On the periods of certain rational integrals: I

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0. Introduction

In these papers we shall discuss the calculus of residues for rational integrals in several complex variables. The initiative came from having a look at a paper of Poincaré (cf. [11], especially pp. 334–358) and trying to understand the questions he took up and which remain unsettled. Our study turns out to lead quite directly to problems on algebraic cycles and their homology classes, and on the inversion of generalized abelian integrals. The main results on these problems are Theorem 13.1 (generic inversion theorem for cycles algebraically equivalent to zero), Theorem 14.1 (homological and algebraic equivalence may be quite different), and Theorem 17.1 (this result gives an analytic invariant of algebraic homolgy classes).

In more detail, we will study rational integrals of the form

$$\int_\Gamma \frac{P(z^1, \cdots, z^n)}{Q(z^1, \cdots, z^n)} dz^1 \wedge \cdots \wedge dz^n = \int_\Gamma \frac{P(z)}{Q(z)} dz$$

where $P(z)$, $Q(z)$ are polynomials and Γ is a closed path of integration. Such integrals are called *periods* and, following an heuristic discussion of the case $n = 1$ in § 1, we will in §§ 2, 3, 4 *reduce the period* $\int_\Gamma \{P(z)/Q(z)\}dz$ *to canonical form* in case the polar locus of the integrand $\omega = \{P(z)/Q(z)\}dz$ is non-singular.

Actually, we should and do consider $\int_\Gamma \omega$ as a rational integral on the *complex projective space* $\mathbf{P}_n$. Our results, if suitably interpreted (cf. § 10), will be valid for rational integrals on a general algebraic manifold; we have chosen to do everything on $\mathbf{P}_n$ because there we can be quite explicit in our methods while encountering the essential difficulties of the general situation.

After putting $\int_\Gamma \omega$ in canonical form, we come to the problem of computing the number of linearly independent periods. This question is trivial for $n = 1$ and can be done for $n = 2$. However, for $n \geqq 3$ we find (cf. § 5, Proposition 5.6 and Theorem 5.8) that *the algebraic cycles lying in the polar variety of the integrand contribute relations among the periods*. Thus we are faced with a Hodge-type problem: Are all relations accounted for by such algebraic cycles? In these papers, especially in Part II, we will focus on the easier question: *Do algebraic cycles give any conditions other than the Hodge conditions*?

Now if all relations are to be accounted for by algebraic cycles, then by considerations of dimension there are some relations which must be trivial. We are able to prove this by finding suitable *Riemann bilinear relations for the periods*. This is done in §§ 6, 7, 8 where, in particular, we relate our periods of rational integrals to the *Hodge theory of harmonic integrals* (cf. (8.6) in § 8).

In the last section (§ 9) we let the periods depend on parameters. Thus we are looking at $\int_\Gamma (P(z; \lambda)/Q(z; \lambda))dz$ where P, Q are polynomials in z, λ, and we prove a *local Torelli theorem* (cf. Theorem 9.8) to the effect that the periods locally determine the polar locus $Q(z; \lambda) = 0$ up to linear transformation, with only the exception of the cubic surface.

In Part II we continue discussing the effect which algebraic cycles lying in the polar locus have on the periods $\int_\Gamma \{P(z)/Q(z)\}dz$. The motivation here was again a paper of Poincaré (Ann. école norm. sup., 27 (1910), 55–108).

In § 10 we give a treatment using sheaf theory of the reduction of periods and the relation to harmonic integrals of our rational integrals. The reason is partly to complete one argument from § 5, but mainly to have available the mechanism for discussing $\int_\Gamma P(z)dz/Q(z)$ in case the polar locus $Q(z) = 0$ has simple singularities.

In § 11 we introduce an intermediate jacobian variety $T(S)$ where $S \subset \mathbf{P}_{2m}$ is the polar locus of the integrand $P(z)dz/Q(z)$ and $\dim S = 2m - 1$ is odd. If $\Theta(S)$ is the *group of algebraic* $(m - 1)$*-cycles of degree zero* on S, then there is an *Abel-Jacobi homomorphism* $\varphi: \Theta(S) \to T(S)$ having nice properties

(§§ 11, 12). In case $m = 1$, $T(S)$ is the usual jacobian variety of the curve S and φ is the map given by abelian sums. This intermediate jacobian is generally *not* the one introduced using harmonic theory by Weil; rather the definition of $T(S)$ and its properties will be done entirely using complex analysis without reference to C^∞ differentials.

At this juncture events take a non-classical turn. Namely, we prove in § 13 that, with finitely many exceptions on the degree of $S \subset \mathbf{P}_{2m}$, if $m > 1$ and S is generic, then *the Abel-Jacobi map takes the group* $\Sigma(S)$ *of* $(m-1)$*-cycles algebraically equivalent to zero into zero* (the same result holds for Weil's jacobians). Thus there is induced a map $\varphi: \Theta(S)/\Sigma(S) \rightarrow T(S)$, and the *Jacobi inversion theorem takes place on at most a countable subgroup of the torus* $T(S)$. There remains the possibility that $\Theta(S) = \Sigma(S)$, i.e., that homology and algebraic equivalence coincide and the inversion theorem is completely trivial. However, we prove in § 14 that if S is given for example by

$$w_1^5 + w_2^5 + w_3^5 + w_4^5 = 1 + (\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4)^5$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ generic, then $\Theta(S)/\Sigma(S)$ contains an infinite cyclic subgroup which is mapped isomorphically into $T(S)$ by the Abel-Jacobi map.

The method of proof is to take a non-singular hypersurface $V \subset \mathbf{P}_{2m+1}$ given in affine coordinates by $Q(z^1, \cdots, z^{2m+1}) = 0$. We may assume that the pencil of hyperplane sections $|S_\lambda|_{\lambda \in \mathbf{P}_1}$ given by $z^1 = \lambda$ is generic. Then the intermediate jacobians $T(S_\lambda)$ vary holomorphically with λ and can be analyzed at the tangencies of the pencil (cf. §§ 15, 16).

On V the essential algebraic m-cycles are the *primitive cycles*, which are the cycles Z such that $Z = Z \cdot S_\lambda$ is of degree zero on $S_\lambda \subset \mathbf{P}_{2m}(\lambda)$. Given such a primitive cycle Z, we may use the Abel-Jacobi maps to obtain a holomorphic cross-section ν_Z of the family $\mathcal{T} = \bigcup_{\lambda \in \mathbf{P}_1} T(S_\lambda)$ of intermediate jacobians.

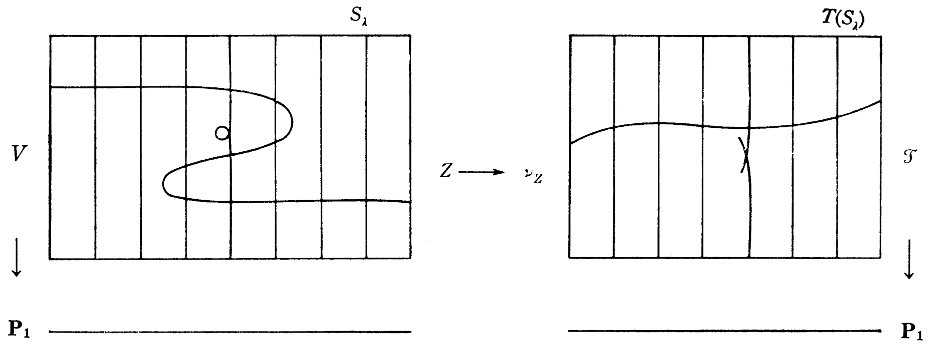


FIG. 0

Following Poincaré-Lefschetz we prove that ν_Z depends only on the homology class $\mathfrak{Z} \in H_{2m}(V, \mathbf{Z})$ of Z. Using this plus the fact that, if $\deg V \geqq 2 + 3/(m-1)$

and $m>1$, φ maps $\Sigma(S_\lambda)$ to zero in $T(S_\lambda)$ for λ generic, we find that the assumption $\Theta(S_\lambda) = \Sigma(S_\lambda)$ leads to a contradiction.

The main result of Part II, which is proved in § 17, is this:

(i) the group Hom $(\mathbf{P}_1, \mathfrak{L})$ of holomorphic cross-sections of $\mathfrak{L} \to \mathbf{P}_1$ is a finitely generated abelian group;

(ii) the cross-section ν_Z associated to a primitive m-cycle Z on V depends only on the homology class of Z.

This result requires the most technical work of any in the paper because we must give a fairly precise discussion of the *generalized intermediate jacobians* $T(S_{\lambda_\alpha})$ at those *critical points* $\lambda_1, \cdots, \lambda_N$ where the hyperplane section S_λ becomes singular, as well as discussing the cross-section ν_Z near such critical points.

As general references to other discussions of residues in several variables, we suggest in addition to the paper of Poincaré [11], the two volumes of Picard-Simart ("Traité des fonctions algébriques de deux variables"—especially the first volume), the monograph of Zariski ("Algebraic surfaces"—especially Ch. VII), the appendix of the Lefschetz-Borel tract [8], and more recently the papers of Atiyah-Hodge (Ann. of Math. **62** (1955), 56–91), Leray (Bull. Soc. Math. France, **87** (1959), 81–180), and Grothendieck [6].

1. Formulation of the problem

We will discuss integrals of the form

$$\int_\Gamma \frac{P(z^1, \cdots, z^n)}{Q(z^1, \cdots, z^n)} dz^1 \cdots dz^n \tag{1.1}$$

where P and Q are polynomials in the complex variables $z^1, \cdots, z^n$ and Γ is a closed n-dimensional path of integration not intersecting the algebraic set $Q(z^1, \cdots, z^n) = 0$. The integral (1.1) will be referred to briefly as a *period*, and we shall discuss such problems as putting the period in canonical form, finding the number of independent periods, and examining how the periods vary with Q.

The simplest case is when $n = 1$ so that (1.1) becomes a familiar *contour integral* $\int_\Gamma P(z)dz/Q(z)$ with Γ being typically a closed curve in the complex plane winding around some of the poles of ω (Fig. 1). By the *residue theorem* we find in this case that

$$\int_\Gamma \frac{P(z)dz}{Q(z)} = \frac{1}{2\pi i}\left\{\sum_{j=1}^N \operatorname{Res}_{p_j}\left(\frac{P(z)}{Q(z)}\right)\right\},$$

and a discussion of the residue theorem might be given by the following steps.

(i) We consider $P(z)dz/Q(z) = \omega$ as a rational differential on the

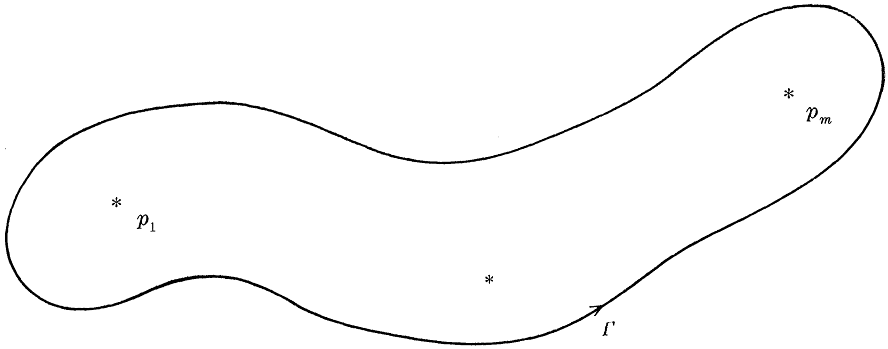
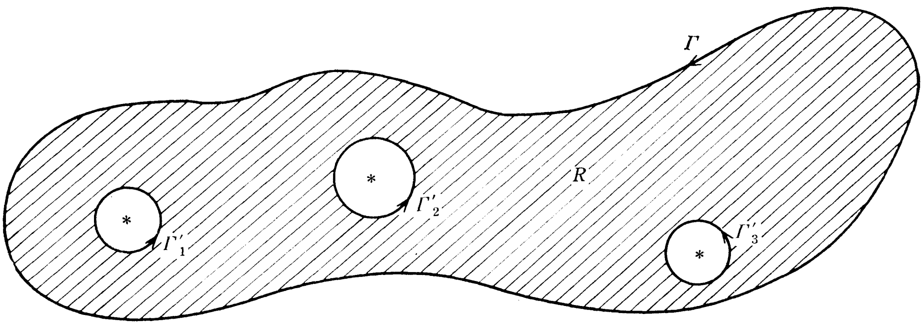


FIG. 1

Riemann sphere $\mathbf{P}_1 = \mathbf{C} \cup \{\infty\}$. To describe ω near $z = \infty$, we make the substitution $w = 1/z$ so that $\omega = -P(1/w)dw/Q(1/w)w^2$, and this expression near $w = 0$ tells us about $P(z)dz/Q(z)$ near infinity. Let $V = \{p_1, \cdots, p_m\}$ be the polar locus of ω; V consists of the zeros of $Q(z)$ plus infinity if $\deg Q < \deg P + 2$.

(ii) The period $\int_\Gamma \omega$ is the same as $\int_{\Gamma'} \omega$ for any closed curve Γ' which is homologous to Γ in $\mathbf{P}_1 - V$:



$$\begin{cases} \Gamma' = \Gamma'_1 + \Gamma'_2 + \Gamma'_3 \\ \text{and } \Gamma - \Gamma' = \partial R. \end{cases}$$

FIG. 2

(iii) Let Γ_j be a small circle winding positively around p_j. Then for any contour Γ we have a homology $\Gamma \sim \sum_{j=1}^m \mu_j(\Gamma)\Gamma_j$ where $\mu_j(\Gamma)$ is the *winding number* of Γ around p_j. Thus $\int_\Gamma \omega = \sum_{j=1}^m \mu_j(\Gamma)\int_{\Gamma_j} \omega$ so that to know $\int_\Gamma \omega$ we need only to know the expressions $\int_{\Gamma_j} \omega$.

We remark that $\sum_{j=1}^m \Gamma_j \sim 0$, and this is the generating relation among the Γ_j.

(iv) The period $\int_\Gamma \omega$ is the same as $\int_\Gamma \omega + d\alpha$ where α is any rational function whose poles lie in V.

(v) Given ω, we can find a rational function α with poles on V such that $\omega + d\alpha$ has only first order poles at each point of V. To see this, suppose for instance that ω has a pole of order two at $z = 0$ and let $\omega(z) = (a/z^2 + b/z + \cdots)dz$ be the Laurent series expansion of ω near $z = 0$. If $\alpha = a/z$, then $\omega + d\alpha = (b/z + \cdots)dz$ has only a first order pole at the origin, and proceeding in this manner we can reduce ω to have simple poles at all points of V.

Putting this all together, to know $\int_\Gamma \omega$ we need only know $\int_{\Gamma_j} \psi(z)$ where $\psi(z)$ has a first order pole at p_j. In this way we may bring $\int_\Gamma \omega$ into a sort of canonical form. These five steps will be generalized to several complex variables and the integral (1.1) brought into a similar canonical form.

To calculate the number of independent integrals $\int_\Gamma \omega$, we use linearity together with the following remarks.

(a) $\int_\Gamma \omega = 0$ for all differentials ω with poles on $V \Rightarrow \sum_{j=1}^m a_j \mu(\Gamma_j) = 0$ for all $a_1, \cdots, a_m$ with $\sum_{j=1}^m a_j = 0$, and so the winding numbers $\mu(\Gamma_j)$ are all the same integer μ and $\Gamma \sim \mu\left(\sum_{j=1}^m \Gamma_j\right) \sim 0$;

(b) $\int_\Gamma \omega = 0$ for all closed contours $\Gamma \Rightarrow \omega$ is the total differential of a rational function α with poles on V $\left(\text{set } \alpha(z) = \int_{z_0}^z \omega \text{ where } z_0 \text{ lies outside } V\right)$. We will see below that (b) generalizes, but that a proper understanding of (a) involves knowledge of the algebraic cycles lying on an algebraic manifold.

To study how the period $\int_\Gamma P(z)dz/Q(z)$ varies with Q, we first look at the special case $Q(z, \lambda) = z(z - \lambda)$ and $P(z) = 1$. Then the period $\pi(\lambda) = \int_\Gamma dz/z(z - \lambda)$ is a function of λ. To find the derivative $\pi'(\lambda)$, we differentiate under the integral sign to find that $\pi'(\lambda) = \int_\Gamma dz/z(z - \lambda)^2$. Then

$$\lambda\pi'(\lambda) + \pi(\lambda)$$
$$= \int_\Gamma \left\{\frac{\lambda}{z(z-\lambda)^2} + \frac{1}{z(z-\lambda)}\right\}dz = \int_\Gamma \frac{dz}{(z-\lambda)^2} = \int_\Gamma d\left(\frac{1}{\lambda - z}\right) = 0 ,$$

so that $\pi(\lambda)$ satisfies the *linear differential equation with regular singular points* $\lambda\pi'(\lambda) + \pi(\lambda) = 0$.

More generally, suppose that $Q(z, \lambda)$ is an arbitrary polynomial in z and λ and $\omega = P(z)dz/Q(z, \lambda)$ is the integrand in the contour integral $\int_\Gamma P(z)dz/Q(z, \lambda)$. The polar locus $V_\lambda = \{z_1(\lambda), \cdots, z_m(\lambda)\}$ where the $z_j(\lambda)$ are algebraic functions of λ giving the roots of the equation $Q(z, \lambda) = 0$. If $\omega^{(j)} = \partial^j \omega / \partial \lambda^j$, then $\omega^{(j)}$ has poles on V_λ also, and there will be a minimal linear relation

$$P_k(\lambda)\omega^{(k)} + \cdots + P_1(\lambda)\omega^{(1)} + P_0(\lambda)\omega = d\alpha \qquad (k \leqq m) \,, \tag{1.2}$$

where the $P_j(\lambda)$ are polynomials and $\alpha = R(z, \lambda)$ is a rational function in z and λ having poles on V. Letting $\pi(\lambda) = \int_\Gamma \omega$, the relation (1.2) gives the differential equation

$$P_k(\lambda)\frac{d^k \pi(\lambda)}{d\lambda^k} + \cdots + P_0(\lambda)\pi(\lambda) = 0 \tag{1.3}$$

satisfied by the period $\pi(\lambda)$. Since (1.2) was the minimal linear relation, all solutions of (1.3) are linear combinations of periods $\int_{\Gamma'} \omega$ for suitable contours Γ'.

The fact that the variable periods of (1.1) satisfy an ordinary differential equation (1.3) will be discussed below, and these differential equations will be proved to be of *fuchsian type*.

2. Rational differentials on projective space

In studying the integral (1.1) in case $n = 1$ we saw that it was convenient to think of $\omega = P(z)dz/Q(z)$ as a rational differential on the closed Riemann sphere $\mathbf{P}_1 = \mathbf{C} \cup \{\infty\}$. Similarly we will consider

$$\omega = P\big((z^1, \cdots, z^n)/Q(z^1, \cdots, z^n)\big)dz^1 \cdots dz^n$$

as a *rational differential form* of degree n on the complex projective space $\mathbf{P}_n$. Now we may think of $\mathbf{P}_n$ as $\mathbf{C}_n$ with the hyperplane $\mathbf{P}_{n-1}$ at infinity added on, or equivalently as the space of lines through the origin in $\mathbf{C}^{n+1}$. The former interpretation is the more convenient geometrically, while the second is better for computational purposes and will be discussed now.

Let $\xi^0, \cdots, \xi^n$ be linear coordinates on $\mathbf{C}^{n+1}$ and $\xi = (\xi^0, \cdots, \xi^n)$ a non-zero point. The line passing through ξ is given parametrically by the points $\lambda\xi = (\lambda\xi^0, \cdots, \lambda\xi^n)$ as λ varies over the usual complex line. Thus, thinking of $\mathbf{P}_n$ as the lines in $\mathbf{C}^{n+1}$, we see that $\mathbf{C}^{n+1} - \{0\}$ lies over $\mathbf{P}_n$ with the equivalence relation $\xi \sim \lambda\xi$ for $\lambda \neq 0$. Observe that the fibres of this mapping $\mathbf{C}^{n+1} - \{0\} \rightarrow \mathbf{P}_n$ are precisely the integral curves of the *Euler vector field* $\theta = \sum_{j=0}^n \xi^j \partial/\partial\xi^j$. In fact, the one-parameter-group generated by θ is just the group of linear transformations $\xi \rightarrow \lambda\xi$ on $\mathbf{C}^{n+1}$.

The embedding $\mathbf{C}^n \subset \mathbf{P}_n$ is given by $(z^1, \cdots, z^n) \rightarrow [1, z^1, \cdots, z^n]$, where $[\ ,\cdots,\ ]$ is the usual notation for *homogeneous coordinates*. Thus the hyperplane at infinity is given by $\xi_0 = 0$, and the inverse mapping $\mathbf{P}_n - \mathbf{P}_{n-1} \rightarrow \mathbf{C}^n$

is just $[\xi^0, \cdots, \xi^n] \to (\xi^1/\xi^0, \cdots, \xi^n/\xi^0)$.

Let φ be a rational differential form of degree k on $\mathbf{C}^n$. Then φ has an expression $\varphi = \{1/B(z)\}\{\sum_J A_J(z)dz^J\}$ where the $A_J(z)$ and $B(z)$ are polynomials, and where we have used the abbreviations $z = (z^1, \cdots, z^n)$, $J = (j_1, \cdots, j_k)$ runs over k-tuples with $j_1 < \cdots < j_k$, and $dz^J = dz^{j_1} \wedge \cdots \wedge dz^{j_k}$. Now φ gives a rational k-form on $\mathbf{C}^{n+1} - \{0\}$ by letting $z^j = \xi^j/\xi^0$ and $dz^j = (\xi^0 d\xi^j - \xi^j d\xi^0)/(\xi^0)^2$; expressing φ in terms of the ξ^i and $d\xi^j$ we can easily tell about its behavior near the hyperplane at infinity. Writing things out we get a formula

$$\varphi = \frac{1}{B(\xi)}\{\textstyle\sum_J A_J(\xi)d\xi^J\} \tag{2.1}$$

where the $A_J(\xi)$ and $B(\xi)$ are homogeneous polynomials with $\deg B = \deg A_J + k$.

Conversely, a rational k-form φ on $\mathbf{C}^{n+1}$ is given by an expression (2.1), and we want to know when φ comes from $\mathbf{P}_n$. To answer this, we define the *contraction* $\langle\theta, \varphi\rangle$ of φ with the Euler vector field θ to be the rational $k - 1$ form on $\mathbf{C}^{n+1}$ given by the following prescription.

(i) $\langle\theta, \varphi\rangle = (1/B(\xi))\{\sum A_J(\xi)\langle\theta, d\xi^J\rangle\}$ (*linearity*);

(ii) $\langle\theta, d\xi^J\rangle = \sum_{1\leq\alpha\leq k} (-1)^{\alpha-1}\langle\theta, d\xi^{j_\alpha}\rangle d\xi^{j_1} \wedge \cdots \wedge \widehat{d\xi^{j_\alpha}} \wedge \cdots \wedge d\xi^{j_k}$ (*derivation property*); and

(iii) $\langle\theta, d\xi^j\rangle = \xi^j$.

PROPOSITION 2.2. *The rational k-form φ on $\mathbf{C}^{n+1}$ given by* (2.1) *comes from a k-form on $\mathbf{P}_n$ if, and only if,*

(a) $\deg B = \deg A_J + k$ *and*

(b) $\langle\theta, \varphi\rangle = 0$.

PROOF. The condition (a) is clear and we need only concern ourselves with (b). If φ comes from $\mathbf{P}_n$, then

$$\varphi = \frac{1}{B(\xi/\xi^0)}\{\textstyle\sum_J A_J(\xi/\xi^0)d(\xi^{j_1}/\xi^0) \wedge \cdots \wedge d(\xi^{j_k}/\xi^0)\}$$

and, since

$$\begin{aligned}\langle\theta, d(\xi^j/\xi^0)\rangle &= \frac{1}{(\xi^0)^2}\langle\theta, \xi^0 d\xi^j - \xi^j d\xi^0\rangle \\ &= \xi^j\xi^0 - \xi^j\xi^0 = 0\ ,\end{aligned}$$

we see that $\langle\theta, \varphi\rangle = 0$. The proof of the converse will be based on the following

LEMMA 2.3.

$$d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_k}$$
$$=(\xi^0)^k d\Big(\frac{\xi^{i_1}}{\xi^0}\Big)\wedge\cdots\wedge d\Big(\frac{\xi^{i_k}}{\xi^0}\Big)+\frac{d\xi^0}{\xi^0}\wedge\langle\theta, d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_k}\rangle .$$

PROOF. If $k=1$ we have $d\xi^i=\xi^0 d(\xi^i/\xi^0)+(d\xi^0/\xi^0)\langle\theta, d\xi^i\rangle$ so that the formula holds in this case. Using induction we have

$$\begin{aligned} d\xi^{i_1}&\wedge\cdots\wedge d\xi^{i_k}\\ &=\{(\xi^0)^{k-1}d(\xi^{i_1}/\xi^0)\wedge\cdots\wedge d(\xi^{i_{k-1}}/\xi^0)+d\xi^0/\xi^0\wedge\langle\theta, d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_{k-1}}\rangle\}\\ &\quad\cdot\{\xi^0 d(\xi^{i_k}/\xi^0)+\xi^{i_k}d\xi^0/\xi^0\}\\ &=(\xi^0)^k d(\xi^{i_1}/\xi^0)\wedge\cdots\wedge d(\xi^{i_k}/\xi^0)+d\xi^0/\xi^0\wedge\langle\theta, d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_{k-1}}\rangle\wedge d\xi^{i_k}\\ &\quad+(-1)^{k-1}d\xi^0/\xi^0\wedge\xi^{i_k}d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_{k-1}}\\ &=(\xi^0)^k d(\xi^{i_1}/\xi^0)\wedge\cdots\wedge d(\xi^{i_k}/\xi^0)+d\xi^0/\xi^0\wedge\langle\theta, d\xi^{i_1}\wedge\cdots\wedge d\xi^{i_k}\rangle .\quad\text{Q.E.D.}\end{aligned}$$

Taking φ to be given by (2.1), we use (2.3) to find that

$$\varphi=\frac{(\xi^0)^k}{B(\xi)}\{\textstyle\sum_{j_1<\cdots<j_k}A_J(\xi)d(\xi^{j_1}/\xi^0)\wedge\cdots\wedge d(\xi^{j_k}/\xi^0)\}+d\xi^0/\xi^0\wedge\langle\theta,\varphi\rangle ;$$

thus, if $\langle\theta,\varphi\rangle=0$, we see that φ comes from $\mathbf{P}_n$ as desired.

We want to use Proposition (2.2) to get an explicit description of the forms (2.1) with fixed denominator $B(\xi)$ which come from $\mathbf{P}_n$. For this purpose it is convenient to rewrite (2.1) as

$$\varphi=\frac{1}{B(\xi)}\{\textstyle\sum_{i_1<\cdots<i_l}(-1)^{i_1+\cdots+i_l}A_{i_1\cdots i_l}(\xi)(\cdots\widehat{d\xi^{i_1}}\cdots\widehat{d\xi^{i_l}}\cdots)\}\tag{2.4}$$

where φ now has degree $n+1-l$ and

$$(\cdots\widehat{d\xi^{i_1}}\cdots\widehat{d\xi^{i_l}}\cdots)=d\xi^0\wedge\cdots\wedge\widehat{d\xi^{i_1}}\wedge\cdots\wedge\widehat{d\xi^{i_l}}\wedge\cdots\wedge d\xi^n .$$

LEMMA 2.5.

$$\begin{aligned}\langle\theta,\varphi\rangle=\frac{1}{B(\xi)}\{&\textstyle\sum_{j_1<\cdots<j_{l+1}}(-1)^{j_1+\cdots+j_{l+1}}\\ &\cdot[\textstyle\sum_{\alpha=1}^{l+1}(-1)^\alpha\xi^{j_\alpha}A_{j_1\cdots\hat{j}_\alpha\cdots j_{l+1}}(\xi)](\cdots\widehat{d\xi^{j_1}}\wedge\cdots\wedge\widehat{d\xi^{j_{l+1}}}\cdots)\} .\end{aligned}$$

The proof is a direct calculation. Using (2.5), the condition $\langle\theta,\varphi\rangle=0$ now becomes

$$\textstyle\sum_{\alpha=1}^{l+1}(-)^\alpha\xi^{j_\alpha}A_{j_1\cdots\hat{j}_\alpha\cdots j_{l+1}}(\xi)=0\tag{2.6}$$

for all $j_1<\cdots<j_{l+1}$.

To put (2.6) in a more workable form, we let E be a complex vector space with basis $e_0,\cdots,e_n$, $\mathcal{F}_a$ the vector space of homogeneous polynomials in $\xi^0,\cdots,\xi^n$ of degree a, and $\mathcal{F}_a^l=\mathcal{F}_a\otimes\Lambda^l E$. Then η belonging to $\mathcal{F}_a^l$ is written $\eta=\sum_{i_1<\cdots<i_l}A_{i_1\cdots i_l}(\xi)\otimes e_{i_1}\wedge\cdots\wedge e_{i_l}$. Comparing this with (2.4) and taking $a=\deg B-(n+1-l)$ we may identify $\mathcal{F}_a^l$ with the forms (2.4) having a fixed

denominator $B(\xi)$. By (2.6) the condition $\langle\theta, \eta\rangle = 0$ becomes $\xi\cdot\eta = 0$ where $\xi = \sum_{j=0}^{n}\xi^j e_j$ belongs to $\mathcal{F}_1^1$, $\xi\cdot e_I = \sum_{j=0}^{n}\xi^j e_j \wedge e_I(I = (i_1, \cdots, i_l))$, and ξ is defined to be linear over the polynomials.

This suggests that we consider the complex

$$(2.7)\qquad 0 \longrightarrow \mathcal{F}_a^0 \xrightarrow{\ \xi\ } \mathcal{F}_{a+1}^1 \xrightarrow{\ \xi\ } \mathcal{F}_{a+2}^2 \xrightarrow{\ \xi\ } \cdots \xrightarrow{\ \xi\ } \mathcal{F}_{a+n+1}^{n+1} \longrightarrow 0 .$$

LEMMA 2.8. *The complex* (2.7) *is exact.*

PROOF. We will construct a homotopy operator $\partial\colon \mathcal{F}_{a+l}^l \to \mathcal{F}_{a+l-1}^{l-1}$ with $(\partial\xi + \xi\partial)\eta = (n+1+a)\eta$. Letting $e_0^*, \cdots, e_n^*$ be a dual basis of E^*, we define ∂ by linearity and $\partial(A \otimes e_{j_1} \wedge \cdots \wedge e_{j_l}) = \sum_{j=0}^{n}\partial A/\partial\xi^j \otimes \langle e_j^*, e_{j_1} \wedge \cdots \wedge e_{j_l}\rangle$ where $\langle e_j^*, \ \rangle\colon \Lambda^l E \to \Lambda^{l-1}E$ is the contraction operator. The proof of $(\partial\xi + \xi\partial)\eta = (n+1+a)\eta$ is now straightforward using the Euler relation $\sum_{j=0}^{n}\xi^j\partial A/\partial\xi^j = (\deg A)A$.

Combining (2.2), (2.6), and (2.8) we conclude

THEOREM 2.9. *The rational $n+1-l$ forms on $\mathbf{P}_n$ may all be written as*

$$(2.10)\qquad \varphi = \frac{1}{B(\xi)}\{\textstyle\sum_{j_1<\cdots<j_l}(-1)^{j_1+\cdots+j_l}[\sum_{\alpha=1}^{l}(-1)^{\alpha}\xi^{j_\alpha}A_{j_1\cdots\hat{j}_\alpha\cdots j_l}(\xi)] \cdot(\cdots d\hat{\xi}^{j_1}\cdots d\hat{\xi}^{j_l}\cdots)\}$$

where $\deg B = \deg A_{j_1\cdots\hat{j}_\alpha\cdots j_l} + (n+2-l)$.

COROLLARY 2.11. *The rational n-forms on $\mathbf{P}_n$ are all written as* $\varphi = P(\xi)\Omega/Q(\xi)$ *where* $\Omega = \sum_{\alpha=0}^{n}(-1)^\alpha\xi^\alpha(\cdots d\hat{\xi}^\alpha\cdots)$ *and* $\deg Q = \deg P + (n+1)$.

The proof of (2.11) may be given directly as follows. Using

$$d(\xi^1/\xi^0) \wedge \cdots \wedge d(\xi^n/\xi^0) = (1/(\xi^0)^{n+1})\Omega ,$$

we have that

$$(P(z)dz^1 \wedge \cdots \wedge dz^n)/Q(z) = (\xi^0)^{\deg P}P(\xi)\Omega/(\xi^0)^{\deg Q+n+1}Q(\xi) .$$

3. Topological reduction of the periods

We return to consideration of the period (1.1). Writing

$$\omega = (P(z^1, \cdots, z^n)/Q(z^1, \cdots, z^n))dz^1 \wedge \cdots \wedge dz^n$$

as a rational n-form on $\mathbf{P}_n$ (cf. § 2 above), we have $\omega = (P(\xi)/Q(\xi))\Omega$ where $\Omega = \sum_{j=0}^{n}(-1)^j\xi^j(\cdots d\hat{\xi}^j\cdots)$ and $P(\xi)$, $Q(\xi)$ are homogeneous polynomials with $\deg Q = \deg P + (n+1)$. The *polar locus* V of ω is the algebraic variety in $\mathbf{P}_n$ defined by $Q(\xi) = 0$. We make throughout the basic

Assumption 3.1. V is a *non-singular* hypersurface in $\mathbf{P}_n$.

With this assumption, we may rewrite ω as $\omega = P(\xi)\Omega/Q(\xi)^k$ where $Q(\xi) = 0$ is the *minimal* defining equation of V and P, Q are relatively prime.

(Note that V is irreducible if $n > 1$, and in this case the principal ideal (Q) is a prime ideal.) The differential form ω then has a pole of order k along V; since $k \deg Q = \deg P + (n+1)$ we have necessarily that $k \geqq 1$.

The path of integration Γ in the period $\int_\Gamma \omega$ represents a *piecewise smooth singular homology class* in $\mathbf{P}_n - V$, which we write as $\Gamma \in H_n(\mathbf{P}_n - V, \mathbf{Z})$. By analogy with (ii) in § 1, we have as a consequence of *Stokes's theorem* that

(3.2) The period $\int_\Gamma \omega$ depends only on the homology class of $\Gamma \in H_n(\mathbf{P}_n - V, \mathbf{Z})$.

By using homology we want now to put the n-cycle Γ in a standard form. For this, we will make auxiliary use of a riemannian metric on $\mathbf{P}_n$; for example, the usual non-euclidean distance function

$$d(\xi, \eta) = \frac{|\xi|^2 |\eta|^2 - |(\xi, \eta)|^2}{|\xi|^2 |\eta|^2} \qquad (\xi, \eta \in \mathbf{C}^{n+1} - \{0\})$$

will serve the purpose.

We first claim that Γ is homologous to zero in $\mathbf{P}_n$. In case $n = 2m+1$ is odd this follows simply from $H_{2m+1}(\mathbf{P}_n, \mathbf{Z}) = 0$. If $n = 2m$ is even we let $\mathbf{P}_{m+1} \subset \mathbf{P}_{2m}$ be a linear subspace in general position with respect to V and $W = \mathbf{P}_{m+1} \cdot V$. Then W is a non-singular m-dimensional subvariety of $\mathbf{P}_{2m}$, and in homology $W \sim (\deg Q)\gamma_m$ where $\gamma_m \in H_{2m}(\mathbf{P}_{2m}, \mathbf{Z})$ is the generator. Since the *intersection number* $\Gamma \cdot \gamma_m = (1/\deg Q)\Gamma \cdot W = 0$, we find that $\Gamma \sim 0$ in $\mathbf{P}_{2m}$ (cf. Lefschetz [8]).

Write now $\Gamma = \partial R$ where R is an $n+1$ chain in $\mathbf{P}_n$. We may assume that R is in *general position* with respect to V, and thus R meets V *transversely* in an $n-1$ cycle γ lying on V (since $\partial(R \cdot V) = \partial R \cdot V \pm R \cdot \partial V = 0$). Let $T_\varepsilon(\gamma)$ be the points in R with $d(z, \gamma) < \varepsilon$. Because of transversality, for sufficiently small ε a point z in $T_\varepsilon(\gamma)$ is given uniquely by a pair (v, w) where v lies on γ and w belongs to the *normal ε-disc* to V at v.

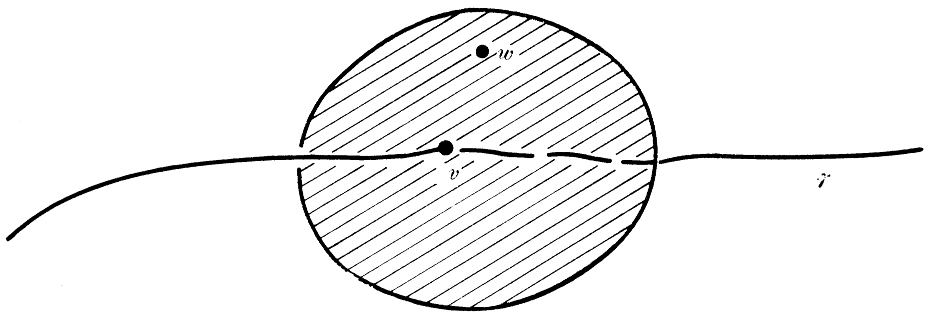


FIG. 3

Thus $T_\varepsilon(\gamma)$ is a *solid tube of discs* surrounding γ and the boundary $\partial T_\varepsilon(\gamma) = \tau_\varepsilon(\gamma)$ is a family of disjoint circles lying in $\mathbf{P}_n - V$ and parametrized by γ. Clearly $\partial\big(R - T_\varepsilon(\gamma)\big) = \Gamma - \tau_\varepsilon(\gamma)$ so that $\Gamma = \tau_\varepsilon(\gamma)$ in $H_n(\mathbf{P}_n - V, \mathbf{Z})$. We call $\tau_\varepsilon(\gamma)$ the *ε-tube lying over* γ, and $\tau(\gamma)$ will be any $\tau_\varepsilon(\gamma)$ for ε sufficiently small (these cycles are all homologous). The argument just given shows that

(3.3) Any n-cycle Γ in $\mathbf{P}_n - V$ is homologous to a tube over an $n-1$ cycle γ on V.

We may re-phrase all of this as follows. For a singular chain γ on V we let $\tau(\gamma)$ be the chain in $\mathbf{P}_n - V$ consisting of the normal ε-discs lying over γ. Then $\partial\big(\tau(\gamma)\big) = \tau(\partial\gamma)$ so that there is induced a $\mathbf{Z}$-linear mapping

(3.4) $$H_{n-1}(V, \mathbf{Z}) \xrightarrow{\tau} H_n(\mathbf{P}_n - V, \mathbf{Z}) ,$$

given geometrically by taking tubes over cycles. Then (3.3) says that τ is onto in (3.4) (cf. Fig. 2 in § 1).

We now determine the kernel of τ in (3.4). If $\gamma \in H_{n-1}(V, \mathbf{Z})$ and $\tau_\varepsilon(\gamma) = \partial R$ for some $n+1$ chain R in $\mathbf{P}_n - V$, then $\Gamma = -R + T_\varepsilon(\gamma)$ is an $(n+1)$-cycle on $\mathbf{P}_n$ with $\Gamma \cdot V = \gamma$. If n is even, then $H_{n+1}(\mathbf{P}_n, \mathbf{Z}) = 0$ and $\Gamma = \partial C$ for some $(n+2)$-chain on $\mathbf{P}_n$. We may assume that C meets V transversely in an n-chain $c = C \cdot V$, and then $\partial c = \partial(C \cdot V) = \Gamma \cdot V = \gamma$ so that $\gamma = 0$ in $H_{n-1}(V, \mathbf{Z})$ and τ is an injection in this case. If $n = 2m - 1$ is odd, then $\Gamma \in H_{2m}(\mathbf{P}_{2m-1}, \mathbf{Z})$ and so $\Gamma \sim l\mathbf{P}_m$ where $\mathbf{P}_m \subset \mathbf{P}_{2m-1}$ is a linear subspace. Thus $\Gamma - l\mathbf{P}_m = \partial C$ and we may repeat the above argument to find that $\gamma - l(\mathbf{P}_m \cdot V) = \partial(C \cdot V)$ so that $\gamma = l(\mathbf{P}_m \cdot V)$ in $H_{2m-2}(V, \mathbf{Z})$. In summary:

PROPOSITION 3.5. *The tube mapping* (3.4) *is always surjective and is injective if n is even. In case $n = 2m - 1$ is odd the kernel of τ is infinite cyclic and consists of the integral multiples of the homology class of a general linear section $\mathbf{P}_m \cdot V$.*

4. Analytic reduction of the periods

In our study of the period (1.1), we have seen that $\int_\Gamma \omega$ depends only on the homology class of Γ in $H_n(\mathbf{P}_n - V, \mathbf{Z})$, and we have put Γ in a sort of canonical form by writing it as a tube over a cycle $\gamma \in H_{n-1}(V, \mathbf{Z})$. We now want to reduce the integrand ω, and this reduction will be based again on the following consequence of Stokes's theorem

(4.1) $\int_\Gamma \omega = \int_\Gamma \omega + d\varphi$ where φ is a rational $(n-1)$-form on $\mathbf{P}_n$ with polar locus V.

The following two theorems give our reduction.

THEOREM 4.2. *Given a rational n-form ω on $\mathbf{P}_n$ with polar locus V, there exists a rational $(n-1)$-form φ with polar locus V such that $\omega + d\varphi$ has a pole of order n along V.*

THEOREM 4.3. *If ω as above has a pole of order k along V, and if there is a φ such that $\omega + d\varphi$ has a pole of order $k-1$ along V, then we can find ψ with a pole of order $k-1$ along V such that $\omega + d\psi$ has a pole of order $k-1$ along V.*

We may re-phrase (4.2) and (4.3) as follows.

(4.2′) *We can always reduce ω to have a pole of order n or less;*

(4.3′) *If we can reduce the order of pole of ω from k to $k-1$, then we can do so by an exact form $d\psi$ where ψ has a pole of order $k-1$.*

A rational $n-1$ form φ with a pole of order l along V is, by Theorem 2.9, written as

$$(4.4) \qquad \varphi = \frac{1}{Q(\xi)^l}\{\textstyle\sum_{i<j}(-1)^{i+j}[\xi^i A_j(\xi) - \xi^j A_i(\xi)](\cdots d\hat{\xi}^i \cdots d\hat{\xi}^j \cdots)\}$$

where $l \deg Q = \deg A_j + n$. We may then compute $d\varphi$ to find that

$$(4.5) \qquad d\varphi = \frac{P\Omega}{Q^{l+1}} \text{ where } P(\xi) = l\Big(\textstyle\sum_{j=0}^n A_j(\xi)\dfrac{\partial Q(\xi)}{\partial \xi^j}\Big) - Q(\xi)\Big(\textstyle\sum_{j=0}^n \dfrac{\partial A_j(\xi)}{\partial \xi^j}\Big).$$

From (4.5) we conclude

PROPOSITION 4.6. *Given $\omega = P\Omega/Q^k$ with a pole of order k along V, we can reduce the order of the pole of ω to $k-1$ by a form $d\varphi$ where φ has a pole of order $k-1$ if, and only if, P belongs to the ideal $(\partial Q/\partial\xi^0, \cdots, \partial Q/\partial\xi^n)$.*

PROPOSITION 4.7. *If φ given by (4.4) is such that $d\varphi$ has only a pole of order l along V, then*

$$\textstyle\sum_{j=0}^n A_j(\xi)\partial Q(\xi)/\partial\xi^j = S(\xi)Q(\xi)$$

for some homogeneous polynomial $S(\xi)$ with $\deg S = l \deg Q - (n+1)$.

According to (2.10) a rational $(n-2)$-form ψ with a pole of order $l-1$ along V is given by

$$(4.8) \qquad \begin{aligned}\psi = \frac{1}{Q(\xi)^{l-1}}\{\textstyle\sum_{i<j<k}(-1)^{i+j+k}[\xi^i B_{jk} - \xi^j B_{ik} + \xi^k B_{ij}]\\ \cdot(\cdots d\hat{\xi}^i \cdots d\hat{\xi}^j \cdots d\hat{\xi}^k \cdots)\}.\end{aligned}$$

Then $d\psi = \varphi$ where φ is given by (4.4) and

$$(4.9) \qquad A_j \equiv (l-1)\Big\{\textstyle\sum_{k<j} B_{kj}\dfrac{\partial Q}{\partial\xi^k} - \sum_{l>j} B_{jl}\dfrac{\partial Q}{\partial \xi^l}\Big\} \qquad (\text{mod } Q).$$

Using this we conclude

PROPOSITION 4.10. *Let* φ *be given by* (4.4). *Then there exists a rational* $n-2$ *form* ψ *such that* $\varphi + d\psi$ *has a pole of order* $l-1$ *along* V *if there exists a skew-symmetric matrix* $C_{ij}(\xi)$ *of homogeneous polynomials with*

$$A_i(\xi) \equiv \sum_{j=0}^{n} C_{ij}(\xi)\frac{\partial Q(\xi)}{\partial \xi^j} \qquad (\text{mod } Q) .$$

We want to prove Theorem 4.2 using Proposition 4.6. Following a suggestion of David Mumford we will use the following

(4.11) THEOREM OF MACAULAY [9]. *Let* $R_0, \cdots, R_n$ *be homogeneous forms of degrees* $r_0, \cdots, r_n$ *such that the equations* $R_0(\xi) = 0, \cdots, R_n(\xi) = 0$ *have no common solution in* $\mathbf{P}_n$. *Then the radical* rad $(R_0, \cdots, R_n) = (\xi^0, \cdots, \xi^n)$ *is the maximal ideal* $\boldsymbol{m}$ *of* $0 \in \mathbf{C}^{n+1}$, *and the module of forms* R *such that* $R \cdot \boldsymbol{m}^l \subset (R_0, \cdots, R_n)$ *is precisely* $(R_0, \cdots, R_n) + \boldsymbol{m}^{\rho - l}$ *where* $\rho = \sum_{j=0}^{n}(r_j - 1) + 1$.

For our application we let $R_j = \partial Q/\partial \xi^j$ so that $r_j = q - 1$ where $q = \deg Q$. The fact that rad $(R_0, \cdots, R_n) = \boldsymbol{m}$ is obviously equivalent to the non-singularity of V. Since $\rho = (n+1)(q-2) + 1$ we get that

$$P(\xi) \in \left(\frac{\partial Q}{\partial \xi^0}, \cdots, \frac{\partial Q}{\partial \xi^n}\right) \qquad \text{if } \deg P \geqq (n+1)(q-2) + 1 . \tag{4.12}$$

Now suppose that $\omega = P\Omega/Q^k$ has a pole of order k along V. From Proposition 4.6 we have

PROPOSITION 4.13. *We can reduce the order of the pole of* ω *to* $k - 1$ *if* $k \geqq \big((q-1)(n+1) + 1\big)/q$.

PROOF. Use that $\deg P = kq - (n+1)$ in (4.12).

In particular since $n + 1 \geqq \big((q-1)(n+1)+1\big)/q$, we obtain Theorem 4.2. However, Proposition 4.13 gives somewhat better information; for each n ($= \dim \mathbf{P}_n$) and q ($=$ degree of $V \subset \mathbf{P}_n$), we let $n(q)$ be least integer such that every ω can be reduced to have a pole of order $n(q)$ along V. Then Theorem 4.2 says that $n(q) \leqq n$ for all q, whereas more precise data is given in Table 1 on the next page.

For example, the first row in Table 1 says that every form on $\mathbf{C}^n$ is exact (obvious); the fifth row $\boxed{3 \mid 2, 3 \mid 2}$ says that if $V \subset \mathbf{P}_3$ is a surface of degree 2 or 3, then every form can be reduced to have a second order pole along V; and so forth. We can easily see that the numbers given by Proposition 4.13 are the best possible.

We now prove Theorem 4.3 using Propositions 4.7 and 4.10. For this we use the following result of Dwork [3, p. 36].

n	q	$n(q)$
$1, 2, 3, \cdots$	1	0
1	$2, 3, \cdots$	1
2	2	1
2	$3, 4, \cdots$	2
3	2, 3	2
3	$4, 5, \cdots$	3
4	2	2
4	3, 4	3
4	$5, 6, \cdots$	4
5	2	3
5	3, 4, 5	4
5	$6, 7, \cdots$	5
n	n	$n-1$

TABLE 1

PROPOSITION 4.14. *Let $R_0, \cdots, R_n$ be homogeneous forms as in Theorem* 4.11; *thus* $\operatorname{rad}(R_0, \cdots, R_n) = \boldsymbol{m}$. *Let $A \subset \{1, \cdots, n\}$ and suppose that $P_\alpha (\alpha \in A)$ is a set of forms such that $\sum_{\alpha \in A} P_\alpha R_\alpha = 0$. Then there exists a skew-symmetric matrix $S_{\alpha\beta}$ of homogeneous forms such that $\sum_{\beta \in A} S_{\alpha\beta} R_\beta = P_\alpha$. Furthermore if* $\deg(P_\alpha R_\alpha) = k$ *then we may assume that* $\deg S_{\alpha\beta} = k - \deg(R_\alpha R_\beta)$.

PROOF OF THEOREM 4.3. We must show that, if φ is a rational $n-1$ form (4.4) with a pole of order l along V such that $d\varphi$ has a pole of order l along V, then we can find a rational $n-2$ form ψ as in (4.8) such that $\varphi + d\psi$ has a pole of order $l-1$ along V.

From the assumption that $d\varphi$ has a pole of order l, it follows from Proposition 4.7 that

$$\sum_{j=0}^{n} \{A_j(\xi) - \xi^j S(\xi)\} \cdot \partial Q / \partial \xi^j = 0 \, .$$

Using Proposition 4.14 we may find a skew-symmetric matrix $C_{ij}(\xi)$ of homogeneous polynomials of the correct degree such that

$$A'_i(\xi) = A_i(\xi) - \xi^i S(\xi) = \sum_{j=0}^{n} C_{ij}(\xi) \partial Q / \partial \xi^j \, .$$

Since the $A'_i(\xi)$ define the same $n-1$ form φ as the $A_i(\xi)$ by (4.4), we are done if we apply Proposition 4.10.

5. On the number of independent periods

We have now put the integral (1.1) in canonical form. Namely, we have a homology $\Gamma \sim \tau(\gamma)$ where $\gamma \in H_{n-1}(V, \mathbf{Z})$, and we may find a rational $n-1$ form ψ such that $\omega + d\psi = \varphi$ has a pole along V of minimal order $k \leqq n$. Then $\int_\Gamma \omega = \int_{\tau(\gamma)} \varphi$.

Let $\mathcal{H}_k(V)$ be the vector space of rational n-forms with a pole of order k along V and taken modulo the forms $d\psi$ where ψ is a rational $n-1$ form with a pole of order $k-1$ along V. Symbolically, if $A_l^q(V)$ are the rational q-forms on $\mathbf{P}_n$ with a pole of order l along V, then

$$\mathcal{H}_k(V) = A_k^n(V)/dA_{k-1}^{n-1}(V) . \tag{5.1}$$

This $\mathcal{H}_k(V)$ is a sort of *de Rham cohomology group* constructed from rational differentials keeping close track of the order of poles of the differentials.

By using the reductions in §§ 3 and 4, to know $\int_\Gamma \omega$ we need to know the linear mappings

$$\pi_k\colon H_{n-1}(V, \mathbf{Z}) \longrightarrow \mathcal{H}_k(V)^* \tag{5.2}$$

given by $\langle \pi_k(\gamma), \varphi \rangle = \frac{1}{2\pi i}\int_{\tau(\gamma)} \varphi$ for $\gamma \in H_{n-1}(V, \mathbf{Z})$, $\varphi \in \mathcal{H}_k(V)$. Now π_k is a $\mathbf{Z}$-linear mapping from a $\mathbf{Z}$-module to a complex vector space, and to know $\int_\Gamma \omega$ we must in particular know the rank and co-rank of π_k. For the co-rank we have the following analogue of the one variable situation.

THEOREM 5.3. *π_k is onto; that is, if $\varphi \in A_k^n(V)$ and $\int_{\tau(\gamma)} \varphi = 0$ for all cycles $\gamma \in H_{n-1}(V, \mathbf{Z})$, then $\varphi = d\psi$ for some form $\psi \in A_{k-1}^{n-1}(V)$.*

PROOF. By Proposition 3.5, if $\int_{\tau(\gamma)} \varphi = 0$ for all $\gamma \in H_{n-1}(V, \mathbf{Z})$ then it follows that $\int_\Gamma \varphi = 0$ for all $\Gamma \in H_n(\mathbf{P}_n - V, \mathbf{Z})$. This in turn implies that $\varphi = d\eta$ for some rational $n-1$ form η with poles on V (of unspecified order); cf. the appendix in Lefschetz [8] and Grothendieck [6]. We shall also give a proof of this in § 10 in Part II of this paper. Using now Theorem 4.3 it follows that $\varphi = d\psi$ for some $\psi \in A_{k-1}^{n-1}(V)$ and so $\varphi = 0$ in $\mathcal{H}_k(V)$.

Remark. In the present context we should re-phrase Theorems 4.2 and 4.3 as follows. Let $A^q(V) = \lim_{k\to\infty} A_k^q(V)$ be the vector space of rational q-forms on $\mathbf{P}_n$ with poles of arbitrary order along V. If $\mathcal{H}(V) = A^n(V)/dA^{n-1}(V)$ is the *rational de Rham cohomology group*, then there are obvious maps $\mathcal{H}_k(V) \to \mathcal{H}(V)$ and (4.2) and (4.3) become respectively:

$$\mathcal{H}_k(V) \longrightarrow \mathcal{H}(V) \longrightarrow 0 \qquad \text{for } k \geqq n , \tag{5.4}$$

$$0 \longrightarrow \mathcal{H}_k(V) \longrightarrow \mathcal{H}(V) \qquad \text{for all } k . \tag{5.5}$$

Finding the kernel of $\pi_k\colon H_{n-1}(V, \mathbf{Z}) \to \mathcal{H}_k(V)^*$ turns out to be an entirely different sort of problem. To see what possibilities can arise, let $n = 2m + 1$

be odd so that $\dim V = 2m$ is even, and suppose that $Z_m \subset V_{2m}$ is an irreducible m-dimensional algebraic subvariety. Thus $Z \subset \mathbf{P}_{2m+1}$ is given by equations $Q(\xi) = 0, P_1(\xi) = 0, \cdots, P_s(\xi) = 0$ where the homogeneous ideal $(Q, P_1, \cdots, P_s)$ of Z is primary. Using the *triangulation theorem* for algebraic sets (cf. [2]; we could also use the Borel-Haefliger theory at this point [1]), we see that the oriented complex Z carries a *fundamental cycle* $[Z] \in H_{2m}(V, \mathbf{Z})$, and we have

PROPOSITION 5.6. $\int_{\tau[z]} \omega = 0$ *for all* $\omega \in \mathcal{H}_m(V)$.

PROOF. We have $\int_{\tau(Z)} \omega = \lim_{\varepsilon\to 0} \int_{\tau_\varepsilon(Z)} \omega$ since all tubes are homologous, and we want to examine the integral $\int_{\tau_\varepsilon(Z)} \omega$ locally near a simple point on Z. We choose local holomorphic coordinates $(u^1, \cdots, u^m, v^1, \cdots, v^m, w) = (u, v, w)$ on $\mathbf{P}_{2m+1}$ so that V is given locally by $w = 0$ and Z by $w = 0, v = 0$. The tube $\tau_\varepsilon(Z)$ is given parametrically by

$$(x, \theta) \longrightarrow (x + \varepsilon f, \varepsilon g, \varepsilon h e^{i\theta})$$

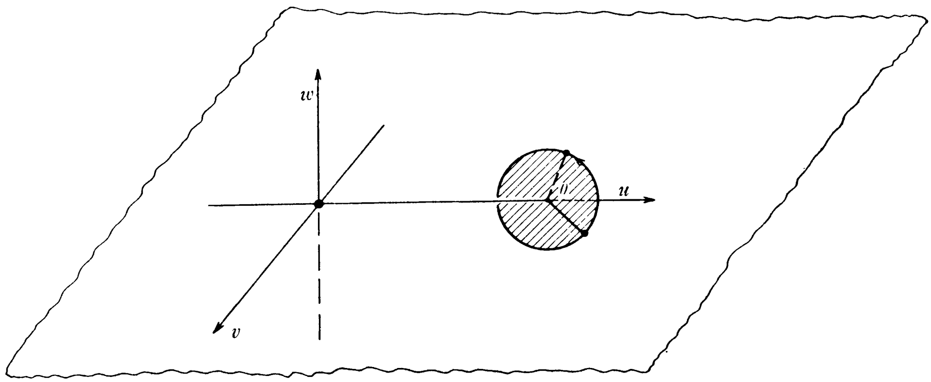


FIG. 4

where $x = (x^1, \cdots, x^m)$, $f = f(x, \bar{x}, \theta)$ and g are vector-valued smooth functions, and h is a smooth scalar function which is close to 1. The form $\omega \in A_m^{2m+1}(V)$ has a pole of order m along V and so locally $\omega = F(u, v, w)du \wedge dv \wedge dw/w^m$ where F is holomorphic, $du = du^1 \wedge \cdots \wedge du^m$, etc. On $\tau_\varepsilon(Z)$ we have $u = x + \varepsilon f$, $v = \varepsilon g$, $w = \varepsilon h e^{i\theta}$ so that $du \equiv dx(\varepsilon)$, from which it follows that $\omega = \varepsilon^{m+1} G dx \wedge d\bar{x} \wedge d\theta / w^m$ where $G(x, \bar{x}, \theta)$ is smooth. Thus $\omega = \varepsilon H dx \wedge d\bar{x} \wedge d\theta$ with H a smooth function, and then obviously $\lim_{\varepsilon \to 0} \int_{\tau_\varepsilon(Z)} \omega = 0$.

COROLLARY 5.7. $\int_{\tau(\gamma)} \omega = 0$ *if* $\omega \in \mathcal{H}_m(V)$ *and* γ *is an algebraic cycle.*

PROOF. By definition, $\gamma \sim \sum_{j=1}^N k_j[Z_j]$ where k_j is an integer and $Z_j \subset V$ is an irreducible, m-dimsional subvariety. The corollary then follows from linearity and Proposition 5.6.

To state the general form of Proposition 5.6 we must first define what it means for $\gamma \in H_{n-1}(V, \mathbf{Z})$ to be a *cycle of rank* q. This shall mean that there exists a subvariety $W_{n-1-q} \subset V_{n-1}$ of codimension q such that γ is in the image of $H_{n-1}(W, \mathbf{Z}) \to H_{n-1}(V, \mathbf{Z})$ (i.e., γ *is homologous to a cycle lying on* W). If $n = 2m + 1$ as above, then γ is of rank m means that γ is an algebraic cycle.

THEOREM 5.8. $\int_{\tau(\gamma)} \omega = 0$ *if* γ *is of rank* q *and* $\omega \in \mathcal{H}_q(V)$.

The proof is essentially the same as Proposition 5.6.

At this point we see that our original problem on discussing the integral (1.1) has led us to the following

(5.9) *Question.*

If $\int_{\tau(\gamma)} \omega = 0$ for all $\omega \in \mathcal{H}_q(V)$, does it follow that some non-zero multiple $k\gamma$ is of rank q?

This is a problem of the same sort as the *Hodge conjecture* (cf. Hodge [7]) and, after some work, can be seen to be a special case thereof (cf. § 8 below). Observe that, if γ is of rank q, then γ is homologous to an $n-1$ cycle lying on a $W_{n-1-q} \subset V_{n-1}$ and so $2(n-1-q) \geqq n-1$ or $q \leqq (n-1)/2$. In other words, the first special case of (5.9) would be to prove

THEOREM 5.10. *If* $\int_{\Gamma} \omega = 0$ *for all* $\omega \in \mathcal{H}_{q(n)}(V)$ *where*

$$q(n) = \begin{Bmatrix} m+1 & \textit{in case} \quad n = 2m+1 & \textit{is odd} \\ m & \textit{in case} \quad n = 2m & \textit{is even} \end{Bmatrix},$$

then $\Gamma \sim 0$ *in* $H_n(\mathbf{P}_n - V, \mathbf{Q})$.

This theorem can be proved, but not so far as we know by the elementary methods now in use. Consequently the proof will be given in § 6 after we have discussed the relationship between our integrals (1.1) and the *Hodge theory of harmonic integrals on algebraic manifolds* (cf. Hodge [7]).

6. Intersection matrices and bilinear relations for the periods of odd dimensional hypersurfaces

We want ultimately to discuss Theorem 5.10, and will take up first the case when $n = 2m$ is even. Then $V_{2m-1} \subset \mathbf{P}_{2m}$ is a non-singular hypersurface and the *Lefschetz theorem* (cf. Lefschetz [8]) tells us that $H_k(V, \mathbf{Z}) \simeq H_k(\mathbf{P}_{2m}, \mathbf{Z})$ for $0 \leqq k \leqq 2m-2$. Consequently, using *Poincaré duality* we find that $H_*(V, \mathbf{Z})$ is a free abelian group, $H_{2k}(V, \mathbf{Z}) \simeq \mathbf{Z}$ for $k = 0, 1, \cdots, 2m-1$, and $H_{2k-1}(V, \mathbf{Z}) = 0$ for $k \neq m$. Furthermore, the *intersection pairing* $H_{2m-1}(V, \mathbf{Z}) \otimes H_{2m-1}(V, \mathbf{Z}) \to \mathbf{Z}$ is skew-symmetric and non-singular (using Poincaré duality again). Thus $H_{2m-1}(V, \mathbf{Z})$ is a free abelian group of even rank $2p$; we may think of p as a sort of *generalized genus* of V.

Let $\gamma_1, \cdots, \gamma_{2p}$ be an integral basis for $H_{2m-1}(V, \mathbf{Z})$. The *intersection numbers* $\gamma_\rho \cdot \gamma_\sigma$ give an integral skew-symmetric $2p \times 2p$ matrix with $\det(\gamma_\rho \cdot \gamma_\sigma) = 1$. Now it is well known that we can make a unimodular substitution $\tilde{\gamma}_\rho = \sum_{\sigma=1}^{2p} M_\rho^\sigma \gamma_\rho$ (M_ρ^σ integral and $\det(M_\rho^\sigma) = \pm 1$) such that the new intersection matrix $(\tilde{\gamma}_\rho \cdot \tilde{\gamma}_\sigma) = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$ where $T = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_p \end{pmatrix}$ is a diagonal matrix with positive integer entries. Since $(\tilde{\gamma}_\rho \cdot \tilde{\gamma}_\sigma) = M(\gamma_\rho \cdot \gamma_\sigma)^t M$, it follows that $\det T = 1$ and so $T = I$ and the matrix $(\tilde{\gamma}_\rho \cdot \tilde{\gamma}_\sigma) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. A choice of basis for $H_{2m-1}(V, \mathbf{Z})$ for which the intersection matrix has the canonical form $Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ will be called a *canonical basis* for $H_{2m-1}(V, \mathbf{Z})$; any two canonical bases $\{\gamma_\rho\}$ and $\{\tilde{\gamma}_\sigma\}$ are related by a unimodular transformation $\gamma_\rho = \sum_{\sigma=0}^{2p} \Lambda_\rho^\sigma \tilde{\gamma}_\sigma$ where $\Lambda Q^t \Lambda = Q$, i.e., by an integral symplectic transformation. Henceforth we will use only such canonical bases for $H_{2m-1}(V, \mathbf{Z})$.

Note that ${}^t Q^{-1} = Q$ for our particular Q.

We recall now the vector spaces $\mathcal{H}_k(V) = A_k^n(V)/dA_{k-1}^{n-1}(V)$ introduced in § 5 (cf. (5.1)). From (5.4) and (5.5) we have the sequence of inclusions

$$\mathcal{H}_1(V) \subset \mathcal{H}_2(V) \subset \cdots \subset \mathcal{H}_{2m}(V) = \mathcal{H}_{2m+1}(V) = \cdots \tag{6.1}$$

so that the subspaces $\mathcal{H}_k(V)$ $(1 \leqq k \leqq 2m)$ give a *filtration* of $\mathcal{H}(V)$ $(= \mathcal{H}_{2m+l}(V)$ for all $l \geqq 0)$. Choose bases $\omega^1, \cdots, \omega^{\delta_k}$ for $\mathcal{H}_k(V)$ which constitute a *basis for the filtration* (6.1); this means that $\omega^1, \cdots, \omega^{\delta_1}$ is a basis for $\mathcal{H}_1(V)$, $\omega^1, \cdots, \omega^{\delta_1}, \cdots, \omega^{\delta_2}$ is a basis for $\mathcal{H}_2(V)$, etc. Letting $\delta_k = \dim \mathcal{H}_k(V)$, an *admissible change of basis* for $\mathcal{H}_k(V)$ is given by $\tilde{\omega}^\rho = \sum_{\sigma=1}^{\delta_k} A_\sigma^\rho \omega^\sigma$ where the matrix $A = (A_\sigma^\rho)$ is a non-singular $\delta_k \times \delta_k$ matrix having the following block form.

$$A = \begin{array}{c} \left(\begin{array}{cccc} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdot & \cdot \\ \vdots & \vdots & & \\ \underbrace{A_{k1}}_{\delta_1} & \underbrace{A_{k2}}_{\delta_2-\delta_1} & \cdots & \underbrace{A_{kk}}_{\delta_k-\delta_{k-1}} \end{array}\right) \end{array} \begin{array}{l} \}\delta_1 \\ \}\delta_2 - \delta_1 \\ \\ \}\delta_k - \delta_{k-1} \end{array} . \tag{6.2}$$

Having so chosen bases for $H_{2m-1}(V, \mathbf{Z})$ and $\mathcal{H}_k(V)$, we form the *period matrix*

$$\Omega_k(V) = \frac{1}{2\pi i} \underbrace{\begin{pmatrix} \int_{\tau(\gamma_1)} \omega^1 & \cdots & \int_{\tau(\gamma_{2q})} \omega^1 \\ \vdots & & \vdots \\ \int_{\tau(\gamma_1)} \omega^{\delta_k} & \cdots & \int_{\tau(\gamma_{2p})} \omega^{\delta_k} \end{pmatrix}}_{2p} \Bigg\} \delta_k \quad ; \tag{6.3}$$

this $\delta_k \times 2p$ period matrix is simply the matrix of the linear transformation π_k in (5.2). Changing admissible bases for $H_{2m-1}(V, \mathbf{Z})$ and $\mathcal{H}_k(V)$ induces the substitution $\Omega_k(V) \to A\Omega_k(V)\Lambda$ where A is a non-singular $\delta_k \times \delta_k$ matrix having the block form (6.2) and Λ is a $2p \times 2p$ integral matrix which satisfies $\Lambda Q {}^t\Lambda = Q$.

Our main result is

THEOREM 6.4. *For $k \leqq m$, the period matrix $\Omega_k = \Omega_k(V)$ of $V \subset \mathbf{P}_{2m}$, satisfies the bilinear relations*

$$(6.5) \qquad \begin{cases} \Omega_k Q {}^t\Omega_k = 0 \qquad (Q = {}^tQ^{-1}) , \\ i\Omega_k Q {}^t\bar{\Omega}_k = H_k \ \ \textit{is a non-singular hermitian matrix} . \end{cases}$$

Furthermore $\delta_m = p$ is the generalized genus of V and $\delta_{2m} = 2p$ is the rank of $H_{2m-1}(V, \mathbf{Z})$.

Proof of Theorem 5.10 *in case $n = 2m$ is even.* We take the $p \times 2p$ period matrix $\Omega_m = \Omega_m(V)$ and consider the $2p \times 2p$ matrix $\begin{pmatrix} \Omega_m \\ \bar{\Omega}_m \end{pmatrix}$. From $i\begin{pmatrix} \Omega_m \\ \bar{\Omega}_m \end{pmatrix} Q \, {}^t\begin{pmatrix} \Omega_m \\ \bar{\Omega}_m \end{pmatrix} = \begin{pmatrix} 0 & H_m \\ -{}^tH_m & 0 \end{pmatrix}$ it follows that $\operatorname{rank}\begin{pmatrix} \Omega_m \\ \bar{\Omega}_m \end{pmatrix} = 2p$.

If $\Gamma = c^1\tau(\gamma_1) + \cdots + c^{2p}\tau(\gamma_{2p})$ is a cycle in $H_{2m}(\mathbf{P}_{2m} - V, \mathbf{Q})$ (thus the $c^\rho \in \mathbf{Q}$) such that $\int_\Gamma \omega^\alpha = 0$ for $\alpha = 1, \cdots, \delta_m = p$, then $0 = \Omega_m c = \overline{\Omega_m c} = \bar{\Omega}_m c$ where $c = {}^t(c^1, \cdots, c^{2p})$. It follows that $c = 0$, which proves Theorem 5.10 in this case.

Remark. The same argument shows that, if $c = {}^t(c^1, \cdots, c^{2p})$ is a *real* column vector satisfying $\Omega_m c = 0$, then $c = 0$. In other words, if we write $\Omega_m = (\pi_1, \cdots, \pi_{2p})$ where the column vectors $\pi_\rho \in \mathbf{C}^p$, then the vectors $\sum_{\rho=1}^{2p} n^\rho \pi^\rho$ ($n^\rho \in \mathbf{Z}$) form a *lattice* $\Gamma(V)$ in $\mathbf{C}^p$. This gives

PROPOSITION 6.6. *The image of $H_{2m-1}(V, \mathbf{Z})$ in $\mathcal{H}_m(V)^*$ is a lattice $\Gamma(V)$.*

To each non-singular hypersurface $V \subset \mathbf{P}_{2m}$ we have canonically associated the complex torus $\mathcal{H}_m(V)^*/\Gamma(V) = T(V)$, which we will call the *intermediate jacobian variety* of V. This torus is *not* in general the one introduced by Weil [12] but will turn out to be the one used in [5]. Observe that we may identify the holomorphic 1-forms on $T(V)$ with $\mathcal{H}_m(V)$ and the first homology group $H_1(T(V), \mathbf{Z})$ with $H_{2m-1}(V, \mathbf{Z})$.

We will give the proof of Theorem 6.4 in § 8 below; for motivation we now give the argument in the case $m = 1$. The hypersurface $V \subset \mathbf{P}_2$ is then a *non-singular plane curve* given in affine coordinates (x, y) by an equation $q(x,y) = 0$. Topologically V is a familiar *Riemann surface* of genus p.

In Fig. 5 at the top of the next page we have drawn in a typical canonical basis for $H_1(V, \mathbf{Z})$ (in fact, they all look like this, cf. Mumford [10]).

For the vector spaces of differentials $\mathcal{H}_k(V)$ we have only two

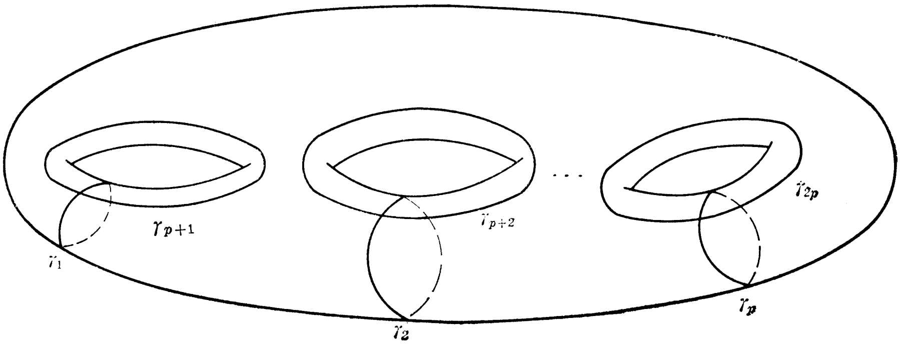


FIG. 5

$\mathcal{H}_1(V) \subset \mathcal{H}_2(V)$ (cf. (6.1)). Since $\mathcal{H}_1(V)$ is the space of rational 2-forms on $\mathbf{P}_2$ with a first order pole along V modulo forms $d\psi$ where ψ is a rational 1-form with a pole of order zero along V (cf. (5.1)), and since there are no holomorphic 1-forms on $\mathbf{P}_2$ (this follows immediately from (2.10)), we find that $\mathcal{H}_1(V)$ is just the vector space of rational 2-forms

$$\omega = \frac{p(x, y)dxdy}{q(x, y)}, \qquad \deg p \leqq \deg q - 3 . \tag{6.7}$$

Now it is quite classical that the *holomorphic differentials* on V are all written as $p(x, y)dx/((\partial q/\partial y)(x, y))$ (restricted to $q(x, y) = 0$) where $\deg p \leqq \deg q - 3$. We agree to call $pdx/(\partial q/\partial y)$ the *Poincaré residue* of $\omega = pdxdy/q$ along V and denote it by $R(\omega)$ (cf. Poincaré [11] and § 8 below). The linear mapping $\omega \to R(\omega)$ thus induces an isomorphism of $\mathcal{H}_1(V)$ with the vector space of holomorphic differentials on V. It is also easy to see, and will be formally proved in § 8 below, that

$$\int_\gamma R(\omega) = \frac{1}{2\pi i}\int_{\tau(\gamma)} \omega \qquad \text{for all cycles } \gamma \in H_1(V, \mathbf{Z}) .$$

It follows then that the period matrix $\Omega_1(V)$ in (6.3) is simply the usual $p \times 2p$ period matrix of the compact Riemann surface V. The bilinear relations $\Omega_1 Q^t\Omega_1 = 0$, $i\Omega_1 Q^t\bar{\Omega}_1 > 0$ are the well-known *Riemann bilinear relations*, and are just restatements of the relations $\int_V R(\omega) \wedge R(\omega') = 0$ $(\omega, \omega' \in \mathcal{H}_1(V))$, $i\int_V R(\omega) \wedge \overline{R(\omega)} > 0$ if $\omega \neq 0$. Also the *intermediate jacobian* of V is in this case just the *ordinary jacobian* variety of the non-singular curve V, and the generalized genus reduces to the classical genus.

This covers all of Theorem 6.4 except the equality $\dim \mathcal{H}_2(V) = 2p$. Now $\dim \mathcal{H}_2(V) = \dim \mathcal{H}_1(V) + \dim (\mathcal{H}_2(V)/\mathcal{H}_1(V))$. Letting $\mathcal{F}_l$ be the vector space of homogeneous polynomials $P(x, y, z)$ of degree l, we have from Theorem 4.3 that

$$(6.8)\qquad \begin{cases}\dim \mathcal{H}_1(V) = \dim \mathcal{F}_{n-3} = p\ , \\ \dim\big(\mathcal{H}_2(V)/\mathcal{H}_1(V)\big) = \dim\left(\mathcal{F}_{2n-3}/\mathcal{F}_{2n-3}\cap\left(\dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial y}, \dfrac{\partial Q}{\partial z}\right)\right)\end{cases}$$

where $Q(x, y, z) = 0$ is the homogeneous equation of V, $\deg Q = n$, and $(\partial Q/\partial x, \partial Q/\partial y, \partial Q/\partial z)$ is the homogeneous ideal generated by $\partial Q/\partial x, \partial Q/\partial y, \partial Q/\partial z$. We shall check the equality

$$p = \dim \mathcal{F}_{n-3} = \dim\left(\mathcal{F}_{2n-3}/\mathcal{F}_{2n-3}\cap\left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}\right)\right)$$

in two special cases and defer the general proof to § 8.

Example 1. $Q(x, y, z) = x^3 + y^3 + z^3$ so that V is a plane cubic. Then $n = 3$ and the genus $p = 1 = \dim \mathcal{F}_0$. Now $(\partial Q/\partial x, \partial Q/\partial y, \partial Q/\partial z) = (x^2, y^2, z^2)$ so that $\mathcal{F}_3/\mathcal{F}_3 \cap (\partial Q/\partial x, \partial Q/\partial y, \partial Q/\partial z)$ has the monomial xyz as basis and everything is verified.

Example 2. $Q(x, y, z; \lambda) = zy^2 - x(x - z)(x - \lambda z)$ so that V is again a plane cubic with the familiar affine equation $y^2 = x(x - 1)(x - \lambda)$. Using (6.7) a basis for $\mathcal{H}_1(V)$ is given by the rational 2-form $\omega = dxdy/\{y^2 - x(x-1)(x-\lambda)\}$ and the Poincaré residue $R(\omega) = dx/y = dx/\sqrt{x(x - 1)(x - \lambda)}$ is the usual *elliptic integral*. Now $(\partial Q/\partial x, \partial Q/\partial y, \partial Q/\partial z)$ is the somewhat complicated ideal

$$\big(-3x^2 + 2(1 + \lambda)xz - \lambda z^2, zy, y^2 + (1 + \lambda)x^2 - 2\lambda xz\big)$$

and, after some work, it can be checked that for $\lambda \neq 0, 1$ the form $x^2z - xz^2$ gives a basis of

$$\mathcal{F}_3/\mathcal{F}_3 \cap \left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}\right).$$

7. Intersection matrices and bilinear relations for the periods of even dimensional hypersurfaces

We want to discuss the intersection matrix and bilinear relations for the periods in the remaining case when $V_{2m} \subset \mathbf{P}_{2m+1}$ is an even-dimensional hypersurface of *degree* q given by $Q(\xi^0, \cdots, \xi^{2m+1}) = 0$ ($\deg Q = q$). In this case the *tube mapping* $H_{2m}(V, \mathbf{Z}) \xrightarrow{\tau} H_{2m+1}(\mathbf{P}_{2m+1} - V, \mathbf{Z}) \to 0$ is onto with kernel consisting of the integral multiples lh where $\boldsymbol{h} = [\mathbf{P}_{m+1}\cdot V]$ is the homology class of a linear section of V (cf. Proposition 3.5). Using the Lefschetz and Poincaré duality theorems as before, we find that $H_{2k+1}(V, \mathbf{Z}) = 0$ for all k, $H_{2k}(V, \mathbf{Z}) \simeq \mathbf{Z}$ for $k \neq m$, and $H_{2m}(V, \mathbf{Z})$ is a free abelian group. The intersection pairing $H_{2m}(V, \mathbf{Z}) \otimes H_{2m}(V, \mathbf{Z}) \to \mathbf{Z}$ is symmetric, non-singular, and $\boldsymbol{h}\cdot\boldsymbol{h} = q$. We claim that there exists a class $\boldsymbol{\gamma} \in H_{2m}(V, \mathbf{Z})$ such that $\boldsymbol{h}\cdot\boldsymbol{\gamma} = 1$. (*Proof.* Using the Lefschetz theorem again, the mapping $H_{2m}(V, \mathbf{Z}) \to H_{2m}(\mathbf{P}_{2m+1}, \mathbf{Z}) \to 0$ is onto. Thus there exists a $2m$-cycle $\boldsymbol{\gamma}$ on V such that $\boldsymbol{\gamma} \sim [\mathbf{P}_m]$ in $\mathbf{P}_{2m+1}$ and then

$\boldsymbol{h} - q\boldsymbol{\gamma} \sim 0$ in $\mathbf{P}_{2m+1}$. Now $\boldsymbol{h}\cdot(\boldsymbol{h} - q\boldsymbol{\gamma}) = \int_{\boldsymbol{h}-q\boldsymbol{\gamma}} \omega^m = 0$ where

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\log(|\xi^0|^2 + \cdots + |\xi^{2m+1}|^2)$$

is the cohomology class of the divisor $\mathbf{P}_{2m}$ on $\mathbf{P}_{2m+1}$. This gives that $\boldsymbol{h}\cdot\boldsymbol{\gamma} = 1$.) Note that $\boldsymbol{\gamma}$ is unique modulo the kernel of $H_{2m}(V, \mathbf{Z}) \to H_{2m}(\mathbf{P}_{2m+1}, \mathbf{Z})$; the cycles in this kernel will be called *vanishing cycles* (cf. Lefschetz [8]). We denote the space of vanishing cycles by $H_{2m}(V, \mathbf{Z})_0$ and observe that $H_{2m}(V, \mathbf{Z})_0$ consists of the classes $\gamma \in H_{2m}(V, \mathbf{Z})$ which satisfy $\boldsymbol{h}\cdot\gamma = 0$.

Because of the existence of $\boldsymbol{\gamma} \in H_{2m}(V, \mathbf{Z})$ with $\boldsymbol{h}\cdot\boldsymbol{\gamma} = 1$, we can choose an *integral basis* $\gamma_1, \cdots, \gamma_b$ for $H_{2m}(V, \mathbf{Z})_0$ which is part of an integral basis for $H_{2m}(V, \mathbf{Z})$. Now $\gamma_1, \cdots, \gamma_\rho, \boldsymbol{h}$ forms a basis for the rational homology group $H_{2m}(V, \mathbf{Q})$ for which the intersection matrix is

$$\begin{pmatrix} & 0 \\ (\gamma_\rho\cdot\gamma_\sigma) & \vdots \\ & 0 \\ 0 \cdots 0 & q \end{pmatrix}.$$

Thus $(\gamma_\rho\cdot\gamma_\sigma)$ is an integral, symmetric, non-singular matrix. Since the change of *rational* basis from $\gamma_1, \cdots, \gamma_b, \boldsymbol{\gamma}$ to $\gamma_1, \cdots, \gamma_b, \boldsymbol{h}$ is given by a matrix

$$M = \begin{pmatrix} 1 & \cdots & & 0 \\ & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ c^1 & \cdots & c^b & q \end{pmatrix} \qquad \left(\text{where } \boldsymbol{h} = q\boldsymbol{\gamma} + \textstyle\sum_{\rho=1}^{b} c^\rho\gamma_\rho\right),$$

we have

$$\det(\gamma_\rho\cdot\gamma_\sigma)q = \det\left\{M\begin{pmatrix} & & \gamma_\rho\cdot\boldsymbol{\gamma} \\ & (\gamma_\rho\cdot\gamma_\sigma) & \vdots \\ \boldsymbol{\gamma}\cdot\gamma_\rho & \cdots & \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} \end{pmatrix}{}^tM\right\} = \pm q^2 .$$

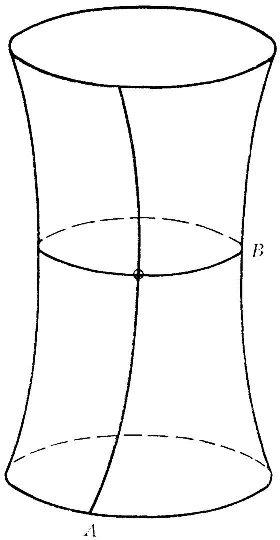
This gives the

PROPOSITION 7.1. *Let $\gamma_1, \cdots, \gamma_b$ be a basis for the free $\mathbf{Z}$-module $H_{2m}(V, \mathbf{Z})_0$ of vanishing cycles. Then $\gamma_1, \cdots, \gamma_b$ is part of a $\mathbf{Z}$-basis for $H_{2m}(V, \mathbf{Z})$, the intersection pairing $H_{2m}(V, \mathbf{Z})_0 \otimes H_{2m}(V, \mathbf{Z})_0 \to \mathbf{Z}$ is symmetric, non-singular, and* $\det(\gamma_\rho\cdot\gamma_\sigma) = \pm q$. *The tube mapping*

$$H_{2m}(V, \mathbf{Z})_0 \xrightarrow{\tau} H_{2m+1}(\mathbf{P}_{2m+1} - V, \mathbf{Z})$$

is injective and maps $H_{2m}(V, \mathbf{Z})_0$ onto a subgroup of finite index in $H_{2m+1}(\mathbf{P}_{2m+1} - V, \mathbf{Z})$.

Example 7.2. The simplest example is when $q = 2$ and so V is the *quadric* in $\mathbf{P}_{2m+1}$ given by $\xi^0\xi^{m+1} + \cdots + \xi^m\xi^{2m+1} = 0$. We let $A \subset V$ be the linear $\mathbf{P}_m$ given by $\xi^0 = \cdots = \xi^m = 0$, $B \subset V$ the $\mathbf{P}_m$ given by $\xi^{m+1} = \cdots = \xi^{2m} = \xi^m = 0$. Then $A \cap B$ is the point $[0, \cdots, 0, 1] \in \mathbf{P}_{2m+1}$ and so $A \cdot B = 1$.



This is the picture when $m = 2$; A and B are the usual *generators* of the quadric V.

FIG. 6

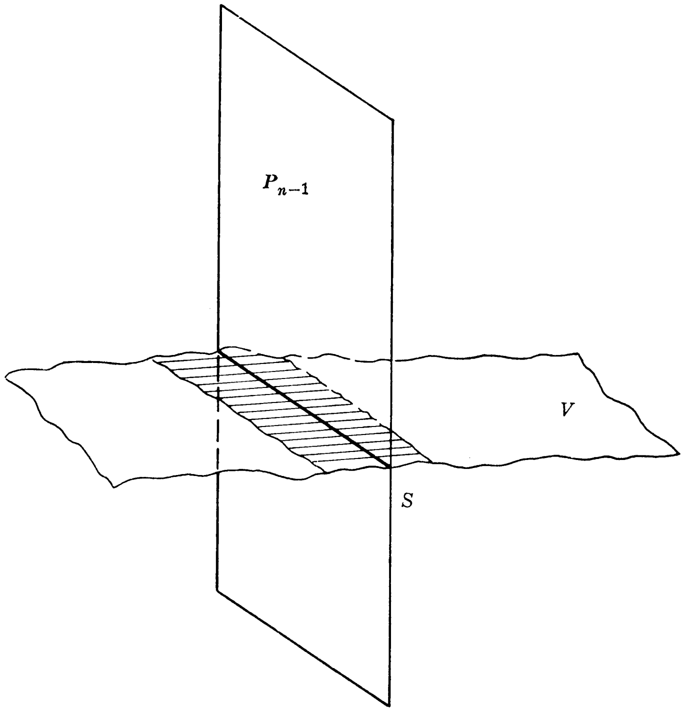
Letting α, β be the homology classes of A, B we have $\alpha \cdot \beta = 1$, $\boldsymbol{h} = \alpha + \beta$, and α, β give a $\mathbf{Z}$ basis of $H_{2m}(V, \mathbf{Z})$. We may take $\boldsymbol{\gamma} = \beta$ (or α would do); then $\alpha - \beta$ generates $H_{2m}(V, \mathbf{Z})_0$ and $\alpha - \beta, \beta$ gives a basis of $H_{2m}(V, \mathbf{Z})$ of the sort considered in Proposition 7.1. The intersection matrix for this basis is $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$, and for the rational basis $\alpha - \beta, \alpha + \beta$ is $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

We could also give V by the equation $(\xi^1)^2 + \cdots + (\xi^{2m+1})^2 = (\xi^0)^2$, whose affine counterpart is $(z^1)^2 + \cdots + (z^{2m+1})^2 = 1$. Then the cycle $\operatorname{Im} z^j = 0$ $(j = 1, \cdots, 2m + 1)$ is an ordinary $2m$-sphere δ; the homology class of δ is $\alpha - \beta$ since $\delta \cdot \boldsymbol{h} = 0$ and $\delta \cdot \delta = -2$. This illustrates the following definition and proposition.

For a general non-singular hypersurface $V \subset \mathbf{P}_n$ we let V_f be the *finite part* of V; i.e., V_f is V minus the intersection of V with a general $\mathbf{P}_{n-1}$. A homology class $\gamma \in H_k(V, \mathbf{Z})$ is *finite* if γ is in the image of $H_k(V_f, \mathbf{Z}) \rightarrow H_k(V, \mathbf{Z})$.

PROPOSITION 7.3. *The finite cycles are precisely the vanishing cycles.*

PROOF. We let S be the section of $V \subset \mathbf{P}_n$ by a $\mathbf{P}_{n-1}$ in general position and let N be a *tubular neighborhood* of S in V:



(The shaded region is N.)

FIG. 7

Since N is a *disc bundle* lying over S, we may use the *Thom isomorphism* to have $H_k(\bar{N}, \partial\bar{N}; \mathbf{Z}) \simeq H_{k-2}(S, \mathbf{Z})$. In particular, taking $k = n - 1$ and using that $S \subset \mathbf{P}_{n-1}$ is a non-singular hypersurface, we have

$$(7.4) \qquad \begin{cases} H_{2m-1}(\bar{N}, \partial\bar{N}; \mathbf{Z}) = 0 & \text{in case } n = 2m \text{ is even ,} \\ H_{2m}(\bar{N}, \partial\bar{N}; \mathbf{Z}) \simeq H_{2m-2}(S, \mathbf{Z}) & \text{in case } n = 2m + 1 \text{ is odd .} \end{cases}$$

Using excision in the form $H_{n-1}(V, V - N; \mathbf{Z}) \simeq H_{n-1}(\bar{N}, \bar{N}; \mathbf{Z})$ and the Thom isomorphism, the *exact homology sequence* of the pair $(V - N, V)$ becomes

$$(7.5) \qquad \begin{array}{c} H_{n-1}(V - N, \mathbf{Z}) \longrightarrow H_{n-1}(V, \mathbf{Z}) \xrightarrow{\theta} H_{n-3}(S, \mathbf{Z}) \, . \\ \Vert \\ H_{n-1}(V_f, \mathbf{Z}) \end{array}$$

Observe that the map θ in (7.5) takes a cycle $\gamma \in H_{n-1}(V, \mathbf{Z})$ and intersects it with S (cf. Lefschetz [8]).

In case $n = 2m$ is even we conclude from (7.5) and (7.4) that all of $H_{2m-1}(V, \mathbf{Z})$ is both vanishing and finite. Since a finite cycle is clearly vanishing, we must prove that $\theta(\gamma) = 0$ in (7.5) in case $\gamma \in H_{2m}(V, \mathbf{Z})_0$ is a vanishing cycle and $n = 2m + 1$ is odd.

Write $S = S_1 = V \cdot \mathbf{P}_{2m}$, $S_2 = V \cdot \mathbf{P}_{2m-1}$, $\cdots$, $S_m = V \cdot \mathbf{P}_{m+1}$ where

$\mathbf{P}_{m+1} \subset \cdots \subset \mathbf{P}_{2m} \subset \mathbf{P}_{2m+1}$ is in general position with respect to V. Then we have a diagram

$$\begin{tikzcd} H_{2m}(V, \mathbf{Z}) \arrow[r, "\theta"] \arrow[dddr, "\psi"'] & H_{2m-2}(S_1, \mathbf{Z}) \arrow[d] \\ & H_{2m-4}(S_2, \mathbf{Z}) \arrow[d] \\ & \vdots \arrow[d] \\ & H_0(S_m, \mathbf{Z}) \end{tikzcd} ,$$

and the fact that γ is a vanishing cycle means that the intersection number $\gamma \cdot h = \psi(\gamma) = 0$. Using this it will suffice to show that the vertical arrows in the above diagram are injective. But S_j is a non-singular hypersurface in $\mathbf{P}_{2m-j+1}$ and so $H_{2m-2j}(S_j, \mathbf{Z}) \simeq H_{2m-2j}(\mathbf{P}_{2m-j+1}, \mathbf{Z}) \simeq \mathbf{Z}$ by the Lefschetz theorem. From this it follows that the vertical arrows are isomorphisms and we are done.

On the free $\mathbf{Z}$-module $H_{2m}(V, \mathbf{Z})_0$ of vanishing cycles or finite cycles—they amount to the same—there is the intersection pairing $H_{2m}(V, \mathbf{Z})_0 \otimes H_{2m}(V, \mathbf{Z})_0 \rightarrow \mathbf{Z}$, which is a non-singular, symmetric quadratic form of determinant $\pm q$. We can write $H_{2m}(V, \mathbf{Z})_0$ as an orthogonal direct sum $K_0 \oplus K_1 \oplus \cdots \oplus K_r$ so that the intersection matrix becomes

$$P = \begin{pmatrix} P_0 & & & 0 \\ & \cdot & & \\ & & P_1 & \\ & & & \ddots \\ 0 & & & P_r \end{pmatrix}$$

where P_0 is an *anisotropic* symmetric integral matrix (i.e., ${}^t\xi P_0 \xi \neq 0$ for all integral vectors ξ) and

$$P_1 = \begin{pmatrix} 0 & d_1 \\ d_1 & 0 \end{pmatrix}, \cdots, P_r = \begin{pmatrix} 0 & d_r \\ d_r & 0 \end{pmatrix}.$$

A basis of $H_{2m}(V, \mathbf{Z})_0$ for which the intersection pairing has this form will be called a *standard basis*. As before we agree to use only standard basis for $H_{2m}(V, \mathbf{Z})_0$.

For the non-singular hypersurface $V \subset \mathbf{P}_{2m+1}$ we consider the *filtration* (cf. (6.1))

$$(7.6) \qquad \mathcal{H}_1(V) \subset \mathcal{H}_2(V) \subset \cdots \subset \mathcal{H}_{2m+1}(V) = \mathcal{H}_{2m+2}(V) = \cdots = \mathcal{H}(V)$$

of $\mathcal{H}(V)$. We choose bases $\omega^1, \cdots, \omega^{\delta_k}$ for $\mathcal{H}_k(V)$ which constitute a basis for the filtration (7.6) as was done below (6.1). An admissible change of basis is then given by a matrix A having the form (6.2). As in (6.3) we may now form the period matrix

$$\Omega_k(V) = \frac{1}{2\pi i}\left(\int_{\tau(\gamma_\rho)} \omega^\alpha\right) \qquad (\alpha = 1, \cdots, \delta_k;\ \rho = 1, \cdots, b)$$

where the cycles $\gamma_1, \cdots, \gamma_b$ give a standard basis for $H_{2m}(V, \mathbf{Z})_0$. As before, $\Omega_k(V)$ is determined up to a substitution $\Omega_k(V) \to A\Omega_k(V)\Lambda$ where A has the form (6.2) and Λ is an integral matrix satisfying ${}^t\Lambda P\Lambda = P$. The analogue of Theorem 6.4 is

THEOREM 7.7. *For $k \leqq m$ the period matrix $\Omega_k = \Omega_k(V)$ of $V \subset \mathbf{P}_{2m+1}$ satisfies the bilinear relations*

$$(7.8) \qquad \begin{cases} \Omega_k Q {}^t\Omega_k = 0 & (Q = {}^tP^{-1})\,, \\ \Omega_k Q {}^t\bar{\Omega}_k = H_k \text{ is a non-singular hermitian matrix.} \end{cases}$$

Furthermore, $\Omega_{m+1} Q {}^t\Omega_m = 0$ and the hermitian matrix $\Omega_{m+1} Q {}^t\bar{\Omega}_{m+1}$ has the maximal possible rank δ_{m+1}.

Example 7.9. We continue with the quadric given in Example 7.2 above. The $2m + 1$ forms with a pole of order k along V are the $\omega = P\Omega/Q^k$ where $\deg P = 2k - 2m - 2$ (cf. (2.11)). Thus $\mathcal{H}_k(V) = 0$ if $k < m + 1$ and $\mathcal{H}_{m+1}(V)$ is one dimensional with generator the differential Ω/Q^{m+1}. The ideal

$$(\partial Q/\partial\xi^0, \cdots, \partial Q/\partial\xi^{2m+1}) = (\xi^0, \xi^1, \cdots, \xi^{2m+1}) = \boldsymbol{m}$$

is the maximal ideal so that, by Proposition 4.6, we have $\mathcal{H}_{m+1}(V) = \mathcal{H}_{m+2}(V) = \cdots = \mathcal{H}(V)$. It follows that the period matrix $\Omega_{m+1}(V)$ is the 1×1 matrix with entry

$$(7.10) \qquad \frac{1}{2\pi i}\int_{\tau(\delta)} \frac{dz^1 \wedge \cdots \wedge dz^{2m+1}}{(z\cdot z - 1)^{m+1}}$$

where δ is the $2m$ sphere $x\cdot x = 1$ embedded in V. We will evaluate the integral in (7.10) and prove

$$\frac{1}{2\pi i}\int_{\tau(\delta)} \frac{dz^1 \wedge \cdots \wedge dz^{2m+1}}{(z\cdot z - 1)^{m+1}} = \frac{(m - 1/2)!}{4m!}\mu_{2m}$$

where μ_{2m} is the volume of the standard $2m$ sphere.

PROOF. For λ real and positive we set $z = x + iy$ and

$$\pi(\lambda) = \frac{1}{2\pi i}\int_{\tau(\delta_\lambda)} (dz^1 \wedge \cdots \wedge dz^{2m+1})/(z\cdot z - \lambda)$$

where δ_λ is given by $x\cdot x = \lambda$. Then the m^{th} derivative

$$d^m\pi(\lambda)/d\lambda^m = \frac{m!}{2\pi i}\int_{\tau(\delta_\lambda)}(dz^1\wedge\cdots\wedge dz^{2m+1})/(z\cdot z-\lambda)^{m+1}$$

so that the integral in (7.10) is equal to

$$\frac{1}{m!}\frac{d^m\pi(\lambda)}{d\lambda^m}\bigg|_{\lambda=1}.$$

On the other hand, since $dz^1\wedge\cdots\wedge dz^{2m+1}/(z\cdot z-\lambda)$ has a pole of first order along $z\cdot z-\lambda=0$, it is easy to see that

$$\frac{1}{2\pi i}\int_{\tau(\delta_\lambda)}(dz^1\wedge\cdots\wedge dz^{2m+1})/(z\cdot z-\lambda)=\int_{\delta_\lambda}(dx^1\wedge\cdots\wedge dx^{2m})/2x^{2m+1}$$
$$=\frac{\lambda^{m-1/2}}{2}\int_{y\cdot y=1}dx^1\wedge\cdots\wedge dx^{2m}/x^{2m+1}=(\lambda^{m-1/2}/2)\mu_{2m}$$

Taking m^{th} derivatives gives our result. Note that this proves Theorem 7.7 for this particular example.

The proof of Theorem 5.10 for the case $n=2m+1$ follows from Theorem 7.7 by an argument similar to that below Theorem 6.4.

8. Residues and the Hodge filtration

We want to prove the bilinear relations (6.5) and (7.8). The idea already appears in the example of curves worked out at the end of § 6. If $V\subset\mathbf{P}_n$ is our non-singular hypersurface we let $A_k^n(V)$ be the space of rational n-forms on $\mathbf{P}_n$ with poles of order k along V. Each ω in $A_k^n(V)$ gives a linear form $R(\omega)$ on $H_{n-1}(V,\mathbf{Z})$ by $\langle R(\omega),\gamma\rangle=\frac{1}{2\pi i}\int_{\tau(\gamma)}\omega$. Thus ω gives a *cohomology class* $R(\omega)\in H^{n-1}(V,\mathbf{C})\simeq\operatorname{Hom}_{\mathbf{Z}}\big(H_{n-1}(V,\mathbf{Z}),\mathbf{C}\big)$ which, for obvious reasons stemming from the case $n=1$, we will call the *residue* of ω. The *cup product* $H^{n-1}(V,\mathbf{C})\otimes H^{n-1}(V,\mathbf{C})\to\mathbf{C}$ then induces a bilinear form $\mathcal{H}_k(V)\otimes\mathcal{H}_l(V)\to\mathbf{C}$, and the bilinear relations (6.5) and (7.7) will be statements about this bilinear form.

To describe the mapping $R\colon\mathcal{H}(V)\to H^{n-1}(V,\mathbf{C})$ we must speak of the *Hodge filtration* of $H^{n-1}(V,\mathbf{C})$ and of the *primitive part* $H^{n-1}(V,\mathbf{C})_0$ of $H^{n-1}(V,\mathbf{C})$.

First we let $A^{q,k}(V)$ be the C^∞ differential forms of type

$$(q,0)+\cdots+(q-k,k)$$

on V; a form $\varphi\in A^{q,k}(V)$ is written as $\varphi=\varphi_{q,0}+\cdots+\varphi_{q-k,k}$ where each $\varphi_{r,s}$ is a C^∞ form of type (r,s) on V. The exterior derivative $d\colon A^{q,k}(V)\to A^{q+1,k+1}(V)$ and we set $\mathbf{F}^{q,k}(V)=Z^{q,k}(V)/dA^{q-1,k-1}(V)$ where $Z^{q,k}(V)$ is the space of closed forms in $A^{q,k}(V)$. Thus a class $\varphi\in\mathbf{F}^{q,k}(V)$ is represented by a closed form $\varphi=\varphi_{q,0}+\cdots+\varphi_{q-k,k}$ modulo $d\eta$ where $\eta=\eta_{q-1,0}+\cdots+\eta_{q-k,k-1}$.

Observe that there are natural maps $\mathbf{F}^{q,k}(V) \to H^q(V, \mathbf{C})$ (using *de Rham's theorem*).

THEOREM 8.1. *There are natural maps* $\mathcal{H}_k(V) \xrightarrow{R} \mathbf{F}^{n-1,k-1}(V)$ *such that the following diagram commutes*

$$
(8.2)\qquad
\begin{array}{ccccccc}
\mathcal{H}_1(V) & \longrightarrow & \mathcal{H}_2(V) & \longrightarrow \cdots \longrightarrow & \mathcal{H}_n(V) & = & \mathcal{H}(V) \\
\downarrow R & & \downarrow R & & \downarrow R & & \downarrow R \\
\mathbf{F}^{n-1,0}(V) & \longrightarrow & \mathbf{F}^{n-1,1}(V) & \longrightarrow \cdots \longrightarrow & \mathbf{F}^{n-1,n-1}(V) & = & H^{n-1}(V, \mathbf{C}) .
\end{array}
$$

Next we recall that a class $\psi \in H^{n-1}(V, \mathbf{C})$ is *effective* or *primitive* if $\omega \cdot \psi = 0$ in $H^{n+1}(V, \mathbf{C})$ where $\omega \in H^2(V, \mathbf{C})$ is the *Poincaré dual* of the homology class $[\mathbf{P}_{n-1} \cdot V]$ of a hyperplane section (cf. [7], [12]). The differential form representing ω was given at the beginning of § 7, and we see that cup product with ω gives a map $\mathbf{F}^{q,k}(V) \to \mathbf{F}^{q+1,k+1}(V)$, so that we may define what it means for a class in $\mathbf{F}^{n-1,k-1}(V)$ to be primitive.

THEOREM 8.3. *The natural maps* $R: \mathcal{H}_k(V) \to \mathbf{F}^{n-1,k-1}(V)$ *map* $\mathcal{H}_k(V)$ *to the primitive part* $\mathbf{F}^{n-1,k-1}(V)_0$ *of* $\mathbf{F}^{n-1,k-1}(V)$, *and* $R: \mathcal{H}(V) \to H^{n-1}(V, \mathbf{C})_0$ *is an isomorphism.*

COROLLARY 8.4. *The maps* $\mathbf{F}^{n-1,k-1}(V)_0 \to H^{n-1}(V, \mathbf{C})_0$ *are injective.*

We thus have a filtration

$$
(8.5)\qquad \mathbf{F}^{n-1,0}(V)_0 \subset \mathbf{F}^{n-1,1}(V)_0 \subset \cdots \subset \mathbf{F}^{n-1,n-1}(V)_0 = H^{n-1}(V, \mathbf{C})_0 ,
$$

which we call the *Hodge filtration* of $H^{n-1}(V,\mathbf{C})_0$. We may combine (8.1) and (8.3) into the statement

(8.6.) *The residue maps* $R: \mathcal{H}_k(V) \to \mathbf{F}^{n-1,k-1}(V)_0$ *are isomorphisms which take the filtration* (6.1) *of* $\mathcal{H}(V)$ *obtained from the order of poles of cohomology classes along* V *onto the Hodge filtration of* $H^{n-1}(V, \mathbf{C})_0$.

PROOF OF THEOREM 8.1. We let $B^{q,k}(l)$ be the space of differential forms φ on $\mathbf{P}_n$ which are of type $(q,0) + \cdots + (q-k,k)$, are C^∞ on $\mathbf{P}_n - V$, and which have the local property that $f^l\varphi$ and $f^{l-1}df \wedge \varphi$ are C^∞ if $f = 0$ is a local holomorphic defining equation for V. If $g = 0$ is another defining equation for V, then $g = uf$ where u is a non-vanishing holomorphic function and the relations $g^l\varphi = u^l f^l \varphi$, $g^{l-1}dg \wedge \varphi = u^l f^{l-1} df \wedge \varphi + u^{l-1} f^l du \wedge \varphi$ show that $B^{q,k}(l)$ is well-defined.

It is easy to see that the exterior derivative $d: B^{q,k}(l) \to B^{q+1,k+1}(l)$ and we let $Z^{q,k}(l)$ be the closed forms in $B^{q,k}(l)$.

LEMMA 8.7. *If* $\varphi \in B^{q,k}(l)$ *and* $l > 1$, *then locally* $\varphi = d\psi + \eta$ *where* ψ, η *have poles of order* $l-1$ *along* V *and* ψ *is of type* $(q-1,0) + \cdots + (q-1-k,k)$.

PROOF. We may choose local holomorphic coordinates $z^1, \cdots, z^{n-1}, w$ on $\mathbf{P}_n$ such that $w = 0$ is the defining equation of V. Since $w^l \varphi$ is C^∞, we may write $\varphi = (\alpha \wedge dw)/w^l + \beta'/w^l$ where α, β' are C^∞ and do not involve dw. On the other hand $w^{l-1} dw \wedge \varphi = (\beta' \wedge dw)/w$ is C^∞ so that $\beta = \beta'/w$ is C^∞ and we have locally

$$\varphi = \frac{\alpha \wedge dw}{w^l} + \frac{\beta}{w^{l-1}} \qquad \text{where } \alpha, \beta \text{ do not involve } dw\,. \tag{8.8}$$

Now (8.8) is valid for $l \geqq 1$; if $l > 1$, then we may take $\psi = \big(\pm 1/(l-1)\big)(\alpha/w^{l-1})$ to get our Lemma.

LEMMA 8.9. *If $\varphi \in B^{q,k}(l)$ and $l > 1$, then we have globally $\varphi = d\psi + \eta$ where ψ, η have poles of order $l - 1$ along V and ψ is of type $(q-1, 0) + \cdots + (q-1-k, k)$.*

PROOF. Let $\{U_\alpha\}$ be a finite open covering of $\mathbf{P}_n$ such that we have $\varphi = d\psi_\alpha + \eta_\alpha$ as in Lemma 8.7 in each U_α. If $\{\rho_\alpha\}$ is a *partition of unity* subordinate to $\{U_\alpha\}$, then we set $\psi = \sum_\alpha \rho_\alpha \psi_\alpha$. Now

$$\varphi - d\psi = \varphi - \textstyle\sum_\alpha \rho_\alpha d\psi_\alpha - \sum_\alpha d\rho_\alpha \psi_\alpha = \sum_\alpha (\rho_\alpha \eta_\alpha - d\rho_\alpha \psi_\alpha) = \eta$$

as required.

LEMMA 8.10. *If $\varphi \in Z^{q,k}(l)$ and $l > 1$ we write $\varphi = d\psi + \eta$ according to Lemma 8.9. Then $\eta \in Z^{q,k+1}(l-1)$ and the map $\varphi \to \eta$ is a well-defined linear mapping*

$$T\colon Z^{q,k}(l) \longrightarrow Z^{q,k+1}(l-1)/dB^{q-1,k}(l-1)\,.$$

PROOF. If $\varphi = d\psi + \eta = d\psi' + \eta'$ according to Lemma 8.9, then $d(\psi - \psi') = \eta' - \eta$ and $\psi - \psi' \in B^{q-1,k}(l-1)$.

LEMMA 8.11. *If $\theta \in B^{q-1,k-1}(l)$, then $T(d\theta) \in dB^{q-1,k}(l-1)$ in Lemma* 8.10.

PROOF. Write $d\theta = d\psi + \eta$ according to Lemma 8.9, and $\theta = d\xi + \zeta$ according also to Lemma 8.9. Then ψ, η, ξ, ζ all have poles of order $l - 1$ along V and ψ is of type $(q-1, 0) + \cdots + (q-1-k, k)$, ξ is of type $(q-2, 0) + \cdots + (q-k-1, k-1)$. Thus $T(d\theta) = \eta = d(\zeta - \psi)$ where $\zeta - \psi$ has a pole of order $l - 1$ along V; i.e., $T(d\theta) \in dB^{q-1,k}(l-1)$.

Combining Lemmas 8.10 and 8.11 we have for $l > 1$

$$Z^{q,k}(l)/dB^{q-1,k-1}(l) \xrightarrow{T} Z^{q,k+1}(l-1)/dB^{q-1,k}(l-1) \tag{8.12}$$

where T is given by Lemma 8.10. Putting these together we have a diagram

(8.13)

$$\begin{array}{l} A_k^n(V)/dA_{k-1}^{n-1}(V) \xrightarrow{T} Z^{n,1}(k-1)/dB^{n-1,0}(k-1) \xrightarrow{T} \cdots \xrightarrow{T} Z^{n,k-1}(1)/dB^{n-1,k-2}(1) \\ \qquad \Vert \\ \quad \mathcal{H}_k(V) \end{array}$$

which we collapse into

$$A_k^n(V)/dA_{k-1}^{n-1}(V) \xrightarrow{T} Z^{n,k-1}(1)/dB^{n-1,k-2}(1) \tag{8.14}$$

with the following *prescription*:

(8.15) Given $\omega \in A_k^n(V)$, there exist forms $\psi_1, \cdots, \psi_{k-1}$ where ψ_α is of type $(n-1,0)+\cdots+(n-\alpha, \alpha-1)$ and has a pole of order $k-\alpha$ along V such that

$$\omega - d(\psi_1 + \cdots + \psi_{k-1}) = \eta$$

is of type $(n, 0) + \cdots + (n-k+1, k-1)$ and has a pole of order 1 along V. Then $T(\omega) = \eta$ in (8.14).

Suppose now that $\varphi \in B^{q,k}(1)$. According to (8.8) we have locally $\varphi = (\alpha \wedge dw)/w + \beta$; we set $R(\varphi) = \alpha \mid V$ and call $R(\varphi)$ the *Poincaré residue* of φ.

PROPOSITION 8.16. *The Poincaré residue operator* R: $B^{q,k}(1) \to A^{q-1,k}(V)$ *is well-defined and satisfies*

(a) $R(d\lambda) = dR(\lambda)$ *for* $\lambda \in B^{q-1,k-1}(1)$; *and*

(b) $\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{\tau_\varepsilon(\gamma)} \varphi = \int_\gamma R(\varphi)$ *for* $\varphi \in B^{n,k}(1)$ *and any* $n-1$ *chain* γ *on* V.

COROLLARY 8.17. *The Poincaré residue operator*

$$R\colon Z^{n,k-1}(1)/dB^{n-1,k-2}(1) \longrightarrow Z^{n-1,k-1}(V)/dA^{n-2,k-2}(V)$$

is well-defined and satisfies $\frac{1}{2\pi i}\int_{\tau(\gamma)} \eta = \int_\gamma R(\eta)$ *for any cycle* $\gamma \in H_{n-1}(V, \mathbf{Z})$.

If we have proved Proposition 8.16, then we may combine (8.18) and (8.17) to have

$$\mathcal{H}_k(V) \xrightarrow{R} \mathbf{F}^{n-1,k-1}(V) \tag{8.18}$$

with the following properties:

(i) Given $\omega \in A_k^n(V)$, we write $\omega - d(\psi_1 + \cdots + \psi_{k-1}) = \eta$ according to prescription (8.15), and then $R(\omega) = R(\eta)$; and

(ii) $\frac{1}{2\pi i}\int_{\tau(\gamma)} \omega = \int_\gamma R(\omega)$ for $\gamma \in H_{n-1}(V, \mathbf{Z})$.

The maps in (8.18) then give a proof of Theorem 8.1.

PROOF OF PROPOSITION 8.16. Locally we write $\varphi = (\alpha \wedge dw)/w + \beta$ and have set $R(\varphi) = \alpha \mid V$. If $\varphi = (\alpha' \wedge dw)/w + \beta'$, then $(\alpha - \alpha') \wedge dw/w$ is C^∞ and so $\alpha - \alpha' = w\xi$ and $\alpha \mid V = \alpha' \mid V$ so that $R(\varphi)$ is independent of the representation (8.8). If $w' = 0$ is another local equation for V, then $w' = uw$ $(u \neq 0)$ and $\varphi = (\alpha' \wedge dw')/w' + \beta' = (\alpha' \wedge dw)/w + \big((\alpha' \wedge du)/u + \beta'\big)$ so that $R(\varphi)$ is independent of the local defining equation of V. Thus $R\colon B^{q,k}(1) \to A^{q-1,k}(V)$ is well-defined.

Since $d\varphi = (d\alpha \wedge dw)/w + d\beta$ we see that R commutes with exterior derivative which proves (a). The proof of (b) is more or less clear and will be omitted.

Example 8.19. Suppose that $V \subset \mathbf{P}_2$ is a non-singular plane curve given in affine coordinates by $q(x, y) = 0$. From (6.7) we see that the rational 2-forms $\omega \in A_1^2(V)$ with a first order pole along V are written as

$$\omega = p(x, y)dxdy/q(x, y) \qquad (\deg p \leqq \deg q - 3)\ .$$

Since $\omega = pdqdy/(\partial q/\partial x)q = pdxdq/(\partial q/\partial y)q$, the Poincaré residue $R(\omega) = -pdy/(\partial q/\partial x) = pdx/(\partial q/\partial y)$ is a holomorphic differential on V (cf. the end of § 6).

Before proving Theorem 8.3 we observe

LEMMA 8.20. *All of $H^{n-1}(V, \mathbf{C})$ is primitive in case $n = 2m$ is even. In case $n = 2m + 1$ is odd, the condition that $\psi \in H^{2m}(V, \mathbf{C})$ be primitive is that $\int_h \psi = 0$ where $h = [\mathbf{P}_{m+1} \cdot V]$ is the homology class of a linear section.*

The proof of Theorem 8.3 now follows from Proposition 3.5, Theorem 5.3, and (ii) below (8.18).

PROOF OF THEOREMS 6.4 AND 7.7. We shall use the following formula. Let $\gamma_1, \cdots, \gamma_b$ be an integral basis for the primitive homology $H_{n-1}(V, \mathbf{Z})_0$, and set $Q = {}^t(\gamma_\rho \cdot \gamma_\sigma)^{-1}$; that is, Q is the transposed inverse of the intersection matrix. Let $\varphi, \psi \in H^{n-1}(V, \mathbf{C})_0$ be *primitive cohomology classes* (cf. (8.20)) and set $\varphi^\rho = \int_{\gamma_\rho} \varphi$, $\psi^\rho = \int_\gamma \psi$. Then we have

$$\int_V \varphi \wedge \psi = \sum\nolimits_{\rho,\sigma} \varphi^\rho Q_{\rho\sigma} \psi^\sigma\ . \tag{8.21}$$

We now choose our basis $\omega^1, \cdots, \omega^{\delta_1}; \omega^{\delta_1+1}, \cdots, \omega^{\delta_2}, \cdots, \omega^{\delta_{n-1}+1}, \cdots, \omega^{\delta_n}$ for the flag $\mathcal{H}_1(V) \subset \mathcal{H}_2(V) \subset \cdots \subset \mathcal{H}_n(V)$ such that $R(\omega^1), \cdots, R(\omega^{\delta_1})$ gives a basis for the *primitive harmonic space* $H^{n-1,0}(V)_0$, $R(\omega^{\delta_1+1}), \cdots, R(\omega^{\delta_2})$ gives a basis for the primitive harmonic space $H^{n-2,1}(V)_0, \cdots, R(\omega^{\delta_{n-1}+1}), \cdots, R(\omega^{\delta_n})$ gives a basis for the primitive harmonic space $H^{0,n-1}(V)_0$ (cf. Hodge [7]). Since $\int_V \varphi \wedge \psi = 0$ for $\varphi \in H^{n-1-r,r}(V)_0$, $\psi \in H^{s,n-1-s}(V)_0$ unless $r = s$, and since $\bar{H}^{n-1-r,r}(V)_0 = H^{r,n-1-r}(V)_0$, we may use (8.21) to obtain the first bilinear relations in (6.5) and (7.8) and see that

$$(8.22)\qquad i\Omega_m \Omega^t \bar{\Omega}_m = \begin{pmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & & \\ \vdots & & \ddots & \\ 0 & & & H_m \end{pmatrix} \qquad (n = 2m \text{ is even})$$

$$(8.23)\qquad \Omega_m Q^t \bar{\Omega}_m = \begin{pmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & & \\ \vdots & & \ddots & \\ 0 & & & H_m \end{pmatrix} \qquad (n = 2m + 1 \text{ is odd})$$

where H_1 is $\delta_1 \times \delta_1$, H_2 is $(\delta_2 - \delta_1) \times (\delta_2 - \delta_1)$, etc. To finish the proof it will suffice to show that

(8.24) $(-1)^{j-1}H_j$ is positive definite. This follows from the results in Hodge [7]; cf. also [12].

9. Infinitesimal period relations and non-singularity of the period mapping

In this section we want to study the variation of period matrix when our hypersurface $V \subset \mathbf{P}_n$ is perturbed in $\mathbf{P}_n$. For definiteness suppose we consider the case $n = 2m$ so that $\dim V = 2m - 1$. Then according to Theorem 6.4 and (8.24) the period matrix $\Omega = \Omega_m(V)$ is a $p \times 2p$ matrix which satisfies the bilinear relations

$$(9.1)\qquad \begin{cases} \Omega Q^t \Omega = 0 \, , \\ i\Omega Q^t \bar{\Omega} > 0 \end{cases} \qquad Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where the second relation in (9.1) means that $i\Omega Q^t \bar{\Omega} = H$ is an hermitian matrix whose first $\delta_1 \times \delta_1$ block is positive definite, whose first $\delta_2 \times \delta_2$ block has δ_1 positive and $\delta_2 - \delta_1$ negative eigenvalues, whose first $\delta_3 \times \delta_3$ block has $\delta_1 + \delta_3 - \delta_2$ positive and δ_2 negative eigenvalues, etc. on up to $\delta_m = p$. Also Ω is determined up to the equivalence relations

$$(9.2)\qquad \Omega \sim A\Omega, \quad A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & \\ \underbrace{A_{m1}}_{\delta_1} & \underbrace{\cdots}_{\delta_2 - \delta_1} & & \underbrace{A_{mm}}_{\delta_m - \delta_{m-1}} \end{pmatrix},$$

$$(9.3)\qquad \Omega \sim \Omega\Lambda, \quad \Lambda \text{ integral and } \Lambda Q^t \Lambda = Q.$$

Using the equivalence relation (9.2) we may find A such that

$$iA\Omega Q^t(\overline{A\Omega}) = \begin{pmatrix} I_{\delta_1} & 0 & \cdots \\ 0 & -I_{\delta_2-\delta_1} & \\ \vdots & & \ddots \end{pmatrix}.$$

Let $\mathfrak{D}$ be the open complex manifold of $p \times 2p$ matrices Ω which satisfy the two bilinear relations (9.1) and $\mathfrak{X}$ the manifold of those Ω which satisfy the first relation $\Omega Q^t\Omega = 0$. The group G of all $p \times p$ matrices A having the block form in (9.2) operates on both $\mathfrak{D}$ and $\mathfrak{X}$ and we let $D = G\backslash\mathfrak{D}$ and $X = G\backslash\mathfrak{X}$ be the quotient spaces.

Now the group $\mathbf{G}$ of all $2p \times 2p$ matrices T with complex entries which satisfy $TQ^tT = Q$ operates on X by $T(\Omega) = \Omega T^{-1}$, and the subgroup $G \subset \mathbf{G}$ of real matrices $T = \bar{T}$ acts on D. Obviously $\mathbf{G}$ is a simple complex Lie group and $G \subset \mathbf{G}$ is a *real form*. In [4] it is proved that

(9.4) X is a *projective homogeneous algebraic manifold* which is acted on transitively by $\mathbf{G}$;

(9.5) D is an open *homogeneous complex manifold* $H\backslash G$ where the isotropy group H is compact;

(9.6) the natural embedding $D \subset X$ is G-equivariant and D is the G-orbit of the origin in X.

We call D the *period matrix space* for V and refer to [4] for a discussion of its properties. Observe that by (9.5) the *arithmetic group* $\Gamma \subset G$ of integral matrices which preserve Q acts properly discontinuously on D so that the quotient space $M = D/\Gamma$ is an *analytic space* which we call the *modular variety* associated to $V \subset \mathbf{P}_{2m}$.

Now V is given by an equation $Q(\xi) = 0$ where

$$Q(\xi) = \sum\nolimits_{i_1,\cdots,i_q} \lambda_{i_1\cdots i_q}\xi^{i_0}\cdots\xi^{i_q}$$

is a homogeneous form in $\xi^0, \cdots, \xi^{2m}$ of degree q. Using the coefficients of Q as homogeneous coordinates $[\cdots, \lambda_{i_1\cdots i_q}, \cdots]$ in a big $\mathbf{P}_N$ we see that the set of all non-singular hypersurfaces $V \subset \mathbf{P}_{2m}$ of degree q is parametrized by a *Zariski open set* $\Sigma = \Sigma_{2m,q}$ in $\mathbf{P}_N$. This parametrization is *not effective* since the projective group $PG(2m)$ operates on Σ.

By assigning to each non-singular $V \subset \mathbf{P}_{2m}$ its period matrix as computed by the formula (6.3) we have defined the *period mapping*

$$\Phi\colon \Sigma \longrightarrow D/\Gamma\ . \tag{9.7}$$

THEOREM 9.8. (a) *The period mapping Φ is holomorphic and satisfies the infinitesimal bilinear relations*

$$\begin{cases} d\Omega_m Q^t\Omega_{m-1} = 0 & (\textit{in case } n = 2m \textit{ is odd}) \\ d\Omega_m Q^t\Omega_m \quad = 0 & (\textit{in case } n = 2m+1 \textit{ is even})\ . \end{cases}$$

(b) *The differential of the period mapping is injective modulo the projective group* $PG(n)$, *provided that* $\deg Q > 2$ *or* $\deg Q > 3$ *in case* $n = 3$.

Remark. For a discussion of the period mapping for algebraic manifolds we refer to [4], where also (a) is proved in general and (b) is proved for a few special examples. We should explain that (b) means that if $\Phi_*(\tau) = 0$ for a tangent vector τ to Σ at Q, then τ is tangent to the $PG(n)$ orbit passing through Q; in other words, (b) is a sort of *local Torelli theorem*. Observe that the exceptions in (b) are the *quadrics* ($q = 2$) and the *cubic surface* ($q = 3$, $n = 3$), which are both rational and will have no periods.

The remainder of this section will be devoted to proving Theorem 9.8. Our hypersurface $V \subset \mathbf{P}_{2m}$ is given by $Q(\xi) = 0$ and the hypersurfaces V_λ close to V have an equation $Q(\xi) + \lambda R(\xi) = 0$ where $R(\xi)$ is a homogeneous polynomial of degree $q = \deg Q$ and λ is small. All such V_λ are non-singular if λ is sufficiently small, and the numbers $\dim \mathcal{H}_k(V_\lambda)$ will all be the same (by Proposition 4.6, Theorem 4.11, and Theorem 4.3). In particular, if $P_1(\xi), \cdots, P_p(\xi)$ are homogeneous polynomials with $\deg P_\alpha = qm - 2m - 1$ such that the differentials $\omega_\alpha = P_\alpha(\xi)\Omega/Q(\xi)^m$ give a basis for $\mathcal{H}_m(V)$, then the differentials

$$\omega_\alpha(\lambda) = P_\alpha(\xi)\Omega/\{Q(\xi) + \lambda R(\xi)\}^m \tag{9.9}$$

will give a basis for $\mathcal{H}_m(V_\lambda)$.

The *canonical basis* $\gamma_1, \cdots, \gamma_{2m-1}$ for $H_{2n-1}(V, \mathbf{Z})$ will displace uniquely to a canonical basis $\gamma_1(\lambda), \cdots, \gamma_{2p}(\lambda)$ on a nearby V_λ, and it is quite clear that

$$\int_{\tau(\gamma_\rho(\lambda))} \omega_\alpha(\lambda) = \int_{\tau_\varepsilon(\gamma_\rho)} \omega_\alpha(\lambda) \tag{9.10}$$

provided that V_λ is within distance $\varepsilon/2$ of V. From (9.10) it follows that

$$\frac{\partial}{\partial\lambda}\left\{\int_{\tau(\gamma_\rho(\lambda))} \omega_\alpha(\lambda)\right\} = \int_{\tau(\gamma_\rho)} \partial\omega_\alpha(\lambda)/\partial\lambda\ .$$

Thus the period matrix

$$\Omega_m(\lambda) = \begin{pmatrix} \int_{\tau(\gamma_1(\lambda))} \omega_1(\lambda) \cdots \int_{\tau(\gamma_{2p}(\lambda))} \omega_1(\lambda) \\ \vdots \qquad\qquad\qquad \vdots \\ \int_{\tau(\gamma_1(\lambda))} \omega_p(\lambda) \cdots \int_{\tau(\gamma_{2p}(\lambda))} \omega_p(\lambda) \end{pmatrix}$$

varies holomorphically with λ and $(\partial\Omega_m(\lambda)/\partial\lambda)Q\Omega_{m-1}(\lambda) = 0$ since $(\partial\omega_\alpha/\partial\lambda)(\lambda) = -RP_\alpha\Omega/(Q + \lambda R)^{m+1}$ lies in $\mathcal{H}_{m+1}(V_\lambda)$ and the cup product between $R(\mathcal{H}_{m+1}(V_\lambda))$ and $R(\mathcal{H}_{m-1}(V_\lambda))$ is zero (cf. (8.21) and Theorem 8.1). This completes the proof of (a) in Theorem 9.8.

To prove (b) we observe from (9.2) that $\Phi_*(\partial/\partial\lambda)$ (taken at $\lambda = 0$) is the zero tangent vector to the period matrix space D at $\Phi(V)$ if, and only if, $\partial\Omega_m(\lambda)/\partial\lambda\,|_{\lambda=0} = A\Omega_m(0)$ for some matrix A given in (9.2) (A need not be invertible). This implies that $\partial\omega_\alpha(\lambda)/\partial\lambda\,|_{\lambda=0}$ lies in $\mathcal{H}_m(V)$ for $\alpha = 1, \cdots, p$ which, by (9.9) and Proposition 4.6, means that

(9.11) $RP \in (\partial Q/\partial\xi^0, \cdots, \partial Q/\partial\xi^{2m})$ for all polynomials P of degree $qm - 2m - 1$.

From Macaulay's theorem (4.11) we find that $R \in (\partial Q/\partial\xi^0, \cdots, \partial Q/\partial\xi^{2m})$ if $(m + 1)(q - 2) > q - 2$. The simultaneous inequalities $qm \geqq 2m + 1$ and $(m + 1)(q - 2) > q - 2$ are satisfied if $q > 2$, so that in this case we have from (9.11) that there is a constant matrix (M_{ij}) such that

$$R(\xi) = \sum_{i,j=0}^{2m} M_{ij}\xi^i\partial Q/\partial\xi_j\ . \tag{9.12}$$

Then the infinitesimal projective transformation $\sum_{i,j} M_{ij}\xi^i\partial/\partial\xi^j$ has on Σ the same tangent at $Q \in \Sigma$ as the curve $Q + \lambda R$ passing through Q, which proves (b) in case $n = 2m$ is even. The other case $n = 2m + 1$ leads to the inequalities $(m + 2)(q - 2) + 1 > q - 2$ and $qm \geqq 2m + 2$; the rest of the argument proceeds just as before.

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(Received February 5, 1969)