n_2}$, denotes the canonical basis of $\mathfrak{gl}_{\text{sub}}$, i.e. in this example $B_1 := E_{4,1}$, $B_2 := E_{4,2}$, $B_3 := E_{4,3}$, $B_4 := E_{5,1}$, $B_5 := E_{5,2}$, $B_6 := E_{5,3}$.

Using this basis $\{B_1, \dots, B_6\}$ of $\mathfrak{gl}_{\text{sub}}$, we find the corresponding matrix of the adjoint action, given by

$$\Psi = \begin{pmatrix} 0 & 1 & x & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -x & 0 & 0 & 0 & 0 & 1 \\ -x & 0 & 0 & 0 & 1 & x \\ 0 & -x & 0 & -1 & 0 & 0 \\ 0 & 0 & -x & -x & 0 & 0 \end{pmatrix}.$$

The eigenring contains only the identity, which shows that $\mathfrak{gl}_{\text{sub}}$ is Ψ-indecomposable. The characteristic polynomial of Ψ has two factors.

$$\chi_\Psi(\lambda) = \left(\lambda^2 + x\right)\left(\lambda^4 + 2\lambda^2 x^2 + x^4 + 2\lambda^2 x - 2x^3 + 2\lambda^2 + 3x^2 - 2x + 1\right).$$

[4] The reader may also find a pdf version at http://www.unilim.fr/pages_perso/jacques-arthur.weil/DreyfusWeilReductionExamples.pdf.

The corresponding generalized eigenspaces (over $\mathbf{k}$) are:

$$E_1 = \left\langle \begin{pmatrix} 0\\0\\0\\0\\-x\\1 \end{pmatrix}, \begin{pmatrix} 0\\-x\\1\\0\\0\\0 \end{pmatrix} \right\rangle \text{ and } E_2 = \left\langle \begin{pmatrix} 0\\0\\0\\0\\\frac{1}{x}\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\\frac{1}{x}\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix} \right\rangle.$$

For each of them, the associated Ψ-space, see Definition 2.19, is the whole $\mathfrak{gl}_{\text{sub}}$. We conclude that $\mathfrak{gl}_{\text{sub}}$ is Ψ-irreducible. The “flag” only has one level in this case:

$$\mathfrak{gl}_{\text{sub}} = \boxed{\langle B_1, B_2, B_3, B_4, B_5, B_6 \rangle}$$

Remark 2.20. In the spirit and notations of Remark 2.9, we note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are irreducible modules so $\mathcal{M}_1^\star \otimes \mathcal{M}_2$ is a completely reducible module. As $\mathfrak{gl}_{\text{sub}}$ is an indecomposable Ψ-space, this shows that it is actually irreducible.

2.5.2. The “$SO_3 \times B_2$” example

Consider the system $[A_{diag}]$ given by:

$$A_{diag}(x) := \left(\begin{array}{ccc|cc} 0 & 1 & x & 0 & 0\\ -1 & 0 & 0 & 0 & 0\\ -x & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & x & 1\\ 0 & 0 & 0 & 0 & -x \end{array}\right) = \left(\begin{array}{c|c} A_1(x) & 0\\ \hline 0 & A_2(x) \end{array}\right).$$

The diagonal blocks are respectively in the Lie algebras $\mathfrak{so}_3$ of the 3-dimensional special orthogonal group and in the Lie algebra

$$\mathfrak{b}_2 := \mathrm{Vect}_{\mathcal{C}} \left\{ \begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix} \right\}$$

of the two-dimensional Borel group. The matrix $A_{diag}(x)$ is in reduced form as we may see using Aparicio-Monforte et al. (2013); Barkatou et al. (2016). Using the canonical basis of $\mathfrak{gl}_{\text{sub}}$ as previously, the matrix for the adjoint action $\Psi = [A_{\text{diag}}, \bullet]$, acting on $\mathfrak{gl}_{\text{sub}}$, is

$$\Psi = \left(\begin{array}{ccc|ccc} x & 1 & x & 1 & 0 & 0\\ -1 & x & 0 & 0 & 1 & 0\\ -x & 0 & x & 0 & 0 & 1\\ \hline 0 & 0 & 0 & -x & 1 & x\\ 0 & 0 & 0 & -1 & -x & 0\\ 0 & 0 & 0 & -x & 0 & -x \end{array}\right) =: \Psi_0 + x\Psi_1.$$

As above, we let $\mathcal{R} := \mathcal{C}[\Psi_0, \Psi_1]$. Computation shows that the eigenring $\mathrm{End}_{\mathcal{R}}(\mathfrak{gl}_{\text{sub}})$ is spanned by the identity. So $\mathfrak{gl}_{\text{sub}}$ is Ψ-indecomposable. Looking at the matrix Ψ, we immediately see that the space spanned by the first three vectors is a Ψ-space. Let us recover that using the algorithm in §2.4.2 to illustrate the method.

If a subspace $V \subset \mathfrak{gl}_{\text{sub}}$ is a Ψ-space in $\mathfrak{gl}_{\text{sub}}$ then $V \otimes_{\mathcal{C}} \mathbf{k}$ is invariant. Such an invariant subspace is found from the generalized eigenspaces of Ψ. The characteristic polynomial $\chi_\Psi(\lambda)$ of Ψ has four factors $f_1(x) = (\lambda - x)$, $f_2(x) = (\lambda + x)$, $f_3(x) = \left(\lambda^2 - 2\lambda x + 2x^2 + 1\right)$ and $f_4(x) = \left(\lambda^2 + 2\lambda x + 2x^2 + 1\right)$. The corresponding generalized eigenspaces in $\mathfrak{gl}_{\text{sub}} \otimes_{\mathcal{C}} \mathbf{k}$ are respectively

$$E_1 = \left\langle \begin{pmatrix} 0 \\ -x \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \qquad E_2 = \left\langle \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2x} \\ 0 \\ -x \\ 1 \end{pmatrix} \right\rangle,$$

$$E_3 = \left\langle \begin{pmatrix} 0 \\ \frac{1}{x} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad E_4 = \left\langle \begin{pmatrix} 0 \\ \frac{-1}{2x^2} \\ \frac{-1}{2x} \\ 0 \\ \frac{1}{x} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{-1}{2x} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$$

We compute the smallest subspace $\overline{V}_1$ of $\mathfrak{gl}_{\mathrm{sub}}$ such that $E_1 = \overline{V}_1 \otimes_{\mathcal{C}} \mathbf{k}$: it is found from a Wei-Norman decomposition of the generator of E_1. Now we let V_1 be the orbit of $\overline{V}_1$ under $\mathcal{R}$. We find that $V_1 =< B_1, B_2, B_3 >$. Proceeding similarly with E_2, E_3 and E_4, we find respectively $V_3 = V_1$ and $V_2 = V_4 = \mathfrak{gl}_{\mathrm{sub}}$. Note that $V_1 \otimes_{\mathcal{C}} \mathbf{k} = E_1 \oplus E_3$. As the dimension of V_1 is minimal, it is Ψ-irreducible.

We let $B_i^{[1]} := B_i$, for $i = 1, 2, 3$, $W^{[1]} := V_1$ and then $B_1^{[2]}, B_2^{[2]}, B_3^{[2]} = B_4, B_5, B_6$ to obtain the flag $\mathfrak{gl}_{\mathrm{sub}} = W^{[2]} \supsetneq W^{[1]} \supsetneq \{0\}$:

$$\mathfrak{gl}_{\mathrm{sub}} = \boxed{\begin{array}{c} \boxed{\langle B_1^{[2]}, B_2^{[2]}, B_3^{[2]} \rangle} \\ \downarrow \\ \boxed{\langle B_1^{[1]}, B_2^{[1]}, B_3^{[1]} \rangle} \end{array}}$$

This example is continued in §3.2.

2.5.3. The "$B_3 \times B_2$" example

Let us consider

$$A_{diag}(x) := \left(\begin{array}{ccc|cc} 1 & x & 0 & 0 & 0 \\ 0 & -x-1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ \hline 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & -x \end{array} \right).$$

The associated Lie algebra has dimension 4 and it turns out that this system is in reduced form. First, the diagonal $\mathrm{Diag}(1, -x-1, x, x, -x)$ is in reduced form: its associated Lie algebra has dimension 2 while the associated Picard-Vessiot extension is generated over $\mathbb{C}(x)$ by e^x and $e^{\frac{x^2}{2}}$, which are algebraically independent; the remaining reduction (applying the full algorithm at the end of this paper) is a simple integration exercise.[5]

[5] This gives, in turn, a proof that e^x, $e^{\frac{x^2}{2}}$, $\int^x e^{\frac{t^2}{2}}\,\mathrm{d}t$ and $\int^x \mathrm{e}^{-\frac{1}{2}(t+2)^2}\,\mathrm{d}t$ are algebraically independent.

The matrix of the adjoint action $\Psi = [A_{\text{diag}}, \bullet]$ in the canonical basis is

$$\Psi = \begin{pmatrix} x-1 & 0 & 0 & 1 & 0 & 0 \\ -x & 2x+1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -x-1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2x \end{pmatrix}.$$

Using the eigenring decomposition algorithm from §2.4.2, we find a decomposition $\mathfrak{gl}_{\text{sub}} = W_1 \oplus W_2$ as a direct sum of two indecomposable subspaces W_i of respective dimensions 4 and 2. The restrictions of Ψ to these subspaces have respective matrices

$$\Psi|_{W_1} = \begin{pmatrix} 1 & -x & 0 & 0 \\ 0 & -1-x & 0 & 0 \\ 1 & 0 & 2x+1 & -x \\ 0 & 1 & 0 & x-1 \end{pmatrix} \text{ and } \Psi|_{W_2} = \begin{pmatrix} 0 & 1 \\ 0 & -2x \end{pmatrix}.$$

We have $W_1 = \langle C_1, C_2, C_3, C_4 \rangle$ and $W_2 = \langle C_5, C_6 \rangle$, with $C_1 = B_5$, $C_2 = B_4$, $C_3 = B_2$, $C_4 = B_1$, $C_5 = B_6$, $C_6 = B_3$. The characteristic polynomial of $\Psi|_{W_1}$ is $(\lambda - 1)(-2x - 1 + \lambda)(-x + 1 + \lambda)(x + 1 + \lambda)$. For the factor $f_1(\lambda) := \lambda - 2x - 1$, the eigenspace is $V_1 := E_1 = \langle C_3 \rangle$. It has a constant basis and hence its generator spans a Ψ-space. For the factor $f_2(\lambda) := \lambda - 1$, we have $E_2 = \langle -2xC_1 + C_3 \rangle$. The associated Ψ-space is

$$V_2 := \langle C_1, C_3 \rangle .$$

Note that V_2 is a *reducible* Ψ-space, even though E_2 was an irreducible $\mathbf{k}[\Psi]$-module: we have $V_1 \subsetneq V_2$. Continuing in this way, we find a basis for the flag on W_1:

$$B_1^{[1]} := C_3, B_1^{[2]} := C_1, B_1^{[3]} := C_3 + C_4, B_1^{[4]} := C_2.$$

Similarly, the flag on W_2 is given by $B_1^{[2]} := C_5$, $B_1^{[1]} := C_6$. The matrix of Ψ in this new basis is

$$\left(\begin{array}{cccc|cc} 2x+1 & 1 & -x & 0 & 0 & 0 \\ 0 & 1 & 0 & -x & 0 & 0 \\ 0 & 0 & x-1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -x-1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2x \end{array}\right).$$

To summarize, our isotypical flag[6] $\mathfrak{gl}_{\text{sub}} = W_1 \oplus W_2$ in this "$B_3 \times B_2$" example is given by:

[6] We stress the fact that there is another possible choice of flag in this example. All choices are equivalent, by the Krull-Schmidt theorem, see Proposition 2.13, so our choice is essentially cosmetic but does not influence the complexity of the computations.

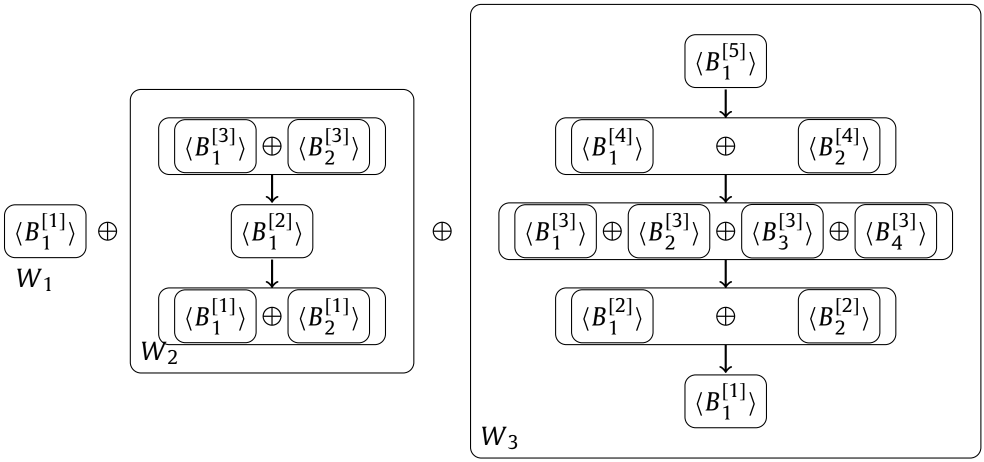
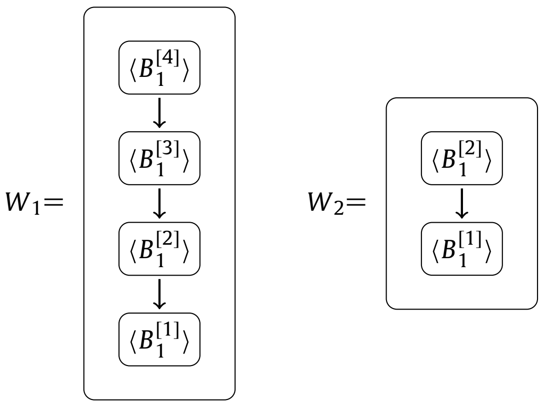


Fig. 1. The isotypical flag of the nilpotent example.



2.5.4. A nilpotent example

Let us consider

$$A_{diag}(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

As shown in the maple worksheet, see Dreyfus and Weil (2020), or direct computation, see Dreyfus and Weil (2021), Example 3, the system is in reduced form. The matrix Ψ of the adjoint action in the canonical basis of $\mathfrak{gl}_{\mathrm{sub}}$ is $\Psi := \frac{1}{x}\Psi_0 + \frac{1}{x-1}\Psi_1$, see the worksheet (Dreyfus and Weil, 2020). It turns out that Ψ is nilpotent and that its minimal polynomial is $\chi_\Psi(\lambda) = \lambda^3$. The eigenring has dimension 32. The eigenring decomposition algorithm provides a decomposition of $\mathfrak{gl}_{\mathrm{sub}}$ as a direct sum of three indecomposable Ψ-spaces W_1, W_2, W_3 of respective dimensions 1, 5, and 10 (Fig. 1).

The matrix of Ψ acting on the 5-dimensional block W_2 is given in Dreyfus and Weil (2020). The flag reduction method provides a new basis on which the matrix of $\Psi|_{W_2}$ is

$$\Lambda_2 = \left(\begin{array}{cc|c|cc} 0 & 0 & \frac{1}{x-1} & 0 & 0 \\ 0 & 0 & \frac{1}{x} & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{x} & \frac{1}{x-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We note that, although W_2 is an indecomposable Ψ-space, it has decomposable quotients and subspaces, namely $W_2/\mathrm{Vect}_{\mathcal{C}}\{B_1^{[2]}, B_1^{[1]}, B_2^{[1]}\}$ and $\mathrm{Vect}_{\mathcal{C}}\{B_1^{[1]}, B_2^{[1]}\}$ are decomposable.

Similarly, the flag reduction method provides a new basis on which the matrix of $\Psi|_{W_3}$ is

$$\Lambda_3 = \left(\begin{array}{c|cc|cccc|cc|c} 0 & \frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{x} & -\frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

3. Examples of reduction on an isotypical flag

As the next part of the algorithm is a bit technical and may be cumbersome to read, we start by performing our reduction technique on the above four examples. They are presented in increasing order of complexity. They are chosen so that the phenomena can be better understood before stating the general reduction procedure.

3.1. The "$SO_3 \times SL_2$" example continued

This example is our simplest. Note that the Berman-Singer algorithm (Berman and Singer, 1999; Berman, 2002) applies to this example (and leads to the same conclusion). Let

$$A(x) := \left(\begin{array}{ccc|cc} 0 & 1 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{x} + \frac{1}{x-1} & 1 - \frac{1}{x^2} & x & 0 & 1 \\ x + \frac{1}{(x-1)^2} & 1 - \frac{1}{x-1} & -1 - \frac{1}{x-1} & -x & 0 \end{array}\right).$$

In §2.5.1, we have seen that $\mathfrak{gl}_{\mathrm{sub}}$ has dimension 6 and is Ψ-irreducible. We recall the notation $\mathfrak{g}_{\mathrm{sub}} := \mathfrak{gl}_{\mathrm{sub}} \cap \mathfrak{g}$. Since $\mathfrak{gl}_{\mathrm{sub}}$ has no proper Ψ-space here, we have either $\mathfrak{g}_{\mathrm{sub}} = \mathfrak{gl}_{\mathrm{sub}}$ (in which case $A(x)$ is already in reduced form) or $\mathfrak{g}_{\mathrm{sub}} = \{0\}$ (in which case $A_{diag}(x)$ is a reduced form of $[A(x)]$, as we had assumed that $A_{\mathrm{diag}}(x)$ was in reduced form).

We look for a reduction matrix of the form $P = \mathrm{Id} + \sum_{i=1}^{6} f_i(x) B_i$ such that $P[A] = A_{\mathrm{diag}}$. Writing down this equality, see Proposition 2.7, we find that the vector of coefficients $\vec{F} := (f_1(x), \ldots, f_6(x))^T$ must be a rational solution of the system

$$Y' = \Psi.Y + \vec{b}, \quad \text{where } \vec{b} := \begin{pmatrix} -\frac{1}{x} + \frac{1}{x-1} \\ 1 - \frac{1}{x^2} \\ x \\ x + \frac{1}{(x-1)^2} \\ 1 - \frac{1}{x-1} \\ -1 - \frac{1}{x-1} \end{pmatrix}.$$

Using the Maple implementation of the Barkatou algorithm in the package `IntegrableConnections`[7] from Barkatou et al. (2022), we find a unique rational solution

$$\vec{F} = \begin{pmatrix} 1 \\ \frac{1}{x} \\ 0 \\ -\frac{1}{x-1} \\ 0 \\ 0 \end{pmatrix}$$

and it follows that $A_{\text{diag}}(x)$ is a reduced form of $[A(x)]$ with reduction matrix equal to

$$P(x) := \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & \frac{1}{x} & 0 & 1 & 0 \\ -\frac{1}{x-1} & 0 & 0 & 0 & 1 \end{array}\right).$$

3.2. The "$SO_3 \times B_2$" example, continued

We continue with the example from §2.5.2.

$$A_{diag}(x) := \left(\begin{array}{ccc|cc} 0 & 1 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & -x \end{array}\right), \; A_{sub}(x) := \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline x+3 & 0 & -1 & 0 & 0 \\ -x^2-3x & x^2 & x^2+1 & 0 & 0 \end{array}\right).$$

In §2.5.2, we have found that $\mathfrak{gl}_{\text{sub}}$ has dimension 6, that it is indecomposable, that it admits only one proper subspace and a flag $\mathfrak{gl}_{\text{sub}} = W^{[2]} \supsetneq W^{[1]} \supsetneq \{0\}$:

$$\mathfrak{gl}_{\text{sub}} = \boxed{\begin{array}{c} \boxed{\langle B_1^{[2]}, B_2^{[2]}, B_3^{[2]} \rangle} \\ \big\downarrow \\ \boxed{\langle B_1^{[1]}, B_2^{[1]}, B_3^{[1]} \rangle} \end{array}}$$

This means that the only proper Ψ-subspace of $\mathfrak{gl}_{\text{sub}}$ is $W^{[1]}$. So there are three possibilities for the reduced matrix. We start by trying to perform reduction on the first level of the flag, namely

[7] The Maple command is `RationalSolutions([Psi],[x],['rhs',[B]]);`.

$W^{[2]}/W^{[1]}$. We look for a gauge transformation of the form $P^{[2]} = \mathrm{Id} + \sum_{i=1}^{3} f_i(x) B_i^{[2]}$. There is a (partial) reduction if and only if we can find $f_i(x) \in \mathbf{k}$ such that $P^{[2]}[A]$ has no component in $W^{[2]}/W^{[1]}$. This means that $\vec{F} := (f_1(x), f_2(x), f_3(x))^T$ must be a rational solution of the linear differential system

$$Y' = \begin{pmatrix} -x & 1 & x \\ -1 & -x & 0 \\ -x & 0 & -x \end{pmatrix} Y + \begin{pmatrix} -x^2 - 3x \\ x^2 \\ x^2 + 1 \end{pmatrix}.$$

Using again `IntegrableConnections` from Barkatou et al. (2022), we find a unique rational solution and we derive an intermediate reduction matrix $P^{[2]}$ given by

$$P^{[2]}(x) := \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ -1 & x & x+1 & 0 & 1 \end{array}\right).$$

We let $A^{[2]}(x) := P^{[2]}(x)[A(x)] = \left(\begin{array}{ccc|cc} 0 & 1 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ \hline 2+x & x & x & x & 1 \\ 0 & 0 & 0 & 0 & -x \end{array}\right)$. We now try to reduce the level $W^{[1]}$ of the flag. So we look for a gauge transformation of the form $P^{[1]} = \mathrm{Id} + \sum_{i=1}^{3} f_i(x) B_i^{[1]}$. As $W^{[1]}$ is irreducible, there will be (partial reduction) if and only if $P^{[1]}[A^{[2]}(x)]$ has no components in $W^{[1]}$. This means that $\vec{F} := (f_1(x), f_2(x), f_3(x))^T$ must be a rational solution of the linear differential system

$$Y' = \begin{pmatrix} x & 1 & x \\ -1 & x & 0 \\ -x & 0 & x \end{pmatrix} Y + \begin{pmatrix} 2+x \\ x \\ x \end{pmatrix}.$$

As there is no such rational solution, we find that $A^{[2]}$ cannot be reduced any further so that it is in reduced form. Computing $\mathrm{Lie}(A^{[2]})$ shows that it has dimension 8.

To summarize, the flag of the "$SO_3 \times B_2$" example after the reduction is as follows; the red rectangles correspond to the part we have deleted via the reduction matrix, and the blue rectangles correspond to the non removable part, i.e. the reduced matrix:

$$\begin{tikzcd} \boxed{\cancel{\langle B_1^{[2]}, B_2^{[2]}, B_3^{[2]} \rangle}} \arrow[d] \\ \boxed{\langle B_1^{[1]}, B_2^{[1]}, B_3^{[1]} \rangle} \end{tikzcd}$$

3.3. The "$B_3 \times B_2$" example continued

We continue with the example from §2.5.3. We have

$$A_{diag}(x) := \left(\begin{array}{ccc|cc} 1 & x & 0 & 0 & 0 \\ 0 & -x-1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ \hline 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & -x \end{array}\right), \; A_{sub}(x) := \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -x & 1 & -1-x & 0 & 0 \\ x+1 & \frac{x+1}{x} & 2x^2+1 & 0 & 0 \end{array}\right).$$

The isotypical flag has the form $\mathfrak{gl}_{\text{sub}} = W_1 \oplus W_2$ with the following flag structures on the W_i:

$$W_1 = \boxed{\begin{tikzcd} \boxed{\langle B_1^{[4]} \rangle} \arrow[d] \\ \boxed{\langle B_1^{[3]} \rangle} \arrow[d] \\ \boxed{\langle B_1^{[2]} \rangle} \arrow[d] \\ \boxed{\langle B_1^{[1]} \rangle} \end{tikzcd}} \qquad W_2 = \boxed{\begin{tikzcd} \boxed{\langle B_1^{[2]} \rangle} \arrow[d] \\ \boxed{\langle B_1^{[1]} \rangle} \end{tikzcd}}$$

We take the matrix of Ψ in the adapted basis computed in §2.5.3.

We first perform the reduction on W_2, with its adapted flag basis. We look for a gauge transformation $P^{[2]} := \mathrm{Id} + f_2(x)B_1^{[2]}$ to remove $B_1^{[2]}$ from $P^{[2]}[A_1]$. We find that f_2 should be a rational solution of $1 - 2xf_2(x) + 2x^2 - f_2'(x) = 0$. The only rational solution is x so $P^{[2]} := \mathrm{Id} + xB_1^{[2]}$. Similarly, we look for the last gauge transformation $P^{[1]} := \mathrm{Id} + f_1(x)B_1^{[1]}$. We find $f_1(x) = -x + c_1$ for an arbitrary constant parameter c_1. So the reduction matrix on W_2 is $P_{W_2} = P^{[2]}.P^{[1]}$. The reduction on W_2 is then of the form

$$A_2 := P_{W_2}[A] = \left(\begin{array}{ccc|cc} 1 & x & 0 & 0 & 0 \\ 0 & -1-x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ \hline -x & 1 & 0 & x & 1 \\ x+1 & \frac{x+1}{x} & 0 & 0 & -x \end{array}\right)$$

We now perform the reduction on W_1, with its adapted flag basis. We look for a gauge transformation $P^{[4]} = \mathrm{Id} + f_4(x)B_1^{[4]}$ so that $B_1^{[4]}$ would be absent from $P^{[4]}[A]$. The condition is $-(1+x)f_4(x) + 1 + x - f_4'(x) = 0$. This equation has the unique rational solution 1 so we take $P^{[4]} = \mathrm{Id} + B_1^{[4]}$ and let $A^{[3]} := P^{[4]}[A_2]$. Now we want to remove $B_1^{[3]}$ from $A^{[3]}$ via $P^{[3]} = \mathrm{Id} + f_3(x)B_1^{[3]}$. The condition is $-(1-x)f_3(x) + 1 - x - f_3'(x) = 0$ which admits the unique rational solution $f_3(x) = 1$ so $P^{[3]} = \mathrm{Id} + B_1^{[3]}$. We now set $P^{[2]} = \mathrm{Id} + f_2(x)B_1^{[2]}$, look at the condition for $B_1^{[2]}$ to vanish from $P^{[2]}[A^{[2]}]$. We obtain $f_2'(x) = f_2(x) - \frac{x^2-x-1}{x}$. This equation has no rational solution so we see that we can no further reduce on W_1. So we let $P_{W_1} := P^{[4]}.P^{[3]}$ be the reduction matrix on W_1.

Finally, letting $P := P_{W_1} P_{W_2}$, we find

$$A_{\text{red}} := P[A] = \left(\begin{array}{ccc|cc} 1 & x & 0 & 0 & 0 \\ 0 & -1-x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ \hline 0 & -x+1 & 0 & x & 1 \\ 0 & -x+1+\frac{1}{x} & 0 & 0 & -x \end{array}\right).$$

Note that A_{red} does not depend upon c_1. Now $\text{Lie}(A_{\text{red}})$ has dimension 6 and its off-diagonal part is spanned by the matrices of $B_1^{[1]}$ and $B_1^{[2]}$ of W_1. Our construction shows that any gauge transformation of the form $\text{Id} + M$ with $M \in \text{Vect}_{\mathcal{C}}(B_1^{[1]}, B_1^{[2]})$ will keep $B_1^{[2]}$ (and then $B_1^{[1]}$) in the Lie algebra so Corollary 2.6 shows that $[A_{\text{red}}]$ is in reduced form.

To summarize, the flag of the "$B_3 \times B_2$" example after the reduction is as follows; the red rectangles correspond to the part we have deleted via the reduction matrix, and the blue rectangles correspond to the non removable part, i.e. the reduced matrix:

$$W_1 = \boxed{\begin{array}{c} \boxed{\cancel{\langle B_1^{[4]}\rangle}} \\ \downarrow \\ \boxed{\cancel{\langle B_1^{[3]}\rangle}} \\ \downarrow \\ \boxed{\langle B_1^{[2]}\rangle} \\ \downarrow \\ \boxed{\langle B_1^{[1]}\rangle} \end{array}} \qquad W_2 = \boxed{\begin{array}{c} \boxed{\cancel{\langle B_1^{[2]}\rangle}} \\ \downarrow \\ \boxed{\cancel{\langle B_1^{[1]}\rangle}} \end{array}}$$

3.4. The nilpotent example, continued

We continue with the example of §2.5.4. We let $(\tilde{N}_i)_{i=1\ldots16}$ denote the adapted basis of $\mathfrak{gl}_{\text{sub}}$ found in §2.5.4. We recall that we have three indecomposable Ψ-spaces of dimensions 1, 5 and 10 respectively. We will study

$$A_{\text{sub}}(x) := \sum_{i=1}^{16} f_i(x) \tilde{N}_i,$$

where the $f_i(x)$ are the following

$$f_1(x) := 0,$$

$$f_2(x) := \tfrac{1}{x^2}, \quad f_3(x) := 0, \quad f_4(x) := \tfrac{2-x}{2x^2}, \quad f_5(x) := \tfrac{1-x}{x^2},$$
$$f_6(x) := \tfrac{3-x}{x^2},$$

$$f_7(x) := 0, \quad f_8(x) := -\tfrac{1}{2x}, \quad f_9(x) := -\tfrac{1}{2(x-1)}, \quad f_{10}(x) := \tfrac{1}{x},$$
$$f_{11}(x) := -\tfrac{1}{2x}, \quad f_{12}(x) := 0, \quad f_{13}(x) := -\tfrac{1}{2(x-1)}, \quad f_{14}(x) := -\tfrac{1}{2(x-1)},$$
$$f_{15}(x) := \tfrac{1}{x^2} - \tfrac{1}{2(x-1)}, \quad f_{16}(x) := \tfrac{2}{x^2} + \tfrac{1}{x-1}.$$

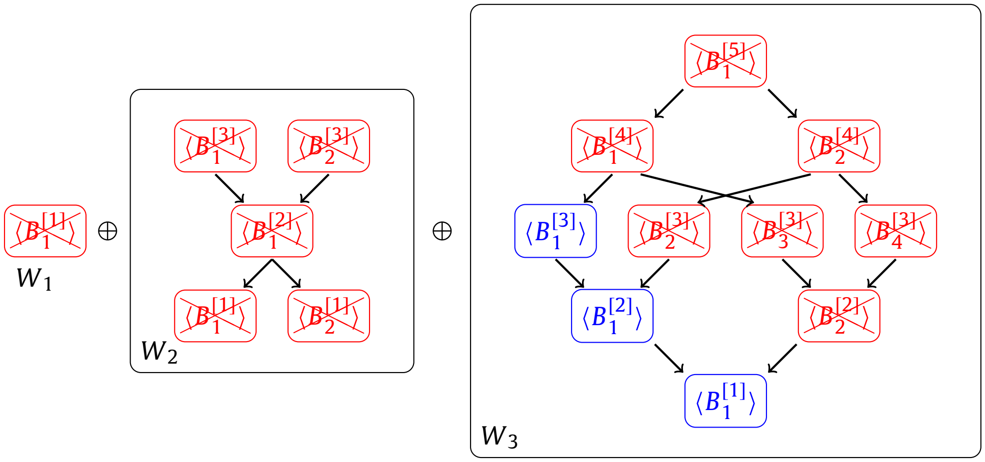


Fig. 2. The action of the adjoint map on the isotypical flag of the nilpotent example §3.4. The spaces W_1 (left), W_2 (center) and W_3 (right) satisfy $\mathfrak{gl}_{\mathrm{sub}} = W_1 \oplus W_2 \oplus W_3$. The red rectangles correspond to the part that we get rid of via the reduction matrix, and the blue rectangles correspond to what will remain in the reduced matrix.

Since the coefficient in front of $\tilde{N}_1$ is 0, the reduction to the 1-dimensional block is already completed (Fig. 2).

$\boxed{\textbf{Reduction of the 5-dimensional block.}}$

To remove all of W_2, it would be enough to have a rational solution of the system

$$\vec{Y}' = \Lambda_2.\vec{Y} + \vec{b} \text{ with } \Lambda_2 = \left(\begin{array}{cc|c|cc} 0 & 0 & \frac{1}{x-1} & 0 & 0 \\ 0 & 0 & \frac{1}{x} & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{x} & \frac{1}{x-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ and } \vec{b} = \begin{pmatrix} \frac{2}{x} \\ \frac{2x+1}{x^2} \\ \frac{x+1}{x^2} \\ 0 \\ \frac{1}{x^2} \end{pmatrix}.$$

Although it is simple to find such a solution in this case, we detail the calculations to illustrate the general method. The following differential systems give the conditions for reduction at the three levels of the flags.

$$\begin{array}{ll} (W^{[3]}): & \left\{ \begin{array}{lcl} f'_{3,1}(x) & = & \frac{1}{x^2} \\ f'_{3,2}(x) & = & 0 \end{array} \right. \\ (W^{[2]}): & \left\{ \begin{array}{lcl} f'_{2,1}(x) & = & \frac{1}{x-1} f_{3,1}(x) + \frac{1}{x} f_{3,2}(x) + \frac{2-x}{2x^2} \end{array} \right. \\ (W^{[1]}): & \left\{ \begin{array}{lcl} f'_{1,1}(x) & = & \frac{1}{x} f_{2,1}(x) + \frac{1-x}{x^2} \\ f'_{1,2}(x) & = & \frac{1}{x-1} f_{2,1}(x) + \frac{3-x}{x^2}. \end{array} \right. \end{array}$$

We proceed level by level. The first two equations correspond to the first level $W_2^{[3]}$ of the flag. The condition to remove an element from $W_2^{[3]}$ is that there should be a rational solution to the equation $y' = c_1.\frac{1}{x^2} + c_2.0$. We look for a basis of the $\mathcal{C}$-vector space of pairs $(c_1, c_2) \in \mathcal{C}^2$ such that there exists $f \in \mathbf{k}$ with $f' = c_1.\frac{1}{x^2} + c_2.0$. This space is found to be 2-dimensional; for $\underline{c} = (1, 0)$, we have $f_{3,1} := -\frac{1}{x} + c_{3,1}$; for $\underline{c} = (0, 1)$, we have $f_{3,2} := c_{3,2}$, where the $c_{3,i}$ are arbitrary constants (their importance will soon be visible). Our gauge transformation is $P^{[3]} = \mathrm{Id} + f_{3,1} B_1^{[3]} + f_{3,2} B_2^{[3]}$ and $A^{[2]} := P^{[3]}[A]$ does not contain any terms from $W_2^{[3]}$.

Now $W_2^{[2]}$ is 1-dimensional. The equation for the reduction on $W_2^{[2]}$ is

$$y' = -\frac{2}{x-1} f_{3,1} - \frac{2}{x} f_{3,2} + \frac{x+1}{x^2} = -\frac{2}{x-1}\left(-\frac{1}{x} + c_{3,1}\right) - \frac{2}{x} c_{3,2} + \frac{x+1}{x^2}$$
$$= \frac{-2\,c_{3,2} - 1}{x} + \frac{-2\,c_{3,1} + 2}{x-1} + \frac{1}{x^2}.$$

We have necessary and sufficient conditions on the parameters $c_{3,i}$ to have a rational solution, namely $c_{3,1} = 1$, $c_{3,2} = -\frac{1}{2}$ and then a general rational solution $f_{2,1} := \frac{-1}{x} + c_{2,1}$. Our new gauge transformation is $P^{[2]} = \mathrm{Id} + (-\frac{1}{x} + c_{2,1}) B_1^{[2]}$ and $A^{[1]} := P^{[2]}[A]$ does not contain any term from $W_2^{[3]}$ nor from $W_2^{[2]}$.

Now we look for pairs $(c_1, c_2) \in \mathcal{C}^2$ such that there exists $f \in \mathbf{k}$ that is a rational solution of

$$y' = c_1\left(-\frac{2}{x} f_{2,1} + \frac{2x+1}{x^2}\right) + c_2\left(-\frac{2}{x-1} f_{2,1} + \frac{2}{x}\right)$$
$$= c_1\left(-\frac{2}{x}\left(\frac{-1}{x} + c_{2,1}\right) + \frac{2x+1}{x^2}\right) + c_2\left(-\frac{2}{x-1}\left(\frac{-1}{x} + c_{2,1}\right) + \frac{2}{x}\right)$$
$$= \frac{-2c_1 c_{2,1} + 2\,c_1}{x} + \frac{2c_2\left(-c_{2,1} + 1\right)}{x-1} + \frac{3c_1}{x^2}.$$

This integral is rational if and only if both residues are zero. As the solution $c_1 = c_2 = 0$ is not admissible, we see that a necessary and sufficient condition is $c_{2,1} = 1$. The set of desired pairs (c_1, c_2) is of dimension 2. For $\underline{c} = (1, 0)$, we have $f_{1,1} := -\frac{3}{x} + c_{1,1}$; for $\underline{c} = (0, 1)$, we have $f_{1,2} := c_{1,2}$, where the $c_{1,i}$ are constants and can be chosen arbitrarily. For the simplicity of the expression of the gauge transformation, we can choose $c_{1,1} = c_{1,2} = 0$ (but the other choice is valid too). Our last gauge transformation matrix will be $P^{[1]} = \mathrm{Id} - \frac{3}{x} B_1^{[1]}$.

Finally, the reduction matrix on W_2 is

$$P_2 := P^{[3]} P^{[2]} P^{[1]} = \mathrm{Id} + \left(-\frac{1}{x} + 1\right) B_1^{[3]} - \frac{1}{2} B_2^{[3]} + \left(-\frac{1}{x} + 1\right) B_1^{[2]} - \frac{3}{x} B_1^{[1]}$$

and the matrix $A_2 := P_2[A]$ is reduced on W_2.

Reduction of the 10-dimensional block.

We now turn to the 10-dimensional block W_3. The reduction equations are

$$\begin{array}{ll}
(W^{[5]}): & \left\{ f'_{5,1}(x) = 0 \right. \\
(W^{[4]}): & \left\{ \begin{array}{lcl} f'_{4,1}(x) &=& \frac{1}{x} f_{5,1}(x) - \frac{1}{2x} \\ f'_{4,2}(x) &=& \frac{1}{x-1} f_{5,1}(x) - \frac{1}{2(x-1)} \end{array} \right. \\
(W^{[3]}): & \left\{ \begin{array}{lcl} f'_{3,1}(x) &=& \frac{1}{x} f_{4,1}(x) + \frac{1}{x} \\ f'_{3,2}(x) &=& \frac{1}{x} f_{4,2}(x) - \frac{1}{2x} \\ f'_{3,3}(x) &=& \frac{1}{x-1} f_{4,1}(x) \\ f'_{3,4}(x) &=& \frac{1}{x-1} f_{4,2}(x) - \frac{1}{2(x-1)} \end{array} \right. \\
(W^{[2]}): & \left\{ \begin{array}{lcl} f'_{2,1}(x) &=& \frac{1}{x-1} f_{3,1}(x) - \frac{1}{x} f_{3,2}(x) - \frac{1}{2(x-1)} \\ f'_{2,2}(x) &=& -\frac{1}{x-1} f_{3,3}(x) + \frac{1}{x} f_{3,4}(x) - \frac{1}{2(x-1)} + \frac{1}{x^2} \end{array} \right. \\
(W^{[1]}): & \left\{ f'_{1,1}(x) = \frac{1}{x-1} f_{2,1}(x) + \frac{1}{x} f_{2,2}(x) + \frac{2}{x^2} + \frac{1}{x-1}. \right.
\end{array}$$

The first equation gives $f_{5,1} = c_{5,1} \in \mathcal{C}$. The equations on $W^{[4]}$ both have rational solutions if and only if $c_{5,1} = \frac{1}{2}$. We then have $f_{4,i} = c_{4,i} \in \mathcal{C}$. Letting $y := \sum_{i=1}^{4} c_i . f_{3,i}$ for unknown c_i, the equations on $W^{[3]}$ are

$$y' = \frac{c_1(c_{4,1}+1)+c_2(c_{4,2}-1/2)}{x} + \frac{c_3c_{4,1}+c_4(c_{4,2}-1/2)}{x-1},$$

and we investigate values of $\underline{c} := (c_1,\ldots,c_4)$ (and $c_{4,i}$) for which this may have a rational solution. Of course, this has a rational solution if and only if both residues are zero. The algebraic conditions for both residues to be zero are

$$\begin{cases} c_1(c_{4,1}+1)+c_2(c_{4,2}-1/2) &= 0 \\ c_3c_{4,1}+c_4(c_{4,2}-1/2) &= 0. \end{cases} \tag{3.1}$$

We view (3.1) as a linear system in the $\underline{c}$ in coefficients in $\mathcal{C}(c_{4,i})$. We study for which $c_{4,i}$ the space of solutions $\underline{c}$ of (3.1) has maximal dimension. Here, it would be 4 if and only if $c_{4,1}+1 = c_{4,1} = c_{4,2}-1/2 = 0$ which cannot occur. We see that it has dimension 3 if and only if either $c_{4,1}+1 = c_{4,2}-1/2 = 0$ or $c_{4,1} = c_{4,2}-1/2 = 0$. Then, the only possibilities are $\{c_3 = 0, c_{4,1} = -1, c_{4,2} = 1/2\}$ and $\{c_1 = 0, c_{4,1} = 0, c_{4,2} = 1/2\}$. We need to study each component separately. It turns out that both lead to the same result, a reduced form. We show how things go on the second component. The computations for the first component may be found in the Maple Worksheet (Dreyfus and Weil, 2020) and are detailed in Dreyfus and Weil (2021). We have $c_{4,1} = 0$ and $c_{4,2} = \frac{1}{2}$. The set of $\underline{c}$ for which the equation has a rational solution is a 3-dimensional $\mathcal{C}$-vector space; it is generated by $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. We have thus have $f_{3,i}(x) = c_{3,i} \in \mathcal{C}$ for $i = 2,3,4$ and $f_{3,1}$ remains unknown: the equation $f'_{3,1}(x) = -\frac{1}{x}$ has no rational solution. So we cannot remove $B_1^{[3]}$ from the result. However, the constant $c_{3,1}$ will play a role in the reduction process.

Remark 3.1. As $B_1^{[3]}$ appears in $\mathfrak{g}$, the adjoint action of A_{diag} implies that $B_1^{[2]}$ and $B_1^{[1]}$ will be present in $\mathfrak{g}$, even if we found transformations which might seem to remove them from the reduced matrix. In the matrices of $\mathfrak{g}$ given at the end of the computation, the third one has $B_1^{[3]}$ as its off-diagonal part, the fourth one has $B_1^{[2]}$ and the fifth one is $B_1^{[1]}$.

Letting $y = c_1.f_{2,1} + c_2.f_{2,2}$, the family of reduction equations on W_2 is now:

$$y' = \frac{-c_1c_{3,2}+c_2c_{3,4}}{x} + \frac{c_1(c_{3,1}-1/2)-c_2(c_{3,3}+1/2)}{x-1} + \frac{c_2}{x^2}.$$

The condition for both residues to be zero gives again a linear system on c_1 and c_2

$$\begin{cases} - \; c_1c_{3,2} & + \; c_2c_{3,4} &= 0 \\ + \; c_1(c_{3,1}-1/2) & - \; c_2(c_{3,3}+1/2) &= 0. \end{cases}$$

The space of solutions (c_1,c_2) has maximal dimension 2 when $c_{3,2} = c_{3,4} = 0$, $c_{3,1} = 1/2$ and $c_{3,3} = -\frac{1}{2}$. Now, for $\underline{c} = (1,0)$, we obtain $f_{2,1} = -\frac{1}{x} + c_{2,1}$; for $\underline{c} = (0,1)$, we find $f_{2,2} = c_{2,2}$. The last equation is

$$f'_{1,1}(x) = \frac{1}{x-1}\left(-\frac{1}{x}+c_{2,1}\right) + \frac{c_{2,2}}{x} + \frac{2}{x^2} + \frac{1}{x-1} = \frac{1}{x} - \frac{1}{x-1} + \frac{c_{2,1}}{x-1} + \frac{c_{2,2}}{x} + \frac{2}{x^2} + \frac{1}{x-1}.$$

This imposes $c_{2,1} = 0$ and $c_{2,2} = -1$ and $f_{1,1}(x) = \frac{1}{x} + c_{1,1}$, where $c_{1,1}$ is a constant that can be chosen arbitrary. We obtain the reduced form

$$A_{\mathrm{red}}(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{x} & 1 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 1 \end{array}\right).$$

The associated Lie algebra is spanned by

$$\left(\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{array}\right), \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0 \end{array}\right), \left(\begin{array}{cccc|cccc} 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&0&-1&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right),$$

$$\left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&1/2&0&0&0&0&0\\ 0&0&0&-1/2&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right), \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right).$$

This gives us the Lie algebra $\mathfrak{g} = \mathrm{Lie}(A_{\mathrm{red}}(x))$ of the differential Galois group. It is 5-dimensional, whereas the Lie algebra associated to the original matrix $A(x)$ had dimension 14. This shows that the Picard-Vessiot extension is obtained from K_{diag} by performing only one integral.

Remark 3.2. We recall that $\mathfrak{g}_{\mathrm{diag}}$ is the Lie algebra associated to $A_{\mathrm{diag}}(x)$ and (cf. the proof of Theorem 2.4)

$$\mathfrak{g}_{\mathrm{diag}} = \left\{ \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array} \right) \middle| \, \exists\, S, \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array} \right) \in \mathfrak{g} \right\}.$$

Let us set

$$\mathfrak{g}_s := \mathfrak{g}/\mathfrak{g}_{\mathrm{diag}} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right) \middle| \, \exists\, D_1, D_2, \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array} \right) \in \mathfrak{g} \right\}.$$

Note that neither $\mathfrak{g}_{\mathrm{diag}}$ nor $\mathfrak{g}_s$ are subalgebras of $\mathfrak{g}$. Here, $\mathfrak{g}$ has dimension 5, $\mathfrak{g}_{\mathrm{diag}}$ has dimension 4 and $\mathfrak{g}_{sub} := \mathfrak{g} \cap \mathfrak{gl}_{\mathrm{sub}}$ has dimension 1. However, we see that the set $\mathfrak{g}_s$ of "off-diagonal" parts of elements of $\mathfrak{g}$ has dimension 3 so that $\mathfrak{g} \subsetneq \mathfrak{g}_{\mathrm{diag}} \oplus \mathfrak{g}_s$. Our reduction process computes a subalgebra $\mathfrak{g}$ of $\mathrm{Lie}(A)$ such that its $\mathfrak{g}_s$ has minimal dimension. Now, an odd phenomenon occurs; in the course of the reduction, an off-diagonal element in $\mathrm{Lie}(A)$ may be "absorbed" as the triangular part of an element that was present. For example, in the third matrix, an element of $\mathfrak{gl}_{\mathrm{sub}}$ has a coefficient $-\frac{1}{x}$ after reduction so it becomes the lower triangular part of the constant matrix from $\mathrm{Lie}(A_{\mathrm{diag}})$ corresponding to $\frac{1}{x}$. So the minimization of the dimension of $\mathfrak{g}_s$ is a necessary, but a priori not sufficient condition to reduce the system. To prove that the reduction process is complete, we need to show that there are no gauge transformation which send the last matrix to nilpotent elements whose coefficients are in the Wei-Norman decomposition of A_{diag}. A simple computation shows that the last element $\mathfrak{g}_{\mathrm{sub}}$ cannot be "absorbed" as the off-diagonal part of an element of $\mathfrak{g}_{\mathrm{diag}}$ so our system is indeed in reduced form.

4. Computation of the reduction matrix on an isotypical flag

Let $\mathfrak{h} := \mathrm{Lie}(A)$. As above, we let

$$\mathfrak{h}_{\mathrm{diag}} := \left\{ \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array} \right) \middle| \, \exists\, S \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array} \right) \in \mathfrak{h} \right\},$$

$$\mathfrak{h}_s := \mathfrak{h}/\mathfrak{h}_{\mathrm{diag}} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right) \middle| \, \exists\, D_1, D_2 \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array} \right) \in \mathfrak{h} \right\} \text{ and}$$

$$\mathfrak{h}_{\mathrm{sub}} := \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right) \in \mathfrak{h} \right\}.$$

We have $\mathfrak{h}_{\mathrm{diag}} = \mathrm{Lie}(A_{\mathrm{diag}})$. Neither $\mathfrak{h}_{\mathrm{diag}}$ nor $\mathfrak{h}_s$ are subalgebras of $\mathfrak{h}$. We have $\mathfrak{h} \subset \mathfrak{h}_{\mathrm{diag}} \oplus \mathfrak{h}_s$ but the nilpotent example, see §3.4, shows that the inclusion may be strict. Our reduction strategy will consist of three steps.

$\boxed{1}$ Find a gauge transformation $P = \mathrm{Id} + B$, where[8] $B \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$, and $\tilde{\mathfrak{h}} := \mathrm{Lie}(P[A])$ such that $\tilde{\mathfrak{h}}_s$ has minimal dimension. The result will depend on parameters.

$\boxed{2}$ Look for an eventual gauge transformation $P = \mathrm{Id} + B$, where $B \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})$ which, maps each element of $\tilde{\mathfrak{h}}_{\mathrm{sub}}$ to an element of $\tilde{\mathfrak{h}}_{\mathrm{sub}}$ whose coefficients are in the Wei-Norman decomposition of A_{diag}. This depends again on parameters.

$\boxed{3}$ Compute, with a Groebner basis, conditions on the remaining parameters to have a Lie algebra $\mathrm{Lie}(P[A])$ of minimal dimension.

Step $\boxed{1}$ is the main part of the algorithm. It consists in trying to eliminate as many generators as possible in $\mathfrak{h}_s$. Heuristically, Step $\boxed{1}$ seems to be always sufficient to obtain a reduced form. However, Steps $\boxed{2}$ and $\boxed{3}$ are necessary to have a mathematically guaranteed procedure. We use the isotypical decomposition $\mathfrak{gl}_{\mathrm{sub}} = \bigoplus_{i=1}^{\kappa} W_i$ of Proposition 2.13. Proposition 2.7 tells us that in the reduction process, we may (and will) perform a reduction on each isotypical block W_i independently. We consider an isotypical block $W = V_1 \oplus \cdots \oplus V_\nu$ where the V_i are indecomposable pairwise isomorphic Ψ-spaces. We follow the construction above Definition 2.18 to obtain a Ψ-isotypical flag on W:

$$W = W^{[\mu]} \supsetneq W^{[\mu-1]} \supsetneq \cdots \supsetneq W^{[1]} \supsetneq W^{[0]} = \{0\},$$

with $W^{[k]} = \bigoplus_{j=1}^{\nu_k} V_j^{[k]}$ and we have a Ψ-isomorphism $\phi_j : V_1^{[k]} \to V_j^{[k]}$. Recall that $W^{[k]}/W^{[k-1]}$ is a direct sum of pairwise isomorphic irreducible Ψ-spaces. Before we continue, we need to enrich our toolbox with the following fundamental classical algorithm.

4.1. Differential systems with a parametrized right-hand side

We recall a classical computational lemma on rational solutions of differential systems with a parametrized right-hand side. We reprove it here for self-containedness though it is well known to specialists.

Lemma 4.1 *(Singer (1991); Barkatou (1999); Berman (2002)). Let $m \in \mathbb{N}^*$. Given a matrix $\Lambda(x) \in \mathcal{M}_m(\mathbf{k})$ and t vectors $\vec{b}_i(x) \in \mathbf{k}^m$, we consider the differential system with parametrized right-hand-side*

$$Y'(x) = \Lambda(x)Y(x) + \sum_{i=1}^{t} s_i \vec{b}_i(x),$$

where the s_i are scalar parameters. The set of tuples $(F(x), (c_1, \ldots, c_t)) \in \mathbf{k}^m \times \mathcal{C}^t$, such that the differential system $Y'(x) = \Lambda(x)Y(x) + \sum_{i=1}^{t} c_i \vec{b}_i(x)$ admits a rational solution $Y(x) = F(x)$ is a finite-dimensional $\mathcal{C}$-vector space. Furthermore, with our assumptions on $\mathbf{k}$, one can effectively compute a basis of this vector space.

Proof. We give a short proof of this well known fact, from Berman (2002), page 889.

A vector $F(x) = (f_1(x), \ldots, f_m(x))^T \in \mathbf{k}^m$ is a rational solution of the differential system $Y'(x) = \Lambda(x)Y(x) + \sum_{i=1}^{t} c_i \vec{b}_i(x)$, for given constants c_i, if and only if the vector $Z(x) := (f_1(x), \ldots, f_m(x), c_1, \ldots, c_t)^T$ is a rational solution of the homogeneous first order system

$$Z'(x) = \left(\begin{array}{c|c} \Lambda(x) & \vec{b}_1(x), \ldots, \vec{b}_t(x) \\ \hline 0 & 0 \end{array}\right) Z(x).$$

The rational solutions of the latter form a $\mathcal{C}$-vector space which, by Assumption ②, see §1.1, can be explicitly computed. □

[8] Here, it would actually be enough to take $B \in \mathfrak{h}_s(\mathbf{k})$ but our choice simplifies both the exposition and the implementation for a minor additional cost.

Remark 4.2. Regarding the proof of Lemma 4.1, we see that we may replace $\mathbf{k}$ by any field which satisfies Assumption ② of Section 1.1. By Singer (1991), Lemma 3.5, we may thus replace the base field $\mathbf{k}$ by $\mathbf{k}(t_1,\dots,t_\ell)$ where $t_i'=0$ and the new constant field $\mathcal{C}(t_1,\dots,t_\ell)$ is a transcendental extension of $\mathcal{C}$.

4.2. Reduction on a level $W^{[k]}/W^{[k-1]}$ of the isotypical block W

Let us fix $k\in\{1,\dots,\mu\}$. Assume that we have performed reductions on the levels $W^{[\mu]}/W^{[\mu-1]}$, $\dots$, $W^{[k+1]}/W^{[k]}$ (this assumption being void if $k=\mu$) and that we want to perform reduction on the level $W^{[k]}/W^{[k-1]}$. Our matrix A has thus been transformed into a matrix $A^{[k]}$. As in the nilpotent example, see §3.4, the reduction on the previous levels may have introduced a set T_k of parameters in the off-diagonal coefficients of this matrix $A^{[k]}$, and an affine variety $\mathcal{T}_k$ defined by the algebraic conditions satisfied by these parameters. So, at this stage the matrix $A^{[k]}(x,\underline{t})$ has off-diagonal coefficients in $\mathbf{k}(T_k)$ with the constraint $\underline{t}\in\mathcal{T}_k$. The construction below will show how T_k, $\mathcal{T}_k$, and $A^{[k]}$ are built with a decreasing recursion from $T_\mu=\varnothing$, $\mathcal{T}_\mu=\varnothing$, and $A^{[\mu]}=A$. The matrix Ψ of the adjoint action of A_{diag} is unchanged at each step and does not depend on the parameters.

Let $B_{1,1},\dots,B_{r,1}$ be a basis of $V_1^{[k]}/V_1^{[k-1]}$. It induces a basis of $V_1^{[k]}/V_1^{[k-1]}\otimes_{\mathcal{C}}\mathcal{C}(T_k)$. The isomorphism $\phi_j: V_1^{[k]}\to V_j^{[k]}$ induces an isomorphism $\tilde{\phi}_j: V_1^{[k]}/V_1^{[k-1]}\to V_j^{[k]}/V_j^{[k-1]}$ so we may define $B_{i,j}:=\tilde{\phi}_j(B_{i,1})$ to obtain an adapted basis of $V_j^{[k]}/V_j^{[k-1]}$ and, hence, of $V_j^{[k]}/V_j^{[k-1]}\otimes_{\mathcal{C}}\mathcal{C}(T_k)$. In this basis, the restriction of Ψ to each $V_j^{[k]}/V_j^{[k-1]}\otimes_{\mathcal{C}}\mathcal{C}(T_k)$ will have the same matrix, which we call $\Lambda^{[k]}$. This matrix $\Lambda^{[k]}$ has coefficients in $\mathbf{k}$.

Let $\Psi^{[k]}:=\Psi|_{W^{[k]}/W^{[k-1]}}$ be the restriction of Ψ to $W^{[k]}/W^{[k-1]}$. We still call $\Psi^{[k]}$ the restriction to $(W^{[k]}/W^{[k-1]})\otimes_{\mathcal{C}}\mathbf{k}(T_k)$. In our adapted basis of $W^{[k]}/W^{[k-1]}$, the matrix of $\Psi^{[k]}$ is block diagonal with all blocks equal to $\Lambda^{[k]}$.

For any matrix $B\in W^{[k]}$, we have

$$\Psi(B)=\Psi^{[k]}(B)+\widetilde{B},\quad \text{with } \widetilde{B}\in W^{[k-1]}. \tag{4.1}$$

Simplifying notations, let us set $\nu=\nu_k$. We decompose the matrix $A^{[k]}$ of our system at this stage as

$$A^{[k]}(x,\underline{t})=\overline{A}(x,\underline{t})+\sum_{j=1}^{\nu}\left(\sum_{i=1}^{r}a_{i,j}(x,\underline{t})B_{i,j}\right).$$

The coefficients $a_{i,j}(x,\underline{t})$ are in $\mathbf{k}(T_k)$; the matrix $\overline{A}(x,\underline{t})$ represents the remaining components of parts of $A^{[k]}$, including the components $W^{[\ell]}/W^{[\ell-1]}\otimes_{\mathcal{C}}\mathbf{k}(T_k)$ with $\ell\neq k$.

We look for a gauge transformation of the form

$$P(x,\underline{t})=\mathrm{Id}_n+B \text{ where } B=\sum_{j=1}^{\nu}\left(\sum_{i=1}^{r}f_{i,j}(x,\underline{t})B_{i,j}\right), \tag{4.2}$$

with $f_{i,j}(x,\underline{t})\in\mathbf{k}(T_k)$. We apply Proposition 2.7 and (4.1) to obtain the existence of $\widetilde{B}_{i,j}\in W^{[k-1]}$ (the part of $\Psi(B)$ which is sent to $W^{[k-1]}$) such that

$$P[A^{[k]}]=\left[\overline{A}(x,\underline{t})+\sum_{j=1}^{\nu}\left(\sum_{i=1}^{r}f_{i,j}(x,\underline{t})\widetilde{B}_{i,j}\right)\right] + \underbrace{\sum_{j=1}^{\nu}\sum_{i=1}^{r}\Big(a_{i,j}(x,\underline{t})B_{i,j}+f_{i,j}(x,\underline{t})\Psi^{[k]}(B_{i,j})-f_{i,j}'(x,\underline{t})B_{i,j}\Big)}_{\text{components of } P[A^{[k]}] \text{ on } W^{[k]}/W^{[k-1]}}. \tag{4.3}$$

Suppose we hoped to remove all the $B_{i,j}$. We would have to remove all of the second sum in (4.3). For each $j \in \{1, \dots, \nu\}$, let

$$\vec{Y}_j := \begin{pmatrix} f_{1,j}(x,\underline{t}) \\ \vdots \\ f_{r,j}(x,\underline{t}) \end{pmatrix} \text{ and } \vec{b}_j := \begin{pmatrix} a_{1,j}(x,\underline{t}) \\ \vdots \\ a_{r,j}(x,\underline{t}) \end{pmatrix}.$$

The elimination conditions would become

$$\begin{cases} \vec{Y}_1' &= \Lambda^{[k]}\vec{Y}_1 &+ \vec{b}_1 \\ &\vdots & \\ \vec{Y}_\nu' &= \Lambda^{[k]}\vec{Y}_\nu &+ \vec{b}_\nu. \end{cases}$$

However, some of these systems may have no rational solution whereas some combination of the $\vec{Y}_i$ could be rational and lead to (partial) reduction. Indeed, by performing reduction, we are trying to eliminate irreducible Ψ-subspaces of $W^{[k]}/W^{[k-1]} \otimes_{\mathcal{C}} \mathcal{C}(T_k)$. By Goursat's Lemma, see Lemma 2.17, these are of the form $\{\sum_{i=1}^{\nu} c_j\tilde{\phi}_i(v),\ v \in V_1^{[k]}/V_1^{[k-1]} \otimes_{\mathcal{C}} \mathcal{C}(T_k)\}$ with $c_i \in \mathcal{C}(T_k)$. In order to perform reduction, we thus need to look for constants $\underline{c} = (c_1, \dots, c_\nu)$ with $c_i \in \mathcal{C}(T_k)$ such that the following system has a nonzero rational solution in $\mathbf{k}(T_k)$:

$$\vec{Y}' = \Lambda^{[k]}\vec{Y} + \sum_{i=1}^{\nu} c_i\vec{b}_i. \tag{4.4}$$

Note that $\Lambda^{[k]}$ does not depend on the parameters and Lemma 4.1 stays valid with the field $\mathbf{k}$ replaced by $\mathbf{k}(T_k)$, see Remark 4.2. To decide when the system has nonzero rational solutions, one first finds bounds on valuations at the poles and at infinity (this is possible because $\Lambda^{[k]}$ does not depend on the parameters); this reduces the problem to solving a system of linear equations whose right hand side depends linearly on the parameters c_i. The compatibility conditions for this system (obtained, for example, by gaussian elimination) yield a matrix $M(\underline{t})$ with coefficients in $\mathcal{C}(T_k)$ so that the system (4.4) has a rational solution if and only if

$$M(\underline{t}).\begin{pmatrix} c_1 \\ \vdots \\ c_\nu \end{pmatrix} = 0. \tag{4.5}$$

We want $M(\underline{t})$ with $\underline{t} \in \mathcal{T}_k$ to have a kernel of maximal dimension, as this kernel allows us to compute irreducible Ψ-subspaces that can be removed in the reduction process. Let $\mathcal{V}_k$ denote the algebraic conditions on T_k which encode the fact that $M(\underline{t})$ with $\underline{t} \in \mathcal{T}_k$ has minimal rank $\nu - d$; this can be computed for example with a Groebner basis, see Cox et al. (2007). We set $\mathcal{T}_{k-1} := \mathcal{T}_k \cap \mathcal{V}_k$. Now $\mathcal{T}_{k-1}$ is a finite union of irreducible algebraic varieties; this can again be computed with a Groebner basis (Cox et al., 2007). For each of these irreducible varieties, we proceed as follows. Applying these conditions to the matrix $M(\underline{t})$, we choose a basis $\mathcal{B}$ of $\ker(M(\underline{t}))$ in $\mathcal{C}(T_k)^\nu$. For $\underline{c}_j$ in $\mathcal{B}$, we compute the corresponding general rational solution $\vec{F}_j$ to the system $Y' = \Lambda^{[k]}Y + \sum_{i=1}^{\nu} c_{i,j}\vec{b}_i$. Note that $[\Lambda^{[k]}]$ may have rational solutions (this was the case in our nilpotent example, see §3.4) hence the need for a general solution. Note that at this stage it may be necessary to introduce additional parameters in order to express this general solution, in which case we add these new additional parameters to T_k to form T_{k-1}.

Remark 4.3. As we saw in the nilpotent example §3.4, the process of passing from $\mathcal{T}_k$ to $\mathcal{T}_{k-1}$ may add constraints that fix the value of a constant, thus withdrawing it from later computations.

Now we have found a new adapted basis $\overline{B}_{i,j}(\underline{t})$ of $W^{[k]}/W^{[k-1]} \otimes_{\mathcal{C}} \mathcal{C}(T_k)$ and a gauge transformation

$$P^{[k]}(x,\underline{t}) = \mathrm{Id}_n + \sum_{j=1}^{d}\left(\sum_{i=1}^{r} f_{i,j}(x,\underline{t})\overline{B}_{i,j}(\underline{t})\right).$$

Applying this gauge transformation will remove from $W^{[k]}/W^{[k-1]}$ the Ψ-spaces spanned by the $\overline{B}_{i,j}(\underline{t})$ with $j \leq d$. Because of the condition on minimality of the rank of M and the nature of the Ψ-subspaces of $W^{[k]}/W^{[k-1]} \otimes_{\mathcal{C}} \mathcal{C}(T_k)$, no other matrix in $W^{[k]}/W^{[k-1]} \otimes_{\mathcal{C}} \mathcal{C}(T_k)$ can be removed using a gauge transformation as in (4.2).

4.3. The full reduction

We now perform the reduction on the whole isotypical block W with its isotypical flag

$$W = W^{[\mu]} \supsetneq W^{[\mu-1]} \supsetneq \cdots \supsetneq W^{[1]} \supsetneq W^{[0]} = \{0\}.$$

Step 1. We start from a set of parameters $T_\mu = \varnothing$ and algebraic conditions $\mathcal{T}_\mu = \varnothing$. The matrix of the system is $A^{[\mu]} := A$. We perform the reduction process of Section 4.2 on $W^{[\mu]}/W^{[\mu-1]}$; we obtain a gauge transformation $P^{[\mu]}$ and $A^{[\mu-1]} := P^{[\mu]}(A^{[\mu]})$. If $\mu > 1$, we go down to $W^{[\mu-1]}$ and iterate until the level $W^{[1]}$. The complete gauge transformation used for the successive reductions on W is $P_W(x,\underline{t}) := \prod_{k=1}^{\mu} P^{[k]}(x,\underline{t})$. It contains a set $T_W := T_0$ of parameters t_i, subject to the set $\mathcal{T}_W := \mathcal{T}_0$ of algebraic conditions. Note that, by construction, the matrices $P^{[k]}$ all commute pairwise so the product $P_W(x,\underline{t})$ is well defined. The same remark will hold for the reduction matrix of Theorem 4.4 below.
Step 2. As explained in Remark 3.2, in the course of this reduction some "off-diagonal" element of the Lie algebra may be "absorbed" by turning a diagonal element of $\mathfrak{g}_{\mathrm{diag}}$ into a triangular one. Let $g_1(x), \dots, g_\delta(x) \in \mathbf{k}$ be the $\mathcal{C}$ linearly independent elements appearing in the Wei-Norman decomposition of $A_{\mathrm{diag}}(x)$. Let $B_1, \dots, B_k$ be a basis of $\mathfrak{h}_W$, the Ψ-space obtained after this step of reduction process. Let $A_W(x,\underline{t}) := P_W(x,\underline{t})[A]$. We have to compute the set of $f_i \in \mathbf{k}(T_W)$, $C_1, \dots, C_\delta \in \mathcal{M}_n(\mathcal{C}(T_W))$ such that $\widetilde{P}_W[A_W] = A_W + C_1 g_1 + \cdots + C_\delta g_\delta$, where $\widetilde{P}_W(x) = \mathrm{Id}_n + \sum_{i=1}^{k} f_i B_i$. By Proposition 2.7, this is equivalent to solving an inhomogeneous linear differential equation in the same form as the one in Lemma 4.1 in the field $\mathbf{k}(T_W)$. This provides a new set of parameters that we must add to T_W and additional algebraic constraints $\mathcal{T}_W$. Using again a Groebner basis, compute an element $\underline{t}_0 \in \mathcal{T}_W$ such that $Lie(\widetilde{P}_W(x,\underline{t}_0)[A_W(x,\underline{t}_0)])$ has minimal dimension (this is a rank optimization computation). Finally, set $P_W(x) := \widetilde{P}_W(x,\underline{t}_0)$.

Theorem 4.4. *For each isotypical block W_i in the isotypical decomposition $\mathfrak{gl}_{\mathrm{sub}} = \bigoplus_{i=1}^{\kappa} W_i$, let $P_{W_i}(x)$ denote the (partial) reduction matrix constructed in the above paragraph. Now let $P(x) := \prod_{i=1}^{\kappa} P_{W_i}(x)$ and $A_{red}(x) := P(x)[A(x)]$. Then the system $[A_{red}(x)]$ is in reduced form and $P(x)$ is the corresponding reduction matrix.*

Remark 4.5. In many situations, like the first three examples, no parameters are required to reduce the system. In that case, no Groebner bases are needed and only linear algebra is used in the reduction process, making the algorithm quite effective.

Proof. In virtue of Theorem 2.4, we deduce that there exists a gauge transformation $Q \in \left\{\mathrm{Id}_n + B(x), B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\}$, such that $[Q[A_{red}]]$ is in reduced form. We have $Q[A_{red}] = QP[A]$, where $QP \in \left\{\mathrm{Id}_n + B(x), B(x) \in \mathfrak{gl}_{\mathrm{sub}}(\mathbf{k})\right\}$. Let $\mathfrak{h} := \mathrm{Lie}(A_{red})$. As above, for $\star \in \{\mathfrak{g}, \mathfrak{h}\}$, we let

$$\star_{diag} := \left\{ \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right) \middle| \ \exists S \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array}\right) \in \star \right\},$$

$$\star_s := \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array}\right) \middle| \ \exists D_1, D_2 \text{ such that } \left(\begin{array}{c|c} D_1 & 0 \\ \hline S & D_2 \end{array}\right) \in \star \right\} \text{ and}$$

$$\star_{sub} := \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right) \in \star \right\}.$$

Since $[Q[A_{red}]]$ is in reduced form, its Lie algebra is $\mathfrak{g}$. By Remark 1.9, $\mathfrak{g} \subset \mathfrak{h}$. Then, $\mathfrak{g}_s \subset \mathfrak{h}_s$. By construction, the gauge transformation P minimizes $\mathfrak{h}_s$, so $\mathfrak{h}_s \subset \mathfrak{g}_s$ and we have $\mathfrak{h}_s = \mathfrak{g}_s$. By Proposition 2.7, $\mathfrak{h}_{diag} = \mathfrak{g}_{diag}$. Now, as explained in Remark 3.2, we still could have a strict inclusion $\mathfrak{g} \subsetneq \mathfrak{h}$. If that were the case, the condition $\mathfrak{g} \subsetneq \mathfrak{h}$ would imply by minimality of $\mathfrak{g}_s$, that a gauge transformation in $\mathfrak{h}_{\mathrm{sub}}(\mathbf{k})$ would transform the coefficient of an off-diagonal element into one that is present in the Wei-Norman decomposition of A_{diag} (the "absorption" mechanism described in Remark 3.2, turning a diagonal element of $\mathfrak{h}$ into a triangular one). By the second minimality condition, see Step 2 above, this phenomenon does not occur and we conclude that $\mathfrak{g} = \mathfrak{h}$ and $[A_{red}]$ is in reduced form. $\square$

5. A general algorithm for reducing a differential system

We now have tools to put general linear differential systems, i.e. those whose diagonal part may have more than two diagonal blocks, into reduced form.

5.1. An iteration lemma

In order to iterate the reduction process of §4 to block triangular systems, we need the following lemma.

Lemma 5.1. *Let $n_1, n_2, n_3 \in \mathbb{N}^*$, and for $i \in \{1, 2, 3\}$, let $A_i \in \mathcal{M}_{n_i}(\mathbf{k})$. Assume that the differential systems with respective matrices (in what follows, S is an $n_3 \times n_2$ matrix with coefficients in $\mathbf{k}$)*

$$\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & A_3 \end{array} \right) \text{ and } \left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array} \right)$$

are in reduced form. Then, letting

$$A := \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & S & A_3 \end{array} \right),$$

the system $[A]$ is in reduced form.

Proof. By our first assumption, we may apply the reduction process of §4, see Theorem 4.4, to the system with matrix

$$\left(\begin{array}{cc|c} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ \hline 0 & S & A_3 \end{array} \right).$$

Let $n := n_1 + n_2 + n_3$. Due to Theorem 2.4, there exists a reduction matrix of the form

$$P := \left(\begin{array}{c|c|c} \mathrm{Id}_{n_1} & 0 & 0 \\ \hline 0 & \mathrm{Id}_{n_2} & 0 \\ \hline P_1 & P_2 & \mathrm{Id}_{n_3} \end{array} \right) \in \mathcal{M}_n(\mathbf{k}).$$

We then find

$$\begin{aligned} \Psi\big((P_1 \mid P_2) \big) &= -(P_1 \mid P_2) \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right) + A_3 (P_1 \mid P_2) \\ &= (A_3 P_1 - P_1 A_1 \mid A_3 P_2 - P_2 A_2). \end{aligned}$$

With Proposition 2.7, we find that

$$P[A]=\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline A_3P_1-P_1A_1-P_1' & S+A_3P_2-P_2A_2-P_2' & A_3 \end{array}\right).$$

Since the latter is reduced, we know that $\mathrm{Lie}(P[A])=\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the differential Galois group of $[A]$. By Remark 1.9, we find $\mathrm{Lie}(P[A])=\mathfrak{g}\subset \mathrm{Lie}(A)$. By construction of $\mathrm{Lie}(A)$, any matrix in $\mathrm{Lie}(A)$ must have the form $\left(\begin{array}{c|c|c} \star & 0 & 0 \\ \hline 0 & \star & 0 \\ \hline 0 & \star & \star \end{array}\right)$. As $\mathfrak{g}\subset \mathrm{Lie}(A)$, the same holds for $\mathfrak{g}$. Since P_1 acts only on the bottom left block of $P[A]$, we thus find that without loss of generality, we may assume $P_1=0$. Then we see that the reduction matrix will have no effect on the A_1 block but will only act on the block $\boxed{\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array}}$. As the latter is in reduced form, we find, see Proposition 1.7, that for all P_2, the Lie algebra of

$$\left(\begin{array}{c|c} \mathrm{Id}_{n_2} & 0 \\ \hline P_2 & \mathrm{Id}_{n_3} \end{array}\right)\left[\left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array}\right)\right]=\left(\begin{array}{c|c} A_2 & 0 \\ \hline S+A_3P_2-P_2A_2-P_2' & A_3 \end{array}\right)$$

contains the Lie algebra of $\left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array}\right)$. It is now clear that we have the inclusion $\mathrm{Lie}(A)\subset \mathrm{Lie}(P[A])=\mathfrak{g}$. By Remark 1.9 $\mathfrak{g}\subset \mathrm{Lie}(A)$ and we find $\mathrm{Lie}(A)=\mathfrak{g}$, i.e. $[A]$ is in reduced form. $\square$

5.2. The algorithm

Let us now describe the global reduction process. Let $\mathcal{A}(x)\in\mathcal{M}_n(\mathbf{k})$ and consider the linear differential system $Y'(x)=\mathcal{A}(x)Y(x)$. The contribution of this paper to this general algorithm is part (4) below.

(1) Factor the linear differential system, see e.g. Compoint and Weil (2004); Barkatou (2007); van der Hoeven (2007) and references therein. We obtain a matrix $A(x)\in\mathcal{M}_n(\mathbf{k})$ such that the system $Y'(x)=\mathcal{A}(x)Y(x)$ is equivalent to $Y'(x)=A(x)Y(x)$, where

$$A(x)=\left(\begin{array}{c|c|c|c} A_1(x) & & & 0 \\ \hline & \ddots & & \\ \hline & S_{i,j}(x) & \ddots & \\ \hline & & & A_k(x) \end{array}\right),$$

and each diagonal block $Y'(x)=A_\ell(x)Y(x)$, $\ell=1\dots k$, is irreducible.

(2) Using for example Barkatou et al. (2016); Aparicio-Monforte et al. (2013), compute a reduced form of the block-diagonal system

$$Y'(x)=\left(\begin{array}{c|c|c} A_1(x) & & 0 \\ \hline & \ddots & \\ \hline 0 & & A_k(x) \end{array}\right)Y(x).$$

Note that the reduced form $\left(\begin{array}{c|c|c} A_{1,red}(x) & & 0 \\ \hline & \ddots & \\ \hline 0 & & A_{k,red}(x) \end{array}\right)$ may have entries in a finite algebraic extension $\mathbf{k_0}$ of $\mathbf{k}$. Let $P_{diag}(x)\in\mathrm{GL}_n(\mathbf{k_0})$ be the corresponding gauge transformation.

(3) Compute

$$A_{diag,red}(x) := P_{diag}(x)[A(x)] = \left(\begin{array}{c|c|c|c} A_{1,red}(x) & & & 0 \\ \hline & \ddots & & \\ \hline \vdots & \ddots & A_{k-1,red}(x) & \\ \hline \mathfrak{S}_{k,1}(x) & \dots & \mathfrak{S}_{k,k-1}(x) & A_{k,red}(x) \end{array}\right).$$

(4) Let $\mathcal{A}_k := A_{k,red}$ and $\ell := k$. While $\ell \geq 2$ do
  (a) Apply the reduction process of §4, see Theorem 4.4 with $\mathbf{k}$ replaced by $\mathbf{k_0}$, see Remark 5.2, to compute a reduced form of

$$\left(\begin{array}{c|c} A_{\ell-1,red}(x) & 0 \\ \hline \mathfrak{S}_{\ell-1}(x) & \mathcal{A}_{\ell}(x) \end{array}\right), \text{ where } \mathfrak{S}_{\ell-1}(x) := \begin{pmatrix} \mathfrak{S}_{\ell,\ell-1}(x) \\ \vdots \\ \mathfrak{S}_{k,\ell-1}(x) \end{pmatrix},$$

  is the block column below $A_{\ell-1,red}(x)$ in $A_{diag,red}(x)$.
  (b) Let $\mathcal{A}_{\ell-1}(x)$ be this new reduced form. Let $\ell := \ell - 1$ and iterate.
  End do.

The correctness of Step 4a is ensured by Lemma 5.1. It follows that the resulting system $Y'(x) = \mathcal{A}_1(x)Y(x)$ is a reduced form of $Y'(x) = A(x)Y(x)$.

Remark 5.2. In Step 4a, we may have to introduce an algebraic extension $\mathbf{k_0}$ of $\mathbb{C}(x)$ and compute solutions in $\mathbf{k_0}$ of linear differential systems with coefficients in $\mathbf{k_0}$ and a parameterized right hand side. This can be reduced (see Singer, 1991) to computing solutions in $\mathbf{k}$ of a system of bigger dimension. From §1.1, we see that it is still possible. Although it would require some extra work, it would not be a practical obstacle.

Example 5.3. When $k = 3$, Step 4 performs the following. Consider the matrix given by Step 3.

$$A_{diag,red}(x) = \left(\begin{array}{c|c|c} A_{1,red}(x) & 0 & 0 \\ \hline \mathfrak{S}_{2,1}(x) & A_{2,red}(x) & 0 \\ \hline \mathfrak{S}_{3,1}(x) & \mathfrak{S}_{3,2}(x) & A_{3,red}(x) \end{array}\right).$$

We start by reducing the matrix $\left(\begin{array}{c|c} A_{2,red}(x) & 0 \\ \hline \mathfrak{S}_{3,2}(x) & A_{3,red}(x) \end{array}\right)$ to obtain $\mathcal{A}_2(x)$. Then we reduce

$$\left(\begin{array}{c|c} A_{1,red}(x) & 0 \\ \hline \mathfrak{S}_{3}(x) & \mathcal{A}_{2}(x) \end{array}\right),$$

with $\mathfrak{S}_3(x) := \begin{pmatrix} \mathfrak{S}_{2,1}(x) \\ \mathfrak{S}_{3,1}(x) \end{pmatrix}$, to obtain the final reduced matrix.

6. Computation of the Lie algebra of the differential Galois group

We consider a differential system $[\mathcal{A}(x)] : Y'(x) = \mathcal{A}(x)Y(x)$ with $\mathcal{A}(x) \in \mathcal{M}_n(\mathbf{k})$. In this section, we review how to compute the Lie algebra $\mathfrak{g}$ of the differential Galois group G. We first assume that $[\mathcal{A}(x)]$ is in reduced form; the non-reduced case is addressed in Remark 6.1. We stress the fact that, from now on, all the results are mostly well known and are included for completeness. We can find a Wei-Norman decomposition of $\mathcal{A}(x)$. We compute the smallest $\mathcal{C}$-vector space containing its generators and stable under the Lie bracket. Let $B_1, \dots, B_\sigma$ be a basis of this space. We know that the smallest *algebraic* Lie algebra containing the B_i is $\mathfrak{g}$.

An algorithm for computing the smallest algebraic Lie algebra containing the B_i can be found in Fieker and de Graaf (2007). In order to be self-contained, we are going to summarize this work.

It follows from Chevalley (1951), Chapter II, Theorem 14, that the Lie algebra generated by a finite family of algebraic Lie algebras is algebraic. Therefore, to compute $\mathfrak{g}$ it is sufficient to be able to compute $\mathfrak{g}_i := \mathrm{Lie}(B_i)$ for all $i \in \{1, \dots, \sigma\}$.

To be able to compute $\mathfrak{g}$, we are thus reduced to the following problem: given a matrix $B \in \mathcal{M}_n(\mathcal{C})$, compute $\mathrm{Lie}(B)$. Let $B = D + N$ be the additive Jordan decomposition of B, where D is diagonalizable, N is nilpotent, and $DN = ND$. From Chevalley (1951), Chapter II, Theorem 10, we deduce that

$$\mathrm{Lie}(B) = \mathrm{Lie}(D) \oplus \mathrm{Lie}(N).$$

Let us compute $\mathrm{Lie}(N)$. The matrix N is nilpotent. As we can see in Chevalley (1951), Chapter II, §13, Proposition 1, the $\mathcal{C}$-vector space spanned by N is an algebraic Lie algebra, with corresponding algebraic group $\{\exp(\alpha N), \alpha \in \mathcal{C}\}$, which is a vector group. Therefore,

$$\mathrm{Lie}(N) = \mathrm{Vect}_{\mathcal{C}}(N).$$

Let us compute $\mathrm{Lie}(PD_0P^{-1})$, where $PD_0P^{-1} = D$, P is an invertible matrix, and $D_0 = \mathrm{Diag}(d_1, \dots, d_n)$ is a diagonal matrix. Set

$$\Delta := \left\{ (e_1, \dots, e_n) \in \mathbb{Z}^n \middle| \sum_{\ell=1}^{n} e_\ell d_\ell = 0 \right\}.$$

By Chevalley, see for instance Fieker and de Graaf (2007), Theorem 2, we obtain

$$\mathrm{Lie}(D_0) = \left\{ \mathrm{Diag}(a_1, \dots, a_n) \in \mathcal{C}^n \middle| \sum_{\ell=1}^{n} e_\ell a_\ell = 0, \forall (e_1, \dots, e_n) \in \Delta \right\},$$

and

$$\mathrm{Lie}(D) = P\mathrm{Lie}(D_0)P^{-1}.$$

Remark 6.1. If we start from a system $[\mathcal{A}(x)]$ which is not in reduced form, the algorithm presented in §5 will compute a finite field extension $\mathbf{k}_0$ of $\mathbf{k}$ and a matrix $\mathcal{A}_{red}(x) \in \mathcal{M}_n(\mathbf{k}_0)$ such that $[\mathcal{A}_{red}(x)]$ is a reduced form of $[\mathcal{A}(x)]$. Let $G_{\mathbf{k}_0}$ be the differential Galois group over $\mathbf{k}_0$. Since the gauge transformation that performs the reduction has entries in $\mathbf{k}_0$, and the Galois group is invariant under gauge transformation, $G_{\mathbf{k}_0}$ is the differential Galois group of $[\mathcal{A}_{red}(x)]$ over $\mathbf{k}_0$. By Lemma 32 in Aparicio-Monforte et al. (2013), $G_{\mathbf{k}_0}$ is connected. Note that by the Galois correspondence, see van der Put and Singer (2003), Proposition 1.34, $G/G_{\mathbf{k}_0}$ is finite, which means that $G_{\mathbf{k}_0}$ is the connected component of the identity of G. So the Lie algebras of G and $G_{\mathbf{k}_0}$ coincide and we may apply the above construction to obtain $\mathrm{Lie}(\mathcal{A}_{red}(x))$ and hence $\mathfrak{g}$.

7. Computation of the differential Galois group of a reduced form

Let $\mathcal{A}(x) \in \mathcal{M}_n(\mathbf{k})$; let G be the differential Galois group of $[\mathcal{A}(x)]$ and $\mathfrak{g}$ be the Lie algebra. We now know, using §6, how to compute $\mathfrak{g}$. The goal of this section is to explain how, theoretically, one may recover G from $\mathfrak{g}$ when $[\mathcal{A}(x)]$ is in reduced form. The problem of recovering a connected group from its Lie algebra is solved in de Graaf (2009). In this section, we propose a solution, based on ideas from (Derksen et al., 2005, Section 3), but we do not claim originality nor algorithmic efficiency in what follows; this section is included for completeness.

Since $[\mathcal{A}(x)]$ is in reduced form, by Lemma 32 in Aparicio-Monforte et al. (2013), we obtain that G is connected. Let $B_1, \dots, B_\sigma$ be a basis of the $\mathcal{C}$-vector space $\mathfrak{g}$. As G is connected, it is the smallest algebraic group containing $\exp(\mathfrak{g})$. It follows that

$$G = \overline{\langle \exp(B_1), \dots, \exp(B_\sigma) \rangle}$$

it is the smallest algebraic group that contains the matrices $\exp(B_i)$.

So let us compute $\overline{\langle \exp(B_1), \dots, \exp(B_\sigma) \rangle}$. This problem has been solved in full generality in (Derksen et al., 2005, Section 3). It is simpler here, since the algebraic group we are looking for is connected. We start by a classical observation taken from (Derksen et al., 2005, Section 3.1).

Lemma 7.1 *(Derksen et al. (2005), Section 3.1). Let V_1, V_2 be affine varieties over $\mathcal{C}$ and $\psi : V_1 \to V_2$ be a morphism of affine varieties. Let $X \subset V_1$ be a Zariski closed subset. If we have generators for the vanishing ideal $\mathfrak{f} \subset \mathcal{C}[V_1]$ of X, then we may compute $\overline{\langle \psi(X) \rangle}$.*

Proof. For self-containedness, we reproduce the proof from (Derksen et al., 2005, Section 3.1). The morphism $\psi : V_1 \to V_2$ corresponds to a homomorphism $\psi^\star : \mathcal{C}[V_2] \to \mathcal{C}[V_1]$ of the coordinate rings (see Cox et al. (2007), Proposition 8 in Chapter 4]). Given generators of the vanishing ideal $\mathfrak{f} \subset \mathcal{C}[V_1]$ of X, one can compute generators of the ideal $(\psi^\star)^{-1}(\mathfrak{f})$ using a Groebner basis. The latter are the generators of $\overline{\langle \psi(X) \rangle}$. $\square$

We begin by computing the Zariski closure of the group generated by a single matrix $M := \exp(B)$, with $B \in \mathfrak{g}$. We have a Dunford decomposition $B = S + N$ with S semi-simple, N nilpotent and $[S, N] = 0$. So $\exp(B) = D \cdot U$ with $D := \exp(S)$ diagonalizable and $U := \exp(N)$ unipotent.

As $[D, U] = 0$, we find that $\overline{\langle M \rangle} = \overline{\overline{\langle D \rangle}.\overline{\langle U \rangle}}$. Using Lemma 7.1, if we are able to compute $\overline{\langle D \rangle}$ and $\overline{\langle U \rangle}$, we see that we may compute $\overline{\langle M \rangle} = \overline{\overline{\langle D \rangle}.\overline{\langle U \rangle}}$. Thus, what is left for us to do is to treat the cases where M is unipotent or diagonalizable.

We start with the unipotent case. As N is nilpotent, the map

$$\psi : t \mapsto \exp(tN)$$

is an algebraic map from $V_1 := \mathcal{C}$ to $V_2 := \mathrm{GL}_n(\mathcal{C})$; moreover, $\exp(tN)$ is a linear combination of a finite number of powers of N. Pick a matrix $M = (x_{i,j})$ of indeterminates and eliminate t from the equations $M - \exp(tN) = 0$; this makes sense because $\exp(tN)$ is polynomial in t. As shown in Lemma 7.1, this allows us to recover the Zariski closure of the image of $\mathcal{C}$ under ψ, which is $\overline{\langle U \rangle}$.

We now treat the diagonalizable case. We have $D = \exp(S)$ with S semi-simple. We may diagonalize S so that, letting

$$\mathcal{D} := \mathrm{Diag}(\lambda_1, \dots, \lambda_n), \quad D_0 := \exp(\mathcal{D}) = \mathrm{Diag}(d_1, \dots, d_n) \text{ with } d_i = \exp(\lambda_i),$$

we have $S = P\mathcal{D}P^{-1}$ and $D = PD_0P^{-1}$ for some $P \in \mathrm{GL}_n(\mathcal{C})$. In order to compute $\overline{\langle D \rangle}$, it is sufficient to understand the algebraic relations between the eigenvalues of D. This will be done in the same way as in the computation of the Lie algebra of a diagonal matrix. As we see in Derksen et al. (2005), Section 3.3., the ideal that generates $\overline{\langle D_0 \rangle}$ will be obtained from the $e_1, \dots, e_n \in \mathcal{C}$ such that $(d_1)^{e_1} \cdots (d_n)^{e_n} = 1$ with $d_i = \exp(\lambda_i)$. So, we set

$$\Delta' := \left\{ (e_1, \dots, e_n) \in \mathbb{Z}^n \middle| \sum_{\ell=1}^{n} e_\ell \lambda_\ell = 0 \right\},$$

and we find

$$\overline{\langle D_0 \rangle} = \left\{ (d_1, \dots, d_n) \in (\mathcal{C}^*)^n \middle| \prod_{\ell=1}^{n} (d_\ell)^{e_\ell} = 1, \forall (e_1, \dots, e_n) \in \Delta' \right\}.$$

As the map $X \mapsto PXP^{-1}$ is algebraic, Lemma 7.1 tells us that we may compute $\overline{\langle D \rangle}$ from the relation

$$\overline{\langle D \rangle} = P\overline{\langle D_0 \rangle}P^{-1}.$$

Now that we know how to compute the $\overline{\langle \exp(B_i) \rangle}$, for every $i \in \{1, \dots, \sigma\}$, Lemma 7.1 shows that we may compute $\overline{\langle \exp(B_1), \dots, \exp(B_\sigma) \rangle} = G$ and we are done.

Remark 7.2. If we start from a system $[\mathcal{A}(x)]$ which is not in reduced form, the algorithm presented in §5 will compute a finite field extension $\mathbf{k}_0$ of $\mathbf{k}$ and a matrix $\mathcal{A}_{red}(x) \in \mathcal{M}_n(\mathbf{k}_0)$ such that $[\mathcal{A}_{red}(x)]$ is a reduced form of $[\mathcal{A}(x)]$. By Remark 6.1, the differential Galois groups over $\mathbf{k}_0$ of $[\mathcal{A}_{red}(x)]$ and $[\mathcal{A}(x)]$ coincide and are equal to G°, the connected component of the identity of G. The defining ideal of G° gives the algebraic relations over $\mathbf{k}_0$ inside the Picard-Vessiot extension.

Example 7.3. We have given in §3.4, the generators of the Lie algebra of the nilpotent example. A Zariski-dense subgroup of the differential Galois group is generated by $t_1\mathrm{Id}_8$ with $t_1 \in \mathcal{C}^*$, and the family of the following matrices, with $t_2, \ldots, t_5 \in \mathcal{C}$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline t_2 & 0 & 0 & 0 & 1 & 0 & t_2 & 0 \\ 0 & -t_2 & 0 & 0 & 0 & 1 & 0 & -t_2 \\ 0 & 0 & -t_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t_2 & 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ t_3 & 0 & 0 & 0 & 0 & 1 & t_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -t_3 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2t_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2t_4 & 1 & 0 & 0 & 0 & 0 \\ \hline -t_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t_4 & 0 & 0 & 2t_4 & 1 & 0 & 0 \\ 0 & 0 & -t_4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t_4 & 0 & 0 & 2t_4 & 1 \end{array}\right), \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t_5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Using the above procedure, one may recover the equations of the Galois group from the data of these generators.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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