

ON A CLASS OF HYPERGEOMETRIC DIAGONALS

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ABSTRACT. We prove that the diagonal of any finite product of algebraic functions of the form

$$(1 - x_1 - \cdots - x_n)^R, \quad R \in \mathbb{Q},$$

is a generalized hypergeometric function, and we provide an explicit description of its parameters. The particular case $(1 - x - y)^R / (1 - x - y - z)$ corresponds to the main identity of Abdelaziz, Koutschan and Maillard in [J. Phys. A 53 (2020), 205201, 16 pp., §3.2]. Our result is useful in both directions: on the one hand it shows that Christol’s conjecture holds true for a large class of hypergeometric functions, on the other hand it allows for a very explicit and general viewpoint on the diagonals of algebraic functions of the type above. Finally, in contrast to the approach of Abdelaziz, Koutschan and Maillard, our proof is completely elementary and does not require any algorithmic help.

1. INTRODUCTION

Let \mathbb{K} be a field of characteristic zero and let $g \in \mathbb{K}[[\mathbf{x}]]$ be a power series in $\mathbf{x} = (x_1, \dots, x_n)$

$$g(\mathbf{x}) = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} g_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{K}[[\mathbf{x}]].$$

The *diagonal* $\text{Diag}(g)$ of $g(\mathbf{x})$ is the univariate power series given by

$$\text{Diag}(g) := \sum_{j \geq 0} g_{j, \dots, j} t^j \in \mathbb{K}[[t]].$$

A power series $h(\mathbf{x})$ in $\mathbb{K}[[\mathbf{x}]]$ is called *algebraic* if there exists a non-zero polynomial $P(\mathbf{x}, T) \in \mathbb{K}[\mathbf{x}, T]$ such that $P(\mathbf{x}, h(\mathbf{x})) = 0$; otherwise, it is called *transcendental*.

If $g(\mathbf{x})$ is algebraic, then its diagonal $\text{Diag}(g)$ is usually transcendental; however, by a classical result by Lipshitz [Lip88], $\text{Diag}(g)$ is D-finite, i.e., it satisfies a non-trivial linear differential equation with polynomial coefficients in $\mathbb{K}[t]$. Equivalently, the coefficients sequence $(g_{j, \dots, j})_{j \geq 0}$ of $\text{Diag}(g)$ is P-recursive, i.e., it satisfies a linear recurrence with polynomial coefficients (w.r.t. j).

When a P-recursive sequence satisfies a recurrence of order 1, we say that it is *hypergeometric*. An important class of power series, whose coefficients sequence

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is hypergeometric by design, is that of generalized hypergeometric functions. Let $p, q \in \mathbb{N}$ and a_1, \dots, a_p and b_1, \dots, b_q be rational numbers such that $b_i + j \neq 0$ for any $i, j \in \mathbb{N}$. The *generalized hypergeometric function* ${}_pF_q$ with parameters a_1, \dots, a_p and b_1, \dots, b_q is the univariate power series in $\mathbb{K}[[t]]$ defined by

$${}_pF_q([a_1, \dots, a_p]; [b_1, \dots, b_q]; t) := \sum_{j \geq 0} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{t^j}{j!},$$

where $(x)_j := x(x+1) \cdots (x+j-1)$ is the rising factorial.

We are interested in this article in the following (dual) questions:

- (i) What are the algebraic power series $g(\mathbf{x})$ whose diagonal $\text{Diag}(g)$ is a generalized hypergeometric function ${}_pF_q$?¹
- (ii) What are the hypergeometric sequences $(a_j)_{j \geq 0}$ whose generating functions $\sum_{j \geq 0} a_j t^j$ can be written as diagonals of algebraic power series?

Already for $n \in \{1, 2\}$ these questions² are non-trivial. The classes of diagonals of bivariate rational power series and of algebraic power series coincide [Pól22, Fur67]. Hence, questions (i) and (ii) contain as a sub-question the characterization of algebraic hypergeometric functions. This problem was only recently solved in a famous paper by Beukers and Heckman [BH89].

Another motivation for studying questions (i) and (ii) comes from the well-known conjecture below, formulated in [Chr87, Chr90] by Christol. Recall that $f \in \mathbb{Q}[[t]]$ is called *globally bounded* if it has finite non-zero radius of convergence and $\beta \cdot f(\alpha \cdot t) \in \mathbb{Z}[[t]]$ for some non-zero $\alpha, \beta \in \mathbb{Z}$.

Christol's conjecture. If $f \in \mathbb{Q}[[t]]$ is D-finite and globally bounded, then $f = \text{Diag}(g)$ for some $n \in \mathbb{N}$ and some algebraic power series $g \in \mathbb{Q}[[x_1, \dots, x_n]]$.

Christol's conjecture is still largely open, even in the particular case when f is a generalized hypergeometric function. In this case, it has been proved [Chr87, Chr90] in two extreme subcases: when all the bottom parameters b_i are integers (case of “minimal monodromy weight”, in the terminology of [Chr15]) and when they are all non-integers (case of “maximal monodromy weight”). In the first extremal case, the proof is based on the observation that

$$(1) \quad {}_pF_q([a_1, \dots, a_p]; [1, \dots, 1]; t) = (1-t)^{-a_1} \star \cdots \star (1-t)^{-a_p},$$

where \star denotes the Hadamard (term-wise) product, and on the fact that diagonals are closed under Hadamard product [Chr88, Prop. 2.6]. In the second extremal case, it is based on the equivalence between being globally bounded and algebraic; this equivalence, proved by Christol [Chr87, Chr90], is itself based on [BH89].

The other cases (of “intermediate monodromy weight”) are widely open. A first explicit example of this kind, itself still open as of today, was given by Christol himself as soon as 1987 [Chr87, §VII]:

¹Note that a necessary condition is that $q = p - 1$, since the radius of convergence must be finite and non-zero.

²From an algorithmic viewpoint, questions (i) and (ii) are very different in nature: while (i) is decidable (given an algebraic power series, one can decide if its diagonal is hypergeometric, for instance by combining the algorithms in [BLS13] and [Pet92]), the status of question (ii) is not known (does there exist an algorithm which takes as input a hypergeometric sequence and outputs an algebraic series whose diagonal is the generating function of the input sequence?).

Is $f(t) = {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right]; \left[1, \frac{1}{3}\right]; t\right)$ the diagonal of an algebraic power series?

Two decades later, Bostan et al. [BBCHM12, BBCHM13] produced a large list of about 100 similar ${}_3F_2$ (globally bounded) functions, which are potential counterexamples to Christol's conjecture (in the sense that, like ${}_3F_2([1/9, 4/9, 5/9]; [1, 1/3]; t)$, they are not easily reducible to the two known extreme cases, via closure properties of diagonals, e.g., with respect to Hadamard products). In 2020, Abdelaziz, Koutschan and Maillard [AKM20, §3] managed to show that two members of that list, namely ${}_3F_2([1/9, 4/9, 7/9]; [1, 1/3]; t)$ and ${}_3F_2([2/9, 5/9, 8/9]; [1, 2/3]; t)$, are indeed diagonals. Precisely,

$$(2) \quad {}_3F_2\left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9}\right]; \left[1, \frac{2}{3}\right]; 27t\right) = \text{Diag}\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right)$$

and

$$(3) \quad {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right]; \left[1, \frac{1}{3}\right]; 27t\right) = \text{Diag}\left(\frac{(1-x-y)^{2/3}}{1-x-y-z}\right).$$

Sec. 3 of [AKM20] also contains the following extension of identities (2) and (3), to any $R \in \mathbb{Q}$:

$$(4) \quad {}_3F_2\left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3}\right]; [1, 1-R]; 27t\right) = \text{Diag}\left(\frac{(1-x-y)^R}{1-x-y-z}\right).$$

A common feature of identities (2) and (3) (and their extension (4)) is that the top parameters are in arithmetic progression, as opposed to Christol's initial example. However, they are the first known examples of generalized hypergeometric functions with intermediate monodromy weight, not trivially reducible to the two known extreme cases, and which are provably diagonals.

Our first result extends identity (4) to a much larger class of (transcendental) generalized hypergeometric functions.

Theorem 1. *Let $R, S \in \mathbb{Q}$ and $n, N \in \mathbb{N}$ such that $S \neq 0$ and $0 \leq n \leq N$. Set $s := N - n$ and $Q := S - R$. Then the generalized hypergeometric function*

$${}_{N+s}F_{N+s-1}\left(\left[\frac{Q}{N}, \frac{Q+1}{N}, \dots, \frac{Q+N-1}{N}, \frac{S}{s}, \dots, \frac{S+s-1}{s}\right]; \left[\frac{Q}{s}, \dots, \frac{Q+s-1}{s}, 1, \dots, 1\right]; N^N t\right)$$

is equal to the diagonal

$$\text{Diag}\left(\frac{(1-x_1-\dots-x_n)^R}{(1-x_1-\dots-x_N)^S}\right).$$

Note that identity (4) corresponds to the particular case $(n, N, S) = (2, 3, 1)$ of Theorem 1. The proof of (4) given in [AKM20, §3.2] relies on an algorithmic technique called *creative telescoping* [Kou13], which works in principle³ on any diagonal of an algebraic function, as long as the number $\max(n, N)$ of indeterminates is *fixed*. Our identity in Theorem 1 contains a number of indeterminates which

³Creative telescoping algorithms, such as the one in [BLS13], compute a linear differential equation for $\text{Diag}(g(\mathbf{x}))$. This equation is converted to a linear recurrence, whose hypergeometric solutions can be computed using Petkovšek's algorithm [Pet92]. Note that the complexity (in time and space) of these algorithms increases with n, N, R and S .

are themselves variable; hence it cannot be proved by creative telescoping in this generality. In §2.5 we offer instead a direct and elementary proof of a natural generalization. We note that Theorem 1 can also be proven by directly multiplying out the argument of the diagonal using the multinomial theorem, collecting needed terms and simplifying using the Chu-Vandermonde identity.

Note that in the theorem above, and similarly in later statements, the restriction on R and S to be rational numbers is actually superfluous. Indeed, from the proofs it is obvious that the identities hold for arbitrary (formal) parameters R, S ; however, we include this condition because we wish $(1 - x_1 - \cdots - x_n)^R (1 - x_1 - \cdots - x_N)^{-S}$ to be an algebraic function.

In §2 we will further generalize Theorem 1 in two distinct directions. The first extension (Theorem 3) shows that the diagonal of the product of an arbitrary number of arbitrary powers of linear forms of the type $1 - x_1 - \cdots - x_m$ is again a generalized hypergeometric function. The second extension (Theorem 4) shows that under a condition on the exponents the same stays true if the product is multiplied with another factor of the form $(1 - x_1 - \cdots - x_{m-2} - 2x_{m-1})^b$. For instance, when restricted to $m = 3$ variables, these results specialize as follows:

Theorem 2. *For any $R, S, T \in \mathbb{Q}$, we have:*

$$(5) \quad \text{Diag} \left((1-x)^R (1-x-y)^S (1-x-y-z)^T \right) = {}_6F_5 \left(\left[\frac{-(R+S+T)}{3}, \frac{1-(R+S+T)}{3}, \frac{2-(R+S+T)}{3}, \frac{-(S+T)}{2}, \frac{1-(S+T)}{2}, -T \right]; \left[\frac{-(R+S+T)}{2}, \frac{1-(R+S+T)}{2}, -(S+T), 1, 1 \right]; 27t \right)$$

and

$$(6) \quad \text{Diag} \left((1-x)^R (1-x-2y)^S (1-x-y-z)^{-1} \right) = {}_4F_3 \left(\left[\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2} \right]; \left[\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1 \right]; 27t \right).$$

Note that (6) generalizes and explains the following two identities observed in [AKM20, Eq. (30)–(31)]

$$(7) \quad {}_3F_2 \left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right]; \left[1, \frac{2}{3} \right]; 27t \right) = \text{Diag} \left(\frac{(1-x-2y)^{2/3}}{1-x-y-z} \right)$$

and

$$(8) \quad {}_3F_2 \left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9} \right]; \left[1, \frac{5}{6} \right]; 27t \right) = \text{Diag} \left(\frac{(1-x-2y)^{1/3}}{1-x-y-z} \right).$$

Once again, our proofs of the (generalizations of) identities (5) and (6) are elementary, and do not rely on algorithmic tools.

One may wonder if other generalizations are possible, for instance whether the coefficient 2 can be replaced by a different one in (6). The following example shows that this is not the case. Let

$$U(t) = \text{Diag} \left(\frac{\sqrt[3]{1-ax}}{1-x-y} \right).$$

Then, the coefficients sequence $(u_j)_{j \geq 0}$ of $U(t)$ satisfies the second-order recurrence relation

$$\begin{aligned} 2a^2(6n+5)(3n+1)u_n - 3(n+1)(a^2+4a-4)n + 2a^2+18a-18)u_{n+1} \\ = 9(1-a)(n+2)(n+1)u_{n+2}. \end{aligned}$$

When $a \in \{0, 1, 2\}$, the sequence $(u_j)_{j \geq 0}$ also satisfies a shorter recurrence, of order 1, as shown by our main results. In these cases, $U(t)$ is a hypergeometric function. When $a \notin \{0, 1, 2\}$, the second-order recurrence is the minimal-order satisfied by $(u_j)_{j \geq 0}$; hence $U(t)$ is not a hypergeometric function. This can be proved either using the explicit identity

$$U(t) = \sqrt[3]{\frac{a/2}{1-4t} + \frac{1-a/2}{(1-4t)^{\frac{3}{2}}}}$$

or by using the general approach in [Bou99, §5].

An apparent weakness of our results is that they only provide examples with parameters in (unions of) arithmetic progressions. This is true, as long as identities are used alone. But symmetries may be broken by combining different identities and using for instance Hadamard products. As an illustration, by taking the Hadamard product in both sides of the following identities

$${}_3F_2\left(\left[\frac{Q}{3}, \frac{Q+1}{3}, \frac{Q+2}{3}\right]; [1, Q]; t\right) = \text{Diag}\left(\frac{(1-\frac{x_1}{3}-\frac{x_2}{3})^{1-Q}}{1-\frac{x_1}{3}-\frac{x_2}{3}-\frac{x_3}{3}}\right)$$

and

$${}_2F_1\left(\left[\frac{Q}{6}, \frac{Q+3}{6}\right]; \left[\frac{Q}{3}\right]; t\right) = \text{Diag}\left(\frac{(1-\frac{x_4}{2})^{1-Q/3}}{1-\frac{x_4}{2}-\frac{x_5}{2}}\right),$$

both particular cases of Theorem 1, one deduces that the *non-symmetric* hypergeometric function

$${}_4F_3\left(\left[\frac{Q}{6}, \frac{Q+3}{6}, \frac{Q+1}{3}, \frac{Q+2}{3}\right]; [1, 1, Q]; t\right)$$

is equal to the diagonal

$$\text{Diag}\left(\left(1-\frac{x_1}{3}-\frac{x_2}{3}\right)^{1-Q}\left(1-\frac{x_4}{2}\right)^{1-\frac{Q}{3}}\left(1-\frac{x_1}{3}-\frac{x_2}{3}-\frac{x_3}{3}\right)^{-1}\left(1-\frac{x_4}{2}-\frac{x_5}{2}\right)^{-1}\right).$$

Similarly, the *non-symmetric* hypergeometric function⁴

$${}_3F_2\left(\left[\frac{Q}{6}, \frac{Q+3}{6}, \frac{2Q+3}{6}\right]; \left[1, \frac{2Q}{3}\right]; t\right)$$

is equal to the diagonal

$$\text{Diag}\left(\left(1-\frac{x_1}{2}\right)^{1-\frac{Q}{3}}\left(1-\frac{x_3}{2}\right)^{1-\frac{2Q}{3}}\left(1-\frac{x_1}{2}-\frac{x_2}{2}\right)^{-1}\left(1-\frac{x_3}{2}-\frac{x_4}{2}\right)^{-1}\right).$$

A natural challenge is to prove (or disprove) that Christol's ${}_3F_2$ can be obtained in such a way.

⁴Amusingly, the above ${}_3F_2$ is not only asymmetric, but it also shares another similarity with Christol's example: the sum of two of the three top parameters is equal to the third one. This pattern occurs in several other examples.

As a final remark, one should not think that every generalized hypergeometric function which is a diagonal needs to have a representation like in our Theorem 1 or 2. For instance, the diagonal $\text{Diag}((1 - (1 + w)(x + y + z))^{-1})$ is equal [BBMW15] to the generalized hypergeometric function

$${}_4F_3\left(\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]; \left[1, 1, \frac{1}{2}\right]; \frac{729}{4}t\right) = 1 + 18t + 1350t^2 + \cdots,$$

which is seemingly not of the form covered by any of our results.

2. GENERAL CASE

This section contains several parts: first we introduce in §2.1 and §2.2 some notation and state the two general Theorems 3 and 4. Then we explain them in §2.3 by means of four examples, showing that both Theorem 1 and Theorem 2 are special cases. Further, we continue in §2.4 with several lemmas and their proofs. Finally, the general theorems are proven in §2.5 and §2.6.

2.1. First statement. Let $N \in \mathbb{N} \setminus \{0\}$ and $b_1, \dots, b_N \in \mathbb{Q}$ with $b_N \neq 0$. We want to prove that the diagonal of

$$(9) \quad R(x_1, \dots, x_N) := (1 + x_1)^{b_1} (1 + x_1 + x_2)^{b_2} \cdots (1 + x_1 + \cdots + x_N)^{b_N}$$

can be expressed as a hypergeometric function. For each $k = 1, \dots, N$ we define the tuple

$$u^k := \left(\frac{B(k)}{N - k + 1}, \frac{B(k) + 1}{N - k + 1}, \dots, \frac{B(k) + N - k}{N - k + 1} \right),$$

where $B(k) := -(b_k + \cdots + b_N)$. For $k = 1, \dots, N - 1$ we set

$$v^k := \left(\frac{B(k)}{N - k}, \frac{B(k) + 1}{N - k}, \dots, \frac{B(k) + N - k - 1}{N - k} \right).$$

Moreover set $v^N := (1, 1, \dots, 1)$ with exactly $N - 1$ ones. It follows by construction that the lengths of the tuples

$$u := (u^1, \dots, u^N) \quad \text{and} \quad v := (v^1, \dots, v^N)$$

are given by $M := N + \cdots + 2 + 1 = N(N + 1)/2$ and $M - 1$ respectively. We have the following generalization of Theorem 1:

Theorem 3. *It holds that*

$$\text{Diag}(R(x_1, \dots, x_N)) = {}_M F_{M-1}([u]; [v]; (-N)^N t).$$

2.2. Second statement. Let $N \in \mathbb{N} \setminus \{0\}$ and $b_1, \dots, b_N \in \mathbb{Q}$. Assume that $b_N \neq 0$ and $b_{N-1} + b_N = -1$. We will prove that, for any $b \in \mathbb{Q}$, we can express

$$(1 + x_1 + \cdots + x_{N-2} + 2x_{N-1})^b \cdot R(x_1, \dots, x_N)$$

as a hypergeometric function as well. Again, let $B(k) := -(b_k + \cdots + b_N)$. For each $k = 1, \dots, N - 2$ we define the tuple

$$\tilde{u}^k := \left(\frac{B(k) - b}{N - k + 1}, \frac{B(k) - b + 1}{N - k + 1}, \dots, \frac{B(k) - b + N - k}{N - k + 1} \right)$$

and set $\tilde{u}^{N-1} := -(b_{N-1} + b_N + b)/2 = (1 - b)/2$ and $\tilde{u}^N := -b_N$. Moreover, for $k = 1, \dots, N - 2$ we set

$$\tilde{v}^k := \left(\frac{B(k) - b}{N - k}, \frac{B(k) - b + 1}{N - k}, \dots, \frac{B(k) - b + N - k - 1}{N - k} \right),$$

and $\tilde{v}^{N-1} := (1, 1, \dots, 1)$ with exactly $N - 1$ ones. It follows by construction that the lengths of the tuples

$$\tilde{u} := (\tilde{u}^1, \dots, \tilde{u}^N) \quad \text{and} \quad \tilde{v} := (\tilde{v}^1, \dots, \tilde{v}^{N-1})$$

are given by $M - 1 = N + \dots + 4 + 3 + 1 + 1 = N(N + 1)/2 - 1$ and $M - 2$ respectively.

Theorem 4. *It holds that*

$$\begin{aligned} \text{Diag}((1 + x_1 + \dots + x_{N-2} + 2x_{N-1})^b \cdot R(x_1, \dots, x_N)) \\ = {}_{M-1}F_{M-2}([\tilde{u}]; [\tilde{v}]; (-N)^N t). \end{aligned}$$

2.3. Examples. Let us list some examples of the general theorems and draw the connection to previous statements.

- (1) First we emphasize that Theorem 1 follows promptly from the more general Theorem 3 by letting all $b_j = 0$ except $b_n = R$ and $b_N = -S$. Clearly, the change $\mathbf{x} \mapsto -\mathbf{x}$ in the algebraic function is reflected by the change $t \mapsto (-1)^N t$ in its diagonal.
- (2) Letting $N = 3$ in Theorem 3 we obtain immediately the first part of Theorem 2.
- (3) If moreover $T = -1$ in the case $N = 3$, we achieve a cancellation of the last parameter and are left with

$$\begin{aligned} \text{Diag}\left(\frac{(1+x)^R(1+x+y)^S}{1+x+y+z}\right) = \\ {}_5F_4\left(\left[\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}, \frac{2-S}{2}\right]; \right. \\ \left. \left[\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1-S, 1\right]; -27t\right). \end{aligned}$$

- (4) Comparing with the similar situation of Theorem 4 in the case $N = 3$ and $b_{N-1} = -1 - b_N = 0$, we see that a family of ${}_4F_3$ functions remains and covers the second statement of Theorem 2.

2.4. Lemmas and proofs. In this section we will state and prove necessary lemmas for the proofs of Theorems 3 and 4.

Lemma 1. *Let N be a positive integer and $b_1, \dots, b_N \in \mathbb{Q}$ such that $b_N \neq 0$. It holds that*

$$\begin{aligned} [x_1^{k_1} \dots x_N^{k_N}] (1+x_1)^{b_1} (1+x_1+x_2)^{b_2} \dots (1+x_1+\dots+x_N)^{b_N} \\ = \binom{b_N}{k_N} \binom{b_{N-1}+b_N-k_N}{k_{N-1}} \dots \binom{b_1+\dots+b_N-k_N \dots -k_2}{k_1}. \end{aligned}$$

This result contains the core identity of the present paper, since it enables the connection between the algebraic functions $R(\mathbf{x})$ of the form (9) and hypergeometric sequences. It can be proven in two ways: a direct approach works by multiplying the left-hand side out using the multinomial theorem, picking the needed coefficient and reducing the sum using the Chu-Vandermonde identity several times. This procedure is rather tedious and not instructive; therefore we present a combinatorially inspired proof.

Proof. Because $(1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_{N-1})^{b_{N-1}}$ does not depend on x_N , we obtain that the left-hand side of the equation is equal to

$$\begin{aligned} & [x_1^{k_1} \cdots x_N^{k_N}] \prod_{i=1}^{N-1} \left(1 + \sum_{j=1}^i x_j\right)^{b_i} \cdot (1+x_1+\cdots+x_{N-1})^{b_N} \\ & \qquad \qquad \qquad \cdot \left(1 + \frac{x_N}{1+x_1+\cdots+x_{N-1}}\right)^{b_N} \\ & = [x_1^{k_1} \cdots x_N^{k_N}] \prod_{i=1}^{N-1} \left(1 + \sum_{j=1}^i x_j\right)^{b_i} \cdot (1+x_1+\cdots+x_{N-1})^{b_N} \\ & \qquad \qquad \qquad \cdot \sum_{k \geq 0} \binom{b_N}{k} \left(\frac{x_N}{1+x_1+\cdots+x_{N-1}}\right)^k \\ & = \binom{b_N}{k_N} [x_1^{k_1} \cdots x_{N-1}^{k_{N-1}}] (1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_{N-2})^{b_{N-2}} \\ & \qquad \qquad \qquad \cdot (1+x_1+\cdots+x_{N-1})^{b_{N-1}+b_N-k_N}. \end{aligned}$$

Now the claim follows by iteration. \square

We remark that the same strategy as above can be used to prove an even more general statement. Let

$$R(\mathbf{x}) = \prod_{i=1}^N \left(1 + \sum_{j \in I_i} x_j\right)^{b_i},$$

for rational numbers b_1, \dots, b_N , such that all variables x_1, \dots, x_N appear in $R(\mathbf{x})$, and sets $I_1, \dots, I_N \subseteq \{1, \dots, N\}$ with the property that $I_1 \cup \cdots \cup I_{n-1} \subsetneq I_1 \cup \cdots \cup I_{n-1} \cup I_n$ for all $n = 1, \dots, N$. Then, similarly to the statement in Lemma 1, the coefficient of $x^{k_1} \cdots x^{k_N}$ in $R(\mathbf{x})$ is a product of binomial coefficients and the diagonal of $R(\mathbf{x})$ is a generalized hypergeometric function. The notation, however, becomes quite cumbersome in this setting and no new ideas are needed; therefore we stick to the more insightful but less general case $I_n = \{1, \dots, n\}$.

Note that Lemma 1 shares some similarities with Straub's Theorem 3.1 in [Str14], which provides explicit expressions of rational power series of the form

$$\begin{aligned} & \left((1+x_1+\cdots+x_{\lambda_1})(1+x_{\lambda_1+1}+\cdots+x_{\lambda_1+\lambda_2}) \cdots \right. \\ & \qquad \qquad \qquad \left. \cdot (1+x_{\lambda_1+\cdots+\lambda_{\ell-1}}+\cdots+x_N) - \alpha \cdot x_1 x_2 \cdots x_N \right)^{-1}. \end{aligned}$$

In Lemma 1, we allow products of linear forms with arbitrary exponents; however there is no term $\alpha \cdot x_1 x_2 \cdots x_N$, while in [Str14, Theorem 3.1] the linear forms have disjoint variables and appear with exponent 1. Setting $\alpha = 0$ in Straub's formula also yields a product of binomial coefficients.

It is legitimate to wonder whether there is a common generalization of Lemma 1 and Thm. 3.1 in [Str14]. For instance, one may ask for which values of α is the diagonal

$$\text{Diag} \left((\sqrt{1-x}(1-y) - \alpha xy)^{-1} \right) = 1 + (\alpha + 1/2)t + (\alpha^2 + 2\alpha + 3/8)t^2 + \cdots$$

hypergeometric? For a general α , the minimal recurrence satisfied by the coefficients of the diagonal is of order 4, for $\alpha = \pm i/2$ it is of order 3, and it seems that the only

rational value of α for which there exists a shorter recurrence is $\alpha = 0$, in which case the diagonal is hypergeometric.

Now we want to verify a similar statement for the situation as in Theorem 4, so the case where we deal with the coefficient sequence of $(1 + x_1 + \cdots + x_{N-2} + 2x_{N-1})^b \cdot R(x_1, \dots, x_N)$. We lay the grounds for a lemma similar to Lemma 1, by starting with a rather surprising identity.

Lemma 2. *Let $k \in \mathbb{N}$ and $b \in \mathbb{Q}$ arbitrary. It holds that*

$$[x^k] \frac{(1+2x)^b}{(1+x)^{k+1}} = 4^k \binom{(b-1)/2}{k}.$$

Proof. First notice that for arbitrary a, b we can compute

$$[x^k](1+2x)^b(1+x)^a = [x^k] \left(\sum_{i \geq 0} 2^i \binom{b}{i} x^i \right) \left(\sum_{j \geq 0} \binom{a}{j} x^j \right) = \sum_{j=0}^k 2^j \binom{b}{j} \binom{a}{k-j}.$$

So we set $a = -(k+1)$ and obtain

$$[x^k] \frac{(1+2x)^b}{(1+x)^{k+1}} = \sum_{j=0}^k 2^j \binom{b}{j} \binom{-k-1}{k-j} = 2^k \sum_{j=0}^k (-1)^j 2^{-j} \binom{b}{k-j} \binom{k+j}{k}.$$

It remains to prove the following identity⁵

$$(10) \quad \sum_{j=0}^k (-2)^{-j} \binom{b}{k-j} \binom{k+j}{k} = 2^k \binom{(b-1)/2}{k}.$$

To do this, we note that

$$\sum_{j=0}^k (-1)^j \binom{b}{k-j} \binom{k+j}{k} u^j = \binom{b}{k} {}_2F_1([-k, k+1]; [b+1-k]; u),$$

and

$${}_2F_1([-k, k+1]; [b+1-k]; 1/2) = \frac{\Gamma((b+1-k)/2)\Gamma((b+2-k)/2)}{\Gamma((b+1-2k)/2)\Gamma((b+2)/2)} = 2^k \frac{\binom{(b-1)/2}{k}}{\binom{b}{k}},$$

by Kummer's identity [Kum36, Eq. 3, p. 134]. \square

The proof above explains the special role of the coefficient $a = 2$ mentioned in §1: it is one of the few values, along with 1 and -1 , for which there exists a closed form expression for the evaluation of a ${}_2F_1([\alpha, 1-\alpha]; [\gamma]; u)$ at $u = 1/a$.

Now we can proceed and prove the essential lemma for Theorem 4. Note that contrary to Lemma 1 the following statement is purely about diagonal coefficients and not for general exponents anymore. Except for the missing factor $\binom{b_{N-1}+b_{N-k}}{k}$ and the new two factors 4^k and $\binom{(b-1)/2}{k}$ the formulas are completely analogous.

⁵Note that identity (10) could alternatively be proven by using Zeilberger's creative telescoping algorithm [Zei91], or derived from identity (3.42) in [Gou72, p. 27] by setting $2n - x = b$, multiplying with 2^k and reverting the summation.

Lemma 3. Let N be a positive integer and $b_1, \dots, b_N \in \mathbb{Q}$ such that $b_{N-1} + b_N = -1$. For any $b \in \mathbb{Q}$ the coefficient of $x_1^k \cdots x_N^k$ in

$$(1 + x_1 + \cdots + x_{N-2} + 2x_{N-1})^b \cdot (1 + x_1)^{b_1} \cdots (1 + x_1 + \cdots + x_N)^{b_N}$$

is given by

$$4^k \binom{(b-1)/2}{k} \binom{b_N}{k} \cdot \binom{b_{N-2} + b_{N-1} + b_N + b - 2k}{k} \cdots \cdot \binom{b_1 + \cdots + b_N + b - (N-1)k}{k}.$$

Proof. By the same argument as in the proof of Lemma 1, the left-hand side is equal to $\binom{b_N}{k}$ multiplied with the coefficient of $x_1^k \cdots x_{N-1}^k$ in

$$(1 + x_1 + \cdots + x_{N-2} + 2x_{N-1})^b \cdot \prod_{j=1}^{N-2} \left(1 + \sum_{i=1}^j x_i\right)^{b_j} \cdot (1 + x_1 + \cdots + x_{N-2} + x_{N-1})^{b_{N-1} + b_N - k}.$$

Because the product in the middle does not depend on x_{N-1} and since we assumed $b_{N-1} + b_N = -1$, we can first compute

$$\begin{aligned} [x_{N-1}^k] \left(1 + 2 \frac{x_{N-1}}{1 + x_1 + \cdots + x_{N-2}}\right)^b & \left(1 + \frac{x_{N-1}}{1 + x_1 + \cdots + x_{N-2}}\right)^{-1-k} \\ & = 4^k \binom{(b-1)/2}{k} (1 + x_1 + \cdots + x_{N-2})^{-k}, \end{aligned}$$

by Lemma 2. Therefore we are left with

$$4^k \binom{(b-1)/2}{k} \binom{b_N}{k} \cdot [x_1^k \cdots x_{N-2}^k] \prod_{j=1}^{N-3} \left(1 + \sum_{i=1}^j x_i\right)^{b_j} \cdot (1 + x_1 + \cdots + x_{N-2})^{b_{N-2} + b - 2k - 1},$$

which is easily computed using Lemma 1. \square

Note that the requirement $b_{N-1} + b_N = -1$ comes from the $+1$ in the denominator of the left-hand side in Lemma 2. Since this identity is itself surprising and does not allow for obvious generalizations, the condition on the relationship of b_{N-1} and b_N is necessary.

2.5. Proof of Theorem 3. For the proof of Theorem 3 we will only use Lemma 1 and algebraic manipulations similar to the proof of Bober's Lemma 4.1 in [Bob09].

By Lemma 1 we obtain the coefficient of t^n for any $n \in \mathbb{N}$ on the left-hand side:

$$\begin{aligned} [t^n] \text{Diag}((1 + x_1)^{b_1} \cdots (1 + x_1 + \cdots + x_N)^{b_N}) \\ = \binom{b_N}{n} \cdots \binom{b_1 + \cdots + b_N - (N-1)n}{n}. \end{aligned}$$

For the right-hand side we use the fact that for all a, b and non-negative integers n it holds

$$(a/b)_n ((a+1)/b)_n \cdots ((a+b-1)/b)_n \cdot b^{bn} = (a)_{bn}.$$

Then

$$U_k := \prod_{i=1}^{N-k+1} (u_i^k)_n = \frac{(-b_k - \cdots - b_N)_{(N-k+1)n}}{(N-k+1)^{(N-k+1)n}},$$

for all $k = 1, \dots, N$. Similarly,

$$V_k := \prod_{i=1}^{N-k} (v_i^k)_n = \frac{(-b_k - \cdots - b_N)_{(N-k)n}}{(N-k)^{(N-k)n}},$$

for all $k = 1, \dots, N-1$. Clearly $V_N := \prod_{i=1}^{N-1} (v_i^N)_n = (n!)^{N-1}$. We deduce that

$$\begin{aligned} [t^n]_M F_{M-1}([u]; [v]; (-N)^N t) &= \frac{(-N)^{nN}}{n!} \prod_{i=1}^N \frac{U_i}{V_i} \\ &= \frac{(-1)^{nN}}{(n!)^N} \prod_{i=1}^N \frac{(-b_i - \cdots - b_N)_{(N-i+1)n}}{(-b_i - \cdots - b_N)_{(N-i)n}} \\ &= \frac{(-1)^{nN}}{(n!)^N} \prod_{i=1}^N (-b_i - \cdots - b_N + (N-i)n)_n. \end{aligned}$$

The claim of Theorem 3 follows from the fact that

$$\begin{aligned} (-1)^n \frac{(-b_k - \cdots - b_N + (N-k)n)_n}{n!} \\ &= (-1)^n \binom{-b_k - \cdots - b_N + (N-k+1)n - 1}{n} \\ &= \binom{b_k + \cdots + b_N - (N-k)n}{n}. \end{aligned}$$

2.6. Proof of Theorem 4. The proof of Theorem 4 is very similar: we will use Lemma 3 and the same reasoning as before. The only difference lies in the fact that because the hypergeometric function has one parameter less, we need to redefine U_{N-1}, V_{N-1} and V_N . Recall that the denominator of U_k was given by $(N-k+1)^{(N-k+1)n}$ and it cancelled with the denominator of V_{k-1} . In the present case, \tilde{U}_{N-1} will have no denominator and therefore 2^{2n} from \tilde{V}_{N-2} survives. This fits with the 4^k in the statement of Lemma 3 and is another indicator for the importance and essence of the constant $a = 2$.

Using Lemma 3 we obtain the coefficient of t^n for any $n \in \mathbb{N}$ on the left-hand side:

$$\begin{aligned} [t^n] \text{Diag}((1 + x_1 + \cdots + 2x_{N-1} + x_N)^b R(x_1, \dots, x_N)) \\ &= 4^n \binom{(b-1)/2}{n} \binom{b_N}{n} \cdot \binom{b_{N-2} + b_{N-1} + b_N + b - 2n}{n} \cdots \\ &\quad \cdot \binom{b_1 + \cdots + b_N + b - (N-1)n}{n}. \end{aligned}$$

By the same reasoning as before, we have for all $k = 1, \dots, N-2, N$

$$\tilde{U}_k := \prod_{i=1}^{N-k+1} (\tilde{u}_i^k)_n = \frac{(-b_k - \cdots - b_N - b)_{(N-k+1)n}}{(N-k+1)^{(N-k+1)n}},$$

and similarly

$$\tilde{V}_k := \prod_{i=1}^{N-k} (\tilde{v}_i^k)_n = \frac{(-b_k - \cdots - b_N - b)_{(N-k)n}}{(N-k)^{(N-k)n}},$$

for $k = 1, \dots, N-2$. Clearly $\tilde{V}_{N-1} := \prod_{i=1}^{N-1} (\tilde{v}_i^{N-1})_n = (n!)^{N-1}$ and we set $\tilde{V}_N := 1$. Moreover, this time we have $\tilde{U}_{N-1} := (\tilde{u}^{N-1})_n = ((1-b)/2)_n$. Altogether, we find

$$\begin{aligned} [t^n]_M F_{M-1}([\tilde{u}]; [\tilde{v}]; (-N)^N t) &= \frac{(-N)^{nN}}{n!} \prod_{i=1}^N \frac{\tilde{U}_i}{\tilde{V}_i} \\ &= \frac{(-1)^{nN} 2^{2n} ((1-b)/2)_n}{(n!)^N} \prod_{i=1}^{N-2} \frac{(-b_i - \cdots - b_N - b)_{(N-i+1)n}}{(-b_i - \cdots - b_N - b)_{(N-i)n}} \cdot \frac{(-b_N - b)_n}{1} \\ &= 4^n \frac{(-1)^n ((1-b)/2)_n}{n!} \prod_{i=1}^{N-2} \frac{(-1)^n (-b_i - \cdots - b_N + (N-i)n)_n}{n!} \cdot \frac{(-1)^n (-b_N - b)_n}{n!}. \end{aligned}$$

Using the same final observation as before we conclude the proof.

3. ALGEBRAICITY AND HADAMARD GRADE

3.1. Algebraic cases. We address here the following question: given $b_1, \dots, b_N \in \mathbb{Q}$, $b_N \neq 0$, when is the diagonal

$$\text{Diag}(R(\mathbf{x})) = \text{Diag}((1+x_1)^{b_1} \cdots (1+x_1 + \cdots + x_N)^{b_N})$$

an algebraic function?

Corollary 1. *Diag($R(\mathbf{x})$) is algebraic if and only if $N = 2$ and $b_2 \in \mathbb{Z}$, or $N = 1$.*

In the proof below we will use several times the following useful fact [Chr15, Thm. 33]: if a generalized hypergeometric function is algebraic, then its monodromy weight is zero, that is the number of integer bottom parameters is at most equal to the number of integer top parameters.

Proof. By Theorem 3 it is sufficient to study the algebraicity of the generalized hypergeometric function $H(t)$ defined by

$${}_{N(N+1)/2}F_{N(N+1)/2-1}([u^1, \dots, u^{N-1}, -b_N]; [v^1, \dots, v^{N-1}, 1, 1, \dots, 1]; t),$$

where u^k and v^k , $k = 1, \dots, N-1$ are defined like in §2.1:

$$u^k := \left(\frac{b}{\ell+1}, \frac{b+1}{\ell+1}, \dots, \frac{b+\ell}{\ell+1} \right) \quad \text{and} \quad v^k := \left(\frac{b}{\ell}, \frac{b+1}{\ell}, \dots, \frac{b+\ell-1}{\ell} \right)$$

for $b = -(b_k + b_2 + \cdots + b_N)$ and $\ell = N - k$.

By definition, $N-1$ of the bottom parameters are ones. We claim that each tuple u^k contains at most one integer and if it does contain one, then v^k does as well. From the definition of u^k it follows that if some $u_i^k \in \mathbb{Z}$ then $b \in \mathbb{Z}$ and $b \equiv -i+1 \pmod{\ell+1}$. This shows that for any $k = 1, \dots, N-1$ at most one $u_i^k \in \mathbb{Z}$. Because of the definition of v^k we see that if $b \in \mathbb{Z}$ and $b \equiv i \pmod{\ell}$ for some $i \in \{1, \dots, \ell\}$, then $v_i^k \in \mathbb{Z}$. This proves the claim.

Henceforth, in order to introduce new integer parameters on the top, while not creating equally many on the bottom, it is only possible to choose $-b_N$ integer. Therefore, in order to achieve monodromy weight zero – a necessary condition for

algebraicity of $H(t)$ – we need to have $N - 1 \leq 1$. From the same argument it follows that in the case $N - 1 = 1$, we need to have $-b_N \in \mathbb{Z}$.

Obviously for $N = 1$ the diagonal is algebraic, so it remains to prove that, conversely, when $-b_2 =: S$ is an integer and $b_1 =: R \in \mathbb{Q}$ arbitrary, then the diagonal in

$$(11) \quad \text{Diag} \left(\frac{(1-x_1)^R}{(1-x_1-x_2)^S} \right) = {}_3F_2 \left(\left[\frac{S-R}{2}, \frac{S-R+1}{2}, S \right]; [1, S-R]; 4t \right)$$

is an algebraic function. If R is an integer too, this follows from [Pól22, Fur67]. In the general case, one can rewrite the ${}_3F_2$ in (11) as the Hadamard product

$$(12) \quad {}_3F_2 \left(\left[\frac{S-R}{2}, \frac{S-R+1}{2}, S \right]; [1, S-R]; t \right) \\ = {}_2F_1 \left(\left[\frac{S-R}{2}, \frac{S-R+1}{2} \right]; [S-R]; t \right) \star (1-t)^{-S}.$$

The ${}_2F_1$ is algebraic as it corresponds to Case I in Schwarz's table [Sch73]. Since S is an integer, $(1-t)^{-S}$ is a rational function. We conclude by applying Jungen's theorem [Jun31, Thm. 8]: the Hadamard product of an algebraic and a rational function is algebraic; see also [Sta99, Prop. 6.1.11]. \square

3.2. Hadamard grade. Recall that the *Hadamard grade* [AMF11] of a power series $S(t)$ is the least positive integer $h = h(S)$ such that $S(t)$ can be written as the Hadamard product of h algebraic power series, or ∞ if no such product exists. Since algebraic power series are diagonals [Fur67, §3], and diagonals are closed under Hadamard product [Chr88, Prop. 2.6], any power series with finite Hadamard grade is a diagonal [AMF11, Thm. 7]. Conversely, it is not clear whether diagonals always have finite Hadamard grade.⁶

A natural question in relation with Corollary 1 is the following: given $b_1, \dots, b_N \in \mathbb{Q}$, determine the Hadamard grade of $\text{Diag}(R(\mathbf{x})) = \text{Diag}((1+x_1)^{b_1} \cdots (1+x_1 + \cdots + x_N)^{b_N})$, or at least decide if it is finite or not. We have the following result as an application of the classification in [BH89] (in particular case 1 in Table 8.3) and our main theorem.

Corollary 2. *The Hadamard grade of $\text{Diag}(R(\mathbf{x}))$ is at most N .*

Proof. Like in the proof of Corollary 1, we define

$$u^k := \left(\frac{b}{\ell+1}, \frac{b+1}{\ell+1}, \dots, \frac{b+\ell}{\ell+1} \right) \quad \text{and} \quad v^k := \left(\frac{b}{\ell}, \frac{b+1}{\ell}, \dots, \frac{b+\ell-1}{\ell} \right)$$

for $b = -(b_k + b_2 + \cdots + b_N)$, $\ell = N - k$ and $k = 1, \dots, N - 1$. Then by Theorem 3 it is sufficient to study the Hadamard grade of

$$H(t) = {}_{N(N+1)/2}F_{N(N+1)/2-1}([u^1, \dots, u^{N-1}, -b_N]; [v^1, \dots, v^{N-1}, 1, 1, \dots, 1]; t).$$

Now notice that

$$H(t) = {}_NF_{N-1}([u^1]; [v^1]; t) \star \cdots \star {}_2F_1([u^{N-1}]; [v^{N-1}]; t) \star {}_1F_0([-b_N]; []; t),$$

and that each hypergeometric function in the Hadamard product is algebraic by [BH89, Thm. 7.1]. \square

⁶There exist diagonals of any prescribed finite grade [RR14, Cor. 1 & 2], assuming the Rohrlisch–Lang conjecture [Wal06, Conj. 22]. If Christol's conjecture also holds, there exist diagonals of infinite grade [RR14, Prop. 1].

For instance, by Theorem 1, when $S = 1$, $R = 1/2$, the diagonal

$$\text{Diag} \left(\frac{(1 - x_1 - x_2)^R}{(1 - x_1 - x_2 - x_3)^S} \right)$$

is a transcendental ${}_2F_1$, which can be written as the Hadamard product of two algebraic functions

$${}_2F_1 \left(\left[\frac{1}{6}, \frac{5}{6} \right]; \left[\frac{1}{2} \right]; t \right) \star (1 - t)^{-1/2}$$

(the ${}_2F_1$ being algebraic by Schwarz's classification [Sch73]), hence its Hadamard grade is 2.

Similarly, the diagonals from (2) and (3) have Hadamard grade 2 due to the identities

$${}_3F_2 \left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9} \right]; \left[1, \frac{2}{3} \right]; t \right) = {}_3F_2 \left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9} \right]; \left[\frac{1}{2}, \frac{2}{3} \right]; t \right) \star (1 - t)^{-1/2}$$

and

$${}_3F_2 \left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right]; \left[1, \frac{1}{3} \right]; t \right) = {}_3F_2 \left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right]; \left[\frac{1}{2}, \frac{1}{3} \right]; t \right) \star (1 - t)^{-1/2}$$

and due to the fact that the two ${}_3F_2$'s on the right-hand side are algebraic by the interlacing criterion [BH89, Thm. 4.8]; see Figure 1 for a pictorial proof, where red points correspond to top parameters, and blue points to bottom parameters (and the additional parameter 1).

More generally, the diagonal from (4) has Hadamard grade 2 due to the identity

$${}_3F_2 \left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3} \right]; [1, 1-R]; t \right) = {}_3F_2 \left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3} \right]; \left[\frac{1}{2}, 1-R \right]; t \right) \star (1 - t)^{-1/2},$$

since the ${}_3F_2$ on the right-hand side is an algebraic function for any $R \in \mathbb{Q}$ (with Fig. 1 replaced by a similar one, containing only interlacing blue right triangles and red equilateral triangles).

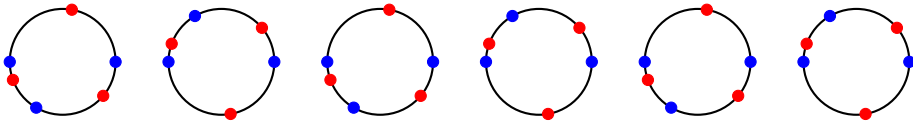


FIGURE 1. A pictorial proof of the algebraicity of ${}_3F_2 \left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3} \right]; \left[\frac{1}{2}, 1-R \right]; t \right)$ for $R \in \{\frac{1}{3}, \frac{2}{3}\}$. There are $\varphi(18) = 6$ conditions to check, which lead to two distinct interlacing configurations.

This observation provides an alternative (and probably the shortest) proof that the hypergeometric functions in (2), (3) and (4) are diagonals of algebraic functions. However, this viewpoint does not yield such a compact diagonal representation as found in [AKM20] and in our main theorem.

The same observation also quickly solves two more cases amongst the 16 cases in the list [BBCHM12, p. 58], namely those ${}_3F_2([N_1/9, N_2/9, N_3/9]; [1, M_1/3]; t)$ for which $(N_1, N_2, N_3; M_1)$ is $(1, 4, 7; 2)$ or $(2, 5, 8; 1)$.

Furthermore, using the interlacing criterion it is easy to see that the hypergeometric function ${}_3F_2([1/9, 4/9, 7/9]; [a, b]; t)$ is algebraic if (a, b) or (b, a) occurs in the set

$$\{(3/4, 1/4), (2/3, 1/3), (2/3, 1/6), (1/2, 1/3), (1/2, 1/6)\}.$$

Similarly, ${}_3F_2([2/9, 5/9, 8/9]; [a, b]; t)$ is algebraic if (a, b) or (b, a) is part of

$$\{(5/6, 1/2), (5/6, 1/3), (3/4, 1/4), (2/3, 1/2), (2/3, 1/3)\}.$$

Moreover, both ${}_3F_2\left(\left[\frac{1}{4}, \frac{3}{8}, \frac{7}{8}\right]; \left[\frac{2}{3}, \frac{1}{3}\right]; t\right)$ and ${}_3F_2\left(\left[\frac{1}{8}, \frac{3}{4}, \frac{5}{8}\right]; \left[\frac{2}{3}, \frac{1}{3}\right]; t\right)$ are algebraic.

The previous analysis proves Corollary 3.

Corollary 3. *The hypergeometric function*

$${}_3F_2([A, B, C]; [1, D]; t)$$

has Hadamard grade 2 (hence is a diagonal) for $(A, B, C; D)$ in the following set

$$\begin{aligned} &\left\{ (1/4, 3/8, 7/8; 1/3), (1/4, 3/8, 7/8; 2/3), (1/8, 5/8, 3/4; 1/3), (1/8, 5/8, 3/4; 2/3), \right. \\ & (1/9, 4/9, 7/9; 1/2), \textcolor{red}{(1/9, 4/9, 7/9; 1/3)}, (1/9, 4/9, 7/9; 1/4), (1/9, 4/9, 7/9; 1/6), \\ & \textcolor{orange}{(1/9, 4/9, 7/9; 2/3)}, (1/9, 4/9, 7/9; 3/4), (2/9, 5/9, 8/9; 1/2), \textcolor{orange}{(2/9, 5/9, 8/9; 1/3)}, \\ & \left. (2/9, 5/9, 8/9; 1/4), \textcolor{red}{(2/9, 5/9, 8/9; 2/3)}, (2/9, 5/9, 8/9; 3/4), (2/9, 5/9, 8/9; 5/6) \right\}. \end{aligned}$$

Note that the authors of [BBCHM12, BBCHM13] produced in 2011 a list of 116 potential counter-examples to Christol's conjecture; they displayed a sublist of 18 cases in the preprint [BBCHM12, Appendix F], of which they selected 3 cases that were published in [BBCHM13, §5.2]. As of today, to our knowledge, the 3 cases in [BBCHM13] are still unsolved,⁷ while 2 of the 18 cases in [BBCHM12] have been solved in [AKM20] (in red, above) and 2 others in the current paper (in orange, above). From the list of 116 cases, only 2 were previously solved, in [AKM20]. Corollary 3 solves 14 cases more, raising the number of solved cases to 16 (out of 116). Finally we note that another 24 cases could be resolved with the ansatz ${}_3F_2([A, B, C]; [1, D]; t) = {}_2F_1([A, B]; [r]; t) \star {}_2F_1([C, r]; [D]; t)$ and finding $r \in \mathbb{Q}$ such that both hypergeometric functions on the right-hand side are algebraic. The remaining 76 cases, however, seem to be much more difficult.

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⁷Rivoal and Roques proved in [RR14, Proposition 1] that one of the 3 cases in [BBCHM13, §5.2], namely ${}_3F_2\left(\left[\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right]; \left[1, \frac{1}{2}\right]; t\right)$, has infinite grade assuming the Rohrllich–Lang conjecture [Wal06, Conj. 22]; the status of the analogous statement for Christol's ${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right]; \left[1, \frac{1}{3}\right]; t\right)$ is still unclear.

REFERENCES

- [AKM20] Y. Abdelaziz, C. Koutschan, and J.-M. Maillard, *On Christol's conjecture*, J. Phys. A **53** (2020), no. 20, 205201, 16, DOI 10.1088/1751-8121/ab82dc. MR4093477
- [AMF11] J.-P. Allouche and M. Mendès France, *Hadamard grade of power series*, J. Number Theory **131** (2011), no. 11, 2013–2022, DOI 10.1016/j.jnt.2011.04.011. MR2825108
- [BBCHM12] A. Bostan, S. Boukraa, G. Christol, S. Hassani, and J.-M. Maillard, *Ising n -fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity*, 2012. [arXiv:1211.6031](#).
- [BBCHM13] A. Bostan, S. Boukraa, G. Christol, S. Hassani, and J.-M. Maillard, *Ising n -fold integrals as diagonals of rational functions and integrality of series expansions*, J. Phys. A, 46(18):185202, 44, 2013.
- [BBMW15] A. Bostan, S. Boukraa, J.-M. Maillard, and J.-A. Weil, *Diagonals of rational functions and selected differential Galois groups*, J. Phys. A **48** (2015), no. 50, 504001, 29, DOI 10.1088/1751-8113/48/50/504001. MR3434836
- [BH89] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function ${}_nF_{n-1}$* , Invent. Math. **95** (1989), no. 2, 325–354, DOI 10.1007/BF01393900. MR974906
- [BLS13] Alin Bostan, Pierre Lairez, and Bruno Salvy, *Creative telescoping for rational functions using the Griffiths-Dwork method*, ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2013, pp. 93–100, DOI 10.1145/2465506.2465935. MR3206345
- [Bob09] Jonathan W. Bober, *Factorial ratios, hypergeometric series, and a family of step functions*, J. Lond. Math. Soc. (2) **79** (2009), no. 2, 422–444, DOI 10.1112/jlms/jdn078. MR2496522
- [Bou99] Delphine Boucher, *About the polynomial solutions of homogeneous linear differential equations depending on parameters*, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC), ACM, New York, 1999, pp. 261–268, DOI 10.1145/309831.309952. MR1802089
- [Chr87] G. Christol, *Fonctions hypergéométriques bornées*, Groupe Travail Anal. Ultramétrique **14** (1986-1987), 1–16, Talk no. 8.
- [Chr88] Gilles Christol, *Diagonales de fractions rationnelles* (French), Séminaire de Théorie des Nombres, Paris 1986–87, Progr. Math., vol. 75, Birkhäuser Boston, Boston, MA, 1988, pp. 65–90. MR990506
- [Chr90] Gilles Christol, *Globally bounded solutions of differential equations*, Analytic number theory (Tokyo, 1988), Lecture Notes in Math., vol. 1434, Springer, Berlin, 1990, pp. 45–64, DOI 10.1007/BFb0097124. MR1071744
- [Chr15] Gilles Christol, *Diagonals of rational fractions*, Eur. Math. Soc. Newsl. **97** (2015), 37–43. MR3409223
- [Fur67] Harry Furstenberg, *Algebraic functions over finite fields*, J. Algebra **7** (1967), 271–277, DOI 10.1016/0021-8693(67)90061-0. MR215820
- [Gou72] Henry W. Gould, *Combinatorial identities*, Henry W. Gould, Morgantown, W.Va., 1972. A standardized set of tables listing 500 binomial coefficient summations. MR0354401
- [Jun31] R. Jungen, *Sur les séries de Taylor n'ayant que des singularités algébriques-logarithmiques sur leur cercle de convergence* (French), Comment. Math. Helv. **3** (1931), no. 1, 266–306, DOI 10.1007/BF01601817. MR1509439
- [Kou13] Christoph Koutschan, *Creative telescoping for holonomic functions*, Computer algebra in quantum field theory, Texts Monogr. Symbol. Comput., Springer, Vienna, 2013, pp. 171–194. MR3616751
- [Kum36] E. E. Kummer, *Über die hypergeometrische Reihe $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$* , (Fortsetzung). J. Reine Angew. Math., 15:127–172, 1836.
- [Lip88] L. Lipshitz, *The diagonal of a D -finite power series is D -finite*, J. Algebra **113** (1988), no. 2, 373–378, DOI 10.1016/0021-8693(88)90166-4. MR929767
- [Pet92] Marko Petkovšek, *Hypergeometric solutions of linear recurrences with polynomial coefficients*, J. Symbolic Comput. **14** (1992), no. 2-3, 243–264, DOI 10.1016/0747-7171(92)90038-6. MR1187234

- [Pól22] G. Pólya, *Sur les séries entières, dont la somme est une fonction algébrique*, Enseign. Math. **22** (1921/1922), 38–47.
- [RR14] T. Rivoal and J. Roques, *Hadamard products of algebraic functions*, J. Number Theory **145** (2014), 579–603, DOI 10.1016/j.jnt.2014.06.015. MR3253320
- [Sch73] H. A. Schwarz, *Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt* (German), J. Reine Angew. Math. **75** (1873), 292–335, DOI 10.1515/crll.1873.75.292. MR1579568
- [Sta99] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, DOI 10.1017/CBO9780511609589. MR1676282
- [Str14] Armin Straub, *Multivariate Apéry numbers and supercongruences of rational functions*, Algebra Number Theory **8** (2014), no. 8, 1985–2007, DOI 10.2140/ant.2014.8.1985. MR3285621
- [Wal06] Michel Waldschmidt, *Transcendence of periods: the state of the art*, Pure Appl. Math. Q. **2** (2006), no. 2, Special Issue: In honor of John H. Coates., 435–463, DOI 10.4310/PAMQ.2006.v2.n2.a3. MR2251476
- [Zei91] Doron Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), no. 3, 195–204, DOI 10.1016/S0747-7171(08)80044-2. MR1103727

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