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Encoding Algebraic Power Series

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Abstract The division algorithm for ideals of algebraic power series satisfying Hironaka's box condition is shown to be finite when expressed suitably in terms of the defining polynomial codes of the series. In particular, the codes of the reduced standard basis of the ideal can be constructed effectively.

Keywords Algebraic power series · Henselization · Power series division · Gröbner and standard bases · Polynomial codes

Mathematics Subject Classification 13P10 · 13J05 · 14Q20 · 16W60 · 30H50 · 32A05 · 32A38

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1 Introduction

Let $H : \mathbb{A}_K^{n+p} \to \mathbb{A}_K^p$ be a polynomial map between affine spaces over a field K. Assume that H satisfies at 0 the assumption of the implicit function theorem,

$$\partial_y H(0,0) \in \mathrm{Gl}_p(K) \text{ and } H(0,0) = 0,$$

where $y = (y_1, \ldots, y_p)$ denote coordinates on $\mathbb{A}_K^p$. Then there is a unique formal power series solution $h = (h_1, \ldots, h_p)$ of the system $H(x, y) = 0$ at 0, say

$$H(x, h(x)) = 0 \text{ and } h(0) = 0.$$

Actually, the components h_i are algebraic power series in the sense that each h_i satisfies a univariate polynomial equation over the polynomial ring $K[x_1, \ldots, x_n]$. Conversely, any algebraic power series h_1 vanishing at 0 arises in this way: there is a system of polynomial equations $H(x, y) = 0$ satisfying the assumption of the implicit function theorem so that the unique solution h has first component h_1. This is known as the Artin–Mazur theorem [2,7,9]. The characterization allows one to encode algebraic power series by a polynomial vector $H \in K[x, y]^p$ as above. The advantage of this code in comparison with taking the minimal polynomial lies in the fact that the latter determines the algebraic series only up to conjugation, so that extra information is necessary to specify the series, typically a sufficiently high truncation of the Taylor expansion. In contrast, the polynomial code H determines the series h_1 completely and is easy to handle algebraically.

Phrased more abstractly, the Henselization of the localization of $K[x_1, \ldots, x_n]$ at the maximal ideal $(x_1, \ldots, x_n)$ can be realized as the inductive limit of essentially étale extensions [5,9,16,38,49]. Therefore any algebraic power series belongs to such an extension—which, by definition, can be described by a code as above.

It is then natural to ask to what extent operations with algebraic power series can be expressed in terms of their code; and, if this is the case for a certain operation, what will be the respective formulation of the operation in terms of the code.

In the present article we answer this question for the division of algebraic power series and for the construction of reduced standard bases of ideals. When just considered for formal power series, the division is an infinite algorithm in the infinitely many coefficients of the series. If the involved series are algebraic and satisfy Hironaka's box condition (to be defined below, see Sect. 3), Lafon in the principal ideal case and Hironaka in general have shown that the remainder of the division is again an algebraic series [27,33], cf. also [9, Theorem 8.2.9, p. 169]. As a consequence, the reduced standard basis of the ideal is also formed by algebraic series. This fact was used for instance by Hironaka in order to construct idealistic exponents of singularities on étale neighborhoods and to control the behavior of the local resolution invariant ν^* under blowup [27,28, chap. III].

The beforementioned box condition is the natural extension to the case of ideals of the notion of x_n-regularity of a series. It postulates the existence of a specific Rees decomposition—namely one which is generated by monomials in an appropriate coordinate system—of the quotient module of the power series ring modulo the given ideal, cf. [43] and section 2. Algorithms to determine Rees decompositions have been proposed by Sturmfels–White [52]. They rely on the construction of (not necessarily reduced) standard bases.

Starting with a system of algebraic generators of an ideal with box condition it is not at all clear how to construct, from the polynomial codes of the generators, the codes of the algebraic series defining the reduced standard basis, or, respectively, the codes of the quotients and the remainder of the division of a given algebraic series by the ideal. This question will be the subject of the article. The main result is the following (see Theorem 11.1 for the precise statement).

Theorem *Let I be an ideal of $K[[x_1, \dots, x_n]]$ generated by algebraic power series $g_1, \dots, g_r$. Assume that the initial ideal $\mathrm{in}(I)$ of I satisfies Hironaka's box condition. There exists a finite algorithm which computes, for any algebraic power series f, from the polynomial family codes of f and $g_1, \dots, g_r$ the family codes of algebraic power series $a_1, \dots, a_r$ and c so that*

$$f = a_1 g_1 + \cdots + a_r g_r + c$$

is the formal power series division of f by $g_1, \dots, g_r$.

For the principal ideal case, i.e., the Weierstrass division, such an algorithm has been proposed and proven to work by Alonso–Mora–Raimondo [2]. This algorithm is already quite complicated. The general case, i.e., the division of one series by several series, is substantially more intricate and resisted for a long time.

In this paper we will describe explicitly how to operate with the codes of algebraic power series in order to perform the division in general. This, of course, reproves Lafon's and Hironaka's existential results. But, more importantly, it provides a precise manual of how to express algebraic operations with algebraic power series in terms of their codes. This is by no means trivial, and the resulting algorithm, when carried out in a concrete example, turns out to have high complexity (we give one explicit computation in the Appendix).

From a logical or operational point of view, the algorithm is very interesting. It is built on two simultaneous inductions, both on the number of variables, which resemble the induction which appears in the proof of the Artin approximation theorem [6]. Coordinates in the affine space and generators of the ideals have to be chosen very carefully so as to make the argument work. But once this is done appropriately, the proofs develop quite naturally. In this sense, we are not only able to codify algebraic power series—we know and understand how this codification mimics their manipulation in the division process.

Behind the curtain, there resides a finiteness principle which is ubiquitous in algebraic geometry and commutative algebra: the Noether normalization lemma, or, phrased differently, the finiteness of certain morphisms. In our context, this finiteness is first met in the notion of x_n-regularity of power series in the Weierstrass division, and then also in Hironaka's box condition and our concept of echelon (which is a Rees decomposition of a prescribed combinatorial type). It is the prerequisite for a subtle induction on the number of variables, but has the drawback that in the induction step one has to consider modules instead of ideals. This aggravates the notation, though modules are the natural context to work with.

The nicest part of our algorithm is what we call *virtual division*, a trick which has already appeared in various disguises in the literature, e.g., in the work of Artin, Malgrange, Mora, Pfister–Popescu and Alonso–Mora–Raimondo: when dividing formal power series expand them with respect to one variable and write the coefficient series in the remaining variables as new unknown variables. If this is done with the necessary caution, the successive operations in the division of the formal power series can be carried out in terms of these virtual series and will then be *finite* processes. To make this approach work in reality, a precise understanding of the structure of the division algorithm is mandatory.

To resume and rephrase the above, our division algorithm for the codes of algebraic power series shows that the division is a finite process once you succeed to interpret certain packages of infinitely many data (i.e., coefficient series) as single objects which undergo a uniform transformation under division. The complexity of the algorithm shows that this encryption is by no means obvious.

The emphasis of the paper is theoretical—actual computations become quickly unfeasible. We rather provide insight and methods of how to manipulate algebraic power series abstractly within finite algorithms. This may turn out to be useful in other situations where one aims at or needs finiteness assertions: passage to étale neighborhoods, Noetherianity, semicontinuity of invariants of complete local rings, recursion theory for generating series, ...

Example Let us briefly explain the method of this paper in the special case of the construction of the code of the Weierstrass normal form of an x_n-regular power series $g(x)$ of order d. Assume for simplicity that g is actually a polynomial, say $g(x) = G(x) \in K[x]$ (capital letters will be reserved throughout for polynomials). Introduce new variables $u_0, \ldots, u_{d-1}$ and define a polynomial $B \in K[x_n, u]$ as $B(x_n, u) = x_n^d + \sum_{j=0}^{d-1} u_j \cdot x_n^j$. This is our candidate presentation for the Weierstrass normal form of G. It then suffices to determine (algebraic) series $u_0(x'), \ldots, u_{d-1}(x') \in K[[x']] = K[[x_1, \ldots, x_{n-1}]]$ such that the series $b(x)$ obtained from B by substitution of u_j by $u_j(x')$, say

$$b(x) = B(x_n, u(x')) = x_n^d + \sum_{j=0}^{d-1} u_j(x') \cdot x_n^j,$$

equals the Weierstrass normal form of G. Instead of constructing the series $u_j(x')$ directly, we shall develop a procedure to determine their code (in the sense described above, see Sect. 6 for details). To do this, observe first that x_n^d is the initial monomial of G with respect to the lexicographic order $<_{lex}$ on $\mathbb{N}^n$ for which $(1, 0, \ldots, 0) > \ldots > (0, \ldots, 0, 1)$, i.e., the exponent of x_n^d is the smallest element with respect to $<_{lex}$ of the support of G (this uses that $u_j(0) = 0$ since b has order d at 0). The usual power series division of the monomial x_n^d by G with respect to this initial monomial then yields a formal power series remainder $r(x) = \sum_{j=0}^{d-1} u_j(x') \cdot x_n^j$ such that $x_n^d - r(x)$ is the Weierstrass normal form of G. This division is in general an infinite process.

The key point now is to view x_n^d alternatively as the *leading* monomial of the polynomial B with respect to a suitable monomial order $<_\omega$ on $\mathbb{N} \times \mathbb{N}^d$, i.e., the exponent of x_n^d becomes the *largest* element with respect to $<_\omega$ of the support of B. Indeed, just take for $<_\omega$ an order such that $u_j \ll x_n$ for $j = 0, \dots, d-1$. Then $u_j \cdot x_n^j < x_n^d$ for $j < d$ and hence x_n^d will be the largest monomial of G with respect to $<_\omega$. This now allows us to divide G by B polynomially with respect to the leading monomial x_n^d, say

$$G = Q \cdot B + R,$$

with quotient a polynomial Q in $K[x, u]$ and with remainder a polynomial R in $K[x, u]$ of the form $R = \sum_{j=0}^{d-1} U_j(x', u) \cdot x_n^j$ for some polynomial coefficients $U_j \in K[x', u]$. If g were not a polynomial but just an algebraic series, one would have to take for G the polynomial code of it, see Sect. 13 for the precise procedure. This polynomial division is, of course, a finite process. A rather tedious computation then shows that the Jacobian matrix $\partial_u U$ of the vector $U = (U_0, \dots, U_{d-1}) \in K[x', u]^d$ with respect to the u-variables is invertible when evaluated at 0. It thus defines, by the implicit function theorem, a unique vector $u(x') = (u_0(x'), \dots, u_{d-1}(x'))$ of algebraic series $u_j(x')$ such that $U(x', u(x')) = 0$. This just means that U is a code for $u(x')$. But, by construction, $R(x, u(x')) = 0$, so that $G(x) = Q(x, u(x')) \cdot B(x, u(x'))$. By comparison of the initial monomials it follows that $Q(x, u(x'))$ is invertible as a power series, hence $b(x) = B(x, u(x'))$ is indeed the Weierstrass normal form of G as required.

This example gives an idea of how the codes of reduced standard bases and of the quotients and the remainder of a division can be constructed. In practice and for the required generality the technicalities become unfortunately much more involved.

At the same time, there remain puzzling mysteries when the involved algebraic series are no longer x_n-regular (in which case the Weierstrass normal form has to be defined as the reduced standard basis of the ideal). For instance, the polynomial $xy - z(x + y + x^2y^2)$ with initial monomial xy has an algebraic series as its normal form, whereas the normal form of $xy - z(x^2 + y^2 + x^2y^2)$ is a transcendental series (over a ground field of characteristic zero; it is a so-called Mahler series). Both facts are easy to prove by direct computation. In contrast, the normal form of $xy - z(1 + y)(1 + x^2y)$, a polynomial which appears in the counting of Gessel walks, is again an algebraic series, but this seems to be very intricate to prove. The algebraicity of the normal form was eventually shown by Bostan–Kauers—a substantial part of their proof relies on heavy computer machinery [10] (see [11,12] for alternative proofs). However, modifying slightly the input polynomial, taking now $xy - z(1 + y)(1 + xy^2)$, it is almost immediate to detect the algebraicity of the normal form using a suitable division.

These examples suggest that there are hidden structural patterns which cause the phenomena to happen and which should explain the occurrence of algebraic or transcendental normal forms. Little seems to be known in this respect. For instance, the classification of the generating functions of lattice walks in the first quadrant, studied among others by Bousquet-Mélou, Mishna and Petkovšek, does not seem to reveal a systematic background [13,15,35].

Organization of the paper After some preliminary recalls on the formal power series and polynomial division covering Sects. 2–5, we introduce and study in Sects. 6–8 codes of algebraic series and of the ideals generated by them. These are polynomial data which completely determine the series and ideals they encode. For later purposes the codification is carried out from the beginning for vectors of algebraic series and the modules they generate.

Section 9 describes how to compute the codes of standard bases of ideals and modules from a given (arbitrary) generator system (Theorem 9.1). This is straightforward, and based on Lazard's homogenization method, respectively, Mora's tangent cone algorithm. Both were refined and extended by Gräbe and Greuel–Pfister. Our two main results (Sects. 10, 11) concern the construction—in terms of the defining codes—of *reduced* standard bases of modules of algebraic power series vectors (Theorem 10.1), and of the quotients and the remainder of an algebraic power series division (Theorem 11.1).

The proofs of these two theorems are mutually interwoven (Sects. 12–15). First, the construction of the reduced standard basis is performed in the x_n-regular case (i.e., in the case where the initial module of the given module of algebraic power series vectors is generated by monomial vectors depending only on the last variable x_n). This is by far the most complicated step. It clearly shows how important it is to codify the series in a very systematic manner. Otherwise it would be hopeless to prove that the resulting polynomial vectors represent again codes (i.e., satisfy the assumption of the implicit function theorem). In the case of principal ideals, the proof provides the code of the Weierstrass normal form of the given series, cf. [2].

The preceding construction of the codes of the reduced standard basis in the x_n-regular case is then used to establish the division of algebraic series on the level of codes in the x_n-regular case. This is not too difficult. It relies on the effectivity of the division algorithm in localizations of polynomial rings, proven by Lazard, Mora, Gräbe and Greuel–Pfister.

Once the two theorems are established in the x_n-regular case, the general case is carried out by induction on the number of variables. It is here that Hironaka's box condition comes into play. One key feature is its persistence under taking hyperplane sections (in a well defined sense), and this is used to know that the associated modules in $n-1$ variables satisfy again the box condition. So induction applies to prove both theorems simultaneously.

In the last section, we illustrate the instances and the complexity of the algorithm in the computation of a concrete example.

2 Monomial Modules

The letters n, p, r, s are reserved for fixed integers in $\mathbb{N}$. The letters i, k and ℓ will generally vary in the ranges $1 \leq i \leq p$, $1 \leq k \leq r$ and $1 \leq \ell \leq s$.

We denote by $K[x_1, \ldots, x_n] = K[x]$ and $K[[x_1, \ldots, x_n]] = K[[x]]$ the polynomial, respectively, formal power series ring in n variables $x = (x_1, \ldots, x_n)$ over a field K. Elements of $K[x]^s$ and $K[[x]]^s$ will be called *polynomial*, respectively, *formal power series vectors*. Capital letters will be reserved for polynomials, lower case letters for power series. We set $x' = (x_1, \dots, x_{n-1})$ and denote by $y = (y_1, \dots, y_p)$ additional variables.

Vectors $g \in K[[x]]^s$ will be expanded into $g = \sum_{\alpha\ell} c_{\alpha\ell} x^\alpha e_\ell$ with $c_{\alpha\ell} \in K$ and $e_\ell = (0, \dots, 0, 1, 0, \dots, 0)$ the canonical K-basis of K^s. The vectors $x^\alpha e_\ell$ are called *monomial vectors*. Note that all their entries but one are zero: a vector all whose entries are monomials will not be considered here as a monomial vector. The *support* of g is the set $\mathrm{supp}(g) = \{(\alpha, \ell) \in \mathbb{N}^n \times \{1, \dots, s\},\ c_{\alpha\ell} \neq 0\}$. We sometimes abbreviate pairs (α, ℓ) by $\alpha\ell$.

Brackets $\langle g_1, \dots, g_r \rangle$ denote submodules of $K[[x]]^s$ generated by power series vectors $g_1, \dots, g_r \in K[[x]]^s$. We abbreviate this by $\langle g_k \rangle$ if the range of k is clear from the context.

A *monomial submodule* of $K[[x]]^s$ is a submodule M of $K[[x]]^s$ generated by monomial vectors. It is a Cartesian product $M = \Pi_{\ell=1}^s M_\ell$ of monomial ideals M_ℓ in $K[[x]]$. The elements of M are the power series vectors with support in $\Sigma = \{(\alpha, \ell) \in \mathbb{N}^n \times \{1, \dots, s\},\ x^\alpha e_\ell \in M\}$. The *canonical direct monomial complement* of a monomial submodule M of $K[[x]]^s$ is the subvector space $\mathrm{co}(M)$ of $K[[x]]^s$ of power series vectors with support in $\Sigma' = (\mathbb{N}^n \times \{1, \dots, s\}) \setminus \Sigma$. This provides the direct sum decomposition of K-vectorspaces $M \oplus \mathrm{co}(M) = K[[x]]^s$.

We say that a monomial submodule M of $K[[x]]^s$ is x_n*-regular* if it is generated by monomial vectors in $K[[x_n]]^s$, say $M = \langle M \cap K[[x_n]]^s \rangle$. We shall then always assume for simplicity—applying if necessary a permutation of the components of $K[[x]]^s$—that M is generated by vectors of the form $x_n^{d_k} \cdot e_k$ with $d_k \geq 0$ and $1 \leq k \leq r$ for some $r \leq s$. In this case the complement $\mathrm{co}(M)$ is a Cartesian product

$$\mathrm{co}(M) = \prod_{k=1}^r \left(\oplus_{j=0}^{d_k-1} K[[x']] \cdot x_n^j \right) \times K[[x]]^{s-r}$$

of a finitely generated free $K[[x']]$-module with a finitely generated free $K[[x]]$-module. We say that M satisfies *Hironaka's box condition* if $\mathrm{co}(M)$ can be written as a Cartesian product of direct sums of finite free monomial $K[[x_1, \dots, x_j]]$-modules

$$\mathrm{co}(M) = \prod_{\ell=1}^s \oplus_{j=0}^n \oplus_{\gamma \in \Gamma_{\ell,j}} K[[x_1, \dots, x_j]] \cdot x^\gamma$$

with finite sets $\Gamma_{\ell,j} \subset \mathbb{N}^n$. Being x_n-regular is a special case of the box condition. For cyclic submodules of $K[[x]]^s$, both notions coincide. They obviously depend on the numbering of the variables $x_1, \dots, x_n$. Notice that for $s = 1$ and $0 \neq M \subsetneq K[[x]]$ a nontrivial ideal, the indices of the boxes F_j run from 1 to $n - 1$. Also notice that the box condition for a monomial submodule $M \subset K[[x]]^s$ is equivalent to the box condition for each of the factors of M (which are monomial ideals in $K[[x]]$). The ideal $\langle xy \rangle$ in the power series ring $K[[x, y]]$ in two variables is the simplest example of a monomial ideal not satisfying the box condition.

W. Seiler informed us that in the case of ideals the box condition is equivalent to his notion of δ-regular coordinates [48]. We say that a monomial submodule M of $K[[x]]^s$ is an *echelon* if it can be written as

$$M = \prod_{\ell=1}^s \oplus_{j=0}^n \oplus_{\delta \in \Delta_{\ell,j}} K[[x_1, \dots, x_j]] \cdot x^\delta$$

with finite sets $\Delta_{\ell,j} \subset \mathbb{N}^n$. This can be rewritten as

$$M = \oplus_{\ell=1}^{s} \ \oplus_{\delta\in\Delta_\ell} K[[x_1,\ldots,x_{n_\delta}]] \cdot x^\delta \cdot e_\ell$$

where $\Delta_\ell = \bigcup_j \Delta_{\ell,j}$ and where, for each δ, the index n_δ takes a value between 0 and n. This notion is a special case of a Rees decomposition of M [43]. We call the collection of monomial vectors $x^\delta \cdot e_\ell$ with $\delta \in \Delta_\ell$ and $1 \le \ell \le s$ a *Janet basis* of the echelon M with *scopes* n_δ (also known as *levels* or *classes*). Our definition differs slightly from Janet's original definition in the sense that we only allow nested groups of variables in the coefficients [30,31], cf. also [44]. We refer to the related notions of Pommaret bases and involutive bases [21,46,47], and the more general concepts of Rees and Stanley decompositions of rings [3,8,43,51].

For the following result, see also Janet [30,31] and Seiler [47].

Theorem 2.1 *Monomial submodules of $K[[x]]^s$ satisfying Hironaka's box condition are echelons.*

Proof Let M be such a module, and let $M_n = \langle M \cap K[[x_n]]^s\rangle$ be the submodule of $K[[x]]^s$ generated by the x_n-pure monomial vectors of M. By definition, M_n is x_n-regular. Let $x_n^{d_k} \cdot e_k$ with $1 \le k \le r$ be a minimal generator system of M_n (after possibly permuting the components of $K[[x]]^s$). Then $M_n = \oplus_{k=1}^r K[[x]] \cdot x_n^{d_k} \cdot e_k$, which shows that the monomial vectors $x_n^{d_k} \cdot e_k$ form a Janet basis of M_n with scopes $n_k = n$. The direct sum decomposition

$$K[[x]]^s = M_n \oplus \left(\oplus_{m=1}^{r} \ \oplus_{j=0}^{d_m-1} K[[x']] \cdot x_n^j \cdot e_m\right) \ \oplus \ \left(\oplus_{m=r+1}^{s} K[[x]] \cdot e_m\right)$$

yields a decomposition $M = M_n \oplus M'$ where M' is now a $K[[x']]$-submodule of the finitely generated free $K[[x']]$-module $\oplus_{m=1}^{r} \ \oplus_{j=0}^{d_m-1} K[[x']] \cdot x_n^j \cdot e_m$. We use here that, because of the box condition, M has zero intersection with $\oplus_{m=r+1}^{s} K[[x]] \cdot e_m$.

It is checked that the box condition persists under the above decomposition, i.e., that M' satisfies it again. By induction on the number of variables, M' is an echelon. Its Janet basis has scopes $\le n-1$. From $M = M_n \oplus M'$ now follows that also M is an echelon. □

Remark Notice that the decomposition of M as an echelon

$$M = \prod_{\ell=1}^{s} \oplus_{j=0}^{n} \oplus_{\delta\in\Delta_{\ell,j}} K[[x_1,\ldots,x_j]] \cdot x^\delta$$

provides a partition of the exponent sets of power series vectors in M. Such partitions are used to make also the quotients of a power series division unique by imposing on them the respective support conditions, cf. [20].

Example The assertion of the theorem does not hold for arbitrary modules as was pointed out by W. Seiler. Take the ideal I of $K[x,y,z]$ generated by the three monomials xy, xz and yz. It is easy to see that it does not satisfy the box condition. And it is not an echelon, since, for instance, among the monomials of I which are not multiples of xy one has monomials $x^d z$ and $y^d z$ of arbitrary degree d in x and y. As the situation is symmetric with respect to any permutation of the variables, I does not admit the required decomposition of an echelon.

Remark Echelons appear as a special instance of nested Artin approximation when restricting to the case of linear equations. Nested means that the components of the solutions only depend on nested sets of variables. In this setting, algebraic series play an important role, as they are a prerequisite in Popescu's nested Artin approximation theorem [40,41,50,52]. In contrast, Gabrielov has given an example with convergent but not algebraic series f and $g_1, \ldots, g_r$ so that $f = \sum \widehat{a}_i g_i$ with formal series $\widehat{a}_i \in K[[x_1, ..., x_{s_i}]]$, for some given $s_i \leq n$, but so that the analogous decomposition of f with convergent coefficients a_i does not exist [19]. The phenomenon does not occur for algebraic f and g_i (for instance, by Popescu's theorem). The deeper reason behind Gabrielov's example is described in [1].

3 Monomial Orders and Initial Modules

Division theorems are based on ordering the summands $c_{\alpha\ell} x^\alpha e_\ell$ of the expansion of a power series vector $g = \sum_{\alpha\ell} c_{\alpha\ell} x^\alpha \cdot e_\ell$ according to the indices $(\alpha, \ell) \in \mathbb{N}^n \times \{1, \ldots, s\}$ with nonzero coefficients $c_{\alpha\ell}$: *a monomial order* on $\mathbb{N}^n \times \{1, \ldots, s\}$ is a total order $<_\eta$ on $\mathbb{N}^n \times \{1, \ldots, s\}$ which is compatible with the semigroup structure of $\mathbb{N}^n$, having 0 as its smallest element, and which is Noetherian. This means that if $(\alpha, \ell) <_\eta (\beta, m)$ then $(\alpha + \gamma, \ell) <_\eta (\beta + \gamma, m)$ for any $\gamma \in \mathbb{N}^n$, and, secondly, that any decreasing sequence becomes stationary. The order is *degree compatible* if $|\alpha| < |\beta|$ implies $(\alpha, \ell) <_\eta (\beta, m)$, where $|\alpha|$ denotes the sum of the components of α. An *extension* of $<_\eta$ is a monomial order $<_\varepsilon$ on $\mathbb{N}^{n+p} \times \{1, \ldots, s\}$ whose restrictions to $\mathbb{N}^n \times \{\delta\} \times \{1, \ldots, s\}$ coincide for all $\delta \in \mathbb{N}^p$ with the order induced by $<_\eta$ on $\mathbb{N}^n \times \{\delta\} \times \{1, \ldots, s\}$. We will always identify monomial orders on $\mathbb{N}^n \times \{1, \ldots, s\}$ with the induced ordering of the monomial vectors in $K[[x]]^s$.

The *initial monomial vector* $\mathrm{in}(g)$ of $g = \sum c_{\alpha\ell} x^\alpha \cdot e_\ell \in K[[x]]^s$ with respect to $<_\eta$ is the vector $x^\alpha \cdot e_\ell$ of the expansion of g for which (α, ℓ) is *minimal* with respect to $<_\eta$. We shall assume that $x^\alpha \cdot e_\ell$ has coefficient 1 in the expansion of g. We then write $g = x^\alpha \cdot e_\ell - \overline{g}$ and call $\overline{g}$ the *tail* of g.

For a submodule I of $K[[x]]^s$, the *initial module* of I with respect to $<_\eta$ is the monomial submodule $\mathrm{in}(I)$ of $K[[x]]^s$ generated by all initial monomial vectors of elements of I. This is a monomial submodule which depends on the choice of $<_\eta$. We denote by $\mathrm{co}(I)$ the canonical direct monomial complement of $\mathrm{in}(I)$ in $K[[x]]^s$. Elements $g_1, \ldots, g_r$ of $K[[x]]^s$ form a *standard basis* w.r.t. $<_\eta$ if their initial monomial vectors generate the initial module $\mathrm{in}(I)$ of the module I generated by $g_1, \ldots, g_r$. They are a *reduced standard basis* if the tails $\overline{g}_k$ belong to $\mathrm{co}(I)$. We do not require that a reduced standard basis is minimal.

We say that a submodule I of $K[[x]]^s$ is *x_n-regular*, respectively, satisfies *Hironaka's box condition*, or is an *echelon* with respect to the monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \ldots, s\}$, if its initial module $\mathrm{in}(I)$ is x_n-regular, respectively, satisfies the box condition, or is an echelon. A *Janet basis* of a submodule I of $K[[x]]^s$ which is an echelon w.r.t. $<_\eta$ is a generator system $g_1, \ldots, g_r$ of I whose initial monomial vectors $\mathrm{in}(g_k)$ form a Janet basis of $\mathrm{in}(I)$.

For a polynomial vector $G \in K[x]^s$, define the *leading monomial vector* $\mathrm{lm}(G)$ as the monomial vector $x^\alpha e_\ell$ of the expansion of G which is *maximal* with respect to the chosen monomial order. Similarly as for initial modules, one obtains now the leading module $\mathrm{lm}(I)$ of a submodule I of $K[x]^s$.

4 Division of Formal Power Series and Polynomials

We recall the division theorem for modules of formal power series of Grauert, Hironaka and Galligo [4,20,24,27,29]. For extensions of this result to more general settings see [3,8,21,47,53].

Theorem 4.1 *Let I be a submodule of $K[[x]]^s$ with initial module* $\mathrm{in}(I)$ *with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Let* $\mathrm{co}(I)$ *be the canonical direct monomial complement of* $\mathrm{in}(I)$ *in $K[[x]]^s$. Then $I \oplus \mathrm{co}(I) = K[[x]]^s$.*

Sketch of proof. The sum $I \oplus \mathrm{co}(I)$ is direct by definition of $\mathrm{co}(I)$. To see that it equals $K[[x]]^s$, choose a standard basis $g_1, \dots, g_r$ of I. It suffices to show that the linear map $u : K[[x]]^r \times \mathrm{co}(I) \to K[[x]]^s$, $(a_1, \dots, a_r, b) \to \sum a_k g_k + b$ is surjective. By definition of standard bases, the map $v : K[[x]]^r \times \mathrm{co}(I) \to K[[x]]^s$, $(a_1, \dots, a_r, b) \to \sum a_k \cdot \mathrm{in}(g_k) + b$ is surjective. Writing $u = v + w$, the assertion follows by restricting v to a direct complement L of its kernel and by showing that $u_{|L} = v_{|L} + w_{|L}$ is an isomorphism with inverse the geometric series $(v_{|L})^{-1} \sum_{j=0}^\infty (-(v_{|L})^{-1} w_{|L})^j$. This series then induces the required linear map $K[[x]]^s \to L$ inverse to $u_{|L}$, see [29, Theorem 5.1] for details.

The division theorem can be formulated more explicitly as follows: If $g_1, \dots, g_r$ generate I, each vector $f \in K[[x]]^s$ has a decomposition $f = \sum_k a_k g_k + h$ with unique $h \in \mathrm{co}(I)$. The power series expansions of the quotients a_k and the remainder h can be computed up to any given degree by a finite algorithm (take the expansion of the geometric series above up to the respective degree). The requirement that h belongs to $\mathrm{co}(I)$ makes the remainder independent of the choice of $g_1, \dots, g_r$ (but it depends on the monomial order $<_\eta$). If $g_1, \dots, g_r$ form a standard basis, the quotients a_k can be made unique by imposing suitable support conditions on them [20]. A reduced standard basis of I is given as $x^\alpha \cdot e_\ell - h_{\alpha\ell}$ with (α, ℓ) varying in some finite subset $V \subset \mathbb{N}^n \times \{1, \dots, s\}$, where the vectors $x^\alpha \cdot e_\ell$ are generators of $\mathrm{in}(I)$ and the vectors $h_{\alpha\ell}$ denote the remainder of the division of $x^\alpha \cdot e_\ell$ by I. □

For modules which are echelons one can formulate a more precise statement:

Theorem 4.2 *Let I be a submodule of $K[[x]]^s$ with initial module* $\mathrm{in}(I)$ *w.r.t. a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Assume that I is an echelon, and let $x^\alpha \cdot e_\ell$ be a Janet basis of* $\mathrm{in}(I)$ *with scopes $n_{\alpha\ell}$, (α, ℓ) varying in some finite set $V \subset \mathbb{N}^n \times \{1, \dots, s\}$. Choose any elements $g_{\alpha\ell}$ of I with initial monomial vectors $x^\alpha \cdot e_\ell$. Then*

$$I \oplus \mathrm{co}(I) = \left(\oplus_{\alpha\ell \in V} K[[x_1, \dots, x_{n_{\alpha\ell}}]] \cdot g_{\alpha\ell}\right) \oplus \mathrm{co}(I) = K[[x]]^s.$$

Proof First notice that $\mathrm{in}(I) = \oplus_{\alpha\ell\in V} K[[x_1, \dots, x_{n_{\alpha\ell}}]] \cdot x^\alpha \cdot e_\ell$ is a direct sum, by the definition of echelons. This allows us to modify the map u from the proof of the division theorem by restricting it to the K-subspace

$$\prod_{\alpha\ell\in V} K[[x_1, \dots, x_{n_{\alpha\ell}}]] \times \mathrm{co}(I).$$

The map v is then by construction an isomorphism, and the same reasoning as before shows that this holds also for u. This proves the claim.

In the polynomial case, the division admits an analogous formulation. The same proof as above applies, because the evaluation of the geometric series $(v_{|L})^{-1} \sum_{j=0}^{\infty}(-(v_{|L})^{-1} w_{|L})^j$ on a polynomial vector $(a_1, \dots, a_r, b) \in K[x]^r \times \mathrm{co}(I)$ terminates at sufficiently large j. □

Theorem 4.3 *Let I be a submodule of $K[x]^s$ with leading module* $\mathrm{lm}(I)$ *with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Let* $\mathrm{co}(I)$ *be the canonical direct monomial complement of* $\mathrm{lm}(I)$ *in $K[x]^s$. Then $I \oplus \mathrm{co}(I) = K[x]^s$.*

Again, there is a more precise version in case the leading module $\mathrm{lm}(I)$ is an echelon.

Theorem 4.4 *Let I be a submodule of $K[x]^s$ with leading monomial module* $\mathrm{lm}(I)$ *with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Assume that* $\mathrm{lm}(I)$ *is an echelon. Let G_k be a polynomial Janet basis of I with leading monomial vectors* $\mathrm{lm}(G_k)$ *of scope n_k. Then any $F \in K[x]^s$ admits a unique division*

$$F = \sum_k A_k G_k + C$$

with $A_k \in K[x_1, \dots, x_{n_k}]$ and $C \in$ $\mathrm{co}(I)$*. The decomposition can be obtained from the polynomial vectors F and G_k by a finite algorithm.*

5 Algebraic Power Series

Algebraic power series are formal power series $h(x) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ in several variables $x = (x_1, \dots, x_n)$ with coefficients in a field K which satisfy an algebraic relation of the form

$$P(x, h(x)) = p_d h^d + p_{d-1} h^{d-1} + \dots + p_1 h + p_0 = 0,$$

where the coefficients $p_i = p_i(x)$ are polynomials, and $p_d \neq 0$. We refer to [6,9,16,32,33,38,39,42,45,55] for the respective background. An algebraic power series vector is a vector in $K[[x]]^s$ whose components are algebraic series.

Typical algebraic series are rational functions as $x \cdot (1+x)^{-1}$, roots of polynomials as $\sqrt{1 + x^2 y}$, inverses f^{-1} of polynomial mappings $f : K^n \to K^n$ satisfying at a point p the assumption of the inverse function theorem as $f(x, y) = (x + x^3, y - xy^2)$ at 0, or solutions $y(x)$ of polynomial equations $f(x, y) = 0$ satisfying at a point p

the assumption of the implicit function theorem with respect to the variables y as $f(x, y) = y + xy + x^3y^2$ at 0.

The ring of algebraic series in n variables is thus the algebraic closure of the polynomial ring $K[x_1, \ldots, x_n]$ inside the formal power series ring $K[[x_1, \ldots, x_n]]$. It can equivalently be interpreted as the Henselization of the polynomial ring at 0.

Note that the minimal polynomial of an algebraic series h determines h only up to conjugacy: there may be other power series solutions to the equation, the conjugates of h, and h can be distinguished from these for instance by a sufficiently high truncation of its Taylor expansion. The simplest example thereof is the equation $y^2 - 2y + x = 0$ with algebraic solutions $h_\pm = 1 \pm \sqrt{1-x}$.

Lafon proved in 1965 that the Weierstrass division preserves the algebraicity of the involved series [33], see also [9]. This was reproven in 2000 by Bousquet-Mélou and Petkovšek working with the recursions defining the coefficients of the series [14]. The result of Lafon was extended by Hironaka in 1977 to the division by ideals with several generators satisfying the box condition [27]. We formulate here the division directly for modules.

Theorem 5.1 *Let I be a submodule of $K[[x]]^s$ generated by algebraic power series vectors. Assume that I satisfies Hironaka's box condition with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \ldots, s\}$. For any algebraic power series vector $f \in K[[x]]^s$ the remainder c of the formal power series division of f by I with respect to $<_\eta$ is an algebraic power series vector.*

The theorem implies in particular that any submodule of $K[[x]]^s$ with box condition which is generated by algebraic power series vectors admits a reduced standard basis consisting of algebraic power series vectors. Without box condition the remainder of the division need not be algebraic. In [27, p. 75], Hironaka cites the following example of Gabber and Kashiwara, which was rediscovered by Bousquet-Mélou and Petkovšk in combinatorics when counting lattice paths [14,15].

Example 5.2 Divide xy by $g = (x - y^2)(y - x^2) = xy - x^3 - y^3 + x^2y^2$ as formal power series with respect to the initial monomial xy. The remainder of the division lies in $\operatorname{co}(xy) = K[[x]] + K[[y]]$ and equals the lacunary series $b = \sum_{k\geq 0}(-1)^k x^{3\cdot 2^k} + \sum_{k\geq 0}(-1)^k y^{3\cdot 2^k}$ which is transcendental. Alternatively, we may write $xy = a \cdot g + r(x) + s(y)$ with series $a \in K[[x, y]]$, $r \in K[[x]]$, $s \in K[[y]]$. The symmetry between x and y in this expression yields $r(x) = s(x)$. Substituting y by x^2 produces $x^3 = a \cdot 0 + r(x) + r(x^2)$ which also gives the expansion of r.

6 Codes of Algebraic Power Series

In this section we introduce the necessary terminology for working effectively with algebraic power series. The variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_p)$ are fixed throughout.

A *mother code* (over x and y) is a polynomial row vector $H = (H_1, \ldots, H_p) \in K[x, y]^p$ with $H(0, 0) = 0$ whose Jacobian matrix $D_y H$ with respect to y is invertible at 0,

$$D_y H(0,0) \in \mathrm{Gl}_p(K).$$

The invertibility of $D_y H(0,0)$ can be rephrased by saying that for any degree compatible monomial order on $\mathbb{N}^p$ the initial ideal of the ideal $\langle H_1(0,y), \ldots, H_p(0,y)\rangle$ of $K[[y]]$ is generated by $y_1, \ldots, y_p$. There then exists a linear coordinate change in the y_i's so that the initial monomials $\mathrm{in}(H_i(0,y))$ of $H_i(0,y)$ equal y_i. For any degree compatible monomial order on $\mathbb{N}^{n+p}$ so that $y_i < x_j$ for all i and j it then follows that the initial monomials $\mathrm{in}(H_i)$ of H_i equal y_i. Instead of changing the y_i's one could also change H by multiplying it from the right with a suitable matrix in $\mathrm{Gl}_p(K)$ making $D_y H(0,0)$ unipotent upper triangular. In the sequel we shall always assume that $\mathrm{in}(H_i) = y_i$ with respect to the chosen monomial order on $\mathbb{N}^{n+p}$.

The *baby series vector* of a mother code $H \in K[x,y]^p$ is the formal power series vector $h = (h_1, \ldots, h_p) \in K[[x]]^p$ vanishing at 0 which is the unique solution of $H(x, h(x)) = 0$. The existence and uniqueness of h are ensured by the implicit function theorem for formal power series. The components h_i of the baby series vector h are algebraic series. This can be seen by the algebraic implicit function theorem [32, p. 91], or Artin's approximation theorem [6], or by the following argument: consider the system $H(x,y) = 0$ as equations for the last variable y_p. After a renumeration of the components of H, the last derivative $\partial_{y_p} H_p(0,0)$ does not vanish. There then exists a unique solution $h_p(x, y_1, \ldots, y_{p-1})$ of $H_p(x, y_1, \ldots, y_{p-1}, y_p) = 0$ vanishing at 0, and h_p is algebraic over $K[x, y_1, \ldots, y_{p-1}]$. By induction on n and the transitivity of algebraicity we conclude that $h = (h_1, \ldots, h_p)$ is algebraic.

An algebraic power series vector $h = (h_1, \ldots, h_p) \in K[[x]]^p$ is *a baby series vector* if it admits a mother code $H \in K[x,y]^p$ defining it.

A *father code* is a vector $G = (G_1, \ldots, G_r)$ of polynomial vectors $G_i \in K[x,y]^s$ (there are no further conditions on the G_i). We consider G as a row vector with entries the column vectors G_i, say as a matrix in $K[x,y]^{s \times r}$.

A *family code* is a pair (H, G) where $H \in K[x,y]^p$ is a mother code and $G \in K[x,y]^{s\times r}$ a father code, both carrying on the same sets of variables. We say that algebraic power series vectors $g_1, \ldots, g_r \in K[[x]]^s$ *have family code* $(H, G) \in K[x,y]^p \times K[x,y]^{s\times r}$ if

$$g_k = G_k(x, h(x))$$

for $1 \le k \le r$, where $h \in K[[x]]^p$ is the baby series vector of the mother code H. The vectors g_k hence belong to $K[x,h]^s \subset K[[x]]^s$. We call h the baby series vector underlying $g_1, \ldots, g_r$, or, the other way round, $g_1, \ldots, g_r$ the algebraic power series vectors produced from h by the father code G.

For convenience (and by abuse of notation) we often use the definite article "the" for codes of algebraic series.

Example 6.1 Take $g_1 = z^3 + z^2 h$, $g_2 = xz^2 + xzh$ with baby series $h = 1 - \sqrt{1-x^2}$, mother code $H = 2y - y^2 - x^2$ and father code $G_1 = z^3 + z^2 y$, $G_2 = xz^2 + xzy$. Notice that the second series solution $1 + \sqrt{1-x^2}$ of $H = 0$ has nonzero constant term and is therefore not considered as a baby series of H.

Example 6.2 Let H be the vector (H_1, H_2) with $H_1 = y_1^2 - 2y_1 - y_2 - x_2$ and $H_2 = x_2y_2^2 - 2y_2 - y_1 - x_1$. The vector H is the mother code of the baby series vector (h_1, h_2) where h_1 and h_2 are related by $h_1 = 1 - \sqrt{1 + x_2 + h_2}$ and $h_2 = (1 - \sqrt{1 + x_2(x_1 + h_1)})/x_2$. The mother code H defines the same baby series vector as the mother code $H' = (H_1', H_2')$ given by

$$H_1' = -x_1 + x_2^3 + 2x_2 - 4x_2y_1^3 + (3 + 4x_2^2)y_1 + (-2x_2^2 - 2 + 4x_2)y_1^2 + x_2y_1^4,$$
$$H_2' = -y_1^2 + 2y_1 + y_2 + x_2.$$

Now, $D_yH'(0, 0)$ is unipotent upper triangular and H_1' does not depend on y_2. Hence, the expansion of the series h_1 can be computed up to any order from the equation $H_1' = 0$. From $H_2' = 0$ we get $h_2 = -x_2 - 2h_1 + h_1^2$.

7 Construction of Codes

Codes of algebraic power series as above were introduced by Alonso, Mora and Raimondo. Their construction is based on an effective version of the Artin–Mazur theorem [7, p. 88, 2, Appendix, 9, Theorem 8.4.4, p. 173].

Theorem 7.1 *For any algebraic series $g \in K[[x]]$ there is a finite algorithm to construct from an algebraic relation $P(x, t) = 0$ satisfied by g and the Taylor expansion of g up to sufficiently high degree a family code $(H, G) \in K[x, y]^p \times K[x, y]$ of g, for some p.*

Proof Let $P(x, g(x)) = 0$ be a minimal hence irreducible algebraic relation for g. Denote by $X \subseteq \mathbb{A}_K^{n+1}$ the zero-set of P in affine $(n + 1)$-space $\mathbb{A}_K^{n+1}$ over K. We assume that $g(0) = 0$ so that $(0, 0) \in X$. Let Y be the normalization of X. Choose an embedding $Y \subset \mathbb{A}^{n+p}$ so that the normalization map $\pi : Y \to X$ is induced by the projection $\mathbb{A}^{n+p} \to \mathbb{A}^{n+1}$, $(x, y) \to (x, y_1)$ on the first $n + 1$ components.

The Taylor expansion of g specifies a unique point $b \in Y$ which maps to $0 \in X$ and through which, by the universal property of normalization, passes a lifting $(x, \widetilde{g}(x))$ of $(x, g(x))$. From Zariski's main theorem [37, p. 209, 56] we know that Y is analytically irreducible at b. But as Y contains the graph of $\widetilde{g}$ and has dimension n, it is smooth at b. By the Jacobian criterion it is therefore possible to choose polynomial equations $H_1, \ldots, H_p$ defining Y in a Zariski neighborhood of b in $\mathbb{A}^{n+p}$ and satisfying at b the assumption of the implicit function theorem, i.e., of a mother code. Let $(h_1, \ldots, h_p)$ be the associated baby series vector. By the special choice of π we get $g = h_1$, say $g = G(h_1, \ldots, h_p)$ with father code $G = y_1$. This proves the theorem. □

The construction of the normalization is effective [18] and implemented for instance in the computer-algebra program Singular [26].

When handling several algebraic power series it is more economic to work with one mother code and several father codes instead of choosing separate mother codes for each series. This goes as follows.

Let be given mother codes $H^j \in K[x, y^j]^{p_j}$ for $j = 1, \ldots, r$ in distinct sets of variables $y^j = (y_1^j, \ldots, y_{p_j}^j)$ defining baby series vector $h^j = (h_1^j, \ldots, h_{p_j}^j) \in K[[x]]^{p_j}$.

The *direct sum* H of the H^j's is given as the row vector $H = (H^1, \ldots, H^r) \in \Pi_{j=1}^r K[x, y]^{p_j} \cong K[x, y]^p$, where y denotes the collection of all y^j and $p = \sum p_j$. This H is again a mother code, because the Jacobian matrix $D_y H(0, 0)$ of H with respect to y at 0 has block diagonal form with invertible blocks equal to $D_{y^j} H^j(0, 0)$ on the diagonal. The vector $h = (h^1, \ldots, h^r)$ obtained by listing all baby series vectors h^j of the mother codes H^j in a row is the baby series vector of H. This passage to direct sums of mother codes allows us to treat several baby series vectors h^j simultaneously as one baby series vector h (with many components). Accordingly, finitely many algebraic series can always be considered as produced by certain father codes from the *same* baby series vector $h = (h_1, \ldots, h_p)$ of *one* mother code $H \in K[x, y]^p$. This allows us to work throughout with vectors in $K[x, h_1, \ldots, h_p]^s$.

Note that mother codes as defined above may require large sets of variables and are thus computationally very expensive.

8 Codes for Modules of Algebraic Series

Let be given algebraic power series vectors $g_1, \ldots, g_r \in K[[x]]^s$ vanishing at 0 with mother code $H \in K[x, y]^p$, baby series vector $h \in K[[x]]^p$ and father code $G \in K[x, y]^{s\times r}$ so that $g_k = G_k(x, h(x))$. The submodule $\langle g_k\rangle$ of $K[[x]]^s$ generated by the series g_k admits the following polynomial description.

Lemma 8.1 *Let $\langle (y_i - h_i) \cdot e_\ell, g_k\rangle$ and $\langle H_i \cdot e_\ell, G_k\rangle$ be the submodules of $K[[x, y]]^s$ generated by the vectors $(y_i - h_i) \cdot e_\ell$ and g_k, respectively, $H_i \cdot e_\ell$ and G_k, for $1 \le i \le p$, $1 \le \ell \le s$, $1 \le k \le r$. Then*

$$\langle (y_i - h_i) \cdot e_\ell, g_k\rangle = \langle H_i \cdot e_\ell, G_k\rangle.$$

Proof We fix a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \ldots, s\}$ and choose an extension $<_\varepsilon$ of $<_\eta$ to $\mathbb{N}^{n+p} \times \{1, \ldots, s\}$ which is degree compatible with respect to $\mathbb{N}^p$ and satisfies $y_i \cdot e_\ell <_\varepsilon x_j \cdot e_\ell$ for all $1 \le i \le p$, $1 \le j \le n$ and $1 \le \ell \le s$. After a suitable multiplication of H with a constant matrix in $\mathrm{GL}_p(K)$ we may assume that $\mathrm{in}(H_i \cdot e_\ell) = y_i \cdot e_\ell$.

The ideal $\langle H_i\rangle$ of $K[[x, y]]$ generated by $H_1, \ldots, H_p$ is contained in the ideal $\langle y_i - h_i\rangle$ because of $H(x, h(x)) = 0$. Take a monomial order $<_\delta$ on $\mathbb{N}^{n+p}$ so that $y_i <_\delta x_j$ for all $1 \le i \le p$ and $1 \le j \le n$. The initial ideals of $\langle H_i\rangle$ and $\langle y_i - h_i\rangle$ coincide because, by the choice of $<_\delta$, they are both generated by $y_1, \ldots, y_p$. By the division theorem for formal power series, the two ideals coincide. As g_k is obtained from G_k by replacing y_i by h_i, the submodules of $K[[x, y]]^s$ generated by $g_1, \ldots, g_r$, respectively, $G_1, \ldots, G_r$ are congruent modulo $\langle y_i - h_i\rangle = \langle H_i\rangle$. This proves the lemma.

We call $\widetilde{I} = \langle H_i \cdot e_\ell, G_k\rangle \subset K[[x, y]]^s$, or, more accurately, its polynomial generators $H_i \cdot e_\ell$ and G_k, the *family code* of the submodule $I = \langle g_k\rangle$ of $K[[x]]^s$. Observe that $\widetilde{I} \cap K[[x]]^s = I$. □

Lemma 8.2 *Let be given a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \ldots, s\}$ and an extension $<_\varepsilon$ of $<_\eta$ to $\mathbb{N}^{n+p} \times \{1, \ldots, s\}$ which is degree compatible with respect to $\mathbb{N}^p$ and* *satisfies* $y_i \cdot e_\ell <_\varepsilon x_j \cdot e_\ell$ *for all* $1 \le i \le p$, $1 \le j \le n$ *and* $1 \le \ell \le s$. *Let* $\widetilde{I} = \langle H_i \cdot e_\ell, G_k \rangle$ *and* $I = \langle g_k \rangle$ *be the respective submodules of* $K[[x, y]]^s$ *and* $K[[x]]^s$. *Then*

$$\mathrm{in}(\widetilde{I}) \cap K[[x]]^s = \mathrm{in}(I).$$

Proof We consider a minimal reduced standard basis of $\widetilde{I}$. Let $\widetilde{g}_k$ be an element of this basis which does not have an initial monomial vector of the form $y_i \cdot e_\ell$. From reducedness it follows that $\widetilde{g}_k$ is independent of $y_1, \dots, y_p$, say $\widetilde{g}_k \in \widetilde{I} \cap K[[x]]^s = I$. In particular, the vectors $\widetilde{g}_k$ form a standard basis of I and hence $\mathrm{in}(\widetilde{I}) \cap K[[x]]^s = \mathrm{in}(I)$. □

9 Construction of Standard Basis

The first construction we need is a direct consequence of Mora's tangent cone algorithm [36], respectively, Lazard's homogenization method [34], cf. also with [2, Theorem 1.3, 17, p. 202, 22, 23, 25, Theorem 6.4.3]. It provides an algorithm to construct a family code of a (not necessarily reduced) standard basis of a module of algebraic power series vectors.

Theorem 9.1 *Let* I *be a submodule of* $K[[x]]^s$ *generated by algebraic power series vectors* $g_1, \dots, g_r \in K[[x]]^s$ *which are given by their family code. Let* $<_\eta$ *be a monomial order on* $\mathbb{N}^n \times \{1, \dots, s\}$. *There is a finite algorithm to compute family codes of the elements of a standard basis of* I *with respect to* $<_\eta$ *from family codes of* $g_1, \dots, g_r$. *In particular, it is possible to compute the initial module* $\mathrm{in}(I)$ *of* I.

Proof Let $g_1, \dots, g_r$ have mother code $H \in K[x, y]^p$, baby series vector $h \in K[[x]]^p$ and father code $G \in K[x, y]^{s \times r}$. Extend $<_\eta$ to a monomial order $<_\varepsilon$ on $\mathbb{N}^{n+p} \times \{1, \dots, s\}$ which is degree compatible with respect to $\mathbb{N}^p$ and satisfies $y_i \cdot e_\ell <_\varepsilon x_j \cdot e_\ell$ for all i, j and ℓ. We assume w.l.o.g. that the initial monomial vectors of $H_i \cdot e_\ell$ with respect to $<_\varepsilon$ are $y_i \cdot e_\ell$.

As $\widetilde{I} = \langle H_i \cdot e_\ell, G_k \rangle$ is generated by polynomial vectors, Mora's tangent cone algorithm or Lazard's homogenization method apply to construct a polynomial standard basis for it. This basis is in general not reduced. We may choose a minimal basis consisting of the vectors $H_i \cdot e_\ell$ with $\mathrm{in}(H_i \cdot e_\ell) = y_i \cdot e_\ell$ and of other polynomial vectors $\widetilde{G}_1, \dots, \widetilde{G}_{r'} \in K[x, y]^s$ with initial monomial vectors in $K[[x]]^s$. The latter form the father code of algebraic power series vectors $\widetilde{g}_1, \dots, \widetilde{g}_{r'} \in K[[x]]^s$, say $\widetilde{g}_k = \widetilde{G}_k(x, h)$. Note that $\widetilde{G}_k$ is congruent to g_k modulo the submodule $\langle H_i \cdot e_\ell \rangle$ of $K[[x]]^s$. By Lemma 8.2, the $\widetilde{g}_k$ form a standard basis of I. This proves the theorem. □

10 Construction of Reduced Standard Basis

The central part in establishing the division algorithm for modules of algebraic power series vectors is the construction of a *reduced* standard basis. The mere existence follows from Hironaka's theorem. The effective part in the special case of principal ideals, i.e., the construction of the code of the Weierstrass form of an x_n-regular algebraic power series, has been established by Alonso, Mora and Raimondo [2, Theorem 5.5]. The general statement is as follows:

Theorem 10.1 *Let I be a submodule of $K[[x]]^s$ generated by algebraic power series vectors. Assume that I satisfies Hironaka's box condition with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Then family codes of a reduced standard basis of I can be computed by a finite algorithm from family codes of any algebraic power series vectors $g_1, \dots, g_r \in K[[x]]^s$ generating I.*

The proof of this result is given in Sects. 13–15. In the formal power series case, a reduced standard basis can be constructed up to any given degree by dividing monomial generators of the initial module by the module itself. For algebraic series, this construction would require to dispose already of an effective division algorithm. To avoid this logical cycle, reduced standard bases have to be constructed in a different way.

The clue relies in the concept of a *virtual reduced standard basis*. Such a basis consists of polynomial vectors whose coefficients are unknown and written as new variables. Upon replacing the variables by suitable series in x, the virtual reduced standard basis will transform into an actual reduced standard basis of the module. The resulting coefficient series of the actual reduced standard basis—more precisely, their codes—are computed by dividing the polynomial generators of the module $\langle (y_i - h_i) \cdot e_\ell, g_k \rangle = \langle H_i \cdot e_\ell, G_k \rangle$ by the virtual basis using the polynomial division algorithm. The definition requires that both the generators and the virtual basis are polynomial vectors, and that the initial monomial vectors of the virtual reduced standard basis can be interpreted as the leading (i.e., maximal) monomial vectors w.r.t. another, suitably chosen monomial order. The choice of this order is rather subtle, see Sect. 13. The remainders of the division then allow us to extract the codes of the required coefficients series.

11 Effective Division for Algebraic Power Series

Our main result asserts that the division by modules of algebraic power series vectors with box condition can be made effective, i.e., can be performed by applying finitely many operations to the codes. The case of principal ideals I, say the effective Weierstrass division theorem for algebraic power series, is due to Alonso, Mora and Raimondo in [2, Theorem 5.6].

Theorem 11.1 *Let I be a submodule of $K[[x]]^s$ generated by algebraic power series vectors. Assume that I satisfies Hironaka's box condition with respect to a monomial order $<_\eta$ on $\mathbb{N}^n \times \{1, \dots, s\}$. Let be given family codes of algebraic power series vectors $g_1, \dots, g_r \in K[[x]]^s$ generating I. There exists a finite algorithm which computes, for any algebraic power series vector $f \in K[[x]]^s$, from a family code of f, family codes of algebraic power series $a_1, \dots, a_r$ in $K[[x]]$ and of an algebraic power series vector $c \in \mathrm{co}(I) \subset K[[x]]^s$ so that*

$$f = \sum\nolimits_{k=1}^{r} a_k g_k + c$$

is the formal power series division of f *by* $g_1, \ldots, g_r$.

The remainder c is unique by the formal division theorem 4.1. It only depends on the chosen monomial order $<_\eta$. The quotients a_k are also unique if one imposes the appropriate support conditions on their exponent sets, cf. [20]. This is made precise in the explicit construction of the series a_k in Sects. 14 and 15. Due to the uniqueness we may call $f = \sum_{k=1}^{r} a_k g_k + c$ *the* formal power series division of f by $g_1, \ldots, g_r$.

We shall prove Theorem 11.1 by first constructing from $g_1, \ldots, g_r$ via Theorems 9.1 and 10.1 family codes of a reduced standard basis of I. The division algorithm for a reduced standard basis will then be established by induction on the number of variables.

12 Logical Structure of the Proofs of Theorems 10.1 and 11.1

Both theorems will be established independently of Hironaka's existential division theorem. We start with establishing Theorem 10.1, the construction of the codes of a reduced standard basis, in the special case of x_n-regular modules. This is the hardest part of the whole story. It relies on introducing the virtual reduced standard basis of the module, which allows us to perform *polynomial* divisions for constructing the required codes. This section is inspired by Mora's tangent cone algorithm and the techniques of Alonso, Mora and Raimondo in [2]. Extracting from the virtual reduced standard basis the actual reduced standard basis uses in an essential way the assumption of x_n-regularity.

From Theorem 10.1 for x_n-regular modules we deduce the division algorithm of Theorem 11.1 for x_n-regular modules. The algorithm uses again a polynomial division, this time by the codes of the reduced standard basis. For its termination it is necessary that the basis is already reduced.

The general cases of Theorems 10.1 and 11.1 are then deduced simultaneously from the special cases by induction on the number of variables and using Hironaka's box condition together with the notion of Janet basis. One selects from the given (not yet reduced) standard basis of the module those elements which are x_n-regular. Such elements exist because of the box condition. Considering the module generated by these elements, one may construct the codes of its reduced standard basis via Theorem 10.1 in the special case. Then Theorem 11.1 allows us to reduce effectively the remaining elements with respect to the first set of elements. By induction on the number of variables, the tails of the first elements can now be divided conversely by the remaining elements, yielding eventually the codes of the whole reduced standard basis of the module. Once this is achieved, it is relatively simple to establish also the division of Theorem 11.1 in the general case.

13 Proof of Theorem 10.1 for x_n-Regular Modules

In the situation of Theorem 10.1, we first treat the case where I is x_n-regular with respect to $<_\eta$. As seen in Lemmata 8.1 and 8.2, it suffices to construct family codes of the elements of a reduced standard basis of the submodule $\widetilde{I} = \langle H_i \cdot e_\ell, G_k \rangle = \langle (y_i - h_i) \cdot e_\ell, g_k \rangle$ of $K[[x, y]]^s$ with respect to the chosen extension $<_\varepsilon$ of $<_\eta$ to

$\mathbb{N}^{n+p} \times \{1, ..., s\}$. Note here that $\widetilde{I}$ is not x_n-regular, since also the $y_i \cdot e_\ell$ appear in the initial module. This is, however, not a serious drawback. The construction of the family codes is somewhat involved and goes in several steps. Let us first specify the setting.

(a) We suppose that the generators g_k of I vanish at 0 for all $1 \leq k \leq r$. Hence this also holds for all $H_i \cdot e_\ell$ and G_k. We may assume by Theorem 9.1 and its proof that the polynomial vectors $H_i \cdot e_\ell$ and G_k form a minimal standard basis of $\widetilde{I}$. As I is x_n-regular and $\mathrm{in}(H_i \cdot e_\ell) = y_i \cdot e_\ell$, the initial module $\mathrm{in}(\widetilde{I})$ is generated by $y_i \cdot e_\ell$ and monomial vectors $x_n^{d_k} \cdot e_{m_k}$ for some $d_k > 0$, $1 \leq m_k \leq s$ and $1 \leq k \leq r$. By the minimality of the standard basis, all m_k are different. Hence $r \leq s$. After a suitable permutation of the components of $K[[x]]^s$ we may assume that $m_k = k$, say $\mathrm{in}(G_k) = x_n^{d_k} \cdot e_k$ for all k. The permutation of the components is only made for notational convenience. It will not affect the induction we shall apply later on when proving Theorems 10.1 and 11.1 in the general case.

The canonical direct monomial complement $\mathrm{co}(\widetilde{I})$ of $\mathrm{in}(\widetilde{I})$ in $K[[x, y]]^s$ is of form

$$\mathrm{co}(\widetilde{I}) = \left(\oplus_{m=1}^{r} \oplus_{j=0}^{d_m-1} K[[x']] \cdot x_n^j \cdot e_m\right) \oplus \left(\oplus_{m=r+1}^{s} K[[x]] \cdot e_m\right).$$

Write the minimal reduced standard basis of $\widetilde{I}$ as

$$b_{i\ell} = y_i \cdot e_\ell - b_{i\ell}^\circ - \sum_{m=1}^{r} \sum_{j=0}^{d_m-1} u_{i\ell mj}(x') \cdot x_n^j \cdot e_m - \sum_{m=r+1}^{s} v_{i\ell m}(x) \cdot e_m,$$
$$b_k = x_n^{d_k} \cdot e_k - b_k^\circ - \sum_{m=1}^{r} \sum_{j=0}^{d_m-1} u_{kmj}(x') \cdot x_n^j \cdot e_m - \sum_{m=r+1}^{s} v_{km}(x) \cdot e_m,$$

with polynomial vectors $b_{i\ell}^\circ$ and b_k° in

$$\left(\oplus_{m=1}^{r} \oplus_{j=0}^{d_m-1} K x_n^j \cdot e_m\right) \oplus \left(\oplus_{m=r+1}^{s} K \cdot e_m\right)$$

and power series $u_{i\ell mj}(x')$, $v_{i\ell m}(x)$, $u_{kmj}(x')$ and $v_{km}(x)$ vanishing at 0. It is necessary here to split off $b_{i\ell}^\circ$ and b_k° because the mother codes of algebraic power series are only defined for series vanishing at 0. Note that $u_{i\ell mj}(x')$ and $u_{kmj}(x')$ do not depend on x_n, and that $b_{i\ell}^\circ$ and b_k° vanish at 0 because the $H_i \cdot e_\ell$ and G_k do. In particular, these vectors have zero entries in the last $s - r$ components. Since $\mathrm{in}_\varepsilon(b_{i\ell}) = y_i \cdot e_\ell$ and $\mathrm{in}_\varepsilon(b_k) = x_n^{d_k} \cdot e_k$ the ℓ-th component of $b_{i\ell}^\circ$ and the k-th component of b_k° are both zero.

The series $b_{i\ell}$ have different shapes according to whether $1 \leq \ell \leq r$ or $r + 1 \leq \ell \leq s$. Namely, for $r + 1 \leq \ell \leq s$, it follows from the x_n-regularity of I that the vectors $(y_i - h_i) \cdot e_\ell$ are already reduced. Hence we have $b_{i\ell} = (y_i - h_i) \cdot e_\ell$ for $r + 1 \leq \ell \leq s$. This will be used later on. We are grateful to D. Wagner for specifying an inaccuracy which appeared at this place in an earlier draft of the paper.

In a first step, we determine the vectors $b_{i\ell}^\circ$ and b_k°. Afterward, family codes of the coefficient series $u_{i\ell mj}(x')$, $v_{i\ell m}(x)$, $u_{kmj}(x')$ and $v_{km}(x)$ will be constructed. This will show in particular that they are algebraic series.

(b) In order to compute $b^\circ_{i\ell}$ and b°_k, one can construct the reduced standard basis of $\widetilde{I}$ up to a sufficiently high degree by applying its formal power series construction modulo a sufficiently high power of the maximal ideal of $K[[x, y]]$. As $b^\circ_{i\ell}$ and b°_k are polynomials, they can thus be read off from the respective truncated expansions.
(c) The series $u_{i\ell mj}(x')$, $v_{i\ell m}(x)$, $u_{kmj}(x')$ and $v_{km}(x)$ will be determined by a trick which has already appeared several times in the literature, see, e.g., [2]: define the *virtual reduced standard basis* of $\widetilde{I}$ as the polynomial vectors

$$B_{i\ell} = y_i \cdot e_\ell - b^\circ_{i\ell} - \sum\nolimits_{m=1}^{r} \sum\nolimits_{j=0}^{d_m-1} u_{i\ell mj} \cdot x_n^j \cdot e_m - \sum\nolimits_{m=r+1}^{s} v_{i\ell m} \cdot e_m,$$
$$B_k = x_n^{d_k} \cdot e_k - b^\circ_k - \sum\nolimits_{m=1}^{r} \sum\nolimits_{j=0}^{d_m-1} u_{kmj} \cdot x_n^j \cdot e_m - \sum\nolimits_{m=r+1}^{s} v_{km} \cdot e_m,$$

where $u_{i\ell mj}$, $v_{i\ell m}$, u_{kmj} and v_{km} are now new variables (to be abbreviated by u and v). From these we shall construct certain polynomials $U_{i\ell mj}$, $V_{i\ell m}$, U_{kmj} and V_{km} in $K[x, y, u, v]$. All these together will constitute a mother code (U, V) of the baby series vector $(u(x'), v(x))$ of components $u_{i\ell mj}(x')$, $u_{kmj}(x')$, respectively, $v_{i\ell m}(x)$, $v_{km}(x)$. And, consequently, $B_{i\ell}$ and B_k will be father codes of the series vectors $b_{i\ell}$ and b_k we were looking for, with $b_{i\ell},\ b_k \in K[x, y, u(x'), v(x)]^s$.

We have noticed above that, for $r + 1 \le \ell \le s$, the vectors $b_{i\ell}$ equal $(y_i - h_i) \cdot e_\ell$. As the polynomial vectors $B_{i\ell}$ are father codes of $b_{i\ell}$ they will therefore have, for $r + 1 \le \ell \le s$, only one nonzero entry, namely in the ℓ's component. Hence we may set all variables $u_{i\ell mj}$, $v_{i\ell m}$ for $r + 1 \le m \le s$ and $m \ne \ell$ equal to 0. This will be used below when proving the independence of $U_{i\ell mj}$ and U_{kmj} on v.
(d) The construction of the codes U and V uses the polynomial division algorithm from Theorem 4.4 with respect to a suitably chosen monomial order. To this end, compare monomial vectors $u^\gamma v^\delta x^\alpha y^\beta \cdot e_m$ by considering the integer vector

$$(\beta, \alpha_n - d_m, \alpha', -m, \gamma, \delta)$$

lexicographically. Here, the tuples γ and δ are taken as ordered vectors, e.g., by choosing some ordering of their components. It is easily checked that this defines a monomial order $<_\omega$ on $\mathbb{N}^{q+n+p} \times \{1, \ldots, s\}$, where q is the number of u and v variables. The leading (= maximal) monomial vectors of $B_{i\ell}$ and B_k w.r.t. $<_\omega$ are $y_i \cdot e_\ell$ and $x_n^{d_k} \cdot e_k$.

We now divide each $H_i \cdot e_\ell$ and G_k polynomially as described in Theorem 4.4 by all $B_{\iota\lambda}$ and B_κ with respect to this monomial order, say, with leading monomial vectors $y_\iota \cdot e_\lambda$ and $x_n^{d_\kappa} \cdot e_\kappa$, and the scopes $n_{\iota\lambda} = q + n + \iota$ and $n_\kappa = q + n$ (with $1 \le \iota \le p$, $1 \le \lambda \le s$ and $1 \le \kappa \le r$). The division yields in finitely many steps remainders $R_{i\ell}$ and R_k in the canonical direct monomial complement

$$\begin{aligned} &K[u, v]\widehat{\otimes}\mathrm{co}(\widetilde{I}) \\ &\quad = \Big(\oplus_{m=1}^r (\oplus_{j=0}^{d_m-1} K[u, v][[x']] \cdot x_n^j) \cdot e_m\Big) \ \oplus\ \big(\oplus_{m=r+1}^s K[u, v][[x]] \cdot e_m\big) \end{aligned}$$

of $K[u, v]\widehat{\otimes}\mathrm{in}(\widetilde{I})$ in $K[u, v][[x, y]]^s$. Expanding these remainders as polynomial vectors in x_n yields

$$R_{i\ell} = \sum_{m=1}^{r} \sum_{j=0}^{d_m-1} U_{i\ell mj} \cdot x_n^j \cdot e_m + \sum_{m=r+1}^{s} V_{i\ell m} \cdot e_m,$$
$$R_k = \sum_{m=1}^{r} \sum_{j=0}^{d_m-1} U_{kmj} \cdot x_n^j \cdot e_m + \sum_{m=r+1}^{s} V_{km} \cdot e_m,$$

with polynomials $U_{i\ell mj}, U_{kmj}$ in $K[u, x']$ and $V_{i\ell m}, V_{km}$ in $K[u, v, x]$. Note here that $U_{i\ell mj}$ and U_{kmj} do not depend on v because $v_{i\ell m}$ and v_{km} only appear in the last $s - r$ components of $B_{i\ell}$ and B_k and because $u_{i\ell mj}$ and $v_{i\ell m}$ can a priori be set equal to 0 for $m \neq \ell$.

(e) We show that U and V have no constant terms. Replacing in $R_{i\ell}$ and R_k the variables u and v by the series $u(x')$ and $v(x)$ produces power series vectors $r_{i\ell}$ and r_k which belong to $\mathrm{co}(\widetilde{I})$ because $u(x')$ does not depend on x_n and U does not depend on v. But, by construction, $r_{i\ell}$ and r_k also belong to $\widetilde{I}$. From the formal power series division it follows that both $r_{i\ell}$ and r_k are identically zero. This in turn implies by the direct sum decomposition of $\mathrm{co}(\widetilde{I})$ that replacing in U and V the variables u and v by $u(x')$ and $v(x)$ gives zero. As $u(x')$ and $v(x)$ have no constant term, also U and V have no constant term.

(f) We show that U and V form a mother code of certain baby series. By the description of mother codes it suffices to find a monomial order $<_\xi$ on $\mathbb{N}^{q+n+p} \times \{1, \ldots, s\}$ such that the respective initial monomials of $U_{i\ell mj}(u, 0)$, $V_{i\ell m}(u, v, 0)$, $U_{kmj}(u, 0)$ and $V_{km}(u, v, 0)$ are $u_{i\ell mj}$, $v_{i\ell m}$, u_{kmj} and v_{km} (recall that q is the number of u and v variables). By taking an order which is compatible with the degree in the u and v variables it suffices to prove the above for the linear parts of $U_{i\ell mj}(u, 0)$, $V_{i\ell m}(u, v, 0)$, $U_{kmj}(u, 0)$ and $V_{km}(u, v, 0)$.

These linear parts are given by the first substitution step of the polynomial division as the coefficients of $x_n^j \cdot e_m$ (with $1 \leq j \leq d_m - 1$, $1 \leq m \leq r$), respectively, e_m (with $r + 1 \leq m \leq s$), when dividing $H_i \cdot e_\ell$ and G_k by the vectors $B_{\iota\lambda}$ and B_κ ($1 \leq \iota \leq p$, $1 \leq \lambda \leq s$, $1 \leq \kappa \leq r$) with leading monomial vectors $y_\iota \cdot e_\lambda$ and $x_n^{d_\kappa} \cdot e_\kappa$ and scopes $q + n + \iota$, respectively, $q + n$. Here, the y variables are ordered naturally $y_1, \ldots, y_p$, so that the scope $q + n + \iota$ of $y_\iota \cdot e_\lambda$ allows multiplication of $B_{\iota\lambda}$ with polynomials in $x_1, \ldots, x_n, y_1, \ldots, y_\iota$ and all u and v variables.

Recall that the polynomial vectors $b^\circ_{\iota\lambda}$ and b°_κ of $K[x]^s$ appearing in $B_{\iota\lambda}$ and B_κ vanish at zero and hence do not contribute to the linear terms of $U_{i\ell mj}(u, 0)$, $V_{i\ell m}(u, v, 0)$, $U_{kmj}(u, 0)$ and $V_{km}(u, v, 0)$.

Before we enter in the (rather laborious) construction of the monomial order $<_\xi$ on $\mathbb{N}^{q+n+p} \times \{1, \ldots, s\}$ we give a heuristic argument of how it is found and why it should exist.

We already mentioned that the required condition on the Jacobian matrices of U and V only refers to the linear part of U and V with respect to the u- and v-variables. Therefore we may set in U and V the x- and y-variables equal to 0. Next, these linear parts are given by subtracting from the vectors $H_i \cdot e_\ell$ and G_k suitable *constant* multiples of the vectors $B_{\iota\lambda}$ and B_κ, for all ι, λ and κ. As $\mathrm{in}(H_i \cdot e_\ell) = \mathrm{in}(B_{i\ell})$, it is clear that the variables $u_{i\ell mj}$ appear in $U_{i\ell mj}$ with coefficient 1, for all m and j. The same holds for the other u- and v-variables and the respective components of U and V.

The starting point for the specification of a suitable order $<_\xi$ is the following observation (we restrict to $U_{i\ell mj}$, the same comments apply to the other components of U and V): if variables $u_{\iota\lambda mj}$ appear, for some ι, λ, in the linear part of $U_{i\ell mj}$, they should be *larger* than $u_{i\ell mj}$, with respect to the prospective $<_\xi$. These variables stem from a suitable subtraction of a constant multiple of $B_{\iota\lambda}$ from $H_i \cdot e_\ell$. As $\mathrm{in}(B_{\iota\lambda}) = y_\iota \cdot e_\lambda$, the subtraction has to eliminate a (constant multiple of the) monomial vector $y_\iota \cdot e_\lambda$ from the expansion of $H_i \cdot e_\ell$. And as $\mathrm{in}(H_i \cdot e_\ell) = y_i \cdot e_\ell$, we know that only those $B_{\iota\lambda}$ will be subtracted from $B_{i\ell}$ for which $y_\iota \cdot e_\lambda$ is larger than $y_i \cdot e_\ell$ with respect to the order $<_\varepsilon$ on $\mathbb{N}^{n+p} \times \{1, \dots, s\}$.

It is therefore appropriate to choose a monomial order $<_\zeta$ on $\mathbb{N}^q$ so that $u_{\iota\lambda mj}$ is larger than $u_{i\ell mj}$ with respect to $<_\zeta$, for all m and j, whenever $\mathrm{in}(H_\iota \cdot e_\lambda) >_\varepsilon \mathrm{in}(H_i \cdot e_\ell)$. Similar arguments apply to the other variables, by repeating the preceding considerations for the divisions of $H_i \cdot e_\ell$ by B_κ and for the divisions of G_k by $B_{\iota\lambda}$ and B_κ. They specify further conditions to be imposed on $<_\zeta$, and one has to show that such an order actually does exist (i.e., that the conditions do not contradict each other). From $<_\varepsilon$ and $<_\zeta$ one then defines the requested monomial order $<_\xi$ on $\mathbb{N}^{q+n+p} \times \{1, \dots, s\}$ via the lexicographic order on $\mathbb{N} \times (\mathbb{N}^{n+p} \times \{1, ..., s\}) \times \mathbb{N}^q$ induced by $<_\mathbb{N}$, $<_\varepsilon$ and $<_\zeta$:

$$(\gamma, \alpha, \beta, \ell) <_\xi (\gamma', \alpha', \beta', \ell') \quad \text{if} \quad (|\gamma|, (\alpha, \beta, \ell), \gamma) <_{lex} (|\gamma'|, (\alpha', \beta', \ell'), \gamma').$$

The details are explained in the next paragraphs for the linear terms of $U_{i\ell mj}$, $V_{i\ell m}$, U_{kmj} and V_{km}.

Linear terms of $U_{i\ell mj}(u, 0)$: these occur after the first substitution step of the polynomial division as the coefficients of $x_n^j \cdot e_m$ (with $1 \le i \le p$, $1 \le \ell \le s$, $1 \le m \le r$, $1 \le j \le d_m - 1$) when dividing $H_i \cdot e_\ell$ by the vectors $B_{\iota\lambda}$ and B_κ with leading monomial vectors $y_\iota \cdot e_\lambda$ and $x_n^{d_\kappa} \cdot e_\kappa$ and scopes $q+n+\iota$, respectively, $q+n$ (where ι, λ and κ vary in the ranges $1 \le \iota \le p$, $1 \le \lambda \le s$, $1 \le \kappa \le r$). Notice that the polynomials $U_{i\ell mj}$ do not depend on y and v.

Let $x^\rho y^\sigma \cdot e_\ell$ be a monomial vector of the expansion of $H_i \cdot e_\ell$, with $\rho \in \mathbb{N}^n$, $\sigma \in \mathbb{N}^p$. If it is a multiple of the leading monomial vectors $y_\iota \cdot e_\lambda$, respectively, $x_n^{d_\kappa} \cdot e_\kappa$, of $B_{\iota\lambda}$, respectively, B_κ, subject to the correct scope conditions, it will be replaced in the polynomial division by the according multiple of the tails $\overline{B}_{\iota\lambda}$, respectively, $\overline{B}_\kappa$. After the substitution we have to look at the coefficient of $x_n^j \cdot e_m$ and set $x = 0$ and $y = 0$. We distinguish three cases.

(i) The substitution of the monomial vector $y_i \cdot e_\ell$ of $H_i \cdot e_\ell$ by $\overline{B}_{i\ell}$ produces in the coefficient of $x_n^j \cdot e_m$ the summand $u_{i\ell mj}$. The order $<_\zeta$ has to be chosen so that this variable is the smallest one among the u- and v-variables appearing linearly in this coefficient (after having set $x = 0$ and $y = 0$).

(ii) A general monomial vector $x^\rho y^\sigma \cdot e_\ell$ of $H_i \cdot e_\ell$ is a multiple of the leading monomial vector $y_\iota \cdot e_\lambda$ of $B_{\iota\lambda}$ with scope $q+n+\iota$ and contributes to the coefficient of $x_n^j \cdot e_m$ (for some $1 \le m \le r$ and $0 \le j \le d_m - 1$, and after having set $x = 0$ and $y = 0$) if and only if $\lambda = \ell$, $\rho = (0, \dots, 0, \rho_n)$ with $\rho_n \le j$ and $\sigma = e_\iota$, say $x^\rho y^\sigma \cdot e_\lambda = x_n^{\rho_n} y_\iota \cdot e_\lambda$. The only contributions can be constant multiples of $u_{\iota\lambda mj'}$ with $j' + \rho_n = j$. Note then that for this to happen we must have $x_n^{\rho_n} y_\iota \cdot e_\lambda >_\varepsilon \mathrm{in}(H_i \cdot e_\lambda) = y_i \cdot e_\ell$ (otherwise this monomial does not appear in $H_i \cdot e_\lambda$) and $j' \le j$. Therefore $<_\zeta$ should satisfy

$$u_{\iota\lambda m j'} >_\zeta u_{i\lambda m j} \qquad \text{for } j' \le j \text{ and } x_n^{\rho_n} y_\iota \cdot e_\lambda >_\varepsilon y_i \cdot e_\lambda,$$

$$\text{say } j' \le j \text{ and } x_n^j \cdot \mathrm{in}(H_\iota \cdot e_\lambda) >_\varepsilon x_n^{j'} \cdot \mathrm{in}(H_i \cdot e_\lambda).$$

(iii) A general monomial vector $x^\rho y^\sigma \cdot e_\ell$ of $H_i \cdot e_\ell$ is a multiple of the leading monomial vector $x_n^{d_\kappa} \cdot e_\kappa$ of B_κ with scope $q+n$ and contributes to the coefficient of $x_n^j \cdot e_m$ (after having set $x=0$ and $y=0$) if and only if $\kappa = \ell$, $\rho = (0, \dots, 0, \rho_n)$ with $\rho_n = d_\kappa + t$ for some $t \ge 0$ and $\sigma = (0, \dots, 0)$, say $x^\rho y^\sigma \cdot e_\lambda = x_n^{\rho_n} \cdot e_\kappa$. The only contributions can be constant multiples of $u_{\kappa m j'}$ with $t + j' = j$. Note then that we must have $x_n^{\rho_n} \cdot e_\kappa >_\varepsilon \mathrm{in}(H_i \cdot e_\kappa) = y_i \cdot e_\kappa$ and $j' \le j$. Therefore $<_\zeta$ should satisfy

$$u_{\kappa m j'} >_\zeta u_{i\kappa m j} \qquad \text{for } j' \le j \text{ and } x_n^{\rho_n} \cdot e_\kappa >_\varepsilon y_i \cdot e_\kappa,$$

$$\text{say } j' \le j \text{ and } x_n^j \cdot \mathrm{in}(G_\kappa) >_\varepsilon x_n^{j'} \cdot \mathrm{in}(H_i \cdot e_\kappa).$$

Linear terms of $V_{i\ell m}(u, v, 0)$: these occur after the first substitution step of the polynomial division as the coefficients of e_m (with $1 \le i \le p$, $1 \le \ell \le s$, $r+1 \le m \le s$) when dividing $H_i \cdot e_\ell$ by the vectors $B_{\iota\lambda}$ and B_κ with leading monomial vectors $y_\iota \cdot e_\lambda$ and $x_n^{d_\kappa} \cdot e_\kappa$ and scopes $q+n+\iota$, respectively, $q+n$ (where ι, λ and κ vary in the ranges $1 \le \iota \le p$, $1 \le \lambda \le s$, $1 \le \kappa \le r$).

Let $x^\rho y^\sigma \cdot e_\ell$ be a monomial vector of the expansion of $H_i \cdot e_\ell$, with $\rho \in \mathbb{N}^n$, $\sigma \in \mathbb{N}^p$. If it is a multiple of the leading monomial vectors $y_\iota \cdot e_\lambda$, respectively, $x_n^{d_\kappa} \cdot e_\kappa$, of $B_{\iota\lambda}$, respectively, B_κ, subject to the correct scope conditions, it will be replaced in the polynomial division by the according multiple of the tails $\overline{B}_{\iota\lambda}$, respectively, $\overline{B}_\kappa$. After the substitution we have to look at the coefficient of e_m and set $x=0$ and $y=0$. We distinguish three cases.

(i) The substitution of the monomial vector $y_i \cdot e_\ell$ of $H_i \cdot e_\ell$ by $\overline{B}_{i\ell}$ produces in the coefficient of e_m the summand $v_{i\ell m}$. The order $<_\zeta$ has to be chosen so that this variable is the smallest one among the u- and v-variables appearing linearly in this coefficient (after having set $x=0$ and $y=0$).

(ii) A general monomial vector $x^\rho y^\sigma \cdot e_\ell$ of $H_i \cdot e_\ell$ is a multiple of the leading monomial vector $y_\iota \cdot e_\lambda$ of $B_{\iota\lambda}$ with scope $q+n+\iota$ and contributes to the coefficient of e_m (after having set $x=0$ and $y=0$) if and only if $\ell = \lambda$, $\rho = (0, \dots, 0)$ and $\sigma = e_\iota$, say $x^\rho y^\sigma \cdot e_\lambda = y_\iota \cdot e_\lambda$. The only contributions can be constant multiples of $v_{\iota\lambda m}$. For this to happen we must have $y_\iota \cdot e_\lambda >_\varepsilon \mathrm{in}(H_i \cdot e_\lambda)$ (otherwise this monomial does not appear in $H_i \cdot e_\lambda$). Therefore $<_\zeta$ should satisfy

$$v_{\iota\lambda m} >_\zeta v_{i\lambda m} \qquad \text{for } y_\iota \cdot e_\lambda >_\varepsilon y_i \cdot e_\lambda,$$

$$\text{say } \mathrm{in}(H_\iota \cdot e_\lambda) >_\varepsilon \mathrm{in}(H_i \cdot e_\lambda).$$

(iii) A general monomial vector $x^\rho y^\sigma \cdot e_\ell$ of $H_i \cdot e_\ell$ is a multiple of the leading monomial vector $x_n^{d_\kappa} \cdot e_\kappa$ of B_κ with scope $q+n$ and contributes to the coefficient of e_m (after having set $x=0$ and $y=0$) if and only if $\kappa = \ell$, $\rho = (0, \dots, 0, \rho_n)$ with $\rho_n = d_\kappa$ and $\sigma = (0, \dots, 0)$, say $x^\rho y^\sigma \cdot e_\lambda = x_n^{d_\kappa} \cdot e_\kappa$. The only contributions can be constant multiples of $v_{\kappa m}$. Note then that we must have $x_n^{d_\kappa} \cdot e_\kappa >_\varepsilon \mathrm{in}(H_i \cdot e_\kappa)$. Therefore $<_\zeta$ should satisfy

$$v_{\kappa m} >_\zeta v_{i\kappa m} \qquad \text{for } x_n^{d_\kappa}\cdot e_\kappa >_\varepsilon \mathrm{in}(H_i\cdot e_\kappa),$$
$$\text{say } \mathrm{in}(G_\kappa) >_\varepsilon \mathrm{in}(H_i\cdot e_\kappa).$$

Linear terms of $U_{kmj}(u,0)$: these occur after the first substitution step of the polynomial division as the coefficients of $x_n^j\cdot e_m$ (with $1\le k\le r, 1\le m\le r, 0\le j\le d_m-1$) when dividing G_k by the vectors $B_{\iota\lambda}$ and B_κ with leading monomial vectors $y_\iota\cdot e_\lambda$ and $x_n^{d_\kappa}\cdot e_\kappa$ and scopes $q+n+\iota$, respectively, $q+n$ (where ι, λ and κ vary in the ranges $1\le\iota\le p$, $1\le\lambda\le s$, $1\le\kappa\le r$).

Let $x^\rho y^\sigma\cdot e_\lambda$ be a monomial vector of the expansion of G_k, with $\rho\in\mathbb{N}^n$, $\sigma\in\mathbb{N}^p$. If it is a multiple of the leading monomial vectors $y_\iota\cdot e_\lambda$, respectively, $x_n^{d_\kappa}\cdot e_\kappa$, of $B_{\iota\lambda}$, respectively, B_κ, subject to the correct scope conditions, it will be replaced in the polynomial division by the according multiple of the tails $\overline{B}_{\iota\lambda}$, respectively, $\overline{B}_\kappa$. After the substitution we have to look at the coefficient of $x_n^j\cdot e_m$ and set $x=0$ and $y=0$. We distinguish three cases.

(i) The substitution of the monomial vector $x_n^{d_k}\cdot e_k$ of G_k by $\overline{B}_k$ produces in the coefficient of $x_n^j\cdot e_m$ the summand u_{kmj}. The order $<_\zeta$ has to be chosen so that this variable is the smallest one among the u- and v-variables appearing linearly in this coefficient (after having set $x=0$ and $y=0$).

(ii) A general monomial vector $x^\rho y^\sigma\cdot e_\kappa$ of G_k is a multiple of the leading monomial vector $y_\iota\cdot e_\lambda$ of $B_{\iota\lambda}$ with scope $q+n+\iota$ and contributes to the coefficient of $x_n^j\cdot e_m$ (after having set $x=0$ and $y=0$) if and only if $\kappa=\lambda$, $\rho=(0,\ldots,0,\rho_n)$ and $\sigma=e_\iota$, say $x^\rho y^\sigma\cdot e_\lambda=x_n^{\rho_n}y_\iota\cdot e_\lambda$. The only contributions can be constant multiples of $u_{\iota\lambda mj'}$ with $\rho_n+j'=j$, say $\rho_n=j-j'$. For this to happen we must have $x_n^{\rho_n}y_\iota\cdot e_\lambda>_\varepsilon \mathrm{in}(G_k)$ (otherwise this monomial does not appear in G_k). Therefore $<_\zeta$ should satisfy

$$u_{\iota\lambda mj'} >_\zeta u_{kmj} \qquad \text{for } j'\le j \text{ and } x_n^{\rho_n}y_\iota\cdot e_\lambda >_\varepsilon x_n^{d_k}\cdot e_k,$$
$$\text{say } j'\le j \text{ and } x_n^j\cdot \mathrm{in}(H_\iota\cdot e_\lambda) >_\varepsilon x_n^{j'}\cdot \mathrm{in}(G_k).$$

(iii) A general monomial vector $x^\rho y^\sigma\cdot e_\kappa$ of G_k is a multiple of the leading monomial vector $x_n^{d_\kappa}\cdot e_\kappa$ of B_κ with scope $q+n$ and contributes to the coefficient of $x_n^j\cdot e_m$ (after having set $x=0$ and $y=0$) if and only if $\rho=(0,\ldots,0,\rho_n)$ with $\rho_n\ge d_\kappa$ and $\sigma=(0,\ldots,0)$, say $x^\rho y^\sigma\cdot e_\lambda=x_n^{\rho_n}\cdot e_\kappa$ with $\rho_n=d_\kappa+t$ for some $t\ge 0$. The only contributions can be constant multiples of $u_{\kappa mj'}$ with $t+j'=j$. Note then that we must have $x_n^{d_\kappa+t}\cdot e_\kappa>_\varepsilon \mathrm{in}(G_k)$ and therefore $<_\zeta$ should satisfy

$$u_{\kappa mj'} >_\zeta u_{kmj} \qquad \text{for } j'\le j \text{ and } x_n^{d_\kappa+t}\cdot e_\kappa >_\varepsilon \mathrm{in}(G_k),$$
$$\text{say } j'\le j \text{ and } x_n^j\cdot \mathrm{in}(G_\kappa) >_\varepsilon x_n^{j'}\cdot \mathrm{in}(G_k).$$

Linear terms of $V_{km}(u,v,0)$: these occur after the first substitution step of the polynomial division as the coefficients of e_m (with $1\le k\le r, r+1\le m\le s$) when dividing G_k by the vectors $B_{\iota\lambda}$ and B_κ with leading monomial vectors $y_\iota\cdot e_\lambda$ and $x_n^{d_\kappa}\cdot e_\kappa$ and scopes $q+n+\iota$, respectively, $q+n$ (where ι, λ and κ vary in the ranges $1\le\iota\le p$, $1\le\lambda\le s$, $1\le\kappa\le r$).

Let $x^\rho y^\sigma \cdot e_\kappa$ be a monomial vector of the expansion of G_k, with $\rho \in \mathbb{N}^n$, $\sigma \in \mathbb{N}^p$. If it is a multiple of the leading monomial vectors $y_\iota \cdot e_\lambda$, respectively, $x_n^{d_\kappa} \cdot e_\kappa$, of $B_{\iota\lambda}$, respectively, B_κ, subject to the correct scope conditions, it will be replaced in the polynomial division by the according multiple of the tails $\overline{B}_{\iota\lambda}$, respectively, $\overline{B}_\kappa$. After the substitution we have to look at the coefficient of e_m and set $x = 0$ and $y = 0$. We distinguish three cases.

(i) The substitution of the monomial vector $x_n^{d_k} \cdot e_k$ of G_k by $\overline{B}_k$ produces in the coefficient of e_m the summand v_{km}. The order $<_\zeta$ has to be chosen so that this variable is the smallest one among the u- and v-variables appearing linearly in this coefficient (after having set $x = 0$ and $y = 0$).

(ii) A general monomial vector $x^\rho y^\sigma \cdot e_\kappa$ of G_k is a multiple of the leading monomial vector $y_\iota \cdot e_\lambda$ of $B_{\iota\lambda}$ with scope $q + n + \iota$ and contributes to the coefficient of e_m (after having set $x = 0$ and $y = 0$) if and only if $\kappa = \lambda$, $\rho = (0, \dots, 0)$ and $\sigma = e_\iota$, say $x^\rho y^\sigma \cdot e_\lambda = y_\iota \cdot e_\lambda$. The only contributions can be constant multiples of $v_{\iota\lambda m}$. For this to happen we must have $y_\iota \cdot e_\lambda >_\varepsilon \mathrm{in}(G_k)$ (otherwise this monomial does not appear in G_k). Therefore $<_\zeta$ should satisfy

$$v_{\iota\lambda m} >_\zeta v_{km} \qquad \text{for } y_\iota \cdot e_\lambda >_\varepsilon \mathrm{in}(G_k),$$
$$\text{say } \mathrm{in}(H_\iota \cdot e_\lambda) >_\varepsilon \mathrm{in}(G_k).$$

(iii) A general monomial vector $x^\rho y^\sigma \cdot e_\kappa$ of G_k is a multiple of the leading monomial vector $x_n^{d_\kappa} \cdot e_\kappa$ of B_κ with scope $q + n$ and contributes to the coefficient of e_m (after having set $x = 0$ and $y = 0$) if and only if $\rho = (0, \dots, 0, \rho_n)$ with $\rho_n = d_\kappa$ and $\sigma = (0, \dots, 0)$, say $x^\rho y^\sigma \cdot e_\kappa = x_n^{d_\kappa} \cdot e_\kappa$. The only contributions can be constant multiples of $v_{\kappa m}$. Note then that we must have $x_n^{d_\kappa} \cdot e_\kappa >_\varepsilon \mathrm{in}(G_k)$ and therefore $<_\zeta$ should satisfy

$$v_{\kappa m} >_\zeta v_{km} \qquad \text{for } x_n^{d_\kappa} \cdot e_\kappa >_\varepsilon \mathrm{in}(G_k),$$
$$\text{say } \mathrm{in}(G_\kappa) >_\varepsilon \mathrm{in}(G_k).$$

This concludes the computation of the required inequalities for the order $<_\zeta$ on $\mathbb{N}^q$. It will be a monomial order on $\mathbb{N}^q$, where q is the number of the variables u and v, and has to be graded lexicographic subject to the following relations

$$\begin{aligned}
u_{\iota\ell mj'} &>_\zeta u_{i\ell mj} && \text{if } j' \le j \text{ and } x_n^j \cdot \mathrm{in}(H_\iota \cdot e_\ell) >_\varepsilon x_n^{j'} \cdot \mathrm{in}(H_i \cdot e_\ell),\\
u_{i\ell mj'} &>_\zeta u_{kmj} && \text{if } j' \le j \text{ and } x_n^j \cdot \mathrm{in}(H_i \cdot e_\ell) >_\varepsilon x_n^{j'} \cdot \mathrm{in}(G_k),\\
u_{ikmj'} &<_\zeta u_{kmj} && \text{if } j' \ge j \text{ and } x_n^j \cdot \mathrm{in}(H_i \cdot e_k) <_\varepsilon x_n^{j'} \cdot \mathrm{in}(G_k),\\
u_{\kappa mj'} &>_\zeta u_{kmj} && \text{if } j' \le j \text{ and } x_n^j \cdot \mathrm{in}(G_\kappa) >_\varepsilon x_n^{j'} \cdot \mathrm{in}(G_k),\\
v_{\iota\ell m} &>_\zeta v_{i\ell m} && \text{if } \mathrm{in}(H_\iota \cdot e_\ell) >_\varepsilon \mathrm{in}(H_i \cdot e_\ell),\\
v_{i\ell m} &>_\zeta v_{km} && \text{if } \mathrm{in}(H_i \cdot e_\ell) >_\varepsilon \mathrm{in}(G_k),\\
v_{ikm} &<_\zeta v_{km} && \text{if } \mathrm{in}(H_i \cdot e_k) <_\varepsilon \mathrm{in}(G_k),\\
v_{\kappa m} &>_\zeta v_{km} && \text{if } \mathrm{in}(G_\kappa) >_\varepsilon \mathrm{in}(G_k).
\end{aligned}$$

The indices vary in the regions

$$1 \le i, \iota \le p,$$
$$1 \le \ell \le s,$$
$$1 \le m \le r,$$
$$1 \le j, j' \le d_m - 1 \text{ and}$$
$$1 \le k, \kappa \le r$$

for the u variables, respectively, in the regions

$$1 \le i, \iota \le p,$$
$$1 \le \ell \le s,$$
$$r + 1 \le m \le s \text{ and}$$
$$1 \le k, \kappa \le r$$

for the v variables. It is checked that the inequalities for $<_\zeta$ do not contradict each other, i.e., that there actually does exist a monomial order $<_\zeta$ fulfilling the eight conditions.

We now extend $<_\varepsilon$ to a monomial order $<_\xi$ on $\mathbb{N}^{q+n+p} \times \{1, \dots, s\}$ defined by

$$(\gamma, \alpha, \beta, \ell) <_\xi (\gamma', \alpha', \beta', \ell') \quad \text{if} \quad (|\gamma|, (\alpha, \beta, \ell), \gamma) <_{lex} (|\gamma'|, (\alpha', \beta', \ell'), \gamma').$$

Here, $<_{lex}$ denotes the lexicographic order on $\mathbb{N} \times (\mathbb{N}^{n+p} \times \{1, \dots, s\}) \times \mathbb{N}^q$, where $|\gamma|$ and $|\gamma'|$ are compared as elements of $\mathbb{N}$ with the natural order, (α, β, ℓ) and (α', β', ℓ') as elements of $\mathbb{N}^{n+p} \times \{1, \dots, s\}$ with the order $<_\varepsilon$, and γ and γ' as elements of $\mathbb{N}^q$ with respect to the order $<_\zeta$. The inequalities which were imposed on $<_\zeta$ ensure that—as shown above—the initial monomials with respect to $<_\xi$ of the linear terms of $U_{i\ell mj}(u, 0)$, $V_{i\ell m}(u, v, 0)$, $U_{kmj}(u, 0)$ and $V_{km}(u, v, 0)$ are $u_{i\ell mj}$, $v_{i\ell m}$, u_{kmj} and v_{km}. This was needed to show that U and V satisfy the properties of a mother code.
(g) We show that $u(x')$ and $v(x)$ are the baby series of U and V. By definition, $u(x')$ and $v(x)$ vanish at zero. We have already seen in part (d) above that $r_{i\ell} = R_{i\ell}(x, u(x'), v(x))$ and $r_k = R_k(x, u(x'), v(x))$ are zero. As $u(x')$ does not depend on x_n and U does not depend on v it follows from the decomposition of $\mathrm{co}(\widetilde{I})$ that $U(x, u(x'))$ and $V(x, u(x'), v(x))$ are zero. This is what had to be shown and concludes the proof of Theorem 10.1 in the x_n-regular case.

14 Proof of Theorem 11.1 for x_n-Regular Modules

We will prove this special case of Theorem 11.1 by first constructing from the family codes of the given generators $g_1, \dots, g_r$ of I—using Theorems 9.1 and 10.1 in the special case—the family codes of the reduced standard basis of I. This is done via the extension $\widetilde{I}$ of I from Lemmata 8.1 and 8.2. The father code F of the power series vector f is then divided by the father codes $B_{i\ell}$ and B_k of the reduced standard basis $b_{i\ell}$ and b_k of the module $\widetilde{I}$ (see Sect. 13 for the notation). The crucial point here is that the monomial orders can be chosen so that the initial monomial vectors of $b_{i\ell}$ and b_k coincide with the leading monomial vectors of $B_{i\ell}$ and B_k. In this way the power series division of f by $b_{i\ell}$ and b_k is transcribed properly into the polynomial division of F by $B_{i\ell}$ and B_k as it appears in Theorem 4.4. From the quotients and the remainder of this polynomial division we get the family codes of the algebraic series a_k and c in the statement of Theorem 11.1.

So let us carry out this program. By Theorem 9.1 we may assume that the module I is given by a minimal standard basis $g_1, \dots, g_r \in K[[x]]^s$ with initial monomial vectors $x_n^{d_k} \cdot e_k$. Let $(H, G) \in K[x, y]^p \times K[x, y]^{s\times r}$ be the family code of $g_1, \dots, g_r$ and let $h = (h_1, \dots, h_p)$ be the baby series vector of the mother code $H = (H_1, \dots, H_p) \in K[x, y]^p$, so that $g_k = G_k(x, h(x))$.

By Lemma 8.1 the submodule $\widetilde{I} = \langle (y_i - h_i) \cdot e_\ell, g_k\rangle$ of $K[[x, y]]^s$ equals $\langle H_i \cdot e_\ell, G_k\rangle$. Let $<_\varepsilon$ be an extension of $<_\eta$ to $\mathbb{N}^{n+p} \times \{1, \dots, s\}$ with $y_i \cdot e_\ell <_\varepsilon x_j \cdot e_\ell$ for all i, j and ℓ as defined in Lemma 8.2. By Theorem 10.1 in the x_n-regular case we may assume that we already dispose of a reduced standard basis $b_{i\ell}$, b_k of $\widetilde{I}$ with initial monomial vectors $y_i \cdot e_\ell$ and $x_n^{d_k} \cdot e_k$ with respect to $\underset{\sim}{<_\varepsilon}$. The father code of $b_{i\ell}$, b_k is given by the virtual reduced standard basis $B_{i\ell}$, B_k of $\widetilde{I}$, the mother code is the vector (U, V) of components $U_{i\ell mj}$, $V_{i\ell m}$, U_{kmj} and V_{km}. We denote by $(u(x'), v(x))$ with components $u_{i\ell mj}(x')$, $v_{i\ell m}(x)$, $u_{kmj}(x')$ and $v_{km}(x)$ the corresponding baby series vector.

We wish to divide an algebraic power series vector $f \in K[[x]]^s$ by the submodule $I = \langle g_k\rangle$ of $K[[x]]^s$. We may assume that f has the same baby series vector h as $g_1, \dots, g_r$. Write $f = F(x, h(x)) \in K[x, h]^s$ with father code $F \in K[x, y]^s$. We divide F by the polynomial vectors $B_{i\ell}$ and B_k according to the polynomial division algorithm (Theorem 4.4) with leading monomial vectors $y_i \cdot e_\ell$ and $x_n^{d_k} \cdot e_k$ and scopes $q + n + i$, respectively, $q + n$ (recall that n is the number of x-variables, q is the number of u- and v-variables). We get a decomposition

$$F = \sum \widetilde{A}_{i\ell} \cdot B_{i\ell} + \sum \widetilde{A}_k \cdot B_k + C$$

with some polynomials $\widetilde{A}_{i\ell}$ in $K[u, v, x, y]$, $\widetilde{A}_k$ in $K[u, v, x]$, and a polynomial vector $C \in K[u, v]\widehat{\otimes}\mathrm{co}(\widetilde{I})$. Replacing in this equation y by $h(x)$, u by $u(x')$ and v by $v(x)$ yields a decomposition

$$f = \sum \tilde{a}_{i\ell} \cdot \tilde{b}_{i\ell} + \sum \tilde{a}_k \cdot b_k + c$$

for some algebraic power series $\tilde{a}_{i\ell}$, $\tilde{a}_k \in K[[x]]$ and an algebraic power series vector $c \in K[[x]]^s$. The vectors $\tilde{b}_{i\ell}$ and b_k are obtained from $B_{i\ell}$ and B_k by the same substitution of the variables. In particular, $\tilde{b}_{i\ell}$ results from $b_{i\ell}$ by setting $y = h(x)$.
(a) The vector c has mother code U and V and father code C. Expand C into

$$C = \sum\nolimits_{m=1}^{r} \sum\nolimits_{j=0}^{d_m-1} C_{mj}(u, x') \cdot x_n^j \cdot e_m + \sum\nolimits_{m=r+1}^{s} C_m(u, v, x) \cdot e_m,$$

with polynomials $C_{mj}(u, x')$ and $C_m(u, v, x)$. Observe that, similarly as in Sect. 13, part (c), the polynomials $C_{mj}(u, x')$ will not depend on v. Substituting in C the variables u and v by $u(x')$ and $v(x)$ we obtain for c the decomposition

$$c = \sum\nolimits_{m=1}^{r} \sum\nolimits_{j=0}^{d_m-1} C_{mj}(u(x'), x') \cdot x_n^j \cdot e_m + \sum\nolimits_{m=r+1}^{s} C_m(u(x'), v(x), x) \cdot e_m.$$

Therefore $c \in \mathrm{co}(I)$ as required.
(b) We will show that the vectors $\tilde{b}_{i\ell}$ belong to the module $\langle b_k \rangle$, thus getting a decomposition

$$f = \sum a_k \cdot b_k + c$$

for some power series $a_k \in K[[x]]$. To this end, recall that $\langle (y_i - h_i) \cdot e_\ell, g_k \rangle = \langle b_{i\ell}, b_k \rangle$ (as submodules of $K[[x, y]]^s$) and that the vectors b_k do not depend on the y-variables. Thus the replacement of y_i by h_i does not affect them and gives $\langle b_k \rangle \subset \langle g_k \rangle$. As the initial modules of these two modules are equal (being generated by $x_n^{d_k} \cdot e_k$ for $1 \le k \le r$), the division theorem for power formal series yields the equality $\langle g_k \rangle = \langle b_k \rangle$. This shows that the $b_{i\ell}$ belong to the submodule $\langle (y_\iota - h_\iota) \cdot e_\lambda, b_k \rangle$ of $K[[x, y]]^s$. Therefore, upon replacing y_i by h_i in $b_{i\ell}$ we get $\tilde{b}_{i\ell} \in \langle b_k \rangle \subset K[[x]]^s$ as claimed.
(c) We finally show that the power series $a_k \in K[[x]]$ are algebraic and that their codes can be computed algorithmically. For this we will express constructively the father codes $B_{i\ell}$ of $\tilde{b}_{i\ell}$ in terms of B_k and $H_i \cdot e_\ell$.

The problem which we have to solve here is the following: assume given a submodule J of $K[[x]]^s$ generated by polynomial vectors $P_1, \ldots, P_r$, and let Q be a polynomial vector. We shall use Algorithm 1.7.6 of [25] computing the polynomial weak normal form of a polynomial with respect to a polynomially generated ideal in a power series ring, together with the comment at the bottom of page 58. By definition of the polynomial weak normal form [25, def. 1.6.5], we get the construction of a decomposition $SQ = \sum W_k P_k + R$ with polynomials S, W_k and R such that $S(0) \neq 0$, where R equals the remainder of the formal power series division of SQ by $P_1, \ldots, P_r$. In case that Q already belongs to the ideal generated by $P_1, \ldots, P_r$ in the power series ring, this decomposition specializes to $Q = \sum \widetilde{W}_k P_k$ with rational coefficients $\widetilde{W}_k = W_k / S$ in the localization of the polynomial ring at 0.

Apply this technique to the polynomial vectors $B_{i\ell}$ and the submodule $J = \langle B_k, H_i \cdot e_\ell, U \cdot e_\ell, V \cdot e_\ell \rangle$ of $K[[x, y, u, v]]^s$ (with the obvious abbreviations for U and V). By definition, J is generated by polynomial vectors. We have to check that $B_{i\ell} \in J$. For this, recall that $\widetilde{I} = \langle H_i \cdot e_\ell, G_k \rangle = \langle b_{i\ell}, b_k \rangle$ as submodules of $K[[x, y]]^s$ and that $\tilde{b}_{i\ell} \in \langle b_k \rangle$ in $K[[x]]^s$. Then, by construction of U and V, we get the equalities

$$\begin{aligned} \langle B_{i\ell}, B_k, H_i \cdot e_\ell, U \cdot e_\ell, V \cdot e_\ell \rangle &= \langle \tilde{b}_{i\ell}, b_k, H_i \cdot e_\ell, U \cdot e_\ell, V \cdot e_\ell \rangle \\ &= \langle b_k, H_i \cdot e_\ell, U \cdot e_\ell, V \cdot e_\ell \rangle \\ &= \langle B_k, H_i \cdot e_\ell, U \cdot e_\ell, V \cdot e_\ell \rangle \\ &= J. \end{aligned}$$

We conclude that $B_{i\ell} \in J$. This shows that we can write $B_{i\ell}$ as a linear combination of the B_k, $H_\iota \cdot e_\lambda$, $U \cdot e_\lambda$, $V \cdot e_\lambda$ with constructible rational power series coefficients, say

$$B_{i\ell} = \sum\nolimits_{i\ell k} W_{i\ell k} B_k \qquad \text{modulo } H, U \text{ and } V,$$

where $W_{i\ell k} \in K[[x, y, u, v]]$ are rational functions. Upon replacing y_i by $h_i(x)$, u by $u(x')$ and v by $v(x)$ only the B_k will subsist (the evaluations of the other polynomial vectors $H_\iota \cdot e_\lambda, U \cdot e_\lambda, V \cdot e_\lambda$ vanish). This shows that the $W_{i\ell k}$ are (rational) father codes of the coefficients $w_{i\ell k}$ in the linear combinations $\tilde{b}_{i\ell} = \sum_{i\ell k} w_{i\ell k} b_k$ expressing $\tilde{b}_{i\ell}$ in terms of b_k. The mother codes are the components of the polynomial vectors H, U and V.

By definition of a_k in terms of $\tilde{a}_{i\ell}$ and $\tilde{a}_k$ it now follows that the series a_k are algebraic and that their family codes can be constructed by a finite algorithm. This establishes Theorem 11.1 for x_n-regular modules.

15 Proofs of Theorems 10.1 and 11.1 in the General Case

The idea for proving both theorems in the general case is to split a given minimal standard basis of I into two groups specified by the variables appearing in their initial monomial vectors. The first group consists of generators whose initial monomial vectors are pure x_n-powers. The remaining generators have initial monomial vectors which involve also some other variable.

So let be given, by Theorem 9.1, vectors $g_1, \ldots, g_r$ which form a minimal standard basis of I. Adding suitable monomial multiples of the g_k we may assume that $g_1, \ldots, g_r$ form a minimal Janet basis of I with scopes $n_1, \ldots, n_r$. We order $g_1, \ldots, g_r$ and permute the components of $K[[x]]^s$ so that, for some $1 \leq t \leq r$, the vectors $g_1, \ldots, g_t$ are x_n-regular with initial monomial vectors $x_n^{d_k} \cdot e_k$, and so that the initial monomial vectors of the remaining $g_{t+1}, \ldots, g_r$ involve at least one of the variables $x_1, \ldots, x_{n-1}$. It is easy to see that the scopes $n_{t+1}, \ldots, n_r$ of $g_{t+1}, \ldots, g_r$ are all $< n$. This implies that

$$I = \sum\nolimits_{k=1}^{t} K[[x]] \cdot g_k + \sum\nolimits_{k=t+1}^{r} K[[x']] \cdot g_k.$$

Therefore no $g_{t+1}, \ldots, g_r$ need to be multiplied in the subsequent divisions by x_n.

By Theorem 10.1 in the x_n-regular case we may assume that $g_1, \ldots, g_t$ form already the *reduced* standard basis of the submodule $I_0 = \langle g_1, \ldots, g_t \rangle$ of $K[[x]]^s$. By Theorem 11.1 in the x_n-regular case we know how to divide $g_{t+1}, \ldots, g_r$ by $g_1, \ldots, g_t$ through a finite algorithm for the respective family codes. This allows us to assume that $g_{t+1}, \ldots, g_r$ belong to

$$M = \mathrm{co}(I_0) = \sum\nolimits_{m=1}^{t} \sum\nolimits_{j=0}^{d_m-1} K[[x']] \cdot x_n^j \cdot e_m + \sum\nolimits_{m=t+1}^{s} K[[x]] \cdot e_m.$$

It follows from the box condition that the initial monomial vectors of $g_{t+1}, \ldots, g_r$ have their nonzero entry in one of the first t components, and hence belong to the subspace

$$M_1 = \sum\nolimits_{m=1}^{t} \sum\nolimits_{j=0}^{d_m-1} K[[x']] \cdot x_n^j \cdot e_m$$

of M. Setting $I' = \sum_{k=t+1}^{r} K[[x']] \cdot g_k$ we have $I' \subset M$ and $\mathrm{in}(I') \subset M_1$. The monomial order on $\mathbb{N}^n \times \{1, \dots, s\}$ induces via the inclusion $M \subset K[[x]]^s$ in a natural way an ordering of the monomial vectors in M.

We may now apply induction on n as follows.

First notice that $\mathrm{in}(I')$, as a submodule of the free finite $K[[x']]$-module M_1, satisfies again Hironaka's box condition with respect to the induced ordering of the variables. Secondly, no division occurs in the second summand $M_2 = \sum_{m=t+1}^{s} K[[x]] \cdot e_m$ of M. Therefore, by induction on the number of variables and discarding the (irrelevant) fact that the summand M_2 is not finitely generated as $K[[x']]$-module, we may assume to know how to construct the *reduced* standard basis of the $K[[x']]$-submodule I' of M by a finite algorithm on the level of codes. Notice that this basis, when considered as vectors in $K[[x]]^s$, remains reduced with respect to $g_1, \dots, g_t$ because its elements belong to $M = \mathrm{co}(I_0)$.

So we may assume that $g_{t+1}, \dots, g_r$ already form a reduced standard basis of I'. By induction on n we may apply the division algorithm of Theorem 11.1 to I' as a submodule of M. Thus we know how to divide effectively algebraic power series vectors in M by I'.

Apply this to the tails $\overline{g}_k = x_n^{d_k} \cdot e_k - g_k$ of $g_1, \dots, g_t$. They belong to M since $g_1, \dots, g_t$ are a reduced standard basis of I_0 and $M = \mathrm{co}(I_0)$. We divide these $\overline{g}_k$ by I'. This allows us to assume from the beginning that $g_1, \dots, g_t$ are reduced with respect to $g_{t+1}, \dots, g_r$, i.e., that $\overline{g}_k \in \mathrm{co}(I')$ for $1 \le k \le t$. As $I' \subset M = \mathrm{co}(I_0)$, the new $g_1, \dots, g_t$ form again a reduced standard basis (the module they generate may be different from I_0, but its initial module is the same). In total, we have found the reduced standard basis $g_1, \dots, g_r$ of I. This proves Theorem 10.1.

As for Theorem 11.1, any algebraic power series vector $f \in K[[x]]^s$ we wish to divide by $I = \langle g_1, \dots, g_r \rangle$ can first be divided by $I_0 = \langle g_1, \dots, g_t \rangle$ using Theorem 11.1 in the x_n-regular case. It thus yields a remainder in $M = \mathrm{co}(I_0)$. Then, using induction on n and the fact that I' satisfies the box condition in M, we may divide this remainder as vector in M by I'. The resulting remainder can be interpreted, via the inclusion of M in $K[[x]]^s$, as a vector in $\mathrm{co}(I) \subset K[[x]]^s$. It will coincide with the remainder of the formal power series division of f by I in $K[[x]]^s$. It does not matter here that the second summand $\sum_{m=t+1}^{s} K[[x]] \cdot e_m$ of M is not finitely generated as $K[[x']]$-module, because no division occurs in the last $s - t$ components of f.

This establishes the division algorithm for algebraic power series vectors f in $K[[x]]^s$ by submodules I with box condition. Theorem 11.1 is proven.

16 Example

In this section we show in a concrete situation how the algorithms of Theorems 10.1 and 11.1 work in practice (for more examples, see [54]). We will consider an ideal in three variables generated by algebraic power series involving a single baby series. Our objective will be the computation of the codes of the reduced standard basis of the ideal. As it will turn out, the reduced standard basis will consist of polynomials, so that, at the end, there will be no mother codes needed and the father codes of the basis coincide with the elements of the basis. Nevertheless, the example is significant, since it is not at all clear how to construct the codes of the reduced standard basis without using the techniques developed in the paper.

The example is chosen so as to illustrate the various aspects of the algorithm (reduction, division, passage to vectors, induction on the number of variables). Some steps could also be performed directly using some ad hoc tricks due to the simplicity of some of the generators of the ideals and modules involved. This will be indicated correspondingly. Nevertheless, all portions of the algorithm will show off at least once.

As a general rule, each step in the computations below will be followed by a renaming of the involved objects so as to keep the presentation as systematic as possible. In the subsequent step, letters will always refer to this renamed object and not to the original object defined at earlier stages of the exposition.

The initial variables will be denoted x, y and z, corresponding to x_1, x_2 and x_3 in the text, with this ordering. This will affect x_n-regularity, being here first z-regularity, then, later, y-regularity and finally x-regularity. Also, the involved polynomial divisions will use this ordering of the variables.

The additional auxiliary variables appearing in the mother codes will be denoted by $t_1, t_2, \dots$ (instead of $y_1, y_2, \dots$ as in the text). The respective baby series will be $h_1, h_2, \dots$

We consider the ideal I in $K[[x, y, z]]$ generated by three power series g_1, g_2, g_3 given as

$$\begin{aligned} g_1 &= z^2 + xyz + \frac{1}{4}xyz^2 + \dots \\ &= z^2 + xyh(z), \\ g_2 &= yz + x^2z + y^2z, \\ g_3 &= y^2 + xyz. \end{aligned}$$

Here,

$$h(z) = 1 - \sqrt{1-z} = \frac{1}{2}z + \frac{1}{8}z^2 + \dots$$

is the only involved baby series. Its mother code H is taken as

$$H = 2t - t^2 + z.$$

So that $h = h(z)$ is the unique formal power series solution of $H(z, t) = 0$ satisfying $h(0) = 0$. Later on, when other mother codes will appear, we shall set $t = t_1$, $h = h_1$ and $H = H_1$. The father codes of g_1, g_2, g_3 are

$$\begin{aligned} G_1 &= z^2 + xyt \\ G_2 &= yz + x^2z + y^2z, \\ G_3 &= y^2 + xyz. \end{aligned}$$

The last two G_2 and G_3 do not involve t because g_2 and g_3 are polynomials and hence $G_2 = g_2$ and $G_3 = g_3$. For our purposes it will be sufficient to have just one generator which is a true series.

We wish to compute family codes of the elements of the reduced standard basis of $I = \langle g_1, g_2, g_3 \rangle \subset K[[x, y, z]]$ with respect to a given monomial order on $\mathbb{N}^3$. We shall choose the graded lexicographic order $<_\eta$ on $\mathbb{N}^3$ with $x > y > z$. This yields the initial monomials

$$\begin{aligned} \mathrm{in}(g_1) &= z^2, \\ \mathrm{in}(g_2) &= yz, \\ \mathrm{in}(g_3) &= y^2. \end{aligned}$$

It will turn out these do not yet generate the initial ideal $\mathrm{in}(I)$ of I. The missing monomial is $x^4 z$, which is the initial monomial of the element

$$g_4 = x^4 z - x^3 y z^2 + x^4 y h(z)$$

of I with father code

$$G_4 = x^4 z - x^3 y z^2 + x^4 y t.$$

Actually, $g_1, \ldots, g_4$ form a standard basis of I with respect to $<_\eta$. This basis is obviously not reduced.

Overview: For the convenience of the reader, let us list the various steps which will appear in the calculations (below, "computation of ..." will always mean "computation of the code of ...".)

Step 1: Computation of a standard basis of I. In addition to g_1, g_2, g_3 we will get a fourth generator g_4 of I, the one from above.

Step 2: Specification of all x_n-regular elements of this basis and computation of the reduced standard basis of the ideal I_1 generated by them. Here, x_n is z; as only g_1 is z-regular, $I_1 = \langle g_1 \rangle$ is principal and its reduced standard basis can be computed with the algorithm of [2, Theorem 5.5] or, equivalently, as described in Theorem 10.1 above in the x_n-regular case for principal ideals. The reduced standard basis of I_1 will again be denoted by g_1. Its tail belongs to $\mathrm{co}(I_1) \cong K[[x, y]]^2$, where $\mathrm{co}(I_1) = K[[x, y]] \oplus K[[x, y]]z$ denotes the canonical monomial direct complement of I_1 in $K[[x, y, z]]$ with respect to the chosen monomial order.

Step 3: Reduction of g_2, g_3, g_4 by $I_1 = \langle g_1 \rangle$. This is the division of g_2, g_3, g_4 by g_1 with the algorithm of [2, Theorem 5.6] or, equivalently, the division as described in Theorem 11.1 above in the x_n-regular case for principal ideals, x_n being here z. The reduced series will again be denoted by g_2, g_3, g_4.

Step 4: Interpretation of g_2, g_3, g_4 as vectors in $\mathrm{co}(I_1) \cong K[[x, y]]^2$ and computation of the reduced standard basis of the submodule $I_2 = \langle g_2, g_3, g_4 \rangle$ of $K[[x, y]]^2$ generated by them. By Step 1, the vectors g_2, g_3, g_4 already form a standard basis of I_2, so they need not be completed again. Step 4 consists of four substeps.

Substep 4A: Specification of all y-regular elements among g_2, g_3, g_4 and computation of the reduced standard basis of the submodule I_3 of $K[[x, y]]^2$ generated by these as described in Theorem 10.1 for the x_n-regular case (only g_2 and g_3 will be y-regular, so that $I_3 = \langle g_2, g_3\rangle$.) The reduced standard basis of I_3 will again be denoted by g_2, g_3. Its tails belong to $\text{co}(I_3) \cong K[[x]]^3$, where $\text{co}(I_3) = (K[[x]] \oplus K[[x]]y) \times K[[x]]$ denotes the canonical monomial direct complement of I_3 in $K[[x, y]]^2$ with respect to the chosen monomial order.

Substep 4B: Reduction of g_4 by $I_3 = \langle g_2, g_3\rangle$. This is the division of g_4 by g_2, g_3 in $K[[x, y]]^2$ as described in Theorem 11.1 above in the x_n-regular case, x_n being now y. The reduced vector will again be denoted by g_4.

Substep 4C: Interpretation of g_4 as a vector in $\text{co}(I_3) \cong K[[x]]^3$ and computation of the reduced standard basis of the submodule I_4 of $K[[x]]^3$ generated by it as described in Theorem 10.1 in the x_n-regular case, x_n being here x. The situation will be so simple that the reduced standard basis of I_4 can be read off directly without using Theorem 10.1. It will again be denoted by g_4.

Substep 4D: Reduction of g_2, g_3 by $I_4 = \langle g_4\rangle$. This is the division of the tails $\overline{g}_2$, $\overline{g}_3$ of g_2, g_3 by g_4 in $K[[x]]^3$ as described in Theorem 11.1 in the x_n-regular case, x_n being here x. Again, the situation will be so simple that the reduction can be read off without using Theorem 11.1. The reduced vectors will again be denoted by g_2, g_3.

The reduced standard basis of I_2 obtained in step 4 is thus g_2, g_3, g_4.

Step 5: Reduction of g_1 by $I_2 = \langle g_2, g_3, g_4\rangle$. This is the division of the tail $\overline{g}_1$ of g_1 by g_2, g_3, g_4 in $K[[x, y]]^2$ as described in Theorem 11.1 in the general case. This step consists of 2 substeps.

Substep 5A: Reduction of g_1 by $I_3 = \langle g_2, g_3\rangle$. This is the division of the tail $\overline{g}_1$ of g_1 by g_2, g_3 in $K[[x, y]]^2$ as described in Theorem 11.1 in the x_n-regular case, x_n being here y. The reduced vector will again be denoted by g_1. Its tail belongs to $\text{co}(I_3) \cong K[[x]]^3$.

Substep 5B: Reduction of g_1 by $I_4 = \langle g_4\rangle$. This is the division of the tail $\overline{g}_1$ of g_1 by g_4 in $K[[x]]^3$ as described in Theorem 11.1 in the x_n-regular case, x_n being here x. The reduced vector will again be denoted by g_1.

Conclusion: The vectors g_1, g_2, g_3, g_4 obtained after step 5 now have to be reinterpreted as power series in $K[[x, y, z]]$. By construction, they form the reduced standard basis of the ideal I we started with.

Computations: We start now with the explicit description of the various stages of the construction of the reduced standard basis of the ideal I.

Step 1: Computation of a minimal standard basis of I.

Let $\widetilde{I} = \langle H, G_k\rangle = \langle t - h, g_k\rangle$ be the ideal of $K[[x, y, z, t]]$ associated to I as in Lemma 8.1 (here, no e_ℓ's appear, since we work with ideals instead of modules; the index k varies between 1 and 3). We may apply Mora's tangent cone algorithm or Lazard's homogenization method. Let u be a homogenizing variable, and denote by H^h, G_k^h the homogenized polynomials of H and G_k in $K[x, y, z, t, u]$.

We extend the monomial order $<_\eta$ on $\mathbb{N}^3$ first to an order $<_\varepsilon$ on $\mathbb{N}^4$ (the set of exponents of series in $K[[x, y, z, t]]$) such that $\text{in}_\varepsilon H = t$ and $\text{in}_\varepsilon G_k = \text{in}_\eta g_k$, and then $<_\varepsilon$ to an order $<_h$ on $\mathbb{N}^5$ (the set of exponents of series in $K[[x, y, z, t, u]]$) such that $\pi(\mathrm{lm}_h(H^h)) = \mathrm{in}_\varepsilon(H)$ and $\pi(\mathrm{lm}_h(G_k^h) = \mathrm{in}_\varepsilon(G_k)$, where lm_h denotes the leading monomial of a polynomial with respect to $<_h$ and $\pi : \mathbb{N}^4 \times \mathbb{N} \to \mathbb{N}^4$.

A polynomial Gröbner basis with respect to $<_h$ on $\mathbb{N}^5$ of the ideal $J \subset K[x, y, z, t, u]$ generated by H^h, G_1^h, G_2^h and G_3^h is given by

$$\begin{aligned}
&ut - \frac{1}{2}uz - \frac{1}{2}t^2, uy^2 + zyx, uzy + zy^2 + zx^2, uz^2 + tyx,\\
&zy^3 - z^2yx + zyx^2, t^2y^2 + 2tzyx - z^2yx, z^2y^2 - ty^2x + z^2x^2,\\
&t^2zy + 2tzy^2 - ty^2x + 2tzx^2, t^2z^2 + 2t^2yx - tzyx,\\
&z^3yx - tzyx^2 + tyx^3, zy^2x^2 - z^2x^3 + zx^4, ty^3x - tzyx^2 + tyx^3,\\
&z^3x^3 - ty^2x^3, t^2yx^3 + 2tzx^4 - z^2x^4, uzx^4 - z^2yx^3 + tyx^4,\\
&tz^2yx^3 - \frac{1}{2}t^2zx^4 + 2tzx^5 - z^2x^5 + \frac{1}{2}tyx^5,\\
&t^3zx^4 - 4t^2zx^5 - 8tzyx^5 + 4z^2yx^5 + 4tzx^6 - z^2x^6.
\end{aligned}$$

Now substitute u by 1 and t by $h(z)$ to get a standard basis of I. It is given by g_1, g_2 and g_3 as above and the series g_4, with

$$\begin{aligned}
g_4 &= x^4z - x^3yz^2 + x^4yh(z) =\\
&= x^4z - x^3yz^2 + x^4y(\frac{1}{2}z + \frac{1}{8}z^2 + \ldots)
\end{aligned}$$

and initial monomial x^4z. This series has as father code the polynomial

$$G_4 = x^4z - x^3yz^2 + x^4yt.$$

The standard basis shows that the ideal I satisfies Hironaka's box condition with respect to a monomial order such that $x < y < z$. The initial ideal is generated by z^2, yz, y^2 and x^4z. Moreover, it can be seen that the series g_1, g_2, g_3, g_4 form a Janet basis of I with scopes 3, 2, 2 and 1, respectively.

Step 2: Computation of the reduced standard basis of the ideal $I_1 = \langle g_1 \rangle$.

Clearly, $g_1 = z^2 + xyh$ is the only z-regular series among $g_1, \ldots, g_4$. We set $I_1 = \langle g_1 \rangle \subset K[[x, y, z]]$. The monomials xyz^m appearing in $xyh(z)$ are multiples of the initial monomial z^2 of g_1, therefore g_1 is not reduced (or in Weierstrass form). Let us apply the algorithm described in Theorem 10.1 for x_n-regular series in order to find a reduced standard basis of the ideal I_1. This algorithm coincides with the algorithm in [2, Theorem 5.5]. The minimal reduced standard basis b_{11}, b_1 of the ideal $\widetilde{I}_1 = \langle H, G_1 \rangle \subset K[[x, y, z, t]]$ has the following form (with the notation of the proof of Theorem 10.1).

$$\begin{aligned}
b_{11} &= t - b_{11}^\circ - u_{1110}(x') - u_{1111}(x')z,\\
b_1 &= z^2 - b_1^\circ - u_{110}(x') - u_{111}(x')z,
\end{aligned}$$

where $b_{11}^{\circ}, b_1^{\circ}$ belong to $K \oplus Kz$, the letter x' stands for the variables (x, y), and $u_{1110}(x')$, $u_{1111}(x')$, $u_{110}(x')$, $u_{111}(x')$ are power series vanishing at 0. To simplify let us write

$$
\begin{aligned}
b &= t - b^{\circ} - u_0(x') - u_1(x')z, \\
c &= z^2 - c^{\circ} - w_0(x') - w_1(x')z.
\end{aligned}
$$

We first compute b° and c° by setting x and y equal to 0 in the ideal $\tilde{I}_1$. We get the ideal

$$
\langle H(0,0,z,t), G_1(0,0,z,t)\rangle \;=\; \langle H, z^2\rangle \;=\; \langle t-h, z^2\rangle \;\subset\; K[[t,z]].
$$

From the mother code of $h(z)$ we can compute its Taylor expansion up to any given degree. In this case we have $h = \frac{1}{2}z + \frac{1}{8}z^2 + \cdots$. It follows that the (minimal) reduced standard basis of the ideal $\langle t-h, z^2\rangle$ is $t - \frac{1}{2}z$ and z^2. This implies that $b^{\circ} = \frac{1}{2}z$ and $c^{\circ} = 0$.

Next we have to find a family code for the series $u_0(x,y)$, $u_1(x,y)$, $w_0(x,y)$, $w_1(x,y)$. We will divide—using the polynomial division—the polynomials H and G_1 by the virtual reduced standard basis

$$
\begin{aligned}
B &= t - b^{\circ} - u_0 - u_1 z = t - \frac{1}{2}z - u_0 - u_1 z, \\
C &= z^2 - c^{\circ} - w_0 - w_1 z = z^2 - w_0 - w_1 z
\end{aligned}
$$

of the ideal $\widetilde{I}_1$ with initial monomials t and z^2, where u_0, u_1, w_0, w_1 are now just unknowns. The remainders R, S of these divisions are

$$
\begin{aligned}
R &= (-2u_1 + u_0 + 2u_0u_1 + \frac{1}{4}w_1 + u_1w_1 + u_1^2w_1)z - 2u_0 + u_0^2 + \frac{1}{4}w_0 \\
&\quad + u_1w_0 + u_1^2w_0, \\
S &= (\frac{1}{2}xy + xyu_1 + w_1)z + xyu_0 + w_0.
\end{aligned}
$$

Let U_1, U_2, respectively, W_1, W_2, be the coefficients of z and 1 in R and S. It is easy to prove that they form a mother code with baby series $u_0(x,y)$, $u_1(x,y)$, $w_0(x,y)$ and $w_1(x,y)$. In the present example the solutions vanishing at 0 of this mother code can be described in an equivalent and more explicit way as follows. From the four equations $U_1 = U_2 = W_1 = W_2 = 0$ we get

$$
\begin{aligned}
u_0(x,y) &= w_0(x,y) = 0, \\
w_1(x,y) &= -\frac{1}{2}xy - xyu_1(x,y), \\
u_1(x,y) &= -\frac{1}{16}xy + \frac{1}{16}x^2y^2 - \frac{67}{1024}x^3y^3 + O(x^4y^4),
\end{aligned}
$$

where the last series is the unique solution vanishing at 0 of the equation

$$H_2(x, y, z, t_2) = 8xyt_2^3 + 12xyt_2^2 + 16(1 + xy)t_2 + xy = 0$$

in a new variable t_2. In this way, H_2 becomes the mother code of the algebraic series $u_1(x, y)$, its father code being the polynomial t_2. The father code of $w_1(x, y)$ is $-\frac{1}{2}xy - xyt_2$.

The reduced standard basis of the ideal $I_1 = \langle g_1 \rangle$ is given by substituting in the polynomial $C = z^2 - w_0 - w_1 z$ the variables w_0 and w_1 by the series $w_0(x, y) = 0$ and $w_1(x, y) = -\frac{1}{2}xy - xyu_1(x, y)$. We get the algebraic series $z^2 + (\frac{1}{2}xy + xyu_1(x, y))z$ with father code $C(0, -\frac{1}{2}xy - xyt_2, x, y, z) = z^2 + (\frac{1}{2}xy + xyt_2)z$. We denote this series in the sequel again by g_1, and call its father code G_1. The corresponding baby series $u_1(x, y)$ is denoted by $h_2(x, y)$ with mother code $H_2(x, y, z, t_2)$ from above. For later reference we collect the new data in a table.

$$\begin{aligned}
g_1 &= z^2 + \left(\frac{1}{2}xy + xyh_2(x, y)\right) z, \\
G_1 &= z^2 + \left(\frac{1}{2}xy + xyt_2\right) z, \\
h_2(x, y) &= -\frac{1}{16}xy + \frac{1}{16}x^2y^2 - \frac{67}{1024}x^3y^3 + \dots, \\
H_2(x, y, z, t_2) &= 8xyt_2^3 + 12xyt_2^2 + 16(1 + xy)t_2 + xy.
\end{aligned}$$

Note here that the original baby series $h = h_1$ has been eliminated.

Step 3: Reduction of g_2, g_3, g_4 by $I_1 = \langle g_1 \rangle$.

We will apply the algorithm described in the proof of Theorem 11.1 for x_n-regular series to divide g_2, g_3, g_4 by g_1. It will be useful to add a new variable t_3 and define

$$H_3(x, y, z, t_1, t_2, t_3) = t_3 + \frac{1}{2}xy + xyt_2.$$

In this setting (H_1, H_2, H_3) is the mother code of the baby series (h_1, h_2, h_3) where $h_1 = h(z)$ and $h_2 = u_1(x, y)$ have been previously defined and where h_3 equals $w_1(x, y)$ from above. It is clear from $\mathrm{in}(I_1) = \langle z^2 \rangle$ that $g_2 = yz + x^2z + y^2z$ and $g_3 = y^2 + xyz$ are already reduced with respect to I_1. Let us reduce g_4. We shall use polynomial division. Let $\widetilde{I}_1 = \langle B, C \rangle$ be the ideal in $K[[x, y, z, t_1, t_2, t_3]]$ associated to I_1 as in Lemma 8.1 (it is checked that this is exactly the ideal of the lemma), with virtual reduced standard basis

$$\begin{aligned}
B &= t_1 - \frac{1}{2}z - t_2 z, \\
C &= z^2 - t_3 z.
\end{aligned}$$

Dividing the father code G_1 of g_1 by B and C with initial monomials t_1 and z^2 we get

$$\begin{aligned} G_4 &= x^4 z - x^3 y z^2 + t_1 x^4 y = \\ &= x^4 y B - x^3 y C + D_4, \end{aligned}$$

where $D_4 = (\frac{1}{2}yx^4 + yx^4 t_2 + x^4 - yx^3 t_3)z$. Let us replace G_4 by D_4 and call it again G_4. It is the father code of a new algebraic series, denoted again by g_4, and defined by $g_4 = G_4(x, y, z, h_1, h_2, h_3)$. We have

$$g_4 = \left(\frac{1}{2}yx^4 + yx^4 h_2 + x^4 - yx^3 h_3\right) z.$$

The series g_2, g_3, g_4 are now reduced with respect to $I_1 = \langle g_1 \rangle$. For later reference we collect the actual data in a table.

$$\begin{aligned} g_1 &= z^2 + (\frac{1}{2}xy + xyh_2(x, y))z, \\ G_1 &= z^2 + (\frac{1}{2}xy + xyt_2)z, \\ g_2 &= G_2 = yz + x^2 z + y^2 z, \\ g_3 &= G_3 = y^2 + xyz, \\ g_4 &= \left(\frac{1}{2}yx^4 + yx^4 h_2 + x^4 - yx^3 h_3\right) z, \\ G_4 &= \left(\frac{1}{2}yx^4 + yx^4 t_2 + x^4 - yx^3 t_3\right) z, \\ h_1 &= \frac{1}{2}z + \frac{1}{8}z^2 + \dots, \\ h_2 &= -\frac{1}{16}xy + \frac{1}{16}x^2 y^2 - \frac{67}{1024}x^3 y^3 + \dots, \\ h_3 &= -\frac{1}{2}xy - xyh_2, \\ H_1 &= 2t_1 - t_1^2 + z, \\ H_2 &= 8xyt_2^3 + 12xyt_2^2 + 16(1 + xy)t_2 + xy, \\ H_3 &= t_3 + \frac{1}{2}xy + xyt_2. \end{aligned}$$

Step 4: Computation of the reduced standard basis of the submodule $I_2 = \langle g_2, g_3, g_4 \rangle$ of $\mathrm{co}(I_1) \cong K[[x, y]]^2$.

The canonical direct monomial complement $\mathrm{co}(I_2)$ equals $K[[x, y]] \oplus K[[x, y]]z$ and is therefore isomorphic to $K[[x, y]]^2$ as $K[[x, y]]$-module. The three series g_2, g_3, g_4 are mapped under this isomorphism onto the vectors

$$\begin{aligned} g_2 &= (0, y + x^2 + y^2), \\ g_3 &= (y^2, xy), \\ g_4 &= (0, x^4 + \frac{1}{2}x^4 y + x^4 y h_2 - x^3 y h_3). \end{aligned}$$

The monomial order $<_\eta$ on $\mathbb{N}^3$ induces via the inclusion $K[[x, y]] \oplus K[[x, y]]z \subset K[[x, y, z]]$ a monomial order, also denoted by $<_\eta$, on $\mathbb{N}^2 \times \{1, 2\}$. The respective initial monomial vectors are

$$\begin{aligned} \mathrm{in}(g_2) &= (0, y),\\ \mathrm{in}(g_3) &= (y^2, 0),\\ \mathrm{in}(g_4) &= (0, x^4). \end{aligned}$$

We see that g_2 and g_3 are y-regular, whereas g_3 is not. By the proof of Theorem 10.1 we first treat the submodule generated by g_2 and g_3.

Substep 4A: Computing the reduced standard basis of the submodule $I_3 = \langle g_2, g_3\rangle$ of $K[[x, y]]^2$.

The vectors g_2, g_3 are not the reduced standard basis of I_3 but form at least a minimal standard basis. The father codes of g_2 and g_3 are $G_2 = (0, y + x^2 + y^2)$ and $G_3 = (y^2, xy)$, respectively. They do not depend on the variables t_i. From the proofs of Theorems 2 and 3, it follows that we have to consider the virtual reduced standard basis of $\widetilde{I}_3 = I_3$. Said differently, we do not need to consider the vectors $B_{i\ell}$. Thus

$$\begin{aligned} B_2 &= y^2 \cdot e_1 - b_2^\circ - \sum\nolimits_{m=1}^{2} \sum\nolimits_{j=0}^{d_m-1} u_{2mj}\, y^j e_m,\\ B_3 &= y \cdot e_2 - b_3^\circ - \sum\nolimits_{m=1}^{2} \sum\nolimits_{j=0}^{d_m-1} u_{3mj}\, y^j e_m, \end{aligned}$$

where $d_1 = 2, d_2 = 1$ and the vectors b_2°, b_3° belong to $(K \times K) \oplus (Ky \times (0))$. The vectors b_2°, b_3° are obtained by specializing x to 0 in G_2 and G_3. From $G_2(0, y) = (y^2, 0)$, $G_3(0, y) = (0, y + y^2)$ we conclude that $b_2^\circ = b_3^\circ = (0, 0)$.

We then apply the polynomial division to reduce G_2 and G_3 by the virtual reduced standard basis B_2 and B_3 of I_3 with initial monomial vectors $y^2 \cdot e_1$ and $y \cdot e_2$. The corresponding remainders are

$$\begin{gathered} ((u_{311}u_{211} + u_{211}u_{220} + u_{211} + u_{210})y + u_{210}u_{220} + u_{210} + u_{310}u_{211},\\ u_{320}u_{211} + u_{220}^2 + u_{220} + x^2),\\ ((u_{211}x + u_{311})y + u_{210}x + u_{310}, u_{220}x + u_{320}). \end{gathered}$$

Therefore, the system

$$\begin{aligned} &u_{311}u_{211} + u_{211}u_{220} + u_{211} + u_{210} = 0,\\ &u_{210}u_{220} + u_{210} + u_{310}u_{211} = 0,\\ &u_{320}u_{211} + u_{220}^2 + u_{220} + x^2 = 0,\\ &u_{211}x + u_{311} = 0,\\ &u_{210}x + u_{310} = 0,\\ &u_{220}x + u_{320} = 0 \end{aligned}$$

is the mother code for the series $u_{210}(x), u_{211}(x), u_{220}(x), u_{310}(x), u_{311}(x), u_{320}(x)$. From this system we get

$$\begin{aligned} u_{210}(x) &= u_{211}(x) = u_{310}(x) = u_{311}(x) = 0, \\ u_{220}(x) &= h_4(x), \\ u_{320}(x) &= -h_4(x)x, \end{aligned}$$

where

$$h_4(x) = -\frac{1}{2} + \sqrt{\frac{1}{4} - x^2} = -x^2 - x^4 - 2x^6 - 5x^8 + O(x^{10})$$

is the unique solution vanishing at 0 of the equation

$$H_4 = t_4^2 + t_4 + x^2 = 0.$$

The reduced standard basis of the submodule $I_3 = \langle g_2, g_3 \rangle$ of $K[[x, y]]^2$ is

$$\begin{aligned} &(0, y - h_4(x)), \\ &(y^2, xh_4(x)). \end{aligned}$$

We denote these vectors again by g_2 and g_3. For later reference we collect the actual data in a table.

$$\begin{aligned} g_1 &= z^2 + (\frac{1}{2}xy + xyh_2(x, y))z, \\ G_1 &= z^2 + (\frac{1}{2}xy + xyt_2)z, \\ g_2 &= G_2 = yz + x^2 z + y^2 z, \\ g_3 &= G_3 = y^2 + xyz, \\ g_4 &= \left(\frac{1}{2}yx^4 + yx^4 h_2 + x^4 - yx^3 h_3\right) z, \\ G_4 &= \left(\frac{1}{2}yx^4 + yx^4 t_2 + x^4 - yx^3 t_3\right) z, \\ h_1 &= \frac{1}{2}z + \frac{1}{8}z^2 + \ldots, \\ h_2 &= -\frac{1}{16}xy + \frac{1}{16}x^2y^2 - \frac{67}{1024}x^3y^3 + \ldots, \\ h_3 &= -\frac{1}{2}xy - xyh_2, \\ h_4 &= -x^2 - x^4 - 2x^6 - 5x^8 + \ldots, \\ H_1 &= 2t_1 - t_1^2 + z, \\ H_2 &= 8xyt_2^3 + 12xyt_2^2 + 16(1 + xy)t_2 + xy, \\ H_3 &= t_3 + \frac{1}{2}xy + xyt_2, \\ H_4 &= t_4^2 + t_4 + x^2 = 0. \end{aligned}$$

Substep 4B: Reduction of g_4 by the submodule $I_3 = \langle g_2, g_3\rangle$ of $K[[x, y]]^2$.

We reduce the vector $g_4 = (0, \frac{1}{2}yx^4 + yx^4h_2 + x^4 - yx^3h_3)$ by $I_3 = \langle g_2, g_3\rangle$. We point out that it is not enough—as the special shape of $g_2 = (0, y - h_4(x))$ may suggest—to replace y by $h_4(x)$ in g_4 because the power series $h_2(x, y)$ and $h_3(x, y)$ depend on x, y.

The virtual reduced standard basis $B_{i\ell}$, B_2, B_3 with $i = 2, 3, 4$, $\ell = 1, 2$, of the submodule $\widetilde{I_3} = \langle H_i \cdot e_\ell, G_2, G_3\rangle$ of $K[[x, y, z, t_2, t_3, t_4]]$ as in Lemma 8.1 is

$$\begin{aligned}
B_{i\ell} &= t_i \cdot e_\ell - b^\circ_{i\ell} - \sum\nolimits_{m=1}^{2} \sum\nolimits_{j=0}^{d_m-1} u_{i\ell mj} \cdot y^j \cdot e_m,\\
B_2 &= (0, y - h_4(x)),\\
B_3 &= (y^2, xh_4(x)),
\end{aligned}$$

using here the computation we made in Substep 4A. To calculate the reduced standard basis of $\widetilde{I_3}$ we use polynomial division: we divide $H_{i\ell}$, $i = 2, 3, 4$, $\ell = 1, 2$, and G_2 and G_3 by $B_{\iota\lambda}$, $\iota = 2, 3, 4$, $\lambda = 1, 2$, and B_2, B_3 with leading monomial vectors $t_\iota \cdot e_\lambda,\ y \cdot e_2,\ y^2 \cdot e_1$, respectively. From the remainders of these divisions we get—by a rather tedious computation—a system defining the mother code for the series $u_{i\ell mj}(x)$. Another, more direct computation then shows that this system can be transformed into an equivalent system of form $H_5 = H_6 = 0$ where

$$\begin{aligned}
H_5 &= 8xt_4t_5^3 + 16xt_4t_5^2 + (16 + 16xt_4)t_5 + xt_4,\\
H_6 &= (16 + 12t_4t_5x + 8t_4t_5^2x + 16xt_4 + \frac{1}{32}t_4^3x^3 - \frac{3}{4}t_4^2x^2 - \frac{1}{2}t_4^2t_5x^2)t_6 +\\
&\quad + x^3t_4 + \frac{1}{512}t_4^3x^5 - \frac{3}{64}t_4^2x^4,
\end{aligned}$$

and where we have set $t_5 = u_{2220},\ t_6 = u_{2120}$. The baby series vector of the mother code (H_5, H_6) will be denoted by (h_5, h_6).

Now we can apply polynomial division to reduce G_4 with respect to the virtual reduced standard basis $B_{i\ell}$, B_2, B_3 of $\widetilde{I_3}$. The division gives

$$\begin{aligned}
G_4 &= \sum \widetilde{A}_{i\ell}B_{i\ell} + \widetilde{A}_2B_2 + \widetilde{A}_3B_3 + C_4,\\
C_4 &= \left(0, \left((t_4 + t_4^2)t_5 + 1 + \frac{1}{2}t_4^2 + \frac{1}{2}t_4\right)x^4\right),
\end{aligned}$$

where C_4 is the father code of the reduction of g_4 by g_2, g_3. We denote this reduction again by g_4. It is the vector obtained from C_4 by substituting the variables t_4, t_5 by the power series h_4 and h_5.

Substep 4C: Computation of the reduced standard basis of the submodule $I_4 = \langle g_4\rangle$ of $\mathrm{co}(I_3) \cong K[[x]]^3$.

By Substep 4B we have achieved that g_4 belongs to the canonical direct monomial complement

$$\mathrm{co}(I_3) = (K[[x]] \oplus K[[x]]y) \times K[[x]]$$

of I_3 in $K[[x, y]]^2$. We will identify $\mathrm{co}(I_3)$ with $K[[x]]^3$ as $K[[x]]$-modules. Thus

$$g_4 = \left(0, 0, \left(\left(h_4 + h_4^2\right)\left(h_5 + \frac{1}{2}\right) + 1\right) x^4\right).$$

The reduced standard basis of I_4 is $(0, 0, x^4)$ since $h_4(0) = 0$ implies that $((h_4 + h_4^2)(h_5 + \frac{1}{2}) + 1)$ is invertible in $K[[x]]$. Here we could also apply the algorithm of Theorem 10.1 to compute the reduced standard basis of I_4. In this case the computations are trivial because the base ring is the principal ideal domain $K[[x]]$. We set again $g_4 = (0, 0, x^4)$ with father code $G_4 = (0, 0, x^4)$.

Substep 4D: Reduction of g_2, g_3 by $I_4 = \langle g_4 \rangle$.

We apply the division algorithm of Theorem 11.1 in order to divide the tails $\overline{g}_2$ and $\overline{g}_3$ of g_2 and g_3 by g_4 (this is sufficient since $\mathrm{in}(g_2)$ and $\mathrm{in}(g_3)$ do not contribute to the remainders.) As $\overline{g}_2, \overline{g}_3$ and g_4 belong to $\mathrm{co}(I_3) = (K[[x]] \oplus K[[x]]y) \times K[[x]]$ we may treat them as vectors in $K[[x]]^3$. We thus have

$$\begin{aligned} g_2 &= (0, 0, h_4), \\ g_3 &= (0, 0, -xh_4), \\ g_4 &= (0, 0, x^4). \end{aligned}$$

Using that $h_4 = -x^2 - x^4 - 2x^6 - \cdots$ it can be seen by inspection that the remainders of the division of $\overline{g}_2$ and $\overline{g}_3$ by g_4 are $(0, 0, -x^2)$ and $(0, 0, x^3)$.

This can also be seen alternatively by applying the polynomial division. Namely, as $\overline{g}_2, \overline{g}_3$ as well as g_4 belong to $(0) \times (0) \times K[[x]]$ we can work with the respective last components in $K[[x]]$. Let us consider the polynomials $\overline{G}_2 = t_4$, $\overline{G}_3 = -xt_4$ and $G_4 = x^4$ as the father codes of the last components of $\overline{g}_2, \overline{g}_3$ and g_4, respectively.

Since the only baby series appearing in g_2, g_3, g_4 is h_4 we have to consider the virtual reduced standard basis B_{41}, B_4 of the ideal $\widetilde{I}_5 \subset K[[x, t_4]]$ generated by $H_4 = t_4^2 + t_4 + x^2$ and $G_4 = x^4$. One has

$$\begin{aligned} B_{41} &= t_4 - u_{4110} - u_{4111}x + u_{4112}x^2 + u_{4113}x^3, \\ B_4 &= x^4 - u_{410} - u_{411}x + u_{412}x^2 + u_{413}x^3. \end{aligned}$$

We get $u_{410} = u_{411} = u_{412} = u_{413} = 0$ since the initial monomial of the baby series b_4 of B_4 should be x^4. On the other hand, the remainder of the polynomial division of H_4 by B_{41} and B_4 is

$$(u_{4113} + 2u_{4111}u_{4112})x^3 + (1 + u_{4112} + u_{4111})x^2 + u_{4111}x,$$

which implies $u_{4111} = u_{4113} = 0$ and $u_{4112} = -1$. The reduced standard basis of $\widetilde{I}_5$ is $b_{41} = t_4 + x^2$ and $b_4 = x^4$. Finally, we have to divide $\overline{G}_2$ and $\overline{G}_3$ by B_{41} and B_4 using the polynomial division with leading monomial vectors t_4 and x^4. One has

$$\overline{G}_2 = t_4 = B_{41} - x^2,$$

$$\overline{G}_3 = -xt_4 = -xB_{41} + x^3.$$

Rephrasing everything as vectors in $K[[x, y]]^2$, the reductions of g_2, g_3 by g_4 are

$$(0, y + x^2),$$
$$(y^2, -x^3).$$

We set again $g_2 = (0, y + x^2)$, $g_3 = (y^2, -x^3)$, rewrite g_4 as $g_4 = (0, x^4)$, together with their father codes $G_2 = (0, y + x^2)$, $G_3 = (y^2, -x^3)$ and $G_4 = (0, x^4)$. This is the reduced standard basis of I_3; it coincides with what we have got at the beginning of this substep.

Conclusion of Step 4: To finish Step 4 we have to rewrite the preceding vectors as algebraic power series in x, y, z in order to obtain the reduced standard basis of the ideal $I_2 = \langle g_2, g_3, g_4\rangle$ of $K[[x, y, z]]$. The corresponding reduced standard basis is given by the polynomials (we write again g_1, g_2 and g_3)

$$g_2 = yz + x^2z,$$
$$g_3 = y^2 - x^3z,$$
$$g_4 = x^4z.$$

They coincide with their father codes.

Step 5: Reduction of g_1 by the submodule $I_2 = \langle g_2, g_3, g_4\rangle$ of $K[[x, y]]^2$.

Recall that $g_1 = z^2 + (\frac{1}{2}xy + xyh_2)z$. It suffices to divide the tail $\overline{g}_1 = -(\frac{1}{2}xy + xyh_2)z$ of g_1 by $I_2 = \langle g_2, g_3, g_4\rangle$. We consider $\overline{g}_1$ and g_2, g_3, g_4 as vectors in $\mathrm{co}(I_1) \cong K[[x, y]]^2$. Their father codes are $\overline{G}_1 = (0, -xyt_2 - \frac{1}{2}xy)$, $G_2 = (0, y + x^2)$, $G_3 = (y^2, -x^3)$, $G_4 = (0, x^4)$, respectively. The computation splits into two parts. Following Theorem 11.1 we will divide first $\overline{g}_1$ by $I_3 = \langle g_2, g_3\rangle$ as vectors in $K[[x, y]]^2$ because g_2, g_3 are the y-regular power series among g_2, g_3, g_4. Afterward, $\overline{g}_1$ will be divided by $I_4 = \langle g_4\rangle$ as a vector in $\mathrm{co}(I_3) \cong K[[x]]^3$.

Notice here that it is necessary to work with power series vectors in the canonical direct monomial complements $\mathrm{co}(I_1)$ and $\mathrm{co}(I_3)$. This is possible because, by the preceding steps, g_1 is reduced with respect to itself (hence $\overline{g}_1$ belongs to $\mathrm{co}(I_1)$), g_2, g_3 and g_4 are reduced with respect to g_1 (hence also belong to $\mathrm{co}(I_1)$), and g_4 is reduced with respect to g_2 and g_3 (hence belongs to $\mathrm{co}(I_3) \subset \mathrm{co}(I_1)$).

Substep 5A: Reduction of g_1 by $I_3 = \langle g_2, g_3\rangle$.

We have to divide $\overline{g}_1$ by g_2 and g_3 as described in Theorem 11.1. We will use the polynomial division to divide the father code $\overline{G}_1$ of $\overline{g}_1$ by the virtual reduced standard basis B_{21}, B_{22}, B_2, B_3 of $\widetilde{I}_3 = \langle H_2 \cdot e_1, H_2 \cdot e_2, G_2, G_3\rangle$ in $K[[x, y, t_2]]^2$. Notice that the only baby series appearing in $\overline{g}_1, g_2, g_3$ is h_2. Therefore, the only mother code appearing in $\widetilde{I}_1$ is H_2. We have

$$B_{21} = (t_2 - u_{2110} - u_{2111}y, -u_{2120}),$$
$$B_{22} = (-u_{2210} - u_{2211}y, t_2 - u_{2220}),$$

$$B_2 = (0, y - u_{220}),$$
$$B_3 = (y^2, -u_{320}),$$

where the form of B_2 and B_3 follows from the computation made in Substep 4D. The remainder of this polynomial division is $R = (0, x^3(u_{2220} + \frac{1}{2}))$. The algebraic series $u_{2220}(x)$ is defined by the mother code

$$H_7 = 8x^3t_7^3 + 12x^3t_7^2 - (16 - 16x^3)t_7 + x^3,$$

where we have set $t_7 = u_{2220}$. This mother code H_7 results from the division of $H_2 \cdot e_\ell$ and G_2, G_3 by $B_{i\lambda}$, B_2, B_3 and an appropriate simplification. Let us write h_7 for the baby series $u_{2220}(x)$ with mother code H_7. It then follows that the reduction of $\overline{g}_1 = (0, -\frac{1}{2}xy - xyt_2)$ with respect to I_3 is $(0, x^3(h_7 + \frac{1}{2}))$. We write this reduction again as $\overline{g}_1 = (0, x^3(h_7 + \frac{1}{2}))$. Note that it belongs to $\mathrm{co}(I_3)$.
Substep 5B: Reduction of g_1 by $I_4 = \langle g_4 \rangle$.

We have to divide the tail $\overline{g}_1$ of g_1 by g_4 as described in Theorem 11.1. For this we will consider $\overline{g}_1$ and g_4 as vectors in $\mathrm{co}(I_3) \cong K[[x]]^3$. We have $\overline{g}_1 = (0, 0, x^3(h_7 + \frac{1}{2}))$ and $g_4 = (0, 0, x^4)$. Since $h_7(0) = 0$ the reduction of $\overline{g}_1$ with respect to g_4 is $(0, 0, \frac{1}{2}x^3)$. As in Substep 4D this reduction can be also computed by using the polynomial division. We omit the details.
Conclusion of example: Starting with the family code $H_1 = t_1^2 - 2t_1 + z$, $G_1 = z^2 + xyt_1$, $G_2 = yz + x^2z + y^2z$, $G_3 = y^2 + xyz$ of the generators g_1, g_2 and g_3 of the ideal $I \subset K[[x, y, z]]$ with baby series $h = 1 - \sqrt{1 - z} = \frac{1}{2}z + \frac{1}{8}z^2 + \dots$ we have found the reduced standard basis of I with respect to $<_\eta$ as the polynomials (denoted again by g_1, g_2, g_3 and g_4)

$$g_1 = z^2 - \frac{1}{2}x^3z,$$
$$g_2 = yz + x^2z,$$
$$g_3 = y^2 - x^3z,$$
$$g_4 = x^4z.$$

They coincide with their father codes, and all baby series and mother codes have disappeared. We leave it as a challenge to the interested reader to find this basis of I directly without using the algorithms of the paper.

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References

1. Alonso, M.E., Castro-Jiménez, F.J., Hauser, H., Koutschan, C.: Gabrielov's counterexample to nested linear Artin approximation. Preprint 2017.
2. Alonso, M.E., Mora, T., Raimondo, M.: A computational model for algebraic power series. J. Pure Appl. Alg. 77 (1992), 1-38.
3. Amasaki, M.: Applications of the generalized Weierstrass preparation theorem to the study of homogeneous ideals. Trans. Amer. Math. Soc. 317 (1990), 1-43.
4. Aroca, J.-M., Hironaka, H., Vicente, J.-L.: The theory of the maximal contact. Memorias Mat. Inst. Jorge Juan Madrid 29 (1975).
5. Artin, M.: Grothendieck topologies. Mimeographed notes, Harvard University 1962.
6. Artin, M.: Algebraic approximations of structures over complete local rings. Publ. Math. I.H.E.S. 36 (1969), 23-58.
7. Artin, M., Mazur, B.: On periodic points. Ann. Math. 81 (1965), 82-99.
8. Baclawski, K., Garsia, A.: Combinatorial decomposition of a class of rings. Adv. Math. 39 (1981), 155-184.
9. Bochnak, J., Coste, M., Roy, M.-F.: Géométrie Algébrique Réelle. Springer 1987.
10. Bostan, A., Kauers, M.: The complete generating function for Gessel walks is algebraic. With an appendix by Mark van Hoeij. Proc. Amer. Math. Soc. 138 (2010), no. 9, 3063-3078.
11. Bostan, A., Kurkova, I., Raschel, K.: A human proof of Gessel's lattice path conjecture. Trans. Amer. Math. Soc. 369 (2017), 1365-1393.
12. Bousquet-Mélou, M.: An elementary solution of Gessel's walks in the quadrant. Adv. Math. 303 (2016), 1171-1189.
13. Bousquet-Mélou, M., Mishna, M.: Walks with small steps in the quarter plane. Contemp. Math. 520 (2010), 1-39.
14. Bousquet-Mélou, M., Petkovšek, M.: Linear recurrences with constant coefficients. Discrete Math. 225 (2000), 51-75.
15. Bousquet-Mélou, M., Petkovšek, M.: Walks confined in a quadrant are not always D-finite. Theor. Comp. Sci. 307 (2003), 257-276.
16. Brieskorn, E., Knörrer, H.: Plane Algebraic Curves. Birkhäuser 1986.
17. Cox, D., Little, J., O'Shea, D.: Using Algebraic Geometry. Springer 1998.
18. De Jong, T., Pfister, G.: Local Analytic Geometry. Springer 2000.
19. Gabrielov, A.: The formal relations between analytic functions. Funk. Anal. Pril. 5 (1971), 64-65.
20. Galligo, A.: A propos du Théorème de Préparation de Weierstrass. Lecture Notes in Math. 409, 543-579. Springer 1973.
21. Gerdt, V., Blinkov, Y.: Involutive bases of polynomial ideals. Simplification of systems of algebraic and differential equations with applications. Math. Comput. Simulation 45 no. 5-6 (1998), 519-541.
22. Gräbe, H.-G.: The tangent cone algorithm and homogenization. J. Pure Appl. Algebra 97 (1994), 303-312.
23. Gräbe, H.-G.: Algorithms in local algebra. J. Symb. Comp. 19 (1995), 545-557.
24. Grauert, H.: Über die Deformation isolierter Singularitäten analytischer Mengen. Invent. Math. 15 (1972), 171-198.
25. Greuel, G.-M., Pfister, G.: A Singular Introduction to Commutative Algebra, 2nd edition. Springer 2007.
26. Greuel, G.M., Pfister, G., Schönemann, H.: Singular. A Computer Algebra System for Polynomial Computations. Universität Kaiserslautern, www.singular.uni-kl.de.
27. Hironaka, H.: Idealistic exponents of singularity. In: Algebraic Geometry, The Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977.
28. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79 (1964), 109-326.
29. Hauser, H., Müller, G.: A rank theorem for analytic maps between power series spaces. Publ. Math. I.H.E.S. 80 (1995), 95-115.
30. Janet, M.: Sur les systèmes d'équations aux dérivées partielles. J. Math. 8^{e} sér., III (1920), 65-151.
31. Janet, M.: Leçons sur les systèmes d'équations aux dérivées partielles. Gauthiers-Villars 1929.
32. Kurke, H., Pfister, G., Rozcen, M.: Henselsche Ringe und algebraische Geometrie. Dt. Verlag der Wissenschaften, Berlin 1975.
33. Lafon, J.-P.: Séries formelles algébriques. C. R. Acad. Sci. Paris 260 (1965), 3238-3241.

34. Lazard, D.: Gröbner bases, Gaussian elimination and resolutions of systems of algebraic equations. In: Computer algebra, London 1983. Lecture Notes Comp. Sci. 162, 146-156. Springer 1983.
35. Mishna, M.: Classifying lattice walks restricted to the quarter plane. J. Combin. Theory Ser. A 116 (2009), no. 2, 460-477.
36. Mora, T.: An algorithm to compute the equations of tangent cones. Proceedings Eurocam 82. Lecture Notes Comp. Sci. 144, 158-165. Springer 1982.
37. Mumford, D.: The Red Book of Varietes and Schemes. Springer, 2nd edition 1999.
38. Nagata, M.: On the theory of Henselian rings. Nagoya Math. J. 5 (1953), 5-57, and 7 (1954), 1-19.
39. Nagata, M.: Local rings. Interscience 1962.
40. Popescu, D.: General Néron desingularization. Nagoya Math. J. 100 (1985), 97-126.
41. Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. 104 (1986), 85-115.
42. Raynaud, M.: Anneaux Locaux Henséliens. Lecture Notes Math. 169. Springer 1970.
43. Rees, D.: A basis theorem for polynomial modules. Proc. Cambridge Phil. Soc. 52 (1956), 12-16.
44. Riquier, C.: Les Systèmes d'Équations aux Dérivées Partielles. Gauthier-Villars 1910.
45. Ruiz, J.: The basic theory of power series. Vieweg 1993.
46. Seiler, W.: A combinatorial approach to involution and δ-regularity I: Involutive bases in polynomial algebras of solvable type. Appl. Algebra Engrg. Comm. Comput. 20, no. 3-4 (2009), 207-259.
47. Seiler, W.: A combinatorial approach to involution and δ-regularity II: Structure analysis of polynomial modules with Pommaret bases. Appl. Algebra Engrg. Comm. Comput. 20, no. 3-4 (2009), 261-338.
48. Seiler, W.: Spencer cohomology, differential equations, and Pommaret bases. In: Gröbner Bases in Symbolic Analysis, 169-216. Radon Ser. Comput. Appl. Math. 2. De Gruyter 2007.
49. Séminaire de Géométrie Algébrique 4, vol. 2 (Artin, M., Grothendieck, A., Verdier, J.L., eds.). Publ. Math. I.H.E.S. 32 (1972).
50. Spivakovsky, M.: A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. J. Amer. Math. Soc. 12 (1999), no. 2, 381-444.
51. Sturmfels, B., White, N.: Computing combinatorial decompositions of rings. Combinatorica 11 (1991), 275-293.
52. Swan, R.: Néron-Popescu desingularization, Algebra and geometry (Taipei, 1995), 135-192, Lect. Algebra Geom., 2, Intern. Press 1998.
53. van der Hoeven, J.: Effective power series computations. Preprint 2014.
54. Wagner, D.: Algebraische Potenzreihen. Diploma Thesis, Univ. Innsbruck 2005.
55. Wilczynski, E.J.: On the form of the power series for an algebraic function. Amer. Math. Monthly 26 (1919), 9-12.
56. Zariski, O.: Analytical irreducibility of normal varieties. Ann. Math. 49 (1948), 352-361.