TAMING APPARENT SINGULARITIES VIA ORE CLOSURE

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Last modification: October 20, 2016. Based on last check-in:
`Id: torsion.tex,v 1.136 2009/12/18 10:21:02 salvy Exp`

Contents

Introduction

The coefficients of power series solutions of a linear differential equation obey a linear recurrence. Such a recurrence can be computed using a simple ring morphism from linear differential operators to linear difference operators. However, the recurrence obtained that way need not have the minimal possible order. For instance, the third-order differential equation

$$2x(x-1)(x+1)y''' + (x^3 - 4x^2 + 5x + 6)y'' - (x^3 + x^2 - 7x + 9)y' - 6y = 0 \tag{1}$$

leads to a linear recurrence of order 4, while another recurrence of order 3 is also satisfied by the coefficients of all its power series solutions. One reason for this non-minimality is the presence of *apparent singularities* of the differential equation. These are points where the differential equation is singular, while none of its solutions is. In our example, both 0 and -1 are apparent singularities. What happens is that there exists another linear differential equation of higher order (6 in our example) that also annihilates the solutions of (1) but does not have its apparent singularities. The coefficients of this alternative differential equation have lower degree and lead to a smaller order recurrence. Besides, the differential equation of higher order has of course more solutions than the original.

A similar example can be given for the converse computation of a linear differential equation satisfied by the generating series of the solutions of a given linear recurrence. In order to deal simultaneously with both these cases, as well as their q-analogues and other related questions, we use the algebraic framework of *skew-polynomial rings* [11, 7, 8]. These are rings of polynomials in a variable ∂ with coefficients in a commutative algebra A over a field k. The product is defined by the relations

$$\partial u = \sigma(u)\partial + \delta(u), \tag{2}$$

for u in A where σ is an algebra endomorphism of A which reduces to the identity on k, and δ is a σ-derivation, which means that it satisfies $\delta(uv) = \sigma(u)\delta(v) + \delta(u)v$. The ring is denoted $A[\partial; \sigma, \delta]$. Examples of skew-polynomial rings with the corresponding σ and δ are given in Tables 1 and 2. The elements of these rings are often called Ore polynomials, after Oystein Ore who studied them in [18] and in particular proved that $A[\partial;\sigma,\delta]$ is left Euclidean when A is a field and σ is injective.

In this setting, the equation we start with, like (1), determines a skew polynomial L that generates a left ideal $\mathfrak{I}$ in a ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$. In this work, we study relations of the form

$$PL = fR, \tag{3}$$

where P and R in $\mathscr{D}[x]$ and f in $k[x]$ are unknown and f does not divide P on the left. The zeroes of f give a generalization of *apparent singularities* in skew-polynomial rings. Algebraically, the problem we are dealing with is an *extension-contraction* problem. The *extension* of $\mathfrak{I}$ is the left ideal $\mathfrak{E} = \mathscr{D}(x)\mathfrak{I} \subset \mathscr{D}(x)$, in the left Euclidean ring $\mathscr{D}(x) = k(x)[\partial;\sigma,\delta]$. The contraction of this latter ideal is $\mathfrak{K} = \mathfrak{E} \cap \mathscr{D}[x]$, which is in general larger than $\mathfrak{I}$. This ideal contains the polynomials R from (3). In the context of the univariate Weyl algebra, this has been called the *Weyl closure* of the ideal [19]. In our more general setting, we call $C(\mathfrak{I}) = \mathfrak{K}$ the *Ore closure* of $\mathfrak{I}$. In this work we provide algorithms computing generators of this closure for a large variety of skew-polynomial rings.

These algorithms depend on the morphism σ in several ways. It turns out that the set of f's for which Equation (3) has a non-trivial solution has a composite structure and that it is sufficient to patch up local spaces computed for coprime f's. Then, we show in Proposition 1.8 that f must divide one of the $\sigma^d(\operatorname{lc}(L))$, $d \geq 0$, where $\operatorname{lc}(L)$ denotes the leading coefficient of L. Depending on σ this set of $\sigma^d(\operatorname{lc}(L))$ can be finite or not. In most cases of interest, further arguments make it possible to determine a finite set of prime factors of the possible f, or at least a finite way to generate this set. Next, for a prime p, the solutions to (3) with $f = p^\nu$ can in most cases be generated by solutions corresponding to a finite set of small values of ν. The maximal exponent in this set also depends on σ. We show that in several cases this exponent is 1. This is a well-known result in the special case of the Weyl algebra, usually attributed to Kashiwara [15]. In other cases including difference equations, the bound can be arbitrary, but can often be determined algorithmically. In those cases, the apparent singularities cannot always be made to disappear completely. However, their multiplicity is reduced as much as possible, whence the word "taming" in our title.

This work is inspired by that of Tsai's in the differential case [19], which we generalize to skew-polynomial rings. The difference case has been treated in large part in [4]. In both cases, we improve these algorithms by exploiting specialized, faster algorithms for finding the polynomial solutions of a certain dual equation, instead of relying on general-purpose linear algebra algorithms.

Our contributions are as follows. On the theoretical level, we define the Ore closure and provide general algorithms for its computation, described on the level of skew-polynomial rings, with highlight on (q-)differential, (q-)shift, and Mahlerian settings. In the differential case, this is very close to Tsai's work, with better efficiency by interpreting Ore closure in terms of polynomial solutions and duality. Beside the case of pure skew-polynomial rings generated by ∂ over $k[x]$, we also consider (when technical conditions permit) cases of the larger ring generated by both ∂ and a formal inverse ∂^{-1}. The typical example here is that of forward and backward shifts. Our main application there is an algorithm for obtaining minimal-order recurrences satisfied by formal series solutions.

This article is structured as follows. In Section 1, we recall the basic definitions and properties of skew-polynomial rings and discuss properties of the Ore closure that do not depend on σ.

PhD C'est faux.

In Section 2, we deal with the case of factors p of f such that $\sigma(p)$ divides p. We say that the apparent singularity *is fixed* by σ. This contains the important special cases of differential operators and, under a technical constraint, of q-differential operators. We discuss these cases in detail. We also prove there that our algorithms lead to recurrences of minimal order for the coefficients of series solutions.

PhD Je ne vois pas où c'est.

Section 3 is devoted to infinite orbits, i.e., factors such that p divides $\sigma^d(p)$ for no $d \geq 1$. In particular, difference and q-difference equations are covered in this section, at least when q is not a root of unity. Section 6 deals with the remaining case, where there exist a finite orbit: $\sigma^d(p)$ divides p for some $d > 1$. This covers q-difference equations when q is a root of unity and the difficult part of the class of Mahlerian equations. This case is more delicate and our results there are less complete.

Note to the programmer. Although the results are expressed using an algebraic vocabulary, we have isolated our algorithms is small boxes that are ready to be implemented without any reference to the rest of the text.

BS À vérifier

Prelude: Apparent singularities of differential operators

In this preliminary section, we show that in the differential setting, an operator with apparent singularity can be desingularized, in the sense that one can find a left multiple of it without the given apparent singularity. The results of this section can be found in [20, Lemma 9.2] and [1, Prop. 6 and 7]. However, we give a proof that is much in the spirit of our subsequent treatment of the general problem and that should serve as a guide to the reader.

For this section, we assume k to be an algebraically closed field of characteristic 0, and that $\mathscr{D}[x]$ is $k[x][\partial;\sigma,\delta]$ with $\sigma(x) = x$ and $\delta(x) = 1$. In other words, $\mathscr{D}[x]$ is the associative algebra $k\langle x, \partial; \partial x - x\partial - 1\rangle$ known as the first Weyl algebra, and is isomorphic to the algebra of linear ordinary differential operators with polynomial coefficients.

Fix $L \in \mathscr{D}[x]$, written $L = \ell_0 + \cdots + \ell_r\partial^r$, with $\ell_i \in k[x]$ and $\ell_r \neq 0$ called the *leading coefficient* of L. The series $\hat{y} \in k[[x-\alpha]]$ is a *formal solution* of L at α if $L\hat{y} = 0$. It is well known that if $\ell_r(\alpha) \neq 0$, then L admits a set of r formal solutions $\hat{y}_1, \ldots, \hat{y}_r$ which are linearly independent over k. If, on the contrary, the leading coefficient of L has a root at α, there will not exist, in general, r linearly independent formal solutions of L in $k[[x-\alpha]]$. In the special case where $\ell_r(\alpha) = 0$ but a basis of r formal solutions exists, it is classically said that the point $x = \alpha$ is an *apparent singularity* of L.

Our goal is to show that a necessary condition for a point $x = \alpha$ to be an apparent singularity of L is that there exists a multiple of L which is not singular at it. As a consequence, the left module $\mathscr{D}[x]/\mathscr{D}[x]L$ should have some non-trivial local torsion at $x - \alpha$, in the sense that there are elements mapped to 0 by multiplication by a power of $x - \alpha$. This may be properly stated as follows, where without loss of generality we restrict to $\alpha = 0$.

Proposition 0.1. *Fix $L \in \mathscr{D}[x]$, written $L = \ell_0 + \cdots + \ell_r\partial^r$ with $\ell_i \in k[x]$, $\mathrm{lc}(L) = \ell_r \neq 0$ and $\ell_r(0) = 0$. Assume that L admits r formal solutions $\hat{y}_1, \ldots, \hat{y}_r$ in $k[[x]]$ which are linearly independent over k. Then, there exists $P \in \mathscr{D}[x]$ such that the product PL takes the form $x^\nu R$ for some $R \in \mathscr{D}[x]$ whose leading coefficient is not a multiple of x and $\nu > 0$. The product PL can be constructed explicitly as a least common left multiple of L and another skew polynomial.*

Proof. Let μ_i denote the valuation of $\hat{y}_i$ with respect to x. Up to a change of basis of the solution space of L, we can assume that the non-negative integers μ_i are in strictly increasing order. Let T be the product of the $x\partial - v$ over all non-negative integers v less than the largest valuation μ_r and different from the μ_i, and let M be the least common left multiple of L and T in $\mathscr{D}[x]$.

The solution space of M is the vector space generated by the $\hat{y}_i$ for $1 \leq i \leq r$ and the monomials x^v for the integers v used in the definition of T above. By construction, the valuations of the basis elements range exhaustively from 0 to μ_r. After sorting the basis by increasing valuations and a change of basis, we obtain a basis consisting of formal power series $\hat{s}_i$ for $0 \leq i \leq \mu_r$ where $\hat{s}_i \in x^i + (x^{\mu_r+1})$. Thus, in the Wronskian matrix $\mathrm{W}(\hat{s}_0, \ldots, \hat{s}_{\mu_r})$, the terms above the diagonal are in (x), while the ith term on the diagonal is in $(i-1)! + (x)$. As a consequence, the Wronskian determinant is non-zero modulo (x), and is thus invertible in $k[[x]]$. Write W for the differential operator with coefficients in $k[[x]]$ that represents the linear ordinary differential equation

$$\det \mathrm{W}(y, \hat{s}_0, \ldots, \hat{s}_{\mu_r}) / \det \mathrm{W}(\hat{s}_0, \ldots, \hat{s}_{\mu_r}) = 0$$

in the unknown y. By comparison of the solution sets, W is a multiple of M with same order $\mu_r + 1$ and coefficients in $k[[x]]$, and it is monic with regard to ∂. Since M is also a left multiple PL of L, we have the relation $PL = M = \mathrm{lc}(M)W$, and since x divides $\mathrm{lc}(L)$, it divides $\mathrm{lc}(M)$. Factoring the maximal power x^ν out of $\mathrm{lc}(M)$ leads to the result. $\square$

Operator		$\partial \cdot f(x)$	$x \cdot f(x)$
Differentiation	$\frac{d}{dx}$	$f'(x)$	$xf(x)$
Shift	S	$f(x+1)$	$xf(x)$
Difference	Δ	$f(x+1)-f(x)$	$xf(x)$
q-Dilation	Q	$f(qx)$	$xf(x)$
Continuous q-difference		$f(qx)-f(x)$	$xf(x)$
q-Differentiation	$\delta^{(q)}$	$\frac{f(qx)-f(x)}{(q-1)x}$	$xf(x)$
q-Shift	S_q	$f(x+1)$	$q^x f(x)$
Discrete q-difference	Δ_q	$f(x+1)-f(x)$	$q^x f(x)$
Eulerian operator	Θ	$xf'(x)$	$xf(x)$
Mahlerian operator	M	$f(x^b)$	$xf(x)$
Divided differences		$\frac{f(x)-f(a)}{x-a}$	$xf(x)$

TABLE 1. Some common Ore operators

1. ORE CLOSURE OF AN IDEAL

In this work, all rings and fields are supposed to be of characteristic zero, and k always denotes a commutative field. Appendix E summarizes the notation introduced in this section and used throughout the text.

1.1. **Ore operators and skew-polynomial rings.** Our motivation for using skew-polynomial rings [11, 7, 8] is the convenient polynomial representation of linear operators which they offer. We now recall the basic definitions and facts we use.

Derivations. Let A be a commutative algebra over k, and σ an algebra endomorphism that induces the identity over k. A k-linear endomorphism δ of A is called a *σ-derivation* if it satisfies the skew Leibniz rule $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all a and b in A.

Skew-polynomial rings. The associative ring generated over A by a new indeterminate ∂ and the relations $\partial a = \sigma(a)\partial + \delta(a)$ for all $a \in A$ is called a *(left) skew-polynomial ring* and denoted by $A[\partial; \sigma, \delta]$. It is a $k[x]$-module.

Ore operators. A skew polynomial $L \in A[\partial; \sigma, \delta]$ acts on left $A[\partial; \sigma, \delta]$-modules—in most of our applications, these are modules of functions, power series, or sequences. Solving then means finding an element y in such a module such that $Ly = 0$. In this perspective, skew polynomials are often called *Ore operators*. Tables 1 and 2 illustrate some common types of Ore operators when $A = k[x]$, together with the values of σ and δ which define the associated skew-polynomial ring.

Constants. For any skew-polynomial ring $A[\partial; \sigma, \delta]$, the elements of A that are fixed under σ and mapped to zero by δ are called the *constants* of the ring. They constitute a sub-ring $\mathscr{C}$ of A, they commute with ∂, and therefore with any element of $A[\partial; \sigma, \delta]$.

Operator	σ	δ	∂x
Differentiation	$\mathrm{Id}_{k[x]}$	$\frac{d}{dx}$	$x\partial + 1$
Shift	S	0	$(x+1)\partial$
Difference	S	Δ	$(x+1)\partial + 1$
q-Dilation	Q	0	$qx\partial$
Cont. q-difference	Q	$Q - \mathrm{Id}_{k[x]}$	$qx\partial + (q-1)x$
q-Differentiation	Q	$((q-1)x)^{-1}(Q - \mathrm{Id}_{k[x]})$	$qx\partial + 1$
q-Shift	Q	0	$qx\partial$
Discr. q-difference	Q	$Q - \mathrm{Id}_{k[x]}$	$qx\partial + (q-1)x$
Eulerian operator	$\mathrm{Id}_{k[x]}$	$x\frac{d}{dx}$	$x\partial + x$
Mahlerian operator	M	0	$x^b\partial$
Divided differences	$f(a)$	$x \mapsto \frac{f(x)-f(a)}{x-a}$	$a\partial + 1$

TABLE 2. Corresponding skew-polynomial rings and their commutation rules

Skew polynomials over $k[x]$ *and* $k(x)$. In view of their importance in this work, we denote $k[x][\partial;\sigma,\delta]$ by $\mathscr{D}[x]$ and similarly $\mathscr{D}(x) = k(x)[\partial;\sigma,\delta]$. They are characterized by the values of $\sigma(x)$ and $\delta(x)$. When δ is 0, we simply write $k[x][\partial;\sigma]$ in place of $k[x][\partial;\sigma,0]$. The structure of skew-polynomial rings is studied in Appendix A.

The order of a skew polynomial $L \in \mathscr{D}[x]$ is the degree of L as a polynomial in ∂. The content of a non-zero skew polynomial $L = \ell_0 + \cdots + \ell_r\partial^r \in \mathscr{D}[x]$ is the greatest common divisor of its coefficients $\ell_0, \ldots, \ell_r$ and L is primitive if its content is 1.

Hypothesis about σ. Henceforth, we only consider concrete cases where σ is injective on $k[x]$. In other words, $\sigma(x)$ is a polynomial of degree at least 1. This only rejects the case of divided differences in Tables 1 and 2. Under this condition, the maps σ and δ extend in a unique way to $k(x)$: one readily proves the necessary relations $\sigma(f^{-1}) = \sigma(f)^{-1}$ and $\delta(f^{-1}) = -\sigma(f)^{-1}\delta(f)f^{-1}$ for any non-zero $f \in k[x]$. The new maps on $k(x)$ are still denoted σ and δ, and $\mathscr{D}[x]$ extends uniquely into the skew-polynomial ring $\mathscr{D}(x)$ over $k(x)$. This does not mean that we always assume the hypothesis, but eventually we arrive always to a situation where the hypothesis is satisfied.

Gröbner basis. In all the cases we shall consider, the algebra $\mathscr{D}[x]$ is not left principal, while $\mathscr{D}(x)$ is. As a consequence, the Ore closure that we compute in this algebra ultimately requires several generators to be described. In the frequent case when $\sigma(x)$ is a polynomial of degree at most 1, the presentation of Ore closures can be normalized by appealing to a theory of Gröbner bases adapted to this setting. This theory was first developped for the generality of skew-polynomial rings we are interested in here in [16]; an extended theory that we shall not require before Section 4.4 is to be found in [10].

1.2. **Singularities and their geometry.**

1.2.1. *Singularities of a skew polynomial.*

Definition. Let $L = \ell_0 + \cdots + \ell_r \partial^r \in \mathscr{D}[x]$ be a primitive skew polynomial of order r. We call a prime $p \in k[x]$ a *singular point* of L if there exists $m \geq r$ such that $\sigma^m(p)$ and ℓ_r are not coprime. We also say that the polynomial L has a *singularity* at p or at its zeroes α in the algebraic closure $\bar{k}$ of k whenever p is a singular point. Primes $p \in k[x]$ that are not singular, as well as their zeroes, are called *ordinary points* of L. The definitions are extended to non-zero skew polynomials L by considering the quotient of L by its content.

***Example* 1.1.** In the differential case, this is the classical definition. Singularities correspond to points where the power series solutions of the differential equation do not span a vector space of dimension as high as the order.

***Example* 1.2.** In the algebra $\mathscr{D}[x]$ defined by the backward shift σ and $\delta = 0$, Euler's Γ function is solution of the operator $(x-1)\partial - 1$. Our definition makes all of $0, -1, -2, \dots$ singular points, in accordance with the singularities of Γ.

1.2.2. *Pullbacks.* The definition of singular points leads to the study of those polynomials p such that $\sigma^m(p)$ vanishes at a zero of ℓ_r. This leads us to consider the action of σ on prime polynomials. It must be noticed that the definitions which follow assume implicitely $\deg \sigma(x) \geq 1$, that is σ is injective. Otherwise the pullback of a prime polynomial reduces to the null ideal. Henceforth the consideration of the orbits is always made under the hypothesis that σ is injective.

The *pullback* of the ideal $\mathfrak{I} = (f) \subset k[x]$ under σ, denoted $\sigma^{-1}(\mathfrak{I})$, is the ideal of the $g \in k[x]$ such that f divides $\sigma(g)$. We write $\sigma^{-j}(\mathfrak{I})$ for the pullback of $\sigma^{-(j-1)}(\mathfrak{I})$. This is also $(\sigma^j)^{-1}(\mathfrak{I})$. The pullback of a prime ideal is prime. For any given prime ideal $\mathfrak{p} = (p) \subset k[x]$ the pullback thus defines successive prime ideals $\mathfrak{p}_0 = \mathfrak{p}$, $\mathfrak{p}_1 = \sigma^{-1}(\mathfrak{p}_0)$, $\mathfrak{p}_2 = \sigma^{-1}(\mathfrak{p}_1)$, and so on. Since $k[x]$ is principal, we can write $\mathfrak{p}_j = (p_j)$ with $p_j \in k[x]$ prime and monic. We call *orbit* of $p = p_0$ the sequence $(\mathfrak{p}_0, \mathfrak{p}_1, \dots)$, or the equivalent sequence $(p_0, p_1, \dots)$ of generators. With $p = x - \alpha$ and $\alpha \in k$ (or with p prime and α a zero of p in $\bar{k}$), it is equivalent to consider the sequence $\big(\alpha, \tilde{\sigma}(\alpha), \tilde{\sigma}^2(\alpha), \dots\big)$ as the orbit of α, where $\tilde{\sigma}$ is the polynomial function defined by the polynomial $\sigma(x)$.

A combinatorial description of those orbits that contain a prime factor of ℓ_r is obtained by considering a graph with vertices the primes contained in those orbits and edges between each prime and its pullback. The general structure is then an oriented cycle (possibly reduced to a single point), on whose nodes are attached oriented rooted trees (possibly empty). This geometric intuition leads to the following terminology:

(1) If no two of the $\mathfrak{p}_i$ are equal, we say that p defines an *infinite orbit*;
(2) If there exist integers $t_0 \geq 0$, $t \geq 1$ such that $\mathfrak{p}_{t_0} = \mathfrak{p}_{t_0+t}$, we say that p defines an *ultimately periodic orbit*. We call t a period of p or say that p has period t. When t is minimal, we call it *the period* and if t_0 is minimal for the period, we call it *the delay*;
(3) If p defines an ultimately periodic orbit with zero delay, we say that p defines a *(purely) periodic orbit*;
(4) If p defines a periodic orbit of period 1, we say that the point defined p is a *fixed point.*

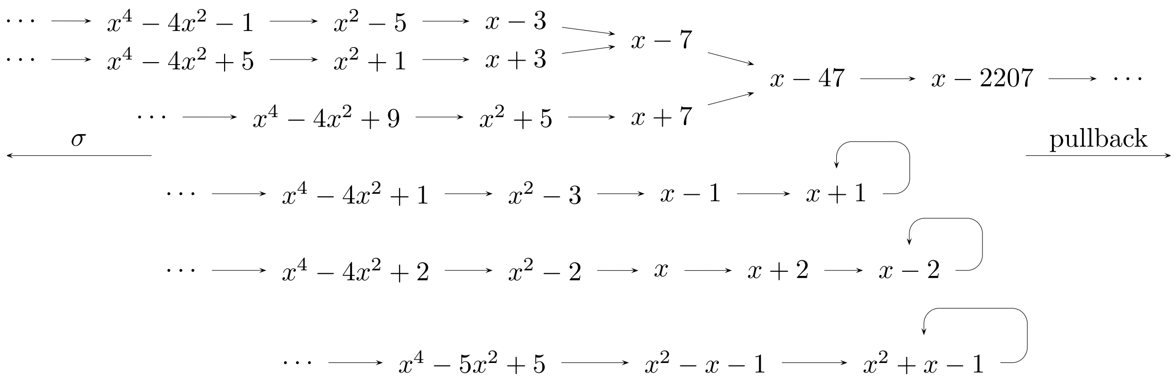

FIGURE 1. Geometry of the substitution σ defined by $\sigma(x) = x^2 - 2$, the ground field being $k = \mathbb{Q}$. An arrow $p \to q$ means that the ideal (q) is the pullback of the prime ideal (p); or equivalently that p is a prime factor of $\sigma(q)$.

For most cases in Table 2, it is quite straightforward to determine the orbits of a prime polynomial, but cases like the q-shift operator with q a root of unity or the Mahlerian operator show that some care is needed in general.

***Example* 1.3.** Figure 1 depicts the orbits and pullbacks of several polynomials in the skew-polynomial ring $\mathscr{D}[x] = \mathbb{Q}[x][\partial; \sigma]$ defined by $\sigma(x) = x^2 - 2$, $\delta = 0$. Both an infinite and several ultimately periodic orbits with fixed points are shown.

1.2.3. *Apparent singularities.* Skew polynomials mimic operators on functional spaces. In that setting, solutions of $Ly = 0$ are also solutions of $QLy = 0$ for any Q. There may be points that are singular for L but not singular for QL, and these cases will be of particular interest to us. We name them after the classical notion of the differential case that they extend.

Definition. The singular point $p \in k[x]$ of $L = \ell_0 + \cdots + \ell_r \partial^r \in \mathscr{D}[x]$ is called *apparent* if there is a multiple $QL \in \mathscr{D}[x]$ with $Q \in \mathscr{D}(x)$ for which p is not singular.

In the differential case, our notions of singular point and apparent singularity match the classical ones. On the other hand, the singularities apparent by our definition contain but are not restricted to apparent singularities in the sense of [1]. In the case of a general σ, our choice avoids the counter-intuitive situation of an apparent singularity α of an operator L whose removal in a multiple PL would introduce a non-apparent singularity on the orbit of α.

1.3. **Ore closure.**

Definition. The *Ore closure* of an ideal $\mathfrak{I}$ of $\mathscr{D}[x]$ is the ideal $C(\mathfrak{I}) = \mathscr{D}(x)\mathfrak{I} \cap \mathscr{D}[x]$. obtained by successively extending and contracting the ideal $\mathfrak{I}$. The notation is extended to skew polynomials into $C(L) = C(\mathscr{D}[x]L)$.

Apparent singularities or Ore closure reflect the relationship between the respective annihilating ideals in $\mathscr{D}[x]$ and $\mathscr{D}(x)$ of a given element of a $\mathscr{D}(x)$-module. For simplicity, consider a cyclic $\mathscr{D}(x)$-module $\mathscr{D}(x)y$. The left ideal $\mathfrak{K}$ of $\mathscr{D}[x]$ consisting of the Q such that $Qy = 0$ is a subset of the left ideal $\mathfrak{E}$ with the same property in $\mathscr{D}(x)$. Now, $\mathscr{D}(x)$ is left principal so that there exists $L \in \mathscr{D}[x]$ with $\mathfrak{E} = \mathscr{D}(x)L$. Clearly, the inclusion $\mathfrak{I} = \mathscr{D}[x]L \subset \mathfrak{K}$ holds, but equality is not true in general. In our applications, the element y—a function—is prescribed by its defining operator L, so we get $\mathfrak{E}$ for free. Our aim is to compute a description of $C(L) = C(\mathfrak{I}) = \mathfrak{K} = \mathfrak{E} \cap \mathscr{D}[x] = \mathscr{D}(x)L \cap \mathscr{D}[x]$.

1.3.1. *Ore closure and polynomial torsion.* The gap between $\mathscr{D}[x]L$ and its closure can be interpreted as a torsion module.

Definition. The *torsion* of $\mathscr{D}[x]/\mathscr{D}[x]L$ is the $k[x]$-module of classes $R + \mathscr{D}[x]L$ for which there exist $f \in k[x]$, $f \neq 0$, and $P \in \mathscr{D}[x]$ satisfying the relation $PL = fR$. We denote it $T(\mathscr{D}[x]/\mathscr{D}[x]L)$ or $T(L)$.

Theorem 1.4. *The relation between Ore closure and torsion module is given by*
$$T(L) = C(L)/\mathscr{D}[x]L.$$

Proof. A non-zero R belongs to $C(L)$ if and only if there exists $f \neq 0$ in $k[x]$ such that $fR \in \mathscr{D}[x]L$. In turn, this holds if and only if the class $R + \mathscr{D}[x]L$ is in $T(L)$. ☐

1.3.2. *Local closure and torsion with prescribed support.* In view of the intended calculations, it is convenient to decompose the global torsion module $T(L)$ further into smaller $k[x]$-modules according to the annihilating polynomials f.

Definition. The *local Ore closure at* $f \in k[x]$, $f \neq 0$, of an ideal $\mathfrak{I}$ of $\mathscr{D}[x]$ is the ideal $C_f(\mathfrak{I}) = k[x, f^{-1}, \sigma(f)^{-1}, \sigma^2(f)^{-1}, \dots][\partial; \sigma, \delta]\mathfrak{I} \cap \mathscr{D}[x]$. The notation is extended into $C_f(L) = C_f(\mathscr{D}[x]L)$.

In the applications or proofs other representations of torsion spaces may be more natural to consider.

Definition. We denote by $\ker_L f$ the $k[x]$-module of the classes $R + \mathscr{D}[x]L$ for which there exists $P \in \mathscr{D}[x]$ satisfying $PL = fR$. Similarly $\ker_f L$ is the $k[x]$-module of $P + f\mathscr{D}[x]$ for which there exists $R \in \mathscr{D}[x]$ satisfying $PL = fR$.

Clearly, $C(L) = \bigcup_{f \in k[x] \setminus \{0\}} C_f(L)$. Again, we have $T(L) = \sum_{f \in k[x] \setminus \{0\}} \ker_L f$. We now decompose the torsion module.

Definition. The *local torsion at* f, $T_f(L)$, is the $k[x]$-module
$$T_f(L) = \bigcup_{j, \nu_0, \dots, \nu_j \geq 0} \ker_L \left(f^{\nu_0} \sigma(f)^{\nu_1} \cdots \sigma^j(f)^{\nu_j} \right).$$

The local analogue of Proposition 1.4 holds.

Lemma 1.5. *The relation between local Ore closure and local torsion space is given by*
$$T_f(L) = C_f(L)/\mathscr{D}[x]L.$$

Proof. A non-zero R belongs to $C_f(L)$ if and only if there exist non-negative integers j, ν_0, …, ν_j such that $f^{\nu_0}\sigma(f)^{\nu_1}\cdots\sigma^j(f)^{\nu_j}R \in \mathscr{D}[x]L$. In turn, this holds if and only if the class $R+\mathscr{D}[x]L$ is in $T_f(L)$. □

1.3.3. *Global closure and local torsion.* The following turns out to be more convenient for computations.

Definition. The *local pseudo-torsion at f*, denoted $\Gamma_f(\mathscr{D}[x]/\mathscr{D}[x]L)$ or in short $\Gamma_f(L)$, is the $k[x]$-module of the $R+\mathscr{D}[x]L$ that are annihilated by some f^ν,

$$\Gamma_f(\mathscr{D}[x]/\mathscr{D}[x]L) = \Gamma_f(L) = \bigcup_{\nu\geq 0} \ker_L f^\nu.$$

As long as we consider a fixed point, the local pseudo-torsion is the local torsion. It turns out that computing the local pseudo-torsion spaces is sufficient to determine the global torsion. The proof relies on the next technical lemma.

Lemma 1.6. *For coprime polynomials f and g, the submodule $\Gamma_{fg}(L)$ decomposes as $\Gamma_{fg}(L) = \Gamma_f(L) \oplus \Gamma_g(L)$.*

Proof. First, we show that $\Gamma_f(L)$ and $\Gamma_g(L)$ are in direct sum. Take $m \in \Gamma_f(L)\cap\Gamma_g(L)$, then there exist μ and ν such that $f^\mu m = g^\nu m = 0$; by co-primeness, there exist cofactors a and b from $k[x]$ such that $1 = af^\mu + bg^\nu$, so that $m = 0$ upon application to m.

The inclusion $\Gamma_f(L) \subset \Gamma_{fg}(L)$ follows from multiplying by g^ν the identity $f^\nu m = 0$. This gives the inclusion of the right-hand side in Γ_{fg}. Conversely, start from a relation $(fg)^\nu m = 0$, and write $1 = af^\nu + bg^\nu$ for polynomials a and b. Then $bg^\nu m$ belongs to Γ_f since it is annihilated by f^ν, while $af^\nu m$ belongs to Γ_g, therefore m belongs to $\Gamma_f + \Gamma_g$. □

Theorem 1.7. *The global torsion $T(L)$ is the sum of the local torsion spaces $T_p(L)$ and the direct sum of the local pseudo-torsion spaces $\Gamma_p(L)$ over all primes p of $k[x]$,*

$$T(L) = \sum_{p \text{ prime}} T_p(L) = \bigoplus_{p \text{ prime}} \Gamma_p(L),$$

where the sums are indexed by prime and monic polynomials. As a consequence, the global closure is the sum of the local closures,

$$C(L) = \sum_{p \text{ prime}} C_p(L).$$

Proof. From the definitions and Lemma 1.6, follows the chain of inclusion

$$T(L) = \sum_{f\neq 0} \ker_L f \subset \sum_{f\neq 0} \Gamma_f(L) = \bigoplus_{p \text{ prime}} \Gamma_p(L) \subset \sum_{p \text{ prime}} T_p(L) \subset T(L).$$

□

The immediate consequence of this theorem is that in the sequel we always compute local pseudo-torsion spaces.

1.4. **Candidate primes, dispersion.** We now try to reduce both infinite sums in the previous lemma to finite sums. The first progress for determining global torsion is to exclude situations where local pseudo-torsion must be trivial. It turns out that candidate primes for non-trivial torsion will be derived from leading and trailing coefficients of the skew polynomial under consideration. Indeed, any nonzero $L \in \mathscr{D}[x]$ may be uniquely written in the form $L = \ell_s\partial^s + \cdots + \ell_r\partial^r$ for integers s and r with $s \leq r$ and $\ell_s\ell_r \neq 0$. The leading coefficient ℓ_r is denoted by $\mathrm{lc}(L)$, and the trailing coefficient ℓ_s by $\mathrm{tc}(L)$. The integer r is called the *order* of L, while s is called its *valuation* with respect to ∂. The next assertion provides a denumerable set of candidates for non-trivial torsion.

Theorem 1.8 (Candidate primes)**.** *Let $L = \ell_s\partial^s + \cdots + \ell_r\partial^r \in \mathscr{D}[x]$ be a skew polynomial of order r and valuation s, and $p \in k[x]$ be prime. If the pseudo-torsion space $\Gamma_p(L)$ is non-trivial, then there exist $n \in \mathbb{N}$ such that p divides $\sigma^n(\ell_r)$. If additionally $\delta = 0$, then p also divides $\sigma^m(\ell_s)$ for some $m \leq n$. A prime p with these properties is called a* candidate prime*.*

Proof. Since $\Gamma_p(L) \neq 0$, there exist $R \notin \mathscr{D}[x]L$ and an integer $\nu \geq 1$ such that the relation $PL = p^\nu R$ holds for some $P \in \mathscr{D}[x]$. Keeping ν fixed, choose R so that P has minimal order n and maximal valuation m for that degree. Note that p^ν does not divide P on the left, for otherwise R would be a multiple of L. We contend that $p^\nu \nmid \mathrm{lc}(P)$. Otherwise, writing $\mathrm{lc}(P) = p^\nu a$ and setting $P_1 = P - p^\nu a\partial^n$, $R_1 = R - a\partial^n L$ would give a new relation, $P_1L = p^\nu R_1$. Here, P_1 cannot be 0 for otherwise P would have been divisible by p^ν. This new P_1 contradicts the minimality of n. Since $\mathrm{lc}(PL) = \mathrm{lc}(P)\sigma^n(\ell_r)$, we deduce that $p \mid \sigma^n(\ell_r)$ using the fact that p is prime.

The case when $\delta = 0$ is dealt with by an analogous argument considering coefficients at the valuation term instead of the degree term. □

In the frequent case when σ is injective, $\sigma^n(\ell_r) \neq 0$ for all integer n and we are thus able to define a set $\mathcal{C}$ of candidate primes where the torsion may be non-trivial. Note that the factors of $\sigma(p)$ for any $p \in \mathcal{C}$ are still in $\mathcal{C}$. This one-to-many relation is just the reverse of the pullback (Section 1.2.2) induced by σ on the level of monic primes. The simplest prototypical operator types are as follows:

- Algebra of differential operators, $\mathscr{D}[x] = k[x][\partial; \mathrm{Id}_{k[x]}, d/dx]$: a prime p is a candidate if it divides the leading coefficient $\mathrm{lc}(L) = \ell_r$ of L.
- Algebra of finite forward difference operators, $\mathscr{D}[x] = k[x][\partial; S, S - \mathrm{Id}_{k[x]}]$: a prime p is a candidate if it divides the shifted leading coefficient $\mathrm{lc}(L)(x+n) = \ell_r(x+n)$ for some non-negative integer n.
- Algebra of forward shift operators, $\mathscr{D}[x] = k[x][\partial; S]$: a prime p is a candidate if it divides the shifted leading coefficient $\mathrm{lc}(L)(x+n) = \ell_r(x+n)$ for some non-negative integer n and moreover it divides the shifted trailing coefficient $\mathrm{tc}(L)(x+m) = \ell_s(x+m)$ for some non-negative integer $m \leq n$.

With the notations of Theorem 1.8, the differences $n - m$ of interest are related to the notion of *dispersion*.

Definition. For f and g in $k[x]$, the *dispersion* $\operatorname{disp}_\sigma(f,g)$ is the set of non-negative integers d such that $\sigma^d f$ and g have a non-trivial common divisor.

A natural way to compute the dispersion is based on factorisation of polynomials [17]. But if $\sigma^d(x)$ is a polynomial with respect to d, as in the shift case where $\sigma^d(x) = x + d$, a better way is available: the dispersion of f and g can be computed as the set of non-negative integer roots of the resultant with respect to x of the polynomials $f(\sigma^d(x))$ and $g(x)$, which is a polynomial in d.

The dispersion leads to a characterization of the set $\mathcal{C}$ of candidate primes.

Lemma 1.9. *When $\delta = 0$, a prime is a candidate prime if and only if it is a factor of a $\sigma^m(\gcd(\sigma^d(\ell_r), \ell_s))$ for some $d \in \operatorname{disp}_\sigma(\ell_r, \ell_s)$ and $m \geq 0$.*

***Example* 1.10.** This continues Example 1.3. We consider the skew-polynomial ring $\mathscr{D}[x] = \mathbb{Q}[x][\partial;\sigma]$ defined by $\sigma(x) = x^2 - 2$, $\delta = 0$, and the skew polynomial

$$\begin{aligned} L = {} & (x^2-49)\partial^2 \\ & - x\left(687\,x^7 + 540\,x^6 - 5083\,x^5 - 3560\,x^4 + 10763\,x^3 + 5520\,x^2 - 6367\,x - 580\right)\partial \\ & \qquad + (x-1)\,(x+2)\left(x^4 - 4\,x^2 + 5\right)\left(687\,x^3 - 834\,x^2 - 647\,x - 1046\right). \end{aligned}$$

According to the definition of Section 1.2, the singularities of L are the primes p such that $\ell_2 = (x-7)(x+7)$ and $\sigma^m(p)$ are not coprime, for some $m \geq 2$. In other words, the singularities are the elements of the orbit of $x - 2207$ under the action of the pullback, that is the elements on the right of $x - 2207$ in Figure 1. By Lemma 1.9, the candidate primes are to be searched *on the left* of $x - 7$ and $x + 7$.

The trailing coefficient has four prime factors. Both factors $x-1$ and $x+2$ are not related to $x \pm 7$ by the pullback. Neither is $687\,x^3 - 834\,x^2 - 647\,x - 1046$ (as may verified by considering the degrees). It appears that $x+7$ is not a factor of some $\sigma^n(x^4-4x^2+5)$ even if both polynomials have $x - 47$ in their orbits. Hence $x + 7$ cannot give a non-trivial local pseudo-torsion. Finally, only the factors $x - 7$ and $x^4 - 4x^2 + 5$ can provide a non-trivial local pseudo-torsion. We find (Fig. 1) $\operatorname{disp}_\sigma(\ell_2, \ell_0) = \{3\}$. Thus all candidate primes have been isolated into the set of factors of $\sigma^m(x^4 - 4x^2 + 5)$, $m \in \mathbb{N}$.

In practice, it will be convenient to replace the usually infinite set $\mathcal{C}$ of candidates with a finite set $\mathcal{DC}$ of distinguished candidate primes. How this is done depends on the combinatorial structure of $\mathcal{C}$ in terms of cycles and trees under the pullback σ^{-1}. We will return to this point in Section 4.5, while the use of the dispersion for the special case where σ is invertible is dealt with in Section 4.4.

1.5. **Multiplicity bounds.** For a candidate prime $p \in k[x]$, the computation of the local pseudo-torsion is faced with another source of infiniteness, namely the infinite union of $\ker_L p^\nu$ in the definition of $\Gamma_p(L)$. Clearly, if p^ν annihilates some $R + \mathscr{D}[x]L$, so does $p^{\nu+1}$, whence the inclusions $\ker_L p^\nu \subset \ker_L p^{\nu+1}$. In this section, we characterize a large class for which this sequence is ultimately stationary and we give an upper bound on the last strict inclusion. This helps the computation of a finite basis of $\Gamma_p(L)$. A relevant quantity is the multiplicity:

Definition. For a skew polynomial L, the *multiplicity* of a torsion element of $\Gamma_p(L)$ is the minimal integer $\nu \in \mathbb{N}$ such that p^ν annihilates it.

We note $\omega_p(f)$ the *valuation* of a polynomial f with respect to some prime polynomial p, i.e., the largest integer ω such that p^ω divides f. By convention, $\omega_p(0) = +\infty$.

Lemma 1.11 (Bound on multiplicity: general σ and δ)**.** *Let $\mathscr{D}[x] = k[x][\partial; \sigma, \delta]$ be a skew-polynomial ring with σ injective. Consider $P = p_0 + \cdots + p_d\partial^d \in \mathscr{D}[x]$ and $L = \ell_0 + \cdots + \ell_r\partial^r$ with $p_d\ell_r \neq 0$. Fix p, a prime of $k[x]$, and set $v_j = \omega_p\big(\sigma^j(\ell_r)\big)$ for $j \geq 0$. Suppose that $(v_j)_{j\geq 0}$ has a finite and nonempty support, and let m be the maximum index $j \geq 0$ for which $v_j \neq 0$. Then, whenever $PL = p^\nu R$ for a polynomial $R \in \mathscr{D}[x]$ and a positive integer ν such that P is reduced modulo p^ν on the left, the order d of P is such that $v_d \neq 0$, so in particular $d \leq m$, and ν is bounded by:*

$$\nu \leq \min_{j=0}^{d} \omega_p(p_j) + v_j + \cdots + v_d. \tag{4}$$

Note that $(v_j)_{j\geq 0}$ has a nonempty support means that p is a candidate prime.

This result can always be used for infinite orbits. Indeed, such an orbit $(p_k)_{k\geq 0}$ with $p_0 = p$ consists of pairwise coprime polynomials, since a prime is then mapped to a prime by the pullback. Therefore, each $p_j^{v_j}$ divides ℓ_r, so that the existence of m, the maximum index j such that $v_j \neq 0$, is always granted.

Proof. We start by analyzing products from the point of view of valuations.

Lemma 1.12. *With the same notation, let $U = PL$. Then whenever $0 \leq j \leq d$, the following inequality holds:*

$$\omega_p(p_j) + v_j + \cdots + v_d \geq \min_{k=j}^{d} \big(\omega_p(u_{r+k}) + v_{k+1} + \cdots + v_d\big). \tag{5}$$

Proof. Because σ is injective, no $\sigma^i(\ell_r)$, $i \geq 0$, is zero. Writing $\partial^i L = \sigma^i(\ell_r)\partial^{r+i} + \sum_{k=0}^{r+i-1} c_{i,k}\partial^k$, we get $u_{r+j} = p_j\sigma^j(\ell_r) + \sum_{i=j+1}^{d} p_i c_{i,r+j}$. Upon multiplying by $\sigma^{j+1}(\ell_r)\cdots\sigma^d(\ell_r)$ and introducing $\tilde{p}_i = \sigma^i(\ell_r)\cdots\sigma^d(\ell_r)\, p_i$ and $\tilde{u}_{r+i} = \sigma^{i+1}(\ell_r)\cdots\sigma^d(\ell_r)\, u_{r+i}$, one obtains

$$\tilde{u}_{r+j} = \tilde{p}_j + \sum_{i=j+1}^{d} \sigma^{j+1}(\ell_r)\cdots\sigma^{i-1}(\ell_r)\, \tilde{p}_i\, c_{i,r+j}$$

and

$$\tilde{p}_j \in (\tilde{u}_{r+j},\, \tilde{p}_{j+1},\, \tilde{p}_{j+2},\, \ldots\, \tilde{p}_d)\,.$$

Since $\tilde{u}_{r+d} = u_{r+d} = \sigma^d(\ell_r)p_d = \tilde{p}_d$, one has $\tilde{p}_d \in (\tilde{u}_{r+d})$, so it follows by descending induction that the relation $\tilde{p}_j \in (\tilde{u}_{r+j}, \ldots, \tilde{u}_{r+d})$ holds whenever $0 \leq j \leq d$. Taking valuations yields the announced inequalities. $\square$

Notons d'abord les relations

$$v_j > 0 \iff p \mid \sigma^j(\ell_r) \iff p_j \mid \ell_r.$$

On peut avoir un support fini dans le cas ultimement périodique si les p_j qui divisent ℓ_r sont sur la queue du ρ. Cela pourrait peut-être donner un exemple, d'autant plus intéressant que le cas ultimement périodique va passer à la trappe. Reprendre l'exemple $\sigma(x) = x^2-2$.

Assume $PL = p^\nu R$, whose leading coefficient is $p_d\sigma^d(\ell_r)$ and thus has to be divisible by p^ν. To reach a contradiction, suppose that v_d is 0; p^ν thus divides p_d, in conflict with the hypothesis of reduction modulo p^ν. Therefore, the valuation v_d is non-zero. Applying Inequality (5) of Lemma 1.12 to $U = p^\nu R$ shows that for each $j \in [0, d]$ there exists a $k \geq j$ such that $\omega_p(p_j) + v_j + \cdots + v_d \geq \nu + v_{k+1} + \cdots + v_d$, that is $\nu \leq \omega_p(p_j) + v_j + \cdots + v_k$, hence $\nu \leq \omega_p(p_j) + v_j + \cdots + v_d$ and the inequality announced in Lemma 1.11. ☐

L'hypothèse que le support de $(v_j)_{j\geq 0}$ est fini permet de borner l'ordre de P par $d \leq m$, ce qui fait que la borne

$$\nu \leq \min_{j=0}^{d} \omega_p(p_j) + v_j + \cdots + v_d$$

est effective (si l'on a m). Si l'on enlève l'hypothèse que le support est fini, par exemple pour une orbite ultimement périodique, la majoration demeure. Elle exprime que pour augmenter la multiplicité, il faut augmenter l'ordre, ce qui semble une trivialité.

A better bound is available when σ is injective and δ is 0.

Lemma 1.13 (Bound on multiplicity: σ injective and $\delta = 0$). *Let $\mathscr{D}[x] = k[x][\partial; \sigma]$ be a skew-polynomial ring with σ injective and $\delta = 0$. Fix $p \in k[x]$, a prime polynomial. Introduce $L = \ell_s\partial^s + \cdots + \ell_r\partial^r \in \mathscr{D}[x]$ of order r and valuation s, and set $v_j = \omega_p\big(\sigma^j(\ell_r)\big)$ and $v'_j = \omega_p\big(\sigma^j(\ell_s)\big)$ for $j \geq 0$. Suppose that both sequences $(v_j)_{j\geq 0}$ and $(v'_j)_{j\geq 0}$ have a finite and non empty support. Let m and m' be respectively the maximum index $j \geq 0$ for which $v_j \neq 0$ and the minimum index $j \geq 0$ for which $v'_j \neq 0$. Consider $P = p_{d'}\partial^{d'} + \cdots + p_d\partial^d \in \mathscr{D}$, with $d' \leq d$ and $p_{d'}p_d \neq 0$, and reduced modulo p^ν on the left. For a relation of the form $PL = p^\nu R$ to hold with $R \in \mathscr{D}[x]$, and a positive integer ν, it is then necessary that $m' \leq d' \leq d \leq m$, and inequalities*

$$\nu \leq \omega_p(p_j) + v_j + v_{j+1} + \cdots + v_d \tag{6}$$

and

$$\nu \leq \omega_p(p_j) + v'_j + v'_{j-1} + \cdots + v'_{d'} \tag{7}$$

must be satisfied for $d' \leq j \leq d$.

Proof. Equation (6) is merely an explicit reformulation of (4) in Lemma 1.11. Besides the upper bound on the degree of the higher term in P, and the bound on ν in terms of $v_j = \omega_p\big(\sigma^j(\ell_r)\big)$, a symmetric treatment yields a lower bound for the degree of the least term in P, and a bound on ν in terms of $v'_j = \omega_p\big(\sigma^j(\ell_s)\big)$. The reasoning is similar to those in the proof of Lemma 1.12, with $\tilde{u}_{s+i} = \sigma^{d'}(\ell_s)\cdots\sigma^{i-1}(\ell_s)u_{s+i}$, $\tilde{p}_i = \sigma^{d'}(\ell_s)\cdots\sigma^{i}(\ell_s)p_i$. Concluding as in the proof of Lemma 1.11 proves $d' \geq m'$ and (7). ☐

The preceding results will be used in situations where P is not divisible by p on the left. Then, there exists an index j, $d' \leq j \leq d$, for which the valuation $\omega_p(p_j)$ is zero. Then, Inequality (4) in the case of a general σ, and Inequalities (6) and (7) in the case of an injective σ and $\delta = 0$ may be relaxed into the next inequalities, less precise but more practical to apply.

Corollary 1.14. *With the notations and hypotheses of Lemma 1.11, the relation $PL = p^\nu R$ with $R \in \mathscr{D}[x]$, $\nu > 0$, $P \in \mathscr{D}[x]$, and P not divisible by p implies*

$$\nu \leq \sum_{j=m'}^{m} v_j.$$

Corollary 1.15. *With the notations and hypotheses of Lemma 1.13, the relation $PL = p^\nu R$ with $R \in \mathscr{D}[x]$, $\nu > 0$, $P \in \mathscr{D}[x]$, and P not divisible by p implies*

$$\nu \leq \sum_{j=m'}^{m} v_j \qquad \text{and} \qquad \nu \leq \sum_{j=m'}^{m} v'_j.$$

1.6. **Algebraic Extensions.** When the multiplicity is 1, k is not algebraically closed and p is a candidate prime with $\deg p = d > 1$, the result can be reconstructed from one computation in $k[x]/(p)$ as follows.

Lemma 1.16. *Consider a prime $p \in k[x]$ of degree $d > 1$ and let α be a root of p in $\bar{k}$. Suppose*

$$P^{(\alpha)}L = (x-\alpha)R^{(\alpha)}, \qquad P^{(\alpha)} = p_0(\partial) + \alpha p_1(\partial) + \cdots + \alpha^{d-1}p_{d-1}(\partial), \tag{8}$$

for $L \in \mathscr{D}[x] = k[x][\partial;\sigma,\delta]$, and $P^{(\alpha)}$, $R^{(\alpha)} \in \mathscr{D}(\alpha)[x] = k(\alpha)[x][\partial;\sigma,\delta]$, and $p_\ell \in k[\partial]$. Then

$$\big(p_0(\partial) + xp_1(\partial) + \cdots + x^{d-1}p_{d-1}(\partial)\big)L \in p\mathscr{D}[x]. \tag{9}$$

Conversely, any relation $PL \in p\mathscr{D}[x]$ for given $P, L \in \mathscr{D}[x]$ provides a relation of the form (8).

Proof. The Galois group of the splitting field K of p over k acts transitively on the roots of p and formula (8) is valid not only for one root α but for all roots β of p. For $0 \leq \ell < d$, $(x^\ell - \beta^\ell)/(x-\beta)$ is in $K[x]$ and the sum of such terms is in $k[x]$. Thus with $p' = dp/dx$

$$\frac{p'}{p}x^\ell = \sum_{p(\beta)=0} \frac{x^\ell}{x-\beta} \in \sum_{p(\beta)=0} \frac{\beta^\ell}{x-\beta} + k[x].$$

Multiplying by $p_\ell(\partial)$ on the right, then summing over ℓ yields

$$\left(p' \sum_{\ell=0}^{d-1} x^\ell p_\ell(\partial)\right) L \in p\mathscr{D}[x].$$

Since p and p' are coprime, we can find u and v such that $up' = 1 - pv$. Multiplying the previous equation by u on the left gives (9). Conversely, given a relation $PL \in p\mathscr{D}[x]$, write $p = (x-\alpha)q$ for $q \in k(\alpha)[x]$, then a Bézout relation $(x-\alpha)u + qv = 1$. Then, $vp = (x-\alpha)\big(1-(x-\alpha)u\big)$, so that $(vP)L \in (x-\alpha)\mathscr{D}(\alpha)[x]$. ☐

1.7. **Torsion by linear algebra.** Given the skew polynomial L, the prime p and the integer ν, we consider the torsion equation $PL = p^\nu R$, for unknown P and R. The equation rewrites $PL = 0 \bmod p^\nu \mathscr{D}[x]$ and this emphasizes the linear character of the problem. We provide two ways of exploiting this depending on two different choices of bases.

1.7.1. *Powers of* ∂. Write

$$P = \sum_i p_i(x)\partial^i.$$

An equation $PL = U$, expanded with respect to the powers of ∂ may be rewritten as an equation $\mathsf{PL} = \mathsf{U}$, where P is the infinite row vector of the coefficients p_i, $i \geq 0$, of P, and L is some infinite square matrix. This matrix has all its entries null for $j > i + r$. Moreover in case $\delta = 0$ it is a band matrix. This point of view is useful in Section 6.

***Example* 1.17.** We consider the skew polynomial $L = (x-2)^\mu\partial - x^\mu$ of the ring $k[x][\partial; S]$, where μ is a positive integer.

The dispersion being 2, Lemma 1.9 shows that the candidate primes are of the form $x+k$, $k \in \mathbb{N}$. With $p = x$, in the notations of Lemma 1.13, we have $m = 2$ and $m' = 0$. Then this lemma gives a bound $d \leq 2$ on the order of P and $n \leq \mu$ on its multiplicity. We are thus looking for three nonzero polynomials p_0, p_1, p_2 such that

$$\begin{pmatrix} p_0(x) & p_1(x) & p_2(x) \end{pmatrix} \begin{pmatrix} -x^\mu & (x-2)^\mu & 0 \\ 0 & -(x+1)^\mu & (x-1)^\mu \\ 0 & 0 & -(x+2)^\mu \end{pmatrix} = 0 \bmod x^\mu.$$

Such a solution exists since $(x-1)^\mu$ and $(x-2)^\mu$ are invertible modulo x^μ. This shows that the bounds given by Lemma 1.13 are tight.

1.7.2. *Powers of* p. Now we write

$$P = \sum_{i=0}^{\nu-1} p^i(x)P_i(x,\partial) + p(x)^\nu\mathscr{D}[x], \qquad L = \sum_{i=0}^{\nu-1} p^i(x)L_i(x,\partial) + p(x)^\nu\mathscr{D}[x],$$

where the P_i and L_i are polynomials in x and ∂ whose degree in x is less than that of p. The L_i are known and we are looking for P_i such that $PL \in p(x)^\nu\mathscr{D}[x]$. For simplicity of exposition, we restrict here to a prime p with degree 1, so that the coefficients P_i and L_i are in $k[\partial]$. We note P the column vector $(P_0, \ldots, P_{\nu-1})$. Right multiplication of $P \in \mathscr{D}[x]/p^\nu\mathscr{D}[x]$ by p defines a lower triangular matrix $\mathsf{\Lambda}$ such that $\mathsf{Q} = \mathsf{\Lambda}\mathsf{P}$ has entries defined by

$$Pp + p^\nu\mathscr{D}[x] = \mathsf{Q}_0 + p\mathsf{Q}_1 + \cdots + p^{\nu-1}\mathsf{Q}_{\nu-1} + p^\nu\mathscr{D}[x]. \tag{10}$$

Thus the equation $PL = p^\nu R$ gives rise to a finite-dimensional system

$$\left(\sum_{i=0}^{\nu-1} L_i(\partial)\mathsf{\Lambda}^i\right)\mathsf{P} = 0, \tag{11}$$

where P is the unknown. This is a triangular system of functional equations between the coefficients of P. These equations have coefficients in $k[\partial]$ and the solutions have to be searched for in $k[\partial]$ as well.

In the following sections, the nature of $\mathsf{\Lambda}$ and System (11) will be made so explicit in several specialized contexts that the functional equations will be algorithmically solvable for polynomial solutions in $k[\partial]$.

1.8. **Generic algorithms for torsion, road map for common Ore operators.** For reference, and indeed to summarize the discussion to this point, we outline two generic algorithms: one for global torsion (Algorithm 1.18) and one for local pseudo-torsion (Algorithm 1.19).

Algorithm 1.18 (Global torsion)**.**

Input: A skew-polynomial ring $\mathscr{D}[x]$ with σ injective and $L \in \mathscr{D}[x]$
Output: A set of generators for the global torsion $T(L)$

(1) Determine a finite set $\mathfrak{DC}$ of distinguished candidate primes p
(2) For each prime p in this set, determine a set of generators of the local pseudo-torsion $\Gamma_p(L)$ (Algorithm 1.19)
(3) Return the union over the primes p of the sets of generators (possibly after the post-process of computing a Gröbner basis)

Algorithm 1.19 (Local pseudo-torsion at p)**.**

Input: A skew-polynomial ring $\mathscr{D}[x]$ with σ injective, $L \in \mathscr{D}[x]$, and p a prime of $k[x]$
Output: A set of generators for the local pseudo-torsion $\Gamma_p(L)$

(1) Determine the multiplicity ν of the prime p
(2) Solve the corresponding system of linear functional equations (11) for polynomial solutions
(3) Return generators (possibly after the post-process of computing a Gröbner basis)

In the following sections we detail aspects of these algorithms that can be given a special treatment depending on properties of the Ore operators. A summary of the behaviour of the most common operators is given in Table 3.

Operator	Bound on ν	P is a polynomial solution of a	Notes
Differentiation $\frac{d}{dx}$	1 (Thm. 2.4)	Hom. differential eqn	§ 2.3
Euler's $\vartheta = x\frac{d}{dx}$	1 (Thm. 2.4)	Differential eqn ($\alpha \neq 0$)	No torsion if $\alpha = 0$
Shift S	Thm. 1.13	Triang. differential sys	§ 4.1.1
Difference Δ	*Reduces to Shift by Theorem 3.1*		
q-Differentiation $\delta^{(q)}$			
($q^N \neq 1$)			
$\alpha = 0$	1 (Thm. 2.5)	Hom. q-Difference eqn	
$\alpha \neq 0$	*Reduces to q-Dilation by Theorem 3.1*		
($q^N = 1$)	?	?	§ 6.3
q-Dilation Q			
($q^N \neq 1$)	Thm. 1.13	Triang. q-Difference sys	§ 4.2
($q^N = 1$)	factors of determinant	finite linear system	§§ 2.1, 6.3
q-Difference Δ_q	*Reduces to q-Shift by Theorem 3.1*		
Mahlerian operator M	no	?	§§ 2.1,

TABLE 3. Some common Ore operators and the torsion relation $PL = (x - \alpha)^\nu R$

2. SINGULARITIES FIXED UNDER THE ACTION OF σ

Curieusement cette section n'a pas d'introduction. Mais il n'est pas encore temps de l'écrire. PhD

2.1. **Dilation and Mahlerian operators.** We deal here with cases where σ is injective but distinct from $\mathrm{Id}_{k[x]}$. It turns out that in such a situation the torsion at a fixed point is often trivial, substitution operators being a special case ($\delta = 0$).

Theorem 2.1. *Dilation operators defined by $\sigma(x) = qx$, with $q \neq 1$, have no torsion at the origin. Similarly Mahlerian operators, defined by $Mx = x^b M$ for $b \geq 2$, have no torsion at the origin. Moreover Mahlerian operators have no torsion at a cyclotomic polynomial Φ_a, where a is coprime with b.*

This theorem is actually a corollary of the following more general result; the relation to cyclotomic polynomials is postponed to Section 6.1, where fixed points are described as particular periodic points.

Proposition 2.2. *Let $\mathscr{D}[x] = k[x][\partial;\sigma]$ be a skew-polynomial ring with σ injective but distinct from $\mathrm{Id}_{k[x]}$. Let L be a primitive skew polynomial L in $\mathscr{D}[x]$, and p a prime polynomial fixed by σ. If p divides $\delta(p)$, the local torsion $T_p(L)$ at p is trivial. The hypothesis $p \mid \delta(p)$ is satisfied in each of both following cases:*

- *$\sigma(x) - x \mid \delta(x)$, and particularly $\delta = 0$;*
- *p and $\sigma(x) - x$ are coprime.*

Proof. Since p is fixed by σ, we have that $p \mid \sigma^k(p)$ for any $k \geq 0$. The condition $p \mid \delta(p)$ therefore implies that for any $A \in \mathscr{D}[x]$, p is a left divisor of Ap.

We prove by contradiction that the torsion is trivial. Assume it is not and let P be such that p is a left divisor of PL, with P not divisible on the left by p. By Theorem 1.8, p divides some $\sigma^n(\ell_r)$ for $n \in \mathbb{N}$ and therefore also ℓ_r, since p is fixed by σ (§1.2.2). Thus p is a left-divisor of $P(L - \ell_r\partial^r)$. Continuing in this way, we conclude that all coefficients of L are divisible by p, a contradiction since L is primitive.

For the last statement, we use Lemma A.1 according to which, for a fixed prime p and assuming that σ is not $\mathrm{Id}_{k[x]}$, we have

$$\delta(p) = \delta(x)\frac{\sigma(p) - p}{\sigma(x) - x} = \delta(x)\frac{pq - p}{\sigma(x) - x} = p\delta(x)\frac{q - 1}{\sigma(x) - x}$$

that is $\delta(p)(\sigma(x) - x) = p\delta(x)(q - 1)$. If p is coprime with $\sigma(x) - x$, we see that p divides $\delta(p)$ by Gauss theorem. We reach the same conclusion if $\sigma(x) - x$ divides $\delta(x)$. ☐

***Example* 2.3.** Consider $\mathscr{D}[x] = \mathbb{Q}[x][\partial;\sigma,\delta]$ with $\sigma(x) = x^2 - 2$. Then $p = x^3 - 3x + 1$ is fixed by σ ($\sigma(p) = (x^3 - 3x + 1)(x^3 - 3x - 1)$) and coprime with $\sigma(x) - x = x^2 - x - 2$. (The modular period, defined in Section 6.1, is $t = 3$.) Hence p is not an apparent singularity.

2.2. The bound $\nu \leq 1$.

In the differential case, the multiplicity is bounded by 1.

Theorem 2.4. *Let $\mathscr{D}[x] = k[x][\partial; \mathrm{Id}_{k[x]}, d/dx]$ be the skew-polynomial ring of classical derivation and $p = x - \alpha$ a prime of $k[x]$. For $L \in \mathscr{D}[x]$ a non-zero skew polynomial, the local torsion $T_p(L)$ is a $\mathscr{D}[x]$-module generated by finitely many $R + \mathscr{D}[x]L$ that solve a torsion relation $pR = PL$ with exponent $\nu = 1$.*

Proof. Letting $T = \mathscr{D}[x]\ker_L(p)$, we first prove by induction on i that $\ker_L(p^i) \subset T$. For $i = 1$ this is clear. Assuming $i \geq 1$ and $\ker_L(p^i) \subset T$, let $R + \mathscr{D}[x]L \in \ker_L(p^{i+1})$. Then from $\partial p^{i+1} = p^i(p\partial + i + 1)$ and $p^{i+1} = p^i p$ and we deduce that both $(p\partial + i + 1)R$ and pR belong to T, therefore so does $\partial pR = (p\partial + 1)R$ and by subtraction iR and finally R since $i \geq 1$.

We thus have $\ker_L(p) \subset T_p(L) \subset \mathscr{D}[x]\ker_L(p)$. The proof is concluded by showing that $T_p(L)$ is a $\mathscr{D}[x]$-module. Left multiplication by x is clear, while $\partial\ker_L(p^i) \subset \ker_L(p^{i+1})$ follows from $p^{i+1}\partial = (\partial p - i - 1)p^i$.

The finiteness property comes from the noetherianity of $\mathscr{D}[x]$ (Lemma A.5). ☐

This result is an elementary version in terms of torsion modules of a theorem by Kashiwara [15]. A classical proof is to be found in [12], and an alternative one in [5].

The proof above extends to the q-differential case when $p = x$, the relevant commutation rule being:

$$\partial x^{i+1} = x^i \left(q^{i+1} x \partial + \frac{q^{i+1} - 1}{q - 1} \right).$$

Thus, we also have.

Theorem 2.5. *Let $\mathscr{D}[x] = k[x][\partial; Q, \delta^{(q)}]$ be the skew-polynomial ring of q-differentiation. The local torsion $T_x(L)$ is a $\mathscr{D}[x]$-module generated by finitely many $R + \mathscr{D}[x]L$ that solve a torsion relation $xR = PL$ with exponent $\nu = 1$.*

These two theorems may be extended slightly by changes of indeterminates as follows.

Corollary 2.6. *Let $\mathscr{D}[x] = k[x][\partial; \sigma, \delta]$ be a skew-polynomial ring such that δ strictly decreases the degree of non-constant polynomials. Let $p = x - \alpha$ be a prime of $k[x]$ such that $\sigma(p)/p \in k$ is not a proper root of unity and $\delta(p)$ is invertible modulo p. Let L be a non-zero element of $\mathscr{D}[x]$. Then the local torsion $T_p(L)$ is generated by finitely many $R + \mathscr{D}[x]L$ that solve a torsion relation $pR = PL$ with exponent $\nu = 1$ for suitable P.*

Proof. Note that by Lemma A.4, $\sigma(x) = qx + r$ and $\delta(x) = \lambda \in k \setminus \{0\}$. Additionally $\sigma(p)/p \in k$, so $qx + r - \alpha$ is proportional to $x - \alpha$. Thus, up to a change of generators of $k[x]$, we can assume that $\sigma(x) = qx$, $p = x$. According to Lemma A.1 either $q = 1$ and $\delta = \lambda\, d/dx$, or q is not a root of unity and $\delta = \lambda \delta^{(q)}$, where $\delta^{(q)}$ is the usual q-differentiation with respect to x. Up to another change of generators of $\mathscr{D}[x]$, this time varying ∂, we may assume $\lambda = 1$. It suffices now to apply Theorem 2.4 or Theorem 2.5. □

2.3. **Differential operators.** In the differential case, we now have the set $\mathcal{C}$ of candidate primes from Thm. 1.8 (factors of the leading coefficient) and the bound 1 on their multiplicity from Thm. 2.4.

Using this, we now give an algorithm that computes the local closure. We consider a skew polynomial L of the Weyl algebra $\mathscr{D}[x] = k[x][\partial; \mathrm{Id}_{k[x]}, d/dx]$ and for the moment k is an algebraically closed field. We consider a torsion relation of the form $PL = pR$ for some $p \in \mathcal{C}$, say $p = x - \alpha$.

Since $\nu = 1$, the system (11) is reduced to one equation in one unknown $P \in k[\partial]$. The operator $\Lambda_{0,0}$ can be made very explicit. Since $\partial^i p \in i\partial^{i-1} + p\mathscr{D}[x]$, p acts like a differentiation operator. We use a Fourier-like duality * that sends x to $\partial + \alpha$ and ∂ to x. It is an anti-automorphism of the Weyl algebra. As a consequence, for the prime $p = x - \alpha$ the torsion relation $PL \equiv 0 \bmod p\mathscr{D}[x]$ translates into the the equation $L^* P^* \equiv 0 \bmod \mathscr{D}[x]\partial$. The problem of computing local torsion at p is reduced to finding polynomial solutions P_0 of $L^* P_0(x) = 0$. These can be found using algorithms from [2, 3]. This leads to the torsion element $R + \mathscr{D}[x]L$ for $R = (x - \alpha)^{-1} P_0(\partial) L$ and we are done in the case when k is algebraically closed.

The following algorithm is essentially a reformulation of the algorithm by Tsai [19], where we isolate more explicitly the relation to polynomial solutions. See also [1, Appendix] for another algorithm.

Algorithm 2.7 (Local closure for differential operators).

Input: $L(x,\partial) \in \mathscr{D}[x]$ the first Weyl algebra, $p \in k[x]$ prime which divides $\operatorname{lc}(L)$
Output: A set $\mathcal{R}$ of generators for the local torsion $T_p(L)$

(1) Let $\alpha \in \bar{k}$ be such that $p(\alpha) = 0$. Find polynomial solutions P to $L(\partial+\alpha, x)P(\alpha, x) = 0$
(2) Return $\{\, p^{-1}P(x,\partial)L : P \text{ is a solution}\,\}$

Note that one only needs to represent α as a root of p in Step 1, using a formal extension $k[x]/(p) \simeq k(\alpha)$, without having to compute α by factoring p over an algebraic closure of k.

Example **2.8.** With $\mathscr{D}[x] = \mathbb{Q}[x][\partial; \mathrm{Id}_{\mathbb{Q}[x]}, d/dx]$, the example from Eq. (1) in the introduction is

$$L(x,\partial) = 2x(x-1)(x+1)\partial^3 + (2x^3 - 4x^2 + 6x + 8)\partial^2 - (x-1)(x^2+3x-8)\partial - 12.$$

It follows from Theorem 1.8 that the set of candidate primes $\mathcal{C}$ is $\{x-1, x, x+1\}$. First, for the prime $p = x-1$, $L(\partial+1, x)P_1(x) = 0$ does not have any non-zero polynomial solution. Hence, the local torsion is trivial, $T_{x-1}(L) = \mathscr{D}[x]L$. Second, for the prime $p = x$, $L(\partial, x)P_0(x) = 0$ admits $P_0(x) = x^4 - 8x^3 + 12x^2$ as a polynomial solution. Hence, $T_x(L) = \mathscr{D}[x]R_0 + \mathscr{D}[x]L$ with

$$\begin{aligned} R_0 = x^{-1}P_0(\partial)L = 2(x-1)(x+1)\partial^7 &- (14x^2 - 20x - 22)\partial^6 \\ &+ (7x^2 - 90x - 21)\partial^5 + (32x^2 - 44x - 56)\partial^4 \\ &- (12x^2 - 192x + 144)\partial^3 - (72x - 192)\partial^2 - 72\partial. \end{aligned}$$

Third, for the prime $p = x+1$, $L(\partial-1, x)P_{-1}(x) = 0$ admits $P_{-1}(x) = x$ as a polynomial solution. Hence, $T_{x+1}(L) = \mathscr{D}[x]R_{-1} + \mathscr{D}[x]L$ with

$$R_{-1} = (x+1)^{-1}P_{-1}(\partial)L = 2(x-1)x\partial^4 + 2(x^2+3)\partial^3 - (x^2 - 5x + 2)\partial^2 - (3x+1)\partial.$$

It follows from Algorithm 1.18 that $T(L) = T_x(L) + T_{x+1}(L)$. The Weyl closure is given as

$$C(L) = \mathscr{D}[x]L + \mathscr{D}[x]R_0 + \mathscr{D}[x]R_{-1}.$$

A Gröbner basis of this ideal for the lexicographical ordering satisfying $\partial > x$ has elements L, R_0, and

$$2(x-1)\partial^7 - (8x-22)\partial^6 - (19x+15)\partial^5 + (21x-114)\partial^4 + (28x-1)\partial^3 - (12x-80)\partial^2 - 36\partial.$$

In the leading coefficient of the last element, only the factor $x-1$ remains. Thus all apparent singularities have been removed.

2.4. **q-Differential operators.** We consider the only possible fixed prime $p = x$. By Thm. 2.5, its multiplicity is bounded by 1. The situation is a q-analogue of the differential case treated above. The commutation $\partial^i x \in (1 + q + \cdots + q^{i-1})\partial^{i-1} + x\mathscr{D}[x]$ shows that x acts on the right as a q-differentiation operator. As in the differential case, we consider the duality exchanging x and ∂ so that the relation $PL \equiv 0 \bmod x\mathscr{D}[x]$ is solved by computing polynomial solutions P_0 of L^*P_0. The algorithm is summarized below.

Algorithm 2.9 (Local closure for q-differential operators at 0).
Input: $L(x,\partial) \in \mathscr{D}[x] = k[x][\partial; Q, \delta^{(q)}]$, such that x divides $\operatorname{lc}(L)$
Output: A set $\mathcal{R}$ of generators for the local torsion $T_x(L)$

(1) Find polynomial solutions P to
$L(\partial, x)P(x) = 0$
(2) Return $\{\, x^{-1}P(\partial)L : P \text{ is a solution} \,\}$

***Example* 2.10.** We consider $L(x,\partial) = x\partial^2 + 1$. Then $L(\partial, x) = x^2\partial + 1$ does not have any polynomial solution (it increases the degree) and thus the torsion at 0 is trivial.

***Example* 2.11.** The polynomial $L(x,\partial) = x\partial - (1+q+\cdots+q^i)$ is fixed under *. It has x^i as a polynomial solution. We thus obtain $x^{-1}\partial^i L = \partial^{i+1}$ where the singularity has been removed.

Un exemple avec des q-Legendre ? BS

3. Singularities along infinite orbits

We now consider the torsion at singularities with an infinite orbit. The only hypothesis here is that σ is injective (and necessarily different from identity so that an infinite orbit can exist). In Section 3.1, we give a reduction which considerably simplifies our analyses: there is an equivalent problem expressed in a skew-polynomial ring with $\delta = 0$. This applies in particular to difference operators, where all primes define infinite orbits, and to q-difference operators when q is not a root of unity, where all primes different from x define infinite orbits. In these two cases, σ is bijective, which brings new properties and problems that are dealt with in the next section. The more general case is exemplified by the Mahlerian operators (Section 3.2), for which σ is injective but not invertible.

3.1. **Reduction to the case $\delta = 0$.** The reduction is as follows.

Theorem 3.1. *Let $\mathscr{D}[x] = k[x][\partial; \sigma, \delta]$ and $\mathscr{E}[x] = k[x][\vartheta; \sigma]$. Assume $\sigma \neq \mathrm{Id}_{k[x]}$ is injective and $\delta \neq 0$, in which case there exist co-prime polynomials u and v in $k[x]$ such that $v(\sigma(x) - x) = u\delta(x)$. Given $L \in \mathscr{D}[x]$ and $p \in k[x]$ prime defining an infinite orbit, let $d_L \in k[x]$ be of minimal degree such that $\tilde{L}(x,\vartheta) := d_L L(x, u^{-1}(\vartheta - v))$ is in $\mathscr{E}[x]$. Then the local pseudo-torsions of L and $\tilde{L}$ are related by*

$$\Gamma_p(L) := \{R(x, u\partial + v) + \mathscr{D}[x]L \mid R(x,\vartheta) + \mathscr{E}[x]\tilde{L} \in \Gamma_p(\tilde{L})\}.$$

This theorem generalizes the fact that a finite difference equation can be viewed as a recurrence equation, and makes explicit the relations between their pseudo-local torsions.

***Example* 3.2.** Consider the q-differential equation

$$q^2(q-1)x^2(x-q^2)\left(\frac{f(qx)-f(x)}{(q-1)x}\right) + \big(q(q+1)x^2 - (q^4+q+1)x + 1\big)f(x) = 0$$

at $x = q^2$. The number q is assumed to be nonzero and not a root of unity, so that the orbit of $p = x - q^2$ is infinite. We use $\mathscr{D}[x] = k[x][\partial; Q, \delta^{(q)}]$, where Q is the q-dilation and $\delta^{(q)}$ the q-differentiation (Table 1), and

$$L = q^2(q-1)x^2(x-q^2)\partial + q(q+1)x^2 - (q^4+q+1)x + 1.$$

The polynomials u and v are given by $u = (q-1)x$ and $v = 1$. We find

$$\tilde{L} = q^2x(x-q^2)\vartheta + (x-1)(qx-1)$$

by a straightforward computation, since the least common denominator is 1 in this example. We have merely translated a q-differential equation into a q-dilation equation.

A generator $X + \mathscr{E}[x]\tilde{L}$ of $\Gamma_{x-1}(\tilde{L})$ will be computed in Section 4.2, namely

$$X = q^6\vartheta^2 + (q^6+q^5-q^3-q^2)\vartheta + q^5 - q^3 - q^2 + 1.$$

Applying Theorem 3.1, we conclude that $\Gamma_{x-1}(L)$ is generated $R + \mathscr{D}[x]L$, where

$$R = q^7(q-1)^2x^2\partial^2 + q^2(q-1)(q+1)(q^4+q^3-1)x\partial + 2q^6 + 2q^5 - 2q^3 - 2q^2 + 1.$$

The corresponding q-differential equation,

$$q^7(q-1)^2x^2\left(\frac{f(q^2x) - (1+q)f(qx) + qf(x)}{(q-1)^2qx^2}\right)$$
$$+ q^2(q-1)(q+1)(q^4+q^3-1)x\left(\frac{f(qx)-f(x)}{(q-1)x}\right) + (2q^6+2q^5-2q^3-2q^2+1)f(x) = 0,$$

no longer has any singularity at powers of q; only the non-apparent singularity at the origin remains.

BS Qu'est-ce qui reste ici et qu'est-ce qui va dans l'annexe B?

Proof of Theorem 3.1. The inclusion of $\phi\big(\Gamma_p(\tilde{L})\big)$ into $\Gamma_p(L)$ is given by Lemma B.3. Let us show the converse inclusion. According to Lemma B.1, we have $v(\sigma(x) - x) = u\delta(x)$ and the least common denominator d_L of $\psi(L)$ lies in the multiplicatively stable part S_u generated by the prime factors of the $\sigma^j(u)$ for $j \geq 0$. Now, let $R + \mathscr{D}[x]L \in \Gamma_p(L)$ be given and write the characteristic property of R as $PL = p^\nu R$, with R and P assumed primitive. Use Corollary B.2 to write $d_PP = \phi(\tilde{P})$ and $d_RR = \phi(\tilde{R})$, with d_P, $d_R \in S_u$. The least common denominator of $\tilde{P}d_L^{-1}$, viewed as an element of $\mathscr{E}(x)$, yields $g \in S_u$ such that $g\tilde{P} = Q'd_L$ for some $Q' \in \mathscr{E}[x]$. With $f = gd_Rd_P \in S_u$, and $X = gd_P\tilde{R}$, $Y = d_RQ' \in \mathscr{E}[x]$, we find $\phi(X) = gd_Pd_RR = fR$, $\phi(Y)d_L = \phi(d_RQ'd_L) = \phi(d_Rg\tilde{P}) = d_Rgd_PP = fP$. Multiplying $PL = p^\nu R$ by f and applying ψ gives $Y\tilde{L} = p^\nu X$, i.e., $X + \mathscr{E}[x]\tilde{L} \in \Gamma_p(\tilde{L})$. To conclude, it suffices to show that $R \in b\phi(X) + \mathscr{D}[x]L$ for some $b \in k[x]$. Now, by construction $fR = \phi(X)$ and since $p \nmid f \in S_u$ by Lemma 3.3, there exist b, $h \in k[x]$ such that $1 = bf + p^\nu h$, implying $R = b\phi(X) + hPL \in b\phi(X) + \mathscr{D}[x]L$. $\square$

The reduction is based on Appendix B and we use the notations d_L, ψ, ϕ from that section. We consider the original skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ and parallely a new skew-polynomial ring $\mathscr{E}[x] = k[x][\vartheta;\sigma]$ with $\delta = 0$. Both maps ψ and ϕ are injective morphisms respectively from $\mathscr{D}[x]$ into $\mathscr{E}(x)$ and from $\mathscr{E}[x]$ into $\mathscr{D}[x]$. With d_L the least common denominator of the coefficients of $\psi(L) \in \mathscr{E}(x)$, we have the formulae $\phi(\tilde{L}) = d_L L$, $\psi(L) = d_L^{-1}\tilde{L}$. We will require the following technical lemma stating the main property of the polynomial d_L with respect to a prime $p \in k[x]$ defining an infinite orbit: p divides $\sigma^j(d_L)$ for no L and no $j \geq 0$.

Lemma 3.3. *Let $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ be a skew-polynomial ring and let u and v be coprime in $k[x]$ satisfying $v\big(\sigma(x)-x\big) = u\delta(x)$. Assume that $p \in k[x]$ is a prime defining an infinite orbit. Then, p divides none of the $\sigma^j(u)$ for $j \geq 0$.*

Proof. We show that $p \mid \sigma^j(u)$ for some $j \geq 0$ leads to a contradiction. Applying σ^j to a Bézout relation for u and v yields that $\sigma^j(u)$ and $\sigma^j(v)$ are coprime as well. Then, applying σ^j to the hypothesis $v\,\big(\sigma(x)-x\big) = u\,\delta(x)$, we conclude that $p \mid \sigma^{j+1}(x) - \sigma^j(x)$. Let $(p_k)_{k\geq 0}$ be the orbit of $p = p_0$ according to the definition of Section 1.2.2. The previous relation means that $\sigma(x) - x$ is in the pullback of order j of p, or in other words that p_j divides $\sigma(x) - x$. With the help of Lemma 6.1 and Corollary 6.2, we conclude that p_j is a fixed point of σ and the orbit of p is finite. □

***Example* 3.4.** Let us consider the skew-polynomial ring $\mathscr{D}[x] = \mathbb{Q}[x][\partial;\sigma,\delta]$ defined by $\sigma(x) = x^2 - 2$, $\delta(x) = x - 1$. We apply the reduction to $\delta = 0$ for the skew polynomial

$$L = (x^2-49)\partial^2 - \frac{1}{20}\left(687x^4 + 540x^3 - 1648x^2 - 880x - 245\right)\partial + (x-47)$$

With the notations of the proof of Lemma B.1, and $w = x$, we have $u = \sigma(w) - w = (x+1)(x-2) \neq 0$, $v = \delta(w) = x - 1 \neq 0$ and ψ is defined by

$$\psi(x) = x, \qquad \psi(\partial) = \frac{1}{x^2-x-1}(\vartheta - x + 1).$$

The application of ψ to L gives $d_L = (x-1)(x+1)^2(x+2)(x-2)^2$ and

$$\begin{aligned}\tilde{L} = {}& (x^2-49)\vartheta^2 \\ & - x\left(687\,x^7 + 540\,x^6 - 5083\,x^5 - 3560\,x^4 + 10763\,x^3 + 5520\,x^2 - 6367\,x - 580\right)\vartheta \\ & \qquad + (x-1)\,(x+2)\left(x^4 - 4\,x^2 + 5\right)\left(687\,x^3 - 834\,x^2 - 647\,x - 1046\right)\end{aligned}$$

The skew polynomial $\tilde{L}$ has been considered in Example 1.10. Obviously the leading coefficient is the same for L and $\tilde{L}$, hence we anew have to consider the primes $x \pm 7$ in order to find the candidate primes. Theorem 1.8 provides as candidates for L the primes which divide some $\sigma^n(x \pm 7)$ with $n \in \mathbb{N}$. But with the reduction to $\delta = 0$ and the result of Example 1.10, it appears that a candidate prime is a factor of some $\sigma^m(x^4 - 4x^2 + 5)$ with $m \in \mathbb{N}$. See Figure 1.

3.2. **Mahlerian-like operators.** We deal here with skew-polynomial rings $\mathscr{D}[x] = k[x][\partial;\sigma]$ where σ is injective, but not bijective. The skew-polynomial ring $\mathscr{D}[x] = k[x][\partial, M]$ with $Mx = x^bM$, $b \geq 2$, is a typical example of such a situation. Note that, owing to Theorem 3.1, more complicated cases with $\delta \neq 0$ can be reduced to this one.

Section 1.7 encoded local torsion as the solution of the linear functional system (11). This reformulation involves general k-linear endomorphisms of the polynomial ring $k[\partial]$. But Lemma 1.11 provides a bound for the order d of the involved polynomials P in the equation $PL = p^\nu R$. Hence the search for the local torsion becomes a problem of matrix algebra. We prefer to see it as a problem of functional equation.

Il faut argumenter les avantages de ce point de vue (on exploite mieux la structure, ce qui mène à des algorithmes plus efficaces). Je veux bien le croire pour le *shift*, mais je ne pense pas que ce soit valable dans le cas général.

We make more explicit the idea of Section 1.7 of using linear operators on $k[\partial]$ and we begin with the computation of a product PL under the sole hypothesis $\delta = 0$. For $\alpha \in k$, let us define the family of coefficients $(c_{k,\ell,m})_{l,\ell,m\geq 0}$ by

$$\sigma^\ell((x-\alpha)^k) = \sum_{m\geq 0} c_{k,\ell,m}(x-\alpha)^m, \qquad k,\, \ell,\, m \geq 0 \tag{12}$$

and the family of formal power series $(H_{k,m})_{k,m\geq 0}$ by

$$H_{k,m} = \sum_{\ell \geq 0} c_{k,\ell,m}\partial^\ell, \qquad k, m \geq 0. \tag{13}$$

The Hadamard product $H_{k,m} \odot Q$ of $H_{k,m}$ and a polynomial $Q = \sum_\ell q_\ell \partial^\ell$ is the polynomial $\sum_\ell c_{k,\ell,m} q_\ell \partial^\ell$. Let us introduce the functional linear operators Λ_m defined through a Hadamard product using $L = \sum_j (x-\alpha)^j L_j(\partial)$

$$\Lambda_m Q = \sum_k L_k\, H_{k,m} \odot Q, \qquad m \geq 0,\, Q \in k[\partial]. \tag{14}$$

With these notations, a product PL may be expressed as follows.

Lemma 3.5. *Let $p = x - \alpha$ be a prime of degree 1, and*

$$P = \sum_i (x-\alpha)^i P_i(\partial), \qquad L = \sum_j (x-\alpha)^j L_j(\partial)$$

in $\mathscr{D}[x] = k[x][\partial;\sigma]$ with $\delta = 0$ and σ injective. Then the product PL writes

$$PL = \sum_j (x-\alpha)^j \sum_m \Lambda_m P_{j-m}(\partial).$$

Proof. The lemma is only a mere computation, namely

$$\begin{aligned} PL &= \sum_{i,k}(x-\alpha)^i P_i(\partial)(x-\alpha)^k L_k(\partial) = \sum_i (x-\alpha)^i \sum_{k,\ell} p_{i,\ell}\partial^\ell (x-\alpha)^k L_k(\partial) \\ &= \sum_i (x-\alpha)^i \sum_{k,\ell} p_{i,\ell}\,\sigma^\ell\big((x-\alpha)^k\big)\partial^\ell L_k(\partial) = \sum_{i,m}(x-\alpha)^{i+m}\sum_{k,\ell} p_{i,\ell}c_{k,\ell,m}\partial^\ell L_k(\partial) \\ &= \sum_j (x-\alpha)^j \sum_{k,m} L_k(\partial)\sum_\ell p_{j-m,\ell}c_{k,\ell,m}\partial^m = \sum_j (x-\alpha)^j \sum_m \Lambda_m P_{j-m}(\partial). \end{aligned}$$

□

The system of functional equations (11) of Section 1.7 translates into the following assertion.

Proposition 3.6. *Let $\mathscr{D}[x] = k[x][\partial;\sigma]$ be a skew-polynomial ring with σ injective and $\delta = 0$, and $\alpha \in k$. Consider*

$$P = \sum_{j=0}^{\nu-1}(x-\alpha)^j P_j(\partial), \qquad L = \sum_{j=0}^{\mu}(x-\alpha)^j L_j(\partial), \tag{15}$$

for $P_j, L_j \in k[\partial]$. Then, the skew polynomial P defines a torsion element in $\ker_{(x-\alpha)^\nu} L$ if and only if the polynomials $P_0, \ldots, P_{\nu-1}$ satisfy the following triangular system of functional equations in the independent variable ∂:

$$\begin{cases} \Lambda_0 P_0 = 0, \\ \Lambda_1 P_0 + \Lambda_0 P_1 = 0, \\ \vdots \\ \Lambda_{\nu-1}P_0 + \cdots + \Lambda_0 P_{\nu-1} = 0, \end{cases} \tag{16}$$

where the Λ_m are the linear operators defined on $k[\partial]$ by Eq. (14).

For a general σ the way to solve such a system is the method of indeterminate coefficients. This process is not well efficient, but we will see in the next section that for certain σ there exist efficient algorithms.

***Example* 3.7.** Let us consider

$$L = (x-16)^3\partial^2 - 49(x-2)^2(x-4) \in \mathscr{D}[x] = \mathbb{Q}[x][\partial; M]$$

with $Mx = x^2M$ and the prime $p = x-2$. It divides $\sigma^2(\ell_2)$ where $\ell_2 = (x-16)^3$ is the leading coefficient of L, and both ℓ_0, and $\sigma(\ell_0)$ where $\ell_0 = -49(x-2)^2(x-4)$ is the trailing coefficient of L. So it is a candidate prime, according to Lemma 1.8. Moreover Corollary 1.15 provides the bound $\nu \le 2$ for its multiplicity and Lemma 1.13 gives a bound $d \le 2$ for the order of a $P \in \ker_L p^\nu$ not divisible by p. With the notations of Lemma 3.5, we have

$$L_0 = -2744\,\partial^2, \quad L_1 = 588\,\partial^2, \quad L_2 = -42\,\partial^2 + 98, \quad L_3 = \partial^2 - 49, \qquad L_k = 0 \quad \text{for } k \ge 4.$$

We need the operators Λ_0 and Λ_1, hence the polynomials $H_{k,m}$ for $0 \le k \le 3$, $0 \le m \le 1$. They are given in the following table.

$H_{k,m}$	$m = 0$	$m = 1$
$k = 0$	$1 + \partial + \partial^2 + \partial^3$	0
$k = 1$	$2\,\partial + 14\,\partial^2 + 254\,\partial^3$	$1 + 4\,\partial + 32\,\partial^2 + 1024\,\partial^3$
$k = 2$	$4\,\partial + 196\,\partial^2 + 64516\,\partial^3$	$16\,\partial + 896\,\partial^2 + 520192\,\partial^3$
$k = 3$	$8\,\partial + 2744\,\partial^2 + 16387064\,\partial^3$	$48\,\partial + 18816\,\partial^2 + 198193152\,\partial^3$

In the last row the coefficients 2744 et 18816 come from the expansion

$$\sigma^2((x-2)^3) = (x^4-2)^3 = 2744 + 18816(x-2) + O((x-2)^2).$$

The equation $\Lambda_0 P_0 = 0$ writes

$$-2744\,\partial^2 H_{0,0} \odot P_0 + 588\,\partial^2 H_{1,0} \odot P_0 - (42\,\partial^2 - 98) H_{2,0} \odot P_0 + (\partial^2 - 49) H_{3,0} \odot P_0 = 0.$$

With $P_0 = p_{0,0} + p_{0,1}\partial + p_{0,2}\partial^2$, this amounts to a little linear system which solves into $P_0 = p_{0,2}(\partial^2 - 42)$. In other words the kernel of Λ_0 is generated by $\partial^2 - 42$. With $P^0 = P_0$, we find $P^0 L = (x-2) R^0$ where

$$\begin{aligned} R^0 = {} & (x-2)^2\,(x+2)^3\,\left(x^2+4\right)^3\,\partial^4 \\ & - \big(49\,x^{11} + 98\,x^{10} + 196\,x^9 + 392\,x^8 + 392\,x^7 + 784\,x^6 \\ & \quad + 1568\,x^5 + 3136\,x^4 + 7252\,x^3 + 14546\,x^2 + 27076\,x + 86408\big)\,\partial^2 \\ & + 2058\,(x-2)\,(x-4) \end{aligned}$$

In the same manner we deal with the equations $\Lambda_0 P_0 = 0$ and $\Lambda_0 P_1 + \Lambda_1 P_0 = 0$. We still have $P_0 = p_{0,2}(\partial^2 - 42)$ and we find $P_1 = p_{1,2}(\partial^2 - 42) - 313 p_{0,2}$. In other words $\Lambda_1(\partial^2 - 42)$ is in the image of Λ_0, a particular solution of the equation $\Lambda_0 P_1 = -\Lambda_1(\partial^2 - 42)$ is -313, and the kernel of Λ_0 is still the line $k(\partial^2 - 42)$. This gives the following skew polynomial

$$P^1 = P_0 + (x-2) P_1 = p_{0,2}\left(\partial^2 - 313\,x + 584\right) + p_{1,2}\,(x-2)\left(\partial^2 - 42\right),$$

from which we deduce $P^1 L = (x-2)^2 R^1$ and $R^1 = p_{0,2} R^1_0 + p_{1,2} R^0$, with

$$\begin{aligned} R^1_0 = {} & (x-2)\,(x+2)^3\,\left(x^2+4\right)^3\,\partial^4 \\ & - \big(49\,x^{10} + 196\,x^9 + 588\,x^8 + 1568\,x^7 + 3528\,x^6 + 7840\,x^5 \\ & \quad + 17248\,x^4 + 37632\,x^3 + 82829\,x^2 + 165180\,x + 597820\big)\,\partial^2 \\ & + 49\,(x-4)\,(313\,x - 584)\,. \end{aligned}$$

Finally the local pseudo-torsion $\Gamma_{x-2}(L)$ is generated in the $k[x]$-module $\mathscr{D}[x]/\mathscr{D}[x]L$ by the classes $R + \mathscr{D}[x]L$ of the skew polynomials R respectively equal to R^0 and R^1_0.

Je continue sur cet exemple dans l'espoir de donner un sens à ce calcul. En même temps je constate que la notion de singularité est mal définie : la définition est correcte mais ne donne pas ce que nous pensons. À dire vrai, je ne sais plus trop ce que je pense.

Je considère le polynôme tordu

$$L = (x-16)^3 \, DD^2 - 49 \, (x-2)^2 \, (x-4) \, .$$

Il définit une récurrence sur les valeurs d'une fonction de variable complexe y en les points d'une orbite sous l'action de la substitution $x \mapsto x^2$, qui s'écrit

$$(x-16)^3 \, y\left(x^4\right) - 49 \, (x-2)^2 \, (x-4) \, y\,(x) = 0.$$

Je déroule la récurrence en regardant ce qui se passe sur l'orbite de 2. Précisément je substitue successivement $x = 2, 4, 16, \ldots$ dans la relation et cela me donne

$$\begin{aligned}
2 &: -2744 \, y\,(16) = 0 \\
4 &: -1728 \, y\,(256) = 0, \\
16 &: -115248 \, y\,(16) = 0, \\
256 &: -796643568 \, y\,(256) + 13824000 \, y\,(4294967296) = 0, \\
65536 &: -1379059029586660 8 \, y\,(65536) \\
&\qquad + 281268868608000 \, y\,(18446744073709551616) = 0, \\
4294967296 &: -388217995596782886939857728228 8 \, y\,(4294967296) \\
&\qquad + 79228161628820625354020352000 \, y\,(3402823669209384634633746074317682 \\
18446744073709551616 &: -30757798503394735729456299390416777672945037985771991282404 8 \, y\,(184467 \\
&\qquad + 627710173538668074750223581100262018402747373634809600000 0 \, y\,(11579
\end{aligned}$$

Je peux regarder ce calcul de deux manières. Ou bien je considère que la récurrence s'écrit

$$(x_n - 16)^3 \, y\,(x_{n+2}) - 49 \, (x_n - 2)^2 \, (x_n - 4) \, y\,(x_n) = 0$$

pour $n \geq 0$ avec $x_0 = 2$ et $x_{n+1} = x_n^2$ pour tout n. Dans ce cas les valeurs $y(x_0)$ et $y(x_1)$ sont arbitraires, toutes les suivantes de rang impair sont nulles, toutes les suivantes de rang pair au moins égal à 4 sont déterminées par $y(65536)$ et l'espace des solutions sur l'orbite est de dimension 3, ce qui fait que 2 ou encore $x-2$ n'est pas singulier, enfin peut-être car la dimension n'est pas exactement 2. Ou bien je vois la récurrence comme

$$(x_{n-2} - 16)^3 \, y\,(x_n) - 49 \, (x_{n-2} - 2)^2 \, (x_{n-2} - 4) \, y\,(x_{n-2}) = 0$$

pour $n \geq 0$ avec $x_0 = 16$ et $x_{n+1} = x_n^2$ pour tout n. Dans ces conditions l'espace des solutions est de dimension 1, puisque j'ai $y(16) = y(256) = 0$, tous le termes suivants de rang impair nuls et tous les termes de rang pair au moins égal à 2 dépendent de la valeur de $y(65536)$. Alors $x - 16$ est singulier. Je note aussi que si je reprends la première interprétation en regardant l'orbite de 16 et en opérant les substitutions avec $x = 16$, $x = 256$, ... je trouve $y(16) = 0$ puis je vois que les valeurs sur l'orbite sont déterminées par les valeurs de $y(256)$ et $y(65536)$. L'espace des solutions est donc de dimension 2 et $x - 16$ n'est pas singulier.

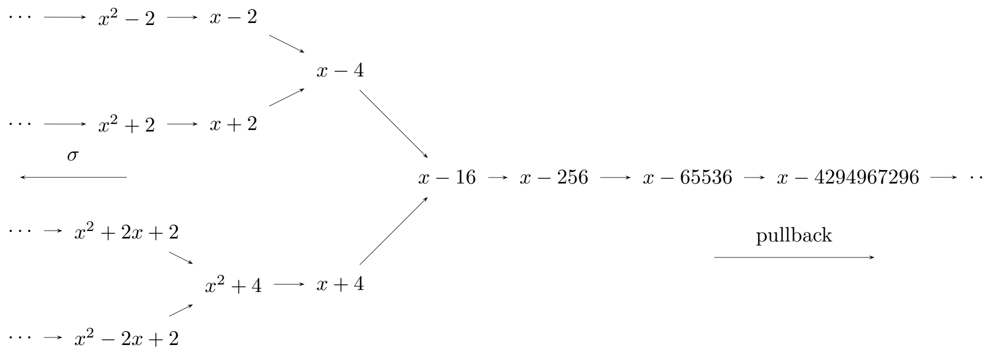

FIGURE 2. The effect of the substitution σ defined by $\sigma(x) = x^2$ and its pullback on the polynomial $x - 16$, the ground field being $k = \mathbb{Q}$.

Appliquons la définition que nous avons donnée. Un polynôme premier p est singulier si $\sigma^m(p)$, avec $m \geq 2$, et $\ell_2 = (x-16)^3$ ne sont pas étrangers. Autrement dit un tel p est à droite de $x-16$ dans le graphe du pullback et à une distance au moins 2 de $x-16$. Le premier qui nous tombe sous la main est $x - 2^{16}$, c'est-à-dire le nombre 65536. Dans cette vision le nombre 16 n'est pas une singularité. Cependant si nous appliquons la première interprétation à l'orbite de 2^{16}, le calcul démarre comme suit

$$\begin{aligned} 65536 :& -13790590295866608\, y\,(65536) \\ &\quad + 281268868608000\, y\,(18446744073709551616) = 0, \\ 4294967296 :& -38821799559678288693985772282288\, y\,(4294967296) \\ &\quad + 79228161628820625354020352000\, y\,(340282366920938463463374607431768211456) = 0 \end{aligned}$$

et l'espace des solutions sur l'orbite de 2^{16} est de dimension 2, une solution étant déterminée par ses valeurs en 2^{16} et 2^{32}. Si nous utilisons la deuxième interprétation, nous démarrons le calcul avec $x = 16$ et nous trouvons encore un espace de dimension 2. Ceci fait que 2^{16} n'est pas singulier et nous pataugeons dans l'incohérence. Cependant il ne faut pas oublier que nous avons fait le choix suivant : dès qu'un point est singulier, tous les points de son orbite sont singuliers. L'argument est que nous ne voulons pas voir surgir des singularités en des points qui étaient considérés comme ordinaires.

Je n'y comprends rien mais cela ne m'arrête pas. Je reste avec la définition qui a été donnée d'une singularité. Je veux comprendre ce qu'apporte le calcul de la pseudo-torsion locale $\Gamma_{x-2}(L)$.

La figure 2 montre l'effet de la substitution σ définie par $\sigma(x) = x^2$ et du *pullback* associé sur le polynôme $x-16$. La figure 3 fournit un support à la discussion suivante sur les singularités et les candidats premiers liés au polynôme tordu L. Le schéma du haut correspond au polynôme L lui-même. Le premier candidat premier qui se présente est $x-4$.

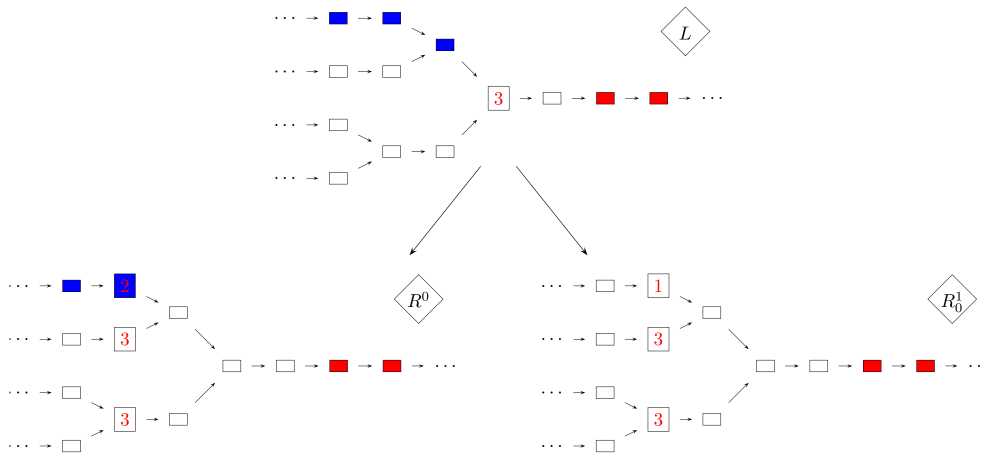

FIGURE 3. This figure is based on Figure 2, with the polynomials shrinked into boxes. It shows for three skew polynomials L, R^0, and R_0^1 the singularities (red boxes) and the candidate primes (blue boxes). Moreover the boxes with a number show the prime factors of the leading coefficient and their multiplicities for each skew polynomials. The skew polynomials R^0 and R_0^1 provide a basis of the local pseudo-torsion of L at $p = x - 2$.

Cependant le calcul montre que $\Gamma_{x-4}(L)$ est réduit à zéro. Le calcul de $\Gamma_{x-2}(L)$, que nous avons mené plus haut, est plus fructueux. Il nous fournit deux polynômes tordus R^0 et R_0^1, pour lesquels les singularités sont les mêmes que pour L. Cependant les candidats premiers ne sont pas les mêmes. Pour R^0, les candidats premiers sont $x-2$, x^2-2, ... Le calcul montre que $\Gamma_{x-2}(R^0)$ est réduit à zéro. Il faudrait poursuivre le calcul des pseudo-torsions locales en les $x^{2^k}-2$ pour en savoir plus. Pour R_0^1, il n'y a pas de candidat premier.

Nous voyons aussi que les multiplicités ne sont pas les mêmes. Insistons sur la différence qu'il y a entre L et R_0^1. Nous avons repoussé vers la gauche les facteurs premiers du terme de tête et nous avons légèrement baissé les multiplicités. Imaginons que nous ayons une fonction y analytique de variable complexe définie au voisinage de l'orbite de 2 mais peut-être pas en les points de l'orbite et que cette fonction soit solution de l'équation fonctionnelle associée à L,

$$(x-16)^3\, y\left(x^4\right) - 49\,(x-2)^2\,(x-4)\, y(x) = 0.$$

Ajoutons l'hypothèse que y est analytique au voisinage de 2 et 16. Au voisinage de l'orbite nous pouvons définir une fonction racine quatrième $z \mapsto z^{1/4}$ qui est analytique et qui envoie les réels positifs sur les réels positifs. L'équation fonctionnelle fournit pour z voisin de 65536 avec $x = z^{1/4}$

$$\begin{aligned} y(z) = {} & \frac{y(x)}{(z-65536)^3}\times \\ & \bigl(49\,x^{12} + 1960\,x^{11} + 57428\,x^{10} + 1451184\,x^9 + 23946496\,x^8 + 347217920\,x^7 \\ & \quad + 4026925056\,x^6 + 24868028416\,x^5 + 154551713792\,x^4 \\ & - 236760072192\,x^3 - 15573551415296\,x^2 + 57243324121088\,x - 53876069761024\bigr)\,. \end{aligned}$$

Nous constatons que y est méromorphe au voisinage de 65536 avec un pôle d'ordre au plus 3.

Utilisons maintenant le polynôme tordu R^0. Il nous fournit une équation fonctionnelle

$$y(x^{16}) = \frac{P_{11}(x)}{(x-2)^2(x+2)^3(x^2+4)^3}y(x^4) - 2058\frac{x-4}{(x-2)(x+2)^3(x^2+4)^3}y(x)$$

dans laquelle P_{11} est un polynôme de degré 11 (et la fraction dont il est le numérateur est irréductible). Nous utilisons une branche de la fonction racine seizième $z \mapsto z^{1/16}$ qui envoie $(0,+\infty)$ sur lui-même et est analytique au voisinage de cette demi-droite. En écrivant

$$\begin{aligned} \frac{P_{11}(x)}{(x-2)^2(x+2)^3(x^2+4)^3} &= \frac{1}{(x^4-16)^2}\frac{P_{11}(x)}{(x+2)(x^2+4)} \\ &= \frac{1}{(x^{16}-65536)^2}\frac{P_{11}(x)(x^{12}+16x^8+256x^4+4096)^2}{(x+2)(x^2+4)} \end{aligned}$$

et en procédant de manière similaire pour le coefficient de $y(x)$ dans l'équation fonctionnelle, nous arrivons pour z voisin de 65536 et avec $x = z^{1/16}$ à

$$y(z) = \frac{1}{(z-65536)^2}Q_1(x)y(x^4) + \frac{1}{z-65536}Q_0(x)y(x)$$

avec $Q_1(x)$ et $Q_0(x)$ des fonctions rationnelles en $x = z^{1/16}$ qui sont analytiques au voisinage de $z = 65536$. Du coup avec l'hypothèse que y est analytique au voisinage de 2 et 16, nous concluons qu'elle est méromorphe au voisinage de 65536 avec en ce point un pôle d'ordre au plus 2.

Nous pouvons procéder de la même manière avec R^1_0, qui fournit une équation fonctionnelle

$$y(x^{16}) = \frac{P_{10}(x)}{(x-2)(x+2)^3(x^2+4)^3}y(x^4) - 49\frac{(x-4)(313x-584)}{(x-2)(x+2)^3(x^2+4)^3}y(x)$$

et donc

$$y(z) = \frac{1}{z-65536}Q_1(x)y(x^4) + \frac{1}{z-65536}Q_0(x)y(x)$$

avec cette fois-ci encore $Q_1(x)$ et $Q_0(x)$ rationnels en $x = z^{1/16}$ et analytiques au voisinage de $z = 65536$. Nous voyons que y est méromorphe au voisinage de 65536 avec en ce point un pôle d'ordre au plus 1.

Je suis un peu épaté que ce soit l'exposant le plus petit qui gagne. Cela tient au fait que l'on fabrique un $\tilde{\sigma}^{-1}$ qui pointe vers le pôle de plus petite multiplicité.

Un lecteur non aveugle remarque que nous avons déjà obtenu $y(16) = 0$ et en dérivant sans finesse le membre gauche de l'équation fonctionnelle fournie par L nous trouvons aussi $y'(16) = 0$. Cela fait qu'en utilisant seulement l'équation fonctionnelle associée à L nous pouvons conclure que le pôle en 65536 est d'ordre au plus 1. Notre artillerie paraît donc bien lourde pour ce petit problème de singularité. Nous répondons que premièrement il est évident que tout vient du polynôme tordu L et que deuxièmement nous ne recherchons pas des astuces mais des méthodes.

L'impression que je retire de cette histoire est que pour définir la notion de singularité nous ne voulons pas considérer les valeurs d'une *solution* sur l'orbite d'un point mais des germes de fonctions méromorphes définis au voisinage des points de l'orbite.

Il faut un algorithme.

3.3. **Distinguished candidate primes.** The idea of candidate primes has reduced the set of primes under consideration to a denumerable set, but for an algorithmic approach it is necessary to reduce anew this set $\mathcal{C}$ to a finite set of distinguished candidate primes $\mathcal{DC}$. With this goal in mind, we consider the local pseudo-torsion spaces along the orbits of the pullback.

Theorem 3.8. *Let L be a non-zero skew polynomial in $\mathscr{D}[x] = k[x][\partial; \sigma, \delta]$ with σ injective. Let $(p_j)_{j\in\mathbb{Z}}$ be a doubly-infinite orbit of the pullback. The sequence $(\Gamma_{p_j}(L))_{j\in\mathbb{Z}}$ is non increasing, eventually null to the positive infinity, and stationnary in size at the negative infinity.*

Proof. According to the definition of candidate primes, the local pseudo-torsion $\Gamma_p(L)$ is not reduced to zero only if the prime p divides some $\sigma^n(\ell_r)$ for some $n \geq 0$, where ℓ_r is as usual the leading coefficient of L. But the set of prime factors of ℓ_r is finite and the prime factors of the $\sigma^n(\ell_r)$ with $n \geq 0$ either do not lie on the doubly-infinite orbit of the pullback under consideration or are distributed on a semi-orbit bounded on the right. (We see the pullback as pushing the primes on the righthand side, and σ giving prime factors on the lefthand side. See Figure 1 in Section 3.1, or Figure 2 in Section 3.2.) As a consequence $\Gamma_{p_j}(L) = \mathscr{D}[x]L/\mathscr{D}[x]L = \{0\}$ for j positive and sufficiently large.

As long as we consider an infinite orbit, we may assume $\delta = 0$. More precisely, according to Theorem 3.1, the local pseudo-torsion spaces associated to the same prime in the process of reduction to $\delta = 0$ are isomorphic. Let q be a prime on the orbit and p be its image by the pullback. This means that we have $\sigma(p) = qg$ for some $g \in k[x]$. If a skew polynomial R is in $\ker_L p^i$, we may write $PL = p^i R$ for some skew polynomial P. Multiplying by ∂ on the left, we obtain $\partial PL = q^i g^i \partial R$, so that $g^i\partial R$ is in $\ker_L q^i$. The map $R \mapsto g^i\partial R$ induces an injective map from $\ker_L p^i$ into $\ker_L q^i$.

The elements of $\ker_{p^i} L$ are represented by skew polynomials $P = P_0(\partial) + pP_1(\partial) + \cdots + p^{i-1}P_{i-1}(\partial)$. According to Corollary 1.15, the degrees d and valuations d' of all the polynomials P_0, …, P_{i-1} are bounded by $m' \leq d' \leq d \leq m$, where m and m' are defined from the valuation with respect to p of the images by σ of the leading and trailing coefficients ℓ_r and ℓ_s of L. This provides a bound for the dimension of the k-vector space $\ker_{p^i} L$. Let us assume that there is no factor of ℓ_r or ℓ_s on the left of p. When we go from p to q the only change comes from the fact that m and m' are shifted by 1, but the difference remains the same and the bound on the dimension too. The same bound is valid for $\ker_L p^i$ and $\ker_L q^i$. We have seen that $\ker_L q^i$ is not smaller in size than $\ker_L p^i$ for we have an injective map from $\ker_L p^i$ into $\ker_L q^i$. Because of the obtained bound, when we follow backward the pullback the size of the kernels $\ker_L p^i$ cannot grow indefinitely and ultimately the size is constant. As a result the injective map $R + \mathscr{D}[x]L \mapsto g^i\partial R + \mathscr{D}[x]L$ from $\ker_L p^i$ into $\ker_L q^i$ becomes onto.

Corollary 1.15 gives an absolute bound for the multiplicity, roughly the sum of the exponents of the prime factors of ℓ_r which are on the doubly-infinite orbit under consideration. With the previous bound, this gives a bound on the dimension of the $\Gamma_{p_j}(L)$, $j \in \mathbb{Z}$, as k-vector spaces.

Je ne suis pas fier de la rédaction de cette preuve et mon style est bien obscur, indépendamment du niveau de l'anglais. Cependant je pense que l'argument est correct. Mes chers coauteurs vont devoir jeter de la lumière.

Y a-t-il moyen de prouver que dès que l'application $R \mapsto g^i\partial R$ envoie $\ker_L p^i$ sur $\ker_L q^i$, alors cela est vrai pour tous les suivants ? Je ne vois pas comment le résultat obtenu va se traduire sur le plan algorithmique.

□

Corollary 3.9. *With the notations of the previous Theorem 3.8, for j sufficiently negative and for all $i \geq 0$, the kernel $\ker_L p_j^i$ is obtained from $\ker_L p_{j+1}^i$ by the injective map $R + \mathscr{D}[x]L \mapsto g^i\partial R + \mathscr{D}[x]Lif$ $\sigma(p_{j+1})$ writes $\sigma(p_{j+1}) = p_j g$.*

***Example* 3.10.** Let us consider in $\mathscr{D}[x] = k[x][\partial;\sigma]$, with $\sigma(x) = x^2 - 2$ and $\delta = 0$, the skew polynomial

$$L = (x-47)\partial^2 + 160(x-2)\partial - 90(x-7)^2.$$

The candidat primes are the prime factors of the $\sigma^m(x-7)$, $m \geq 0$. Applying the method of Section 3.2, we find $\Gamma_{x-7}(L) = \mathscr{D}[x]L + k[x]R_0$, where R_0 is obtained from $P_0L = (x-7)R_0$

with $P_0 = \partial + 180$. More explicitely we have

$$
\begin{aligned}
(\partial + 180)L = (x-7) \times &\\
&\big((x+7)\,\partial^3 + (160\,x + 1300)\,\partial^2 \\
&\quad - \big(90\,x^3 + 630\,x^2 + 2790\,x - 9270\big)\,\partial - (16200\,x - 113400)\big)\,.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\partial(\partial + 180)L = (x-3)(x+3) \times &\\
&\big(\big(x^2+5\big)\,\partial^3 + \big(160\,x^2 + 980\big)\,\partial^2 \\
&- \big(90\,x^6 + 90\,x^4 + 1350\,x^2 - 13050\big)\,\partial - 16200\,(x-3)\,(x+3)\big) \times \partial,
\end{aligned}
$$

$$
\begin{aligned}
\partial^2(\partial + 180)L = (x^2-5)(x^2+1) \times &\\
&\big(\big(x^4 - 4\,x^2 + 9\big)\,\partial^3 + \big(160\,x^4 - 640\,x^2 + 1620\big)\,\partial^2 \\
&- \big(90\,x^{12} - 1080\,x^{10} + 5490\,x^8 - 15120\,x^6 + 25110\,x^4 - 25560\,x^2 - 450\big)\,\partial \\
&\qquad - 16200\,\big(x^2-5\big)\,\big(x^2+1\big)\,\partial^2\big) \times \partial^2.
\end{aligned}
$$

It may be shown that $\Gamma_{x-3}(L)$, $\Gamma_{x+3}(L)$, $\Gamma_{x^2-5}(L)$, $\Gamma_{x^2+1}(L)$ have essentially no other generator. (Figure 1 illustrates the relation between these prime polynomials.) In other words, we have $\Gamma_{x-3}(L) = \mathscr{D}[x]L + k[x]R_1^+$ with $R_1^+ = (x+3)\partial R_0$, $\Gamma_{x+3}(L) = \mathscr{D}[x]L + k[x]R_1^-$ with $R_1^- = (x-3)\partial R_0$, $\Gamma_{x^2-5}(L) = \mathscr{D}[x]L + k[x]R_2^+$ with $R_2^+ = (x^2+1)\partial R_1^+$, and so on.

À ce point, il faudrait définir la notion de candidat distingué. Il manque clairement un ingrédient, à savoir que le graphe de la suite des dimensions $\dim_k \Gamma_{p_j}(L)$, $j \in \mathbb{Z}$, ne comporte pas de palier entre les deux plateaux infinis, mais seulement une descente stricte. Même dans le cas du *shift*, évoqué à la section 4.5, cela n'est pas réellement traité.

4. Singularities along infinite orbits with σ invertible

We continue with the shift operator (Section 4.1) to give a flavour of what happens in general for invertible σ. This is followed by comments on how to immediately modify the algorithm to handle the q-shift case (Section 4.2). As shown by this both examples, in many cases σ is actually an invertible function, and this is a property that we exploit. However, to work in an algebraic setting with σ and its inverse, we must take care to preserve key features of the skew-polynomial ring environment, which cannot readily accomodate a multiplicative inverse of ∂. The approach we take here is to use a new kind of ring, Laurent-Ore algebras, presented in Appendix C. The definition of closure extends to this setting (Section 4.3) and allows us to find torsion spaces which generate the full (local) torsion over the Laurent-Ore algebra (Section 4.4).

PhD Je ne comprends pas cette dernière phrase.

One advantage is increased flexibility when choosing distinguished candidate primes. We bring this section to a close by returning to the skew-polynomial context (Section 4.6) and show how to compute maximal local torsion modules.

PhD L'adjectif *maximal* tombe du ciel. Il est sûrement lié à la notion de premier distingué.

4.1. **Shift operators.** For the rest of this section, we deal only with an invertible σ and infinite orbits. Our goal is to show how this hypothesis greatly simplifies the computations. For the sake of an explicit presentation, we first present the typical case of shift operators. The next, shorter Section 4.2 then deals with q-dilation operators in a formally parallel way.

4.1.1. *Linear differential system and local torsion.* As in the previous Section 3.2 about Mahlerian operators, we use the formalism of Section 1.7 to encode the search for the local torsion into a linear system whose unknowns are polynomials in $k[\partial]$. Here, we make this idea explicit on the example of shift operators. This setting turns out to have a straightforward expression of the linear system adapted to solving for torsion, in terms of derivatives with respect to the indeterminate ∂.

Consider the skew-polynomial ring $\mathscr{D}[x] = k[x][\partial; S]$, where $\partial x = (x+1)\partial$, that is, S is the shift operator on $k[x]$, satisfying $Sf(x) = f(x+1)$. The next proposition considers the simpler case of prime polynomials of degree 1. The general situation is postponed to Propositions 4.4 and 4.6 in Section 4.1.3. Recall that for a polynomial $f \in k[x]$, according to Section 1.3.2, $\ker_f L$ is the set of $P + f\mathscr{D}[x]$ such that there exists $R \in \mathscr{D}[x]$ satisfying $PL = fR$.

Proposition 4.1. *Let $\mathscr{D}[x] = k[x][\partial; S]$, $\alpha \in k$, and consider*

$$P = \sum_{j=0}^{\nu-1} (x-\alpha)^j P_j(\partial), \qquad L = \sum_{j=0}^{\mu} (x-\alpha)^j L_j(\partial), \tag{17}$$

for $P_j, L_j \in k[\partial]$. Define a k-linear endomorphism ϑ of $\mathscr{D}[x]$ by extending the Eulerian operator $\partial\,(d/d\partial)$ from $k[\partial]$ to $\mathscr{D}[x]$ by $\vartheta x = x\vartheta$. Then, the skew polynomial P corresponds to a torsion element in $\ker_{(x-\alpha)^\nu} L$ if and only if the polynomials $P_0, \ldots, P_{\nu-1}$ satisfy the following triangular system of differential equations in the independent variable ∂:

$$\begin{cases} \Lambda_0 P_0 = 0, \\ \Lambda_1 P_0 + \Lambda_0 P_1 = 0, \\ \vdots \\ \Lambda_{\nu-1} P_0 + \cdots + \Lambda_0 P_{\nu-1} = 0, \end{cases} \qquad \text{where} \quad \Lambda_m = \sum_{j=m}^{\mu} \binom{j}{m} L_j(\partial)\vartheta^{j-m} \quad \text{for } 0 \le m < \nu. \tag{18}$$

Proof. We apply Proposition 3.6, and we compute

$$(x+\ell-\alpha)^k = \sum_m \binom{k}{m}(x-\alpha)^m \ell^{k-m}$$

hence

$$c_{k,\ell,m} = \binom{k}{m}\ell^{k-m}, \qquad \text{and} \qquad H_{k,m} = \binom{k}{m}\sum_\ell \ell^{k-m}\partial^\ell.$$

We immediatly see the formula

$$H_{k,m} \odot Q = \binom{k}{m}\vartheta^{k-m} Q \qquad \text{for } Q \in k[\partial],$$

and we conclude readily. The use of the Eulerian operator ϑ is natural in the context in view of the relation

$$\partial^j x = (x+j)\partial^j = (x+\vartheta)\partial^j \qquad \text{if } j \geq 0. \tag{19}$$

This leads to a more direct proof. The commutation $Ax^i = (x+\vartheta)^i A$ follows for all $A \in k[\partial]$. Now, for P and L as in (17), the product PL rewrites

$$PL = \sum_{\substack{0\leq i<\nu \\ 0\leq j\leq\mu}} (x-\alpha)^i \left((x-\alpha+\vartheta)^j P_i\right) L_j = \sum_{\substack{0\leq i<\nu \\ 0\leq m\leq j\leq\mu}} (x-\alpha)^{i+m}\binom{j}{m}\left(\vartheta^{j-m}P_i\right) L_j.$$

Thus, $PL \in (x-\alpha)^\nu \mathscr{D}[x]$ if and only if for all ℓ such that $0 \leq \ell < \nu$,

$$\sum_{m=0}^{\ell}\left(\sum_{j=m}^{\mu}\binom{j}{m}L_j\vartheta^{j-m}\right)P_{\ell-m} = 0,$$

which is equivalent to (18). □

Compare the role of a dual of L played by $\Lambda_0 = \sum_{j=0}^{\mu} L_j\vartheta^j$ with the explicit dualization of Section 2.3. Be cautious that Λ_0 is not obtained by only substituting ϑ for x in L: the order of coefficients and indeterminates has to be reversed in addition to this substitution, because, even if the action is a right action, maps are written on the left of their arguments.

4.1.2. *Algorithm for local torsion.*

Il faut parler d'efficacité.

System (18) can be solved iteratively, by successively considering and solving nonhomogeneous equations $\Lambda_0 P_i = Q_i$ for parametrized polynomials Q_i. More explicitly, once a basis $(S_1,\dots,S_d)$ of the nullspace $\mathcal{S}_0$ of Λ_0 is determined, P_0 can be expressed as a linear combination $\lambda_1 S_1 + \dots + \lambda_d S_d$. Then, solving for P_1 amounts, first, to determining for which values of the parameters λ_i a solution P_1 of $\Lambda_0 P_1 = -\Lambda_1(\lambda_1 S_1 + \dots + \lambda_d S_d)$ exists. The implied constraint on the λ_i is linear and determines a subspace $\mathcal{S}_1$ of $\mathcal{S}_0$, for which one determines a basis. Then, for each basis element, a particular solution P_1 is computed.

The general solution P_1 is obtained by adding a generic element of $\mathcal{S}_0$. The process is repeated in an analogous way for the next equations.

An important note is that the systems (18) obtained for increasing ν extend one another. Consequently, all torsion spaces $\ker_{x-\alpha} L$, $\ker_{(x-\alpha)^2} L$, ..., $\ker_{(x-\alpha)^\nu} L$ are computed during the solving process just described: the iteration step of the calculation consists in restricting the torsion space $\ker_{(x-\alpha)^i} L$ to only those elements that can be refined to elements of $\ker_{(x-\alpha)^{i+1}} L$.

Thus, finding the Ore closure reduces to finding polynomial solutions of parametrized differential equations. Effective algorithms originally developped to solve the creative telescoping problem were based on [2] to do precisely this. See an explicit description in [3] or the direct refinement of [2] used in [9]. We collect the considerations above into Algorithm 4.2, before giving an example.

PhD Il faut écrire cet algorithme.

Algorithm 4.2 (Local pseudo-torsion for the shift)**.**

Input: L, a skew polynomial in $k[x][\partial; S]$ and a candidate prime $p = x - \alpha$
Output: A set $\mathcal{R}$ of generators for the local pseudo-torsion $\Gamma_p(L)$

(1) Determine a bound for the multiplicity ν according to Corollary 1.15
(2) Write $L = \sum_{j=0}^{\mu} (x-\alpha)^j L_j(\partial)$
(3) Solve the functional equation $\Lambda_0 P_0 = 0$ with unknown $P_0 \in k[\partial]$ with
$$\Lambda_0 = \sum_{j=0}^{\mu} L_j(\partial)\vartheta^j.$$
This defines a space of solutions $\mathcal{S}_0$.
(4) For m from 1 to $\nu - 1$, solve iteratively the functional equation
$$\Lambda_m P_0 + \cdots + \Lambda_0 P_m = 0$$
for the unknowm $P_m \in k[\partial]$ with
$$\Lambda_m = \sum_{j=m}^{\mu} \binom{j}{m} L_j(\partial)\vartheta^{j-m} \qquad \text{for } 0 \le m < \nu.$$
This produces

***Example* 4.3.** Let us study the torsion related to the skew polynomial
$$L = (x-3)(x-2)^3\partial - x^3(x-1) \in \mathscr{D}[x] = k[x][\partial; S].$$
By the discussion in Section 1.4, the dispersion $\operatorname{disp}(\ell_1, \ell_0) = \{1, 2, 3\}$ and the set $\mathcal{C}$ of candidate primes is given by primes in the orbits under S of those primes which are a

common factor of $\ell_1 = (x-3)(x-2)^3$ and some $\sigma^i(\ell_0) = (x+i)^3(x+i-1)$ for $i \in \operatorname{disp}(\ell_1, \ell_0)$. In other words the set of candidate primes is $\mathcal{C} = \{x-3, x-2\} + \mathbb{N} = x - 3 + \mathbb{N}$.

Here, we choose to study $p = x$. Results for other primes in $\mathcal{C}$ would be obtained analogously and will be given in Example 4.15. To proceed, we look for systems of generators of the $\ker_L x^i$ as $k[x]$-modules. On the other hand, since x^i annihilates $\ker_L x^i$, the latter is a k-vector space of finite dimension,

Je ne comprends pas cette phrase.

and we also provide explicit finite k-vector space bases. Lemma 1.13 provides a bound on the relevant multiplicities: we compute $m = 3$ and $m' = 0$, and so the bound of Corollary 1.15 on ν is 4. Next, in order to determine bases of the vector spaces $\ker_{x^i} L$ for $1 \le i \le 4$ we use Proposition 4.1 to set up the systems.

First, set $\nu = 1$ and consider the vector space $\ker_x L$. By Proposition 4.1, this is generated by solutions to (18), which becomes

$$\big(\partial(\vartheta-3)(\vartheta-2)^3 - \vartheta^3(\vartheta-1)\big)\, P_0(\partial) = 0,$$

where ∂ is the indeterminate and ϑ the Eulerian derivative with respect to ∂. Written as an explicit differential equation, this is

$$(\partial^5-\partial^4)\frac{d^4P_0}{d\partial^4} - (3\partial^4+5\partial^3)\frac{d^3P_0}{d\partial^3} + (10\partial^3-4\partial^2)\frac{d^2P_0}{d\partial^2} - 22\partial^2\frac{dP_0}{d\partial} + 24P_0 = 0.$$

The sole monic polynomial solution to this equation is $P_0 = \partial^2 + 4\partial$ and the space $\ker_x L$ is generated by $P_0 + x\mathscr{D}[x]$. Correspondingly, $\ker_L x$ is generated by $R_0 + \mathscr{D}[x]L$ with

$$R_0 = x^{-1}P_0L = x^2\,(x-1)\,\partial^3 + \big(3x^3 - 27x^2 + 18x - 48\big)\,\partial^2 - 4\,(x+1)^2\,\partial.$$

Next, to determine $\ker_L x^2$, we compute a basis of $\ker_{x^2} L$, consisting of elements of the form $(P_0 + xP_1) + x^2\mathscr{D}[x]$ for the skew polynomial P_0 that has just been computed and solutions P_1 of the second equation in (18). This step of the calculation corresponds to the restriction from $\mathcal{S}_0$ to $\mathcal{S}_1$ explained in the beginning of the present Section 4.1.2. As here $\ker_x L = \mathcal{S}_0 + x\mathscr{D}[x]$ has k-dimension 1, $\mathcal{S}_1$ has to be either $\mathcal{S}_0$ or $\{0\}$. It turns out that $\Lambda_1 P_0 + \Lambda_0 P_1 = 0$, viewed as an equation in P_1, admits the particular solution $P_1 = 24\partial + 1/6$. Thus, $\ker_{x^2} L$ is the $k[x]$-module generated by $(P_0 + xP_1) + x^2\mathscr{D}[x]$; as a k-vector space, it has dimension 2 with basis

$$\big((P_0 + xP_1) + x^2\mathscr{D}[x], xP_0 + x^2\mathscr{D}[x]\big)\,.$$

Correspondingly, $\ker_L x^2$ is generated as a $k[x]$-module by $R_1 + \mathscr{D}[x]L$; as a k-vector space, it admits the basis $\big(R_1 + \mathscr{D}[x]L, (xR_0) + \mathscr{D}[x]L\big)$. We continue in this manner to determine

$$P_0 = \partial^2 + 4\partial, \quad P_1 = 24\partial + \frac{1}{6}, \quad P_2 = 75\partial + \frac{65}{36}, \quad P_3 = -\frac{1}{54}\partial^3 + 168\partial + \frac{2101}{216}, \tag{20}$$

from which we compute the corresponding R_i in the form $R_i = x^{-i}(P_0 + \cdots + x^iP_i)L$:

$$R_1 = x(x-1)\partial^3 + (24x^3 - 117x^2 + 189x - 150)\partial^2 \\ - \frac{1}{6}(143x^3 + 465x^2 + 474x + 260)\partial - \frac{1}{6}x^2(x-1)6,$$

$$R_2 = (x-1)\partial^3 + (75x^3 - 351x^2 + 558x - 336)\partial^2 \\ - \frac{1}{36}(2635x^3 + 9543x^2 + 8940x + 8404)\partial - \frac{1}{36}x(x-1)(65x+6),$$

$$R_3 = -\frac{1}{54}(x+1)^3\partial^4 + \frac{1}{54}(x^3 + 11x^2 + 45x + 135)\partial^3 + (168x^3 - 765x^2 + 1161x - 618)\partial^2 \\ - \frac{1}{216}(34187x^3 + 143583x^2 + 103092x + 182372)\partial - \frac{1}{216}(x-1)(2101x^2 + 390x + 36).$$

The set of the $R_i + \mathscr{D}[x]L$, $0 \le i \le 3$, generates the $k[x]$-module $\Gamma_x(L)$.

4.1.3. *Torsion at a general prime polynomial.* The next two propositions generalize Proposition 4.1 to primes of arbitrary degree. First, we show how the case of a general p when $\nu = 1$ reduces to Proposition 4.1 by an explicit algebraic extension; next, a direct approach using ϑ again describes the torsion for general p and ν.

Proposition 4.4. *Let $\mathscr{D}[x] = k[x][\partial; S]$, $p \in k[x]$ be prime of degree $d > 1$, and consider $L \in \mathscr{D}[x]$. Then, any torsion element $Q_0 + p\mathscr{D}[x]$ in $\ker_p L$ can be obtained from some torsion element $P_0 + (x-\alpha)\mathscr{D}(\alpha)[x]$ for the desingularization problem viewed in $\mathscr{D}(\alpha)[x] = k(\alpha)[x][\partial; S]$ where α is a root of p in $\bar{k}$. The relationship between P_0 and Q_0 is simply $Q_0(x,\partial) = q_0(\partial) + xq_1(\partial) + \cdots + x^{d-1}q_{d-1}(\partial)$ if $P_0(\partial) = q_0(\partial) + \alpha q_1(\partial) + \cdots + \alpha^{d-1}q_{d-1}(\partial)$.*

Proof. Let P_0 be a torsion element in $\mathscr{D}(\alpha)[x]$. Write P_0 as the evaluation at $x = \alpha$ of a skew polynomial $Q_0(x,\partial)$, that is $P_0(\partial) = Q_0(\alpha,\partial)$. Then, apply Lemma 1.16 to see that $Q_0(x,\partial)$ admits p as a left factor. The converse is proved using a Bézout relation as in Lemma 1.16. ☐

***Example* 4.5.** Let us consider the skew polynomial $L = (x^2 - 5x + 5)\partial - (x^2 - x - 1)$ in $\mathscr{D}[x] = \mathbb{Q}[x][\partial; S]$. It writes $L = S^{-2}(p)\partial - p$ with $p = x^2 - x - 1$. The dispersion is $\{2\}$ and the set $\mathcal{C}$ of candidate primes is the set of the $S^m(p)$ with $m \in \mathbb{N}$. The associated recurrence has a degenerate hypergeometric solution $y = S^{-1}(p)S^{-2}(p)$, and we expect that the singularities are apparent. Let ϕ be a root of p. The skew polynomial L writes

$$L = (6 - 4\,\phi)\,\partial + (1 - 2\,\phi + (2\,\phi - 5)\,\partial)\,(x - \phi) + (\partial - 1)\,(x - \phi)^2$$

According to Proposition 4.1 and Equation 18, we define

$$\Lambda_0 = (6 - 4\,\phi)\,\partial + (1 - 2\,\phi + (2\,\phi - 5)\,\partial)\,\vartheta + (\partial - 1)\,\vartheta^2,$$

where ϑ is the Eulerian operator. We are searching for the polynomial solutions $P_0(\partial)$ of $\Lambda_0 P_0 = 0$ that is (we have supressed a factor ∂)

$$\partial\,(\partial - 1)\,\frac{d^2 P_0}{d\partial^2} + 2\,((\phi - 2)\,\partial - \phi)\,\frac{dP_0}{d\partial} + (6 - 4\,\phi)\;P_0 = 0.$$

We find, up to a constant,

$$P_0(\partial) = \partial^2 - (3\,\phi + 2)\,\partial + 21\,\phi + 13.$$

and

$$P_0 L = (x-\phi)\left((x+\phi-1)\,\partial^3 - 3\,((\phi+1)\,x-\phi)\,\partial^2 \right.\\ \left. + 3\,((8\,\phi+5)\,x-(21\,\phi+13))\,\partial - ((21\,\phi+13)\,x+13\,\phi+8)\right).$$

Next, applying the previous Proposition 4.4, we consider

$$Q_0(x,\partial) = \partial^2 - (3\,x+2)\,\partial + 21\,x + 13.$$

The product $Q_0 L$ has value

$$Q_0 L = \left(x^2-x-1\right)\partial^3 - 3\left(x^3-2\,x^2+1\right)\partial^2 \\ + 3\left(8\,x^3-29\,x^2+13\,x+21\right)\partial - \left(21\,x^3-8\,x^2-34\,x-13\right),$$

that is $Q_0 L = pR$ with

$$R = \partial^3 - 3(x-1)\partial^2 + 3(8\,x-21)\partial - (21\,x+13).$$

Besides the bound of Corollary 1.15 is $\nu \le 1$. Hence the local pseudo-torsion $\Gamma_p(L)$ is generated by R as a $k[x]$-module.

The method could be extended to higher values of ν by computing and introducing suitable polynomials Q_ν in order tomake $Q_\nu \tilde{P}$ divisable by p^ν on the left. We prefer the more direct approach in the proposition that follows.

Proposition 4.6. *Let $\mathscr{D}[x] = k[x][\partial; S]$, $p \in k[x]$ be prime with degree d, and consider*

$$P = \sum_{\substack{0\le j<\nu\\0\le j'<d}} p^j x^{j'} P_{j,j'}(\partial), \qquad L = \sum_{\substack{0\le j\le\mu\\0\le j'<d}} p^j x^{j'} L_{j,j'}(\partial),$$

for $P_{j,j'}, L_{j,j'} \in k[\partial]$. Then, the skew polynomial P corresponds to a torsion element $P + p^\nu \mathscr{D}[x]$ in $\ker_{p^\nu} L$ if and only if the νd polynomials $P_{j,j'}$ satisfy a certain system of νd differential equations that only depends on L and ν.

The linear system, analogous to (18), implied by the statement lacks an aesthetic explicit form; it is the system of the $E_{h,h'} = 0$ obtained by the procedure in the following proof.

Proof. As in the proof of Proposition 4.1, first observe the commutation $Ap^i = \big(p\,(x+\vartheta)\big)^i A$ for all $A \in k[\partial]$, which comes from Equation (19). Then PL rewrites

$$PL = \sum_{\substack{0\le i<\nu\\0\le i'<d}} \sum_{\substack{0\le j\le\mu\\0\le j'<d}} p^i x^{i'} p(x+\vartheta)^j (x+\vartheta)^{j'} P_{i,i'} L_{j,j'}.$$

Recalling that x and ϑ commute, the polynomial in front of $P_{i,i'}$ can be reexpressed as a linear combination of monomials $p^h x^{h'} \vartheta^{h''}$, for $h, h'' \ge 0$ and $0 \le h' < d$. Thus, PL rewrites as a sum of terms of the form $p^h x^{h'} E_{h,h'}$ where $E_{h,h'}$ is a linear combination of the $\vartheta^{h''} P_{i,i'}$

with polynomial coefficients that involve the $L_{j,j'}$. Finally, $PL \in p^\nu \mathscr{D}[x]$ if and only if $E_{h,h'} = 0$ for $0 \le h < \nu$ and $0 \le h' < d$. □

[EXAMPLE NEEDED. ANYONE WELCOME.]

4.2. **Dilation operators.** The results developped in this section for q-dilation operators are stated and proved very analogously to those in previous Section 4.1. We only highlight the differences from the shift case to the q-dilation case, and state results without proofs.

We now examine torsion relations in the skew-polynomial ring $\mathscr{D}[x] = k[x][\partial; Q]$, where $\partial x = qx\partial$, that is, Q is the q-dilation operator on $k[x]$, satisfying $Qf(x) = f(qx)$. We assume that q is *not* a root of unity and disregard singularities at 0, so as to consider infinite orbits. (The case of a singularity at 0 was dealt with in Section 2.1 and we postpone the discussion on roots of unity and finite orbits to Section 6.3.)

The results of this section make use of an analogue η of ϑ in Proposition 4.1. To define it, observe

$$\partial^j x = q^j x \partial^j = x\eta\partial^j \qquad \text{if } j \ge 0$$

where η denotes the q-dilation operator with respect to ∂ on $k[\partial]$. Furthermore, the nice representation of the Λ_m in (22) is made possible by the dissymmetry in the representation of P and L in (21).

Proposition 4.7. *Let $\mathscr{D}[x] = k[x][\partial; Q]$, $q \in k$ not a root of unity, $\alpha \in k$ different from 0, and consider*

$$P = \sum_{j=0}^{\nu-1} (1 - \alpha x)^j P_j(\partial), \qquad L = \sum_{j=0}^{\mu} x^j L_j(\partial), \tag{21}$$

for $P_j, L_j \in k[\partial]$. Define a k-linear endomorphism η of $\mathscr{D}[x]$ by extending the q-dilation operator $f(\partial) \mapsto f(q\partial)$ from $k[\partial]$ to $\mathscr{D}[x]$ by $\eta x = x\eta$. Then, the skew polynomial P corresponds to a torsion element $P + (1 - \alpha x)^\nu \mathscr{D}[x]$ in $\ker_{(1-\alpha x)^\nu} L$ if and only if the polynomials $P_0, \dots, P_{\nu-1}$ satisfy the following triangular system of q-recurrence equations in the independent variable ∂:

$$\begin{cases} \Lambda_0 P_0 = 0, \\ \Lambda_1 P_0 + \Lambda_0 P_1 = 0, \\ \vdots \\ \Lambda_{\nu-1} P_0 + \cdots + \Lambda_0 P_{\nu-1} = 0, \end{cases} \qquad \text{where} \quad \Lambda_m = (-1)^m \sum_{j=m}^{\mu} \alpha^{-j} \binom{j}{m} L_j \eta^j \quad \text{for } 0 \le m < \nu. \tag{22}$$

Here again, Λ_0 plays the role of a dual of L; compare with Sections 2.3 and 4.1.1. We collect these considerations into Algorithm 4.8, before giving an example that illustrates the proposition.

Algorithm 4.8 (Sort).
Input: L, a list
Output: the sorted list
(1) X
(2) Y
(3) Z

Example **4.9.** Let us study the torsion related to the skew polynomial

$$L = q^2x(q^2 - x)\partial - (1 - x)(1 - qx) \in \mathscr{D}[x] = k[x][\partial; Q].$$

Lemma 1.9 applies, and, as $\operatorname{disp}_q(\ell_1, \ell_0) = \{2, 3\}$, the set $\mathcal{C}$ of candidate primes is the union of the orbits under Q of $1 - x$ and $1 - qx$, that is, $\mathcal{C} = \{\, 1 - q^j x : j \in \mathbb{N}\,\}$. Any torsion space for a prime not in $\mathcal{C}$ is zero. We pick up the prime $p = 1 - x$; by Corollary 1.15, a bound on the multiplicities to analyse is 1. (We find $m' = 1$, $m = 2$, $v'_0 = 1$, $v_2 = 1$, and the other values of v'_j and v_j are 0.)

In order to apply Proposition 4.7, we rewrite L in the form

$$L = x^2(-q^2\partial - q) + x(q^4\partial + q + 1) - 1.$$

The q-difference equation resulting from (22) when $\nu = 1$ is

$$-(q^2\partial + q)P_0(q^2\partial) + (q^4\partial + q + 1)P_0(q\partial) - P_0(\partial) = 0.$$

The only polynomial solutions to this equation are multiples of

$$P_0(\partial) = q^6\partial^2 + q^2(q + 1)(q^3 - 1)\partial + (q^2 - 1)(q^3 - 1).$$

From this we determine that the $k[x]$-module $\Gamma_p(L)$ is generated by $R_0 + \mathscr{D}[x]L$ where

$$\begin{aligned} R_0 = (1 - x)^{-1}P_0L = q^{12}x\partial^3 &+ q^6\big((q^5 + q^4 + q^3 - q - 1)x - 1\big)\partial^2 \\ &+ q^2(q + 1)(q^3 - 1)\big((q^3 + q - 1)x - 1\big)\partial - (q^2 - 1)(q^3 - 1)(1 - qx). \end{aligned}$$

4.3. **Closure in a Laurent-Ore algebra.**

Revoir l'introduction en faisant référence à l'appendice C. PhD

Recall that closure of an ideal generated by $L \in \mathscr{D}[x]$ is given by $C(L) = \mathscr{D}(x)L \cap \mathscr{D}[x]$. We thus analogously define closure over the Laurent-Ore algebra $\mathscr{L}[x] = \mathscr{D}[x][\partial^{-1}]$ of an ideal generated by $L \in \mathscr{L}[x]$ as

$$K(L) = \mathscr{L}(x)L \cap \mathscr{L}[x].$$

We can deduce natural connections between these two closures.

Proposition 4.10. *Let $\mathscr{L}$ be a Laurent-Ore algebra associated to the skew-polynomial ring $\mathscr{D}[x]$ and let $L \in \mathscr{D}[x]$. Then $K(L) = \mathscr{D}[x][\partial^{-1}]C(L)$.*

Proof. Let $R \in K(L) = \mathscr{L}(x)L \cap \mathscr{L}[x]$. Then, $R = f(x)^{-1}\partial^{-r}PL$ for some non-negative integer r, some polynomial f, and some $P \in \mathscr{D}[x]$. As $\partial^r f(x)^{-1} = g(x)^{-1}\partial^r$ for some other polynomial g, the skew polynomial $\partial^r R$ is in $\mathscr{D}(x)L \cap \mathscr{L}[x] = \mathscr{D}(x)L \cap \mathscr{D}(x) \cap \mathscr{L}[x] = \mathscr{D}(x)L \cap \mathscr{D}[x] = C(L)$ for $\mathscr{D}(x)L \subset \mathscr{D}(x)$ and because $\mathscr{D}(x) \cap \mathscr{L}[x] \subset \mathscr{D}[x]$. Thus, R is in $\mathscr{D}[x][\partial^{-1}]C(L) = \mathscr{L}[x]C(L)$. Conversely, $\mathscr{D}[x][\partial^{-1}]C(L) = \mathscr{D}[x][\partial^{-1}]\big(\mathscr{D}(x)L \cap \mathscr{D}[x]\big)$ is a subset of $\mathscr{L}(x)L \cap \mathscr{L}[x] = K(L)$. □

PhD Je ne comprends pas ce qu'est la torsion locale dans le cas d'une algèbre de Laurent-Ore.

PhD Dois-je introduire la notation $K_p(L)$?

In the case of Laurent-Ore algebras, the local torsion $T_p(\mathscr{L}[x]/\mathscr{L}[x]L)$ has a simple structure, because there exist isomorphisms between all $\Gamma_{\sigma^j(p)}(\mathscr{L}[x]/\mathscr{L}[x]L)$ for $j \in \mathbb{Z}$.

PhD Où est la preuve de cette dernière assertion ?

Proposition 4.11. *For any $L \in \mathscr{D}[x]$ and any prime $p \in k[x]$, the pseudo-torsion space $\Gamma_p(\mathscr{L}[x]/\mathscr{L}[x]L)$ generates $T_p(\mathscr{L}[x]/\mathscr{L}[x]L)$ as an $\mathscr{L}[x]$-module, and*

$$T_p(\mathscr{L}[x]/\mathscr{L}[x]L) = \mathscr{L}[x]\Gamma_p(\mathscr{L}[x]/\mathscr{L}[x]L) = \bigoplus_{j\in\mathbb{Z}} \partial^j\Gamma_p(\mathscr{L}[x]/\mathscr{L}[x]L). \tag{23}$$

Proof. Observe that an isomorphism from $\Gamma_p(\mathscr{L}[x]/\mathscr{L}[x]L)$ to $\Gamma_{\sigma^{-1}(p)}(\mathscr{L}[x]/\mathscr{L}[x]L)$ is given by multiplication by ∂, with inverse given by multiplication by ∂^{-1}. Indeed, any relation $PL = pR$ in $\mathscr{L}[x]$ implies both $(\partial P)L = \sigma(p)(\partial R)$ and $(\partial^{-1}P)L = \sigma^{-1}(p)(\partial^{-1}R)$. This justifies the existence of the multiplication maps above, which are readily seen to be injective and inverse of one another. The direct-sum expression come from the fact that the $\sigma^j(p)$ are two by two coprime. □

4.4. **Local torsion in a Laurent-Ore algebra.**

PhD Il faut une introduction bien mieux détaillée pour rappeler les ingrédients de l'algorithme.

We can now reformulate the steps given so far into an algorithm to compute local torsion in $\mathscr{L}[x] = \mathscr{D}[x][\partial^{-1}]$. The following algorithm outputs a linear basis for $\Gamma_p(L)$, which by Proposition 4.11 is also a generating set for $T_p(L)$. This calculation reduces to computing $\ker_L p^\nu$ for finitely many ν, which is done using a system similar to those in Propositions 4.1 and 4.7. We contrast this to the forthcoming Algorithm 4.16 to compute complete local torsion spaces in the skew-polynomial ring $\mathscr{D}[x]$, that is, $T_p(L) = \bigoplus_{j\in\mathbb{Z}} \Gamma_{\sigma^j(p)}(L)$, which really requires computing $\Gamma_{\sigma^j(p)}(L)$ for several j.

Il faut proprifier le bout qui suit sur la dispersion dans le cas inversible. PhD

[INTRO: TWO CASES, SPR VS LO.] FC

A different but simpler situation is when σ is invertible, for then, the geometry of primes consists of (pure) cycles and (pure bilaterally) infinite lines. This is the case for example when considering torsion and closure with regard to the algebra of both forward and backward shift operators, as in Section 4.4, Example 4.13.

In this case, and if the application commands working in the Laurent-Ore algebra, [NEW DEFINITION] Then, $\mathcal{C}$ partitions into subsets closed under σ and σ^{-1} (up to normalization to keep primes monic), and all primes in the same subset play the same role, therefore, taking any of them as the distinguished prime will do.

Je trouve que la distinction entre candidats premiers et premiers distingués n'est pas assez clairement explicités, pas assez mise en valeur. Elle est mieux exprimé par *In the preceding example, the prime $p = x$, as opposed to $x - 1$, is* distinguished, *in the sense that it permits one to (completely) compute $\Gamma_{x-1}(L)$, $\Gamma_{x-2}(L)$, etc, from a vector basis of it.* dans la section 4.6. J'ai envie d'employer l'expression *distinguished candidates* pour que l'on sente qu'il y a un deuxième niveau de sélection. Où faut-il placer cette notion de candidat distingué ? PhD

Notion of σ-equivalence and of shift-equivalence *HERE*. FC

Given such a dispersion set, the set of candidate primes is given by

$$\mathcal{C} = \left\{\, [p] : \text{prime } p = \gcd\bigl(\sigma^d(\mathrm{lc}(L)), \mathrm{tc}(L)\bigr), d \in \mathrm{disp}_\sigma(\mathrm{lc}(L), \mathrm{tc}(L)) \,\right\}$$

where $[p]$ is the σ-equivalence class $\{\, \sigma^m(p) : m \in \mathbb{Z} \,\}$.

Ça ne me plaît pas le coup des classes d'équivalences. Ne pourrait-on pas parler de *forward and backward orbit* opposée à l'orbite usuelle qui est une *forward orbit* ? Et même terminologie pour la dispersion. *bilateral dispersion* ? PhD

Algorithm 4.12 (Local torsion in a Laurent-Ore algebra).

Input: $L \in k[x][\partial;\sigma]$ and candidate prime $p \in k[x]$
Output: Generators of $T_p(\mathscr{L}[x]/\mathscr{L}[x]L)$ given as a linear basis of $\Gamma_p(\mathscr{L}[x]/\mathscr{L}[x]L)$

(1) Compute integers $m_{\min}$ and $m_{\max}$ such that $m_{\min} \leq m \leq m_{\max}$ for m satisfying $\omega_p\big(\sigma^m(\operatorname{lc} L)\big) \neq 0$; ????? using σ-dispersion ?????
(2) Compute a bound ν on the possible multiplicities of p using both inequalities

$$\nu \leq \sum_{i=m_{\min}}^{m_{\max}} \omega_p\big(\sigma^i(\operatorname{lc} L)\big), \qquad \nu \leq \sum_{i=m_{\min}}^{m_{\max}} \omega_p\big(\sigma^i(\operatorname{tc} L)\big).$$

(3) For i from 1 to ν compute a basis for $\ker_{p^i} L$. In the shift and q-dilation cases the system to solve is given by Propositions 4.1 and 4.7 respectively
(4) Convert the basis of $\ker_{p^i} L$ to a basis for $\ker_L p^i$ by multiplication by L on the right and division by p^i on the left
(5) GB or return directly?

Example **4.13.**

PhD Il ne semble pas que les exemples aient un titre.

Continuing Example 4.3 about local torsion in the Laurent-Ore algebra of shift operators, let us evaluate the local torsion spaces $T_p(L)$ for all primes p on the example

$$L = (x-3)(x-2)^3\partial - x^3(x-1) \in \mathscr{D}[x] = k[x][\partial; S].$$

By the discussion in Section 1.4, $\operatorname{disp}(\ell_1, \ell_0) = \{1, 2, 3\}$. The common prime factors of $\ell_1 = (x-3)(x-2)^3$ and some $\sigma^i(\ell_0)$ with $\ell_0 = -x^3(x-1)$ and $1 \leq i \leq 3$ are $x-2$ and $x-3$. The set $\mathcal{C}$ of candidate primes is the union of forward and backward orbits of those primes, that is $\mathcal{C} = x + \mathbb{Z}$. As we are working in the Laurent-Ore algebra, any choice of prime in $x + \mathbb{Z}$ is suitable to determine all local torsion spaces in $\mathscr{L}[x]/\mathscr{L}[x]L$. For simplicity we choose $p = x$.

Steps (3) and (4) have already been performed in Example 4.3. Therefore, the family of the $R_i + \mathscr{D}L$, $0 \leq i \leq 3$, computed in that example generates the $\mathscr{L}[x]$-module $T_x(L)$. If a normalized output is preferred, a Gröbner basis calculation for the ordering $\operatorname{lex}(x < \partial)$ reveals

PhD Couic ! ?

Example **4.14.** Continuing Example 4.9 about local torsion in the Laurent-Ore algebra of q-dilation operators, let us evaluate the local torsion spaces $T_p(L)$ for all primes $p \neq x$

on the example

$$L = q^2x(q^2 - x)\partial - (1 - x)(1 - qx) \in \mathscr{D}[x] = k[x][\partial; Q].$$

The set $\mathcal{C}$ consists in the only forward and backward orbit of $1-x$, that is $\{ 1-q^jx : j \in \mathbb{Z} \}$.

Il manque la fin de l'exemple. PhD

4.5. **Distinguished candidates.** Considering a relation $PL = pR$ in $\mathscr{D}[x]$, P factors as $P = \partial P'$ with $P' \in \mathscr{D}[x]$ if and only if $\partial^{-1}PL = \sigma^{-1}(p)\partial^{-1}R$ is in $\mathscr{D}[x]$, that is, if and only if R also factors as $R = \partial R'$ with $R' \in \mathscr{D}[x]$. The map from $\Gamma_{\sigma(p)}(L)$ to $\Gamma_p(L)$ obtained by multiplication by ∂^{-1} is thus only a partial map. It follows that, in the context of skew-polynomial rings, computing the local pseudo-torsion $\Gamma_p(L)$ is not, in general, sufficient to compute the local torsion $T_p(L)$ completely. Questions that arise naturally are:

(1) how to choose among candidate primes to ensure that the local torsion at that point is determined;
(2) whether $\Gamma_{\sigma^{-1}(p)}(L)$ can be generated from known generators of $\Gamma_p(L)$.

The following example illustrates a phenomenon whose understanding is the key to solving both problems. It introduces the idea of distinguished primes.

***Example* 4.15.** We revisit Example 4.3. Recall that the candidate primes were given by the set $x + \mathbb{Z}$.

C'est de l'arnaque l'ensemble des candidats premiers était $x - 3 + \mathbb{N}$. PhD

For any p in this set, we can compute a basis for $\Gamma_p(L)$ using Proposition 4.1. The following table presents the dimension of $\Gamma_p(L)$ for several p.

Candidate prime p	...	$x-3$	$x-2$	$x-1$	x	$x+1$	$x+2$	...
Size of the basis of $\Gamma_p(L)$	...	0	0	1	4	4	4	...

This table indicates, first, that the torsion is trivial for polynomials $x+j$ with $j < -1$, and secondly, that mapping the generator of $\Gamma_{x-1}(L)$ to $\Gamma_x(L)$ generates only a subspace of it. Recall the generator $G = (P_0 + xP_1 + x^2P_2 + x^3P_3) + \mathscr{D}[x]$ of the $k[x]$-module $\Gamma_x(L)$ in our example, where the P_i were given in (20). We determine a generating set for $\Gamma_{x-1}(L)$ from the generators of $\Gamma_x(L)$ by combining elements so that a factor ∂ appears. For example, we see immediately that P_0 factors as $\partial(\partial + 4)$, and thus $\partial + 4$ is a generator of $\Gamma_{x-1}L$. It might be possible that combinations of generators could factor. Indeed we recover the entire set by a simple Gröbner basis calculation with respect to the order $1 > \partial > \partial^2 > \partial^3$. The complete local torsion of x in the skew-polynomial ring is now fully described:

$$T_x(L) = \bigoplus_{j\in\mathbb{Z}} \Gamma_{x+j}(L) = \Gamma_{x-1} \oplus \bigoplus_{j\in\mathbb{N}} \partial^j \Gamma_x(L).$$

PhD

Ici on a un vrai commentaire sur l'idée de *premier distingué*. Par contre je ne comprends pas bien le jeu sur $T_p(L)$ et $\Gamma_p(L)$ et surtout l'adverbe *completely*. Jusqu'ici *complete* était employé pour opposer torsion et pseudo-torsion.

In the preceding example, the prime $p = x$, as opposed to $x - 1$, is *distinguished*, in the sense that it permits one to (completely) compute $\Gamma_{x-1}(L)$, $\Gamma_{x-2}(L)$, etc, from a vector basis of it. (For that matter, any of $x + 1$, $x + 2$, etc, would do as well, and would be accepted as distinguished.) Generically, a prime p is distinguished when the torsion $T_p(L)$ decomposes as a sum,

$$T_p(L) = \bigoplus_{j=1}^{D} \Gamma_{\sigma^{-j}(p)}(L) \oplus \bigoplus_{j\in\mathbb{N}} \partial^j \Gamma_p(L),$$

in which the first direct sum is an irregularity and the second is made of isomorphic summands.

4.6. **Local torsion in a skew-polynomial ring.** We now address the less direct calculations of torsion in skew-polynomial rings, a opposed to Laurent-Ore algebras.

First, let us make explicit the differences in the structure of local torsion. Recall the simple structure of the local torsion in the Laurent-Ore algebra, as described by (23) in Proposition 4.11. A similar argument gives an injective map from $\Gamma_p(\mathscr{D}/\mathscr{D}L)$ to $\Gamma_{\sigma(p)}(\mathscr{D}/\mathscr{D}L)$ by multiplication by ∂.

PhD

Cet argument devrait faire l'objet d'une section intitulée *Local torsion along orbits*.

By contrast,

We are ready to give the formal algorithm and address how to choose a distinguished prime among candidate primes to ensure that a complete local torsion is generated.

PhD

La phrase suivante est une généralité. *Thus, if we have a basis for $\Gamma_p(L)$, we determine the basis of the subspace $\Gamma_{\sigma^{-1}(p)}(L)$ by examining combinations of the generators P in $\Gamma_p(L)$ which factor into $P = \partial P'$, $P' \in \mathscr{D}[x]$.*

As we have remarked in the above example, starting from the definition of the torsion module

$$T_p(L) = \bigoplus_{j\in\mathbb{Z}} \Gamma_{\sigma^j(p)}(L),$$

we know [HOW COME WE DO?] that $\Gamma_{\sigma^j(p)}(L)$ is zero for large negative j, is stationary on a limit for large positive j, varies monotonously inbetween.

Il devrait y avoir un lemme ou même une proposition pour cette assertion. PhD

La nullité pour j très négatif vient de la notion de candidat premier. PhD

Thus, first we determine a minimal value $j = D$ for which $\Gamma_{\sigma^D(p)}(L)$ is not zero, and a $j = B$ which indicates where the stationary limit begins. It then suffices to generate the bases for the finite number of $\Gamma_{\sigma^j(p)}(L)$ inbetween.

Il y a toujours une arnaque puisqu'on met $j \in \mathbb{Z}$. PhD

Algorithm 4.16 (Computing maximal local torsion)**.**

Input: $L \in k[x][\partial;\sigma]$, $p \in k[x]$
Output: $T_p(L) = \bigoplus_{j\in\mathbb{Z}} \Gamma_{\sigma^j(p)}(L)$

(1) Compute
$$m = \max\left\{\, i \in \mathbb{Z} : \omega_p\big(\sigma^i(\mathrm{lc}(L))\big) \neq 0 \,\right\}$$
and
$$m' = \min\left\{\, i \in \mathbb{Z} : \omega_p\big(\sigma^i(\mathrm{tc}(L))\big) \neq 0 \,\right\}$$
(2) Replace p with $\sigma^{-m'}(p)$. By this process, (m, m') becomes $(m - m', 0)$
(3) Determine bases for the vector spaces $V_i = \ker_{p^i} L, i = 1, \dots, \nu_{\max}$
(4) View $\mathscr{D}[x]$ as a left $k[x]$-module, so as to write the basis elements as row vectors of their coefficients in the ∂^j. Let $j_{\max}$ denote the maximum degree in ∂ in the bases
(5) For each $i = 1, \dots, \nu_{\max}$, compute a module Gröbner basis of V_i with respect to the order $1 > \partial > \partial^2 > \dots$
(6) For j from 1 to $j_{\max}$ take the elements of each V_i which can be pulled back by σ^{-j}, pull them back, and let B_j be the $k[x]$-basis of the set
(7) Return the pairs (j, B_j) for non-trivial bases B_j
(8) Return $\mathscr{D}[x](B_0, \dots, B_D, L)$

5. Desingularization and recurrences of minimal order

In this section we examine the reward for our theoretical work, practical applications that help improve the efficiency of recurrence manipulations. In this section we deal with simultaneous rings for different operators, and thus we adopt the convention that the indeterminate ∂ appears in differential algebras, and the indeterminate ϑ appears in shift, and related operators.

5.1. **Forward and backward desingularization of recurrences.** Integer roots in either the leading or trailing coefficient of a recurrence may impede calculation when the recurrence is unrolled. Ore closure can be used to pretreat a recurrence in a process called *desingularization* which removes those singularities of the recurrence which are not singularities of any solution.

Our approach uses Laurent-Ore algebras (Appendix C), notably $\mathscr{E}[n] = k[n][\vartheta, \vartheta^{-1}; S]$, for the forward shift operator S. When we consider an element

$$E = e_s(n)\vartheta^s + \cdots + e_r(n)\vartheta^r, \qquad r,\, s \in \mathbb{Z},\, s \le r,\, e_s e_r \neq 0,$$

we can take the point of view that it represents a recurrence of order $r - s$ with *leading coefficient* $\mathrm{lc}(E) = e_r(n)$, and *trailing coefficient* $\mathrm{tc}(E) = e_s(n)$, or we can consider it as a polynomial in $\mathscr{E}$, for which there is a leading monomial with respect to some monomial ordering. Indeed we take both views in the course of this section.

Definition. The skew polynomial $E \in \mathscr{E}[n] = k[n][\vartheta, \vartheta^{-1}; S]$ is a maximally desingularized multiple of $L \in \mathscr{D}[n] = k[n][\vartheta; S]$ (in *forward* desingularization) if and only if $\deg_n(\mathrm{lc}(E)) = \min_{F \in K(L)} \deg_n(\mathrm{lc}(F))$.

The definition is justified by the the fact that he number of roots of the leading coefficient corresponds with its degree, if the ground field is algebraically closed. There is a corresponding definition for maximally desingularized trailing coefficient, a result of the *backwards* desingularization process.

5.1.1. *Algorithm.* Here we find recurrence operators which have respectively maximally desingularized leading, and trailing coefficients by means of a Gröbner basis calculation. We only use the fact that σ is invertible and that δ is 0, hence this algorithm works for more general situations, thus we state and prove it on this level of generality. This will give us analogous result for q-shift case at the same time. We use the algorithm of Wu for Gröbner bases in Laurent-Ore algebra recalled in Section C.2.

Algorithm 5.1 (Desingularization of recurrences)**.**
Input: $L \in \mathscr{D}[n] = k[n][\vartheta; \sigma]$ with σ invertible (and $\delta = 0$)
Output: E_ℓ and $E_t \in \mathscr{E}[n] = k[n][\vartheta, \vartheta^{-1}; \sigma]$ which respectively have maximally desingularized leading and trailing coefficients

(1) Determine a set $\mathcal{R}$ of generators of $K(L)$
(2) Determine $\mathcal{G}$ the Gröbner basis of the ideal generated by $\mathcal{R}$ in the Laurent-Ore algebra with respect to the ordering $\mathrm{lex}(\vartheta^{-1} > \vartheta > n)$
(3) Output the elements with leading coefficient of minimal degree, and trailing coefficient of minimal degree in G

Il semble que la base de Gröbner demande un article défini.

PhD

Remark, in our Maple implementation the second step is achieved by working in the Ore algebra $k[n][\vartheta; \sigma][\vartheta'; \sigma^{-1}]$, and adding the element $\vartheta\vartheta' - 1$ to the ideal before taking the

Gröbner basis with respect to the ordering $\operatorname{lex}(\vartheta' > \vartheta > n)$. Please see Section D for more details.

Les détails ne sont pas là. PhD

5.1.2. *Correctness of the algorithm.* The proof of the algorithm relies on the following.

Theorem 5.2. *Let L be element in the skew-polynomial ring $k[n][\vartheta;\sigma]$ with σ invertible and $\delta = 0$. Then, the Gröbner basis $\mathcal{G}$ of $K(L)$ with respect to the ordering $\operatorname{lex}(\vartheta^{-1} > \vartheta > n)$ contains a unique E_t (respectively E_ℓ) that has a maximally desingularized trailing (resp. leading) coefficient.*

Proof. First we note that if E has a maximally reduced trailing or leading coefficient, then so do ϑE and $\vartheta^{-1}E$, since δ is zero, and σ does not affect the degree of n. Thus, we are at our leisure to choose some $E \in K(L)$ which has a maximally desingularized trailing coefficient and is of the form $E = e_r(n)\vartheta^r + \cdots + e_0(n) + e_{-1}(n)\vartheta^{-1}$ such that $e_{-1} \neq 0$, $r \geq 0$. This implies that if we set $\deg(e_{-1}(n)) = D$, then

$$D = \min_{F \in K(L)} \deg_n(\operatorname{tc}(F)).$$

By the defining property of a Gröbner basis, there is some $G \in \mathcal{G}$ such that the leading monomial $\operatorname{lm}(G)$ divides the leading monomial $\operatorname{lm}(E)$. According to the monomial order given, $\operatorname{lm}(E) = n^D\vartheta^{-1}$. Now, we know that $\operatorname{lm}(G)$ is not a polynomial in only n by the hypotheses given on L,

Quelles hypothèses ? PhD

hence for this reason, and because it divides $n^D\vartheta^{-1}$, we have $\operatorname{lm}(G) = n^{D'}\vartheta^{-1}$ for some $D' \leq D$. By definition, $D \leq \deg_n(\operatorname{tc}(G)) = \deg_n(\operatorname{lm}(G))$ and it stands that $D = D'$. Thus we conclude that G is *also* an element with maximally desingularized trailing coefficient. If the Gröbner basis is reduced, this element is unique, and we set E_t to G. The argument to show that there is an element with a maximally desingularized leading coefficient is essentially the same. ☐

In fact, we can say more about the structure of this Gröbner basis. For any $G \in \mathcal{G}$ with leading coefficient $g(n)$ and order r we can show that $\deg_n(g(n))$ is minimal over all operators in $K(L)$ with order r. Write $g(n) = a(n)b(n)$ where $a(n)$ divides $\sigma^j(\ell_{r-j}(n))$ and $\gcd(b(n), \sigma^j(\ell_{r-j}(n))) = 1$. Then, there exists polynomials u and v such that $u(n)b(n) + v(n)\sigma^j\ell_{r-j}(n) = 1$. Consider the element $uG + va\vartheta^jL \in K(L)$. The coefficient of ϑ^r is

$$u(n)a(n)b(n) + v(n)a(n)\sigma^j(\ell_{r-j}(n)) = a(n).$$

The minimality of the degree of $g(n)$ beyond elements of this form implies that $\deg_n(a(n)) = \deg_n(g(n))$. Since $a(n)$ divides $g(n)$, this implies that they are equal (up to a scalar) and thus $g(n)$ divides $\sigma^j(\ell_{r-j}(n))$.

Je n'ai rien compris. PhD

***Example* 5.3.**

PhD Les exemples ne constituent pas des sous-sous-sections.

Consider the simple recurrence

$$(n-7)(2n-3)a(n+1)-(n-5)a(n)=0, \qquad a(0)=1.$$

We see immediately that we cannot compute $a(8)$ from this recurrence (and hence no further term). However, if we apply Algorithm 5.1 to $L=(n-7)(2n-3)\vartheta-(n-5)$, we find $E=63(1+2n)\vartheta^2-7(3+4n)\vartheta+(2n+7)-\vartheta^{-1}$. This has both maximally desingularized leading and trailing coefficient. We now can use the original recurrence to compute two additional initial terms, $a(1)=-5/21$, $a(2)=10/63$, and from there, the recurrence is well defined, and we can compute $a(8)=1/654885$.

PhD Il faudrait un exemple avec le q-shift. Peut-être dans Andrews.

5.1.3. *Other approaches.* The problem of desingularizing linear recurrences has been addressed in at least two different ways before. In [1] and [4], the authors provide algorithms and implementations which offer desingularized operators. One advantage of using the closure, however, is that we obtain a basis for all such operators, and this offers some freedom in selecting operators with certain properties. Indeed, as we provide a Gröbner basis, we can be assured that our operator is of minimal order, and that the coefficients of the interior terms are maximally reduced.

Furthermore, following the construction of [1], we can build an operator with simultaneously maximally desigularized leading and trailing coefficients by taking a judicious sum of multiples of the results returned by the algorithm. This seems to be the best possible approach, as to desingularlize one end of a recurrence, one must generally relinquishes control on the other end. We provide further comment on a comparison of strengths of the different implementations in Appendix D.

PhD Éh bien, il est faiblard l'appendice D.

5.2. **Reduction of recurrence order.** As we remarked in the introduction, the coefficient sequence f_n of a power series $u=\sum_n u_n x^n$ satisfies a linear recurrence with polynomial coefficients precisely when u satisfies a linear differential equation with polynomial coefficients. One classic way to determine such a recurrence is to use the isomorphism μ from the differential algebra $\mathscr{D}[x]=k[x][\partial;\mathrm{Id}_{k[x]},d/dx]$ to the dual shift algebra $\mathscr{E}[n]=k[n][\vartheta,\vartheta^{-1};S]$ known as the *formal Mellin transform.*

L'isomorphisme emploie-t-il $\mathscr{D}[x]$ ou $\mathscr{D}(x)$? PhD

It comes from the usual Mellin transform of functions applied to formal power series. The isomorphism acts on the generators by $\mu(x) = \vartheta^{-1}$, $\mu(x^{-1}) = \vartheta$, and $\mu(\partial) = \vartheta n$. The inverse image is straightforward. The image of $L \in \mathscr{D}[x]$ under μ is a $E \in \mathscr{E}[n]$ which represents a recurrence satisfied by the coefficient sequences of all series solutions u to $Lu = 0$. It is natural to ask if this is the recurrence of smallest order, and if not, how one might obtain such a recurrence. Such a recurrence could be more economical from a space point of view when unrolling the recurrence.

Clearly, from the isomorphism we can see that if there are extraneous powers of x lingering in L, then the recurrence will be sub-optimal. This suggests that the closure is a natural candidate for a tool in such a search. Indeed, we can determine a recurrence of minimal order which annihilates all coefficient sequences of solutions to a given linear differential equation using the closure.

Algorithm 5.4 (Reduction of recurrence order)**.**
Input: $L \in \mathscr{D}[x] = k[x][\partial; \mathrm{Id}_{k[x]}, d/dx]$
Output: $E \in \mathscr{E}[n] = k[n][\vartheta, \vartheta^{-1}; S]$ such that E annihilates the coefficient sequence of any $u \in k[[x]]$ satisfying $Lu = 0$, and E is of minimal order amongst all such recurrences.

(1) Determine a set $\mathcal{R}$ of generators of $C(L)$
(2) Compute the image $\mathcal{E} = \mu(\mathcal{R})$ by the formal Mellin transform
(3) Determine the Gröbner basis $\mathcal{G}$ of the ideal generated by $\mathcal{E}$ in the Laurent-Ore algebra $k[n][\vartheta, \vartheta^{-1}; S]$ with respect to the ordering $\operatorname{lex}(\vartheta^{-1} > \vartheta > n)$
(4) Output the element $E \in \mathcal{G}$ of smallest degree in ϑ

***Example* 5.5.** Consider the algebraic equation $y(x) = x + xy(x)^2(1 + y(x))$ which arises in the study of rooted 2-3 trees. If we consider $y(x)$ as a formal power series solution to this, we can determine a homogeneous linear differential equation that it satisfies, and hence a recurrence satisfied by the coefficients of the series expansion.

Faut-il une référence pour le passage algébrique–différentiellement fini ? PhD

The formal power series $y(x)$ is annihilated by the differential operator

$$L = 2x(9x+1)(x+3)(31x^3+18x^2-x-4)\partial^3 \\ + 12(9x+1)(31x^4+122x^3+49x^2-3x-5)\partial^2 + 12x(9x+1)(31x^2+133x+42)\partial.$$

The recurrence that results from a direct application of μ to L is of order 4:

$$\begin{aligned}0 = 31n(n+1)(n+2)\,a(n) + 3(37n+133)(n+2)(n+1)\,a(n+1)\\ + (n+2)(53n^2+347n+546)\,a(n+2) - (7n+25)(n+3)(n+2)\,a(n+3)\\ - 6(n+4)(n+3)(2n+9)\,a(n+4).\end{aligned}$$

If we apply Algorithm 5.4 to L, we find three elements in the basis of $C(L)$, and once we apply μ to these elements, and reduce, we find the recurrence of order 3:

$$\begin{aligned}0 = -31n(26n+79)(n+2)(n+1)\,a(n) - 6(n+2)(n+1)(78n^2+354n+371)\,a(n+1)\\ +(n+2)(n+1)(26n^2+157n+228)\,a(n+2) + 2(2n+7)(26n+53)(n+3)(n+2)\,a(n+3).\end{aligned}$$

Both recurrences describe the sequence 1, 0, 1, 1, 2, 5, 8, 21, 42, ...

PhD Référence Sloane A001005.

5.2.1. *Correctness of the algorithm.* Algorithm 5.4 is correct, as proves the next theorem.

Theorem 5.6. *Given $L \in \mathscr{D}[x] = k[x][\partial; \mathrm{Id}_{k[x]}, d/dx]$, let α be a regular point of the differential equation $Ly(x) = 0$. Define $\mathcal{S}_\alpha L$ to be the vector space of power series y around α which are solutions of $Ly = 0$. Then the Gröbner basis $\mathcal{G}$ of the ideal $\mu(K(L))$, image of the Laurent-Ore algebra closure by the formal Mellin transform, with respect to the order $\operatorname{lex}(\vartheta^{-1} > \vartheta > n)$ contains an E which represents a recurrence annihilating the coefficient sequence of any $u \in \mathcal{S}_\alpha L$, and E is of minimal order amongst all such recurrences.*

Proof. Let $\mathfrak{A} \subset k[n][\vartheta, \vartheta^{-1}; S]$ be the ideal of skew polynomials representing recurrences which annihilate coefficient sequences of elements in $\mathcal{S}_\alpha L$. Clearly $\mu(L)$ is in $\mathfrak{A}$. In fact we contend that $\mathfrak{B} = \mu(K(L))$ is also contained in $\mathfrak{A}$. For any $R \in K(L)$ there is a $p \in k[x]$, $p \neq 0$, and a $P \in \mathscr{D}$ such that $PL = pR$. Thus, if $Ly = 0$, then $0 = PLy = pRy$, hence $Ry = 0$.

PhD Utilité de la citation [8, Corollary 1] ?

Vérifier la définition de la clôture.

Consequently $\mu(R)$ annihiliates the coefficient sequence of y for all $y \in \mathcal{S}_\alpha L$.

Let D be the minimum order of all non-trivial elements in $\mathfrak{A}$, and let $\mathcal{C}$ be the set of the $E \in \mathfrak{A}$ whose order is D. Remark that this set is shift-invariant.

PhD D'où sort cette invariance ?

In order to prove the theorem, it is sufficient to show that $\mathfrak{B} \cap \mathcal{C}$ is non-empty; the minimality of the order is sufficient to establish that it is in the Gröbner basis of $\mathcal{B}$. Consider some $E_0 \in \mathcal{C}$ such that $E_0 = a_0(n) + a_1(n)\vartheta^{-1} + \cdots + a_{D-1}\vartheta^{-D}$, with $a_i(n) \in k[n]$. Again, we can make such a choice by the invariance of $\mathcal{A}$ under σ. Remark that $\mu^{-1}(E_0) \in \mathscr{D}$ since it contains no positive power of ϑ. Set $M = \mu^{-1}(E_0)$. Recall, E_0 annihilates any coefficient sequence coming from a $u \in \mathcal{S}_\alpha L$, hence $M \cdot u = 0$ for such u.

Thus, as a vector space, the solution space of L is a subspace of the solution space of M. We can imitate the Wronskian construction of Proposition 0.1 on basis elements in $\mathcal{S}_\alpha M$ in the orthogonal complement of the basis of L to construct P, a finite differential operator with power series coefficients such that $M = PL$. If we write these three operators as $M = m_0(x) + \cdots + m_i(x)\partial^i$, $P = p_0(x) + \ldots + p_j(x)\partial^j$, and $L = \ell_0(x) + \ldots + \ell_k(x)\partial^k$, then in resulting multiplication the coefficient of highest order gives that $m_i(x) = p_j(x)\ell_k(x)$. Since m_i and ℓ_k are both polynomial, $p_j(x)$ is rational. Similarly, since the coefficients of P satisfy a triangular system, using the fact that the $m_n(x), \ell_n(x)$ are polynomial for all n, and that $p_{K+1}(x), p_{K+2}, \ldots, p_k(x)$ are rational, we can deduce that $p_K(x)$ is also rational, and inductively show that $P \in \mathscr{D}(x)$.

Thus, L is a right divisor of M over $\mathscr{D}(x)$. Thus $M \in \mathscr{D}(x)L \cap \mathscr{D}$, that is, $M \in C(L)$, hence $E_0 \in \mathcal{B}$, and the proof is complete. □

q-mellin? Marni

6. Singularities along periodic orbits of σ

Relire introduction. PhD

We begin in Section 6.3 by treating q-dilation in the case q is a tth root of unity. There, we not only describe distinguished primes but also get their multiplicity in a simple way: In this situation, the endomorphism $\tau = \sigma^t$ is the identity on $k[x]$ and the commutation rule is merely $x\vartheta = \vartheta x$. We can thus view the commutative algebra $k[x, \vartheta]$ as embedded in $\mathscr{D}$. This completes the treatment of q-calculus operators and furthermore constitutes an introduction to the more general, also more sophisticated situation of Section 6.4.

In Section.., we have studied the case... FC

- this section put in context wrt other sections
- what are the difficulties, what fails in previous thms (cand. primes, multiplicities?)
- why $\delta = 0$

Consider a skew-polynomial ring $\mathscr{D} = k[x][\partial; \sigma]$ and recall from Section 1.2.2 that a prime polynomial p has period t with respect to σ if and only if the pullback $\sigma^{-t}\big((p)\big)$ of the principal ideal (p) under σ^t is equal to (p) itself. In this case, the polynomial $\sigma^t(p)$ takes the form gp for some $g \in k[x]$.

Two examples of this periodic behaviour are given by q-dilation when q is a non-trivial tth root of unity and by Mahlerian operators at roots of unity. We consider both of these cases in detail. In each case, computing the local torsion $\Gamma_p(L)$ at p for a skew polynomial L in $\mathscr{D}$ reduces to a linear algebra problem of the form $\mathsf{PL} = p^\nu \mathsf{R}$, for a known matrix L and unknown row vectors P and R. In the periodic q-case, the problem reduces to a commutative one, and the candidate primes are prime factors of the determinant of L; in the periodic Mahlerian case, the primes p are cyclotomic, so that closed forms are available to make the non-commutative situation tractable.

The complexity of our approach directly depends on the polynomial g in the relation $\sigma^t(p) = gp$ mentioned above. The case of a q-dilation operator when q is a non-trivial tth root of unity leads to $g = 1$. Conversely, the assumption $g = 1$ implies the presence of a q-dilation operator for a non-trivial root of unity q, as a comparison of degrees in the equality $\sigma^t(p) = p$ shows (possibly after an affine change of the variable x; compare with Lemma A.2). The case of the Mahlerian operator at roots of unity leads to polynomials g that are invertible modulo p. We have not investigated other situations with an invertible g, but we would expect them to be amenable to the same treatment, albeit with less aesthetically pleasing results. On the other hand, we admit that we do not know how to handle the case of a non-invertible g.

6.1. **Periodic primes and their periods.** In this section, we provide more properties of periods and give conditions under which a prime polynomials can be periodic. We start by making explicit the link between the periodicity of a prime and the action of σ to monomials. This refines the notion of period to a notion of *modular period* to be defined below.

Lemma 6.1. *A prime polynomial p is purely periodic if and only if p divides the polynomial $\sigma^t(x) - x$ for some positive integer t. It is ultimately periodic if and only if p divides the polynomial $\sigma^{t_0+t}(x) - \sigma^{t_0}(x)$ for some non-negative integer t_0 and some positive integer t.*

Proof. Let us first assume that the relation $p \mid \sigma^t(x) - x$ holds for some positive integer t. It follows that p divides $\sigma^t(x)^i - x^i = \sigma^t(x^i) - x^i$ for any $i \in \mathbb{N}$, as well as $\sigma^t(f) - f$ for every polynomial f. Hence, p divides $\sigma^t(f)$ if and only if it divides f, and the equality $\sigma^{-t}\big((p)\big) = (p)$ holds.

Conversely, let the relation $\sigma^{-t}\big((p)\big) = (p)$ hold for some positive integer t. Let π_p denote the projection from $k[x]$ into $k[x]/(p)$. From our hypothesis, the kernel of the composed map $\pi_p \circ \sigma^t$ is (p), so this map induces an injective homomorphism σ_p^t from $k[x]/(p)$ into $k[x]/(p)$. This map is then an automorphism, since the k-vector space $k[x]/(p)$ is finite-dimensional. Hence, it is an element of the finite Galois group $\mathrm{Gal}\big(k[x]/(p) : k\big)$. Denoting by s the order of this automorphism, $(\sigma_p^t)^s$ is then the identity on $k[x]/(p)$. This means that p divides $\sigma^{st}(x) - x$. The proof is similar for ultimate periodicity. $\square$

For future reference, let us note that from the hypothesis $p \mid \sigma^{t_0+t}(x)-\sigma^{t_0}(x)$ we conclude $p \mid \sigma^{t_0+nt}(x) - \sigma^{t_0}(x)$ for all non-negative integers n: Indeed let p_0 be a generator for the ideal $\sigma^{-t_0}\big((p)\big)$. This generator p_0 divides $\sigma^t(x) - x$ and, by induction, $\sigma^{nt}(x) - x$. Applying σ^{t_0} to both sides of this relation yields the result. This and the previous lemma motivate the introduction of the following notion.

Definition. A *modular period* of a periodic prime $p \in k[x]$ is any positive integer t such that $\sigma^t(x) \equiv x \bmod p$. We also say that p *has modular period* t in this case. The minimal modular period of p is also designated as *the* modular period of p.

With this terminology, we get the following corollary of Lemma 6.1.

Corollary 6.2. *The modular period of a periodic prime $p \in k[x]$ is indeed a period. Furthermore, the (minimal) period divides any modular period. All primes of a periodic orbit have the same modular period. If p has degree 1, then the modular period is the period.*

Proof. Only the two last assertions need a proof. Let p be a periodic prime with the period t, and p_1 its pullback. They admit the modular periods kt and k_1t respectively, and these notations give $p_1 \mid \sigma^{k_1t}(x) - x$, hence $p \mid \sigma^{k_1t+1}(x) - \sigma(x)$. But the smallest positive T such that $p \mid \sigma^{T+1}(x) - \sigma(x)$ is kT so that $kt \mid k_1t$. With a round of the orbit, we see that all elements have the same modular period.

If $p = x - \alpha$, the proof of Lemma 6.1, uses $k[x]/(x - \alpha) \simeq k$ and the order of the Galois group is $s = 1$, so that the period is a modular period. □

The last point of Corollary 6.2 means that if we have a fixed point p of modular period t, it becomes a cycle of length t in the splitting field of p.

Proposition 6.3. *For the q-dilation operator with q a primitive tth root of unity, a prime polynomial p in $\mathbb{C}[x]$ has modular period t and period a divisor of t, except in case $p = x$ where the modular period and the period are* 1*.*

Proof. The q-dilation σ with q a primitive tth root of unity satisfies $Q^t = \mathrm{Id}_{k[x]}$, so every prime polynomial p in $\mathbb{C}[x]$ is periodic with a period that divides t. Indeed, if p is x, it has period 1. If not, its constant term is non-zero and is unaffected by application of σ. Therefore, for any period τ, the iterate $\sigma^\tau(p)$, which a priori is a constant times p, is precisely p. It follows that the period of p divides t. It is exactly t if and only if p involves a monomial with degree coprime with t, and, more generally, it is the smallest positive integer τ such that p is of the form $f(x^{t/\tau})$ for some polynomial f.

Moreover, the prime x has modular period 1 since it divides $(q - 1)x$ and other primes have modular period t as they do not divide $(q^i - 1)x$ when $0 < i < t$ and yet do divide $(q^t - 1)x = 0$. □

The orbits of prime polynomials are evident in most cases cited in Tables 1 and 2, but the case of a Mahlerian operator $M : f(x) \mapsto f(x^b)$ with an integer $b \geq 2$ is less known. We deal with two ground fields, the field $\mathbb{Q}$ of rational numbers, and the field $\mathbb{C}$ of complex numbers.

Proposition 6.4. *For the Mahlerian operator $\sigma(x) = x^b$, $b \geq 2$, an ultimately periodic prime polynomial in $\mathbb{Q}[x]$ is x or a cyclotomic polynomial. A cyclotomic polynomial Φ_a is purely periodic if and only if a is coprime with b. In that case Φ_a is a fixed point.*

Proof. Let $p \in \mathbb{Q}[x]$ be a monic, irreducible, ultimately periodic polynomial for the Mahlerian operator. A complex root α of p must satisfy $\alpha^{b^{t_0+t}} = \alpha^{b^{t_0}}$ for some $t_0 \geq 0$ and $t \geq 1$, hence α has to be 0 or a root of unity. As a consequence p is x or a cyclotomic polynomial Φ_a. We borrow from [13, Prop. 4, p. 14] (see also [6]) the formula

$$\Phi_a(x^b) = \prod_{b'|b,\ \gcd(a,b')=1} \Phi_{ab/b'}(x), \tag{24}$$

which shows that a cyclotomic polynomial is ultimately periodic with respect to σ (consider $b' = b/\gcd(a,b)$) and that it is purely periodic if and only if a and b are coprime. In order to prove the last point, let G_b denote the Gräffe operator, which maps a polynomial f to the resultant of $f(u)$ and $u^b - x$ with respect to u. For coprime integers a and b, the pullback of (Φ_a) is the ideal $\big(G_b(\Phi_a)\big)$, the map $x \mapsto x^b$ permutes the roots of Φ_a and so, both monic polynomials $G_b(\Phi_a)$ and Φ_a coincide. In the purely periodic case, the period is exactly 1. In contrast, in the general periodic case, the least t such that $p \mid \sigma^t(x) - x$ is a divisor of the totient function $\varphi(a)$ (for related questions see [6]). ☐

Proposition 6.5. *Consider the Mahlerian operator with exponent $b \geq 2$. A prime p of the form $x - \alpha$, where $\alpha \in \mathbb{C}$ is ultimately periodic if and only if α is 0 or a root of unity. It is purely periodic if and only if α is 0 or a primitive ath root of unity and a is coprime with b. In that case, p is periodic with a period that is a divisor of Euler's totient function $\varphi(a)$.*

Proof. A point α is ultimately periodic if and only if it satisfies an equation of the forme $\alpha^{b^{n_1}} = \alpha^{b^{n_2}}$ with $n_1 \neq n_2$, hence the first assertion. If α is a periodic root of unity, so is its minimal polynomial over $\mathbb{Q}$ and the previous proposition gives the second assertion. The number of primitive ath root of unity is $\varphi(a)$, hence the last assertion, because of the Lagrange theorem. ☐

***Example* 6.6.** With $a = 15$ and $b = 2$, we get the period $t = 4$ which divides $\varphi(15) = 8$. Here, we base on the geometric interpretation of periods in Section 1.2.2 to obtain the period. That is, we iterate $\tilde{\sigma}$ on α until we get a cycle: $\tilde{\sigma}(\alpha) = \alpha^2 \neq \alpha$, $\tilde{\sigma}^2(\alpha) = \alpha^4 \notin \{\alpha, \alpha^2\}$, $\tilde{\sigma}^3(\alpha) = \alpha^8 \notin \{\alpha, \alpha^2, \alpha^4\}$, $\tilde{\sigma}^4(\alpha) = \alpha^{16} = \alpha$.

Theorem 3.1 about reduction to the case $\delta = 0$ was asserted in Section 3 in the context of infinite orbits. Nevertheless the result is yet valid for ultimately periodic orbits, but with an additional condition.

Theorem 6.7. *Assume $\sigma \neq \mathrm{Id}_{k[x]}$ is injective and $\delta \neq 0$. Let $L \in \mathscr{D}[x]$ and $p \in k[x]$ be a utltimately periodic prime such that the primes of the associated periodic orbit have a modular period $t > 1$. Let u and v be coprime in $k[x]$ satisfying $v\big(\sigma(x) - x\big) = u\delta(x)$, ϕ the skew-polynomial ring morphism from $\mathscr{E}[x] = k[x][\vartheta; \sigma]$ into $\mathscr{D}[x]$ defined by $\phi(x) = x$ and $\phi(\vartheta) = u\partial + v$, and ψ the skew-polynomial ring morphism from $\mathscr{D}[x]$ to $\mathscr{E}(x) = k(x)[\vartheta; \sigma]$*

defined by $\psi(x) = x$ *and* $\psi(\partial) = u^{-1}(\vartheta - v)$. *The image by* ϕ *of the pseudo-torsion space* $\Gamma_p(\mathscr{E}[x]/\mathscr{E}[x]\tilde{L})$ *equals* $\Gamma_p(\mathscr{D}[x]/\mathscr{D}[x]L)$ *as a module over* $k[x]$.

Proof. The writing above reminds of the notations of Lemma B.1 and Corollary B.2. A careful reading of Lemma 3.3 shows that the used reduction at absurdum leads to a contradiction because there is in the orbit of p a prime p_j with modular period $t = 1$. With the additional hypothesis, we anew arrive at a contradiction. This is the key point. The proof of Theorem 3.1 applies without modification. □

***Example* 6.8.**

Eventually the only case where the reduction to $\delta = 0$ does not work is the case where the orbit ends with a fixed point with modular period $t = 1$.

If we extend the ground field k into the splitting field K of a prime p with modular period $t > 1$, we obtain a cycle of period $t > 1$. Hence the reduction to $\delta = 0$ works except in the case where we consider a prime whose orbit ends at a fixed point in $\mathbb{C}$.

6.2. **Periodic candidate primes.**

6.2.1. *Linear algebra reformulation.*

σ injective ? Oui, sûrement, sinon comment τ pourrait-il être un isomorphisme ?

PhD

The key tool which allow us to study the local torsion at a periodic polynomial is section operations. For fixed t, the *sections* with regard to ∂ of a finite sum $A = \sum_{i\geq 0} a_i\partial^i \in \mathscr{D}[x]$ are defined by

$$A_u = \sum_{j\geq 0} a_{jt+u}\partial^{tj} \qquad \text{for } 0 \leq u < t;$$

they satisfy $A = \sum_{0\leq u<t} A_u\partial^u$. The sub-$k[x]$-algebra of $\mathscr{D}[x]$ generated by ∂^t, denoted $k[x]\langle\partial^t\rangle$, plays a central role in what follows. To reflect this, consider the endomorphism $\tau = \sigma^t$ of $k[x]$ and introduce the skew-polynomial ring $\mathscr{E}[x] = k[x][\vartheta; \tau]$. The $k[x]$-algebra morphism from $\mathscr{E}[x]$ to $k[x]\langle\partial^t\rangle$ that maps ϑ to ∂^t is an algebra isomorphism, as shown by the commutation $\partial^t x = \tau(x)\partial^t$. We thus identify ϑ and ∂^t from now on. It will prove convenient to extend the ring morphism σ to $\mathscr{D}[x]$ by $\sigma(\partial) = \partial$, so that, for A as above,

$$\sigma(A) = \sum_{i\geq 0} \sigma(a_i)\partial^i.$$

We are now ready to compute the sections of a product $U = PL$ with P and L taken from $\mathscr{D}[x]$. First, collecting terms with respect to ∂ yields the relation

$$U = \sum_{v,w} P_v\partial^v L_w\partial^w = \sum_{v,w} P_v\sigma^v(L_w)\partial^{v+w},$$

which re-writes the skew polynomial U in terms of the skew polynomials P_v and L_w from $\mathscr{E}[x]$. Taking sections then yields

$$U_u = \sum_{v+w=u} P_v \sigma^v(L_w) + \sum_{v+w=u+t} P_v \sigma^v(L_w)\vartheta.$$

Let P and U be the row vectors of the sections associated with P and U: $\mathsf{P} = (P_0, \dots, P_{t-1})$, $\mathsf{U} = (U_0, \dots, U_{t-1})$. Multiplication on the right by L in $\mathscr{D}[x]$ translates into the mutiplication on the right by a $t \times t$ square matrix L with entries in $\mathscr{E}[x]$, namely

(25)

$$\mathsf{L} = \begin{pmatrix} L_0 & L_1 & \dots & L_{t-2} & L_{t-1} \\ \sigma(L_{t-1})\vartheta & \sigma(L_0) & \dots & \sigma(L_{t-3}) & \sigma(L_{t-2}) \\ \vdots & \vdots & & \vdots & \vdots \\ \sigma^{t-2}(L_2)\vartheta & \sigma^{t-2}(L_3)\vartheta & \dots & \sigma^{t-2}(L_0) & \sigma^{t-2}(L_1) \\ \sigma^{t-1}(L_1)\vartheta & \sigma^{t-1}(L_2)\vartheta & \dots & \sigma^{t-1}(L_{t-1})\vartheta & \sigma^{t-1}(L_0) \end{pmatrix} = \left(\sigma^{i-1}(L_{j-i \bmod t})\vartheta^{[[i>j]]}\right),$$

where the exponent $[[i > j]]$ is 1 whenever $i > j$, zero otherwise.

***Example* 6.9.** When $t = 2$, U is given by the relations

$$U_0(x,\vartheta) = P_0(x,\vartheta)L_0(x,\vartheta) + P_1(x,\vartheta)L_1\big(\sigma(x),\vartheta\big)\vartheta,$$
$$U_1(x,\vartheta) = P_0(x,\vartheta)L_1(x,\vartheta) + P_1(x,\vartheta)L_0\big(\sigma(x),\vartheta\big),$$

and the mutiplication matrix is given by

$$\mathsf{L} = \begin{pmatrix} L_0 & L_1 \\ \sigma(L_1)\vartheta & \sigma(L_0) \end{pmatrix}.$$

PhD Commenter. On est passé de la matrice infinie de la section 1.7 à une matrice finie.

6.2.2. *Periodic primes and determinant.* Let us consider a purely periodic prime polynomial p and one of its modular periods t. Then p divides $\tau(p)$ for $\tau = \sigma^t$. As a consequence, the map τ induces an automorphism of $k[x]/(p)$, because the pullback of the prime ideal (p) by τ is (p) itself. Let λ be the image of x in the quotient field $k[x]/(p)$. Because of the condition $\tau(x) \equiv x \mod p$, the commutation rule $\vartheta x = \tau(x)\vartheta$ of $\mathscr{E}[x]$ reduces to $\vartheta\lambda = \lambda\vartheta$ in $\big(k[x]/(p)\big)[\vartheta;\tau] = k(\lambda)[\vartheta;\tau]$. Thus, the skew-polynomial ring $k(\lambda)[\vartheta;\tau]$ is merely the commutative polynomial ring in ϑ with coefficients in the algebraic field $k(\lambda)$, that is $k(\lambda)[\vartheta]$. Observe that the construction of $\mathscr{E}[x]$ works for any period t, whereas obtaining the commutativity of the ring $k(\lambda)[\vartheta;\tau]$ requires t to be a modular period.

Let us assume that we have a non-trivial solution (P, R) to the equation $PL = p^\nu R$. We immediately obtain the congruence $PL \equiv 0 \mod p$. By a common argument of linear algebra in $k(\lambda)[\vartheta]$, we obtain that the determinant $\det \mathsf{L}$ vanishes modulo p. This remark provides us with a necessary condition for a prime p to provide non-trivial torsion.

Lemma 6.10. *For a given integer t, any candidate prime p with modular period t divides the determinant of the $t \times t$ matrix L.*

***Example* 6.11.** We continue Example 2.3, with $\mathscr{D}[x] = \mathbb{Q}[x][\partial; \sigma, \delta]$, and $\sigma(x) = x^2 - 2$. A fixed prime is $p = x + 1$ and the modular period is $t = 1$. Hence the matrix L is simply the skew polynomial L. The condition $\det \mathsf{L} = 0 \bmod p$ writes $L(-1, \partial) = 0$, which means that all coefficients of L have a factor $x + 1$ and this is impossible if L is primitive. Hence a singularity at -1 cannot be removed. The same reasoning works for $p = x - 2$.

6.2.3. *Anew leading and trailing coefficients.* As we have just seen, the candidates primes can be found as factors of the determinant of a commutative matrix whose size is given by their *modular* period. But, except in the case of q-calculus of the next Section 6.3, we do not know a priori the modular period. As we shall see, this is no real obstacle. Let us assume that $L \in \mathscr{D}[x]$ has degree r with respect to ∂ and fix some integer t greater than r. Next, fix any prime polynomial p which admits t as a (non-necessarily minimal) modular period. Then, the sections of L are merely the coefficients of L, that is, the equality $L_u = \ell_u \in k[x]$ holds for $0 \leq u < t$. Due to the special form of the matrix L in (25), the determinant of L has a degree not greater than $t - 1$ in ϑ. More precisely, if L has degree r and valuation s with respect to ∂, then $\det \mathsf{L}$ also has degree r and valuation s. Up to signs, the coefficients of ϑ^r and ϑ^s in this determinant are the products $\ell_r\sigma(\ell_r) \cdots \sigma^{t-1}(\ell_r)$ and $\ell_s\sigma(\ell_s) \cdots \sigma^{t-1}(\ell_s)$, respectively. This yields anew Theorem 1.8, but with an independent proof.

On regarde les facteurs périodiques de ℓ_r et ℓ_s. Il faut que je trouve la bonne expression. PhD

Proposition 6.12. *If $L \in \mathscr{D}[x]$ has degree r and valuation s with respect to ∂, a periodic prime p of modular period t can contribute a non-trivial torsion $T_p(L)$ only if there exist non-negative integers i and j such that $0 \leq i, j < t$ and the following equivalent conditions are satisfied:*

Proof. Continuing with the setting and notation introduced before the proposition statement, both products must be divisible by p for L to be singular modulo p. This yields the condition by the definition of pullbacks. ☐

The search for the periodic prime which appear in the periodic prime factors of ℓ_r and ℓ_s determines the possible periods.

6.3. **Dilation operators with q a root of unity.** In case of a dilation operator with q a root of unity, the application of the Lemma 6.10 is particularly easy because the modular period is known in advance. Moreover there is no need to consider a quotient field.

Theorem 6.13 (Candidate primes and their multiplicity for the q-calculus)**.** *Suppose q is a tth root of unity and $\mathscr{D}$ is the skew-polynomial ring $k[x][\partial; \sigma]$ associated with the q-dilation σ, so that $\sigma(x) = qx$. Any prime p such that a relation $PL = p^\nu R$ exists with $P \notin p\mathscr{D}$ divides the determinant of the matrix L associated to L. Additionally, in order that there exists torsion of multiplicity ν at a prime p, the polynomial p^ν must divide the determinant of the matrix L.*

Proof. In the case under consideration, the morphism $\tau = \sigma^t$ is the identity function of $k[x]$. Consequently, the relation $\vartheta x = x\vartheta$ for $\vartheta = \partial^t$ is satisfied without having to reduce the equation modulo p. The ring $k[x][\vartheta;\tau]$ is the polynomial commutative ring $k[x,\vartheta]$. Besides Lemma 6.10 applies: if $PL = p^\nu R$ with $P \notin p\mathscr{D}[x]$, then p divides $\det \mathsf{L}$. Now, writing L^+ for the transpose of the comatrix of L, we get $\mathsf{P} \cdot \det \mathsf{L} = \mathsf{P}\mathsf{L}\mathsf{L}^+ = 0 \mod p^\nu$. From $P \notin p\mathscr{D}$ follows that at least one coordinate of P is not divisible by p, and is thus invertible modulo p^ν. Therefore $\det \mathsf{L}$ is zero modulo p^ν. ☐

***Example* 6.14.** Let $q = -1$, which gives $t = 2$, and $L = (x+2)^2 + (x+1)(x-2)^2\partial$. Then

$$\mathsf{L} = \begin{pmatrix} (x+2)^2 & (x+1)(x-2)^2 \\ (1-x)(x+2)^2\vartheta & (x-2)^2 \end{pmatrix}.$$

The determinant of this matrix is $\det \mathsf{L} = (x-2)^2(x+2)^2\big((x^2-1)\vartheta+1\big)$, thus the candidate primes are $x-2$ and $x+2$, both of multiplicity at most 2.

We continue with the computation of the torsion spaces $\Gamma_{x-2}(L)$ and $\Gamma_{(x-2)^2}(L)$. To this end, we set $p = x-2$. The matrix L rewrites, in p-adic expansion,

$$\mathsf{L} = \mathsf{L}_0 + \mathsf{L}_1 p + \mathsf{L}_2 p^2 + \cdots = \begin{pmatrix} 16 & 0 \\ -16\vartheta & 0 \end{pmatrix} + \begin{pmatrix} 8 & 0 \\ -24\vartheta & 0 \end{pmatrix} p + \begin{pmatrix} 1 & 3 \\ -9\vartheta & 1 \end{pmatrix} p^2 + \begin{pmatrix} 0 & 1 \\ -\vartheta & 0 \end{pmatrix} p^3.$$

To solve $\mathsf{P}\mathsf{L} = 0$, we write in the same way $\mathsf{P} = \sum_{i\geq 0} \mathsf{P}_i p^i$, where $\mathsf{P}_i = (a_i, b_i)$, solve modulo p first, then lift modulo p^2, etc. The first steps yield $a_0 = b_0\vartheta$. As a consequence, $\mathsf{P}_0 = \begin{pmatrix} u\vartheta & u \end{pmatrix}$ for $u \in k[\vartheta]$ and

$$\mathsf{R} = p^{-1}\mathsf{P}\mathsf{L} = u\begin{pmatrix} -16\vartheta - 8\vartheta p - \vartheta p^2 & (3\vartheta+1)p + \vartheta p^2 \end{pmatrix}.$$

Therefore,

$$\Gamma_{x-2}(L) = k[x,\vartheta]R_1 \qquad \text{with} \qquad R_1 = (x-2)\partial - 8x\partial^2 + (x^2-x-2)\partial^3.$$

We then assume $\mathsf{P} \equiv (a_0 + a_1 p, b_0 + b_1 p) \mod p^2$. Solving $\mathsf{P}\mathsf{L} \equiv 0 \mod p^2$ yields $a_0 = b_0\vartheta$ and $a_1 = (b_0+b_1)\vartheta$. As a consequence, $\mathsf{P} = \begin{pmatrix} u\vartheta + (u+v)\vartheta p & u + vp \end{pmatrix}$ for u and v from $k[\vartheta]$, and therefore

$$\mathsf{R} = p^{-2}\mathsf{P}\mathsf{L} = \begin{pmatrix} -16v\vartheta - 8v\vartheta p - v\vartheta p^2 & u(3\vartheta+1) + (v + 3v\vartheta + 4u\vartheta)p + (u+v)\vartheta p^2 \end{pmatrix}.$$

Therefore,

$$\Gamma_{(x-2)^2}(L) = k[x,\vartheta]R_{2,1} + k[x,\vartheta]R_{2,2}$$

with

$$R_{2,1} = \partial + (x-1)(x+1)\partial^3, \qquad R_{2,2} = (x-2)\partial - (x+2)^2\partial^2 + (x+1)(x-2)\partial^3.$$

FC THERE REMAINS TO COMPARE $\Gamma_{p^\nu}(L)$ AND $\Gamma_{\sigma(p)^\nu}(L)$ ON THIS EXAMPLE.

PhD

Cette question doit être traitée d'un point de vue général. En fait il devrait y avoir une section là-dessus. Une section pour comparer ce qui se passe sur une orbite. Une section sur la multiplicité. Une section sur Mahler. Et aussi une section avec un algorithme.

6.4. **General periodic situation.**

Pouvons nous utiliser *Torsion by linear algebra* ?

Il faut voir s'il est possible d'utiliser le lemme 1.16 pour passer des points fixes aux orbites périodiques. Par exemple sachant que pour Mahler, les polynômes cyclotomiques Φ_a où a est premier avec b sont fixes et que la torsion est triviale, ne puis-je en déduire qu'il en est de même pour les $x-\alpha$ où α est une racine primitive a^{e} de l'unité ? Plus il faudrait relier les deux torsions entre un point fixe et un cycle obtenu par extension du corps de base. On a envie d'utiliser le ppcm des conjugués pour revenir de l'extension vers le corps de base. PhD

When p is a periodic candidate, there always exists an integer h such that the pullbacks $\sigma^{-i}\big((p)\big)$ and $\sigma^{-j}\big((p)\big)$ in the previous proposition are related by

$$\sigma^{-h}\Big(\sigma^{-j}\big((p)\big)\Big) = \sigma^{-i}\big((p)\big).$$

Moreover, all primes in the cycle of p are candidates as well. In order to distinguish fewer candidate primes, we can thus arbitrarily select those candidate primes that divide the leading coefficient of L. This corresponds to imposing $i = 0$ in the previous proposition. We thus get the smaller set

$$\mathcal{DC} = \{\, p \in k[x] : p \text{ prime and periodic},\ \exists i \geq 0,\ p \mid \operatorname{lc}(L),\ \operatorname{tc}(L) \in \sigma^{-i}\big((p)\big) \,\}.$$

Given a skew polynomial $L \in \mathscr{D}[x]$, we want to select finitely many distinguished candidate primes that enable us to compute the part of the torsion $C(L)$ contributed by periodic primes.

Cette phrase exprime que non seulement on veut les candidats premiers, mais qu'en plus on veut en choisir certains qui vont suffire pour avoir la bonne information. PhD

Pour le cas periodic ∂^t défini un endomorphisme injectif du $k[x]$-module $T_p(L)$ pour chaque p de l'orbite. Ceci fait monter le degré en ∂. On a envie de diviser par L, dans $\mathscr{D}(x)$. PhD

Idée qui semble vouée à l'échec : pour étudier la torsion en $p = x - \alpha$, on étend k par une indéterminée a d'où une orbite infinie, puis on substitue α à a. PhD

Appendix A. Structure of skew-polynomial rings

The skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ is generated over $k[x]$ by the indeterminate ∂ and the relations $\partial f = \sigma(f)\partial + \delta(f)$ for all $f \in k[x]$. Both morphisms σ and δ which define the skew-polynomial ring are themselves related, since δ is a σ-derivation and moreover they are completely determined by their values on x. More precisely, we have the following relation [8, 11].

Lemma A.1 (Classification of skew-polynomial rings)**.** *The morphisms σ and δ which define a skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ are related by the formula*

$$\delta = \begin{cases} \delta(x)\dfrac{d}{dx}, & \text{if } \sigma(x) = x, \text{ i.e., } \sigma = \mathrm{Id}_{k[x]}, \\ \delta(x)\dfrac{\sigma - \mathrm{Id}_{k[x]}}{\sigma(x) - x}, & \text{otherwise.} \end{cases}$$

Proof. By linearity, it is sufficient to prove the identity on monomials x^i, $i \in \mathbb{N}$. The proof is by induction. The equality $\delta(1\cdot 1) = \sigma(1)\delta(1)+\delta(1) = 2\delta(1)$ shows that $\delta(1) = 0$ and this is the formula for $i = 0$. Then, δ being a σ-derivation, we have $\delta(x^{i+1}) = \delta(x)x^i+\sigma(x)\delta(x^i)$. In case $\sigma(x) = x$, with the hypothesis $\delta(x^i) = \delta(x)ix^{i-1}$, we obtain $\delta(x^{i+1}) = (i+1)\delta(x)x^i$ as desired. Otherwise, we get $\delta(x^{i+1}) = \delta(x)x^i + \sigma(x)\delta(x)(\sigma(x^i) - x^i)/(\sigma(x) - x)$ whence the result since $\sigma(x)\sigma(x^i) = \sigma(x^{i+1})$. ☐

Lemma A.2. *Let $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ be a skew-polynomial ring. Then, there exists $t \in \mathbb{N}$ such that $\sigma^t = \mathrm{Id}_{k[x]}$ if and only if the identity $\sigma(\tilde{x}) = q\tilde{x}$ holds for a t-th root of unity q and a generator $\tilde{x}$ of $k[x]$.*

Proof. If $\sigma^t = \mathrm{Id}_{k[x]}$ for a positive integer t, then by considering degrees, we get that $\sigma(x) = qx+r$ for some non-zero constant q. From the relation $\sigma^t(x) = q^tx+(1+q+\cdots+q^{t-1})r = x$ we get that q is a tth root of unity. Taking $\tilde{x} = x$ if $q = 1$ and $\tilde{x} = x - r/(1-q)$ otherwise concludes the proof. The converse claim is obvious. ☐

The sub-ring of constants is the ring of those elements that commute with ∂. The following result makes this more explicit.

Corollary A.3 (Characterization of constants)**.** *The ring $\mathscr{C}$ of constants of a skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ is given by*

$$\mathscr{C} = \begin{cases} k[((q-1)x+r)^b] & \text{when } \sigma(x) = qx+r, \text{ with } q^b = 1,\ b \neq 1 \text{ and } r \in k; \\ k[x] & \text{when } \sigma = \mathrm{Id}_{k[x]} \text{ and } \delta = 0; \\ k & \text{otherwise.} \end{cases}$$

Proof. That $k \subset \mathscr{C}$ follows from the definition of σ-derivations. If $\sigma = \mathrm{Id}_{k[x]}$; then $\mathscr{C} = \delta^{-1}(0)$. If $\delta = 0$, then $\mathscr{C} = k[x]$; otherwise by Lemma A.1, $\mathscr{C} = \ker d/dx$ and the last case follows as k is assumed to be of characteristic zero.

Otherwise, $\sigma \neq \mathrm{Id}_{k[x]}$. If $c(x) \in \mathscr{C}$, then $\sigma(c) = c$ (according to the definition of Section 1.1), which means $c(\sigma(x)) = c(x)$. If there exists a constant $c(x)$ with $\deg c > 0$, then degree considerations show that $\sigma(x) = qx + r$ for some q and r in k with $q \neq 0$.

Considering the coefficient of highest degree then shows that $q^{\deg c} = 1$. Thus q is a root of unity. Now let $\tilde{x} = (q-1)x + r$ and observe that $\sigma(\tilde{x}) = q\tilde{x}$ and therefore $\sigma(\tilde{x}^i) = q^i\tilde{x}^i$ for all non-negative integers i. Anew with the hypothesis $\deg c > 0$, we rule out the case $q = 1$: otherwise $c(x+r) = c(x)$ implies $r = 0$ and then $\sigma = \mathrm{Id}_{k[x]}$, contrary to our assumption. Then c can be written in the basis $(\tilde{x}^i)_{i\geq 0}$ and the first case of the assertion follows from identifying coefficients of identical degrees. ☐

Lemma A.4. *Assume that the derivation $\delta \neq 0$ of a skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ strictly decreases the degree of non-constant polynomials. Then $\sigma(x) = qx + r$ for some q and r in k, and δ (given by Lemma A.1) is such that $\delta(x) \in k$ is non-zero.*

Proof. First note that $\delta(x) = 0$ would imply $\delta = 0$. Now, the hypothesis implies that $\delta(x^2)$ has degree at most 1. At the same time, $\delta(x^2) = \sigma(x)\delta(x) + \delta(x)x = \delta(x)\big(\sigma(x) + x\big)$ and $\sigma(x)$ re-writes in the form $qx + r$ for some q and r in k. ☐

The skew-polynomial ring $\mathscr{D}[x]$ is non-necessarily Noetherian.

Lemma A.5. *The skew-polynomial ring $\mathscr{D}[x] = k[x][\partial;\sigma,\delta]$ is Noetherian if an only if* $\deg\sigma(x) \leq 1$.

Proof. The positive case comes from [16, Thm. 3.5.12]. The "extended axiom" [16, Def. 3.2.2] used here is the fact that k is a field.

Je ne comprends pas. l'hypothèse n'est pas utilisée. PhD

The example provided in [21] of the ideal generated by the $x\partial^k$, $k \geq 0$, in the case of the Mahlerian operators, extends to the case $\deg\sigma(x) \geq 2$ and shows that such a skew-polynomial ring is not Noetherian. ☐

Analogues to the results in this section when $k[x]$ and $k(x)$ are respectively replaced with $k[[x]]$ and $k((x))$ can be derived by similar arguments.

Appendix B. Reduction to the case $\delta = 0$

Working in skew-polynomial rings with $\delta = 0$ considerably simplifies computations: as can be seen from Theorem 1.8 (and its direct consequence Lemma 1.9) and Lemma 1.13, we can indeed perform additional reductions when $\delta = 0$ in order to simplify the search for candidate primes and improve the bounds used to compute local torsion spaces.

In order to identify a skew-polynomial ring with non-zero δ as a subring of a skew-polynomial ring where $\delta = 0$, and to be able to related torsion spaces in both situations, we proceed to introduce injective maps between the two contexts. To this end, let us call a skew-polynomial ring morphism from a skew-polynomial ring $\mathscr{S}_1 = A_1[\partial_1;\sigma_1,\delta_1]$ to another skew-polynomial ring $\mathscr{S}_2 = A_2[\partial_2;\sigma_2,\delta_2]$, over respective commutative k-algebras A_1 and A_2, a k-algebra morphism from A_1 to A_2 which extends to a k-algebra morphism μ from $\mathscr{S}_1$ to $\mathscr{S}_2$ satisfying $\mu\circ\sigma_1 = \sigma_2\circ\mu$ and $\mu\circ\delta_1 = \delta_2\circ\mu$. When $A_1 = A_2 = k[x]$, a skew-polynomial ring morphism μ is therefore just a k-algebra morphism satisfying $\mu(x) \in k[x]$

and $\partial_1' x' = \sigma_2(x')\partial_1' + \delta_2(x')$ for $x' = \mu(x)$ and $\partial_1' = \mu(\partial_1)$. The key to the reduction to $\delta = 0$ is given by the following construction.

Lemma B.1. *Let $\mathscr{D}[x]$ be $k[x][\partial;\sigma,\delta]$. Assume that σ is an injective map but $\sigma \neq \mathrm{Id}_{k[x]}$ and that $\delta \neq 0$. Then, for any coprime u and v in $k[x]$, the following properties are equivalent:*

1. *$v\big(\sigma(x) - x\big) = u\,\delta(x)$;*
2. *the skew-polynomial ring morphism ϕ from $\mathscr{E}[x] = k[x][\vartheta;\sigma]$ into $\mathscr{D}[x]$ given by $\phi(x) = x$ and $\phi(\vartheta) = u\partial + v$ is well defined and injective;*
3. *the skew-polynomial ring morphism ψ from $\mathscr{D}[x]$ to $\mathscr{E}(x) = k(x)[\vartheta;\sigma]$ given by $\psi(x) = x$ and $\psi(\partial) = u^{-1}(\vartheta - v)$ is well defined and injective.*

Additionally, such a pair (u, v) exists.

Proof. The morphism σ is injective, so that $\mathscr{E}(x) = k(x)[\vartheta;\sigma]$ is well defined (Section 1.1). The existence claim follows from Lemma A.1, since $\sigma \neq \mathrm{Id}_{k[x]}$. More precisely, there exists a polynomial w such that $u = \sigma(w) - w \neq 0$, and according to this lemma $\delta(w)(\sigma(x) - x) = \delta(x)(\sigma(w) - w)$ and $v = \delta(w) \neq 0$ because $\delta(x) \neq 0$ necessarily. Property (1) is equivalent to both commutation relations $(u\partial + v)x = \sigma(x)(u\partial + v)$ and $\big(u^{-1}(\vartheta - v)\big)x = \sigma(x)(u^{-1}(\vartheta - v)\big) + \delta(x)$. In turn, this is equivalent to the fact that, as defined, both ϕ and ψ respect the skew-polynomial ring structures: $\phi(\vartheta)\phi(x) = \sigma\big(\phi(x)\big)\phi(\vartheta)$ and $\psi(\partial)\psi(x) = \sigma\big(\psi(x)\big)\psi(\partial) + \delta\big(\psi(x)\big)$. Eventually ϕ and ψ are injective maps because the degree in ∂ is left unchanged as well as the leading coefficient up to a factor which depends only on u . ∎

Once a pair (u, v) is fixed, as well as ϕ and ψ as in the lemma, the denominators in $\psi(L)$ for $L \in \mathscr{D}[x]$ are constrained. Indeed, define d_L to be the monic least common denominator of the coefficients of $\psi(L)$ viewed as a polynomial in ϑ. By construction, this divides a product $\prod_{j\geq 0} \sigma^j(u)^{e_j}$ with $e_j \geq 0$; hence, $\tilde{L} = \psi(d_L L) \in \mathscr{E}[x]$. Since $\psi \circ \phi$ is the inclusion map of $\mathscr{E}[x]$ into $\mathscr{E}(x)$, we get $0 = \psi\big(d_L L - \phi(\tilde{L})\big)$, and $d_L L \in \phi(\mathscr{E}[x])$ follows from the fact that ψ is injective. Moreover the skew polynomial $\tilde{L}$ is primitive if the skew-polynomial L has this property.

Corollary B.2. *Let L be a primitive skew polynomial in $\mathscr{D}[x]$ and let d_L be the least common denominator of the coefficients of $\psi(L) \in \mathscr{E}(x)$. Then $\tilde{L} = \psi(d_L L)$ is a primitive skew polynomial in $\mathscr{E}[x]$, and L and $\tilde{L}$ are related by $\phi(\tilde{L}) = d_L L$, $\psi(L) = d_L^{-1}\tilde{L}$.*

The skew-polynomial ring $\mathscr{E}[x]$ of Lemma B.1 belongs to the class of skew-polynomial rings with $\delta = 0$. If we are interested in computing the local pseudo-torsion space $\Gamma_p(L)$ of some $L \in \mathscr{D}[x]$, this lemma constructs an $\tilde{L} \in \mathscr{E}[x]$ and $d_L \in k[x]$ such that $d_L L = \phi(\tilde{L})$. Since ϕ respects the skew-polynomial ring structures, it induces a map also denoted ϕ from $\mathscr{E}[x]/\mathscr{E}[x]\tilde{L}$ into $\mathscr{D}[x]/\mathscr{D}[x]L$.

Lemma B.3. *The map ϕ defines an injective map, also denoted ϕ, from $\mathscr{E}[x]/\mathscr{E}[x]\tilde{L}$ into $\mathscr{D}[x]/\mathscr{D}[x]L$. For any prime polynomial p, it sends the local pseudo-torsion $\Gamma_p(\tilde{L})$ into the local pseudo-torsion $\Gamma_p(L)$.*

Proof. Let π be the projection from $\mathscr{D}[x]$ into $\mathscr{D}[x]/\mathscr{D}[x]L$. The kernel of $\pi \circ \phi$ is $\mathscr{E}[x] \cap \mathscr{E}(x)d_L^{-1}\tilde{L}$, but $\mathscr{E}(x)d_L^{-1} = \mathscr{E}(x)$ and the kernel is $\mathscr{E}[x]\tilde{L}$. Hence $\pi \circ \phi$ factorizes through the quotient $\mathscr{E}[x]/\mathscr{E}[x]\tilde{L}$ into an injective map, again denoted ϕ.

Let S in $\mathscr{E}[x]$ be such that $Q\tilde{L} = p^\nu S$ for some $Q \in \mathscr{E}[x]$ and $\nu \geq 0$. Applying ϕ we obtain $\phi(Q)d_L L = p^\nu \phi(S)$. We see that the induce map ϕ from $\mathscr{E}[x]/\mathscr{E}[x]\tilde{L}$ into $\mathscr{D}[x]/\mathscr{D}[x]L$ sends a torsion element $S + \mathscr{E}[x]\tilde{L} \in \Gamma_p(\tilde{L})$ onto a torsion element $\phi(S) + \mathscr{D}[x]L \in \Gamma_p(L)$. □

The search for a description of $\phi(\Gamma_p(\tilde{L}))$ is made only in the case of an infinite orbit (Section 3.1).

Appendix C. Laurent-Ore algebra for invertible σ

In this section the endomorphism σ of the basic commutative ring A, usually $k[x]$ or $k(x)$, is assumed to be a bijective (one-to-one) map. Moreover the endomorphism δ is the zero map. As a consequence it is tempting to manage the following computation for $a \in A$: $\partial a = \sigma(a)\partial$, hence $a = \partial^{-1}\sigma(a)\partial$, $a\partial^{-1} = \partial^{-1}\sigma(a)$, therefore, interverting the roles of a and $\sigma(a)$, the formula $\partial^{-1}a = \sigma^{-1}(a)\partial^{-1}$. This leads to two equivalent definitions of Laurent-Ore algebras. Then, with an eye forward to the applications, we describe how to use Gröbner bases in our enlarged setting. The description we present here also comes from the thesis work of Wu [22]. An alternative approach can be found in [23].

C.1. **Definitions of Laurent-Ore algebras.** For a given skew-polynomial ring $\mathscr{S} = A[\partial;\sigma]$ over some commutative ring A, such that σ is invertible and $\delta = 0$, we incorporate the inverse of σ into a generalization of skew-polynomial rings by using Laurent-Ore algebras. As a first way to do this, we consider the associative algebra generated by ∂ and ∂^{-1} over A subject to the relations

$$\partial\partial^{-1} = \partial^{-1}\partial = 1, \qquad \partial a = \sigma(a)\partial, \qquad \partial^{-1}a = \sigma^{-1}(a)\partial^{-1}, \qquad \text{for all } a \in A.$$

Elements of this ring are finite sums of the form $\sum a_i\partial^i$, with $a_i \in A$ and the index i runs through $\mathbb{Z}$. Following Wu, we call this ring a *Laurent-Ore algebra* and we denote it by either $\mathscr{L} = A[\partial,\partial^{-1};\sigma]$ or $\mathscr{L} = \mathscr{S}[\partial^{-1}]$. Similarly to our habit for $\mathscr{D}[x]$ and $\mathscr{D}(x)$, we denote $\mathscr{L}[x] = k[x][\partial,\partial^{-1};\sigma]$ and $\mathscr{L}(x) = k(x)[\partial,\partial^{-1};\sigma]$ for $A = k[x]$ and $A = k(x)$ respectively.

We may employ an equivalent construction of Laurent-Ore algebras with the goal of defining Gröbner bases. The construction uses Ore algebras, which are a natural multivariate extension of skew-polynomial rings. These algebras, presented formally by Chyzak and Salvy [10], allow us to represent the inverse by a formal non-commutative indeterminate. For our purpose, they are skew-polynomials rings $k[x][\partial_1;\sigma_1,\delta_1]\cdots[\partial_r;\sigma_r,\delta_r]$ constrained by the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\partial_i\partial_j = \partial_j\partial_i$ for $i \neq j$, and $\sigma_i(\partial_j) = \partial_j$, $\delta_i(\partial_j) = 0$ for $i > j$. A Laurent-Ore algebra $\mathscr{L} = A[\partial,\partial^{-1};\sigma]$ may be wiewed as an image of an Ore Algebra. Obviously we consider a skew-polynomial ring $\mathscr{S} = A[\partial;\sigma]$, where σ is invertible and δ is 0. We define the associated Ore algebra by $\mathscr{O} = \mathscr{S}[\vartheta;\tau]$, where τ denotes the extension of σ^{-1} from A to $\mathscr{S}$ by the relation $\tau(\partial) = \partial$. We remark that $(\partial\vartheta)a = a(\partial\vartheta)$ for any $a \in \mathscr{S}$, because we have $\tau(a) = \sum \tau(a_k)\tau(\partial^k) = a$ if $a = \sum a_k\partial^k$. In other words, $\partial\vartheta$

is in the center of $\mathscr{O}$. This implies that $\mathfrak{T} = \mathscr{O}\,(\partial\vartheta - 1)$ is a two-sided ideal. Now, consider the quotient ring $\mathscr{L} = \mathscr{O}/\mathfrak{T} = A[\bar\partial, \bar\vartheta]$, where $\bar\partial$ and $\bar\vartheta$ stand for the canonical images of ∂ and ϑ respectively. This ring exhibits the desired behaviour: $\bar\partial\bar\vartheta = \bar\vartheta\bar\partial = 1$. This is, up to variable renaming, the Laurent-Ore algebra $\mathscr{L} = A[\partial, \partial^{-1}; \sigma]$.

C.2. **Gröbner bases.** In the setting of Ore algebras, we can algorithmically determine Gröbner bases (building upon earlier work by Kredel [16]). Again, it is the interaction with the inverse of ∂^{-1} that prevents us from working directly in a Laurent-Ore algebra. First we describe Gröbner bases in Ore algebras, and then in Laurent-Ore algebras.

We compute Gröbner bases using a non-commutative analogue of Buchberger's algorithm (see [10] and the references therein). As in the commutative case, the definition is with respect to an admissible monomial ordering, where monomials just means products of indeterminates (commutative and with coefficient 1). A polynomial is then a linear combination of monomials with scalar coefficients. An admissible monomial ordering is a total ordering $\leq$ of the set of monomials in the algebra with the 1 as the least element. Here we consider lexicographical ordering with respect to the degree vector of a monomial taken in the order specified. Thus with respect to the ordering $\mathrm{lex}(\partial < \bar\partial < n)$, we have $n^5\partial\bar\partial < n^2\bar\partial^3\partial$, but the order is reversed with respect to the ordering $\mathrm{lex}(\partial < n < \bar\partial)$. The *leading monomial* $\mathrm{lm}(p)$ of a polynomial p is the largest monomial with non-zero coefficient in the sum according to the ordering. Using this notion, we also demand, for a monomial ordering to be admissible, a compatibility with multiplication in the algebra. This is, for any t, u, and v in the algebra, $\mathrm{lm}(tu) \leq \mathrm{lm}(tv)$, whenever $\mathrm{lm}(u) \leq \mathrm{lm}(v)$. Remark that the product of two monomials is not necessarily a monomial, due to the presence of derivation operators δ. A polynomial p with leading monomial s is said to be reducible by the element q (with respect to a fixed order), if $\mathrm{lm}(q) = t$ and t divides s as commutative polynomials. If $s = \mathrm{lm}(t't)$, then s reduces to p' where $p' = p - at'p$, where the scalar factor a ensures that the monomial s appears with coefficient zero in p'. The main property of Gröbner bases that we use in the applications is that for any u in the ideal generated by the basis, there is some v in the Gröbner basis such that $\mathrm{lm}(v)$ divides $\mathrm{lm}(u)$ as commutative polynomials.

Gröbner bases in Laurent-Ore algebras are achieved by defining an order compatible with the positive and negative powers of the variables: the ordering on $\mathscr{O}$ must imply $1 < \partial < \partial^2 < \cdots$ and $1 < \partial^{-1} < \partial^{-2} < \cdots$ by the canonical projection π from $\mathscr{O}$ to $\mathscr{L}$. A set is a Gröbner basis of an ideal in the Laurent-Ore algebra $\mathscr{L}$ if its pre-image under π is a Gröbner basis of the corresponding ideal in the Ore algebra $\mathscr{O}$ with respect to the lifted order. Precise definitions and statements are given by [22, Section 3.2].

Appendix D. Implementation

Algorithms to compute global torsion have been implemented in Maple for the the differential, shift, and q-shift cases, by Ha Le, and the differential case has been implemented in Magma by Alexa van der Waall. The Maple code is available at http://algo.inria.fr/software.

FIGURE 4. TIMINGS OF TEST SYSTEM 1 (LEFT) AND 2 (RIGHT). *OC indicates the Ore Closure algorithm, vH indicates the van Hoeij implementations, and vH2 indicates the auxiliary van Hoeij implementation. The x-axis indicates the n parameter, and the y-axis indicates the time in seconds to desingularize.*

There are Maple implementations by Mark van Hoeij of the algorithms described in [4] [1], [1] [2]. van Hoeij also provides an improved implementation for the trailing coefficient[3]. We now provide timing and aesthetic comparisons of these algorithms and ours. We are not aware of any comparable routines in Magma.

D.1. **Difference equation timing comparisons.**

We compute more! PhD

The principal difference between the desingularization methods presented here, and those mentioned above is that our algorithms determine generators for the closure ideal, not simply a single operator. This flexibility allows us some choice in representative, and speeds up some other common maninpulations, like simultaneous desingularization of leading and trailing terms, but can mean slow down on particular examples. On average, across the tests our routine is about three to four times faster to desingularize the leading coefficient but slower on the trailing. For a combined desingularization our algorithms are faster.

We first test the following family of systems, which are generalizations of the examples presented in [1]. Figure 4 summarizes the results. The timings were computed using Maple 11.

Test	L	Range
1	$(n-3)(n-2)^i E_n - n^i(n-1)$	$10 \le i \le 30$
2	$(n-3)^{i+1}(n-2)^{i-2}(n-4/3)^{i+5}(n-7/3)E_n + (n-1)^i n^{i-1}(n-4/3)^2(n-1/3)^{i+3}$	$5 \le i \le 20$

D.1.1. *Aesthetic considerations.*

À jeter. PhD

Obviously deciding what constitutes a “nice” answer is a subjective judgement. However, it is true that since our method returns a set generators for the closure, the user has some control over what determining some aspects of the form of the desingularized solution, essentially by generating the Grobener basis with respect to the term ordering of choice. For example, in Test 1, we can compute the average degree in n of a coefficient across all tests. The average from the simultaneous leading and trailing desingularization algorithm of van Hoeij is 15 , while the corresponding Ore Closure implementation has an average of

[1] http://www.math.fsu.edu/~hoeij/papers/desing/DifferentialCase/desing
[2] http://www.math.fsu.edu/~hoeij/papers/desing/DifferenceCase/desing
[3] http://www.math.fsu.edu/~hoeij/papers/desing/DifferenceCase/tdesing

10. The fast, auxilary trailing coefficient desingularization algorithm of van Hoeij results in an average coefficient degree of 77.

D.2. **Differential equation timing comparison.** Next we can compare with the code of van Hoeij for differential singularization on a system which is conjectured to be satisfied by the generating function of punctured staircase polynomials [14][4]. Understanding which singularities are apparent are important for the asymptotic analysis which is performed. The initial system is of order 8, and the leading coefficient is of degree 37. Our implementation technique can desingularize in 200 seconds, although the van Hoeij algorithm can desingularize in 10.

Appendix E. Summary of notation

PhD Revoir toutes les notations, y compris celles qui sont cachées.

Basic rings and fields.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ as usual;
- k a commutative field of characteristic 0;
- x the usual commutative indeterminate;
- $k[x]$, $k(x)$ as usual;
- A a commutative algebra over k, usually $k[x]$ or $k(x)$.

Morphisms.

- ∂ the usual non-commutative indeterminate;
- σ an algebra endomorphism of A inducing the identity over k, and δ a k-linear endomorphism of A, both related by $\partial a = \sigma(a)\partial + \delta(a)$ for $a \in A$,
- derivation d/dx, shift S, difference operator Δ, dilation operator Q, ... Mahlerian operator M, all k-linear endomorphisms of $k[x]$ (see Table 1).

Rings.

- $A[\partial; \sigma, \delta]$ the skew-polynomial ring, that is the associative ring generated over A by δ and the relations cited above;
- $\mathscr{D}[x]$ the skew-polynomial ring $k[x][\partial; \sigma, \delta]$;
- $L = \ell_s\partial^s + \cdots + \ell_r\partial^r$ the generic element of $\mathscr{D}[x]$;
- $k[x][\partial; \sigma]$ the skew-polynomial ring $k[x][\partial; \sigma, \delta]$ when $\delta = 0$;
- $\mathscr{D}(x)$ the skew-polynomial ring $k(x)[\partial; \sigma, \delta]$ when σ is injective;
- $\mathscr{S}$ a generic skew-polynomial ring;
- $\mathscr{L} = \mathscr{S}[\partial^{-1}] = A[\partial, \partial^{-1}; \sigma]$ the localization by ∂ of the skew-polynomial ring $\mathscr{S} = A[\partial; \sigma]$ when σ is bijective and $\delta = 0$; attention, it is not a skew-polynomial ring anymore but a Laurent-Ore algebra;
- $\mathscr{L}[x]$ the Laurent-Ore algebra $k[x][\partial, \partial^{-1}; \sigma]$;
- $\mathscr{L}(x)$ the Laurent-Ore algebra $k(x)[\partial, \partial^{-1}; \sigma]$.

[4]The full system is given in the article.

Closure.

- $C(L)$ the global polynomial closure $\mathscr{D}(x)L \cap \mathscr{D}[x]$;
- $C_f(I)$ the local polynomial closure at f that is $k[x, f^{-1}, \sigma(f)^{-1}, \sigma^2(f)^{-1}, \dots][\partial; \sigma, \delta]L \cap \mathscr{D}[x]$;
- $K(L)$ the global Laurent-Ore algebra closure that is $\mathscr{L}(x)L \cap \mathscr{L}[x]$;
- $K_f(L)$ the local Laurent-Ore algebra closure that is ????

Je ne sais pas ce qu'est le dernier article (mais il est effectivement défini à la section 4.3). PhD

Torsion.

- $\ker_L f$ the set of the classes $R + \mathscr{D}[x]L$ for which there exists $P \in \mathscr{D}[x]$ satisfying $PL = fR$;
- $\ker_f L$ the set of the classes $P + f\mathscr{D}[x]$ for which there exists $R \in \mathscr{D}[x]$ satisfying $PL = fR$;
- $T(L)$ the torsion space of $\mathscr{D}[x]/\mathscr{D}[x]L$ that is the set of classes $R + \mathscr{D}[x]L$ for which there exist $f \in k[x]$ and $P \in \mathscr{D}[x]$ such that $PL = fR$;
- $T_f(L)$ the local torsion space at $f \in k[x]$ that is the set of classes $R + \mathscr{D}[x]L$ for which there exist integers $j \geq 0$, $\nu_0 \geq 0$, $\dots$ $\nu_j \geq 0$, and a skew polynomial $P \in \mathscr{D}[x]$ such that $PL = f^{\nu_0}\sigma(f)^{\nu_1} \cdots \sigma^j(f)^{\nu_j} R$;
- $\Gamma_f(L)$ the local pseudo-torsion space at $f \in k[x]$ that is the set of classes $R + \mathscr{D}[x]L$ for which there exist an integer $\nu \geq 0$ and a skew polynomial $P \in \mathscr{D}[x]$ such that $PL = f^\nu R$.

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