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# A theorem of the complement and some new $o$-minimal structures

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## 0. Introduction

The subject of $o$-minimality is a branch of model theory, but it has potential geometrical interest. Two excellent surveys now exist: [3] is intended for mathematical logicians while [5] is aimed at geometers. Indeed, the latter was written at the request of W. Schmid and K. Vilonen so that they could translate some recent model theoretic results ([13], [4]) into geometrical terms and thereby apply it in their paper [12]. Specifically, although they were ultimately only interested in objects defined over the subanalytic category of sets and maps, some of their constructions forced them into looking at sets of the form

$$Z^{\log}(P) := \{\langle x_1, \ldots, x_n\rangle \in \mathbb{R}^n : x_1 > 0, \ldots, x_n > 0, P(\log x_1, \ldots, \log x_n) = 0\}$$

where $P(X_1, \ldots, X_n)$ is a real polynomial. Clearly such sets are, in general, not subanalytic at the origin. However, it follows from [4] that one can extend the subanalytic category to include all sets of this form while preserving most of its hereditary, geometrical and finiteness properties. This then allowed Schmid and Vilonen to observe that all their arguments could be carried out in this larger category (which is where [5] came in) with the advantage now that the behaviour of the sets $Z^{\log}(P)$ at the origin introduced no special difficulties.

The extended category here (as well as the subanalytic category) is an example of a *geometric category*, for which precise axioms are set out in [5]. Throughout this paper, however, I prefer to work with the equivalent notion (see [5] section 3) of *o-minimal structure (over the real field)* which I now describe.

**Definitions.** A *pre-structure* is simply a sequence $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$. It is called a *structure (over the real field)* if, for all $n, m \geq 1$, the following four conditions are satisfied:

(S1) $\mathcal{S}_n$ is a boolean algebra (under the usual set-theoretic operations);
(S2) $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;
(S3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;
(S4) if $m \geq n$, $A \in \mathcal{S}_m$, then $\pi[A] \in \mathcal{S}_n$ where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates.

If, in addition, we have:

(S5) the boundary of every set in $\mathcal{S}_1$ is finite, then $\mathcal{S}$ is called an *o-minimal* structure (over the real field).

As an example we may take $\mathcal{S} = \mathbb{R}_{an}$ in which, by definition, $\mathcal{S}_n$ is the collection of *globally* subanalytic subsets of $\mathbb{R}^n$, i.e., those subanalytic subsets of $\mathbb{R}^n$ which are also subanalytic when considered as subsets of $\mathbb{P}^n(\mathbb{R})$ (under the identification $\langle x_1, \ldots, x_n \rangle \mapsto \langle 1 : x_1 : \cdots : x_n \rangle$). To see that (S1)–(S5) hold for $\mathbb{R}_{an}$ note that the compactness of $\mathbb{P}^n(\mathbb{R})$ for each $n$ guarantees that (S4) and (S5) are satisfied, and the fact that any semi-algebraic subset of $\mathbb{R}^n$ is also semi-algebraic "at infinity", hence also subanalytic there, implies that (S2) is satisfied. Further, one may easily deduce (S3) and the fact that each $\mathcal{S}_n$ is closed under taking unions and intersections from the corresponding (obvious) properties of the class of subanalytic sets. Finally, closure under complementation also follows from the same property of the class of subanalytic sets but this, of course, is a much deeper fact, namely Gabrielov's famous theorem ([6]; but see also [2] for a model-theoretic treatment).

The point of [5] is to show that much of the topological and geometric theory of the class of globally subanalytic sets can be established from properties (S1)–(S5) alone and hence may be inferred for arbitrary *o*-minimal structures. For example, if $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is such a structure, then "every finite collection of $\mathcal{S}$-sets can be compatibly $\mathcal{S}$-triangulated". More precisely, if $A_1, \ldots, A_m \in \mathcal{S}_n$, then there exists a (finite, not necessarily closed) simplicial complex $K$ in $\mathbb{R}^n$ and a homeomorphism $\Theta : |K| \to \bigcup_{i=1}^m A_i$ with $\Theta$ an $\mathcal{S}$-function (i.e., the graph of $\Theta$ is a set in $\mathcal{S}_{2n}$) and with each $\Theta^{-1}[A_i]$ a union of $K$-simplices.

An example of an *o*-minimal structure which is different from $\mathbb{R}_{an}$ is $\mathbb{R}_{\exp}$. Here $\mathcal{S}_n$ is defined to be the collection of those subsets of $\mathbb{R}^n$ of the form $\pi(f^{-1}(0))$ where, for some $m \geq n$, $f : \mathbb{R}^m \to \mathbb{R}$ is an *exponential polynomial* (i.e., $f(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m, e^{x_1}, \ldots, e^{x_m})$ for some real polynomial $Q(X_1, \ldots, X_{2m})$) and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates. Indeed, the fact that (S5) is satisfied follows from a theorem of Khovanskii (see [8]) which states that sets of the form $f^{-1}(0)$, with $f : \mathbb{R}^n \to \mathbb{R}$ a *Pfaffian* function[1], have only finitely many

[1] A $C^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ is called *Pfaffian* if there exist $C^1$ functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$, with $f = f_k$, such that for each $i, j$ with $1 \leq i \leq k$ and $1 \leq j \leq n$, $\frac{\partial f_i}{\partial x_j}$ is expressible as a polynomial in $x_1, \ldots, x_n, f_1, \ldots, f_i$. It is clear that exponential polynomials are Pfaffian.

connected components. Moreover, as with the case of $\mathbb{R}_{an}$, (S4) has been built into the definition of $\mathbb{R}_{\exp}$ and (S2) and (S3) are also easily verified. The major difficulty again comes with (S1), specifically closure under complementation, and I established this in [13].

It is, incidentally, an easy exercise to show that $\mathbb{R}_{\exp}$ includes all sets of the form $Z^{\log}(P)$. However, this alone was not quite sufficient for the application that Schmid and Vilonen had in mind. They required that the pre-structure, now known as $\mathbb{R}_{an,\exp}$, generated by $\mathbb{R}_{\exp}$ *and* $\mathbb{R}_{an}$ under the set operations implicit in (S1), (S3) and (S4) be an $o$-minimal structure — i.e., satisfy (S5). This is indeed the case, as was shown in [4]. It thus follows, for example, that any globally subanalytic set may be triangulated compatibly with any given finite collection of $Z^{\log}(P)$'s.

These introductory remarks are intended to suggest that it is of some interest to find reasonable conditions on a pre-structure which imply that the structure it generates is $o$-minimal. Although one might only be interested in applying $o$-minimal technology to sets in the pre-structure, this can only be done if they lie in some $o$-minimal structure. It is the aim of this paper to provide such a set of conditions (Theorem 1.8). As with the examples above it is the operation of complementation that causes most problems and hence the title of the paper. Rather than state the main result here, we conclude this section with an application of it.

In view of Khovanskii's theorem mentioned above, several authors have wondered whether one can generalize my result concerning $\mathbb{R}_{\exp}$ to the pre-structure $\mathbb{R}_{\text{Pfaff}}$ in which the defining functions $f$ are allowed to be arbitrary Pfaffian functions rather than just exponential polynomials. It is still not known, however, whether $\mathbb{R}_{\text{Pfaff}}$ is closed under complementation. The methods of this paper, unlike all previous methods for establishing $o$-minimality, allow us to bypass the difficulty and deduce the following:

**Theorem.** *The structure* generated *by* $\mathbb{R}_{\text{Pfaff}}$ *is o-minimal.*

## 1. The main theorem

The obvious way of generating a structure from a given pre-structure is awkward because the set-forming operations implicit in (S1) and (S4) can conspire, when iterated sufficiently many times, to produce unmanageably wild sets out of innocent looking pre-structures. So we first weaken the notion of structure.

**1.1. Definition.** A pre-structure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is called a *weak structure* if, for all $n, m \geq 1$, the following four conditions are satisfied:

(WS1) if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$;

(WS2) $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

(WS3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;
(WS4) if $A \in \mathcal{S}_n$, then $\sigma[A] \in \mathcal{S}_n$ where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection.

The reader may easily check that (WS4) follows from (S1)–(S4), so every structure is a weak structure.

In model-theoretic terminology a structure consists exactly of the definable (with parameters) subsets in some expansion of the ordered ring of real numbers, and conversely, whereas the collection of sets definable (with parameters) in such an expansion by *quantifier-free* formulas provides an example of a weak structure.

We now need to strengthen the *o*-minimality condition because (S5) has very little force in the absence of (S1) and (S4), even in the presence of (WS1)–(WS4).

**1.2. Definitions.**

(i) Suppose $n \geq 1$ and $A \subseteq \mathbb{R}^n$. Then $\gamma(A)$ denotes the smallest natural number $N$ with the following property: for any affine subspace $X$ of $\mathbb{R}^n$ we have $A \cap X = A_1 \cup \ldots \cup A_N$ for some *connected* subsets $A_1, \ldots, A_N$ of $\mathbb{R}^n$. If no such $N$ exists we write $\gamma(A) = \infty$.
(ii) A weak structure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is called *o-minimal* if, in addition to (WS1)–(WS4), it satisfies the following two conditions:

(WS5) for all $n \geq 1$ and $A \in \mathcal{S}_n$, $\gamma(A) < \infty$;
(WS6) for all $n \geq 1$ and $A \in \mathcal{S}_n$ there exists $m \geq n$ and a *closed* set $B \in \mathcal{S}_m$ such that $A = \pi[B]$ where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates.

There is no ambiguity in terminology here. It follows from results in the first papers in the subject, namely [11] and [9], that *o*-minimal structures — as defined in the previous section — satisfy (WS5) and (WS6).

Before stating the main result we should point out that the definitions above are due to Charbonnel ([1]). Indeed, a special case of our main result appears in Charbonnel's paper, but the proof contains many obscurities. However, much of the basic theory of *o*-minimal weak structures can be rescued from [1] and the details have been carefully presented, and the whole theory expanded by S. Maxwell in [10]. We shall present the crucial results in this and the next section leaving the reader to consult [10] for the proofs.

**1.3. Definitions** (Charbonnel)**.** Let $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ be a pre-structure. Then the pre-structures $\mathcal{S}^u, \mathcal{S}^{pr}$ and $\mathcal{S}^{cl}$ are defined as follows:

(i) $\mathcal{S}^u = \langle \mathcal{S}^u_n : n \geq 1 \rangle$ where $\mathcal{S}^u_n := \{\bigcup_{i=1}^p A_i : p \geq 1, A_1, \ldots, A_p \in \mathcal{S}_n\}$;
(ii) $\mathcal{S}^{pr} = \langle \mathcal{S}^{pr}_n : n \geq 1 \rangle$ where $\mathcal{S}^{pr}_n := \{\pi[A] : m \geq n, A \in \mathcal{S}_m$ and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates$\}$;
(iii) $\mathcal{S}^{cl} = \langle \mathcal{S}^{cl}_n : n \geq 1 \rangle$ where $\mathcal{S}^{cl}_n := \big\{A_0 \cap \bigcap_{i=1}^p \bar{A}_i : p \geq 0, A_0, A_1, \ldots, A_p \in \mathcal{S}_n\big\}$.

(Here, and henceforth, for $n \geq 1$ and $S \subseteq \mathbb{R}^n$, $\bar{S}$ denotes the closure of $S$ in $\mathbb{R}^n$.)

**1.4. Lemma** (Charbonnel)**.** *If $\mathcal{S}$ is a weak structure, then so are $\mathcal{S}^u, \mathcal{S}^{pr}$ and $\mathcal{S}^{cl}$. If, further, $\mathcal{S}$ is o-minimal, then $\mathcal{S}^u, \mathcal{S}^{pr}$ and $\mathcal{S}^{cl}$ are too.*

Of the eighteen statements making up this lemma only the proof that $\mathcal{S}^{cl}$ satisfies (WS5) (if $\mathcal{S}$ is $o$-minimal) presents any difficulty. One first proves the following result which I shall also need later.

**1.5. Lemma** (Charbonnel)**.** *If $\mathcal{S}$ is a weak structure satisfying* (WS6)*, $n \geq 1$ and $A \in \mathcal{S}_n^{cl}$, then there exist $m \geq n$, $B \in \mathcal{S}_m$ and an affine subspace $Z$ of $\mathbb{R}^m$ such that $A = \pi(\bar{B} \cap Z)$, where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates.*

Before proceeding further it will be useful to introduce some abbreviations. Suppose that $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is a pre-structure and $A \subseteq \mathbb{R}^n$ for some $n$. Then we often write $A \in \mathcal{S}$ for $A \in \mathcal{S}_n$. Thus if $\mathcal{S}' = \langle \mathcal{S}'_n : n \geq 1 \rangle$ is another pre-structure, then $\mathcal{S} \subseteq \mathcal{S}'$ means $\mathcal{S}_n \subseteq \mathcal{S}'_n$ for all $n \geq 1$, and similarly for other set-theoretic terminology. Also, if $f$ is a function, then $f \in \mathcal{S}$ means that for some $n, m \geq 1$, $\mathrm{dom}(f) \subseteq \mathbb{R}^n$, $\mathrm{ran}(f) \subseteq \mathbb{R}^m$ and $\mathrm{graph}(f) \in \mathcal{S}_{n+m}$.

Clearly $\mathcal{S} \subseteq \mathcal{S}^u$, $\mathcal{S} \subseteq \mathcal{S}^{pr}$ and $\mathcal{S} \subseteq \mathcal{S}^{cl}$.

**1.6. Definition** (Charbonnel)**.** For $\mathcal{S}$ a pre-structure, $\mathcal{S}^{(o)} := \mathcal{S}$, $\mathcal{S}^{(i+1)} := ((\mathcal{S}^{(i)u})^{pr})^{cl}$ (for $i \geq 0$), and $\tilde{\mathcal{S}} := \cup_{i \geq 0} \mathcal{S}^{(i)}$.

Thus if $\mathcal{S}$ is an $o$-minimal, weak structure, then, using 1.4, $\tilde{\mathcal{S}}$ is the smallest weak structure which both contains $\mathcal{S}$ and is closed under taking finite unions, projections and closures. Moreover, $\tilde{\mathcal{S}}$ is $o$-minimal. It is not hard to show that $\tilde{\mathcal{S}}$ is still contained within the *structure* generated by $\mathcal{S}$. Clearly, if $\tilde{\mathcal{S}}$ were closed under taking complements, then $\tilde{\mathcal{S}}$ would actually be the structure generated by $\mathcal{S}$. We shall prove such a theorem of the complement under an extra hypothesis on $\mathcal{S}$ (essentially, a strengthening of (WS6)) and leave the general case as an open problem which we shall discuss briefly in the final section of this paper.

**1.7. Definition.** A pre-structure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is *determined by its smooth functions* (DSF) if, for each $n \geq 1$ and $A \in \mathcal{S}_n$, there exists $m \geq n$ and a $C^\infty$ function $f : \mathbb{R}^m \to \mathbb{R}$ with $f \in \mathcal{S}$ such that $A = \pi[Z(f)]$ where $Z(f) = \{\bar{x} \in \mathbb{R}^m : f(\bar{x}) = 0\}$ and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first $n$ coordinates.

**1.8. The main theorem.** *Suppose that $\mathcal{S}$ is an o-minimal weak structure. Suppose further that $\mathcal{S}$ is determined by its smooth functions. Then $\tilde{\mathcal{S}}$ is closed under complementation and hence is the smallest structure containing $\mathcal{S}$. It is, moreover, o-minimal.*

As a consequence we have the following model-theoretic result:

**1.9. Theorem.** *Let $\mathcal{M}$ be any expansion of the ordered ring of real numbers by $C^\infty$ functions. Suppose that for each $n \geq 1$ every quantifier-free, $\mathcal{M}$-definable (with parameters) subset $A$ of $\mathbb{R}^n$ satisfies $\gamma(A) < \infty$. Then $\mathcal{M}$ is o-minimal (in* *the usual model-theoretic sense that every $\mathcal{M}$-definable (with parameters) subset of $\mathbb{R}$ has finite boundary).*

*Proof.* Let $\mathcal{S}_n$ be the collection of subsets of $\mathbb{R}^n$ definable in $\mathcal{M}$ by a quantifier-free formula (with parameters). Set $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$. It is clearly sufficient to show that $\mathcal{S}$ satisfies the hypotheses of 1.8. But (WS1)–(WS4) are obvious and (WS5) is one of our hypotheses here. To see that $\mathcal{S}$ is DSF (and hence also satisfies (WS6)) an example will surely suffice.

Suppose that $A = \{\langle x_1, \ldots, x_n \rangle = \bar{x} \in \mathbb{R}^n : (e(\bar{x}) > 0 \wedge f(\bar{x}) = 0) \vee g(\bar{x}) > h(\bar{x})\}$ where $e, f, g, h : \mathbb{R}^n \to \mathbb{R}$ are $C^\infty$-functions given by terms of the language of $\mathcal{M}$. Then $A = \pi(Z(F))$ where $F(\bar{x}, x_{n+1}, x_{n+2}) := ((e(\bar{x}) \cdot x_{n+1}^2 - 1)^2 + f(\bar{x})^2) \cdot ((g(\bar{x}) - h(\bar{x})) \cdot x_{n+2}^2 - 1)$ (so $F \in \mathcal{S}$) and $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is projection onto the first $n$ coordinates. $\square$

Note that 1.9 and Khovanskii's theorem [8] imply the theorem stated at the end of the previous section.

## 2. Some properties of *o*-minimal weak structures

Let us fix an *o*-minimal weak structure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ which we do not need to be DSF for the results of this section. Let $\tilde{\mathcal{S}}$ ($= \langle \tilde{\mathcal{S}}_n : n \geq 1 \rangle$, say) be as defined in 1.6, so that $\tilde{\mathcal{S}}$ is also an *o*-minimal weak structure (by 1.4) which is also closed under taking finite unions, projections and closures.

It is an immediate consequence of (WS6) (for $\tilde{\mathcal{S}}$) that every set in $\tilde{\mathcal{S}}$ is Lebesgue measurable and Charbonnel's key idea was to use an argument based on Fubini's theorem, together with induction on $n$, to establish properties of sets in $\tilde{\mathcal{S}}_n$. For example:

**2.1. Theorem** (Charbonnel)**.** *Suppose $n \geq 1$ and $A \in \tilde{\mathcal{S}}_n$. Then the following are equivalent:*

(i) *$A$ has no interior points;*
(ii) *$A$ has measure zero (for Lebesgue measure on $\mathbb{R}^n$);*
(iii) *$\bar{A}$ has no interior points;*
(iv) *$\bar{A}$ has measure zero.*

Implicit in the proof of 2.1 is the following useful result (which can also be deduced easily from 2.1). $\mathbb{R}_+$ is used to denote the set of positive real numbers.

**2.2. Theorem** (Charbonnel)**.** *Let $n \geq 1$ and suppose that $A \in \tilde{\mathcal{S}}_{n+1}$ and $A \subseteq \mathbb{R}^n \times \mathbb{R}_+$. Let $B \subseteq \mathbb{R}^n$ be defined by the condition $\bar{x} \in B \Leftrightarrow \langle \bar{x}, 0 \rangle \in \bar{A}$ ($\forall \bar{x} \in \mathbb{R}^n$). Then (clearly) $B \in \tilde{\mathcal{S}}_n$ and if $A$ contains no interior points then nor does $B$.*

In [10] several properties of functions in $\tilde{\mathcal{S}}$ are established:

**2.3. Theorem** (Weak selection)**.** *Suppose $n, m \geq 1$, $A \in \tilde{\mathcal{S}}_n$, $B \in \tilde{\mathcal{S}}_{n+m}$ and that $A$ contains an interior point. Suppose further that $\forall \bar{x} \in A\ \exists \bar{y} \in \mathbb{R}^m\ \langle \bar{x}, \bar{y} \rangle \in B$. Then there exists an open set $U \in \tilde{\mathcal{S}}_n$, $U \subseteq A$, and a function $\phi : U \to \mathbb{R}^m$ with $\phi \in \tilde{\mathcal{S}}$ such that $\forall \bar{x} \in U\ \langle \bar{x}, \phi(\bar{x}) \rangle \in B$.*

**2.4. Theorem** (Almost everywhere smoothness of functions)**.** *Suppose $n, m \geq 1$, $N \geq 0$, $U \in \tilde{\mathcal{S}}_n$, $U$ open, and $F : U \to \mathbb{R}^m$ is a function in $\tilde{\mathcal{S}}$. Then there exists a closed set $A \in \tilde{\mathcal{S}}_n$ containing no interior points such that $F$ is $C^N$ on $U \backslash A$.*

**2.5. Remark.** Note that in 2.4 we certainly cannot conclude at present that $U \backslash A \in \tilde{\mathcal{S}}_n$. However, $U \backslash A$ contains open balls which certainly do lie in $\tilde{\mathcal{S}}_n$ (by (WS2) for $\tilde{\mathcal{S}}$). Hence we may strengthen the conclusion of 2.3 by stipulating that $\phi$ is a $C^N$ function for any pre-given $N$.

**2.6. Theorem** (Closure under differentiation)**.** *Suppose $n \geq 1$, $U \in \tilde{\mathcal{S}}_n$ $U$ open, and $F : U \to \mathbb{R}$ is a $C^1$ function that is in $\tilde{\mathcal{S}}$. Then the partial derivatives of $F$ (considered as function from $U$ to $\mathbb{R}$) are also in $\tilde{\mathcal{S}}$.*

**Remark.** One does need $F$ to be $C^1$ in 2.6. Differentiability alone does not seem to be enough.

One consequence of 2.5 and 2.6 is that the conclusion of Sard's theorem holds for $C^1$ functions in $\tilde{\mathcal{S}}$. (Sard's theorem does not apply to $C^1$ functions in general.) Suppose $n \geq m \geq 1$ and that $F : U \to \mathbb{R}^m$ is a $C^1$ function in $\tilde{\mathcal{S}}$, where $U$ is an open subset of $\mathbb{R}^n$ (necessarily in $\tilde{\mathcal{S}}$). We can define the set of singular values of $F$, $\mathrm{Sing}(F)$, as follows:

for $\bar{y} \in \mathbb{R}^m$,

$$\bar{y} \in \mathrm{Sing}(F) \quad \text{iff} \quad \exists \bar{x} \in U, \left[ F(\bar{x}) = \bar{y} \wedge \bigwedge_{\bar{k}} \det \left( \frac{\partial(F_1, \ldots, F_m)}{\partial(x_{k_1}, \ldots, x_{k_m})} \right)(\bar{x}) = 0 \right],$$

where $F = \langle F_1, \ldots, F_m \rangle$, the conjunction is over all increasing $m$-tuples $\bar{k} = \langle k_1, \ldots, k_m \rangle$ from $\{1, \ldots, n\}$, and $\frac{\partial(F_1,\ldots,F_m)}{\partial(x_{k_1},\ldots,x_{k_m})}$ denotes the Jacobian matrix $\left( \frac{\partial F_i}{\partial x_{k_j}} \right)_{1 \leq i,j \leq m}$. Clearly 2.6 (together with the closure properties of $\tilde{\mathcal{S}}$) implies that $\mathrm{Sing}(F) \in \tilde{\mathcal{S}}$.

Now if $\mathrm{Sing}(F)$ contained an open ball, $D$ say, (which, by 2.1, is equivalent to $\mathrm{Sing}(F)$ being non-null) then, by 2.3 and 2.5, there would exist a $C^1$ function $\phi : D' \to \mathbb{R}^n$ (for some open ball $D' \subseteq D$) such that $F(\phi(\bar{y})) = \bar{y}$ and $\bar{y} \in \mathrm{Sing}(F)$ for all $\bar{y} \in D'$. However, by differentiating at any particular point $\bar{a} \in D'$, we obtain that $d_{\phi(\bar{a})} F \circ d_{\bar{a}} \phi$ is the identity linear map on $\mathbb{R}^m$ (considered as the tangent space of $\mathbb{R}^m$ at $\bar{a}$). So $d_{\phi(\bar{a})} F : \mathbb{R}^n \to \mathbb{R}^m$ is certainly surjective, which means that $F(\phi(\bar{a})) \notin \mathrm{Sing}(F)$. But $F(\phi(\bar{a})) = \bar{a} \in \mathrm{Sing}(F)$-contradiction. Thus we have shown

**2.7. Theorem.** *Suppose that $n \geq m \geq 1$ and that $F : U \to \mathbb{R}^m$ is a $C^1$ function in $\tilde{\mathcal{S}}$, where $U$ is an open subset of $\mathbb{R}^n$. Then the subset $\mathrm{Sing}(F)$ of $\mathbb{R}^m$, consisting of the singular values of $F$, is a set in $\tilde{\mathcal{S}}$ containing no interior points.*

We also have the following version of 2.7 for functions defined on $\tilde{\mathcal{S}}$-manifolds.

**2.8. Theorem.** *Suppose that $n > m \geq 1$, $\bar{a} \in \mathbb{R}^m$, $F : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}$. Suppose further that $F$ and $f$ are both $C^1$ functions in $\tilde{\mathcal{S}}$ and that $\bar{a}$ is a regular value of $F$. Then there are at most finitely many $b \in \mathbb{R}$ such that $\langle \bar{a}, b\rangle$ is a singular value of the function $\langle F, f\rangle : \mathbb{R}^n \to \mathbb{R}^{m+1}$.*

*Proof.* Say $F = \langle F_1, \ldots, F_m\rangle$. Then, for $b \in \mathbb{R}$, $\langle \bar{a}, b\rangle$ is a singular value of $\langle F, f\rangle$ iff $\exists \bar{x} \in \mathbb{R}^n$ such that $F(\bar{x}) = \bar{a}$, $f(\bar{x}) = b$ and $d_{\bar{x}}F_1, \ldots, d_{\bar{x}}F_m, d_{\bar{x}}f$ are linearly dependent. This clearly shows (by using 2.6, as above) that the set of $b \in \mathbb{R}$ for which $\langle \bar{a}, b\rangle$ is such a singular value is a set in $\tilde{\mathcal{S}}$ and hence, if infinite, would contain an open interval (by (WS5) for $\tilde{\mathcal{S}}$). It follows from 2.5 and 2.3 that, in such a case, there would exist a $C^1$ function $\phi : (\alpha, \beta) \to \mathbb{R}^n$ (for some $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$) such that (a) $F(\phi(t)) = \bar{a}$, $f(\phi(t)) = t$ and (b) $d_{\phi(t)}F_1, \ldots, d_{\phi(t))}F_m$, $d_{\phi(t)}f$ are linearly dependent, for each $t \in (\alpha, \beta)$. However, differentiation of the two equalities in (a) gives, for each $t \in (\alpha, \beta)$, $d_{\phi(t)}F_i \circ d_t\phi = 0$ for $1 \leq i \leq m$ and $d_{\phi(t)}f \circ d_t\phi = 1$. But this and (b) imply that $d_{\phi(t)}F_1, \ldots, d_{\phi(t)}F_m$ are already linearly dependent — contradicting the assumption that $\bar{a}$ $(= F(\phi(t))$ for any $t \in (\alpha, \beta))$ is a regular value of $F$. □

**2.9. Corollary.** *Suppose that $n, k \geq 1$ and that $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a $C^1$ function in $\tilde{\mathcal{S}}$. Suppose further that $\bar{a} \in \mathbb{R}^k$, that $\bar{a}$ is a regular value of $F$, and that $U$ is an open ball in $\mathbb{R}^n$ with the property that the set $F^{-1}(\bar{a}) \cap (U \times \mathbb{R}^k)$, which we denote by $X$, is non-empty and bounded. Then either (i) $\pi[X] = U$, where $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is the projection map onto the first $n$ coordinates, or (ii) there exists $\eta > 0$ and distinct $i_1, \ldots, i_k \in \mathbb{N}$ with $1 \leq i_1, \ldots, i_k \leq n+k$ such that $\det\left(\frac{\partial(F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_k})}\right)^2 \restriction X$ takes all values in the interval $[0, \eta]$, where $\langle F_1, \ldots, F_k\rangle = F$.*

*Proof.*

**Case 1.** $\pi[X]$ is finite.

Let $\bar{b} \in \pi[X]$. Choose $\bar{c} \in \mathbb{R}^k$ such that $\langle \bar{b}, \bar{c}\rangle \in X$ and let $Y$ be the connected component of the manifold $F^{-1}(\bar{a})$ containing the point $\langle \bar{b}, \bar{c}\rangle$. The fact that $\pi[X]$ is finite clearly implies that $\pi[Y] = \{\bar{b}\}$, so $Y \subseteq X$ and hence, in particular, $Y$ is bounded and closed.

Now since $\bar{a}$ is a regular value of $F$ and $F(\langle \bar{b}, \bar{c}\rangle) = \bar{a}$, it follows that $\det\left(\frac{\partial(F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_k})}\right)(\langle \bar{b}, \bar{c}\rangle) \neq 0$ for some distinct $i_1, \ldots, i_k \in \mathbb{N}$ with $1 \leq i_1, \ldots, i_k \leq n+k$. But the implicit function theorem and the fact that $Y$ is bounded clearly imply that $\det\left(\frac{\partial(F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_k})}\right)(\bar{z}) = 0$ for some $\bar{z} \in Y$. The conclusion (ii) now follows from the connectivity of $Y$.

**Case 2.** $\pi[X]$ is infinite.

Then so is $\pi_i \circ \pi[X]$ for some $i = 1, \dots, n$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection map onto the $i$th coordinate. Applying 2.8 (with $n = n + k$, $m = k$, $f = \pi_i \circ \pi$) choose $b \in \pi_i[U]$ such that $\langle \bar{a}, b \rangle$ is a regular value of the function $\langle F, \pi_i \circ \pi \rangle : \mathbb{R}^{n+k} \to \mathbb{R}^{k+1}$. This easily implies that $\bar{a}$ is a regular value of the function $\hat{F} : \mathbb{R}^{n-1+k} \to \mathbb{R}^k : \langle x_1, \dots, x_{n-1+k} \rangle \mapsto F(x_1, \dots, x_{i-1}, b, x_i, \dots, x_{n-1+k})$. Let $\hat{U} = \{\langle x_1, \dots, x_{n-1} \rangle \in \mathbb{R}^{n-1} : \langle x_1, \dots, x_{i-1}, b, x_i, \dots, x_{n-1} \rangle \in U\}$ (so that $\hat{U}$ is an open ball in $\mathbb{R}^{n-1}$), $\hat{X} = \hat{F}^{-1}(\bar{a}) \cap (\hat{U} \times \mathbb{R})$, and $\hat{\pi} : \mathbb{R}^{\pi-1+k} \to \mathbb{R}^{n-1}$ be the projection map onto the first $n-1$ coordinates.

Now if $\hat{\pi}[\hat{X}]$ is finite (and $n - 1 \geq 1$) we may apply the case 1 argument to $\hat{F}, \hat{U}$ and $\hat{X}$ to obtain the conclusion (ii) for the function $\hat{F}$, which clearly implies the same conclusion for $F$. Otherwise, we apply case 2 to $\hat{F}, \hat{U}, \hat{X}$.

We continue in this way until either conclusion (ii) is reached, or else we have found $\bar{b} \in U$ such that $\langle \bar{b}, \bar{c} \rangle \in X$ and $\det \left( \frac{\partial(F_1,\dots,F_k)}{\partial(x_{n+1},\dots,x_{n+k})} \right) (\langle \bar{b}, \bar{c} \rangle) \neq 0$ for some $\bar{c} \in \mathbb{R}^k$.

Now let $Y$ be the connected component of $X$ containing the point $\langle \bar{b}, \bar{c} \rangle$. Then, by the boundedness of $Y$ and the implicit function theorem (and the fact $Y$ is closed in $U \times \mathbb{R}^k$), it follows that *either* $\det \left( \frac{\partial(F_1,\dots,F_k)}{\partial(x_{n+1},\dots,x_{n+k})} \right) (\bar{z}) = 0$ for some $\bar{z} \in Y$ (and conclusion (ii) follows) *or* $\pi[Y] = U$, from which conclusion (i) follows. $\square$

## 3. Defining boundaries via smooth approximations

My aim in this section is to prove the following result.

**3.1. Theorem.** *Suppose that $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is an o-minimal weak structure. Suppose further that $\mathcal{S}$ is DSF. Then for any $n \geq 1$ and closed set $A \in \tilde{\mathcal{S}}_n$, there exists a closed set $B \in \tilde{\mathcal{S}}_n$ such that $B$ has empty interior and $\partial A \subseteq B$, where $\partial A$ denotes the boundary of $A$.*

Readers familiar with cell decomposition arguments will see that the main theorem (1.8) follows fairly routinely from 3.1. The details of this deduction are carried out in the next section.

The set $B$ in 3.1 will be constructed as a "limit" of smooth submanifolds of $\mathbb{R}^n$. In order to describe the limiting process we require the following notions.

**3.2. Definitions.**

(i) By a *k-modulus* (for $k \in \mathbb{N}$) we mean a sequence $\bar{\mu} = \langle \mu_0, \dots, \mu_k \rangle$ where $\mu_0 \in \mathbb{R}_+$ and $\mu_i : \mathbb{R}_+^i \to \mathbb{R}_+$ for $i = 1, \dots, k$.

(ii) A point $\bar{\varepsilon} = \langle \varepsilon_0, \dots, \varepsilon_k \rangle \in \mathbb{R}_+^{k+1}$ is called $\bar{\mu}$-*bounded*, where $\bar{\mu}$ is a $k$-modulus as in (i), if $\varepsilon_0 < \mu_0$ and $\varepsilon_i < \mu_i(\varepsilon_0, \dots, \varepsilon_{i-1})$ for $i = 1, \dots, k$.

(iii) Suppose $n \geq 1$, $k \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$, $S \subseteq \mathbb{R}^{n+k}$ and that $\bar{\mu}$ is a $k$-modulus. Then we say that

  (a) *$S$ approximates $A$ from below* (mod $\bar{\mu}$), written $S \leq A$ (mod $\bar{\mu}$), if for all $\bar{\mu}$-bounded points $\bar{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_k \rangle \in \mathbb{R}_+^{k+1}$ and all $\bar{x} \in \mathbb{R}^n$, if $\langle \bar{x}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in S$, then there exists $\bar{y} \in A$ with $\|\bar{x} - \bar{y}\| < \varepsilon_0$;

  (b) *$S$ approximates $A$ from above on bounded sets* (mod $\bar{\mu}$), written $A \leq S$ (mod $\bar{\mu}$), if for every $\bar{\mu}$-bounded point $\bar{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_k \rangle \in \mathbb{R}_+^{k+1}$ and for each $\bar{x} \in A$ with $\|\bar{x}\| < \varepsilon_0^{-1}$, there exists $\bar{y} \in \mathbb{R}^n$ with $\|\bar{x} - \bar{y}\| < \varepsilon_0$ and $\langle \bar{y}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in S$.

I should remark that here, and henceforth, we use the norm $\|\bar{x}\|_n := \max\{|x_i| : 1 \leq i \leq n\}$ (for $n \geq 1$, $\bar{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$). I drop the subscript $n$ as it will always be clear from the context.

Let us now fix an $o$-minimal weak structure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ (although, as in section 2, we work mostly with the $o$-minimal weak structure $\tilde{\mathcal{S}}$) which we need not assume to be DSF for the moment.

As usual, $\varepsilon$ (with or without subscripts, etc.) ranges over $\mathbb{R}_+$ if not explicitly stated.

**3.3. Lemma.** *Suppose that $n \geq 1$, $k \in \mathbb{N}$, $A \in \tilde{\mathcal{S}}_n$, $S \in \tilde{\mathcal{S}}_{n+k}$ and that $S$ has empty interior. Suppose further that $\bar{\mu}$ is a $k$-modulus (which, both here and subsequently, is definitely not assumed to be in $\tilde{\mathcal{S}}$) such that $\partial A \leq S$ (*mod $\bar{\mu}$*). Then there exists a closed set $B \in \tilde{\mathcal{S}}_n$ with empty interior such that $\partial A \subseteq B$.*

*Proof.* Fix $n, A$. We use induction on $k$. For $k = 0$ we may take $B = \bar{S}$ (by 2.1).

Suppose the lemma is true for $k$ (for all $S, \bar{\mu}$). Let $\bar{\mu}' = \langle \mu_1', \ldots, \mu_{k+1}' \rangle$ be a $(k+1)$-modulus, let $S \in \tilde{\mathcal{S}}_{n+(k+1)}$ with empty interior, and suppose that $\partial A \leq S$ (mod $\bar{\mu}'$). Define the subset $T$ of $\mathbb{R}^n \times \mathbb{R}_+^k$ by $\langle \bar{x}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in T$ iff $\langle \bar{x}, \varepsilon_1, \ldots, \varepsilon_k, 0 \rangle \in \bar{S}$ (for $\bar{x} \in \mathbb{R}^n$, $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R}_+$). Then, by 2.2, $T \in \tilde{\mathcal{S}}_{n+k}$ and $T$ has empty interior.

Define $\mu_0 := \mu_0'$ and $\mu_i(\varepsilon_0, \ldots, \varepsilon_{i-1}) := \mu_i'(\varepsilon_0/2, \varepsilon_1, \ldots, \varepsilon_{i-1})$ for $i = 1, \ldots, k$, $\varepsilon_0, \ldots, \varepsilon_{k-1} \in \mathbb{R}^+$. Let $\bar{\mu} = \langle \mu_0, \ldots, \mu_k \rangle$. Clearly $\bar{\mu}$ is a $k$-modulus so, by the inductive hypothesis, we shall be done if we show that $\partial A \leq T$ (mod $\bar{\mu}$).

To see this let $\bar{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_k \rangle$ be $\bar{\mu}$-bounded and suppose that $\bar{x} \in \partial A$ and that $\|\bar{x}\| < \varepsilon_0^{-1}$. Now let $\varepsilon$ be an arbitrary positive real.

Define $\eta_\varepsilon := \frac{1}{2} \min\{\varepsilon, \mu_{k+1}'(\varepsilon_0/2, \varepsilon_1, \ldots, \varepsilon_k)\}$. Now the fact that $\bar{\varepsilon}$ is $\bar{\mu}$-bounded immediately implies that $\langle \varepsilon_0/2, \varepsilon_1, \ldots, \varepsilon_k, \eta_\varepsilon \rangle$ is $\bar{\mu}'$-bounded. Also $\|\bar{x}\| < (\varepsilon_0/2)^{-1}$. Hence, since $\partial A \leq S$ (mod $\bar{\mu}'$), there exists $\bar{y}_\varepsilon \in \mathbb{R}^n$ with $\|\bar{x} - \bar{y}_\varepsilon\| < \varepsilon_0/2$ such that $\langle \bar{y}_\varepsilon, \varepsilon_1, \ldots, \varepsilon_k, \eta_\varepsilon \rangle \in S$. Since this holds for all $\varepsilon > 0$ and $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, there exists $\bar{y} \in \mathbb{R}^n$ with $\|\bar{x} - \bar{y}\| \leq \varepsilon_0/2$ such that $\langle \bar{y}, \varepsilon_1, \ldots, \varepsilon_k, 0 \rangle \in \bar{S}$. It follows that $\|\bar{x} - \bar{y}\| < \varepsilon_0$ and $\langle \bar{y}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in T$ as required. □

Thus, in order to establish 3.1 it is sufficient to find, for each $n \geq 1$ and closed $A \in \tilde{\mathcal{S}}_n$, a triple $k, S, \bar{\mu}$ satisfying the hypotheses of 3.3. It turns out that this can be achieved by induction on the level of $A$ in the hierarchy described in 1.6, whereas the statement of 3.1 itself does not seem amenable to this line of attack. It soon becomes apparent, however, that the induction would falter rather quickly unless we also incorporate the condition that $S \leq A \pmod{\bar{\mu}}$ as well as $\partial A \leq S \pmod{\bar{\mu}}$. Before proceeding with the details we require some results on moduli.

**3.4. Lemma.** *Let $k \geq 1$, $A \in \tilde{S}_k$ and suppose that $A$ contains no interior points. Then there exists a $k$-modulus $\bar{\mu}$ such that whenever $\langle \varepsilon_0, \ldots, \varepsilon_k \rangle$ is a $\bar{\mu}$-bounded point, $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle \notin A$.*

*Proof.* By induction on $k$. If $k = 1$, then $A$ is finite (by (WS5) for $\tilde{\mathcal{S}}$) so we just take $\mu_0$ to be the least positive element of $A$ (or 1, say, if no such element exists), and $\mu_1(\varepsilon_0) = \varepsilon_0$ for $\varepsilon_0 \in \mathbb{R}_+$.

Suppose the lemma is true for some $k \geq 1$ and that $A \in \tilde{\mathcal{S}}_{k+1}$ and that $A$ has empty interior. For $\bar{a} \in \mathbb{R}^k$ define $A_{\bar{a}} := \{x \in \mathbb{R} : \langle \bar{a}, x \rangle \in A\}$. By (WS5) (for $\tilde{\mathcal{S}}$) there exists $N > 0$ such that for all $\bar{a} \in \mathbb{R}^k$, if $A_{\bar{a}}$ contains at least $N$ elements then $A_{\bar{a}}$ contains an open interval and hence has non-zero measure.

Define

$$B := \left\{ \bar{a} \in \mathbb{R}^k : \exists x_1 \ldots \exists x_N \left( \bigwedge_{1 \leq i < j \leq N} x_i < x_j \wedge \bigwedge_{1 \leq i \leq N} \langle \bar{a}, x_i \rangle \in A \right) \right\}.$$

Clearly $B \in \tilde{\mathcal{S}}_k$ and, by Fubini's theorem, $B$ must have measure zero (otherwise $A$ would have non-zero measure, contradicting 2.1 and the fact that $A$ has empty interior). Hence, by 2.1, $B$ contains no interior points so, by the inductive hypothesis, there is a $k$-modulus $\bar{\mu}$ such that whenever $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ is $\bar{\mu}$-bounded, then $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle \notin B$ which in turn implies that $A_{\langle \varepsilon_1, \ldots, \varepsilon_k \rangle}$ is finite. Let $\mu_{k+1}(\varepsilon_0, \ldots, \varepsilon_k)$ be the least positive element of $A_{\langle \varepsilon_1, \ldots, \varepsilon_k \rangle}$ (or 1, say, if no such exists). Then $\langle \bar{\mu}, \mu_{k+1} \rangle$ is a $(k+1)$-modulus which clearly has the required property. $\square$

We shall be interested in subsets of $\mathbb{R}^n \times \mathbb{R}_+^k$ (for $n, k \geq 1$) defined by conditions on $\langle x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_k \rangle$ of the form

**3.5.**

$$\exists x_{n+1} \ldots \exists x_{n+k-1} \bigwedge_{i=1}^{k} f_i(x_1, \ldots, x_{n+k-1}) = \varepsilon_i,$$

where $f_1, \ldots, f_k$ are $C^\infty$ functions from $\mathbb{R}^{n+k-1}$ to $\mathbb{R}$ which lie in $\tilde{\mathcal{S}}$.

Clearly this defines a set in $\tilde{\mathcal{S}}_{n+k}$ and it is easy to see that it has no interior points. The proof of this latter assertion will be described at the end of this section.

We aim to show that if $\mathcal{S}$ is determined by its smooth functions, then for each $n \geq 1$ and $A \in \tilde{\mathcal{S}}_n$:

**3.6.** There exists $k \geq 1$ (the *complexity* of $A$), a $k$-modulus $\bar{\mu}$ (the *modulus* of $A$) and a set $S \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$ (the *approximation* of $A$), which is a finite union of sets defined by conditions of the form 3.5 (the *approximating constituents* of $A$) such that $\partial \bar{A} \leq S \pmod{\bar{\mu}}$ and $S \leq \bar{A} \pmod{\bar{\mu}}$.

However, we still do not assume that $\mathcal{S}$ is DSF unless stated otherwise.

The proof proceeds by several lemmas. Consider first, however, the partial ordering on $k$-moduli (for a fixed $k \geq 1$) given by $\bar{\mu}' \preceq \bar{\mu}$ iff $\mu_0' \leq \mu_0$ and, for $i = 1, \ldots, k$ and all $\varepsilon_0, \ldots, \varepsilon_{i-1} \in \mathbb{R}^+$, $\mu_i'(\varepsilon_0, \ldots, \varepsilon_{i-1}) \leq \mu_i(\varepsilon_0, \ldots, \varepsilon_{i-1})$ (where $\bar{\mu} = \langle \mu_0, \ldots, \mu_k \rangle$ and $\bar{\mu}' = \langle \mu_0', \ldots, \mu_k' \rangle$).

Clearly if $\bar{\mu}' \preceq \bar{\mu}$, then any $\bar{\mu}'$-bounded point is also $\bar{\mu}$-bounded and hence $\partial A \leq S \pmod{\bar{\mu}}$ (respectively $S \leq A \pmod{\bar{\mu}}$) implies $\partial A \leq S$ (mod $\bar{\mu}'$, (respectively $S \leq A \pmod{\bar{\mu}'}$) for any $n, k \geq 1$, $A \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$. It follows that if $A_1$ and $A_2$ both satisfy 3.6 with the same complexity, then we may assume that the moduli are also the same — just take the $\preceq$-infimum of the two original moduli. However, it is a triviality that we may always increase complexity (by adding a new existentially quantified variable $x_{n+k}$ in 3.5, taking $f_{k+1}(x_1, \ldots, x_{n+k}) := x_{n+k}$ and lengthening the modulus by an arbitrary function from $\mathbb{R}_+^{k+1}$ to $\mathbb{R}_+$) and so we may assume that the complexities and moduli of $A_1$ and $A_2$ *are* the same. One consequence of this observation is the following.

**3.7. Lemma.** *If $r, n \geq 1$ and $A_1, \ldots, A_r$ are subsets of $\mathbb{R}^n$ for which* 3.6 *holds, then it also holds for $\bigcup_{i=1}^r A_i$.*

*Proof.* By the above we may suppose that the $A_i$ all have the same complexity and moduli. Since $\overline{\bigcup_{i=1}^r A_i} = \bigcup_{i=1}^r \bar{A}_i$ and $\partial \left( \overline{\bigcup_{i=1}^r A_i} \right) \subseteq \bigcup_{i=1}^r \partial \bar{A}_i$ we may take the approximation of $\bigcup_{i=1}^r A_i$ to be the union of the approximations of the $A_i$. □

**3.8. Lemma.** *Suppose that $n \geq 1$ and that $F : \mathbb{R}^n \to \mathbb{R}$ is a $C^\infty$ function in $\tilde{\mathcal{S}}$. Then* 3.6 *holds for $A = Z(F)$. (Recall that $Z(F) := \{\bar{x} \in \mathbb{R}^n : F(\bar{x}) = 0\}$.)*

*Proof.* Define $f_1(x_1, \ldots, x_{n+1}) := ((1 + \sum_{i=1}^n x_i^2)^{1/2} + x_{n+1}^2)^{-1}$ and $f_2(x_1, \ldots, x_{n+1}) := F(x_1, \ldots, x_n)^2$ (for $x_1, \ldots, x_n \in \mathbb{R}$). Clearly $f_1$ and $f_2$ are $C^\infty$ functions in $\tilde{\mathcal{S}}$.

Let $\mu_0 = 1$ and set $\mu_1(\varepsilon_0) = (3n)^{-1}\varepsilon_0$ (for $\varepsilon_0 \in \mathbb{R}_+$). Note that if $\varepsilon_0 < \mu_0$, $\varepsilon_1 < \mu_1(\varepsilon_0)$, $\langle x_1, \ldots, x_{n+1} \rangle \in \mathbb{R}^{n+1}$ and $f_1(x_1, \ldots, x_{n+1}) = \varepsilon_1$, then $\|\langle x_1, \ldots, x_n \rangle\| \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2} < \left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} + x_{n+1}^2 = \varepsilon_1^{-1}$. . . . (∗)

Conversely, if $\varepsilon_0 < \mu_0, \varepsilon_1 < \mu_1(\varepsilon_0), \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ and $\|\langle x_1, \ldots, x_n \rangle\| < 2\varepsilon_0^{-1}$, then $\left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} \leq 1 + 2n\varepsilon_0^{-1} < (3n)\varepsilon_0^{-1} = \mu_1(\varepsilon_0)^{-1} < \varepsilon_1^{-1}$ and hence there exists $x_{n+1} \in \mathbb{R}$ such that $f_1(x_1, \ldots, x_{n+1}) = \varepsilon_1$. (∗∗)

We now define $\mu_2(\varepsilon_0, \varepsilon_1)$ (for $\varepsilon_0, \varepsilon_1 \in \mathbb{R}_+$) as follows:

Let $\mathcal{C}_{\varepsilon_0, \varepsilon_1}$ be a cover of the closed ball $\{\bar{x} \in \mathbb{R}^n : \|\bar{x}\| \leq \varepsilon_1^{-1}\}$ by finitely many open balls of radius $\varepsilon_0/3$. For each $\Delta \in \mathcal{C}_{\varepsilon_0, \varepsilon_1}$ choose $\varepsilon_\Delta > 0$ so small that

(a) if $\Delta \cap \partial Z(F) \neq \emptyset$, then $F^2 \restriction \Delta$ takes every value in the interval $(0, \varepsilon_\Delta)$, and

(b) if $F^2 \restriction \Delta$ takes some value in the interval $(0, \varepsilon_\Delta)$, then $F \restriction \bar{\Delta}$ has a zero.

Clearly such a choice is possible (and, in fact, only the continuity of $F$ is needed).

Now let $\mu_2(\varepsilon_0, \varepsilon_1) = \min\{\varepsilon_\Delta : \Delta \in \mathcal{C}_{\varepsilon_0,\varepsilon_1}\}$.

Thus $Z(F)$ has complexity 2, modulus $\langle \mu_0, \mu_1, \mu_2 \rangle$ and we define the approximation to be the set, $S$ say, with the single constituent given by the condition

$$\exists x_{n+1} \bigwedge_{i=1}^{2} f_i(x_1, \ldots, x_{n+1}) = \varepsilon_i. \tag{$***$}$$

To see that $S \leq Z(F) \pmod{\langle \mu_0, \mu_1, \mu_2 \rangle}$ (note that $Z(F) = \overline{Z(F)}$), let $\bar{x} \in \mathbb{R}^n$ and $\langle \varepsilon_0, \varepsilon_1, \varepsilon_2 \rangle$ be $\langle \mu_0, \mu_1, \mu_2 \rangle$-bounded. Suppose, first, that $\langle \bar{x}, \varepsilon_0, \varepsilon_1, \varepsilon_2 \rangle \in S$, so that $(***)$ holds. Choose $x_{n+1} \in \mathbb{R}$ witnessing $(***)$ and note that, by $(*)$, $\|\bar{x}\| < \varepsilon_1^{-1}$. So $\|\bar{x}\| \in \Delta$ for some $\Delta \in \mathcal{C}_{\varepsilon_0,\varepsilon_1}$. Since $F(\bar{x})^2 = \varepsilon_2 < \mu_2(\varepsilon_0, \varepsilon_1) \leq \varepsilon_\Delta$, it follows from (b) that $F(\bar{y}) = 0$ for some $\bar{y} \in \bar{\Delta}$. Clearly $\|\bar{x} - \bar{y}\| \leq 2\varepsilon_0/3 < \varepsilon_0$.

To see that $\partial Z(F) \leq S \pmod{\langle \mu_0, \mu_1, \mu_2 \rangle}$ let $\bar{x} \in \partial Z(F)$ with $\|\bar{x}\| < \varepsilon_0^{-1}$ and suppose that $\langle \varepsilon_0, \varepsilon_1, \varepsilon_2 \rangle$ is $\langle \mu_0, \mu_1, \mu_2 \rangle$-bounded. Again, $\bar{x} \in \Delta$ for some $\Delta \in \mathcal{C}_{\varepsilon_1,\varepsilon_1}$ and therefore $\bar{x} \in \Delta \cap \partial Z(F)$. It follows from (a) that $F^2 \restriction \Delta$ takes every value in $(0, \varepsilon_\Delta)$, in particular $\varepsilon_2$. So $F(\bar{y})^2 = \varepsilon_2$ for some $\bar{y} \in \mathbb{R}^n$ with $\|\bar{x} - \bar{y}\| < 2\varepsilon_0/3 < \varepsilon_0$. Also $\|\bar{y}\| < \varepsilon_0^{-1} + 2\varepsilon_0/3 < 2\varepsilon_0^{-1}$ (using $\varepsilon_0 < \mu_0 = 1$) so, by $(**)$, $f_1(\bar{y}, y_{n+1}) = \varepsilon_1$ for some $y_{n+1} \in \mathbb{R}$. Hence $\langle \bar{y}, \varepsilon_0, \varepsilon_1, \varepsilon_2 \rangle \in S$. □

**3.10. Lemma.** *Let $n \geq 1$, $A \in \tilde{\mathcal{S}}_{n+1}$ and suppose that* 3.6 *holds for $A$ (with $n+1$ replacing $n$). Then* 3.6 *also holds for $\pi[A]$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map onto the first $n$ coordinates.*

*Proof.* Suppose $A$ has complexity $k$, modulus $\bar{\mu}$ and approximation $S$ $(\subseteq \mathbb{R}^{(n+1)+k})$. Let $\mathcal{E}$ denote the set of approximations constituents of $A$ where, for $T \in \mathcal{E}$, $T$ is defined by the following condition on $\langle x_1, \ldots, x_{n+1}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in \mathbb{R}^{n+1} \times \mathbb{R}_+^k$:

$$\exists x_{n+2} \ldots \exists x_{n+k} \bigwedge_{i=1}^{k} f_i^T(x_1, \ldots, x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = \varepsilon_i. \tag{$\dagger$}_T$$

We claim that there is no loss of generality in assuming that $\bar{\mu}$ has the following property (for each $T \in \mathcal{E}$):

(P) if $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ is any $\bar{\mu}$-bounded point, then $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ is a regular value of the function $\bar{f}^T := \langle f_1^T, \ldots, f_k^T \rangle : \mathbb{R}^{n+k} \to \mathbb{R}^k$.

By 2.7, we know that the set of singular values of $\bar{f}^T$ is a subset of $\mathbb{R}^k$ which lies in $\mathcal{S}$ and which has empty interior. Hence, by 3.4, there is some $k$-modulus, $\bar{\mu}_T$ say, such that whenever $\langle \varepsilon_0, \ldots, \varepsilon_k \rangle$ is $\bar{\mu}_T$-bounded, then $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ is a regular value of $\bar{f}^T$. Now just replace $\bar{\mu}$ by the infimum of $\bar{\mu}$ and all the $\bar{\mu}_T$ (for $T \in \mathcal{E}$) (cf. the discussion immediately following 3.6).

We now define the approximating constituents of $\pi[A]$ to be those subsets of $\mathbb{R}^{n+(k+1)}$ defined by conditions of the form

$$\exists x_{n+1} \dots \exists x_{n+k} \bigwedge_{i=1}^{k+1} f_i^T(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) = \varepsilon_i \qquad (\ddagger)_T$$

where $T \in \mathcal{E}$ and the function $f_{k+1}^T : \mathbb{R}^{n+k} \to \mathbb{R}$ is defined by either

$$\langle x_1, \dots, x_{n+k} \rangle \mapsto \left(1 + \sum_{i=n+1}^{n+k} x_i^2\right)^{-1/2}; \qquad \text{(A)}$$

or

$$\langle x_1, \dots, x_{n+k} \rangle \mapsto \det\left(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_n})}\right)^2 (x_1, \dots, x_{n+k}) \qquad \text{(B)}$$

for some distinct $i_1, \dots, i_k \in \mathbb{N}$ with $1 \leq i_1, \dots, i_k \leq n+k$.

The approximation of $\pi[A]$, denoted by $S'$, is defined to be the union of all these approximating constituents (for all possible choices of $T \in \mathcal{E}$ and $f_{k+1}^T$ as (A) or one of the functions (B)). Note that all the $f_i^T$ (for $T \in \mathcal{E}$, $1 \leq i \leq k+1$) are $C^\infty$ and in $\tilde{\mathcal{S}}$.

Thus $\pi[A]$ has complexity $k+1$. The modulus of $\pi[A]$ will be defined presently. It will have the form $\langle \bar{\mu}', \mu'_{k+1} \rangle$ where $\bar{\mu}'$ is a $k$-modulus satisfying $\bar{\mu}' \preceq \bar{\mu}$ (cf the discussion following 3.6) and $\mu'_{k+1} : \mathbb{R}_+^{k+1} \to \mathbb{R}_+$. Let us just observe for the moment that any such $(k+1)$-modulus will satisfy $S' \leq \overline{\pi[A]} \pmod{\langle \bar{\mu}', \mu'_{k+1} \rangle}$. Suppose that $\langle \varepsilon_0, \dots, \varepsilon_{k+1} \rangle$ is a $\langle \bar{\mu}', \mu'_{k+1} \rangle$-bounded point, $\bar{x} \in \mathbb{R}^n$ and $\langle \bar{x}, \varepsilon_1, \dots, \varepsilon_{k+1} \rangle \in S'$. Then $\langle \bar{x}, \varepsilon_1, \dots, \varepsilon_{k+1} \rangle$ satisfies a condition of the form $(\ddagger)_T$ for some $T \in \mathcal{E}$. So certainly there is some $x_{n+1} \in \mathbb{R}$ such that $\langle \bar{x}, x_{n+1}, \varepsilon_1, \dots, \varepsilon_k \rangle$ satisfies $(\dagger)_T$, and hence $\langle \bar{x}, x_{n+1}, \varepsilon_1, \dots, \varepsilon_n \rangle \in S$. But obviously $\langle \varepsilon_0, \dots, \varepsilon_k \rangle$ is a $\bar{\mu}'$-bounded, and hence $\bar{\mu}$-bounded, point. So since $S \leq \bar{A} \pmod{\bar{\mu}}$ it follows that $\langle \bar{y}, y_{n+1} \rangle \in \bar{A}$ for some $\bar{y} \in \mathbb{R}^n$ and $y_{n+1} \in \mathbb{R}$ with $\|\langle \bar{x}, x_{n+1} \rangle - \langle \bar{y}, y_{n+1} \rangle\| < \varepsilon_0$. It follows that $\bar{y} \in \pi[\bar{A}]$ and $\|\bar{x} - \bar{y}\| < \varepsilon_0$. Since $\pi[\bar{A}] \subseteq \overline{\pi[A]}$ we are done.

So it remains to construct $\bar{\mu}'$ and $\mu'_{k+1}$ satisfying, as well as the constraints stated above, the condition $\partial\overline{\pi[A]} \leq S' \pmod{\langle \bar{\mu}', \mu'_{k+1} \rangle}$.

To do this let $\mathcal{C}_{\varepsilon_0}$ be a cover of the closed ball $\{\bar{x} \in \mathbb{R}^n : \cdot\|\bar{x}\| \leq 2\varepsilon_0^{-1}\}$ by finitely many open balls of radius $\varepsilon_0/5$ (for each $\varepsilon_0 > 0$).

For each $\varepsilon_0 > 0$ and each $\Delta \in \mathcal{C}_{\varepsilon_0}$ choose $h_\Delta(\varepsilon_0) \in \mathbb{R}_+$ such that

(a) if there exists $\bar{x} \in \Delta$ and $x_{n+1} \in \mathbb{R}$ with $\langle \bar{x}, x_{n+1} \rangle \in A$, then they may be chosen so that $|x_{n+1}| < h_\Delta(\varepsilon_0)^{-1}$, and

(b) if there exists $\bar{x} \in \Delta$ with $\bar{x} \in \partial\overline{\pi[A]}$, then $\Delta$ contains an open ball of radius $h_\Delta(\varepsilon_0)$, $\Delta^*$ say, such that $\Delta^* \cap \overline{\pi[A]} = \emptyset$.

Clearly such an $h_\Delta(\varepsilon_0)$ exists since $\overline{\pi[A]}$ is closed and it is exactly for this reason that 3.6 is formulated for closures of sets in $\tilde{\mathcal{S}}$ and not for the sets themselves.

Now set $h(\varepsilon_0) := \min\{h_\Delta(\varepsilon_0) : \Delta \in \mathcal{C}_{\varepsilon_0}\} \cup \{\varepsilon_0/10\}$.

Define $\mu'_1 := \min\{\varepsilon_0, 1\}$ and $\mu'_i(\varepsilon_0, \dots, \varepsilon_{i-1}) := \min\{\mu_i(\varepsilon_0, \dots, \varepsilon_{i-1}),\ \mu_i(h(\varepsilon_0), \varepsilon_1, \dots, \varepsilon_{i-1})\}$ (for $i = 1, \dots, k$ and $\varepsilon_0, \dots, \varepsilon_{k-1} \in \mathbb{R}_+$).

The function $\mu'_{k+1} : \mathbb{R}_+^{k+1} \to \mathbb{R}_+$ is defined as follows: Let $\varepsilon_0, \dots, \varepsilon_k \in \mathbb{R}_+$.

For each $\Delta \in \mathcal{C}_{\varepsilon_0}$ and $T \in \mathcal{E}$, we choose a positive real number $g_{\Delta,T}(\varepsilon_0, \dots, \varepsilon_k)$ satisfying the following conditions:

if there exists $f^T_{k+1}$ of the form (A) or (B) and $\eta \in \mathbb{R}_+$ such that $f^T_{k+1}$ assumes all values in the interval $(0, \eta]$ for arguments $\langle x_1, \dots, x_{n+k}\rangle \in \mathbb{R}^{n+k}$ satisfying both $\langle x_1, \dots, x_n\rangle \in \Delta$ and $\bigwedge_{i=1}^k f^T_i(x_1, \dots, x_{n+k}) = \varepsilon_i$, then $g_{\Delta,T}(\varepsilon_0, \dots, \varepsilon_k)$ is such an $\eta$ (for this $f^T_{k+1}$). Set $g_{\Delta,T}(\varepsilon_0, \dots, \varepsilon_k) = 1$, say, if there is no such $f^T_{k+1}$ and $\eta$.

Set $\mu'_{k+1}(\varepsilon_0, \dots, \varepsilon_k) := \min\{g_{\Delta,T}(\varepsilon_0, \dots, \varepsilon_k) : \Delta \in \mathcal{C}_{\varepsilon_0}, T \in \mathcal{E}\}$, and let $\bar{\mu}'$ be the $(k+1)$-modulus $\langle \mu'_0, \dots, \mu'_{k+1}\rangle$.

Clearly $\bar{\mu}'$ satisfies the constraints, stated above, that ensure that $S' \leq \overline{\pi[A]}$ (mod $\bar{\mu}'$).

We now complete the proof of 3.10 by showing that $\partial\overline{\pi[A]} \leq S'$ (mod $\bar{\mu}'$).

Suppose that $\langle \varepsilon_0, \dots, \varepsilon_{k+1}\rangle$ is a $\bar{\mu}'$-bounded point and that $\bar{x}(\in \mathbb{R}^n)$ satisfies $\bar{x} \in \partial\overline{\pi[A]}$ and $\|\bar{x}\| < \varepsilon_0^{-1}$. Then we may choose $\Delta \in \mathcal{C}_{\varepsilon_0}$ such that $\bar{x} \in \Delta$. Since $\partial\overline{\pi[A]} \subseteq \overline{\pi[A]}$ we have $\bar{x} \in \overline{\pi[A]}$, and since $\Delta$ is open, we have that $\Delta \cap \pi(A) \neq \emptyset$.

It now follows from the definition of $h$ (see (a) above) that there exists $\bar{y} \in \Delta$ and $y_{n+1} \in \mathbb{R}$, with $|y_{n+1}| < h(\varepsilon_0)^{-1}$ such that $\langle \bar{y}, y_{n+1}\rangle \in A$. It also follows from the definition of $h$ (see (b) above) that $\Delta$ contains an open ball $\Delta^*$ (of radius $\geq h(\varepsilon_0)$) such that $\Delta^* \cap \overline{\pi[A]} = \emptyset$. Thus, if we pick any $\bar{z} \in \Delta^*$, then $\langle \bar{z}, y_{n+1}\rangle \notin \bar{A}$ (otherwise $\bar{z} \in \pi[\bar{A}] \subseteq \overline{\pi[A]}$) and so there is some $\bar{w} \in \mathbb{R}^n$ lying on the line in $\mathbb{R}^n$ joining $\bar{y}$ to $\bar{z}$ such that $\langle \bar{w}, y_{n+1}\rangle \in \partial\bar{A}$. Since $\Delta$ is convex (being a $\|\cdot\|$-ball) we have $\bar{w} \in \Delta$.

Now $\langle \varepsilon_0, \dots, \varepsilon_{k+1}\rangle$ is a $\bar{\mu}'$-bounded point. By definition of $\bar{\mu}'$ this clearly implies that $\langle h(\varepsilon_0), \varepsilon_1, \dots, \varepsilon_k\rangle$ is a $\bar{\mu}$-bounded point. Also $|y_{n+1}| < h(\varepsilon_0)^{-1}$ and

$$\begin{aligned} \|\bar{w}\| &< \|\bar{x}\| + \frac{2\varepsilon_0}{5} (\text{as } \bar{w}, \bar{x} \in \Delta) \\ &< \varepsilon_0^{-1} + \frac{2\varepsilon_0}{5} < 2\varepsilon_0^{-1} \ (\text{as } \varepsilon_0 < \mu'_0 \leq 1) \\ &\leq h(\varepsilon_0)^{-1}. \end{aligned}$$

Hence $\|\langle \bar{w}, y_{n+1}\rangle\| < h(\varepsilon_0)^{-1}$. It now follows from the fact that $\partial\bar{A} \leq S$ (mod $\bar{\mu}$) and $\langle \bar{w}, y_{n+1}\rangle \in \partial\bar{A}$ that there exists $\langle v_1, \dots, v_{n+1}\rangle \in \mathbb{R}^{n+1}$ such that (setting $\bar{v} = \langle v_1, \dots, v_n\rangle$) $\|\langle \bar{w}, y_{n+1}\rangle - \langle \bar{v}, v_{n+1}\rangle\| < h(\varepsilon_0)$ and $\langle \bar{v}, v_{n-1}, \varepsilon_1, \dots, \varepsilon_k\rangle \in S$.

Let $U$ be the open ball in $\mathbb{R}^n$ with centre $\bar{x}$ and radius $\varepsilon_0/2$. Then $\Delta \subseteq U$ and also $\bar{v} \in U$ since $\|\bar{x} - \bar{v}\| \leq \|\bar{x} - \bar{w}\| + \|\bar{w} - \bar{v}\| < 2\varepsilon_0/5 + h(\varepsilon_0) \leq \varepsilon_0/2$.

Also, $\langle \bar{v}, v_{n+1}, \varepsilon_1, \ldots, \varepsilon_k \rangle$, being a point in $S$, satisfies $(\dagger)_T$ for some $T \in \mathcal{E}$. Choose such a $T$ and define $X := \{\langle \alpha_1, \ldots, \alpha_{n+k} \rangle \in \mathbb{R}^{n+k} : \langle \alpha_1, \ldots, \alpha_n \rangle \in U$ and $f_i^T(\alpha_1, \ldots, \alpha_{n+k}) = \varepsilon_i$ for $i = 1, \ldots, k\}$. We have shown that $X \neq \emptyset$.

Now clearly the range of the function $X \to \mathbb{R} : \langle \alpha_1, \ldots, \alpha_{n+k} \rangle \mapsto (1 + \sum_{i=n+1}^{n+k} \alpha_i^2)^{-1/2}$ is a set in $\tilde{\mathcal{S}}$ and hence, by (WS5) (for $\tilde{\mathcal{S}}$) either (i) contains an interval $(0, \eta')$ (for some $\eta' \in \mathbb{R}_+$) or (ii) is bounded away from zero. However, for $\bar{\alpha} \in \mathbb{R}^n$, $\bar{\alpha} \in U$ implies $\|\bar{\alpha}\| < \|\bar{x}\| + \varepsilon_0 < \varepsilon_0^{-1} + \varepsilon_0 < 2\varepsilon_0^{-1}$ (as $\varepsilon_0 < \mu_0' \leq 1$) and hence $U$ is covered by balls in $\mathcal{C}_{\varepsilon_0}$. So if (i) holds, the above function must take arbitrarily small values on $X \cap (\Delta' \times \mathbb{R}^k)$ for some such ball $\Delta'$, and hence, by (WS5) (for $\tilde{\mathcal{S}}$) again, all values in $(0, \eta)$ (for some $\eta \in \mathbb{R}_+$). It follows, by the definition of $g_{\Delta',T}$ and the fact that $\varepsilon_{k+1} < \mu_{k+1}'(\varepsilon_0, \ldots, \varepsilon_k) \leq g_{\Delta',T}(\varepsilon_0, \ldots, \varepsilon_k)$, that $\left(1 + \sum_{i=n+1}^{n+k} \alpha_i^2\right)^{-1/2}$ takes the value $\varepsilon_{k+1}$ for some $\alpha_1, \ldots, \alpha_{n+k} \in \mathbb{R}$ with $\langle \alpha_1, \ldots, \alpha_n \rangle \in \Delta'$ and $f_i^T(\alpha_1, \ldots, \alpha_{n+k}) = \varepsilon_i$ for $i = 1, \ldots, k$. But then the point $\langle \alpha_1, \ldots, \alpha_n, \varepsilon_1, \ldots, \varepsilon_k \rangle$ satisfies $(\ddagger)_T$ (with $f_{k+1}^T$ of type $(A)$) and so is in $S'$. Further, since we may obviously suppose that $\Delta'$ meets $U$, we have $\|\bar{x} - \langle \alpha_1, \ldots, \alpha_n \rangle\| < \varepsilon_0/2 + 2\varepsilon_0/5 < \varepsilon_0$, thus completing the proof that $\partial\overline{\pi[A]} \leq S'$ (mod $\bar{\mu}'$) in case (i).

If (ii) holds, then clearly $X$ is bounded. Also $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ is a regular value of $\bar{f}^T$ (by $(P)$). Note also that the projection map from $X$ onto the first $n$ coordinates does not have range $U$. For if $\langle \beta_1, \ldots, \beta_n \rangle$ is the centre of the open ball $\Delta^*$ (which, recall, satisfies $\Delta^* \subseteq \Delta \subseteq U$ and $\Delta^* \cap \overline{\pi[A]} = \emptyset$, and has radius $\geq h(\varepsilon_0)$) and $\langle \beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_{n+k} \rangle \in X$ for some $\beta_{n+1}, \ldots, \beta_{n+k} \in \mathbb{R}$, then $\langle \beta_1, \ldots, \beta_{n+1}, \varepsilon_1, \ldots, \varepsilon_k \rangle$ satisfies $(\dagger)_T$ and therefore $\langle \beta_1, \ldots, \beta_{n+1}, \varepsilon_1, \ldots, \varepsilon_k \rangle \in S$. But $\langle h(\varepsilon_0), \varepsilon_1, \ldots, \varepsilon_k \rangle$ is a $\bar{\mu}$-bounded point and $S \leq \bar{A}$ (mod $\bar{\mu}$), so $\langle \bar{\beta}', \beta_{n+1}' \rangle \in \bar{A}$ for some $\bar{\beta}' \in \mathbb{R}^n$, $\beta_{n+1}' \in \mathbb{R}$ with, in particular, $\|\bar{\beta}' - \bar{\beta}\| < h(\varepsilon_0)$. But then $\bar{\beta}' \in \Delta^*$ and $\bar{\beta}' \in \pi[\bar{A}] \subseteq \overline{\pi[A]}$-contradiction.

We may now invoke 2.9 (with $F = \bar{f}^T$) to conclude that some function of type (B) above takes all values in an interval $(0, \eta)$ (for some $\eta \in \mathbb{R}_+$) for arguments in $X$. The proof is now completed as in case (i). $\square$

**3.11. Corollary.** *Assume that $\mathcal{S}$ is DSF. Let $n \geq 1$ and suppose that $A \in \mathcal{S}_n$. Then* 3.6 *holds for $A$.*

*Proof.* This follows immediately from 1.7, 3.8 and repeated use of 3.10 (and, of course, the fact that $\tilde{\mathcal{S}}$ is closed under projection). $\square$

**3.12. Lemma.** *Suppose that $n \geq 1$, $A \in \tilde{\mathcal{S}}_n$ (where $\mathcal{S}$ is, once again, not assumed to be DSF) and that* 3.6 *holds for $A$. Let $Y$ be an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ such that $\bar{A} \cap Y = \partial(\bar{A}) \cap Y$. Then* 3.6 *also holds for $\bar{A} \cap Y$.*

*Proof.* Let $A$ have complexity $k$, modulus $\bar{\mu}(= \langle \mu_0, \ldots, \mu_k \rangle)$, set of approximating constituents $\mathcal{E}$, and approximation $S = \cup\mathcal{E}$. Let $Y = Z(L)$ where $L : \mathbb{R}^n \to \mathbb{R}$ is linear. For $T \in \mathcal{E}$, suppose that $T$ is defined by the following condition on

$\langle x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_k \rangle \in \mathbb{R}^n \times \mathbb{R}_+^k$:

$$\exists x_{n+1} \dots \exists x_{n+k-1} \bigwedge_{i=1}^{k} f_i^T(x_1, \dots, x_{n+k-1}) = \varepsilon_i \qquad (\dagger)_T$$

where $f_1^T, \dots, f_k^T : \mathbb{R}^{n+k-1} \to \mathbb{R}$ are $C^\infty$ functions in $\tilde{\mathcal{S}}$.

We define the approximating constituents of $\bar{A} \cap Y$ to be those sets defined by a condition (on $\langle x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_{k+2} \rangle \in \mathbb{R}^n \times \mathbb{R}_+^{k+2}$) of the form

$$\exists x_{n+1} \dots \exists x_{n+k+1} \bigwedge_{i=1}^{k+2} g_i^T(x_1, \dots, x_{n+k+1}) = \varepsilon_i \qquad (\ddagger)_T$$

where $T \in \mathcal{E}$ and, for $\langle x_1, \dots, x_{n+k+1} \rangle \in \mathbb{R}^{n+k+1}$,

$$g_1^T(x_1, \dots, x_{n+k+1}) := \left( \left( 1 + \sum_{i=1}^{n} x_i^2 \right)^{1/2} + x_{n+k}^2 \right)^{-1},$$
$$g_2^T(x_1, \dots, x_{n+k+1}) := L(x_1, \dots, x_n)^2 + x_{n+k+1}^2,$$

and

$$g_{i+2}^T(x_1, \dots, x_{n+k+1}) := f_i^T(x_1, \dots, x_{n+k-1}) \quad \text{for} \quad i = 1, \dots, k.$$

Then $g_1, \dots, g_{k+2} : \mathbb{R}^{n+k+1} \to \mathbb{R}$ are certainly $C^\infty$ functions lying in $\tilde{\mathcal{S}}$ and we define the approximation of $\bar{A} \cap Y$, $S'$ say, to be the union of these constituents. Thus $\bar{A} \cap Y$ has complexity $k+2$. In order to define its modulus, $\bar{\mu}' = \langle \mu_0', \dots, \mu_{k+2}' \rangle$ say, first choose $K \in \mathbb{R}$, $K \geq 1$, so that $|L(\bar{x}) - L(\bar{x}')| \leq K \cdot \|\bar{x} - \bar{x}'\|$ for all $\bar{x}, \bar{x}' \in \mathbb{R}^n$ (which is possible since $L$ is linear), and a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that whenever $\varepsilon > 0$, $\bar{u} \in \bar{A}$, $\|\bar{u}\| \leq \varepsilon^{-1}$ and $|L(\bar{u})| < h(\varepsilon)$, then there exists $\bar{z} \in \bar{A} \cap Y$ with $\|\bar{z} - \bar{u}\| < \varepsilon$ (which is also clearly possible by considering a finite covering of the closed ball in $\mathbb{R}^n$ of radius $\varepsilon^{-1}$ by balls in $\mathbb{R}^n$ of radius $\varepsilon/3$ and using the fact that $Y = Z(L)$). We also require that $h(\varepsilon) \leq 1$ for all $\varepsilon \in \mathbb{R}_+$.

Now set $\mu_0' := \min(\mu_0, 1)$, $\mu_1'(\varepsilon_0) = \varepsilon_0/2(n+1)$, $\mu_2'(\varepsilon_0, \varepsilon_1) = \min(\frac{\varepsilon_1}{2}, \frac{1}{4K} h(\frac{\varepsilon_1}{2})^2)$, and for $i = 1, \dots, k$, $\mu_{2+i}'(\varepsilon_0, \dots, \varepsilon_{1+i}) = \min(\mu_i(\varepsilon_2, \dots, \varepsilon_{1+i}), \mu_i(\frac{\varepsilon_2}{K}, \varepsilon_3, \dots, \varepsilon_{1+i}))$ (for $\varepsilon_0, \dots, \varepsilon_{1+k} \in \mathbb{R}_+$).

**Claim 1.** *$S' \leq \bar{A} \cap Y$ (mod $\bar{\mu}'$).*

*Proof.* Suppose that $\langle \varepsilon_0, \dots, \varepsilon_{2+k} \rangle$ is a $\bar{\mu}'$-bounded point, $\bar{x} \in \mathbb{R}^n$ and that $\langle \bar{x}, \varepsilon_1, \dots, \varepsilon_{2+k} \rangle \in S'$. Choose $T \in \mathcal{E}$ such that $\langle \bar{x}, \varepsilon_1, \dots, \varepsilon_{2+k} \rangle$ satisfies $(\ddagger)_T$. Now $\varepsilon_2 < \varepsilon_1 < \varepsilon_0 < \mu_0$ and $\varepsilon_{2+i} < \mu_i(\varepsilon_2, \dots, \varepsilon_{1+i})$ for $i = 1, \dots, k$, so $\langle \varepsilon_2, \dots, \varepsilon_{2+k} \rangle$ in a $\bar{\mu}$-bounded point. Further, it follows from the definition of $g_3^T, \dots, g_{k+2}^T$ that

$\langle \bar{x}, \varepsilon_3, \ldots, \varepsilon_{2+k} \rangle \in T$. Since $T \subseteq S$ and $S \leq \bar{A}$ (mod $\bar{\mu}$), there is some $\bar{y} \in \bar{A}$ with $\|\bar{x}-\bar{y}\| < \varepsilon_2$. Now $|L(\bar{x})-L(\bar{y})| \leq K\cdot\|\bar{x}-\bar{y}\| < K\cdot\varepsilon_2 < K\cdot\mu_2'(\varepsilon_0, \varepsilon_1) < (\frac{1}{2}h(\frac{\varepsilon_1}{2}))^2 < \frac{1}{2}h(\frac{\varepsilon_1}{2})$ and hence $|L(\bar{y})| < |L(\bar{x})| + \frac{1}{2}h(\frac{\varepsilon_1}{2})$. But, by considering the conjunct involving $g_2^T$ in $(\ddagger)_T$, we see that $|L(\bar{x})| \leq \varepsilon_2^{1/2} < (\mu_2'(\varepsilon_0, \varepsilon_1))^{1/2} < \frac{1}{2}h(\varepsilon_1/2)$, and so $|L(\bar{y})| < h(\varepsilon_1/2)$. Also, $\|\bar{x}\| < \varepsilon_1^{-1}$ (by considering the conjunct involving $g_1^T$ in $(\ddagger)_T$) and hence $\|\bar{y}\| < \varepsilon_1^{-1}+\varepsilon_2 < (\varepsilon_1/2)^{-1}$. We now apply the definition of $h$ (with $\bar{u} = \bar{y}$ and $\varepsilon = \varepsilon_1/2$) to conclude that there exists $\bar{z} \in \bar{A} \cap Y$ with $\|\bar{y} - \bar{z}\| < \varepsilon_1/2$. But $\|\bar{x} - \bar{y}\| < \varepsilon_2 < \varepsilon_1/2$, so $\|\bar{z} - \bar{x}\| < \varepsilon_1$. As $\varepsilon_1 < \varepsilon_0$, this proves claim 1.

**Claim 2.** $\partial(\bar{A} \cap Y) \leq S'$ *(*mod $\bar{\mu}'$*)*

*Proof.* Let $\langle \varepsilon_0, \ldots, \varepsilon_{2+k} \rangle$ be a $\bar{\mu}'$-bounded point and suppose that $\bar{x} \in \partial(\bar{A} \cap Y)$ and that $\|\bar{x}\| < \varepsilon_0^{-1}$. Since $\bar{A} \cap Y$ is closed, it follows that $\bar{x} \in \bar{A} \cap Y$ and therefore, by the lemma hypotheses, that $\bar{x} \in \partial(\bar{A}) \cap Y$. Now by definition of $\bar{\mu}'$, it follows that $\langle \varepsilon_2/K, \varepsilon_3, \ldots, \varepsilon_{2+k} \rangle$ is a $\bar{\mu}$-bounded point and that $\|\bar{x}\| < (\varepsilon_2/K)^{-1}$. So, since $\partial(\bar{A}) \leq S$ (mod $\bar{\mu}$) there exists $\bar{y} = \langle y_1, \ldots, y_n \rangle \in \mathbb{R}^n$ such that $\langle \bar{y}, \varepsilon_3, \ldots, \varepsilon_{2+k} \rangle \in S$ and $\|\bar{x} - \bar{y}\| < \varepsilon_2/K$.

Choose $T \in \mathcal{E}$ (defined by $(\dagger)_T$) such that $\langle \bar{y}, \varepsilon_3, \ldots, \varepsilon_{2+k} \rangle \in T$ and choose $y_{n+1}, \ldots, y_{n+k-1} \in \mathbb{R}$ such that $\bigwedge_{i=1}^k f_i^T(y_1, \ldots, y_{n+k-1}) = \varepsilon_{2+i}$. Then $\bigwedge_{i=3}^{k+2} g_i^T(\bar{y}, y_{n+1}, \ldots, y_{n+k+1}) = \varepsilon_i$ for any $y_{n+k}, y_{n+k+1} \in \mathbb{R}$.

Since $\bar{x} \in Y$ we have $|L(\bar{y})| = |L(\bar{x}) - L(\bar{y})| \leq K \cdot \|\bar{x} - \bar{y}\| < \varepsilon_2$, so certainly $|L(\bar{y})|^2 < \varepsilon_2$ and hence $L(\bar{y})^2 + y_{n+k+1}^2 = \varepsilon_2$ for some $y_{n+k+1} \in \mathbb{R}$. Then $g_2^T(\bar{y}, y_{n+1}, \ldots, y_{n+k+1}) = \varepsilon_2$ for any $y_{n+k} \in \mathbb{R}$.

Finally, $\|\bar{y}\| \leq \|\bar{x}\| + \varepsilon_2/K < \varepsilon_0^{-1} + \varepsilon_2/K < 2\varepsilon_0^{-1} < \frac{1}{n+1}\cdot\varepsilon_1^{-1}$ and so $\left(1+\sum_{i=1}^n y_i^2\right)^{1/2} < \varepsilon_1^{-1}$, and hence there exists $y_{n+k} \in \mathbb{R}$ such that $g_1^T(\bar{y}, y_{n+1}, \ldots, y_{n+k}) = \varepsilon_1$.

Thus $\langle \bar{y}, \varepsilon_1, \ldots, \varepsilon_{2+k} \rangle$ satisfies condition $(\ddagger)_T$ (with $\bar{y}$ in place of $\bar{x}$ there) and hence $\langle \bar{y}, \varepsilon_1, \ldots, \varepsilon_{k+2} \rangle \in S'$. Since $\|\bar{x} - \bar{y}\| < \varepsilon_2/K < \varepsilon_0$ claim 2 is proved.

The lemma now follows from the claims since $\bar{A} \cap Y$ is closed (i.e., $\overline{\bar{A} \cap Y} = \bar{A} \cap Y$). $\square$

**3.13. Theorem.** *Assume that $\mathcal{S}$ is DSF. Let $n \geq 1$ and suppose that $A \in \tilde{\mathcal{S}}_n$. Then* 3.6 *holds for $A$.*

*Proof.* Recall from 1.6 that $\tilde{\mathcal{S}} = \bigcup_{i=0}^\infty \mathcal{S}^{(i)}$ where $\mathcal{S}^{(0)} = \mathcal{S}$ and $\mathcal{S}^{(i+1)} = ((\mathcal{S}^{(i)u})^{pr})^{cl}$.

Now by 3.11, 3.6 holds for every $n \geq 1$ and $A \in \mathcal{S}_n^{(0)}$. It also clearly follows from 3.7 and 3.10 that if $i \geq 0$ and 3.6 holds for every $n \geq 1$ and $A \in \mathcal{S}_n^{(i)}$, then it also holds for every $n \geq 1$ and $A \in (\mathcal{S}^{(i)u})^{pr}$. Hence it suffices to show that if $\mathcal{J} = \langle \mathcal{J}_n : n \geq 1 \rangle$ is any *o*-minimal weak structure such that $\mathcal{J} \subseteq \tilde{\mathcal{S}}$ and 3.6 holds for every $n \geq 1$ and $A \in \mathcal{J}_n$, then the same is true of $\mathcal{J}^{cl}$.

To establish this first note that by 1.5 and 3.10 it is sufficient to show that if $n \geq 1$, $B \in \mathcal{J}_n$ and $Y$ is an affine subspace of $\mathbb{R}^n$, then 3.6 holds for $\bar{B} \cap Y$. This is trivial if $Y = \mathbb{R}^n$. Otherwise, pick $(n-1)$-dimensional affine subspaces, $Y_1, \ldots, Y_m$

say (where $1 \leq m \leq n$), of $\mathbb{R}^n$ such that $Y = Y_1 \cap \ldots \cap Y_m$. Say $Y_1 = Z(L)$ where $L : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Observe that

$$\bar{B} \cap Y_1 = \overline{B \cap Y_1} \cup \Big(\overline{B \cap P(L)} \cap Y_1\Big) \cup \Big(\overline{B \cap P(-L)} \cap Y_1\Big)$$

where, for a function $G : \mathbb{R}^n \to \mathbb{R}$, $P(G) := \{\bar{x} \in \mathbb{R}^n : G(\bar{x}) > 0\}$.

Further, $B \cap Y_1 \in \mathcal{J}_n$ (by (WS1) and (WS2) for $\mathcal{J}$) so 3.6 holds for $B \cap Y_1$ — and hence, obviously, for $\overline{B \cap Y_1}$ — by the hypothesis on $\mathcal{J}$. Also, $\overline{B \cap P(\pm L)} \cap Y_1 = \partial(\overline{B \cap P(\pm L)}) \cap Y_1$ (as $Y_1 = \partial(P(\pm L))$), so 3.6 holds for $\overline{B \cap P(\pm L)} \cap Y_1$ by 3.12 (note that $B \cap P(\pm L) \in \mathcal{J}_n$ by (WS1) and (WS2) for $\mathcal{J}$). Hence 3.6 holds for $\bar{B} \cap Y_1$ by 3.7.

But $\bar{B} \cap Y_1$ is closed and has empty interior (in $\mathbb{R}^n$ of course), so it now follows from 3.12 that 3.6 holds for $(\bar{B} \cap Y_1) \cap Y_2$. Continuing we see that 3.6 holds for $\bar{B} \cap Y$ as required. □

The main theorem of this section, Theorem 3.1, now follows from 3.13 and 3.3 provided we can show that any subset of $\mathbb{R}^{n+k}$ (for $n, k \geq 1$) which is a finite union of sets defined by conditions of the form 3.5, has empty interior. For this it is clearly sufficient to consider just sets defined by 3.5 (by using 2.1 and the fact that a finite intersection of open dense subsets of $\mathbb{R}^{n+k}$ is open dense), so suppose $n, k \geq 1$ and $S \subseteq \mathbb{R}^{n+k}$ is defined by 3.5. If $S$ had non-empty interior, then by fixing suitable $x_1, \ldots, x_n \in \mathbb{R}$ we could find (using 2.3 and 2.5) an open ball $B \subseteq \mathbb{R}^k_+$ and $C^1$ functions $\phi_i : B \to \mathbb{R}$ (lying in $\tilde{\mathcal{S}}$) for $i = 1, \ldots, k-1$ such that $\bigwedge_{i=1}^k f_i(x_1, \ldots, x_n, \phi_1(\varepsilon_1, \ldots, \varepsilon_k), \ldots, \phi_{k-1}(\varepsilon_1, \ldots, \varepsilon_k)) = \varepsilon_i$ for all $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle \in B$. We leave the reader to choose his or her favourite route to a contradiction from here and thereby complete the proof of Theorem 3.1.

## 4. Locally closed $\tilde{\mathcal{S}}$-cell decomposition and the proof of Theorem 1.22

Let $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ be any $o$-minimal weak structure that is also DSF (or, indeed, any $o$-minimal weak structure satisfying the conclusion of Theorem 3.1). We must show that $\tilde{\mathcal{S}}$ is closed under complementation and to do this I first show that, for each $n \geq 1$ and each *closed* $A \in \tilde{\mathcal{S}}_n$, there exists an $\tilde{\mathcal{S}}$-cell decomposition of $\mathbb{R}^n$ (see below for the definitions) compatible with $A$. It will follow easily from (WS6) that the hypothesis that $A$ be closed can be removed and hence that $\mathbb{R}^n \backslash A$ (for arbitrary $A \in \tilde{\mathcal{S}}_n$) is a union of $\tilde{\mathcal{S}}$-cells, whence $\mathbb{R}^n \backslash A \in \tilde{\mathcal{S}}_n$ as required. Actually, it is easier to work with bounded cells and use a homeomorphism from $\mathbb{R}^n$ to $(-1, 1)^n$.

### 4.1. Definitions.

(i) For $n \geq 1$, an *$\tilde{\mathcal{S}}$-0-cell* in $\mathbb{R}^n$ is a singleton set $\{\bar{a}\}$ (for $\bar{a} \in \mathbb{R}^n$).

(ii) An *$\tilde{\mathcal{S}}$-1-cell* in $\mathbb{R}$ is an open interval $(a, b)$ where $a, b \in \mathbb{R}$ and $a < b$.

(iii) For $n \geq 2$ and $0 \leq k < n$, an $\tilde{\mathcal{S}}-(k+1)$-*cell* in $\mathbb{R}^n$ is either a set of the form graph$(f)$, where $f : C \to \mathbb{R}$ is a continuous, bounded function in $\tilde{\mathcal{S}}$ and $C$ is an $\tilde{\mathcal{S}}-(k+1)$-*cell* in $\mathbb{R}^{n-1}$, or else a set of the form $(f, g) := \{\langle x_1, \dots, x_n\rangle \in \mathbb{R}^n : \langle x_1, \dots, x_{n-1}\rangle \in C$ and $f(x_1, \dots, x_{n-1}) < x_n < g(x_1, \dots, x_{n-1})\}$ where $C$ is an $\tilde{\mathcal{S}} - k$-cell in $\mathbb{R}^{n-1}$ and $f, g : C \to \mathbb{R}$ are continuous, bounded functions in $\tilde{\mathcal{S}}$ satisfying $f(x_1, \dots, x_{n-1}) < g(x_1, \dots, x_{n-1})$ for all $\langle x_1, \dots, x_{n-1}\rangle \in C$.

(iv) An $\tilde{\mathcal{S}}$-*cell* in $\mathbb{R}^n$ is an $\tilde{\mathcal{S}} - k$-cell in $\mathbb{R}^n$ for some $k \leq n$.

**4.2. Remarks.**

(i) An $\tilde{\mathcal{S}}$-cell is bounded. This is not in accordance with the usual definition but will suffice for our purposes.

(ii) All $\tilde{\mathcal{S}}$-cells are sets in $\tilde{\mathcal{S}}$.

(iii) For $n \geq 1$, an $\tilde{\mathcal{S}} - n$-cell in $\mathbb{R}^n$ is also called an *open* $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$. It is an open connected subset of $\mathbb{R}^n$. If $1 \leq k < n$ and $C$ is an $\tilde{\mathcal{S}} - k$-cell in $\mathbb{R}^n$, then $C$ is a connected subset of $\mathbb{R}^n$ with empty interior. However, there exists a projection map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ (i.e., a map of the form $\langle x_1, \dots, x_n\rangle \mapsto \langle x_{i_1}, \dots, x_{i_k}\rangle$ for some $i_1, \dots, i_k$ with $1 \leq i_1 < \dots < i_k \leq n$) such that $\pi[C]$ is an open cell in $\mathbb{R}^k$ and $\pi \restriction C : C \to \pi[C]$ is a homeomorphism.

**4.3. Definition.** Let $n \geq 1$ and suppose that $A \subseteq \mathbb{R}^n$.

(i) If $n = 1$, by an $\tilde{\mathcal{S}}$-*cell decomposition of* $A$, we mean a finite collection, $\mathcal{D}$ say, of pairwise disjoint $\tilde{\mathcal{S}}$-cells in $\mathbb{R}$ such that $A = \cup\mathcal{D}$.

(ii) If $n \geq 2$, by an $\tilde{\mathcal{S}}$-*cell decomposition of* $A$, we mean a finite collection, $\mathcal{D}$ say, of pairwise disjoint $\tilde{\mathcal{S}}$-cells in $\mathbb{R}^n$ such that $A = \cup\mathcal{D}$ and such that the *set* $\{\pi_n[C] : C \in \mathcal{D}\}$ is an $\tilde{\mathcal{S}}$-cell decomposition of $\pi_n[A]$, where $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map onto the first $n-1$ coordinates.

(iii) If $B \subseteq A$ and $\mathcal{D}$ is an $\tilde{\mathcal{S}}$-cell decomposition of $A$, then $\mathcal{D}$ is *compatible* with $B$, if for all $C \in \mathcal{D}$ either $C \subseteq B$ or $C \cap B = \emptyset$.

**Remark.** If (iii) pertains then clearly $\mathcal{D}$ is also compatible with $A\backslash B$. Hence both $B$ and $A\backslash B$ are finite unions of $\tilde{\mathcal{S}}$-cells (in $\mathcal{D}$) and hence (by 4.2(ii)) both lie in $\tilde{\mathcal{S}}$.

Our aim in this section is to prove the following result.

**4.5. Theorem.** *Let $n \geq 1$ and suppose that $D$ is an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$ and that $A$ is a set in $\tilde{\mathcal{S}}_n$. Suppose further that $A$ is a subset of $D$ which is also closed in $D$. Then there exists an $\tilde{\mathcal{S}}$-cell decomposition $\mathcal{D}$ of $D$ which is compatible with $A$.*

Before proving 4.5 let us see how it implies 1.8.

Consider some $n \geq 1$ and $A \in \tilde{\mathcal{S}}_n$. By (WS6) (for $\tilde{\mathcal{S}}$) there exists $m \geq n$ and a *closed* set $B \in \tilde{\mathcal{S}}_m$ such that $A = \pi[B]$, where $\pi : \mathbb{R}^m \to \mathbb{R}^n : \langle x_1, \dots, x_m\rangle \mapsto \langle x_1, \dots, x_n\rangle$.

Now, for each $p \geq 1$ consider the map $\theta_p : \mathbb{R}^p \to (-1, 1)^p : \langle y_1, \dots, y_p\rangle \mapsto$

$\langle y_1(1+y_1^2)^{-1/2}, \dots, y_p \cdot (1+y_p^2)^{-1/2}\rangle$. Then $\theta_m$ is a semi-algebraic map (hence in $\tilde{\mathcal{S}}$) and a homeomorphism. We also clearly have $\pi \circ \theta_m = \theta_n \circ \pi$, so $\mathbb{R}^n \backslash A = \theta_n^{-1}[\theta_n[\mathbb{R}^n \backslash \pi[B]]] = \theta_n^{-1}[(-1,1)^n \backslash \theta_n \circ \pi[B]] = \theta_n^{-1}[(-1,1)^n \backslash \pi \circ \theta_m[B]]$. Thus, for $\bar{y} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^n \backslash A$ if and only if $\theta_n(\bar{y}) \in (-1,1)^n \backslash \pi \circ \theta_m[B]$ so, in order to show that $\mathbb{R}^n \backslash A \in \tilde{\mathcal{S}}$, it is sufficient to show that $(-1,1)^n \backslash \pi \circ \theta_m[B] \in \tilde{\mathcal{S}}$. To do this note that $\theta_m[B]$ is closed in the $\tilde{\mathcal{S}}$-cell $(-1,1)^m$ so, by applying 4.5 with $m$ in place of $n$, $D = (-1,1)^m$ and $A = \theta_m[B]$, there exists an $\tilde{\mathcal{S}}$-cell decomposition $\mathcal{D}$ of $(-1,1)^m$ compatible with $\theta_m[B]$. But by 4.3 (ii), $\{\pi_m[C] : C \in \mathcal{D}\}$ is an $\tilde{\mathcal{S}}$-cell decomposition of $(-1,1)^{m-1}$ which is clearly compatible with $\pi_m \circ \theta_m[B]$. Repeating this process with the maps $\pi_{m-1}, \dots, \pi_{m-n+1}$, we obtain an $\tilde{\mathcal{S}}$-cell decomposition of $(-1,1)^n$ compatible with $\pi \circ \theta_m[B]$ which, by 4.4, gives the required result.

*Proof of* 4.5. Let $n \geq 1$ and consider the following two statements:

(I)$_n$ If $D$ is an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$ and $A$ is a closed subset of $D$ lying in $\tilde{\mathcal{S}}_n$, then there exists an $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with $A$.

(II)$_n$ If $D$ is an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$, $\mathcal{D}$ an $\tilde{\mathcal{S}}$-cell decomposition of $D$ and, for each $C \in \mathcal{D}$, $\mathcal{E}(C)$ is a finite collection of $\tilde{\mathcal{S}}$-cell decompositions of $C$, then there exists an $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with every $\tilde{\mathcal{S}}$-cell in $\cup_{C \in \mathcal{D}} \cup \mathcal{E}(C)$.

These are proved simultaneously by induction on $n$.

To prove (I)$_1$ suppose that $D = (\alpha, \beta)$ for $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ (the result is trivial if $D$ is an $\tilde{\mathcal{S}} - 0$-cell in $\mathbb{R}$) and that $\{a_1, \dots, a_k\}$ enumerates $\partial A \cap D$ (which is finite by (WS5) for $\tilde{\mathcal{S}}$) in increasing order. Clearly we may take $\mathcal{D} = \{\{a_i\} : 1 \leq i \leq k\} \cup \{(a_i, a_{i+1}) : 0 \leq i \leq k\}$ (where we have set $a_0 = \alpha, a_{k+1} = \beta$) for our $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with $A$. The proof of (II)$_1$ is equally straightforward.

Now suppose that (I)$_n$ and (II)$_n$ hold for some $n \geq 1$. We first prove (I)$_{n+1}$, so suppose that $D$ is an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^{n+1}$ and that $A$ is a closed subset of $D$ lying in $\tilde{\mathcal{S}}_{n+1}$.

If $D$ has the form graph($f$) for some continuous function $f : C \to \mathbb{R}$ lying in $\tilde{\mathcal{S}}$, where $C(= \pi_{n+1}[D])$ in an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$, consider the set $\pi_{n+1}[A]$. It lies in $\tilde{\mathcal{S}}_n$ and is a closed subset of $C$. By (I)$_n$ we may choose an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}$ say, of $C$ compatible with $\pi_{n+1}[A]$. Then $\{\text{graph}(f \restriction E) : E \in \mathcal{D}\}$ is clearly an $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with $A$.

Consider now the case that $D$ has the form $(f, g)$ for some continuous functions $f, g : C \to \mathbb{R}$ lying in $\tilde{\mathcal{S}}$, where $C(= \pi_{n+1}[D])$ is an $\tilde{\mathcal{S}} - k$-cell in $\mathbb{R}^n$ for some $k$ *strictly* less than $n$. Choose $\pi$ as in 4.2(iii) and define the map $\pi'$ by $\pi' : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} : \langle \bar{x}, x_{n+1}\rangle \mapsto \langle \pi(\bar{x}), x_{n+1}\rangle$.

Clearly $\pi'[D]$ is an $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^{k+1}$ and $\pi'[A]$ is a closed subset of $\pi'[D]$ lying in $\tilde{\mathcal{S}}$. Since $k+1 \leq n$ we may apply the induction hypotheses to get an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}$ say, of $\pi'[D]$ compatible with $\pi'[A]$. It is routine to check that $\{\pi'^{-1}[E] \cap (C \times \mathbb{R}) : E \in \mathcal{D}\}$ is an $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with $A$.

This completes the proof of $(\mathrm{I})_{n+1}$ in the case that $D$ is not an open $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^{n+1}$.

Suppose now that $D$ is an open $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^{n+1}$, say $D = (f, g)$ where $f, g : C \to \mathbb{R}$ are continuous functions lying in $\tilde{\mathcal{S}}$ and $C(= \pi_{n+1}[D])$ is an open $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$.

First, note that $(\mathrm{II})_n$ implies a stronger form of $(\mathrm{I})_n$. Namely, if $l \geq 1$ and $A_1, \ldots, A_l$ are closed subsets of $C$ lying in $\tilde{\mathcal{S}}$, then there exists an $\tilde{\mathcal{S}}$-cell decomposition of $C$ simultaneously compatible with each set $A_i$. (Just apply $(\mathrm{II})_n$ with $\mathcal{D} = \{C\}$ and $\mathcal{E}(C)$ consisting of $\tilde{\mathcal{S}}$-cell decompositions compatible with $A_i$, for each $i$, whose existence is guaranteed by $(\mathrm{I})_n$.) Secondly — and this is the point of all the effort so far — I claim that we may suppose that $A$ has empty interior. For by 3.1 there exists a closed set $B \in \tilde{\mathcal{S}}_{n+1}$ with empty interior such that $\partial \bar{A} \subseteq B$. Since $\bar{A} \cap D = A$ and $\tilde{\mathcal{S}}$-cells are connected, it follows that any $\tilde{\mathcal{S}}$-cell decomposition of $D$ that is compatible with $(\bar{A} \cap B) \cap D$ is also compatible with $A$. This justifies the claim.

Now for each $i \geq 1$, consider the set

$$A_i := \Big\{ \bar{x} \in C : \exists y_1, \ldots, y_i \big( y_1 < \cdots < y_i \wedge \bigwedge_{j=1}^{i} \langle \bar{x}, y_j \rangle \in A \big) \Big\}.$$

Then each set $A_i$ lies in $\tilde{\mathcal{S}}_n$ and since $A$ has empty interior (in $\mathbb{R}^{n+1}$) it follows by the "(WS5) and Fubini" argument (see e.g., the proof of 3.4) that $A_N$ has empty interior in $\mathbb{R}^n$ for some $N \geq 1$, and I now fix such an integer $N$.

Now define:

$$\begin{aligned}
H &:= \big\{ \langle \bar{x}, \varepsilon \rangle \in C \times \mathbb{R}_+ : \exists y_1, y_2 ( y_1 < y_2 \wedge \langle \bar{x}, y_1 \rangle \in A \wedge \langle \bar{x}, y_2 \rangle \in \\
&\qquad A \wedge y_2 - y_1 = \varepsilon ) \big\}, \\
H_f &:= \big\{ \langle \bar{x}, \varepsilon \rangle \in C \times \mathbb{R}_+ : \exists y ( \langle \bar{x}, y \rangle \in A \wedge y - f(\bar{x}) = \varepsilon ) \big\}, \\
H_g &:= \big\{ \langle \bar{x}, \varepsilon \rangle \in C \times \mathbb{R}_+ : \exists y ( \langle \bar{x}, y \rangle \in A \wedge g(\bar{x}) - y = \varepsilon ) \big\},
\end{aligned}$$

(recall that $A \subseteq D = (f, g)$) and

$$\tilde{H} \text{ (respectively } \tilde{H}_f, \tilde{H}_g) := \big\{ \bar{x} \in C : \langle \bar{x}, 0 \rangle \in \bar{H} \text{ (respectively } \bar{H}_f, \bar{H}_g) \big\}.$$

Then $\bar{A}_1, \ldots, \bar{A}_N, \tilde{H}, \tilde{H}_f, \tilde{H}_g$ are closed subsets of $C$ lying in $\tilde{S}_n$. So, by our generalization of $(\mathrm{I})_n$, there exists an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}$ say, of $C$ simultaneously compatible with all these sets.

Now suppose that $C'$ is an open $\tilde{\mathcal{S}}$-cell in $\mathcal{D}$. We aim to construct an $\tilde{\mathcal{S}}$-cell decomposition of $(C' \times \mathbb{R}) \cap D$ compatible with $(C' \times \mathbb{R}) \cap A$. If $(C' \times \mathbb{R}) \cap A = \emptyset$ we just take $\{(f \restriction C', g \restriction C')\}$. Otherwise $C' \cap \bar{A}_1 \neq \emptyset$ and we choose $k < N$ maximal such that $C' \cap \bar{A}_k \neq \emptyset$. (Note that $C' \cap \bar{A}_N = \emptyset$ for otherwise

$C' \subseteq \bar{A}_N$ contradicting the fact that $A_N$, and hence (by 2.1) $\bar{A}_N$, has empty interior.)

Now the fact that $C' \cap A_N = \emptyset$ clearly implies that $H \cap (C' \times \mathbb{R})$ has empty interior. Hence, by 2.2, $\{\bar{x} \in C' : \langle \bar{x}, 0\rangle \in \overline{H \cap (C' \times \mathbb{R})}\}$ has empty interior. But, as $C'$ is open, this set is equal to $\tilde{H} \cap C'$ and so $\tilde{H} \cap C' = \emptyset$. Similarly $\tilde{H}_f \cap C' = \tilde{H}_g \cap C' = \emptyset$. We *claim* that $C' \subseteq A_k$.

We certainly have $C' \subseteq \bar{A}_k$ since $C' \cap \bar{A}_k \neq \emptyset$.

Consider a point $\bar{x} \in C'$. For $y \in \mathbb{R}$ with $f(\bar{x}) \leq y \leq g(\bar{x})$ choose an open neighbourhood of $\langle \bar{x}, y\rangle$ (in $\mathbb{R}^{n+1}$), $U_y$ say, as follows. If $f(\bar{x}) < y < g(\bar{x})$ and $\langle \bar{x}, y\rangle \notin A$, choose $U_y$ so that $U_y \cap A = \emptyset$ (recall that $A$ is closed in $D$). If $\langle \bar{x}, y\rangle \in A$ then, since $\bar{x} \notin \tilde{H}$, we may choose $U_y$ so that for each $\bar{x}' \in \pi_{n+1}[U_y]$ there is at most one $y' \in \mathbb{R}$ such that $\langle \bar{x}', y'\rangle \in A \cap U_y$. Similarly if $y = f(\bar{x})$ or $y = g(\bar{x})$ then, since $\bar{x} \notin \tilde{H}_f \cup \tilde{H}_g$, we may choose $U_y$ so that $U_y \cap A = \emptyset$. Now by the compactness of the set $\{\bar{x}\} \times [f(\bar{x}), g(\bar{x})]$, finitely many of the sets $U_y$ suffice to cover it, from which it follows that there exists an open neighbourhood of $\bar{x}$ in $\mathbb{R}^n$, $U$ say, such that whenever $\bar{x}' \in U$, $|(\{\bar{x}'\} \times \mathbb{R}) \cap A| \leq |(\{\bar{x}\} \times \mathbb{R}) \cap A|$. But, as $\bar{x} \in \bar{A}_k$, we may choose $\bar{x}' \in U \cap A_k$ here, from which it follows (using also the maximality of $k$) that both sides of the inequality are equal to $k$. Hence $\bar{x} \in A_k$ and the claim is justified.

Thus, for each $i = 1, \ldots, k$, we may define a function $f_i : C' \to \mathbb{R}$ in $\tilde{\mathcal{S}}$ by $f_i(\bar{x}) = y$ if and only if $\exists y_1, \ldots, y_k (y_1 < \cdots < y_k \wedge \bigwedge_{j=1}^{k} \langle \bar{x}, y_j\rangle \in A \wedge y = y_i)$.

A similar argument to the above shows that each function $f_i$ is continuous (cf. the proof of 2.4).

Thus we now have an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}_{C'}$ say, of $(C' \times \mathbb{R}) \cap D$ compatible with $(C' \times \mathbb{R}) \cap A$, namely $\mathcal{D}_{C'} = \{C \times (f_i, f_{i+1}) : i = 0, \ldots, k\} \cup \{C \times \mathrm{graph}(f_i) : i = 1, \ldots, k\}$, where we have set $f_0 = f$ and $f_{k+1} = g$. This applies to each open $\tilde{\mathcal{S}}$-cell $C'$ of $\mathcal{D}$.

If $C'$ is a non-open $\tilde{\mathcal{S}}$-cell in $\mathbb{R}^n$ then, using the fact that we have established $(\mathrm{I})_{n+1}$ for non-open $\tilde{\mathcal{S}}$-cells in $\mathbb{R}^{n+1}$, we may choose an $\tilde{\mathcal{S}}$-cell decomposition $\mathcal{D}_{C'}$ of $(C' \times \mathbb{R}) \cap D$ compatible with $(C' \times \mathbb{R}) \cap A$, which, of course, is a closed subset of $(C' \times \mathbb{R}) \cap D$ lying in $\tilde{\mathcal{S}}$.

Finally, for each $\tilde{\mathcal{S}}$-cell $C' \in \mathcal{D}$, let $\mathcal{D}'_{C'}$ be the $\tilde{\mathcal{S}}$-cell decomposition of $C'$ induced on $C'$ by $\mathcal{D}_{C'}$. (So, in fact, $\mathcal{D}'_{C'} = \{C'\}$ if $C'$ is open.)

Now using $(\mathrm{II})_n$, let $\mathcal{D}^*$ be an $\tilde{\mathcal{S}}$-cell decomposition of $C$ simultaneously compatible with all $\tilde{\mathcal{S}}$-cells in $\mathcal{D}'_{C'}$ (for all $C' \in \mathcal{D}$). Now set

$$\bar{\mathcal{D}} := \big\{(C'' \times \mathbb{R}) \cap E : C'' \in \mathcal{D}^*, C' \in \mathcal{D} \text{ with } C'' \subseteq C', E \in \mathcal{D}_{C'}\big\}.$$

Clearly $\bar{\mathcal{D}}$ is the required $\tilde{\mathcal{S}}$-cell decomposition of $D$ compatible with $A$.

It only remains to prove $(\mathrm{II})_{n+1}$. So suppose $D, \mathcal{D}$ and $\mathcal{E}(C)$ (for $C \in \mathcal{D}$) are as in the hypothesis of $(\mathrm{II})_{n+1}$.

Let $\mathcal{D}'$ be the $\tilde{\mathcal{S}}$-cell decomposition of $\pi_{n+1}[D]$ induced by $\mathcal{D}$. For each triple $\langle C', C, \mathcal{G}\rangle$ with $C' \in \mathcal{D}'$, $C \in \mathcal{D}$, $\pi_{n+1}[C] = C'$ and $\mathcal{G} \in \mathcal{E}(C)$, let $\mathcal{C}(C', C, \mathcal{G})$ be the $\tilde{\mathcal{S}}$-cell decomposition of $C'$ induced on $C'$ by the $\tilde{\mathcal{S}}$-cell decomposition $\mathcal{G}$ of $C$. For $C' \in \mathcal{D}'$ set $\mathcal{E}'(C') = \{\mathcal{C}(C', C, \mathcal{G}) : C \in \mathcal{D}$ with $\pi_{n+1}[C] = C'$, and $\mathcal{G} \in \mathcal{E}(C)\}$. Then $\mathcal{E}'(C')$ is a collection of $\tilde{\mathcal{S}}$-cell decompositions of $C'$ for each $\tilde{\mathcal{S}}$-cell $C' \in \mathcal{D}'$. By the inductive hypothesis $(\mathrm{II})_n$ it follows that there exists an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}^*$ say, of $\pi_{n+1}[D]$ simultaneously compatible with every $\tilde{\mathcal{S}}$-cell in $\mathcal{E}'(C')$ for all $C' \in \mathcal{D}'$.

Now consider an $\tilde{\mathcal{S}}$-cell $E \in \mathcal{D}^*$. Clearly if $C \in \mathcal{D}$, $\mathcal{G} \in \mathcal{E}(C)$ and $B \in \mathcal{G}$, then either $\pi_{n+1}[B] \cap E = \emptyset$ or $E \subseteq \pi_{n+1}[B]$. In the latter case, $B \cap (E \times \mathbb{R})$ will have the form $(f \restriction E, g \restriction E)$ or graph $(f \restriction E)$ for some continuous functions $f, g : \pi_{n+1}[B] \to \mathbb{R}$ (or function $f : \pi_{n+1}[B] \to \mathbb{R}$) lying in $\tilde{\mathcal{S}}$, namely, the functions (or function) defining $B$. Let $\mathcal{F}(E)$ be the (finite) collection of all such functions (restricted to $E$) occurring in this way for some $C \in \mathcal{D}$, $B \in \mathcal{G} \in \mathcal{E}(C)$ with $E \subseteq \pi_{n+1}[B]$, and let $\mathcal{G}(E)$ be the collection of sets of the form $\{\bar{x} \in E : h_1(\bar{x}) = h_2(\bar{x})\}$ for $h_1, h_2 \in \mathcal{F}(E)$. Then $\mathcal{F}(E)$ is clearly a finite collection of sets that lie in $\tilde{\mathcal{S}}_n$ and are closed in $E$. By the generalized version of $(\mathrm{I})_n$ (which only used the inductive hypothesis $(\mathrm{II})_n$) there exists an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}_E$ say, of $E$ simultaneously compatible with each set in $\mathcal{G}(E)$. Clearly if $h_1, h_2 \in \mathcal{G}(E)$, then $h_1 - h_2$ has constant sign (positive, negative or zero) on each $\tilde{\mathcal{S}}$-cell $E' \in \mathcal{D}_E$ and so $\mathcal{D}_E$ and $\mathcal{G}(E)$ determine an $\tilde{\mathcal{S}}$-cell decomposition, $\bar{\mathcal{D}}_E$ say, of $(E \times \mathbb{R}) \cap D$ which is obviously compatible with $(E \times \mathbb{R}) \cap B$ for every $B \in \mathcal{G} \in \mathcal{E}(C)$ for each $C \in \mathcal{D}$.

It only remains to invoke the inductive hypothesis $(\mathrm{II})_n$ once more for the $\tilde{\mathcal{S}}$-cell decomposition $\mathcal{D}^*$ and the collection $\{\{\mathcal{D}_E\} : E \in \mathcal{D}^*\}$ to obtain an $\tilde{\mathcal{S}}$-cell decomposition, $\mathcal{D}^\dagger$ say, of $\pi_{n+1}[D]$ simultaneously compatible with each $\tilde{\mathcal{S}}$-cell in $\mathcal{D}_E$ for all $E \in \mathcal{D}^*$. The required $\tilde{\mathcal{S}}$-cell decomposition of $D$ is $\{(E' \times \mathbb{R}) \cap B : B \in \mathcal{G} \in \mathcal{E}(C), C \in \mathcal{D}\}$.

## 5. Concluding remarks

I do not know if the main theorem remains true if the DSF condition is removed completely, but Macintyre and Karpinski have observed in [7] that it can be weakened by changing "$C^\infty$" to "$C^N$ for all $N$". In other words the definition of DSF (1.7) is modified by stipulating that $A$ has a representation of the form $\pi[Z(f_N)]$ for each $N$, where $f_N : \mathbb{R}^m \to \mathbb{R}$ is a $C^N$ function in $\mathcal{S}$ (and the $m$ must stay small compared to $N$). This is an extremely useful strengthening of our main result when viewed in conjunction with the zero set theorem for *o*-minimal structures (see [5]) and it allowed Macintyre and Karpinski to deduce a relative version of the theorem stated in section 0: the Pfaffian closure of an *o*-minimal structure is also *o*-minimal.

Perhaps the next step is to establish the conclusion of the main theorem for *o*-minimal weak structures determined by their *continuous* functions.

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