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Algebraic Independence of the Values of Elliptic Function at Algebraic Points

Elliptic Analogue of the Lindemann-Weierstrass Theorem

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Introduction

As it is well known, the only general statement about algebraic independence of n values of the exponential function is the Lindemann-Weierstrass theorem [L1], [L2] (1882):

if algebraic numbers $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

Since then such general results were obtained only for E-functions, Siegel [Si], like $J_0(z)$. However the methods of [Si] use the structure of E-functions as entire functions, satisfying a linear differential equation with rational coefficients and having certain special arithmetical properties of the coefficients of their Taylor expansion.

On the other hand, the general Gelfond-Schneider method [G], [Sc], [L] can be applied to a wide class of functions satisfying algebraic differential equations and/or a law of addition. Nevertheless there are no general results on the algebraic independence of two or more numbers (like the Lindemann-Weierstrass theorem) which are obtained by Gelfond-Schneider method. Of course, in the exponential and elliptic case we have some results on the algebraic independence of numbers of the form α^{β^j}, $\wp(\omega\beta)$, or of some constants connected with the periods [Ch1], [Ch2], [Ch3], [MW2]. This is quite natural because in the Gelfond-Schneider method the problem of algebraic independence is often reduced to an algebro-geometrical problem about the structure of singular points of intersections of hypersurfaces near fixed transcendental points. Unlike Siegel's method, it is impossible to construct independent systems of linear forms and complete the proof by showing that some determinant is not zero.

Fortunately we now have a comparatively elementary method to treat the problem of algebraic independence of several numbers via the Gelfond-Schneider method [Ch3]. In general to treat an arbitrary number of algebraically independent elements, we must inevitably use the ideas underlying Hironaka's resolution of singularities.

Dedicated to 50-th anniversary of A.I. Vinogradov.

In principle, these algebraic methods, combined with some technical improvements in the Gelfond-Schneider method give us the possibility to generalize the Lindemann-Weierstrass result for the elliptic functions. At present, we have not succeeded in overcoming all the technical difficulties required to do this in general. We are able to give here only the first results in this direction.

1. Formulation of the Problem

We use standard notations of transcendental number theory and elliptic functions. For $P(x_1,\dots,x_n)\in\mathbb{C}[x_1,\dots,x_n]$ we denote by $H(P)$, the height of $P(x_1,\dots,x_n)$, i.e. the maximum of the absolute value of the coefficients of $P(x_1,\dots,x_n)$, and by $d(P)$ the total degree of $P(x_1,\dots,x_n)$ in $x_1,\dots,x_n$. By $d_{x_i}(P)$ we denote the degree of $P(x_1,\dots,x_n)$ in x_i. The type $t(P)$ of the polynomial $P(x_1,\dots,x_n)$ is defined as $\log H(P)+d(P)$.

We consider elliptic curves over an algebraic number field. We take a uniformization of an elliptic curve by the Weierstrass elliptic function $\wp(z)$. Thus we are considering a Weierstrass elliptic function $\wp(z)$, satisfying $\wp'^2=4\wp^3-g_2\wp-g_3$, where the invariants g_2, g_3 are algebraic numbers. Let ω_1, ω_2 be a pair of fundamental periods of $\wp(z)$ and $L=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2$ be the corresponding lattice of $\wp(z)$. We write throughout the paper

$$\mathbb{F}=\begin{cases}\mathbb{Q}(\omega_1/\omega_2), & \text{if } \omega_1/\omega_2 \text{ is imaginary quadratic:}\\ \mathbb{Q}, & \text{otherwise.}\end{cases} \tag{1}$$

As usual, if the first case in (1) is satisfied, we say that $\wp(z)$ has complex multiplication.

Now we consider algebraic numbers

$$\alpha_1,\dots,\alpha_n$$

which are linearly independent over $\mathbb{F}$. The problem is the following: are the numbers

$$\wp(\alpha_1),\dots,\wp(\alpha_n)$$

algebraically independent over $\mathbb{Q}$? Of course, the assumption that $\alpha_1,\dots,\alpha_n$ are linearly independent over $\mathbb{F}$ cannot be in general reduced to the assumption of linear independence over $\mathbb{Q}$.

We shall prove below some positive results in this direction. For $n<4$ we give complete results in the case of complex multiplication. Another problem arising here is to estimate the measure of algebraic independence of the numbers $\wp(\alpha_1),\dots,\wp(\alpha_n)$ (whenever they are algebraically independent). In this connection, we recall that for the exponential function e^z, the Lindemann-Weierstrass theorem also has a quantitative version (due to K. Mahler [KM]):

if $\alpha_1,\dots,\alpha_n$ are linearly independent over $\mathbb{Q}$ and $P(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$, $P\not\equiv 0$, then

$$|P(e^{\alpha_1},\dots,e^{\alpha_n})|>H(P)^{-c_1 d(P)^n} \tag{2}$$

for some constant c_1 depending only on $\alpha_1, \dots, \alpha_n$ with $H(P)$ sufficiently large with respect to $d(P)$. Up to the constant c_1 the estimate (2) is the best possible and close to the upper bound given by Dirichlet's box principle.

We may ask the same question for $\wp(\alpha_1), \dots, \wp(\alpha_n)$. For $n=1$ E. Reyssat [ER] obtained the bound

$$|P(\wp(\alpha))| > H(P)^{-c_2 d(P)^3}$$

for any algebraic $\alpha \neq 0$ and $P(x) \in \mathbb{Z}[x]$, $P(x) \not\equiv 0$ (his results also cover the case of independent $H(P)$ and $d(P)$).

We will prove here the best possible results for the case of complex multiplication and $n=1,2$

$$|P_1(\wp(\alpha))| > H(P_1)^{-c_3 d(P_1)}; \tag{3}$$

$$|P_2(\wp(\alpha), \wp(\beta))| > H(P_2)^{-c_4 d(P_2)^2}, \tag{4}$$

where $\wp(z)$ has complex multiplication, α and β are algebraic numbers linearly independent over $\mathbb{F}$, $P_1(x) \in \mathbb{Z}[x]$, $P_2(x,y) \in \mathbb{Z}[x,y]$; $P_1 \not\equiv 0$, $P_2 \not\equiv 0$, $c_3 > 0$, $c_4 > 0$ and $H(P_i)$ is sufficiently large with respect to $d(P_i)$.

The estimate (3) is particularly interesting, as it provides (in the case of complex multiplication) the first example, after e^α, of a value of a classical function at an algebraic point having very bad approximation by algebraic numbers of bounded degree.

Perhaps this indicates that for $\wp(\alpha)$ (as for e^α) there are some special continued fraction expansions, etc.

At the end of this introduction we must say that our results can be generalized in two directions. Firstly, the number n of algebraically independent numbers can be increased without big difficulties (up to $n \leqq 6$). Second of all, these results can be generalized to Abelian functions and their values at algebraic points.

2. Main Results

Let as before $\wp(z)$ have algebraic invariants and field $\mathbb{F}$ of endomorphisms.

Theorem 2.1. *Let $\alpha_1, \alpha_2, \alpha_3$ be three algebraic numbers, linearly independent over $\mathbb{Q}$. Then among*

$$\wp(\alpha_1), \wp(\alpha_2), \wp(\alpha_3)$$

there are two algebraically independent numbers.

Theorem 2.2. *Let $\alpha_1, \dots, \alpha_5$ be five algebraic numbers, linearly independent over $\mathbb{Q}$. Then among*

$$\wp(\alpha_1), \dots, \wp(\alpha_5)$$

there are three algebraically independent numbers.

Corollary. *Assume that* $\mathfrak{P}(z)$ *admits complex multiplication. If* α_1, α_2 *are algebraic numbers which are linearly independent over* $\mathbb{F}$, *then* $\mathfrak{P}(\alpha_1)$, $\mathfrak{P}(\alpha_2)$ *are algebraically independent over* $\mathbb{Q}$.

To deduce the Corollary, we simply apply Theorem 1 to the three numbers $\alpha_1, \alpha_2, \tau\alpha_1$, where τ is an element of $\mathbb{F}$ which is not in $\mathbb{Q}$.

Remark. It is clear that Theorems 2.1–2.2 cannot be improved in general, e.g. in the case of complex multiplication of $\mathfrak{P}(z)$ by τ, for the system of numbers $\alpha_1 = 1$, $\alpha_2 = \tau$, $\alpha_3 = \sqrt{2}$, $\alpha_4 = \tau\sqrt{2}$, $\alpha_5 = \sqrt{3}$, we have

$$\#\{\mathfrak{P}(\alpha_1), \mathfrak{P}(\alpha_2), \mathfrak{P}(\alpha_3)\} = 2;$$
$$\#\{\mathfrak{P}(\alpha_1), \ldots, \mathfrak{P}(\alpha_5)\} = 3.$$

Here $\# S$ for a finite $S \subset \mathbb{C}$ denotes the degree of the transcendence of $\mathbb{Q}(S)$.

In fact, our results are proved parallely to strong transcendence measures for $\mathfrak{P}(\alpha)$ and algebraic non-zero α.

Theorem 2.3. *Assume that* $\mathfrak{P}(z)$ *admits complex multiplication, and let* α *be a non-zero algebraic number. Then there exist positive constants* C_1, C_2, *depending only on* $\mathfrak{P}(z)$ *and* α; *as follows: for all non-zero polynomials* $R(x) \in \mathbb{Z}[x]$ *with degree at most* d *and height at most* H *such that* $\log\log H \geqq C_1 d^3$, *we have*

$$|R(\mathfrak{P}(\alpha))| > H^{-C_2 d}.$$

Finally, our method is capable of considerable generalization. We present, without detailed proofs, a number of more general results we have obtained in §5.

3. The Method of Proof

This was first announced by the author in March 1977 at Oberwolfach, as a more complicated means of establishing the Lindemann-Weierstrass theorem using now the Gelfond-Schneider method. Our motivation for giving such a proof was that we knew it would extend to the elliptic case. This is exactly what we show in the present paper. It seems worthwhile, however, to begin by giving the proof in the exponential case, as the ideas underlying the proof are less buried in technical details.

Let $\alpha_1, \ldots, \alpha_n$ be $n \geqq 1$ algebraic numbers which are linearly independent over $\mathbb{Q}$. Write $\mathbb{I}$ for the ring generated over $\mathbb{Z}$ by the $2n$ numbers α_j, e^{α_j} $(1 \leqq j \leqq n)$, and let $\mathbb{K}$ be the quotient field of $\mathbb{I}$. Suppose that the transcendence degree of $\mathbb{K}$ over $\mathbb{Q}$ is $m \leqq n$, and let us choose generators $\theta_1, \ldots, \theta_m, \upsilon$ of $\mathbb{K}$ over $\mathbb{Q}$ so that (i) $\theta_1, \ldots, \theta_m$ are algebraically independent over $\mathbb{Q}$, and (ii) υ is integral over the ring $\mathbb{Z}[\theta_1, \ldots, \theta_m]$. If β is an element of $\mathbb{K}$, see [MW1] p. 4.5 for the definition of the type of β relative to the fixed set of generators $\theta_1, \ldots, \theta_m$, υ of $\mathbb{K}$. In the following, $c_1, c_2, \ldots$ will denote positive constants which depend only on the choice of $\alpha_1, \ldots, \alpha_n$ and the fixed set of generators $\theta_1, \ldots, \theta_m, \upsilon$.

We denote also

$$\mathbb{J}=\mathbb{Z}[\theta_1,\ldots,\theta_m]. \tag{3.1}$$

We construct now for the transcendence proof auxiliary functions of the form

$$F(x)=P(z,e^z), \tag{3.2}$$

where

$$P(x,y)\in\mathbb{J}[x,y] \tag{3.3}$$

and

$$d_x(P)\leqq L_0, \quad d_y(P)\leqq L_1. \tag{3.4}$$

The choice of parameters in the auxiliary function is subject to the following conditions. We have two parameters L_0 and X (corresponding to $\log H(P)$ and $d(P)$). Let L_1 be a sufficiently large *constant*. Let X be a parameter, which is very large relative to the constant L_1. Finally, let L_0 be an integer parameter which is chosen to satisfy

$$\log L_0\geqq X^{n+1}. \tag{3.5}$$

Finally, we define S by

$$S=\left[\frac{L_0 L_1}{2X^n}\right]. \tag{3.6}$$

Lemma 3.1. *For L_0, L_1, X as above there exists a non-zero polynomial $P(x,y)\in\mathbb{J}[x,y]$ with degree at most L_0 in x and most L_1 in y, such that the function*

$$F(z)=P(z,e^z) \tag{3.7}$$

satisfies

$$F^{(k)}(m_1\alpha_1+\ldots+m_n\alpha_n)=0 \tag{3.8}$$

for rational integers m_i, k: $0\leqq m_i\leqq X-1$ $(1\leqq i\leqq n)$ and $0\leqq k\leqq S-1$. Moreover, the coefficients of $P(x,y)$ as polynomials in $\theta_1,\ldots,\theta_n$ (cf. (3.1)) has types at most

$$c_1 S\log L_0$$

and degrees of the coefficients of $P(x,y)$ as polynomials in $\theta_1,\ldots,\theta_n$ is at most $c_2 L_1 X$.

The proof of the existence of the polynomial $P(x,y)$ is based on the Siegel's Lemma 1.5.5 [MW1]; corresponding arguments in the exponential case are standard (cf. [MW2], [Ch2]).

Now we can apply the Schwarz lemma to the function $F(z)$. Using (3.8) and bounds for the coefficients of $P(x,y)$ we obtain immediately

$$|F^{(k_1)}(z)|<\exp(-c_3 SX^n\log L_0) \tag{3.9}$$

for some $c_3>0$, where $k_1\leqq c_4 S$ and $|z|\leqq c_5 X$ for any $c_4\geqq 1$ if L_1 is sufficiently large with respect to c_4. We use now the Small Value Lemma [T] and we obtain either for some positive numbers m_i', $0\leqq m_i'\leqq c_5 X$ and some $k_0\leqq c_6 S$

$$-c_7 SX^n \log L_0 < \log |F^{(k_0)}(m'_1\alpha_1 + \ldots + m'_n \alpha_n)| < -c_8 SX^n \log L_0, \tag{3.10}$$

or for the coefficients $p_{\lambda_0,\lambda_1} \in \mathbb{I}$ of $P(x,y)$ we have

$$\max |p_{\lambda_0,\lambda_1}| < \exp(-c_9 SX^n \log L_0). \tag{3.11}$$

Taking into account Lemma 3.1, (3.5)-(3.6), (3.10)-(3.11) and routine arguments from [Ch2] we come to

Lemma 3.2. *Let parameters L_0, L_1, X, S satisfy conditions as above (see (3.5)-(3.6)). Then there exists either*

(i) *a polynomial $R(x_1,\ldots,x_m) \in \mathbb{Z}[x_1,\ldots,x_m]$ such that*

$$-c_{10} SX^n \log L_0 < \log |R(\theta_1,\ldots,\theta_m)| < -c_{11} SX^n \log L_0; \tag{3.12}$$

or

(ii) *a system $P_l(x_1,\ldots,x_m) \in \mathbb{Z}[x_1,\ldots,x_m]$: $l \in \mathcal{L}$ of polynomials without common factor such that for $l \in \mathcal{L}$*

$$|P_l(\theta_1,\ldots,\theta_m)| < \exp(-c_{12} SX^n \log L_0), \tag{3.13}$$

where for the types and degrees of polynomials $R(x_1,\ldots,x_n)$, $P_l(x_1,\ldots,x_m)$ we have the following bounds

$$\begin{aligned} &t(R) \leqq c_{13} S \log L_0; \\ &t(P_l) \leqq c_{14} S \log L_0: \ l \in \mathcal{L} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &d(R) \leqq c_{15} L_1 X; \\ &d(P_l) \leqq c_{16} L_1 X: \ l \in \mathcal{L}. \end{aligned} \tag{3.15}$$

Here as above $\theta_1,\ldots,\theta_m$ is the basis of transcendence of $\mathbb{K} = \mathbb{Q}(\alpha_1,\ldots,\alpha_n,e^{\alpha_1},\ldots,e^{\alpha_n})$.

Now it is clear from Gelfond's criterion of transcendence [G] that

$$\deg \cdot \operatorname{tr} \mathbb{K} = m \geqq n$$

for $n=1,2$; from [Ch3] the same is clear for $n=3$.

Absolutely the same kind of methods is applied now for the elliptic function and, in fact, the same kind of arguments are valid for any function, satisfying simultaneously an algebraic differential equation and a generalized law of addition (i.e. law of addition, satisfied by the function and its derivatives) [Ch5].

The proof in elliptic case in §4 depends, however, on one technical trick. Instead of taking derivatives of auxiliary function directly we change a little the elliptic function, namely

$$\frac{d^s}{dz^s} \wp(z)|_{z=n\alpha}$$

is replaced by

$$\frac{d^s}{dw^s} \wp(n^\alpha + w)|_{w=0}.$$

This small change leads to big improvements in the estimates.

4. The Elliptic Case

We now apply similar arguments to the elliptic case (in fact, it will be clear that the same type of reasoning will apply to any meromorphic function satisfying both an algebraic law of addition and an algebraic differential equation, e.g. to Abelian functions). We begin by discussing the additional technical difficulties that arise in the elliptic case. We fix a Weierstrass $\wp$-function $\wp(z)$ whose invariants g_2, g_3 are algebraic numbers. As usual, we can suppose without loss of generality that $\frac{1}{4}g_2$, $\frac{1}{4}g_3$, are algebraic integers. We have $4x^3 - g_2 x - g_3 = 4(x - e_1)\cdot(x-e_2)\cdot(x-e_3)$ and we write $\mathcal{O}$ for the ring of integers of the field $\mathcal{F} \stackrel{\mathrm{def}}{=} \mathbb{Q}(g_2, g_3, e_1, e_2, e_3)$.

Let $\alpha_1, \ldots, \alpha_k$ be k algebraic numbers, linearly independent over $\mathbb{Q}$. We fix from now on any transcendence basis

$$\Theta = (\theta_1, \ldots, \theta_m)$$

of the field

$$\mathbb{K} = \mathcal{F}(\alpha_1, \ldots, \alpha_k, \wp(\alpha_1), \ldots, \wp(\alpha_k), \wp'(\alpha_1), \ldots, \wp'(\alpha_k)). \tag{4.1}$$

Thus $\mathbb{K} = \mathbb{Q}(\theta_1, \ldots, \theta_m, \vartheta)$ for ϑ being algebraic over $\mathbb{Q}[\theta_1, \ldots, \theta_m]$ of degree d, say. We consider also the pure transcendental extension of $\mathcal{O}$ given by

$$\begin{aligned} \mathbb{J} &= \mathcal{O}[\theta_1, \ldots, \theta_m], \\ \mathbb{I} &= \mathbb{Z}[\theta_1, \ldots, \theta_m]. \end{aligned} \tag{4.2}$$

We write in this chapter $c_1, c_2, \ldots$ for positive constants which depend only on $g_2, g_3, \alpha_1, \ldots, \alpha_k, \theta_1, \ldots, \theta_m, \vartheta$.

First of all we present without proofs the following three well-known lemmas (see, for example, [L], [M]).

Lemma 4.1. *For each integer $m \geqq 1$, we have*

$$\wp(mz) = \frac{A_m(\wp(z))}{B_m(\wp(z))} \tag{4.3}$$

where $A_m(X)$, $B_m(X)$ are polynomials of respective degrees m^2 and m^2-1, whose coefficients are algebraic integers in $\mathbb{Q}(g_2, g_3)$ with sizes at most $c_1^{m^2}$.

Lemma 4.2. *Let $\vec{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k \setminus \{\vec{0}\}$ and $z_0 = m_1\alpha_1 + \ldots + m_k\alpha_k$. Then there exist polynomials $T_{\vec{m}}(\Theta, \vartheta)$, $T'_{\vec{m}}(\Theta, \vartheta)$, $U_{\vec{m}}(\Theta, \vartheta)$ from $\mathcal{O}[\Theta, \vartheta]$ of total degree in Θ at most*

$$c_4 \sum_{i=1}^{k} m_i^2,$$

and degree in ϑ *at most* $d-1$, *and of sizes at most*

$$\exp\left(c_5 \sum_{i=1}^{k} m_i^2\right),$$

such that the following conditions are satisfied:

$$\mathfrak{P}(z_0)=\frac{T_{\vec{m}}(\Theta,\vartheta)}{U_{\vec{m}}(\Theta,\vartheta)}, \qquad \mathfrak{P}'(z_0)=\frac{T'_{\vec{m}}(\Theta,\vartheta)}{U_{\vec{m}}(\Theta,\vartheta)}. \tag{4.4}$$

This lemma follows immediately from the law of addition for $\mathfrak{P}(z)$, multiplication formula (4.3) and the definition of Θ.

Remark 4.3. We have simple lower and upper bounds for quantities $|T_{\vec{m}}(\Theta,\vartheta)|$, $|T'_{\vec{m}}(\Theta,\vartheta)|$, $|U_{\vec{m}}(\Theta,\vartheta)|$ from Lemma 4.2. First of all by majoration we have:

$$\max\{|T_{\vec{m}}(\Theta,\vartheta)|, |T'_{\vec{m}}(\Theta,\vartheta)|, |U_{\vec{m}}(\Theta,\vartheta)|\}<\exp\left(c_6 \sum_{i=1}^{k} m_i^2\right). \tag{4.5}$$

For the bound below we use formula (4.4). It follows from the result of [M] that

$$\min_{\Omega\in L}|\alpha-\Omega|>\exp(-c_7\log^2 H(\alpha)) \tag{4.6}$$

for $c_7=c_7(g_2,g_3,d(\alpha))>0$ and algebraic $\alpha\neq 0$ of degree $\leqq d(\alpha)$ and height $H(\alpha)$. From (4.6) it trivially follows

$$\exp\left(-c_8 \sum_{i=1}^{k} m_i^2\right)<|\mathfrak{P}(m_1\alpha_1+\ldots+m_k\alpha_k)| <\exp\left(c_9 \sum_{i=1}^{k} m_i^2\right).$$

Combining this with (4.4) and (4.5) we get

$$\min\{|T_{\vec{m}}(\Theta,\vartheta)|, |T'_{\vec{m}}(\Theta,\vartheta)|, |U_{\vec{m}}(\Theta,\vartheta)|\}>\exp\left(-c_{10} \sum_{i=1}^{k} m_i^2\right). \tag{4.7}$$

Lemma 4.4. *For all integers* $l\geqq 1$ *and* $s\geqq 0$, *we have*

$$\left(\frac{d}{dx}\right)^s \mathfrak{P}^l(z)=D_{l,s}(\mathfrak{P}(z),\mathfrak{P}'(z)),$$

where $D_{l,s}(X,Y)$ *is a polynomial of total degree at most*

$$c_2(l+s), \tag{4.8}$$

whose coefficients are algebraic integers in $\mathbb{Q}(g_2,g_3)$ *with sizes at most* $\exp(c_3 s\log(s+l+1))$.

The estimate (4.8) is too coarse to apply to the construction of our auxiliary function, because the combination of (4.8) together with Lemma 4.1 will lead to the degree sm^2 of $\mathfrak{P}^{(s)}(mz)$ as a rational function in $\mathfrak{P}(z)$, $\mathfrak{P}'(z)$. Part of these difficulties are overcomed by Baker-Coates' lemma (see [ER], [DBe]), but, in addition we need to get rid of the dependence on s in the degree of $\mathfrak{P}^{(s)}(mz)$. For this we introduce the following series of lemmas that we called a "generalized Baker-Coates' lemma". As I understand this generalization of Baker-Coates' lemma was independently discovered by Anderson.

Let L be the period lattice of $\mathfrak{P}(z)$. We fix an element ω of $L\backslash 2L$ and write

$$\varphi(z)=\mathfrak{P}\left(z+\frac{\omega}{2}\right). \tag{4.9}$$

The first lemma is just a rewording of the addition theorem.

Lemma 4.5. *We have*

$$\mathfrak{P}(z+w)=\frac{\Pi_1(\mathfrak{P}(z),\mathfrak{P}'(z);\varphi(w),\varphi'(w))}{\Pi_2(\mathfrak{P}(z),\varphi(w))}=\frac{\Lambda_1(z,w)}{\Lambda_2(z,w)} \tag{4.10}$$

where $\Pi_1(\ldots)$, $\Pi_2(\ldots)$ are polynomials with coefficients in $\mathcal{O}$ and total degrees $\leqq 4$ and $\Pi_1(\mathfrak{P}(z), \mathfrak{P}'(z), \varphi(w), \varphi'(w))=\Lambda_1(z,w)$; $\Pi_2(\mathfrak{P}(z); \varphi(w))=\Lambda_2(z,w)$.

Proof. We use the formulae

$$\mathfrak{P}(z+w)=\frac{1}{4}\left(\frac{\varphi'(z)-\varphi'(w)}{\varphi(z)-\varphi(w)}\right)^2-\varphi(z)-\varphi(w)$$

and

$$\varphi(z)=\frac{e\,\mathfrak{P}(z)+\alpha}{\mathfrak{P}(z)-e}$$

for $\mathfrak{P}\left(\frac{\omega}{2}\right)=e\in\mathcal{O}$, $\alpha\in\mathcal{O}$.

Let $R(x,y)$ be any polynomial from $\mathbb{J}[x,y]$ of degree at most L_0 at x and at most L_1 at y, and

$$F(z)=R(z,\mathfrak{P}(z)).$$

We define the following function $F_1(z,w)$ in two variables z, w:

$$F_1(z,w)=\Lambda_2(z,w)^{L_1}\cdot R(z+w,\mathfrak{P}(z+w))$$

for $\Lambda_2(z,w)$ defined in Lemma 4.5. Then the only astuse we use is the following identity:

$$\begin{aligned}\frac{d^s}{dz^s}F(z)&=\frac{d^s}{dw^s}\cdot R(z+w,\mathfrak{P}(z+w))|_{w=0}\\&=\sum_{s_1=0}^{s}\binom{s}{s_1}\frac{d^{s-s_1}}{dw^{s-s_1}}\{\Lambda_2(z,w)^{-L_1}\}|_{w=0}\,\frac{d^{s_1}}{dw^{s_1}}F_1(z,w)|_{w=0}.\end{aligned} \tag{4.11}$$

Now we can investigate $d^{s_1}/dw^{s_1}\,F_1(x,w)|_{w=0}$.

Lemma 4.6. *Let $R(x,y)$ be an arbitrary polynomial in $\mathbb{J}[x,y]$ with degree at most L_0 in x and at most L_1 in y. Define*

$$F_1(z,w)=\Lambda_2(z,w)^{L_1}\cdot R(z+w,\mathfrak{P}(z+w))$$

then for any $r\geqq 0$ we have

$$\frac{d^r}{dw^r}F_1(z,w)|_{w=0}=Q_r(z,\mathfrak{P}(z),\mathfrak{P}'(z)),$$

where $Q_r(x_1,x_2,x_3)$ is a polynomial in $\mathbb{J}[x_1,x_2,x_3]$ satisfying the following properties.

1) The degree of $Q_r(x_1,x_2,x_3)$ in x_1 is at most L_0; the total degree of $Q_r(x_1,x_2,x_3)$ in x_2,x_3 is at most $4\cdot L_1$;

2) Let

$$R(x,y)=\sum_{\lambda_0,\lambda_1}p_{\lambda_0,\lambda_1}x^{\lambda_0}y^{\lambda_1}$$

and

$$Q_r(x_1,x_2,x_3)=\sum_{i_1,i_2,i_3}Q_{r,i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}.$$

Then the coefficients of $Q_r(x_1,x_2,x_3)$ are linear combinations of the coefficients of $R(x,y)$. Namely,

$$Q_{r,i_1,i_2,i_3}=\sum_{\lambda_0,\lambda_1}p_{\lambda_0,\lambda_1}\cdot C_{\lambda_0,\lambda_1,r,i_1,i_2,i_3},$$

where $C_{\lambda_0,\lambda_1,r,i_1,i_2,i_3}$ are algebraic integers from $\mathcal{O}$ of sizes bounded by

$$\exp\{c_{11}(r+L_1)\log(r+L_1+1)+c_{12}L_0+L_0\log L_0\}.$$

Proof. By the definition of $F_1(z,w)$ and Lemma 4.5 we have

$$F_1(z,w)=\sum_{\lambda_0,\lambda_0}p_{\lambda_0,\lambda_1}(z+w)^{\lambda_0}\Lambda_1(z,w)^{\lambda_1}\Lambda_2(z,w)^{L_1-\lambda_1}.$$

Now $\Lambda(z,w)^{\lambda_1}\cdot\Lambda_2(z,w)^{L_1-\lambda_1}$ is a polynomial in $\mathfrak{P}(z)$, $\mathfrak{P}'(z)$, $\varphi(w)$, $\varphi'(w)$ of total degree at most $4\cdot L_1$. Also

$$\{\mathfrak{P}^l(w)\}^{(k)}|_{w=\frac{\omega}{2}}$$

is an algebraic integer from $\mathcal{O}$ of size at most

$$\exp(c_{13}k\log(k+l+1))$$

by Lemma 4.4. We have

$$\frac{d^r}{dw^r}F_1(z,w)|_{w=0}=\sum_{\lambda_0,\lambda_1}p_{\lambda_0,\lambda_1}\cdot\sum_{r_1=0}^{r}\binom{r}{r_1}\cdot\lambda_0\dots(\lambda-r_1+1)$$
$$\times z^{\lambda_0-r_1}\cdot\frac{d^{r-r_1}}{dw^{r-r_1}}\{\Lambda_1(z,w)^{\lambda_1}\cdot\Lambda_2(z,w)^{L_1-\lambda_1}\}|_{w=0}. \tag{4.12}$$

By the remark above the part 2) is proved. Part 1) also follows from (4.12) since the degree of $d^r/dw^r F_1(z,w)|_{w=0}$ in z is at most L_0 and in $\mathfrak{P}(z)$, $\mathfrak{P}'(z)$ is at most $4 \cdot L_1$.

We can apply this lemma with $z = m_1\alpha_1 + \ldots + m_k\alpha_k$:

Lemma 4.7. *Let, as above, $R(x,y) \in \mathbb{J}[x,y]$ and $d_x(R) \leqq L_0$, $d_y(R) \leqq L_1$. We put*

$$F(z) \doteq R(z, \mathfrak{P}(z)).$$

Let now $m_1, \ldots, m_k$ be rational integers, not all zero, and put

$$z_0 = m_1\alpha_1 + \ldots + m_k\alpha_k. \tag{4.13}$$

If s is the order of zero of $F(z)$ at $z = z_0$, then

$$\frac{d^s}{dz^s} F(z)|_{z=z_0} = A_{\vec{m}}(\Theta, \vartheta)^{-L_1} \times P_{s,\vec{m}}(\Theta, \vartheta), \tag{4.14}$$

where $A_{\vec{m}}(\Theta, \vartheta) \neq 0$ and $P_{s,\vec{m}}(\Theta, \vartheta)$ are polynomials in Θ, ϑ with the coefficients from $\mathcal{O}$ and the following conditions are satisfied.

1. *$A_{\vec{m}}(\Theta, \vartheta)$ is independent of $R(x,y)$; it has total degree at most $c_{13} \sum_{i=1}^{k} m_i^2$, degree in ϑ at most $d-1$; and the size of the coefficients at most*

$$\exp\left(c_{13} \sum_{i=1}^{k} m_i^2\right)$$

and also

$$|A_{\vec{m}}(\Theta, \vartheta)| > \exp\left(-c_{13} \sum_{i=1}^{k} m_i^2\right). \tag{4.15}$$

2. *If $R(x,y) = \sum_{\lambda_0, \lambda_1} r_{\lambda_0, \lambda_1} x^{\lambda_0} y^{\lambda_1}$ and $r_{\lambda_0, \lambda_1} = r_{\lambda_0, \lambda_1}(\Theta) \in \mathbb{J}$ have total degrees in Θ at most $D_{\mathbb{J}}(P)$ and sizes of the coefficients at most*

$$\max_{\lambda_0, \lambda_1} \overline{|H(r_{\lambda_0, \lambda_1})|} \leqq H_{\mathbb{J}}(P),$$

then the total degree of $P_{s,\vec{m}}(\Theta, \vartheta)$ is at most

$$c_{15} L_1 \sum_{i=1}^{k} m_i^2 + D_{\mathbb{J}}(P);$$

and the sizes of the coefficients of $P_{s,\vec{m}}(\Theta, \vartheta)$ are at most

$$\overline{|H(P_{s,\vec{m}})|} \leqq H_{\mathbb{J}}(P) \cdot \exp\left(c_{16} L_1 \sum_{i=1}^{k} m_i^2 + c_{17}(s + L_1)\log(s + L_1 + 1) \right.$$
$$\left. + c_{17} L_0 \log L_0 + L_0 \log(m_1^2 + \ldots + m_k^2)\right).$$

Proof. We use first of all the formula (4.12). Then if s is exactly the order of zero of $F(z)$ at $z=z_0$, we get from (4.12):

$$\frac{d^s}{dz^s}F(z)|_{z=z_0}=\Lambda_2(z_0,0)^{-L_1}\cdot\frac{d^s}{dw^s}F_1(z,w)|_{w=0}. \tag{4.16}$$

Now

$$\Lambda_2(z_0,0)=\Pi_2\left(\wp(z_0),\wp\left(\frac{\omega}{2}\right)\right)\neq 0,$$

because z_0 is non-zero algebraic number and thus $\wp(z_0)$ is transcendental. We use the law of addition in the form of Lemma 4.2. Then by the definition of Θ, ϑ:

$$\Lambda_2(z_0,0)=A_0(\Theta,\vartheta)/B_0(\Theta,\vartheta)$$

where $A_0(\Theta,\vartheta)$, $B_0(\Theta,\vartheta)$ are polynomials in Θ, ϑ of degree at most $c_{18}\cdot\sum_{i=1}^k m_i^2$ with the coefficients from $\mathcal{O}$ of sizes at most $\exp\left\{c_{18}\cdot\sum_{i=1}^k m_i^2\right\}$.

By the Lemma 4.6 we have

$$\left.\frac{d^s}{dw^s}F_1(z,w)\right|_{w=0}=Q_s(z,\wp(z),\wp'(z)), \tag{4.17}$$

with $Q_s(x_1,x_2,x_3)$ satisfying the conditions 1)–2) of Lemma 4.6. Now we set $z=z_0$ and take into account Lemma 4.6. We obtain

$$\wp(z_0)=\frac{T_{\vec{m}}(\Theta,\vartheta)}{U_{\vec{m}}(\Theta,\vartheta)},\qquad \wp'(z_0)=\frac{T'_{\vec{m}}(\Theta,\vartheta)}{U_{\vec{m}}(\Theta,\vartheta)}$$

where $T_{\vec{m}}(\Theta,\vartheta)$, $T'_{\vec{m}}(\Theta,\vartheta)$, $U_{\vec{m}}(\Theta,\vartheta)$ are polynomials in Θ, ϑ of degrees at most

$$c_4\cdot\sum_{i=1}^k m_i^2$$

with algebraic integer coefficients from $\mathcal{O}$ of sizes at most

$$\exp\left(c_5\cdot\sum_{i=1}^k m_i^2\right).$$

We replace in (4.17) $\wp(z_0)$, $\wp'(z_0)$ by $T_{\vec{m}}/U_{\vec{m}}$ and $T'_{\vec{m}}/U_{\vec{m}}$, respectively and z_0 by $m_1\alpha_1+\ldots+m_k\alpha_k$. Then according to (4.17):

$$\left.\frac{d^s}{dz^s}F(z)\right|_{z=z_0}=A_0^{-L_1}\cdot U_{\vec{m}}^{-9L_1}\cdot P_{s,m_1,\ldots,m_k}(\Theta,\vartheta).$$

Now by the 1)–2) of Lemma 4.6 and previous bounds for degrees and sizes of $T_{\vec{m}}(\Theta,\vartheta)$, $T'_{\vec{m}}(\Theta,\vartheta)$, $U_{\vec{m}}(\Theta,\vartheta)$, the polynomial $P_{s,m_1,\ldots,m_k}(\Theta,\vartheta)$ has degree at most

$$c_{19}\cdot L_1\sum_{i=1}^k m_i^2$$

and coefficients from $\mathcal{O}$ of sizes at most

$$\exp\left\{c_{20}\cdot L_1 \sum_{i=1}^{k} m_i^2 + L_0 \log L_0 + L_0 \log(m_1+\ldots+m_k) + c_{21}(s+L_1)\log(s+L_1+1)\right\}.$$

This completes the proof of Lemma 4.7.

All these formulae explain how it is possible to compute the size and degree of values of derivatives of the auxiliary function, with essentially best possible results.

Main Assumption. *Let's assume everywhere below L_0, L_1, S, X are sufficiently large numbers,* $\min\{L_0, L_1, S, X\} > c_{22}$ *and L_1 is a constant depending only on*

$$\alpha_1, \ldots, \alpha_k, g_2, g_3; \tag{4.18}$$

L_0 is sufficiently large with respect to L_1, X:

$$\log L_0 \geqq X^{k+1}. \tag{4.19}$$

Lemma 4.8. *Let L_0, L_1, S, X be sufficiently large numbers satisfying* (4.18)–(4.19), *and define*

$$S = \left[\frac{L_0 L_1}{c_{23}\cdot X^k}\right] \tag{4.20}$$

for c_{23} depending only on $[\mathscr{F}(\alpha_1, \ldots, \alpha_k): \mathbb{Q}]$ and $[\mathbb{K}: \mathbb{Q}(\theta_1, \ldots, \theta_m)]$. Then there exists a polynomial

$$P(x,y) \in \mathbb{I}[x,y], \qquad P(x,y) \not\equiv 0$$

such that

$$d_x(P) \leqq L_0, d_y(P) \leqq L_1$$

and such that the function

$$F(z) = P(z, \wp(z)) \tag{4.21}$$

satisfies

$$F^{(s)}(m_1\alpha_1 + \ldots + m_k\alpha_k) = 0 \tag{4.22}$$

for

$$s = 0, 1, \ldots, S-1; \quad 0 < m_i \leqq X-1:\ i = 1, \ldots, k. \tag{4.23}$$

Moreover for the coefficients $p_{\lambda_0,\lambda_1}(\theta_1, \ldots, \theta_m)$ of $P(x,y)$ as polynomials from $\mathbb{Z}[\Theta]$ we have

$$\max_{\lambda_0,\lambda_1} t(p_{\lambda_0,\lambda_1}) \leqq c_{24}\, S \log L_0; \tag{4.24}$$

$$\max_{\lambda_0,\lambda_1} d(p_{\lambda_0,\lambda_1}) \leqq c_{25}\, L_1 X^2. \tag{4.25}$$

Proof of Lemma 4.8. We consider the polynomial $P(x,y)\in\mathrm{II}[x,y]$ in the following form

$$P(x,y)=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1}(\Theta)\,x^{\lambda_0}y^{\lambda_1},$$

with $p_{\lambda_0,\lambda_1}=p_{\lambda_0,\lambda_1}(\Theta)$ defined as

$$p_{\lambda_0\lambda_1}(\Theta)=\sum_{i_1=0}^{N}\dots\sum_{i_m=0}^{N} p_{\lambda_0,\lambda_1,i_1,\dots,i_m}\theta_1^{i_1}\dots\theta_m^{i_m}.$$

Now the coefficients $p_{\lambda_0,\lambda_1,i_1,\dots,i_m}$ will be rational integers chosen by Siegel's lemma [MW1] to satisfy (4.22). Let

$$F(z)=P(z,\mathfrak{P}(z))$$

and let

$$F_1(z,w)=\Lambda_2(z,w)^{L_1}\cdot P(z+w,\Lambda_1(z,w)/\Lambda_2(z,w))$$

where $\Lambda_1(z,w)$ and $\Lambda_2(z,w)$ are defined above in Lemma 4.5. Then

$$F_1(z,w)=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1}(z+w)^{\lambda_0}\cdot\Lambda_1(z,w)^{\lambda_1}\cdot\Lambda_2(z,w)^{L_1-\lambda_1}.$$

We write

$$T_{\lambda_1}(z,w)=\Lambda_1(z,w)^{\lambda_1}\cdot\Lambda_2(z,w)^{L_1-\lambda_1}:\ 0\leqq\lambda_1\leqq L_1.$$

Now it's possible to apply Lemma 4.7 in the case of $R(x,y)=y^{\lambda_1}$. Let $z_0=m_1\alpha_1+\dots+m_k\alpha_k$, $z_0\neq0$, for algebraic $\alpha_1,\dots,\alpha_k$ and rational integers $m_1,\dots,m_k$. We have by Lemma 4.7 (for $R(x,y)=y^{\lambda_1}$) and the definition of $T_{\lambda_1}(z,w)$:

$$\left.\frac{d^r}{dw^r}T_{\lambda_1}(z,w)\right|_{z=z_0,\,w=0}=A_{\vec m}(\Theta,\vartheta)^{-L_1}\cdot\psi_{r,\lambda_1,\vec m}(\Theta,\vartheta). \tag{4.26}$$

Here $A_{\vec m}(\Theta,\vartheta)\neq0$ is independent of r or λ_1 and $\psi_{r,\lambda_1,\vec m}(\Theta,\vartheta)$ is a polynomial in Θ, ϑ of total degree at most

$$c_{26}\cdot L_1\cdot\sum_{i=1}^{k}m_i^2$$

and with algebraic integer coefficients from $\mathcal{O}$ where the sizes of the coefficients are bounded by

$$\exp\left\{c_{27}\cdot L_1\sum_{i=1}^{k}m_i^2+c_{28}\cdot(r+L_1)\log(r+1)\right\}.$$

We write

$$\psi_{r,\lambda_1,\vec m}(\Theta,\vartheta)=\sum_{j_1=0}^{k}\dots\sum_{j_m=0}^{k}\sum_{j_{m+1}=0}^{d-1}G_{r,\lambda_1\vec m,j_1,\dots,j_{m+1}}\theta_1^{j_1}\dots\theta_m^{j_m}\vartheta^{j_{m+1}}, \tag{4.27}$$

here $G_{r,\lambda_1,\vec{m},j_1,\ldots,j_{m+1}}$ are algebraic integers from $\mathcal{O}$ of sizes at most

$$\exp\left\{c_{29}\cdot L_1\sum_{i=1}^{k} m_i^2+c_{30}\cdot(r+L_1)\log(r+1)\right\}$$

and

$$k\leqq c_{31}\cdot L_1\sum_{i=1}^{k} m_i^2. \tag{4.28}$$

Now

$$F_1(z,w)=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1}(z+w)^{\lambda_0}T_{\lambda_1}(z,w).$$

So we have by (4.26) the following expression for $d^t/dw^t\,F_1(z,w)$ at $w=0$ and $z=z_0$:

$$\begin{aligned}&\frac{d^t}{dw^t}F_1(z,w)\bigg|_{z=m_1\alpha_1+\ldots+m_k\alpha_k,\,w=0}\\&=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}p_{\lambda_0,\lambda_1}\sum_{r=0}^{t}\binom{t}{r}\lambda_0\ldots(\lambda_0-t+r+1)\\&\times(m_1\alpha_1+\ldots+m_k\alpha_k)^{\lambda_0-t+r}\psi_{r,\lambda_1,\vec{m}}(\Theta,\vartheta)\times A_{\vec{m}}(\Theta,\vartheta)^{-L_1}.\end{aligned} \tag{4.29}$$

Now we can use (4.27) and rewrite (4.29) in the powers of Θ, ϑ only.

$$\begin{aligned}&\frac{d^t}{dw^t}F_1(z,w)\bigg|_{z=m_1\alpha_1+\ldots+m_k\alpha_k,\,w=0}\\&=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{i_1=0}^{N}\cdots\sum_{i_m=0}^{N}p_{\lambda_0,\lambda_1,i_1,\ldots,i_m}\\&\times\sum_{j_1=0}^{Z}\cdots\sum_{j_m=0}^{Z}\sum_{j_{m+1}=0}^{d-1}H^{m_1,\ldots,m_k}_{\lambda_0,\lambda_1,i_1,\ldots,i_m,j_1,\ldots,j_{m+1},t}\\&\times\theta_1^{j_1}\ldots\theta_m^{j_m}\vartheta^{j_{m+1}}\times A_{\vec{m}}(\Theta,\vartheta)^{-L_1}\end{aligned}$$

and now $H^{m_1,\ldots,m_k}_{\lambda_0,\lambda_1,i_1,\ldots,i_m,j_1,\ldots,j_m,t}$ are *algebraic numbers* from $\mathcal{O}[\alpha_1,\ldots,\alpha_k]$ of sizes at most

$$\begin{aligned}\exp\Bigg\{&c_{32}\cdot L_1\sum_{i=1}^{k}m_i^2+c_{32}\cdot(t+L_1)\log(t+1)\\&+c_{32}\cdot L_0\log(m_1+\ldots+m_k)+L_0\log L_0\Bigg\}\end{aligned}$$

and for the exponents $j_k\leqq Z$ of $\theta_1,\ldots,\theta_m$ we have a bound

$$Z\leqq c_{33}\cdot L_1\cdot\sum_{i=1}^{k}m_i^2+N.$$

In particular, if $m_i\leqq X-1$: $i=1,\ldots,k$, then

$$Z\leqq Z_0=c_{34}\cdot L_1\cdot X^2+N,\qquad c\geqq 1.$$

We consider the following system of $S((X-1)^k-1)(Z_0+1)^m d$ equations for the $(L_0+1)(L_1+1)(N+1)^m$ unknowns $p_{\lambda_0,\lambda_1,i_1,\dots,i_m}$:

$$\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{i_1=0}^{N}\cdots\sum_{i_m=0}^{N} p_{\lambda_0,\lambda_1,i_1,\dots,i_m} \times H^{m_1,\dots,m_k}_{\lambda_0,\lambda_1,i_1,\dots,i_m,j_1,\dots,j_{m+1},t}=0 \tag{4.30}$$

for all $j_1,\dots,j_m=0,1,\dots,Z_0, j_{m+1}=0,1,\dots,d-1$, $t=0,1,\dots,S-1$ and

$$m_i=1,\dots,X-1: \ i=1,\dots,k.$$

Let $d_1=[\mathscr{F}(\alpha_1,\dots,\alpha_k): \mathbb{Q}]$ and choose N in the following way:

$$N=[(2^{1/m}-1)^{-1}\cdot 2\cdot c_{34}\cdot L_1 X^2],$$

so $(Z_0+1)^m\leqq 2(N+1)^m$. Now according to Siegel's lemma 1.5.5 of [MW1] if

$$L_0 L_1 \geqq 4 d_1 d\cdot SX^k,$$

then there exists a non-trivial solution

$$(p_{\lambda_0,\lambda_1,i_1,\dots,i_m}: \ 0\leqq\lambda_0\leqq L_0, 0\leqq\lambda_1\leqq L_1, 0\leqq i_1,\dots,i_m\leqq N)$$

of (4.30) in rational integers of size at most

$$\exp\{c_{35}\cdot L_1 X^2+c_{35}\cdot S\log S+c_{35}\cdot L_0\log L_0+c_{35}\cdot L_0\log X\}.$$

We can put now $c_{23}=4d_1 d$ and then by the choice of the parameters L_0, L_1, S, X in (4.18), (4.19), (4.20):

$$\max_{\lambda_0,\lambda_1,i_1,\dots,i_m} |p_{\lambda_0,\lambda_1,i_1,\dots,i_m}|\leqq \exp(c_{24}\cdot S\log L_0).$$

Thus we have constructed the polynomial $P(x,y)$. We now show that the function $F(z)=P(z,\mathfrak{P}(z))$ satisfies all the conditions of Lemma 4.8. Let

$$0\leqq s\leqq S-1, \quad 0<m_i\leqq X-1: \quad i=1,\dots,k$$

and $z_0=\sum_{i=1}^k m_i\alpha_i$. Then we have:

$$\begin{aligned}\frac{d^s}{dz^s}F(z)\Big|_{z=z_0}&=\frac{d^s}{dw^s}F(z_0+w)\Big|_{w=0}=\frac{d^s}{dw^s}\{\Lambda_2(z_0,w)^{-L_1}\cdot F_1(z_0,w)\}\Big|_{w=0}\\ &=\sum_{s_1=0}^{s}\binom{s}{s_1}\frac{d^{s-s_1}}{dw^{s-s_1}}\{\Lambda_2(z_0,w)^{-L_1}\}\Big|_{w=0}\cdot\frac{d^{s_1}}{dw^{s_1}}F_1(z_0,w)\Big|_{w=0}.\end{aligned}$$

Now $\Lambda_2(z_0,0)\neq 0$ as $\mathfrak{P}(\alpha)$ is transcendental for $\alpha\in\bar{\mathbb{Q}}$, $\alpha\neq 0$. By the choice of the polynomial $P(x,y)$ and systems (4.30) we have

$$\frac{d^{s_1}}{dw^{s_1}}F_1(z_0,w)\Big|_{w=0}=0 \tag{4.31}$$

for any $s_1 \leqq S-1$. Then (4.31) implies

$$\frac{d^s}{dz^s} F(z)\bigg|_{z=z_0} = 0.$$

Thus Lemma 4.8 is proved.

It is possible to come now directly to the end of the proof because for any auxiliary function of the form

$$F(z) = P(z, f(z))$$

where $f(z)$ is a transcendental function satisfying an algebraic differential equation. The Brownawell-Masser paper [B-M] gives a suitable small value lemma.

The case $f(z) = \wp(z)$ was considered in [B-M], but for our purpose it is enough to use Masser's technique of the separation of variables from [M].

Though the small value lemma for $F(z) = P(z, f(z))$ is almost obvious, we, following the suggestion of the referee, present the corresponding proof here.

Let $\sigma(z)$ be Weierstrass σ-function associated with the lattice L. Then $\sigma(z)$ and $\sigma^2(z)\wp(z)$ are entire functions and

$$G(z) = \sigma(z)^{2L_1} F(z)$$

is also entire.

Lemma 4.9. *Let L_0, L_1, S, X be sufficiently large numbers satisfying* (4.18)–(4.20) *and let*

$$F(z) = P(z, \wp(z))$$

be the function satisfying all the properties of Lemma 4.8.

(i) *Let c be an arbitrary positive constant, and suppose X is sufficiently large with respect to c. Then for all integers $s, m_1, \ldots, m_k$ with*

$$0 \leqq s \leqq cS, \quad 0 < m_i \leqq cX: \quad i = 1, \ldots, k \tag{4.32}$$

we have

$$|F^{(s)}(m_1\alpha_1 + \ldots + m_k\alpha_k)| < \exp(-c_{36} S X^k \log L_0). \tag{4.33}$$

(ii) *Conversely, there exist positive constants c_{38}, c_{39} as follows. If*

$$|F^{(s)}(m_1\alpha_1 + \ldots + m_k\alpha_k)| < \exp(-c_{38} S X^k \log L_0) \tag{4.34}$$

for all integers $s, m_1, \ldots, m_k$ with

$$0 \leqq s \leqq c_{39} S, \quad 0 < m_i \leqq c_{39} X: \; i = 1, \ldots, k$$

then we have

$$H(P) < \exp(-c_{43} S X^k \log L_0) \tag{4.35}$$

where $H(P)$ is the height of the polynomial $P(x, y)$.

Proof. We will apply the Schwarz lemma (see e.g. [MW1], 1.41) to the following entire function

$$G(z) = \sigma(z)^{2L_1} F(z). \tag{4.36}$$

Let's assume that for some $N \geqq 1$ and the constant M sufficiently large with respect to N and L_1 (e.g. $M \geqq N^{k+1}$) the function $F(z)$ satisfies the following conditions

$$|F^{(s)}(n_1 \alpha_1 + \ldots + n_k \alpha_k)| < \exp(-MSX^k \log L_0) \tag{4.37}$$

for all rational integers n_i, $0 < n_i < NX$: $i = 1, \ldots, k$ and integers s, $0 \leqq s < NS$. The condition (4.37) is certainly true for $N = 1$ by the construction of $F(z)$: see (4.22)-(4.23).

We apply (4.37) in conjunction with Schwarz's lemma for $G(z)$ in its traditional form (see [MW1] or [T] or [M] for different versions of Schwarz lemma in the form of Hermite interpolation formula). Then from the bounds for $H(P)$ in (4.24)-(4.25) and from the bounds for $|G|_R$, $|\sigma^2 \mathfrak{P}|_R$ [M] we obtain

$$|G^{(s)}(z)| \leqq \exp(-c_{37} N^{k+1} SX^k \log L_0) \tag{4.38}$$

for $s \leqq cX$ and $|z| \leqq c\sqrt{S \log L_0}$ assuming X to be sufficiently large with respect to c and c_{37} is an absolute constant.

Now (4.33) immediately follows from (4.38) with $N = 1$ assuming L_0 to be sufficiently large with respect to X. Indeed, to pass from (4.38) to (4.33) we use (4.36) and notice that for an algebraic α, $\alpha \neq 0$, we have $\sigma(\alpha) \neq 0$ (this is a simple corollary of Th. Schneider's theorem on transcendence of non-zero elements of the lattice L of $\mathfrak{P}(z)$ [Sc]). Moreover [M] gives us a lower bound for $|\sigma(\alpha)|$:

$$|\sigma(\alpha)| \geqq \exp(-\gamma_0 \cdot \log H(\alpha) [\log \log H(\alpha)]^8)$$

where $\gamma_0 > 0$ depends only on $d(\alpha)$. This bound in combination with (4.36) establishes (4.33) under conditions (4.18)-(4.19) on L_0 and X.

Suppose now that N is sufficiently large in (4.37). This corresponds to the assumption (4.34) with $c_{38} = M$ and $c_{39} = N$. Now we apply the bound (4.38) for $s = 0$ and the following system of numbers z:

$$z(\bar{n}, r) = n_1 \omega_1 + n_2 \omega_2 + \frac{1}{r} \tag{4.39}$$

where n_1, n_2, r are rational integers and

$$0 \leqq n_1, \ n_2 \leqq 2\sqrt{L_0}; \quad r \geqq 1. \tag{4.40}$$

Then inserting such $z(\bar{n}, r)$ from (4.39) into $F(z)$ we obtain:

$$f(z(\bar{n}, r)) = P\left(n_1 \omega_1 + n_2 \omega_2 + \frac{1}{r}, \quad \mathfrak{P}\left(\frac{1}{r}\right)\right). \tag{4.41}$$

From (4.39)-(4.40) and (4.38) we obtain immediately

$$|F(z(\bar{n}, r))| \leqq \exp(-c_{40} N^{k+1} SX^k \log L_0) \tag{4.42}$$

for an absolute constant $c_{40} > 0$. For any $r \geqq 1$, let $P_r(w)$ be the polynomial

$$P_r(w) = P\left(w, \mathfrak{P}\left(\frac{1}{r}\right)\right) = \sum_{\lambda_0=0}^{L_0} w^{\lambda_0} \sum_{\lambda_1=0}^{L_1} P_{\lambda_0, \lambda_1} \mathfrak{P}\left(\frac{1}{r}\right)^{\lambda_1}. \tag{4.43}$$

Then we apply Lemma 1.5 [M] to the polynomials $P_r(w)$ with $w=z(\bar{n},r)$ for fixed r; where the number of different $z(\bar{n},r)$ is $\geqq 4L_0 \geqq 4d_w(P_r(w))$. We obtain

$$H(P_r(w)) = \max_{\lambda_0} \left| \sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1} \mathfrak{P}\left(\frac{1}{r}\right) \right|$$
$$\leqq \exp(-c_{41} N^{k+1} S X^k \log L_0). \tag{4.44}$$

Here we used only Lemma 1.5 [M] and trivial observation that $|x(\bar{n}_1,r) - x(\bar{n}_2,r)| \geqq c_{42}$ for distinct $\bar{n}_1=(n_1^1,n_2^1)$ and $\bar{n}_2=(n_1^2,n_2^2)$ with constant $c_{42} = \min\{|\Omega| : \Omega \in L\}$.

Now for every $\lambda_0 = 0,\dots,L_0$ we apply Lemma 1.5 [M] to each of the polynomials

$$p_{\lambda_0}(z) = \sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1} z^{\lambda_1}$$

at $2L_1$ points of the form $z = \mathfrak{P}\left(\frac{1}{r}\right)$.We obtain immediately

$$H(P) \leqq \exp(-c_{42} N^{k+1} S X^k \log L_0)$$

and Lemma 4.9 is proved.

The following lemma is the outcome of our analytic arguments:

Corollary 4.10. *For sufficiently large L_0, L_1, S, X satisfying* (4.18)–(4.19) *we have either*

(i) *there exists a polynomial $P(x_1,\dots,x_m) \in \mathbb{Z}[x_1,\dots,x_m]$ such that*

$$-c_{44} S X^k \log L_0 < \log|P(\theta_1,\dots,\theta_m)| < -c_{45} S X^k \log L_0; \tag{4.45}$$
$$t(P) < c_{46} S \log L_0; \quad d(P) < c_{47} L_1 X^2, \tag{4.46}$$

or

(ii) *there exists a system $P_l(x_1,\dots,x_m) \in \mathbb{Z}[x_1,\dots,x_m]$, $l \in \mathscr{L}$, of polynomials without common factor such that*

$$|P_l(\theta_1,\dots,\theta_m)| < \exp(-c_{48} S X^k \log L_0) \colon\ l \in \mathscr{L}; \tag{4.47}$$
$$t(P_l) \leqq c_{49} S \log L_0 \colon\ l \in \mathscr{L}; \quad d(P_l) \leqq c_{50} L_1 X^2 \colon\ l \in \mathscr{L}. \tag{4.48}$$

Proof of the Corollary 4.10. Let

$$P(x,y) = \sum_{\lambda_0=0}^{L_0} \sum_{\lambda_1=0}^{L_1} p_{\lambda_0,\lambda_1}(\Theta,\vartheta) x^{\lambda_0} y^{\lambda_1}.$$

We can assume in the construction of the polynomial $P(x,y)$ in Lemma 4.8 that the coefficients $p_{\lambda_0,\lambda_1}(\Theta)$, as polynomials in Θ, don't have a common non-constant factor in $\mathbb{I} = \mathbb{Z}[\Theta]$. Otherwise, we can divide $p_{\lambda_0,\lambda_1}(\Theta)$ by this common factor $r_0(\Theta)$. According to [G] this can change only constants in (4.24)–(4.25). Now applying to Lemma 4.9 we obtain one of the two following possibilities:

a) there exist $s_0 \leqq c_{39} S$, $z_0 = \sum_{i=1}^{k} m_i^0 \alpha_i$, $0 < m_i^0 \leqq c_{39} X$ such that $F(z)$ has zero of multiplicity s_0 at z_0 and

$$\exp(-c_{38} S X^k \log L_0) < |F^{(s_0)}(z_0)| < \exp(-c_{36} S X^k \log L_0) \tag{4.49}$$

or

b)

$$H(P) = \max |p_{\lambda_0,\lambda_1}(\Theta)| < \exp(-c_{43} S X^k \log L_0). \tag{4.50}$$

In the case b) we end up with part ii) above automatically, because the system of polynomials $\{p_{\lambda_0,\lambda_1}(\Theta): 0 \leqq \lambda_i \leqq L_i: i = 0, 1\}$ doesn't have a common factor.

In the case a) we use the generalized Baker-Coates Lemma 4.7. We get

$$F^{(s_0)}(z_0) = A_{\vec{m}_0}(\Theta, \vartheta)^{-L_1} \cdot \xi,$$

where $\xi = R(\Theta, \vartheta)$ is an element of $\mathbb{K}$, being the polynomial in Θ, ϑ. By Lemma 4.7 the total degree of ξ in $\mathbb{K}$ is bounded by

$$d_{\mathbb{K}}(\xi) \leqq c_{51} L_1 X^2 \tag{4.51}$$

and the type of ξ in $\mathbb{K}$ is bounded by

$$t_{\mathbb{K}}(\xi) \leqq c_{51} S \log L_0. \tag{4.52}$$

Now we use bound (4.15) and from a), (4.49) and (4.15) we immediately get

$$\exp(-c_{52} S X^k \log L_0) < |\xi| < \exp(-c_{53} S X^k \log L_0). \tag{4.53}$$

Now according to Lemma 2 [Ch2] we get, starting from ξ in $\mathcal{O}[\Theta, \vartheta]$, an element ζ from $\mathbb{Z}[\Theta]$ satisfying the same bounds (4.51)–(4.53) as ξ but with, possibly different constants. This gives us the case i) with (4.45)–(4.46).

The proof of Corollary 4.10 finished the analytic part of the proof.

5. Conclusion of the Proofs

We now derive the theorems announced at the beginning of the paper from Corollary 4.9.

Proof of Theorem 2.1. We apply Gelfond's lemma with independent growth of the degree and height (see [MW1], [DB]). Indeed, we take $k = 3$ in the notation of §4, and we suppose that, contrary to Theorem 2.1, we have

$$m = \text{transcendence degree of } \bar{\mathbb{Q}}(\wp(\alpha_1), \wp(\alpha_2), \wp(\alpha_3)) \leqq 1.$$

Thus, by a well known theorem of Schneider, we must have $m = 1$. Assuming in Corollary 4.9 that L_1 is constant and L_0 is sufficiently large with respect to x we obtain, in particular, the sequence of polynomials

$$\{P_{L_0}(x): \ L_0 \geqq c_{51}\}$$

such that $P_{L_0}(x)\not\equiv 0$,

$$|P_{L_0}(\theta_1)|<\exp(-c_{52}X^3L_0\log L_0);$$
$$t(P_{L_0})\leqq c_{53}L_0\log L_0;$$
$$d(P_{L_0})\leqq c_{54}X^2.$$

Now we apply Gelfond's lemma in the form due to Brownawell [DB] (see also [MW1], Theorem 5.1.1). It follows that θ_1 is algebraic, contrary to the assumption $m=1$.

Analogously, Theorem 2.3 follows from Corollary 4.5 in the case $m=1$, $k=2$, $\alpha_1=\alpha$, $\alpha_2=\tau\alpha$ using classical results on resultants for polynomial $P(x)$ from Theorem 2.3 with polynomials from Corollary 4.5. (See [G], [MW1].) This application is the "classical" Gelfond-Schneider method.

Proof of Theorem 2.3. It's enough to prove our bound (3) for an irreducible $P(x)$ only. Let's assume that $P(x)\not\equiv 0$ is an irreducible polynomial from $\mathbb{Z}[x]$ and

$$|P(\mathfrak{P}(\alpha))|<H^{-\gamma d}$$

for some sufficiently large constant γ depending only on α and $\mathfrak{P}(z)$). We apply Corollary 4.9 where $k=2$, $\alpha_1=\alpha$, $\alpha_2=\tau\alpha$ and $m=1$, where we can choose θ_1 to be $\mathfrak{P}(\alpha)$ (the transcendental generator of $\mathbb{K}$). We consider the following choice of the parameters X, L_0, S:

$$S[\gamma_1 d^{1/2}], \qquad S=\left[\frac{\gamma_2\log H}{\log\log H}\right],$$
$$L=\left[\frac{\gamma_3 d\log H}{\log\log H}\right]$$

for three parameters γ_1, γ_2, γ_3 independent of d and H, where by (4.18)–(4.20), γ_1 is sufficiently large with respect to L_1 and $L_1\gamma_2^2=c_{32}\gamma_3\gamma_1^2$.

Let's assume that γ is sufficiently large with respect to

$$\max\{c_{44}\gamma_1^2\gamma_2, c_{48}\gamma_1^2\gamma_2\}$$

for c_{44}, c_{48} in (4.45) and (4.47). According to the Corollary 4.5 we have either i) or ii). In the case ii) we take any polynomial $P_l(x)$ relatively prime with $P(x)$ and satisfying (4.47)–(4.48). Such $P_l(x)$ exists because $P(x)$ is an irreducible polynomial. For the resultant $\operatorname{Res}(P,P_l)$ we have (see [G]):

$$\operatorname{Res}(P,P_l)\neq 0, \qquad \operatorname{Res}(P,P_l)\in\mathbb{Z}$$

and by [G] or [MW1], 5.3.1,

$$|\operatorname{Res}(P,P_l)|<\exp(-c_{48}\gamma_1^2\gamma_2 d\log H)$$
$$\times\exp(2c_{49}\gamma_2 d\log H)\times\exp(2L_1\gamma_1^2 d\log H).$$

Now, if both γ_1 and γ_2 are sufficiently large (this is possible since γ_3 is a parameter), then

$$|\operatorname{Res}(P,P_l)|<1,$$

which contradicts our assumption. Suppose next that i) is true, and let $P_0(x)$ be a polynomial satisfying (4.45)-(4.46). Then again by [G] or [MW1], we have

$$|\mathrm{Res}(P, P_0)| < \exp(-c_{45}\gamma_1^2\gamma_2 d\log H) \\ \times \exp(2c_{46}\gamma_2 d\log H) \times \exp(2c_{47}\gamma_1^2 L_1 d\log H)$$

and for large γ_1, γ_2,

$$\mathrm{Res}(P, P_0) = 0.$$

Now $P(x)$ divides $P_0(x)$ and the bound for $|P(\mathfrak{P}(\alpha))|$ implies

$$|P_0(\mathfrak{P}(\alpha))| < H^{-\frac{\gamma}{2}d},$$

whence we conclude from the left hand side of (4.45) that $\gamma \leqq 2c_{44}\gamma_1^2\gamma_2$ contradicting to the choice of γ. Thus Theorem 2.4 has been proven.

We now give a result on diophantine approximations for $\mathfrak{P}(\alpha)$ when $\mathfrak{P}(z)$ does not necessarily have complex multiplication. Thus we consider the case $m=1$, $k=1$, $\alpha_1=1$. We obtain the following result from Corollary 4.9 by the classical Gelfond-Schneider approach.

Theorem 5.1. *Let $\mathfrak{P}(z)$ have algebraic invariants, and let $\alpha \neq 0$ be an algebraic number. Then for $R(x) \in \mathbb{Z}[x]$, $R(x) \not\equiv 0$, $H(R) \leqq H$, $d(R) \leqq d$, we have*

$$|R(\mathfrak{P}(\alpha))| > H^{-c_{42}d^2} \tag{1}$$

if $\log\log H \geqq c_{43}d^2$.

Assuming $R(x)$ to br irreducible we obtain (1) by comparing $R(x)$ with polynomials $P(x)$ from i) or $P_l(x)$ from ii). The choice of parameters in Corollary 4.9 is

$$X = O(d), \quad S = O\left(\frac{d\log H}{\log\log H}\right), \quad L_0 = O\left(\frac{d^2\log H}{\log\log H}\right), \quad L_1 = O(1). \tag{2}$$

For the proof of existence of three or four algebraically independent numbers we need either algebro-geometrical method or an elementary method [Ch3], [Ch6]. In this direction, Theorem 2.2 follows immediately from the methods of [Ch3], §4 and the analytic part §4 (Corollary 4.5).

Again the elementary methods of [Ch3], [Ch6] give us immediately the measure of the algebraic independence of two numbers in the Corollary of §2:

Theorem 5.2. *Let $\mathfrak{P}(z)$ have complex multiplication in $\mathbb{F}$ and α, β be algebraic numbers linearly independent over $\mathbb{F}$. If $P(x, y) \in \mathbb{Z}[x, y]$, $P \not\equiv 0$, $H(P) \leqq H$, $d(P) \leqq d$, then*

$$|P(\mathfrak{P}(\alpha), \mathfrak{P}(\beta))| > H^{-c_{44}d^2},$$

when $\log\log H \geqq c_{45}d^5$ *for constants $c_{44} > 0$, $c_{45} > 0$ depending only on $\mathfrak{P}(z)$ and α, β.*

The elementary method immediately gives us Theorem 5.2 along the same lines as in the proof that type of transcendence of $\frac{\pi}{\omega}, \frac{\eta}{\omega}$ is $3+\varepsilon$ for any $\varepsilon>0$ [Ch3], [Ch6]. Moreover, the proof is even more simple as the degree is always bounded (i.e. types of polynomials involved are sufficiently large with respect to their degrees).

6. Concluding Remarks

To conclude this paper we state results that can be obtained by refinment of our methods and formulate some conjectures.

First of all our methods (we mean the algebraic part, because the analytical part of §4 remains almost without changes) can be considerably generalized. We give only three examples:

Theorem A1. *Let $\wp(z)$ have algebraic invariants and complex multiplication by $\mathbb{Q}(\tau)$. If $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}(\tau)$ and $n \leqq 6$ then*

$$\wp(\alpha_1), \ldots, \wp(\alpha_n)$$

are algebraically independent.

We can naturally generalize the results of §2 for Abelian functions (see [Ch5]):

Theorem A2. *Let $\mathbb{A}$ be a simple Abelian variety defined over $\bar{\mathbb{Q}}$ and $\mathscr{A}_1(\bar{z}), \ldots, \mathscr{A}_d(\bar{z}), \mathscr{B}(\bar{z})$ be corresponding Abelian functions, regular at $\bar{z}=\bar{0}$, where $\mathscr{A}_1(\bar{z}), \ldots, \mathscr{A}_d(\bar{z})$ are algebraically independent and $\dim \mathbb{A}=d$. Let $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ be vectors from $\bar{\mathbb{Q}}^d$ (with algebraic coordinates) linearly independent over $\bar{\mathbb{Q}}$. Let D be a number such that*

$$\frac{n}{2}+1>D$$

and

$$D \leqq 6$$

Then among

$$\mathscr{A}_i(\bar{\alpha}_j): \ i=1, \ldots, d, \quad j=1, \ldots, n$$

there are at least D algebraically independent numbers.

These results becomes especially interesting if in $\bar{\alpha}_j$ all coordinates but one are zero and Abelian variety $\mathbb{A}$ has CM-type:

Theorem A3. *Let $\mathbb{A}$ be a simple CM-variety of dimension d having complex multiplications by an order in a field $\mathbb{K}$, $[\mathbb{K}:\mathbb{Q}]=2d$ and defined over $\bar{\mathbb{Q}}$. Let $\mathscr{A}_1(\bar{z}), \ldots, \mathscr{A}_d(\bar{z}), \mathscr{B}(\bar{z})$ be Abelian functions corresponding to $\mathbb{A}$, regular at $\bar{z}=\bar{0}$ and strongly normalized. If $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ are vectors from $\bar{\mathbb{Q}}^d$, linearly independent over $\mathbb{K}$, such that all coordinates of $\bar{\alpha}_i$ but k_0-th is zero: $i=1, \ldots, n$ for some $k_0 \leqq d$, and*

$$n d \leqq 6,$$

then the n d numbers

$$\mathscr{A}_i(\tilde{\alpha}_j):\ i=1,\dots,d, \qquad j=1,\dots,n$$

are algebraically independent.

For the case of non-CM-type and $d>1$ I don't know even what precise form of conjecture to formulate. For $d=1$ it is easy to do:

Conjecture. *Let $\wp(z)$ have algebraic invariants and no complex multiplication and $\alpha_1,\dots,\alpha_n$ be algebraic numbers, linearly independent over $\mathbb{Q}$. Then*

$$\wp(\alpha_1),\dots,\wp(\alpha_n)$$

are algebraically independent.

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