Prince Rupert's Rectangles

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1. INTRODUCTION. More than three hundred years ago, according to the contemporaneous John Wallis [**11**, pp. 470–471], Prince Rupert[1] (1619–1682) won a wager that a hole can be cut in one of two equal cubes large enough to permit the second cube to pass through.

Nearly a century later the Dutch scientist Pieter Nieuwland (1764–1794) showed that the largest cube that can be so passed through a cube of side one has side $3\sqrt{2}/4 \approx 1.061$. An accessible discussion with enlightening anaglyphs appears in Ehrenfeucht [**8**]. In 1950, D. J. E. Schreck [**10**] gave an interesting, historically based survey of Prince Rupert's problem and Nieuwland's extension. Schreck includes a photograph of a model showing the cube in transit.

Nieuwland's "passage" problem of finding the largest cube that can pass through a unit cube is equivalent to finding the largest square that fits in the unit cube, because once the largest square is located, the hole through the cube having that largest square as its cross section clearly provides the desired passage. In higher dimensions one might seek the side of the largest m-dimensional cube that fits in an n-dimensional cube of side one. Not much is known about this question, which apparently was first raised, at least in the case of a cube in a hypercube, by Martin Gardner (see Gardner [**5**, pp. 172–173] or Guy and Nowakowski [**7**, pp. 967–68]).

The literature on Prince Rupert's and Nieuwland's problems is extensive and we cannot claim to have examined it all, but nowhere have we found any mention of the analogous questions for rectangles and "boxes" (i.e., rectangular parallelepipeds):

Question 1. Find the largest rectangle R with given aspect ratio λ that fits in the unit cube, i.e., for given λ with $0 \le \lambda \le 1$, find the largest L so that an $L \times \lambda L$ rectangle fits in the unit cube.[2]

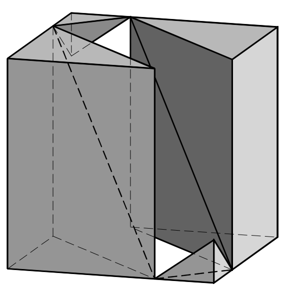
This question is an instance of a general unsolved fitting problem for boxes in higher dimensions: in d-dimensional Euclidean space $\mathbb{E}^d$, find necessary and sufficient conditions on the $k+d$ edges for a k-dimensional box ($k \le d$) with edges $p_1, p_2, \ldots, p_k$ to fit in a d-dimensional box with edges $s_1, s_2, \ldots, s_d$. For $k = 1$ the matter is trivial for each d: a line segment of length p fits in the d-dimensional box precisely when $p \le \sqrt{s_1^2 + s_2^2 + \cdots + s_d^2}$. For $k = d = 2$ the question of when one rectangle fits in another was asked in 1956 by L. R. Ford [**4**], and a necessary and sufficient condition was soon given by W. B. Carver [**2**] (see also Wetzel [**12**]). The problem was posed for $k = d = 3$ by F. M. Garnett in 1923, and an incomplete answer was supplied in 1925 by Carver [**1**].

Question 2. Find the largest box of given shape that can be passed through a suitably perforated unit cube, i.e., for given σ and τ with $0 \le \tau \le \sigma \le 1$, find the largest D so that a $D \times \sigma D \times \tau D$ box can pass through a suitable hole in the unit cube.

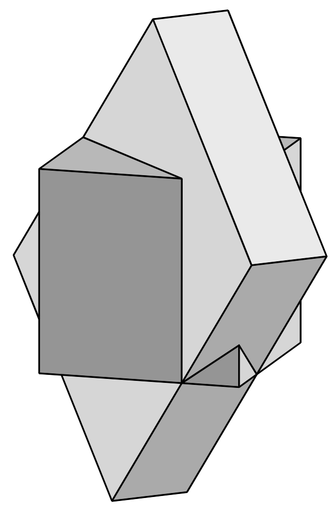
[1] Count Palatine of the Rhine and Duke of Bavaria, son of Frederick V, the Winter King, Elector Palatine, and king of Bohemia, and Elizabeth, daughter of James I of England.

[2] It is convenient to regard a line segment as a rectangle with one side 0, and a rectangle as a box with one edge 0.

This "passage" question bears the same relation to the "fitting" question 1 as Rupert's does to Nieuwland's: once the largest rectangle with aspect ratio $\lambda = \tau/\sigma$ in the unit cube is known, then the hole through the cube having that rectangular cross section provides the desired passage; and no smaller hole with similar cross section suffices. As an example, suppose that $\sigma = 0.7$ and $\tau = 0.2$. The largest $D \times \sigma D \times \tau D$ box that can pass through the unit cube has $D = 2(\sqrt{143} - \sqrt{8})/9 \approx 2.02885$ (see Theorem 5 and its corollary), so that $\sigma D \approx 1.42020$ and $\tau D \approx 0.40577$. Figure 1a shows the largest rectangle with aspect ratio $\lambda = 2/7$ that fits in the unit cube and the cube perforated to accommodate the box. Figure 1b shows the box in transit through the perforated cube.



(a) The perforated unit cube.



(b) The box in transit.

Figure 1. A box through the unit cube.

2. THE FITTING LEMMA.

Let $\boldsymbol{C}$ be the unit cube, and let $\partial\boldsymbol{C}$ denote its boundary, viz., the union of its six closed faces. Once the aspect ratio λ is fixed in [0, 1], there is a largest real number $L_{\max}$ so that the $L_{\max} \times \lambda L_{\max}$ rectangle $R_{\max}$ fits in $\boldsymbol{C}$. The first thing we need to ask is precisely how this largest rectangle $R_{\max}$ is situated in $\boldsymbol{C}$. The answer is given by the "fitting lemma," Lemma 4, whose proof is the objective of this section.

Note first that, because of the central symmetry, any rectangle R that fits in $\boldsymbol{C}$ must also fit in $\boldsymbol{C}$ with its center at the center of symmetry of the cube.

Lemma 1. *If a rectangle $R = ABPQ$ (Figure 2) fits in the unit cube $\boldsymbol{C}$ in any way whatsoever, then it also fits in $\boldsymbol{C}$ with its center at the center of $\boldsymbol{C}$.*

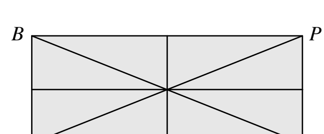


Figure 2. A rectangle with aspect ratio 0.4.

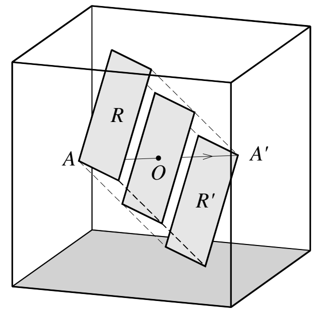


Figure 3. Centering.

Proof. Suppose that the rectangle $R = ABPQ$ fits in $\boldsymbol{C}$ and let A', B', P', Q' be the points symmetric to A, B, P, Q in the center O of $\boldsymbol{C}$ (Figure 3). Then the rectangle $R' = A'B'P'Q'$ is congruent to R and forms with R a parallelepiped that fits in $\boldsymbol{C}$ with center at the center O of $\boldsymbol{C}$. The medial section of this parallelepiped midway between R and R' is a rectangle that is congruent to R and has its center at O. ■

As a consequence of this lemma, we may always assume that a rectangle under consideration is centrally placed in $\boldsymbol{C}$. We call such a rectangle *centered*. A corner of a centered rectangle lies on an edge or face of $\boldsymbol{C}$ precisely when the opposite corner of the rectangle lies on the opposite edge or face of $\boldsymbol{C}$.

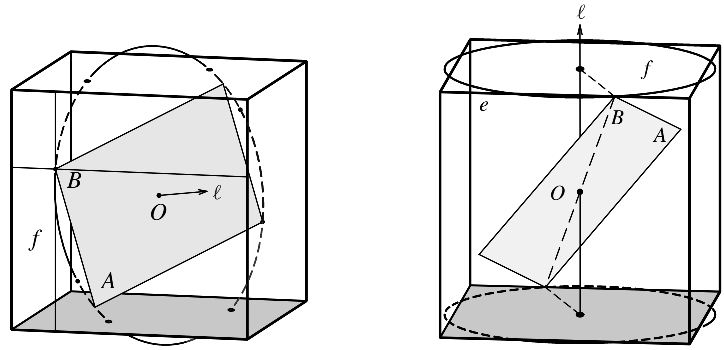
Corners at vertices. If one corner of a centered rectangle R in $\boldsymbol{C}$ lies at a vertex of $\boldsymbol{C}$, then the opposite corner of R lies at the opposite vertex of $\boldsymbol{C}$, and the two remaining corners of R are at opposite vertices of $\boldsymbol{C}$. There are only two possibilities: $\lambda = 0$ and R is a diagonal line segment of $\boldsymbol{C}$ regarded as a $\sqrt{3} \times 0$ rectangle, or $\lambda = 1/\sqrt{2}$ and R is a $\sqrt{2} \times 1$ diagonal cross section of $\boldsymbol{C}$. In the following analysis, we usually suppose that the corners of R are not at the vertices of $\boldsymbol{C}$.

Centered maximal rectangles. How does a centered *maximal* rectangle R fit in $\boldsymbol{C}$? We show next that its corners must lie on the boundary $\partial\boldsymbol{C}$ of $\boldsymbol{C}$.

Lemma 2. *The corners of a centered maximal rectangle R lie on the boundary $\partial\boldsymbol{C}$ of $\boldsymbol{C}$.*

Proof. Our principal tool is the observation that a rectangle that fits in $\boldsymbol{C}$ with all four of its corners inside $\boldsymbol{C}$ is not maximal. In particular, if a *centered* rectangle has two adjacent corners in the interior of $\boldsymbol{C}$, then it is not maximal. It follows that we need examine only those centered rectangles having at least one corner on a face (but not at a vertex) of $\boldsymbol{C}$.

Suppose that one corner A of a centered rectangle $R = ABPQ$ lies inside $\boldsymbol{C}$ and an adjacent corner B lies on a closed face f of $\boldsymbol{C}$ but not at a vertex (Figure 4). If B lies in the interior of a face f, then an appropriate small rotation of R in its circumcircle (Figure 4a) moves B inside $\boldsymbol{C}$ as well; and consequently R is not maximal. If B lies on an edge e of f (Figure 4b), then a suitable small rotation of R about the line ℓ normal to f through O leaves A inside $\boldsymbol{C}$ and moves B off e into f, the case just considered; so again R is not maximal. It follows as claimed that all four corners of a maximal centered rectangle must lie on *faces* of $\boldsymbol{C}$. ■



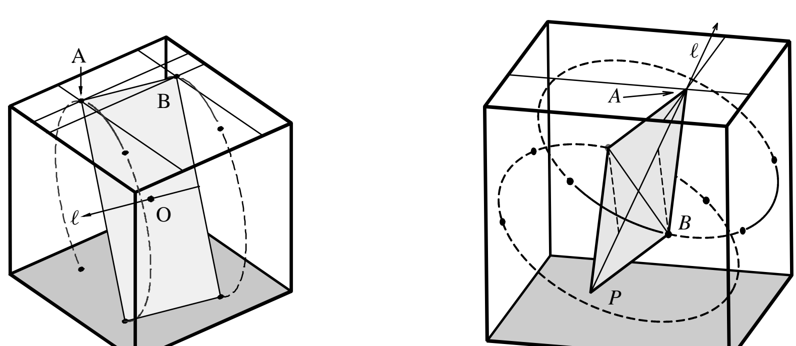
(a) Corner B on an open face. (b) Corner B on an edge.

Figure 4. Corner A inside the cube.

But more is true: the corners of a maximal centered rectangle must be on the edges of $\boldsymbol{C}$.

Lemma 3. *The corners of a centered maximal rectangle R lie on edges of $\boldsymbol{C}$.*

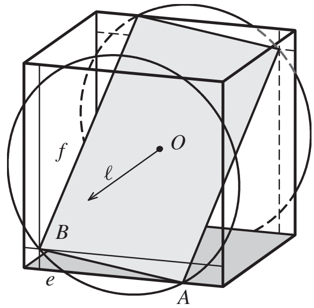
Proof. We show first that at least one of the corners of a centered maximal rectangle R must lie on an edge of $\boldsymbol{C}$. If adjacent corners A and B of R both lie on the same (open) face of $\boldsymbol{C}$ (Figure 5a), then the other two corners P and Q lie on the opposite face of $\boldsymbol{C}$, and a suitable small rotation about the axis ℓ through O parallel to AB moves R completely inside $\boldsymbol{C}$, contrary to Lemma 2. If adjacent corners A and B of R lie on opposite (open) faces of $\boldsymbol{C}$, then the adjacent corners A and Q lie on the same (open) face of $\boldsymbol{C}$, which as we have just seen is a contradiction. If adjacent corners A and B of R lie on adjacent (open) faces of $\boldsymbol{C}$ (Figure 5b), a suitable small rotation of R about the diagonal AP moves B inside $\boldsymbol{C}$, again contrary to Lemma 2. Consequently, at least one corner of R must lie on an edge of $\boldsymbol{C}$; of course, the opposite corner of R then lies on the opposite edge of $\boldsymbol{C}$.



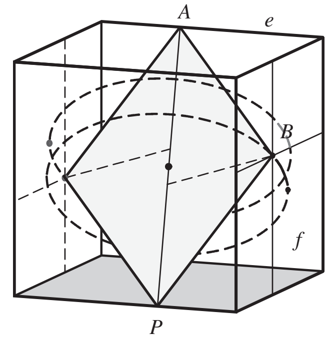
(a) Corners A and B on the same face. (b) Corners A and B on adjacent faces.

Figure 5. Corners A and B on open faces.

Suppose next that a corner A of the centered maximal rectangle R lies on an edge e of $\boldsymbol{C}$. If a corner B adjacent to A lies on an (open) face f of $\boldsymbol{C}$ adjacent to e (Figure 6a),



(a) Corner B on an open face adjacent to e.



(b) Corner B on an open face normal to e.

Figure 6. Corner A on an edge.

then a suitable small rotation of R about the line ℓ through O perpendicular to f moves both A and B into the (open) face f, contrary to the first part of this proof. If a corner B adjacent to A lies on an (open) face f that does not meet e, then the opposite corner Q is adjacent to A and lies on the face f' opposite the face f, which is adjacent to e. Again it follows that R is not maximal. Finally, if a corner A of R lies on an edge e and an adjacent corner B lies on a face f that meets e only at a vertex of $\boldsymbol{C}$ (Figure 6b), the corner P opposite A lies on the edge opposite e, and a suitable small rotation about the diagonal AP of R moves both B and Q into the interior of $\boldsymbol{C}$, once more contrary to Lemma 2.

It follows, as claimed, that all four corners of a centered maximal rectangle lie on edges of $\boldsymbol{C}$. ■

Lemma 4 (Fitting Lemma). *A centered maximal rectangle R whose corners are not at the vertices of C fits in C in exactly one of the following two ways:*

a. two adjacent corners of R lie on the same edge of C (Figure 8); or

b. two adjacent corners of R lie on adjacent edges of C (Figure 9).

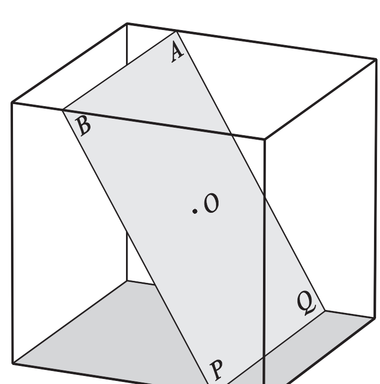


Figure 7. Rectangle is not maximal.

Proof. The four corners of R lie on a sphere centered at the center O of $\boldsymbol{C}$, and each lies at the same distance from the nearest vertex of $\boldsymbol{C}$. If (a) and (b) are both false, then the corners must also lie on four parallel edges of $\boldsymbol{C}$, one on each. It follows that there are two adjacent corners of R (labeled A and B in Figure 7) whose edge is parallel to an edge of $\boldsymbol{C}$. But then a small rotation about an axis through O parallel to AB would move all four vertices off the edges of $\boldsymbol{C}$ into the (open) faces, contrary to Lemma 3. ■

3. THE MAXIMAL RECTANGLES. According to Lemma 4, there are only two possibilities to examine.

Situation (a). Suppose that two adjacent corners of the maximal centered rectangle R lie on the edge e of $\boldsymbol{C}$ (Figure 8). Write a for the distance from A to the nearest vertex V of $\boldsymbol{C}$. Since $AB \le 1 \le AQ$, we see that $L = AQ = \sqrt{2}$ and $\lambda L = AB = 1 - 2a$. Since $0 \le a \le 1/2$, we find that when $0 \le \lambda \le 1/\sqrt{2}$

$$L = \sqrt{2}. \tag{1}$$

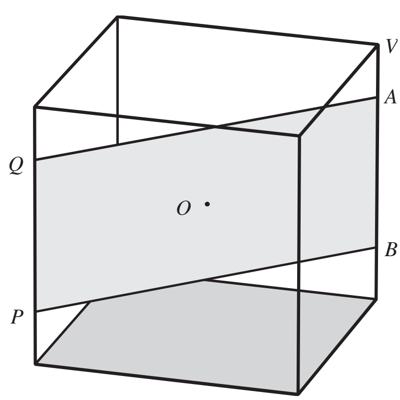


Figure 8. Situation (a).

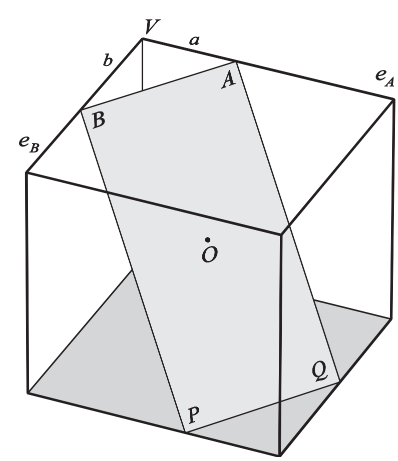


Figure 9. Situation (b).

Situation (b). Suppose that two adjacent corners A and B of a centered maximal rectangle R of aspect ratio λ are on the adjacent open edges e_A and e_B that share a vertex V (Figure 9). Write a for the distance VA and b for the distance VB. Then

$$AB = \sqrt{a^2 + b^2}, \quad BP = \sqrt{1 + (1-a)^2 + (1-b)^2}. \tag{2}$$

Since $OA = OB$, it is easy to see that b must be either a or $1 - a$. Further, AB could be the smaller dimension λL or the larger dimension L of R. Thus situation (b) splits into four apparent subcases, one of which proves to be impossible.

Case b_1: $b = a$ and $AB = \lambda L$. Setting $a = b$ and $\lambda = 1$ in (2), we find that $a = 3/4$ so that $L = 3\sqrt{2}/4$, which is Nieuwland's result. More generally, setting $a = b$, eliminating a, and writing $f_1(\lambda) = L$, we learn that when $0 \le \lambda \le 1$

$$L = \frac{3}{\sqrt{3-\lambda^2}+\sqrt{2}\lambda} = f_1(\lambda) \tag{3}$$

and

$$a = \frac{3}{\sqrt{2}} \frac{\lambda}{\sqrt{3-\lambda^2}+\sqrt{2}\lambda}. \tag{4}$$

Case b_2: $b = 1 - a$ and $AB = \lambda L$. Substituting in (2), eliminating a, requiring that $0 \le a \le 1$, and writing $f_2(\lambda) = L$, we see that for λ satisfying $1/\sqrt{3} \le \lambda \le 1/\sqrt{2}$

$$L = \frac{1}{\sqrt{1-\lambda^2}} = f_2(\lambda). \tag{5}$$

In this case,

$$a = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3\lambda^2-1}{1-\lambda^2}}.$$

Case b_3: $b = a$ and $AB = L$. Eliminating a in (2) leads to the equation

$$(1-\lambda^2)L^2 - 2\sqrt{2}L + 3 = 0.$$

Writing $L = f_3(\lambda)$, we discover that when $1/\sqrt{3} \le \lambda \le 1$

$$L = \frac{3}{\sqrt{2}+\sqrt{3\lambda^2-1}} = f_3(\lambda). \tag{6}$$

(The second root is too large when $\lambda < 1$.) Then

$$a = \frac{2-\sqrt{2}\sqrt{3\lambda^2-1}}{1-\lambda^2}, \tag{7}$$

and to ensure that $a \le 1$ we require additionally that $\lambda \ge 1/\sqrt{2}$. In other words, the appropriate domain for $f_3(\lambda)$ in (6) is described by $1/\sqrt{2} \le \lambda \le 1$.

The apparent possibility with $b = 1 - a$ and $AB = L$ leads to an equation for L having no real solutions, so this possibility can not be realized.

4. CONCLUSIONS. All that remains is to collect our results. We find the following answer to Question 1 of the introduction (see Figure 10).

Theorem 5. *Let $\lambda_1 = 1 - \sqrt{2}/2 \approx 0.29289$ and $\lambda_2 = 1 - \lambda_1 = \sqrt{2}/2 \approx 0.70711$. For each λ with $0 \le \lambda \le 1$, the longer side $L_{\max}(\lambda)$ of the largest rectangle $R_{\max}$ with aspect ratio λ that fits in the cube of side 1 is given as follows:*

$$L_{\max}(\lambda) = \begin{cases} \dfrac{3}{\sqrt{3-\lambda^2}+\sqrt{2}\lambda} & \text{if } 0 \le \lambda \le \lambda_1, \\ \sqrt{2} & \text{if } \lambda_1 \le \lambda \le \lambda_2, \\ \dfrac{3}{\sqrt{2}+\sqrt{3\lambda^2-1}} & \text{if } \lambda_2 \le \lambda \le 1. \end{cases} \tag{8}$$

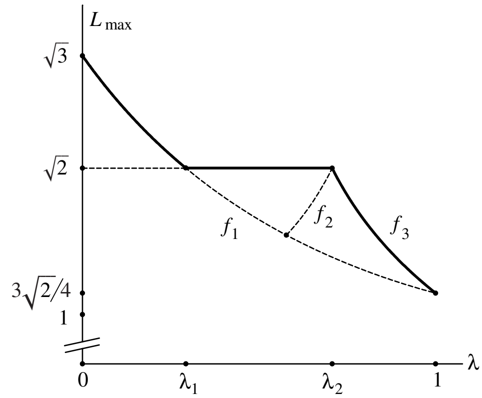
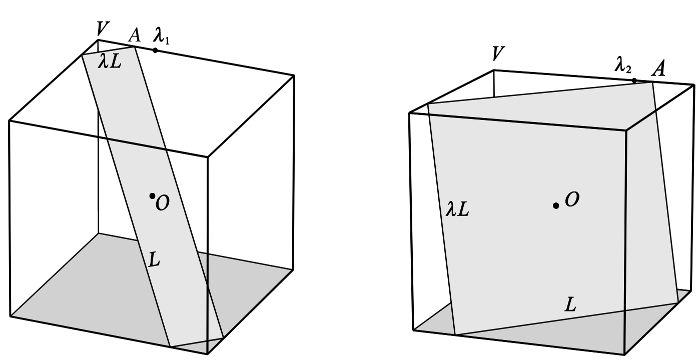


Figure 10. The longer side L_{max} in terms of the aspect ration λ.

Proof. When $0 \leq \lambda \leq \lambda_1$ the longer side $L_{max}(\lambda)$ is the larger of the number $f_1(\lambda)$, where f_1 is the function of case b_1, and $\sqrt{2}$ (from (1)). If $\lambda_1 \leq \lambda \leq \lambda_2$, $L_{max}(\lambda) = \sqrt{2}$, because $f_1(\lambda) \leq \sqrt{2}$ on the entire interval $[\lambda_1, \lambda_2]$ and $f_2(\lambda) \leq \sqrt{2}$ when $\sqrt{3}/3 \leq \lambda \leq \lambda_2$. Similarly $L_{max}(\lambda) = f_3(\lambda)$ for λ satisfying $\lambda_2 \leq \lambda \leq 1$. ■

Figure 11 shows the positions of the maximal rectangles. When $0 \leq \lambda \leq \lambda_1$, there are twelve maximal rectangles, all centered and located as shown in Figure 11a. Their longer dimension L and the distance $a = VA$ are given by formulas (3) and (4), respectively. If $\lambda_1 \leq \lambda \leq \lambda_2$, then $L = \sqrt{2}$, and there are six centered maximal rectangles and infinitely many that are not centered, all located on the diagonal planes of $\boldsymbol{C}$ as shown in Figure 8. If $\lambda_2 \leq \lambda \leq 1$ there are again twelve maximal rectangles, all centered and located as shown in Figure 11b, and their longer dimension L and the distance $a = VA$ are given by the formulas (6) and (7).



(a) The case $0 \leq \lambda \leq \lambda_1$. (b) The case $\lambda_2 \leq \lambda \leq 1$.

Figure 11. Maximal rectangles.

Theorem 5 gives a necessary and sufficient condition for a rectangle to fit in a unit cube: an $a \times b$ rectangle with $a \geq b$ fits into the unit cube $\boldsymbol{C}$ if and only if $a \leq L_{max}(b/a)$, where $L_{max}(\lambda)$ is given by (8).

The answer to Question 2 of the introduction follows immediately from this theorem.

Corollary 6. *Given σ and τ with $0 \le \tau \le \sigma \le 1$, a $D \times \sigma D \times \tau D$ box can pass through a suitable hole in a unit cube $\boldsymbol{C}$ if and only if*

$$D \le \frac{1}{\sigma} L_{\max}\left(\frac{\tau}{\sigma}\right).$$

Proof. The largest $L \times (\tau/\sigma)L$ rectangle that fits in $\boldsymbol{C}$ has $L = L_{\max}(\tau/\sigma)$, so the box can pass through the cube precisely when $\sigma D \le L$. ∎

5. FINAL REMARKS. Greg Huber, a physicist at the University of Massachusetts, and Terry J. Ligocki of Lawrence Berkeley National Laboratory have generated data for the diagonal $d_{\max}(\lambda)$ of $R_{\max}(\lambda)$ by modeling the problem with an iterative numerical procedure known as Boltzmann simulated annealing. Professor Huber emailed us as follows, "The basic idea comes from statistical mechanics. The size of the embedded rectangle is taken to be an "energy," and configurations (orientations of the rectangle) are perturbed and accepted stochastically with respect to a continuously changing "temperature," according to an annealing schedule. Asymptotic convergence to a global extremum is guaranteed under a few strong conditions on the acceptance function and scheduling, but in practice [we] chose heuristic functions." In March 2002 he forwarded to us two hundred computer-generated values of $d_{\max}(\lambda)$ for λ in [0, 1]. The graph of the corresponding values of $L_{\max}(\lambda) = d_{\max}/\sqrt{1+\lambda^2}$ matches closely the function described in Theorem 5 and sketched in bold in Figure 10.

In response to a problem proposed in this MONTHLY by Mauldon [**9**], Chapman [**3**] provided an algebraic answer for the question of finding the largest square that fits in the unit cube, thereby supplying a solution to this well-known problem of intuitive geometry that is "not overly dependent on geometric intuition," as the MONTHLY's 1995 Problem Editors remarked. An algebraic argument independent of the fitting lemma can probably be given for Theorem 5 along the same lines.

Finally, it is a pleasure to acknowledge the assistance of Jonathan P. Green of the UIUC Department of Germanic Languages and Literatures with John Wallis's Latin.

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A limerick

$e^{i\pi} + 1 = 0$
Made the mathematician Euler a hero.
From real to the complex,
With our brains in great flex,
He led us with much zest, but no fearo.

——Submitted by Warren C. Willig, California State University, Northridge