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Computing the Galois Group of a Linear Differential Equation of Order Four

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Abstract. In 1978 J. Kovacic described an efficient algorithm for computing liouvillian solutions of a linear homogeneous differential equation of order two over a field $\mathrm{C}(x)$, where $x' = 1$ and C is an algebraically closed field of characteristic 0. During the years from 1990 to 1994 M. Singer and F. Ulmer published several papers in which they describe efficient algorithms for determining the Galois group of such a differential equation of order two or three and computing liouvillian solutions using this group. In this paper we present results concerning Galois groups of order four linear differential equations. In particular we construct a list of irreducible linear algebraic subgroups of $\mathrm{SL}(4, \mathrm{C})$ where C is an algebraically closed field of characteristic zero. This list is complete up to conjugation, and in the finite primitive case, up to projective equivalence. Then, in keeping with the spirit of the work of Kovacic, Singer and Ulmer we use representation theory to distinguish between the groups in this list.

Keywords: Algebraic Group, Representation, Differential Galois Group, Irreducible, Imprimitive, Primitive, Monomial, Linear Group

1 Introduction

In this paper we give procedures to determine properties of the Galois groups of fourth order linear differential equations, and in many cases, to determine the groups themselves. We shall restrict our work to irreducible fourth order equations since in the case when the differential operator factors one can use the results of [1], [20], [21] and [22]. Furthermore, standard transformations allow one to reduce a linear operator to one whose Galois group is unimodular. Therefore we shall consider irreducible fourth order linear differential operators L over a differential field of characteristic zero with algebraically closed field of constants C and assume that the Galois group is a subgroup of $\mathrm{SL}(4, \mathrm{C})$.

The procedures given in Section 4 are presented as a series of results which allow one to identify a linear algebraic group via the reducibility properties of its representations. These results can be immediately restated in terms of differential equations due to the one-to-one correspondence between the dimensions of $Gal(L)-$submodules of the solutions space V and the orders of factors of $L(y)$. This correspondence is given in detail in Theorem 3.1 of [20] also found in [13] and [18]. This connection between invariant submodules and factorizations of operators can be extended to other modules in addition to the solution space V and is the key to computing the Galois group of a linear differential equation. To see this, consider a vector space W constructed from V via the tools of linear algebra. One may construct from $L(y) = 0$ a new equation which has the $Gal(L)-$module W as its solution space. For Example, given solution space $V \cong \mathrm{C}^m$ of an order m equation, $L(y) = 0$, the equation denoted by $L_{Sym^n}(y) = 0$ is a constructible (see [20]) linear differential equation with solution space isomorphic to $Sym^n\mathrm{V}$. It should be noted that in the case when $Gal(L)$ is not faithful, the equation $L_{Sym^n}(y) = 0$ will not be of maximal order and as a result will not have solution space all of $Sym^n\mathrm{V}$. In order to maintain the effectiveness of a decision algorithm which is dependent on dimension comparisons one would need to choose an equivalent equation having faithful Galois group (see [19] and [22]). Given the equation with solution space isomorphic to $Sym^n\mathrm{V}$, factorizations of $L_{Sym^n}(y)$ correspond to decompositions of $Sym^n\mathrm{V}$ as a $Gal(L)-$module. Due to this direct correspondence between modules and equations the majority of this paper is devoted to distinguishing among the subgroups of SL(4, C) via dimensions of invariant subspaces of auxilliary representations, in particular, those which are linear algebraic costructions of the solution space.

In order to distinguish among the irreducible linear algebraic subgroups of SL(4, C) via representation theory, a complete and detailed list of such groups is needed. Hence, as a secondary result of this paper, one will find a complete list of matrix represenatations of each irreducible unimodular linear algebraic group of degree four, up to conjugation, and in the finite case, up to projective equivalence. The following group theoretic definitions give an initial classification of such groups via their action as modules and constitute the first step in constructing the complete list.

- A subgroup G of $\mathrm{GL}(V)$ is said to act **irreducibly** if the only G$-$invariant subspaces of V are $\{0\}$ and V.
- An irreducible group G is **imprimitive** if, for $k > 1$, there exist subspaces $V_1, \ldots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$ and, for each $g \in \mathrm{G}$, the mapping $\mathrm{V}_i \to g(\mathrm{V}_i)$ is a permutation on the set $S = \{\mathrm{V}_1, \ldots, \mathrm{V}_k\}$.
- The set S is called a **system of imprimitivity of G**.
- If all the subspaces V_i are 1 dimensional, then G is said to be **monomial**.
- An irreducible group $\mathrm{G} \subset \mathrm{GL}(V)$ which is not imprimitive is called **primitive.**

The procedure given in Section 4 is organized in such a way as to allow one to distinguish between a primitive group, an imprimitive monomial group, and an imprimitive nonmomomial group immediately. For example, one is able to distinguish a monomial subgroup from any other subgroup of SL(4, C) via the dimensions of the irreducible components in the decompositions of the modules $\wedge^2$V and Sym^2V. One can distinguish a finite from an infinite primitive group via the modules Sym^3V and Sym^6V and one may determine the dimension of the diagonal of a monomial group via any of the modules $\wedge^2$V, Sym^2V, and Sym^4V. Furthermore, one may also use the representation theory to identify very specific groups as in the following result.

Let G be an irreducible subgroup of SL$(4, \mathbf{C})$. Let V $\cong$ C^4 and denote by π the canonical projection from GL$(4, \mathbf{C})$ onto PGL$(4, \mathbf{C})$. Then $\pi(\mathbf{G}) = \mathbf{G}_{168}$ if and only if $\wedge^2$V is irreducible and Sym^2V has an irreducible G−invariant subspace of dimension 7.

This paper is organized as follows. In Section 2 we construct the groups, giving a complete listing via degree four matrix representations. Section 3 contains the representation theory involved in decomposing auxilliary modules. Included in this section are specific procedures and examples based on the theorem of Clifford for decomposing representations of imprimitive groups. The author computed the irreducible decompositions of the modules $\wedge^2$V, sym^2V, Sym^3V, Sym^4V and Sym^6V under the action of a representative of each class of degree four linear algebraic groups. The results of these computations are tabulated for the reader in Tables 1–5 in the Appendix. In Section 4 information recorded in the tables is translated into a series of results which allow one to identify a group or distinguish between linear algebraic groups via their actions on these modules. In their totality, the results from this section serve as an effective decision procedure for choosing a Galois group.

We used *GAP* for many preliminary calculations and wish to thank the members of the *GAP* Group, in particular Alexander Hulpke, Werner Nickel and Gretchen Ostenheimer, as well as Gene Cooperman and Felix Ulmer, for their help. The calculations presented in this paper were carried out using *MAGMA*. We wish to thank the *MAGMA* administration, in particular John Cannon, Allan Steel, and Bruce Cox for their advice, technical support, and willingness to expand *MAGMA* to meet our needs. Also, we would would like to express appreciation to Jaques-Arthur Weil for bringing to our attention a gap in the construction of the list of the infinite primitive groups. This paper contains the results appearing in the thesis [9]. I would like to extend a special thank you to my advisor, Michael F. Singer, for his numerous comments and suggestions.

2 The Irreducible Unimodular Linear Algebraic Groups of Degree 4

We construct matrix representations of the irreducible, unimodular linear algebraic groups of degree four. The results from this section will serve as a complete list for the purpose of obtaining all possible dimension combinations in the irreducible decomposition of a given module under the action of such a group. We consider the cases, primitive and imprimitive, separately and use these properties of the groups to construct them. For the convenience of the reader, we have tabulated these groups in the **Summary of Group Notation** found in the Appendix. By the definition of imprimitivity, such a group, G, contains a normal subgroup, H, of finite index which is a subdirect product of isomorphic primitive groups. Hence we begin with a discussion of primitive groups.

2.1 The Primitive Groups

Any primitive subgroup of $\mathrm{GL}(n, \mathrm{C})$ projects onto a primitive subgroup of $\mathrm{PGL}(n, \mathrm{C})$ or equivalently $\mathrm{PSL}(n, \mathrm{C})$. A well known application of this fact is that any infinite primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$ is projectively equivalent to $\mathrm{SL}(2, \mathrm{C})$ and hence is a central extension of $\mathrm{SL}(2, \mathrm{C})$. These are the groups $(\mathrm{SL}(2, \mathrm{C}))_n = \{A \in \mathrm{GL}(2, \mathrm{C}) : \det A^n = 1\}$. One may also use this fact to obtain the finite primitive groups of degree four.

The Finite Primitive Groups. Matrix generators of the 30 finite primitive projective groups of degree four have been constructed by Blichfeldt in [3]. Blichfeldt contructs the finite subgroups of $\mathrm{PSL}(n, \mathrm{C})$ via the existence of and reducibility properties of normal subgroups. The groups are organized as such in [3] and range from projective orders of 144 to 25290. For each of Blichfeldt's groups, PG, we search for a set of nonisomorphic primitive subgroups of $\mathrm{SL}(4, \mathrm{C})$ which project onto PG. Any such group G will have a set of generators which are projectively equivalent to a set given by Blichfeldt. Hence, we obtain the set of all determinant one matrices which project onto an element of PG by lifting a generator of PG to the set of four scalar multiples in $\mathrm{SL}(4, \mathrm{C})$. The multiplicative closure of this set it the largest subgroup of $\mathrm{SL}(4, \mathrm{C})$ which projects onto PG and as such contains any projectively equivalent group. We denote this lift by **FPi** in order to correspond to Blichfeldt's group notation i°. A search was done using MAGMA programs to generate subgroups and character tables for nonisomorphic yet projectively equivalent groups. The author then wrote and applied a MAGMA program for decomposing other representations of these groups. All results giving distinct combinations in terms of dimensions of irreducibles are listed in Table 5 of the Appendix. In only one case, that of the group **FP29**, were two projectively equivalent subgroups found giving distinct dimension combinations for the representations considered. The second group, denoted **FP29B** has order twice that of the corresponding projective group 29°. In all other cases, the group **FPi** represented in the table has order 4 times the order of the projective group i°.

The Infinite Primitive Groups. Given an infinite primitive group, G, Proposition 2.3 of [20] implies that the connected component of the identity, G^0, is a semisimple group. In keeping with the spirit of Blichfeldt's construction of the finite primitive groups, we consider two cases separately, that when G^0 is irreducible, in which case we will employ a bit of Lie algebra, and that when G^0 is reducible. In [2] one sees that a finite primitive group G containing a reducible normal subgroup $H \subseteq G' \times G''$ is contained in the tensor product representation of these two, necessarily primitive, groups G' and G''. Although the finite case proofs do not simply carry over to the infinite case, the following lemmas and theorem proven by M.F. Singer give analogous results.

Lemma 2.1.1 *Let G be an infinite primitive subgroup of $SL(4, C)$. Then either G^0 is an irreducible group or G^0 is conjugate to*

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}.$$

Proof. Since G is primitive and hence an irreducible group, G^0 is reductive and hence completely reducible. If G^0 is not irreducible then we can write $C^4 = \bigoplus_{i=1}^{m} V_i$ where each V_i is an irreducible G^0-module. Since G is primitive, the V_i must be isomorphic G^0-modules. If $\dim V_i = 1$ then G^0 would be conjugate to a subgroup of constant matrices in $SL(4, C)$ and hence would be trivial. Therefore G^0 is conjugate to a subgroup of $\left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in GL(2, C) \right\}$. Since $G \subseteq SL(4, C)$ we have that $(\det A)^2 = 1$ and since G^0 is connected, $G^0 \subseteq \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$. Now, let H be the image of G^0 under the map which takes $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ to A. If H is a proper subgroup of $SL(2, C)$ then V_i is not an irreducible G_0-module therefore we must have that $G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$.

Lemma 2.1.2 *Let* $G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$.

1. *The centralizer of G in $GL(4, C)$ is* $\tilde{C} = \left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : (ad - bc) \neq 0 \right\}$.
2. *The normalizer of G in $GL(4, C)$ is $\tilde{C} \cdot G$.*

Proof. Let $G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$.

(Proof of 1) Let $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, where each $u_i \in GL(2, C)$, centralize G. Then $u_i A = A u_i$ for each i and all $A \in SL(2, C)$. Schur's Lemma then implies that

each u_i is a constant matrix and so we may write $\mathrm{U} = \begin{bmatrix} a\mathrm{I} & b\mathrm{I} \\ c\mathrm{I} & d\mathrm{I} \end{bmatrix}$. The determinant of this matrix is $(ad - bc)^2$ and hence this quantity must be equal to 1.

(Proof of 2) Let $\mathrm{U} \in \mathrm{GL}(4, \mathrm{C})$ normalize G. Since G is isomorphic to $\mathrm{SL}(2, \mathrm{C})$ whose automorphism group contains only inner automorphisms, we have that UV^{-1} centralizes G for some $\mathrm{V} \in \mathrm{G}$. The result now follows from the previous lemma.

2.1.1 G^0 reducible

Theorem 2.1.1 *Let* $(SL(2, C))_2 = \{A \in GL(2, C) : det A^2 = 1\}$.

1. *Let G be a primitive subgroup of* $SL(4, C)$ *with* $G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in SL(2, C) \right\}$. *Then there exists a finite primitive subgroup,* $\tilde{H}$, *of* $(SL(2, C))_2$ *such that* $G = HG^0$ *where* $H = \left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \tilde{H}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
2. *Furthermore, for any primitive subgroup* $\tilde{H}$ *of* G_2, HG^0 *is a primitive subgroup of* $SL(4, C)$ *where H is defined as above.*

Proof. (Proof of 1). Let $\tilde{C}$ be as in Lemma 2.1.2 and let $\mathrm{G} \cap \tilde{C} = \mathrm{H}$. Since $\mathrm{G} \subseteq \tilde{C}\mathrm{G}^0$, the group H maps surjectively onto G/G^0. Therefore $\mathrm{HG}^0 = \mathrm{G}$. Since $\tilde{C} \cap \mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix} \right\}$, H is finite. Since $\mathrm{H} \subseteq \mathrm{SL}(4, \mathrm{C})$, we have that $\tilde{\mathrm{H}} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a\mathrm{I} & b\mathrm{I} \\ c\mathrm{I} & d\mathrm{I} \end{bmatrix} \in \mathrm{H} \right\}$ is a finite subgroup of G_2. We will show now that $\tilde{\mathrm{H}}$ is primitive. Assume not and let $g = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \in \mathrm{SL}(2, \mathrm{C})$ satisfy $g\mathrm{H}g^{-1} \subseteq \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -a \\ a^{-1} & 0 \end{bmatrix} : a \in \mathrm{C}^* \right\}$. Then conjugating by $h = \begin{bmatrix} u_1\mathrm{I} & u_2\mathrm{I} \\ u_3\mathrm{I} & u_4\mathrm{I} \end{bmatrix}$ allows us to assume that $\mathrm{G} = \mathrm{HG}^0$ where H is contained in $\left\{ \begin{bmatrix} a\mathrm{I} & 0 \\ 0 & a^{-1}\mathrm{I} \end{bmatrix} : a \in \mathrm{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -a\mathrm{I} \\ a^{-1}\mathrm{I} & 0 \end{bmatrix} : a \in \mathrm{C}^* \right\}$ and $\mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix} : \mathrm{A} \in \mathrm{SL}(2, \mathrm{C}) \right\}$. This would imply that G would be either irreducible or imprimitive, a contradiction.

(Proof of 2) Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of C^4, that is the basis used to define $\mathrm{G}^0 = \left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A} \end{bmatrix} : \mathrm{A} \in \mathrm{SL}(2, \mathrm{C}) \right\}$. Assume G is not primitive and let $\mathrm{C}^4 = \bigoplus_{i=1}^m \mathrm{V}_i$ where G permutes the V_i. Since G^0 is connected, we have that G^0 leaves each V_i invariant. Therefore it must be that $m = 2$, each V_i has

dimension 2 and V_1 and V_2 are isomorphic G^0- modules. Furthermore, the V_1 and V_2 are G^0- isomorphic to the 2-dimensional irreducible $SL(2, C)-$ modules. We therefore can select a basis $\{f_1, f_2\}$ of V_1 and $\{f_3, f_4\}$ of V_2 such that for any $g \in G^0$, the matrices of g with respect to $\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3.f_4\}$ are the same. That is, for some $A \in SL(2, C)$, $g = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ with respect to both bases. This implies that if U is the change of basis matrix then U commutes with G^0 and hence by Lemma 2.1.2, U is of the form $U = \begin{bmatrix} u_1 I & u_2 I \\ u_3 I & u_4 I \end{bmatrix}$. Furthermore, all matrices of UHU^{-1} are either block diagonal or block skew diagonal. Therefore $\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \tilde{H} \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}^{-1}$ is contained in $\left\{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in C^*\right\} \cup \left\{\begin{bmatrix} 0 & -a \\ a^{-1} & 0 \end{bmatrix} : a \in C^*\right\}$ contradicting the fact that $\tilde{H}$ is primitive.

Remarks. (1) Note that the finite primitive subgroups of $GL(2, C)$ of determinant ± 1 are the groups $A_4^{SL_2}$, $S_4^{SL_2}$, and $A_5^{SL_2}$ (in the notation of [20]) or an extension of one of these groups by $\sqrt{-1}I$, where I is the identity in $GL(2, C)$. These extensions will be denoted by $(A_4^{SL_2})_2$, $(S_4^{SL_2})_2$, and $(A_5^{SL_2})_2$.

(2) $GL(2, C) \times GL(2, C)$ acts in a natural way on $C^2 \otimes C^2$ via the maps $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 1 \to \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$ and $1 \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$. If $\tilde{H}$ is any of the above six finite groups, we have a homomorphism of $\tilde{H} \times SL(2, C)$ into $GL(4, C)$. (Note: there is a nontrivial kernel.) The groups described in Theorem 2.1.1 are just the images of the groups $\tilde{H} \times SL(2, C)$.

(3) Note that the image of $SL(2, C) \times SL(2, C)$ in $GL(4, C)$ given by the action on $C^2 \otimes C^2$ is $SO(4, C)$. Therefore, if $\tilde{H} \subseteq SL(2, C)$, then the image of $\tilde{H} \times SL(2, C)$ is in $SO(4, C)$.

2.1.2 G^0 irreducible

Lie algebra considerations yield the following:

Theorem 2.1.2 *Let G be an infinite primitive subgroup of $SL(4, C)$. Then one of the following holds:*

1. $G \subseteq \bigcup_{\omega^4=1} \omega G^0$, *where G^0 is one of*
   (a) $SL(2, C)$ *acting on* $Sym^3 C^2$
   (b) $SP(4, C)$
2. $G \subseteq (\cup_{\omega^4=1}\omega G^0) \bigcup J(\cup_{\omega^4=1}\omega G^0)$, *where* $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *and G^0 is the usual representation of $SO(4, C)$.*

3. *$G = SL(4, C)$ acting on W, where W is one of*
   (a) *the usual representation space, $V \approx C^4$*
   (b) *the dual space, V^*, of V.*

Proof. Let G be an infinite primitive subgroup of SL(4, C). Then the connected component of the identity, G^0, is also an infinite primitive subgroup of SL(4, C) which, by Proposition 2.3 of [20], is a semisimple group. Let $\mathscr{G}$ be the Lie algebra associated with G^0. Then $\mathscr{G}$ is a semisimple Lie subalgebra of sl(4, C) which has an irreducible representation of degree four. Up to isomorphism the simple Lie subalgebras of sl(4, C) which can be found in [10] sections 1 and 19 are:

- sl(4, C) of dimension 15.
- sp(4, C) of dimension 10.
- sl(3, C) of dimension 8.
- sl(2, C) of dimension 3.

Any semisimple Lie algebra is a direct sum of simple ones. Therefore using a dimension argument and the above list one can construct a list of irreducible semisimple Lie subalgebras of sl(4, C). Up to isomorphism these are:

1. sl(4, C) of dimension 15
2. sp(4, C) of dimension 10
3. so(4, C) $\approx$ sl(2, C) $\times$ sl(2, C) of dimension 9
4. sl(3, C) of dimension 8
5. sl(2, C) of dimension 3

Let V denote the standard representation space for each of the listed simple Lie algebras. Then the only four dimensional vector space on which sl(2, C) acts irreducibly is $\mathrm{Sym}^3 V$. sl(3, C) has no such representation. sl(4, C) acts irreducibly and distinctly on both V and the dual space V^*. Finally, the only four dimensional vector space on which sp(4, C) acts irreducibly is V itself. Now any vector space on which a direct sum of infinite Lie algebras acts irreducibly is a tensor product of spaces on which the factors act irreducibly. Therefore the only four dimensional vector space on which so(4, C) $\cong$ sl(2, C) $\times$ sl(2, C) acts irreducibly is $V \otimes V$ where $V \cong C^2$.

There is a one-to-one correspondence between the representations of a Lie algebra and its associated simply connected algebraic group. Therefore we have constructed a complete list of connected irreducible linear algebraic subgroups of SL(4, C). They are:

- SL(4, C) acting on its standard representation, V.
- SL(4, C) acting on the dual, V^*.
- SL(2, C) acting on $\mathrm{Sym}^3 V$ where V is its standard representation.
- SP(4, C) acting on its standard representation, V.
- SO(4, C) is the image of SL(2, C) $\times$ SL(2, C) acting on $V \otimes V$ where V is the standard representation for SL(2, C).

The above are all of the possibilities for the connected component of the identity, G^0, of G when G is an infinite primitive subgroup of $SL(4, C)$. Finally, we will determine all of the possibilities for the group G itself. Recall that any element of G not in G^0 induces an automorphism on G^0 via conjugation. Note that automorphism groups are independent of the representation.

In [11] we find that for any semisimple algebraic group G we have that $Aut(G)$ is the semidirect product of the inner automorphisms of G and the automorphism group of the Dynkin diagram associated to the Lie algebra that corresponds to G. In [10] section 11 one finds that the only two groups out of the list above leading to nontrivial automorphism groups of a Dynkin diagram are $SL(4, C)$ and $SO(4, C)$. The diagram automorphisms associated to these groups are elements of $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$.

Case 1. Suppose G^0 is either $SL(2, C)$ or $SP(4, C)$ Then $Aut(G^0) = Inn(G^0)$. Let G be an algebraic subgroup of $SL(4, C)$ with component of the identity G^0. Note that G^0 is a normal subgroup of finite index in G. By the preceeding remarks, conjugation by any element g of G induces an inner automorphism on G^0. That is for each $g \in G$ there is an $h \in G^0$ such that $gxg^{-1} = hxh^{-1}$ for every $x \in G^0$. That is $h^{-1}g$ centralizes the irreducible group G^0. Schur's Lemma implies that $h^{-1}g$ must be scalar. Since $h^{-1}g \in SL(4, C)$, $h^{-1}g = \xi I$ where ξ is a fourth root of unity.

Therefore we have that $G \subseteq \{I, \omega I, \omega^2 I, \omega^3 I\} \cdot G^0$, where ω is a primitive fourth root of unity.

Case 2. Suppose $G^0 = SO(4, C)$. The automorphism group of G^0 is given by the semidirect product of $Inn(G^0)$ and $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Recall that $SO(4, C) \approx SL(2, C) \times SL(2, C)$. The action of $\mathbf{Z}_2$ interchanges the copies of $SL(2, C)$, so in matrix form this automorphism is equivalent to conjugation by the element $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, where I is the identity in $GL(2, C)$. Call this matrix J. Then, as above, since conjugation by any element of G induces an automorphism on G^0, we have that $G \subseteq (\bigcup_\xi \xi \cdot I \cdot G^0)$, where ξ is either a fourth root of unity or a fourth root of unity times J.

Case 3. Suppose $G^0 = SL(4, C)$. Since we have assumed that $G \subseteq SL(4, C)$ we take $G = SL(4, C)$.

2.2 The Imprimitive Groups

Given any imprimitive group, G, there exists a homomorphism α from G into a permutation group S_k defined by the action of the group G on its system of imprimitivity, $\{V_1, \ldots, V_k\}$. In the order four case, this set will contain either four dimension 1 subspaces or two of dimension 2. By the definition of an imprimitive group, $\alpha(G)$ must be a transitive subgroup of S_k and $H = Ker(\alpha)$, or equivalently $Stab(V_1)$, is a subdirect product of isomorphic primitive subgroups of degree equal to $dim(\mathrm{V}_i)$. Clifford theory then implies that the imprimitive representation of G on V is induced from the irreducible representation of H on V_1. The goal of this section is two-fold: 1) to use the above information to construct a complete list of matrix groups representing a set of conjugacy classes of degree four imprimitive unimodular groups, and 2) to construct this set with the property that for each group G, the set of coset representatives, that is, the transversal of H in G, is as simple as possible thus aiding the later decomposition of various G−modules. We do not include the construction of the imprimitive groups for which H is a subdirect product of projectively finite groups due to the following remarks.

Remarks. (1) As with the primitive groups, Blichfeldt has constructed a list of finite imprimitive groups in [2]. The author constructed the same list using an alternate method prior to her knowledge of this paper. This alternative construction is based upon the knowledge of the finite primitive subgroups of degree two (see [20]) and the lifting of subdirect products of these primitive groups. For the purpose of algorithmically distinguishing between these groups the matrix generators of Blichfeldt were used in a the MAGMA program to decompose representations. Nonetheless, for the interested reader the alternative construction of these groups is found in section 4 of the author's thesis [9].
(2) Any finite monomial group $\tilde{\mathrm{G}}$ corresponding to the permutation group $\mathscr{P}_{\mathrm{G}}$ is a subgroup of an infinite monomial group G with that same permutation action. Therefore the G-invariant summands in the decomposition of a G-module W will also be $\tilde{\mathrm{G}}$-invariant, although they may no longer be irreducible. Nonetheless, this information will be enough to distinguish the monomial groups from the other groups.
Given the above remarks we proceed with the contruction of the infinite imprimitive unimodular groups of degree four.

The Infinite Monomial Groups. Let $\mathrm{G} \subseteq \mathrm{SL}(4, \mathrm{C})$ be an infinite imprimitive group with system of imprimitivity $\{\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathrm{V}_4\}$ where each V_i has dimension 1. Then, G is a monomial group. Let $\alpha : \mathrm{G} \to \mathrm{S}_4$ be associated homomorphism and let H be the kernel of this mapping. Then H is a subdirect product of $(\mathrm{C}^*)^4$. Furthermore, since H is a linear algebraic group and is infinite, its connected component, H^0, is a connected subdirect product of $(\mathrm{C}^*)^4$. Such groups are well known and can be listed via their dimension as algebraic varieties. We will show that those which are contained in $\mathrm{SL}(4, \mathrm{C})$ and which are preserved under the permutation action of some transitive subgroup of S_4 are limited to the following:

- $\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}$

- $\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : \ a, b \in \mathrm{C}^* \right\}$
- diag(SL(4, C))

A degree n extension of one of the first two groups will be denoted by H_1^n or H_2^n respectively, and for consitency we will refer to diag(SL(4, C)) as H_3.

Notes and Notation. We adopt the following notation throughout the remaining discussion of monomial groups.

- We will denote by $\mathscr{P}_{\mathrm{G}}$ the image of G under the homomorphism α and refer to this group as the “permutation action" of G.
- Given any $h \in ker(\alpha)$ and any $\sigma \in \mathscr{P}_{\mathrm{G}}$ we will denote by h^σ the element resulting from permuting the diagonal entries of h according to the permutation σ. Note this action is equivalent to conjugating h by any permutation matrix in GL(4, C) corresponding to σ.

Suppose then that H^0 is a connected linear algebraic subgroup of SL(4, C) which is a subdirect product of $(\mathrm{C}^*)^4$. Then either $\mathrm{H}^0 = \mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$ or the dimension of H^0 is one or two. In the following lemma we use the Lie algebra associated with a linear algebraic group to show that a two dimensional diagonal subgroup H^0 of SL(4, C) which is preserved by the action of some transitive permutation group, $\mathscr{P}_{\mathrm{G}} \subseteq \mathrm{S}^4$, must be conjugate to the group denoted H_2^1. One may apply this method to the case where the dimension of H^0 is one, but there is also an analogous method in [9], proof of Lemma 4.4 where one applies the relations of the permutation group to the factor group G/H with the added restriction that the group H^0 is unimodular. In either way, one can show that a connected one dimensional subgroup H^0 of a monomial group $\mathrm{G} \subseteq \mathrm{SL}(4, \mathrm{C})$ must be conjugate via a permutation matix to the group $\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : \ a \in \mathrm{C}^* \right\}$.

Lemma 2.2.1 *Suppose that H is a connected two dimensional unimodular algebraic subdirect product of $(C^*)^4$ which is closed under conjugation by the permutation matrix P_σ for each σ in an order four subgroup of S_4. Then H is conjugate to*

$$\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : \ a, b \in C^* \right\}.$$

Proof. Consider $Lie(\mathrm{H})$, the Lie algebra associated with H. Theorem 5 in [17] implies that H can be defined via polynomials of the form $x^a y^b z^c w^d = 1$ where (a, b, c, d) is in some subset S of $\mathbf{Z}^4$. Therefore $Lie(\mathrm{H})$ is contained in the solution space of $\{a\alpha + b\beta + c\gamma + d\delta = 0 \ : \ (a, b, c, d) \in \mathrm{S}\}$. Without loss of generality we may allow the powers a, b, c and d to be in $\mathbf{Q}$. Since $\mathrm{H} \subset \mathrm{SL}(4, \mathrm{C})$, $(1, 1, 1, 1) \in \mathrm{S}$. Therefore when looking at an arbitrary element of S other than $(1, 1, 1, 1)$, without loss of generality we may assume that $d = 0$. We may also assume that one of the powers is nonzero, since H is a proper subgroup of $(\mathrm{C}^*)^4$. Hence, given such an equation which vanishes on H, we may assume that it is in the form $x^a y^b z = 1$. That is, $(a, b, 1, 0) \in \mathrm{S}$. Now let $\sigma \in \mathrm{S}_4$ act on elements of S by permuting the entries. Recall that given any $h \in \mathrm{H}$ and $\sigma \in \mathscr{P}_{\mathrm{G}}$, $h^\sigma \in \mathrm{H}$ also. This implies that if $(a, b, 1, 0) \in \mathrm{S}$ then $\sigma(a, b, 1, 0) \in \mathrm{S}$ for each $\sigma \in \mathscr{P}_{\mathrm{G}}$. For each order four subgroup of S_4 we use this fact to show that the set of equations defining H is of a specific form. For example, we show in full detail that when $\mathscr{P}_{\mathrm{G}} =< (1234) >$, the equations defining H include $xz = yw = 1$.

Case 1. Suppose that $\mathscr{P}_{\mathrm{G}}$ is cyclic of order four. For now assume that $\mathscr{P}_{\mathrm{G}} = < (1234) >$. Then $(a, b, 1, 0) \in \mathrm{S}$ implies $(0, a, b, 1), (1, 0, a, b)$, and $(b, 1, 0, a)$ are also in S. Therefore $Lie(\mathrm{H})$ is contained in the nullspace of the following matrix.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & 1 & 0 \\ 0 & a & b & 1 \\ 1 & 0 & a & b \\ b & 1 & 0 & a \end{bmatrix}$$

Since $Lie(\mathrm{H})$ has dimension two, the nullspace of this matrix has dimension ≥ 2. This implies that all 3 by 3 cofactors are zero. Upon choosing various submatrices we have that (a, b, c) is a solution of the following system of equations.

$$(1)\ (a^2 + 1)b = 0 \quad (2)\ 1 - b + b^2 - a = 0$$

Therefore $b = 0$ and $a = 1$ which implies that two equations defining H in this case are $xz = yw = 1$.

$$\text{Therefore} \quad \mathrm{H} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : \ a, b \in \mathrm{C}^* \right\}.$$

Similarly, if $\mathscr{P}_{\mathrm{G}}$ contains either of the permutations (1243) or (1324) then applying the same argument to the matrices corresponding to M above yields defining equations for H of the form $xw = yz = 1$ and $xy = zw = 1$ respectively. Note that the group H defined by $xw = yz = 1$ is conjugate to the group above via $\mathrm{P}_{(34)}$ and the group H defined by $xy = zw = 1$ is conjugate to this same group via $\mathrm{P}_{(23)}$.

Case 2. Assuming that the Klein four group is in $\mathscr{P}_{\mathrm{G}}$ and using the same argument as above we can show that the exponents of a defining equation for H must satisfy the following equations.

$$(1)\ -2ab = 0 \quad (2)\ 1 - b - a = 0$$

Therefore either $b = 0$ and $a = 1$ or $a = 0$ and $b = 1$. Hence, the defining equations for such a group include $xz = yw = 1$, $xy = zw = 1$, or $xw = yz = 1$. Each of the resulting groups is conjugate to $\left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*\right\}$ via a permutation matrix in $\mathrm{GL}(4, \mathrm{C})$.

Given the forms for the connected component H^0 of H in the case where its dimension is one or two, Lemma 2.2.2 nails down the form of H. Theorem 2.2.1 will then in essence show that given such a group H there are, up to conjugation, very few ways in which to extend H to the full monomial group, G.

Lemma 2.2.2 *Suppose that $H \subseteq diag(SL(4, C))$ is a finite extension of one of the following two groups.*

$$H_1 = \left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^*\right\} \quad H_2 = \left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^*\right\}.$$

1. *Then there exists a positive integer, n, such that H is equal to either*

$$H_1^n = \left\{\begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1} a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^*, \xi \in U_n\right\}$$

   or

$$H_2^n = \left\{\begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^*, \xi \in U_n\right\}$$

   respectively, where U_n is the group of n^{th} roots of unity.
2. *Suppose furthermore that $H = ker(\alpha)$ where $\alpha : G \to S_4$ is defined via the action of a monomial group G on its system of imprimitivity and that $Im(\alpha)$ contains either the cyclic group $<(1234)>$ or K, the degree four permutation representation of the Klein four group. Then H is equal to either H_1^1, H_1^2 or H_2^n for some positive integer n.*

Proof. *(Proof of 1).* Let f_1 be the map from diag(GL(4, C)) into GL(2, C) defined by $f_1\left(\begin{bmatrix} a&0&0&0\\ 0&b&0&0\\ 0&0&c&0\\ 0&0&0&d \end{bmatrix}\right) = \begin{bmatrix} ab & 0\\ 0 & cd \end{bmatrix}$. Let f_2 be the map from diag(GL(4, C)) into GL(2, C) defined by $f_2\left(\begin{bmatrix} a&0&0&0\\ 0&b&0&0\\ 0&0&c&0\\ 0&0&0&d \end{bmatrix}\right) = \begin{bmatrix} ac & 0\\ 0 & bd \end{bmatrix}$. Note that each f_i is a group homomorphism with $H_i = ker(f_i)$ and $f_i(H_i) \subseteq SL(2, C)$. In each case since H_i is a normal subgroup of finite index in H, $f_i(H_i)$ is a normal subgroup of finite index in $f_i(H)$. Therefore given $\begin{bmatrix} ab & 0\\ 0 & cd \end{bmatrix} \in f_1(H)$ there exists a positive integer p such that $(ab)^p = (cd)^p = 1$ if $H_1 \subseteq H$, and similarly given $\begin{bmatrix} ac & 0\\ 0 & bd \end{bmatrix} \in f_2(H)$ there exists a positive integer p such that $(ac)^p = (bd)^p = 1$ if $H_2 \subseteq H$. Therefore each $f_i(H)$ is contained in a group of the form $\left\{\begin{bmatrix} \xi & 0\\ 0 & \eta \end{bmatrix} : \xi^p = \eta^p = 1\right\} \bigcap SL(2, C)$. That is each $f_i(H) \subseteq \left\{\begin{bmatrix} \xi & 0\\ 0 & \xi^{-1} \end{bmatrix} : \xi^p = 1\right\}$ which is isomorphic to the finite cyclic group U_p. Since each f_i is a group homomorphism, $f_i(H)$ must be isomorphic to a subgroup of U_p. These are of the form U_n for some n dividing p. Therefore H is equal to one of the groups

$$\left\{\begin{bmatrix} \xi a & 0 & 0 & 0\\ 0 & a^{-1} & 0 & 0\\ 0 & 0 & \xi^{-1}a & 0\\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in C^*, \xi \in U_n\right\} \text{ or}$$

$$\left\{\begin{bmatrix} \xi a & 0 & 0 & 0\\ 0 & \xi^{-1}b & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in C^*, \xi \in U_n\right\}.$$

(Proof of 2). Now suppose further that $H = ker(\alpha)$ where α is the map from an infinite monomial group, G, to S_4. Let $Im(\alpha) = \mathscr{P}_G$. Now since $H \triangleleft G$, $h^\sigma \in H$ for each $\sigma \in \mathscr{P}_G$.

Case 1. H is a finite extension of H_1.

Case 1(a). Assume that $\mathscr{P}_G$ contains $< (1234) >$. Let $\sigma = (1234)$.

Then $h = \begin{bmatrix} \xi a & 0 & 0 & 0\\ 0 & a^{-1} & 0 & 0\\ 0 & 0 & \xi^{-1}a & 0\\ 0 & 0 & 0 & a^{-1} \end{bmatrix} \in H$ implies that $h^\sigma \in H$. Therefore there exists a $c \in \mathrm{C}^*$ and an $\eta \in \mathrm{U}_n$ such that $\begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & \xi a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1}a \end{bmatrix} = \begin{bmatrix} \eta c & 0 & 0 & 0 \\ 0 & c^{-1} & 0 & 0 \\ 0 & 0 & \eta^{-1}c & 0 \\ 0 & 0 & 0 & c^{-1} \end{bmatrix}$.

A computation shows that this implies $\eta = \xi = \pm 1$.

Case 1(b). If we assume that $\mathrm{K} \subseteq \mathscr{P}_{\mathrm{G}}$ then we have that $h \in \mathrm{H}$ implies that $h^\sigma \in \mathrm{H}$ where $\sigma = (12)(34)$. This implies exactly as in case 1 that $\eta = \xi = \pm 1$. The result follows.

Case 2. H is a finite extension of H_2. In this case one can show that given any $h \in \mathrm{H}$ $h^\sigma \in \mathrm{H}$ for each σ in either $<(1234)>$ or K.

Remark. Since both A_4 and S_4 contain the three cycle, (123). It is clear that the action of this element does not normalize H_1^m or H_2^n for any positive integer n. Hence, any monomial group with such permutation action must contain all of diag(SL(4, C)) as its maximal diagonal subgroup.

Theorem 2.2.1 *Let G be an infinite algebraic subgroup of $SL(4, C)$ which acts monomially on C^4. Then G is conjugate to $\bigcup_{\gamma \in \mathscr{S}} \gamma H$ where one of the following holds:*

1. *$H = H_1^1, H_1^2, H_2^n$ or H_3, and $\mathscr{S}$ is a set of determinant one permutation matrices corresponding to one of the following subgroups of S_4: $<(1234)>$, the Klein four group K, or the order eight group, $<(1234), (12)(34)>$.*
2. *$H = H_3$ and $\mathscr{S}$ a set of determinant one permutation matrices corresponding to A_4 or S_4.*

Furthermore, in all cases above, the matrices of $\mathscr{S}$ may be chosen to be unitary.

Proof. Let G be a monomial subgroup of SL(4, C). Then G has a normal subgroup $\mathrm{H} \subseteq$ diag(SL(4, C)) such that G/H is isomorphic to a transitive subgroup of S_4 which we will denote by $\mathscr{P}_{\mathrm{G}}$. The transitive subgroups of S_4 are the three order four cyclic groups, the three order eight groups each of which contains one of the order four cyclic groups, the Klein four group, A_4 and S_4. Since H is a finite cyclic extension of H^0, Lemma 2.2.1 implies that H must be conjugate to $\mathrm{H}_1^n, \mathrm{H}_2^n$ for some positive integer n or H_3. It is easy to show that in order to satisfy the closure property of H under the action of the associated permutation group $\mathscr{P}_{\mathrm{G}}$ that the only choices for the integer n extending H_1 are $n = 1$ and $n = 2$. Hence, given a pair H and $\mathscr{P}_{\mathrm{G}}$ we can write G as $\bigcup_{\sigma \in \mathscr{P}_{\mathrm{G}}} m_\sigma \mathrm{H}$ where m_σ is a determinant one permutation matrix corresponding to σ. All that remains to be shown is that the set of representatives for the transversal, $\mathscr{S}$, of H in G may be chosen to be unitary.

Remark. Note that the groups $<(1243)>$ and $<(1324)>$ are conjugate to $<(1234)>$ via (34) and (23) respectively. Correspondingly, for each possible dimension of this diagonal subgroup H of G, the subgroups H associated with $\mathscr{P}_{\mathrm{G}} =<(1243)>$ and $<(1324)>$ are conjugate to those associated with $\mathscr{P}_{\mathrm{G}} =< (1234) >$ via the respective permutation matrices. Hence any monomial group $\mathrm{G} \subset \mathrm{SL}(4, \mathrm{C})$ where $\mathscr{P}_{\mathrm{G}}$ is an order four cyclic group will be conjugate in $\mathrm{GL}(4, \mathrm{C})$ to one where $\mathscr{P}_{\mathrm{G}} =< (1234) >$. Similarly one can show that any monomial subgroup of $\mathrm{SL}(4, \mathrm{C})$ with permutation action an order eight subgroup of S_4 is conjugate to one with permutation action given by $< (1234), (12)(34) >$. Hence we need only look at the cases where $\mathscr{P}_{\mathrm{G}} =< (1234) >, < (1234), (12)(34) >$, K, A_4 or S_4.

In each case, using the fact that $\mathrm{G}/\mathrm{H} = \mathscr{P}_{\mathrm{G}}$, we find a diagonal matrix, D, such that $\mathrm{D}m_\sigma \mathrm{D}^{-1} \in \gamma \mathrm{H}$ where γ is a unitary matrix with the same permutation action, σ. Furthermore, we will show that such a matrix will actually fix H thus implying that G is conjugate to a group of the desired form. We show the case where $\mathscr{P}_{\mathrm{G}} =< (1234) >$ here. For each of the remaining cases an appropriate matrix D can be found in a similar manner.

Case 1. Suppose $\mathscr{P}_{\mathrm{G}} =< (1234) >$. Let $\sigma_1 = (1234)$, $\sigma_2 = (13)(24)$ and $\sigma_3 = (1432)$. For $i = 1, 2, 3$ let m_{σ_i} be any determinant one permutation matrix corresoponding to σ_i. Since $\mathscr{P}_{\mathrm{G}} =< (1234) >$, in each of the following subcases the matrices m_{σ_i} must satisfy the following conditions:

$$(1)\ m_{\sigma_1}^2 \in m_{\sigma_2}\mathrm{H} \text{ and } (2)\ m_{\sigma_1}m_{\sigma_2} \in m_{\sigma_3}\mathrm{H}.$$

Case 1(a). Suppose that $\mathrm{H} = \mathrm{H}_1^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1}a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$

where $n = 1$ or 2. Since the coordinate projection, π_1, of H is onto C^*, we may assume that the entry in the first column of each m_{σ_i} is a 1. Conditions (1) and (2) then imply that if we choose $m_{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & -(a_1a_2)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \end{bmatrix}$,

there exist $\eta, \xi \in \mathrm{U}_n$ such that

$$m_{\sigma_2} = \begin{bmatrix} 0 & 0 & -\eta a_1^{-2} & 0 \\ 0 & 0 & 0 & -a_2^{-1} \\ \eta^{-1} & 0 & 0 & 0 \\ 0 & a_1^2a_2 & 0 & 0 \end{bmatrix} \text{ and } m_{\sigma_3} = \begin{bmatrix} 0 & \xi a_1a_2 & 0 & 0 \\ 0 & 0 & -a_1^{-2}a_2^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1}a_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & (a_1a_2)^{-1} \end{bmatrix}$. A computation shows that for each i $\mathrm{D}m_{\sigma_i}\mathrm{D}^{-1} \in \gamma_i\mathrm{H}$ where γ_i is a unitary permutation matrix in $\mathrm{SL}(4, \mathrm{C})$ corresponding to σ_i. Hence any monomial group $\mathrm{G} = \bigcup_{\sigma \in \mathscr{P}_{\mathrm{G}}} m_\sigma \mathrm{H}$ where $\mathrm{H} = \mathrm{H}_1^n$ and $\mathscr{P}_{\mathrm{G}} =< (1234) >$ is conjugate to a group of the desired form. Using similar arguments we find the necessary matrix D for each of the following cases:

Case 1(b). $\mathrm{H} = \mathrm{H}_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$.

$$\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix}.$$

Case 1(c). $\mathrm{H} = \mathrm{H}_3 = \mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$. In this case any choice of m_{σ_i} of determinant one is already in the associated coset $\gamma_i \mathrm{H}$ and the result follows.

The Imprimitive Nonmonomial Groups. Let $\mathrm{G} \subseteq \mathrm{SL}(4, \mathrm{C})$ be an imprimitive nonmonomial group. Then G has system of imprimitivity $\{\mathrm{V}_1, \mathrm{V}_2\}$ where each V_i has dimension 2. Let $\alpha : \mathrm{G} \to \mathrm{S}_2$ be the associated nonfaithful permutation representation. Then $\mathrm{H} = Ker(\alpha)$ is a subdirect product of $\mathrm{T} \times \mathrm{T}$, where T is a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$ and there exists a transversal for H in G of the form $\{\mathrm{I}, \gamma\}$, where I is the identity of $\mathrm{GL}(4, \mathrm{C})$ and $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where each $g_i \in \mathrm{GL}(2, \mathrm{C})$. In the case where T is projectively finite, the group G is contained in a product of central extensions of some finite primitive group. As previously indicated the representation theory of these groups follows from the repesentation theory of the finite imprimitive groups outlined in [2] and so we focus on the case where T is not projectively finite. This of course implies that T is projectively equivalent to $\mathrm{SL}(2, \mathrm{C})$. We fix the following notational conventions.

Notation.

- π will denote the canonical projection from $\mathrm{GL}(n, \mathrm{C})$ to $\mathrm{PGL}(n, \mathrm{C})$.
- Corresponding to each nonnegative integer n is a primitive subgroup of $\mathrm{GL}(2, \mathrm{C})$ equal to $\{\mathrm{A} \in \mathrm{GL}(2, \mathrm{C}) : \det(\mathrm{A}^n) = 1\}$. We denote this group by $(\mathrm{SL}(2, \mathrm{C}))_n$.

We proceed now with the list of infinite imprimitive subgroups of $\mathrm{SL}(4, \mathrm{C})$. Given that G is unimodular, H is a subdirect product of $(\mathrm{SL}(2, \mathrm{C}))_n \times (\mathrm{SL}(2, \mathrm{C}))_n$, and H is an index two subgroup of G, we are able to construct the following list of such groups.

Theorem 2.2.2 *Let $G \subseteq SL(4, C)$ be an imprimitive nonmonomial linear algebraic group. Then one of the following holds:*

1. *G contains a normal subgroup H of index two such that $H \subseteq T \times T$ where T is projectively finite.*
2. $\exists\ n = 0, 1, \ldots$ *such that G is conjugate to one of the following algebraic groups.*

$$\left\{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in (SL(2,C))_n, det(AB) = 1\right\}$$
$$\bigcup \left\{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} : A, B \in (SL(2,C))_n, det(AB) = 1\right\}$$
$$\left\{\begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in (SL(2,C))_n, d_B = \frac{1}{det(B)}\right\}$$
$$\bigcup \left\{\begin{bmatrix} 0 & \pm d_B B \\ B & 0 \end{bmatrix} : B \in (SL(2,C))_n, d_B = \frac{1}{det(B)}\right\}$$
$$\left\{\begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in (SL(2,C))_n, d_B = \frac{1}{det(B)}\right\}$$
$$\bigcup \left\{\begin{bmatrix} 0 & d_B B \\ B & 0 \end{bmatrix} : B \in (SL(2,C))_n, d_B = \frac{1}{det(B)}\right\}$$

The following technical lemmas are necessary in the proof of this result. When used in conjunction, they allow us to make very specific choices for the coset representative γ in the transversal $\{I, \gamma\}$ of H in G.

Lemma 2.2.3 *Let T be a primitive subgroup of $GL(m,C)$. Suppose that $g \in GL(m,C)$ is such that for each $t \in T$, conjugation by g produces a scalar multiple of t. Then g is a scalar matrix.*

Proof. Let $\pi : \mathrm{GL}(m, \mathrm{C}) \to \mathrm{PGL}(m, \mathrm{C})$ be the canonical projection. Note that for any $\mathrm{G} \in \mathrm{GL}(m, \mathrm{C})$, the group $< g, \mathrm{T}, >$ is also primitive. The property that $g^{-1}tg$ is equal to a constant multiple of t implies that $t^{-1}g^{-1}tg \in Z(\mathrm{GL}(2,\mathrm{C}))$. Therefore $g \in Z(\pi(< g, \mathrm{T} >))$ and hence also in $Z(< g, \mathrm{T} >)$. Since $< g, \mathrm{T} >$ is primitive, this center is scalar.

Lemma 2.2.4 *Let H be a subgroup of $GL(4,C)$. Let γ be an element such that $\gamma^2 \in H$, $\gamma H \gamma^{-1} = H$, $\gamma \notin H$. Then $H \cup \gamma H$ is a subgroup of $GL(4,C)$.*

Proof. Since γ normalizes H, the set $\{\gamma^i h : h \in \mathrm{H}\}$ forms a group. Since $\gamma^2 \in \mathrm{H}$ and $\gamma \notin \mathrm{H}$, the latter set is $\mathrm{H} \cup \gamma\mathrm{H}$.

Proof of Theorem 2.2.2. By the definition of imprimitivity $\mathrm{G} = \mathrm{H} \cup \gamma\mathrm{H}$ where H is a subdirect product of $\mathrm{T} \times \mathrm{T}$ where T is a primitive subgroup of GL(2, C) and $\{\mathrm{I}, \gamma\}$ is a left transversal of H in G. Assume that T is not projectively finite. Then $\mathrm{T} = (\mathrm{SL}(2,\mathrm{C}))_n$ for some nonnegativeinteger n. Hence given a unimodular algebraic group H which is a subdirect product of $(\mathrm{SL}(2,\mathrm{C}))_n \times (\mathrm{SL}(2,\mathrm{C}))_n$ and an element $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ satisfying $\gamma^2 \in \mathrm{H}$, $\gamma\mathrm{H}\gamma^{-1} = \mathrm{H}$, the group $\mathrm{H} \cup \gamma\mathrm{H}$ is an imprimitive group by Lemma 2.2.4. We show here that given any fixed group $(\mathrm{SL}(2,\mathrm{C}))_n$ there are essentially only three possilbilities for the subdirect product H. Furthermore, given the group H and an element γ

satisfying the above propeties of normality of H in G, there exists an element $y \in \mathrm{GL}(2, \mathrm{C}) \times \mathrm{GL}(2, \mathrm{C})$ fixing H and such that $y^{-1}\gamma y \in \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \cdot \mathrm{H}$. This gives us the desired repesentative group, $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \cdot \mathrm{H}$, corresponding to a given H as stated in the result.

Step 1. We list the possible subgroups H. Recall that H is a subdirect product of $(\mathrm{SL}(2, \mathrm{C}))_n \times (\mathrm{SL}(2, \mathrm{C}))_n$. Let π_1 be the projection of H onto $(\mathrm{SL}(2, \mathrm{C}))_n$. Let $ker(\pi_1) = \{\mathrm{I}\} \times \mathrm{N}$. By Theorem 5.5.1 of [8] N is a normal subgroup of $(\mathrm{SL}(2, \mathrm{C}))_n$ and there is an automorphism f of $((\mathrm{SL}(2, \mathrm{C}))_n)/\mathrm{N}$ such that $(\mathrm{A}, \mathrm{B}) \in \mathrm{H}$ if and only if $f(\mathrm{AN}) = \mathrm{BN}$. Since H is unimodular N is either contained in center of $\mathrm{SL}(2, \mathrm{C})$ or contains $\mathrm{SL}(2, \mathrm{C})$. The automorphisms of $\mathrm{PSL}(2, \mathrm{C})$ and hence those of $(\mathrm{SL}(2, \mathrm{C}))_n/\mathrm{N}$ where N is scalar, are all inner. Therefore in this case the group H is conjugate to either of the following two groups:

- $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \ \mathrm{B} \in (\mathrm{SL}(2, \mathrm{C}))_n, \ d_{\mathrm{B}} = \frac{1}{\det \mathrm{B}}\right\}$
- $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & \pm d_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \ \mathrm{B} \in (\mathrm{SL}(2, \mathrm{C}))_n, \ d_{\mathrm{B}} = \frac{1}{\det \mathrm{B}}\right\}$

If $\mathrm{N} = \mathrm{SL}(2, \mathrm{C})$ then it is clear that any element (A, B) of H satisfies $\mathrm{B} \in \xi_{\mathrm{A}}\mathrm{SL}(2, \mathrm{C})$ where ξ_{A} is a constant depending upon A. Again, since H is a subgroup of $\mathrm{SL}(4, \mathrm{C})$, H must be the group

- $\left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in (\mathrm{SL}(2, \mathrm{C}))_n, \det(\mathrm{AB}) = 1\right\}$

Step 2. We restrict the form of the element γ by requiring the conditions (1) $\gamma^2 \in \mathrm{H}$ and (2) $\gamma \mathrm{H} \gamma^{-1} = \mathrm{H}$ and then find an element $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$, $\mathrm{C}, \mathrm{D} \in \mathrm{GL}(2, \mathrm{C})$, such that $y\mathrm{G}y^{-1}$ is equal to $\mathrm{H} \cup \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \mathrm{H}$.

Case 1. H is conjugate to one of the groups of the form $\left\{\begin{bmatrix} \mathrm{B} & 0 \\ 0 & \xi_{\mathrm{B}}\mathrm{B} \end{bmatrix} : \mathrm{B} \in (\mathrm{SL}(2, \mathrm{C}))_n\right\}$. Let $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_i \in \mathrm{GL}(2, \mathrm{C})$ and $\det g_1 g_2 = 1$. Let $h \in \mathrm{H}$. Conditions (1) and (2) imply that $\gamma = \begin{bmatrix} 0 & g \\ cg & 0 \end{bmatrix}$ where $c = Z(\mathrm{GL}(2, \mathrm{C}))$. Since $\gamma \in \mathrm{SL}(4, \mathrm{C})$, $c \in \left\{\frac{\pm 1}{\det g_1}\right\}$. Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ where $\mathrm{C} = (\sqrt{\det g})\mathrm{I}$ and $\mathrm{D} = (\det g)\mathrm{I}$. Since C and D are scalar matrices and y is in block form it is clear that $y\mathrm{H}y^{-1} = \mathrm{H}$. Also $y\gamma y^{-1} =$

$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\det_g}} g & 0 \\ 0 & \frac{1}{\sqrt{\det_g}} g \end{bmatrix}$. Note that $\det\left(\frac{1}{\sqrt{\det_g}} g\right) = 1$. Therefore $\begin{bmatrix} \frac{1}{\sqrt{\det_g}} g & 0 \\ 0 & \frac{1}{\sqrt{\det_g}} g \end{bmatrix} \in \mathrm{H}$.

Case 2. H is conjugate to $\left\{ \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A}, \mathrm{B} \in (\mathrm{SL}(2, \mathrm{C}))_n \right\}$. Let $\gamma = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}$ where $g_i \in \mathrm{GL}(2, \mathrm{C})$ and $\det g_1 g_2 = 1$. γ satisfies conditions (1) and (2). Let $y = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{D} \end{bmatrix}$ where $\mathrm{C} \in (\mathrm{SL}(2, \mathrm{C}))_n$ and $\mathrm{D} = g_1$. For $h = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} \in \mathrm{H}$ $yhy^{-1} = \begin{bmatrix} \mathrm{CAC}^{-1} & 0 \\ 0 & g_1\xi_{\mathrm{A}}\mathrm{B}g_1^{-1} \end{bmatrix}$ which is clearly in H so again we have $y\mathrm{H}y^{-1} = \mathrm{H}$. Also $y\gamma y^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} g_1 g_2 \mathrm{C}^{-1} & 0 \\ 0 & \mathrm{C} \end{bmatrix}$. Since $\mathrm{C} \in (\mathrm{SL}(2, \mathrm{C}))_n$ and $\det(g_1 g_2) = 1$, $\begin{bmatrix} g_1 g_2 \mathrm{C}^{-1} & 0 \\ 0 & \mathrm{C} \end{bmatrix} \in \mathrm{H}$. Therefore $y\gamma y^{-1} \in \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathrm{H}$.

Using techniques similar to those used in the construction of the above imprimitive groups, the author proved the following result concerning the case where the coordinate group T of H is projectively finite and hence projectively equivalent to one of the groups $A_4^{SL_2}$, $S_4^{SL_2}$, or $A_5^{SL_2}$ as listed in [20]. As indicated in the beginning of this section, Blichfeldt gives the projective counterpart of this list in [2] via generating substitutions. We state the result here without proof as a reference point for the groups named in the decomposition tables found in the appendix.

Theorem 2.2.3 *Let G be an imprimitive, nonmonomial subgroup of $SL(4, C)$ containing a normal subgroup H of index two where $H \subset T \times T$, with T projectively finite. Then there exists a nonnegative integer, n, and a subgroup $F \subset SL(2, C)$ of T such that G is conjugate to one of the following groups.*

- $\left\{ \begin{bmatrix} A & 0 \\ 0 & \xi_A B \end{bmatrix} : A, B \in P_n, B \in AF \right\} \bigcup \left\{ \begin{bmatrix} 0 & A \\ \xi_A B & 0 \end{bmatrix} : A, B \in P_n, B \in AF \right\}$ *where* $\xi_A \in \left\{ \frac{\pm 1}{det(A)} \right\}$ *and one of the following holds.*
  1. $P = A_5$ *and* $\pi(F) \cong \{I\}$ *or* A_5.
  2. $P = S_4$ *and* $\pi(F) \cong \{I\}, K, A_4$ *or* S_4.
  3. $P = A_4$ *and* $\pi(F) \cong \{I\}, K$ *or* A_4.
- $\left\{ \begin{bmatrix} A & 0 \\ 0 & \xi_A B \end{bmatrix} : A, B \in (A_4)_n, B \in AF \right\} \bigcup \left\{ \begin{bmatrix} 0 & A \\ \xi_A B & 0 \end{bmatrix} : A, B \in O_{S_4}(A_4)_n, B \in AF \right\}$ *where* $\pi(F) \cong \{I\}, K$ *or* A_4 *and* $O_{S_4} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$.

3 Representations of the Groups

Given a list of irreducible linear algebraic subgroups of $SL(4, C)$, the next step is to distinguish between them using representation theory. We consider specifically the action of each of the groups on modules of the form $\wedge^n V$ and $Sym^n V$, where $V \cong C^4$. In the degree four case, we have been able to algorithmically distinguish between the groups via the dimensions of the irreducible submodules in the decompositions of $\wedge^2 V$ and $Sym^n V$ for $n = 2, 3, 4$ and 6. These procedures are given in section 4. The proofs of the procedures lie in the observation of the tables found in the Appendix which contain the decompositions of each of these modules under the action of each of the groups.

Also included in this section are effective algorithmic procedures for decomposing G-modules for imprimitive G. These procedures, developed by the author, are based on the theory of Clifford. Examples of the implementation of the decomposition procedures are also in this section. For finite G, we have written and implemented programs using the computer algebra system MAGMA in order to decompose the given G-modules. In all other cases, we present or refer to the already well known decomposition procedures.

3.1 Representations of the Primitive Groups

If G is a primitive subgroup of $SL(4, C)$ then either G is a finite group or the connected component of the identity is a semisimple group.

3.1.1 Representations of the Infinite Primitive Groups

G^0 irreducible. Let G be an infinite primitive subgroup of $SL(4, C)$. Then G^0 is equal to an irreducible degree four representation of $SL(2, C)$, $SO(4, C)$, $SP(4, C)$, or $SL(4, C)$. The irreducible modules for each of these groups are known (see for ex. [7]). For example if $V \cong C^4$, then both $\wedge^m V$ and $Sym^m V$ are irreducible $SL(4, C)$-modules for any m. Using this information one is able to decompose a given G^0-module, W, using the process shown in the following example. Theorem 2.1.2 states that if G^0 is equal to $SL(2, C)$, $SP(4, C)$, or $SL(4, C)$ then G is simply a scalar extension of G^0 and hence any irreducible G^0-module is also an irreducible G-module. Similarly, if $G^0 = SO(4, C) \cong SL(2, C) \times SL(2, C)$ then $G \subseteq H \cup JH$ where H is a finite scalar extension of G^0 and $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. We use classical representation theory to determine the irreducible decomposition of W as an $SO(4, C)$-module and observe that any pair of G^0-irreducible spaces exchanged by J are G-irreducible exactly when the characters are distinct. Hence we can obtain the G-irreducible decomposition of W via the action of J on the $SO(4, C)$-irreducible components.

Example. The decomposition of $Sym^2\mathrm{V}$ as an $\mathrm{SL}(2,\mathrm{C})$-module

Any irreducible $\mathrm{SL}(2,\mathrm{C})$-module of dimension $m+1$ is equal to $\mathrm{V}(m) = Sym^{m-1}\mathrm{C}^2$. A character of $\mathrm{SL}(2,\mathrm{C})$ is completely determined by its value on $\mathrm{diag}(\mathrm{SL}(2,\mathrm{C}))$. Let $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathrm{diag}(\mathrm{SL}(2,\mathrm{C}))$. We can then compute the value of the character of any irreducible $\mathrm{SL}(2,\mathrm{C})$-module at g. Those for $\mathrm{V}(0), \dots, \mathrm{V}(3)$ are

1. $\chi_{\mathrm{V}(0)}(g) = 1$
2. $\chi_{\mathrm{V}(1)}(g) = a + a^{-1}$
3. $\chi_{\mathrm{V}(2)}(g) = a^2 + 1 + a^{-2}$
4. $\chi_{\mathrm{V}(3)}(g) = a^3 + a + a^{-1} + a^{-3}$

Consider now the $\mathrm{SL}(2,\mathrm{C})$-module $\mathrm{W} = Sym^2\mathrm{V}$. Let $\mathrm{V} = span\{v_1, v_2, v_3, v_4\}$. Then the following set is a basis for W.

$$\{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

Recall that G^0 is the irreducible degree four representation of $\mathrm{SL}(2,\mathrm{C})$. Therefore the image of a diagonal element $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ is $\begin{bmatrix} a^3 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-3} \end{bmatrix}$.

The value of the character of W at g is then equal to $a^6 + a^4 + 2a^2 + 2 + 2a^{-2} + a^{-4} + a^{-6}$, which is equal to $\chi_{\mathrm{V}(6)}(g) + \chi_{\mathrm{V}(2)}(g)$. Therefore $\mathrm{W} \cong \mathrm{V}(6) \oplus \mathrm{V}(2)$ as an $\mathrm{SL}(2,\mathrm{C})$-module. Table 4 in the Appendix contains the decompositions of $\wedge^2\mathrm{V}$, $Sym^2\mathrm{V}$, $Sym^3\mathrm{V}$, $Sym^4\mathrm{V}$ and $Sym^6\mathrm{V}$ for each of the possibilities for the connected component G^0 of an infinite primitive group G.

G^0 reducible. Recall from section 2.1.2 that such a group, G, is equal to the image of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ acting on $\mathrm{C}^2 \otimes \mathrm{C}^2$ where $\tilde{\mathrm{H}}$ is one of the finite primitive subgroups $A_4^{SL_2}$, $S_4^{SL_2}$, $A_5^{SL_2}$ of $\mathrm{SL}(2,\mathrm{C})$ or a degree two scalar extension of one of these groups. As with some of the imprimitive groups we will not be able to distinguish between the groups $\tilde{\mathrm{H}}_1 \times \mathrm{SL}(2,\mathrm{C})$ and $\tilde{\mathrm{H}}_2 \times \mathrm{SL}(2,\mathrm{C})$ when $\tilde{\mathrm{H}}_1$ is a scalar extension of $\tilde{\mathrm{H}}_2$. Therefore we will only consider representations of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ where $\tilde{\mathrm{H}}$ is equal to $A_4^{SL_2}$, $S_4^{SL_2}$, or $A_5^{SL_2}$. In this case the image of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ in $\mathrm{SL}(4,\mathrm{C})$ lies in $\mathrm{SO}(4,\mathrm{C})$, the image of $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$. Therefore, if W is an $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$-module constructed from $\mathrm{C}^2 \otimes \mathrm{C}^2$ using tensor products, direct sums, submodules, quotients and duals (for example, $\wedge^2\mathrm{V}$ or $Sym^2\mathrm{V}$) then W is automatically an $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$-module. This gives us the following strategy: first decompose W as an $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$-module and then examine the action of $\tilde{\mathrm{H}} \times \mathrm{SL}(2,\mathrm{C})$ on each $\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C})$-irreducible factor. The following two lemmas will facilitate the application of this strategy. The first is well known but since we could not find a suitable reference, we include the proof.

Lemma 3.1.1 *Let k be any algebraically closed field of characteristic zero and let G and H be reductive groups. Then any irreducible $G \times H$-module is of the form $V \otimes W$ where V is an irreducible G-module and W is an irreducible H-module. Conversely, any such tensor product is a $G \times H$-module.*

Proof. Note that if V is a G-module and W is an H-module then the action of G × H on V ⊗ W is defined by $(g \times h)(v \otimes w) = gv \otimes hw$. Let U be an irreducible G × H-module. We can consider U as a G-module and as such write $\mathrm{U} = \mathrm{U}_1^{(n_1)} \oplus \cdots \oplus \mathrm{U}_r^{(n_r)}$ where the U_i are irreducible G-modules, pairwise nonisomorphic, and each U_i is repeated n_i times in the direct sum. Since H commutes with G, H leaves each $\mathrm{U}_i^{(n_i)}$ invariant (to see this, let $h \in \mathrm{H}$ and note that for any G-invariant projection π, the map $\pi \circ h$ is a G-morphism). Since U is an irreducible G × H-module, we must have that that $\mathrm{U} = F^{(n)}$ for some irreducible $\tilde{\mathrm{H}}$-module F. Proposition 2 of [15] Chapter XVII Section 1 furthermore states that each $h \in \mathrm{H}$ can be associated with a matrix

$$M_h = \begin{bmatrix} b_{11} \dots b_{1n} \\ \vdots \quad\quad \vdots \\ b_{n1} \dots b_{nn} \end{bmatrix}$$

where each $b_{ij} \in End_{\mathrm{G}}(F) = k$ (the last equality is Schur's Lemma). This allows us to associate an H-module M with this representation. Write $U = F_1 \oplus \cdots \oplus F_n$ where the F_j are isomorphic irreducible G-modules and select bases $\{v_{ij} : i = 1, \dots, n,\ j = 1, \dots, \dim(\mathrm{U})\}$ for each summand U such that the matrices of each $g \in \mathrm{G}$ with respect to these bases are identical. Let $g \in \mathrm{G}$ and let (a_{ij}) be this matrix. Then for each j we have $h(v_{ij}) = \sum_{t=1}^{n} a_{it} v_{tj}$. Therefore

$$(g \times h)(v_{ij}) = \sum_t \sum_s a_{it} b_{sj} v_{st}.$$

This is the same as the representation of G × H on $F \otimes M$. Finally, it is clear that M must be irreducible since otherwise U would be irreducible.

We now show the converse. If V is an irreducible G-module and W is an irreducible H− module, then we can write:

$$\begin{aligned} \mathrm{V} \otimes \mathrm{W} &= \mathrm{V}^{(m)} \ \text{ as a G-module} \\ &= \mathrm{W}^{(n)} \ \text{as an H-module} \end{aligned}$$

where m is the dimension of W and n is the dimension of V. If $\mathrm{U} \subseteq \mathrm{V} \otimes \mathrm{W}$ is a G×H-module, then the first part of this proof shows that $\mathrm{U} = \mathrm{V}_1 \otimes \mathrm{W}_1$ where V_1 is an irreducible G-module and W_1 is an irreducible H-module. We can identify V_1 with an irreducible submodule of V ⊗ W and so it must be isomorphic to V. Similarly, W_1 must be isomorphic to W. Counting dimensions we have that $\mathrm{U} = \mathrm{V} \otimes \mathrm{W}$.

Lemma 3.1.2 *Let $W = V(i) \otimes V(j)$ be an irreducible $SL(2, C) \times SL(2, C)$-module and let $\tilde{H}$ be a primitive subgroup of $SL(2, C)$. Then W is reducible as an $\tilde{H} \times SL(2, C)$-module if and only if $V(i)$ is reducible as an $\tilde{H}$-module. Furthermore, if $V(i) = \bigoplus_{k=1}^{n} U_k$ where each U_k is $\tilde{H}$-invariant then W can be written as the sum, $\bigoplus_{k=1}^{n} (U_k \otimes V(j))$, of $\tilde{H} \times SL(2, C)$-modules.*

Proof. Let W be as stated. Note that the action of $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ on W is defined as follows. Let $(g_1, g_2) \in \mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ and let $v_1 \otimes v_2 \in \mathrm{W}$. Then $(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 v_1 \otimes g_2 v_2)$. Since $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$ is a subgroup of $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ W is an $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})-$ module via restriction. By Lemma 3.1.1 any irreducible $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$-module is of the form $\mathrm{U} \otimes \mathrm{V}$ where U is an irreducible $\tilde{\mathrm{H}}$-module and V is an irreducible $\mathrm{SL}(2, \mathrm{C})-$ module. This implies that W is reducible as an $\tilde{\mathrm{H}} \times \mathrm{SL}(2, \mathrm{C})$-module if and only if $\tilde{\mathrm{H}}$ acts reducibly on $\mathrm{V}(i)$. The result then follows from the distributive property of tensor products over direct sums.

Example. The decomposition of Sym^4V as an $A_4^{SL_2} \times \mathrm{SL}(2, \mathrm{C})$-module

Let $\mathrm{W} = Sym^4\mathrm{V}$. Recall that as in the case where $\mathrm{G} = \mathrm{SO}(4, \mathrm{C})$ $\mathrm{V} \cong \mathrm{C}^2 \otimes \mathrm{C}^2$. As an $\mathrm{SO}(4, \mathrm{C})$-module Sym^4V decomposes into the following sum of irreducible submodules. $(\mathrm{V}(0) \otimes \mathrm{V}(0)) \oplus (\mathrm{V}(2) \otimes \mathrm{V}(2)) \oplus (\mathrm{V}(4) \otimes \mathrm{V}(4))$ Table 6 provides the necessary information concerning the decompositions of $\mathrm{V}(2), \ldots, \mathrm{V}(6)$ under the action of each of the finite primitive subgroups of $\mathrm{SL}(2, \mathrm{C})$. Since V(2) is irreducible as a $A_4^{SL_2}$-module and since V(4) is the sum of three $A_4^{SL_2}$-invariant subspaces of dimensions 1,1 and 3, we have the following decomposition of W as an $\mathrm{H} \times \mathrm{SL}(2, \mathrm{C})$-module.

$$\mathrm{Sym}^4\mathrm{V} \Longrightarrow 1, 5^2, 9, 15$$

3.1.2 Representations of the Finite Primitive Groups

Let G be a finite primitive group. One decomposes a given G-module via the character table of G. We again used *MAGMA* to obtain the information found in Table 5 of the Appendix. Careful and thorough observation of this table then gives us the following result for distinguishing among these thirty groups, which we denote simply by FP1 through FP30 to correspond to Blichfeldt's groups denoted by 1° through 30°. Given this relatively large number of groups, we have presented the first step explicitly in this result with a reference to the flow charts found at the end of this paper which should be used to distinguish further among the thirty groups.

Theorem 3.1.1 *Let G be an finite primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then one of the following holds.*

1. *$\wedge^2 V$ has a G-invariant subspace of dimension 1, in which case G can be determined via Flow Chart A.*

2. *$\wedge^2 V$ has a G-invariant subspace of dimension 3, in which case G can be determined via Flow Chart B.*
3. *Neither of the above hold or $\wedge^2 V$ is G-irreducible, in which case G can be determined via Flow Chart C.*

In addition to constructing a decision procedure using the irreducible decompositions of the group modules, one may also use the tables in the Appendix to determine very specific results. For example,

Theorem 3.1.2 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $Gal(L)$ projects onto G_{168} if and only if $L_{\wedge^2}(y)$ is irreducible and $L_{Sym^2}(y)$ has an irreducible factor of order 7.*

Remark. The careful reader will observe that one is not able to use the tables to distinguish between the two groups FP20 and FP21. The second group is generated by the first group and the diagonal matrix with diagonal $(1, 1, 1, -1)$. One would need to use the value of a one dimensional character in order to distinguish between these two groups as the addition of this diagonal element to the generating set will not affect the dimensions of the irreducible components in decomposition of a module $\mathrm{W} = Sym^n\mathrm{V}$. The procedure presented in this paper uses dimension analysis and hence will not help one distinguish between these two groups.

3.2 Representations of the Imprimitive Groups

Recall that an irreducible imprimitive G-module V with system of imprimitivity $\{\mathrm{V}_1, \ldots \mathrm{V}_k\}$ is induced from the irreducible H-module V_1 where $\mathrm{H} = Stab(\mathrm{V}_1)$. We consider here G-modules of the form $Sym^m\mathrm{V}$ and $\wedge^m\mathrm{V}$. Given the decomposition of V as an H-module, one can easily find the H-module decomposition of $Sym^m\mathrm{V}$ and $\wedge^m\mathrm{V}$. Furthermore, G will clearly permute the H-submodules although not necessarily transitively. This implies that in the decomposition of any such space as a G-module, any irreducible summand must be either H-irreducible or imprimitive. Indeed, as noted in [12] page 66, a character of G is primitive if and only if it is not induced from the character of a proper subgroup. Therefore, we may use Clifford theory to extract the irreducible G-submodules of the given module W. We need to fix a bit of terminology before describing this process in detail.

Notation and Definitions

- Given the decomposition $Res_{\mathrm{H}}(\mathrm{W}) = \bigoplus \mathrm{V}_i$ let χ_i be the character associated with V_i. Given such a character χ_i the homogeneous component of W associated with χ_i is the space $\mathrm{U}_i = \bigoplus_{\chi_i=\chi_j} \mathrm{V}_j$.

- Given an H-character χ_i the inertia group I_i associated with this character is the subgroup of G defined by the set $\{g \in \mathrm{G} : \chi_i^g = \chi_i\}$, where χ_i^g is the conjugate character with action on H defined by $\chi_i^g(h) = \chi_i(ghg^{-1})$. Hence, the inertia group associated with a given character of H preserves the H-homogenous components of W via conjugation.
- Given a normal subgroup N of G, if a G-module W is indeed induced from an N-module V, then $\mathrm{W} = \bigcup_{s\in\mathscr{S}} \mathrm{V}s$ where $\mathscr{S}$ is a set of coset representatives or a transversal for N in G.

Given this terminology we are able to outline the general procedure for decomposing a G−module of the form indicated above (where G acts imprimitively on V) using the theory of Clifford. For references on Clifford Theory see [5], [12], or [16].

General Decomposition Procedure for Imprimitive Groups

Step 1. Decompose W as an H-module using classical representation theory.
Step 2. Given any H-module V_i in this decomposition, let $\mathrm{W}_i = \bigcup_{\gamma\in\mathscr{S}} (\mathrm{V}_i)\gamma$ where $\mathscr{S}$ is a transversal for H in G. Note that W_i is G-invariant but not necessarily G-irreducible.
Step 3. In any case, as mentioned above, any irreducible component in the decomposition of W_i as a G-module is either H-irreducible or is imprimitive. Clifford's Theorem then implies that any irreducible imprimitive component is induced from an H-homogeneous component via the action of the associated inertia group. Hence, as a third step we write W_i as a sum of H-homogeneous components $\mathrm{W}_{i,j}$ corresponding to each H character χ_j appearing in W_i and determine the inertia group. Note that the inertia group will contain H as a subgroup of finite index and hence we may represent the inertia group as a subset of the transveral for H in G. Therefore one may think of it as a subgroup of the permutation group, $\mathscr{P}_{\mathrm{G}}$, associated with the original representation of G.
Step 4. Given the inertia group I_j fixing the homogeneous component $\mathrm{W}_{i,j}$, an irreducible imprimitive G-submodule of W_i will be induced from an irreducible I_j-submodule of $\mathrm{W}_{i,j}$. Hence, in step four one must attempt to decompose $\mathrm{W}_{i,j}$ under the action of I_j.

One sees in the examples to come how the specific matrix representations aid in the application of this general procedure.

3.2.1 Representations of Momomial Groups

Recall that an infinite monomial group, G, can be written in the form $\bigcup_{\gamma\in\mathscr{S}} \gamma\mathrm{H}$ where H is an infinite diagonal subgroup of SL(4, C), $\mathscr{S}$ is a transversal of H in G, and $\mathrm{G}/\mathrm{H} \cong \mathscr{P}_{\mathrm{G}}$ is a transitive subgroup of S_4. The following lemmas are evidence that such a representation of a monomial group G is integral in decomposing various G-modules.

Lemma 3.2.1 *Let $G = \bigcup_{\gamma \in \mathscr{S}} \gamma H$ be an imprimitive monomial group. Let W be any G-module with the property that $Res_H(W) = V_1 \oplus \cdots \oplus V_n$, where the dimension of each V_i is one. Let $U = \bigcup_{\gamma \in \mathscr{S}} V_i\gamma$. Suppose that U contains $k > 1$ H-homogeneous components each of dimension m. Then $U = \bigoplus_{i=1}^{m} U_i$ where each U_i is an irreducible G-module of dimension k.*

Proof. Let W and U be as stated. Observe that the group G permutes the set $\{V_i\gamma \ : \ \gamma \in \mathscr{S}\}$ transitively. Hence, any irreducible G-submodule of U either has dimension 1 or is itself a monomial representation of G induced by the inertia group associated with a homogeneous component of U.

Let U_i be a homogenous component of U corresponding to character χ_i. Let I_i be the inertia group of this character. Then U_i must decompose into m one-dimensional I_i submodules. Indeed, I_i is equal to $\bigcup_{\sigma \in T} \sigma H$ where T the subset of $\mathscr{S}$ which fixes the module U_i. Since G acts transitively on the H-submodules of U, I_i acts transitively on those of U_i. Therefore, any irreducible I_i-submodule of U_i of dimension greater than 1 would have to be imprimitive and as such, induced by an H-irreducible component $V_i\gamma$ ([12] Thm. 5.8). Corollary 4.5 of [5] implies that such a representation is irreducible only if the characters of the I_i conjugates of $V_i\gamma$ are distinct. These characters are equal by definition of the inertia group. Therefore U_i must decompose into m one dimensional I_i-submodules each the span of an element α of the form $c_1v_1 + \cdots + c_mv_m$, $v_i \in V_i$ $c_i \neq 0$. Since g permutes the k homogeneous components of U transitively, each one dimensional I_i-invariant subspace of U_i gives rise to an induced G-submodule of U of dimension k (irreducible since it is induced from an irreducible module of the inertia group) and the result follows.

Procedure 1: Decomposing Representations of a Monomial Group

Let G be a monomial group with maximal normal diagonal subgroup H. Let $\mathscr{S}$ be a transversal for H in G corresponding to permutation the action $\mathscr{P}_G$ on its system of imprimitivity. Let W be a G-module. Let $W_1 \oplus \cdots \oplus W_t$ be a decomposition of $Res_H(W)$ into one dimensional H-modules. We decompose W into a sum of irreducible G-submodules using the following procedure.

Step 1. Write W as $U_1 \oplus \cdots \oplus U_t$ where each U_i is equal to $\{(W_j)\gamma \ : \ \gamma \in \mathscr{S}\}$ for some j.

Step 2. For each i write U_i as $U_{i,1} \oplus \cdots \oplus U_{i,k}$ where $U_{i,j}$ is a homogeneous component for H. As indicated in the proof of Lemma 3.2.1, these k subspaces will have equal dimension, say m. It then follows from Lemma 3.2.1 that U_i can be written as a sum of m G-irreducible submodules of dimension k.

Example. The Decomposition of $Sym^2\mathrm{V}$ where $\mathscr{P}_{\mathrm{G}} = \mathrm{K}$

Let V be the usual representation of G with basis $\mathscr{B}_{\mathrm{V}} = \{v_1, v_2, v_3, v_4\}$. Then we can write down the following basis for $\mathrm{W} = Sym^2\mathrm{V}$.

$$\mathscr{B}_{\mathrm{W}} = \{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

Regardless of the form of the diagonal group H, we can write W as the direct sum of the following four G-invariant subspaces, each of which is of the form $\{(\mathrm{W}_i)\gamma : \gamma \in \mathrm{G}\mathscr{S}\}$ where $\mathrm{W}_i = span\{v\}, \quad v \in \mathscr{B}_{\mathrm{W}}$.

- $\mathrm{U}_1 = span\{v_1^2, v_2^2, v_3^2, v_4^2\}$
- $\mathrm{U}_2 = span\{v_1v_2, v_3v_4\}$
- $\mathrm{U}_3 = span\{v_1v_3, v_2v_4\}$
- $\mathrm{U}_4 = span\{v_1v_4, v_2v_3\}$

In order to obtain the decomposition of each U_i as a G-module we determine the homogeneous components for each possible diagonal subgroup H as listed in Theorem 2.2.1.

$$\mathbf{H} = \mathbf{H}_1^1 \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathrm{C}^* \right\}$$

- U_1 has homogeneous components $U_{1,1} = span\{v_1^2, v_3^2\}$ and $\mathrm{U}_{1,2} = span\{v_2^2, v_4^2\}$. Therefore U_1 can be written as the direct sum of two G-invariant subspaces of dimension 2. Furthermore, one can show using the specific elements of the transveral $\mathscr{S}$ as given in Theorem 2.2.1, that they are $span\{u_1^\gamma : \gamma \in \mathscr{S}\}$ and $span\{u_2^\gamma : \gamma \in \mathscr{S}\}$, where $u_1 = v_1^2 + v_3^2$ and $u_2 = v_1^2 - v_3^2$.
- U_2 is itself a homogeneous component of W and hence decomposes into a sum of two one-dimensional G-invariant subspaces. One can easily show using the transversal $\mathscr{S}$ that $v_1v_2 + v_3v_4$ and $v_1v_2 - v_3v_4$ are both semi-invariants for G and hence give us the two modules.
- U_3 has homogeneous components $\mathrm{U}_{3,1} = span\{v_1v_3\}$ and $\mathrm{U}_{3,2} = span\{v_2v_4\}$. Lemma 3.2.1 implies then that U_3 is G$-$irreducible.
- As in the above decomposition of U_2 U_4 can be written as a sum of two dimension one G-modules, in particular, they are the span of the semi-invariants $v_1v_4 + v_2v_3$ and $v_1v_4 - v_2v_3$.

Hence under the action of this group $Sym^2\mathrm{V}$ has G-module decomposition given via dimensions as

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 2^3, 1^4$$

$\mathbf{H} = \mathbf{H}_1^2$ Again, since the H_1^2-homogeneous components of each U_i are the same as the H_1^1-homogeneous components, the decomposition of $Sym^2\mathrm{V}$ under the action of this group is also given by

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 2^3, 1^4$$

$\mathbf{H} = \mathbf{H}_2^1$ or $\mathbf{H}_2^1$ Recall that $\mathrm{H}_2^n = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1}b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}.$

For each i the homogeneous components of U_i for either of these two groups are identical. Hence the decompositions of $Sym^2\mathrm{V}$ under the action of these two groups are the same.

- U_1 has four homogeneous components of dimension 1. These are $\mathrm{U}_{1,j} = span\{v_j^2\}$ for $j = 1, \ldots, 4$. Therefore by Lemma 3.2.1, W_1 is irreducible.
- U_2 is also irreducible by Lemma 3.2.1.
- Consider now U_3. This subspace of $Sym^2\mathrm{V}$ is itself a homogeneous component and hence can be written as a sum of two $\mathrm{G}-$invariant submodules. We may use the transversal $\mathscr{S}$ to determine these submodules. Let $w = v_1v_3 + \alpha v_2v_4,\ \alpha \in \mathrm{C}$. α must satisfy

  1. $w^{\gamma_1} = v_2v_4 + \alpha v_1v_3 = \alpha w$
  2. $w^{\gamma_2} = v_1v_3 + \alpha v_2v_4 = w$
  3. $w^{\gamma_3} = v_2v_4 + \alpha v_1v_3 = \alpha w$

  Any $\alpha \in \{\pm 1\}$ satisfies these conditions. Hence under the action of either of these groups we have

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 4, 2^2, 1^2$$

$\mathbf{H} = \mathbf{H}_2^n, n \neq 1, 2$ Given i, the homogeneous components of U_i for any pair of groups H_2^n and H_2^m where $m, n \notin \{1, 2\}$ are identical. Hence we decompose $Sym^2\mathrm{V}$ here under the action of such a group for arbitrary $n > 2$.

- U_1 again has four H-homogeneous components of dimension 1. These are $U_{1,j} = span\{v_j^2\}$ for $j = 1, \ldots, 4$. Therefore by Lemma 3.2.1, U_1 is irreducible.
- Since $n \neq 1$ or 2, the scalars ξ and ξ^{-1} will be distinct for some elements of H. Therefore each of the spaces $\mathrm{U}_2, \mathrm{U}_3$ and U_4 has two homogeneous components of dimension 1 and hence is also irreducible by Lemma 3.2.1.

Therefore under the action of $\mathrm{G} = \cup_{\gamma \in \mathscr{S}} \gamma \mathrm{H}$ where H is any of the groups $\mathrm{H}_2^n, n \neq 1, 2$ we have

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 4, 2^3$$

H $=$ **H**$_3$ It is easy to see that each subspace W_i has $\dim(W_i)$ H-homogeneous components each of dimension 1. Hence each of the subspaces W_i is irreducible by Lemma 3.2.1. Hence the action of this G on $Sym^2 V$ is also given by

$$\mathrm{Sym}^2 V \Longrightarrow 4, 2^3$$

3.2.2 Representations of the Imprimitive Nonmonomial Groups

Recall that such a group has an index two subgroup H which is a subdirect product of $T \times T$ where T is a primitive subgroup of $GL(2, C)$. T is either projectively finite or it is equal to $(SL(2, C))_n = \{A \in GL(2, C) : \det(A)^n = 1\}$ for some nonnegative integer n. We will show that when T is projectively finite, it is enough to consider representations of certain finite groups. In this case we use *MAGMA* to obtain the irreducible decomposition of a representation of such a group. When $T = (SL(2, C))_n$, we show that one may decompose a G-module as one would a $SL(2, C)$ or $SL(2, C) \times SL(2, C)$-module.

As with the monomial groups, we will first consider the action of H on a given module W. Then, upon observing that G permutes these H−submodules, we use the theory of Clifford in order to obtain a decomposition of W under the action of the group $G = H \cup \gamma H$.

In Section 2 we constructed an infinite list of possibilities for this subgroup $H \subset T \times T$ of a nonmonomial imprimitive group. It is clear that one is not able to decompose a given module under the action of each of these groups. The following lemma allows one to determine all possible dimension combinations in the irreducible decompositions via the actions of only a finite list of groups.

For convenience we make the following convention. We embed $GL(2, C) \times GL(2, C)$ into $GL(4, C)$ by sending (g_1, g_2) to $\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$. This allows us to write $V = C^4$ as a direct sum $V = V_1 \oplus V_2$ where each $V_i \cong C^2$ is invariant under $GL(2, C) \times GL(2, C)$.

Lemma 3.2.2 *Let H_1 and H_2 be reductive subgroups of $(GL(2, C) \times GL(2, C)) \cap SL(4, C)$ such that $H_2 D = H_1 D$, where $D \subseteq Z(GL(2, C)) \times Z(GL(2, C))$. Let $V = C^4$. Then given a nonnegative integer n, one can write $Sym^n V = \bigoplus W_i$ and $\wedge^n V = \bigoplus U_i$ where each W_i and U_i is both an irreducible H_1-module and an irreducible H_2-module.*

Proof. We will prove the result for $Sym^n V$. The proof of the case $\wedge^n V$ is similar. Since $V = V_1 \oplus V_2$ we may write $Sym^n V = \bigoplus_{a+b=n}(Sym^a V_1 \otimes Sym^b V_2)$ and it is clear that each space $Sym^a V_1 \otimes Sym^b V_2$ is both an H_1 and an H_2-submodule. We shall show that any H_1-invariant subspace of $Sym^a V_1 \otimes Sym^b V_2$ is $H_1 D$-invariant. A similar argument will show that any H_2-invariant subspace is of $Sym^a V_1 \otimes Sym^b V_2$ is $H_2 D$-invariant and so any H_1-invariant subspace is H_2-invariant and vice versa. Decomposing each $Sym^a V_1 \otimes Sym^b V_2$ into H_1-irreducible subspaces will yield the lemma.

Let $W \subseteq Sym^a V_1 \otimes Sym^b V_2$ be an H_1- invariant subspace. If $d \in D$, then $d = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & c^{-1} \end{bmatrix}$ or $\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & -c^{-1} & 0 \\ 0 & 0 & 0 & -c^{-1} \end{bmatrix}$ for some $c \in C^*$. If $w_1 \in Sym^a V_1$ and $w_2 \in Sym^b V_2$, then a computation shows that $d(w_1 \otimes w_2) = \pm c^{a-b}(w_1 \otimes w_2)$. Therefore D acts by scalar multiplication on $Sym^a V \otimes Sym^b V$ and so W is $H_1 D-$ invariant.

Corollary 3.2.1 *Let H be a subgroup of $(SL(2, C))_n \times (SL(2, C))_n$ containing $SL(2, C) \times SL(2, C)$. Then $Sym^m V$ and $\wedge^m V$ are each isomorphic to a direct sum of modules of the form $V(k) \otimes V(n)$ for some pair of nonnegative integers n and k and this decomposition is independent of H.*

Proof. Any irreducible $SL(2, C) \times SL(2, C)$ module is of the form $V(n) \otimes V(m)$ for some pair of nonnegative integers n and m. The result follows from the Lemma.

Lemma 3.2.2 and its corollary are made effective in decomposing H-modules as follows: Suppose that $T = (SL(2, C))_n$ for some positive integer n. Given such a group G, the subgroup H is one of the following groups.

1. $H_n^1 = \left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in (SL(2, C))_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$
2. $H_n^2 = \left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in (SL(2, C))_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$
3. $H_n^3 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in (SL(2, C))_n, \det(AB) = 1 \right\}$

Let $D = Z(GL(2, C)) \times Z(GL(2, C))$. Note that $H_n^1 D = H_n^2 D$ $= \left\{ \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} : g \in SL(2, C) \right\} D$. Lemma 3.2.2 then implies that we may decompose modules of the form $Sym^m V$ and $\wedge^k V$ as $SL(2, C)$-modules in order to obtain the subspaces in their decompositions as either H_n^1 or H_n^2- modules. Similarly Corollary 3.2.1 tells us the form of the irreducible H_n^3-modules. Since the character of any G-module where G is either $SL(2, C)$ or $SL(2, C) \times SL(2, C)$ is completely determined by its value on the diagonal elements and each H_n^i above is simply a diagonal extension of one of these two groups, their characters are also completely determined by their values on diagonal elements.

Representations of $G = H \cup \gamma H$. Given the decomposition of $Res_H(W)$ via the means indicated above, one will observe that G permutes the $H-$submodules appearing in $Res_H(W)$ in pairs. Indeed, this follows from the fact that H has index two in G. Given any such pair, W_i, W_j, either G acts reducibly on $W_i \oplus W_j$ or the action of G on this sum is induced from H. Indeed, if the H–characters of W_i and W_j are identical, then the inertia group corresponding to this character is G itself, and Clifford's Theorem implies that G cannot be imprimitive. Certainly G cannot act primitively on $W_i \oplus W_j$ since G permutes these two H-irreducible modules. Therefore, in this case, G acts reducibly and one can show quite easily that the decomposition of $W_i \oplus W_j$ as a G−module is $U_i \oplus U_j$ where $U_i = \{w + \gamma w : w \in W_i\}$ and $\overline{U_i} = \{w - \gamma w : w \in W_i\}$. In the case where the H-characters of W_i and W_j are distinct, H must be the inertia group and hence the representation of G on $W_i \oplus W_j$ is induced from W_i by H. The procedure for decomposing representations of the nonmonomial imprimitive subgroups of SL(4, C) follows directly from these remarks. This procedure is written for the case when T is projectively infinite. As previously noted, in the alternate case one may rely on the matrices of Blichfeldt.

Procedure 2: Decomposing Representations of an Imprimitive Nonmonomial Group

Step 1. Decompose W as an H-module via the classical character theory of SL(2, C) and SL(2, C) × SL(2, C).

Step 2. Determine which pairs of H-submodules are interchanged by G. Since $G = H \cup \gamma H$, it is enough to consider which are exchanged by $\gamma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Note that any H-module W_i of dimension k having the property that $dim(W_j) \neq k$ for all $j \neq i$ must be G-invariant.

Step 3. Since any H-reducible G-module will be induced from an irreducible H−module, any pair of isomorphic H-modules exchanged by G will be reducible as a G-module, with decomposition $U_i \oplus U_j$ as defined previously. Any pair of exchanged modules with distinct H-characters gives the sytem of imprimitivity for an irreducible imprimitive G−submodule of W.

Example. The decomposition of Sym^2V in the case when $\pi_i(H) = (SL(2, C))_n$ H is conjugate to one of the following groups:

- $H_n^1 = \left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in (SL(2, C))_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$
- $H_n^2 = \left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in (SL(2, C))_n, d_B = \frac{1}{\sqrt{\det(B)}} \right\}$
- $H_n^3 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in (SL(2, C))_n, \det(AB) = 1 \right\}$

Let V be the usual representation of G with basis $\mathscr{B}_V = \{v_1, v_2, v_3, v_4\}$. We can write down the following bases for $W = Sym^2V$.

$$\mathscr{B}_W = \{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$$

We determine the dimensions of the components of the irreducible decomposition of W as a G-module for each of the groups G listed in Theorem 2.2.2.

Case 1. Suppose G is equal to $\mathrm{H}_n^1 \cup \gamma \mathrm{H}_n^1$. Consider $Res_{\mathrm{H}}(\mathrm{W})$. Since H is isomorphic to $(\mathrm{SL}(2, \mathrm{C}))_n$ the irreducible H$-$characters are scalar multiples of the irreducible characters of $\mathrm{SL}(2, \mathrm{C})$. Let $h \in \mathrm{diag}(\mathrm{H})$. We may write $h = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \xi a & 0 \\ 0 & 0 & 0 & \xi b \end{bmatrix}$ where $a, b \in \mathrm{C}^*$ and ξ is an n^{th} root of unity. In the above basis h is in diagonal form and hence we can easily determine the value of the character at h. We rewrite this character as a sum of known irreducible characters in essence obtaining the irreducible H-module decomposition of $Sym^2\mathrm{V}$.

$$\begin{aligned}
\chi_{\mathrm{w}}(h) &= a^2 + b^2 + \xi^2 a^2 + \xi^2 b^2 + ab + \xi a^2 + \xi ab + \xi ab + \xi b^2 + \xi^2 ab \\
&= (a^2 + ab + b^2) + \xi(a^2 + ab + b^2) + \xi^2(a^2 + ab + b^2) + \xi ab \\
&= \chi_3(h) + \xi \chi_3(h) + \xi^2 \chi_3(h) + \chi_1(h)
\end{aligned}$$

Therefore as a representation of $\mathrm{H} = \mathrm{H}_n^1$ for arbitrary n, $Sym^2\mathrm{V} = \bigoplus_{i=1}^4 \mathrm{W}_i$ where $\dim(\mathrm{W}_i) = 3$ for $i = 1, 2, 3$ and $\dim(\mathrm{W}_4) = 1$. since $\dim(\mathrm{W}_4)$ is distinct, W_4 is G-invariant. One can see that the space W_2 corresponding to the character $\xi(a^2 + ab + b^2)$ is G$-$invariant as well by noticing that $\{v_1v_3,\ v_1v_4 + v_2v_3,\ v_2v_4\}$ is a basis for W_2. We see by observing $\chi_{\mathrm{w}}(h)$ that the spaces W_1 and W_3 are given by $span\{v_1^2,\ v_1v_2,\ v_2^2\}$ and $span\{v_3^2,\ v_3v_4,\ v_4^2\}$ respectively. These two spaces are interchanged by γ. Let $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_3$. Then U may or may not be reducible depending on the integer n where $\mathrm{T} = (\mathrm{SL}(2, \mathrm{C}))_n$.

Case 1(a). Suppose $n = 1$ or 2. Note that $\chi_{\mathrm{W}_1}(h) = \chi_{\mathrm{W}_3}(h)$ for every $h \in \mathrm{diag}(\mathrm{H})$ and hence for every $h \in \mathrm{H}$. Therefore, $\gamma : \mathrm{W}_1 \to \mathrm{W}_3$ is an H-module isomorphism. This implies that U is the sum of two irreducible G$-$modules of dimension 3. Hence when $n = 1$ or 2 the decomposition of $\mathrm{W} = Sym^2\mathrm{V}$ is given by

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 3^3, 1$$

Case 1(b). Suppose $n \geq 3$. In this case there exists an $h \in \mathrm{H}$ such that $\chi_{\mathrm{W}_1}(h) \neq \chi_{\mathrm{W}_2}(h)$. Therefore Clifford theory implies that $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_3$ affords an irreducible representation of G. Hence we have the following decomposition of $Sym^2\mathrm{V}$.

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 6, 3, 1$$

Case 2. Suppose G is equal to $\mathrm{H}_n^2 \cup \gamma \mathrm{H}_n^2$. Since $\mathrm{H}_n^2\mathrm{D} = \mathrm{H}_n^1\mathrm{D}$, where $\mathrm{D} = Z(\mathrm{GL}(2, \mathrm{C})) \times Z(\mathrm{GL}(2, \mathrm{C}))$. Lemma 3.2.2 implies that the dimensions of the irreducible components of the decomposition of $Sym^2\mathrm{V}$ as an H_n^2- module are identical to those in its decomposition as an H_n^1- module. Therefore when $n = 1$ or 2

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 3^3, 1$$

and when $n \geq 3$

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 6, 3, 1$$

Case 3. Suppose G is equal to $\mathrm{H}_n^3 \cup \gamma\mathrm{H}_n^3$. $\mathrm{H}_n^3\mathrm{D} = (\mathrm{SL}(2,\mathrm{C}) \times \mathrm{SL}(2,\mathrm{C}))\mathrm{D}$. Corollary 3.2.1 then implies that the irreducible H$-$modules are of the form $\mathrm{V}(k) \otimes \mathrm{V}(l)$. The value of a character on a diagonal element $h = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ is equal to the product of $\chi_k(h_1)$ and $\chi_l(h_2)$ where $h_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $h_2 = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ and χ_i is the character of $(\mathrm{SL}(2,\mathrm{C}))_n$ on $\mathrm{V}(i)$. This gives

$$\begin{aligned} \chi_{\mathrm{w}}(h) &= a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \\ &= (a^2 + ab + b^2) + (c^2 + cd + d^2) + (a+b)(c+d) \\ &= \chi_3(h) + \chi_3(h) + \chi_2(h)\chi_2(h) \end{aligned}$$

Therefore as an H_n^3-module for arbitrary n, $Sym^2\mathrm{V} = \bigoplus_{i=1}^3 \mathrm{W}_i$ where $\dim(\mathrm{W}_i) = 3$ for $i = 1, 2$ and $\dim(\mathrm{W}_3) = 4$. Since $\dim(\mathrm{W}_3)$ is distinct, W_3 is G-invariant. We see by observing $\chi_{\mathrm{w}}(h)$ that the spaces W_1 and W_2 are given by $span\{v_1^2, v_1v_2, v_2^2\}$ and $span\{v_3^2, v_3v_4, v_4^2\}$ respectively and are interchanged by γ. Let $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_2$. It is clear that for any n the spaces W_1 and W_2 are not isomorphic as H_n^3-modules. Therefore $\mathrm{U} = \mathrm{W}_1 \oplus \mathrm{W}_2$ is G$-$irreducible and we have the following decomposition of $Sym^2\mathrm{V}$.

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 6, 4$$

Representations of $G = H \cup \gamma H$ where $H \subseteq T \times T$, T projectively finite

Remark. In this case we obtain the irreducible decompositions of the H-modules and G-modules via the character tables of these groups which are generated using *MAGMA*. Hence although we will be able to observe the dimensions and characters of the irreducible H-invariant subspaces, we will not be able to decide if two subspaces of equal dimension are exchanged by γ. Nonetheless, we may still apply the general theory of interchanged modules used in the infinite case in order to obtain a finite list of possible dimensions for a group G containing such an H. For example under the action of imprimitive G where $\mathrm{H} = \left\{\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \xi_{\mathrm{A}}\mathrm{B} \end{bmatrix} : \mathrm{A} \in (\mathrm{A}_4)_n, \mathrm{B} \in \mathrm{A}\{\pm \mathrm{I}\}\right\}$ we get the following list of dimensions of irreducible components in the decomposition of Sym^2V.

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 1, 3^3$$

Given any irreducible H-invariant subspace U of dimension d then the space $\mathrm{U} \cup \gamma\mathrm{U}$ is either an irreducible G-module of dimension d or $2d$, or it is a reducible G-module which decomposes into two irreducible submodules each of dimension d. Hence we may obtain the following list of possible dimensions in the decomposition of Sym^2V under the action of G.

$$\mathrm{Sym}^2\mathrm{V} \Longrightarrow 1, 3^3 \text{ or } 1, 3, 6$$

Table 3 in the Appendix contains the decompositions for each type of imprimitive group having the property that the coordinate projection of H is projectively finite. The following theorm can be generated from this table.

Theorem 3.2.1 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$ with $\pi_i(H) = T$ where T is projectively finite. Let $V \cong C^4$. Then one can identify the group G via Flow Chart D in the Appendix.*

4 Results: Procedures for Determining the Galois Group of a Fourth Order Linear Differential Equation

As stated in the introduction, the following set of results allows one to identify up to conjugation and projective equivalence a degree four irreducible unimodular linear algebraic group via the reducibility properties of modules of the form $\wedge^2$V and Sym^mV where $\mathrm{V} \cong (\mathrm{C}^*)^4$. The motivation for such a procedure lies in the direct connection between factorizations of differential operators and module decompositions of the associated Galois group. Each of these results follows directly from the information found in the representation tables. The organization of this section is significant in that if nothing is known by way of finiteness or primitivity of a given group G, then one would need to begin the procedure at Theorem 4.1.1. Theorem 4.1.1 allows one to make an initial distinction between the primitive groups, the monomial groups, and the imprimitive nonmonomial groups. With the results of Theorem 4.1.1 in hand, Theorems 4.1.2 and 4.1.3 can be used to determine if G is a finite primitive group, or if G is an infinite primitive group, and, in the infinite case, what is the connected component, G^0 of G. In the case when G is a finite primitive group, G projects onto one of the thirty primitive projective groups constructed by Blichfeldt in [3]. Flow charts A, B, C then allow one to completely distinguish among such groups up to projective equivalence. In order to obtain such complete results it was necessary to use the second, third, fourth, and sixth symmetric powers of the usual module as well as the second exterior power.

4.1 Procedure for Identifying a Linear Algebraic Group

STEP 1: Primitive *vs* Monomial *vs* Imprimitive Nonmonomial

Theorem 4.1.1 *Let G be an irreducible subgroup of $SL(4, C)$. Let $V \cong C^4$. Either*

1. *one of the following conditions is satisfied in which case G is an imprimitive monomial group.*
   - *$Sym^2 V$ has a 2 dimensional $G-$invariant subspace.*
   - *$\wedge^2 V$ and $Sym^2 V$ both have an irreducible $G-$invariant subspace of dimension 6.*

   or,
2. *condition 1 does not hold in which case G is either an imprimitive nonmonomial group or a primitive group. Furthermore, G is a primitive group if and only if one of the following is true.*
   - *$\wedge^2 V$ has no $G-$invariant subspace of dimension 2.*
   - *$\wedge^2 V$ has two 1 dimensional $G-$invariant subspaces and $Sym^3 V$ has a $G-$invariant subspace of dimension 6.*

STEP 2A: Finite Primitive *vs* Infinite Primitive

Theorem 4.1.2 *Let G be a primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then G is a finite group if and only if one of the following conditions holds.*

1. *$Sym^3 V$ has a G-invariant subspace of dimension 1.*
2. *$Sym^3 V$ has a G-invariant subspace of dimension 8 and $Sym^6 V$ has a G-invariant subspace of dimension 2,3, or 4.*
3. *$Sym^3 V$ is G-irreducible and $Sym^4 V$ is $G-$reducible.*
4. *$Sym^3 V$ is G-reducible and $Sym^2 V$ is G-irreducible.*
5. *None of the above hold and there exists a $k \neq 5, 7,$ or 21 such that $Sym^6 V$ has more than one $G-$invariant subspace of dimension k.*

STEP 3A: Infinite Primitive Groups

Theorem 4.1.3 *Let G be an infinite primitive subgroup of $SL(4, C)$. Let $V \cong C^4$. Then exactly one of the following holds.*

1. *$\wedge^2 V$ is G-irreducible, in which case $G = SL(4, C)$.*
2. *$\wedge^2 V$ has a one dimensional G-invariant subspace and $Sym^2 V$ is*
   - *G-reducible, in which case $G^0 = SL(2, C)$ or*
   - *G-irreducible, in which case $G^0 = SP(4, C)$.*
3. *None of the above hold (equivalently $Sym^2 V$ has a G-invariant subspace of dimension 1) and one of the following holds.*
   - *$Sym^3 V$ has a G-invariant subspace of dimension 8, in which case $G = A_4^{SL_2} \otimes SL(2, C)$.*
   - *The above does not hold and $Sym^4 V$ has a G-invariant subspace of dimension 10, in which case $G = S_4^{SL_2} \otimes SL(2, C)$.*
   - *Neither of the above hold and $Sym^4 V$ has a G-invariant subspace of dimension 21, in which case $G = A_5^{SL_2} \otimes SL(2, C)$.*
   - *$G^0 = SO(4, C)$.*

STEP 2B: Identify Monomial Group

Theorem 4.1.4 *Let G be an infinite imprimitive monomial subgroup of $SL(4, C)$ with permutation action $\mathscr{P}_G$. Let $V \cong C^4$. Then $\mathscr{P}_G$ is an order four or an order eight subgroup of S_4 if $\wedge^2 V$ is G-reducible. Furthermore, given that $\wedge^2 V$ is reducible the dimension of the maximal diagonal subgroup H of G is determined by one of the following conditions:*

1. *$dim(H) = 1$ if $\wedge^2 V$ has as least four 1 dimensional G-invariant subspaces.*
2. *$dim(H) = 2$ if $\wedge^2 V$ has exactly two 1 dimensional G-invariant subspaces.*
3. *$dim(H) = 3$ if $\wedge^2 V$ does not have a 1 dimensional G-invariant subspace.*

It is expected that one cannot distinguish between finite and infinite monomial groups using representation theory. The reason being that given any representation of the group GL(1, C) there exists a large enough integer n such that the sets of values of the characters of that representation are equal for GL(1, C) and for the group of units U_n. Nonetheless, in the monomial case one can distinguish the infinite groups from the finite groups using the procedure for deciding if the logarithmic derivative of a solution is algebraic and possibly using the generalizations of this procedure as well.

Next, consider the imprimitive nonmonomial groups. Such a group G contains an index two block diagonal subgroup H. The coordinate projections of this subgroup H must be primitive subgroups of GL(2, C). In the case when G is infinite there are essentially three possibilities for unimodular groups. These possibilities can be stated in terms of the first coordinate projection mapping π_1 from H to GL(2, C) as follows: (1) $ker(\pi_1) = \{I\} \times \{I\}$, (2) $ker(\pi_1) = \{I\} \times \{\pm I\}$, or (3) $ker(\pi_1) = \{I\} \times SL(2, C)$. The corresponding block diagonal subgroups H of G are denoted by G_n^1, G_n^2, and G_n^3 respectively.

STEP 2C: Imprimitive Nonmonomial – Finite *vs* Infinite

Theorem 4.1.5 *Let $G = H \cup \gamma H$ be an imprimitive nonmonomial subgroup of $SL(4, C)$ with $H \subseteq T \times T$ and $\pi_i(H) = T$ for $i = 1, 2$. Then T is projectively infinite if and only if $Sym^6 V$ has an irreducible G-invariant subspace of order a multiple of 7. Furthermore, if this condition is satisfied then there exists a nonnegative integer n such that G is conjugate to G_n^1 or G_n^2 if and only if any one of the following* **equivalent conditions** *holds. Otherwise $G = G_n^3$ for some n.*

- *$\wedge^2 V$ has an odd number of one dimensional G-invariant subspaces.*
- *$Sym^2 V$ has a 1 dimensional G-invariant subspace.*
- *$Sym^3 V$ has either a 2 dimensional or 4 dimensional G-invariant subspace.*
- *$Sym^4 V$ has a 1 dimensional G-invariant subspace.*
- *$Sym^6 V$ has either a 7 dimensional or a 1 dimensional G-invariant subspace.*

In a similar manner one is able to work with the imprimitive nonmonomial groups having the property that the coordinate projections of the block diagonal subgroup H are finite. The finite primitive subgroups and their representations are well known. This details of this case are presented in section 3 of this paper.

As previously mentioned, each of the above results can be restated to give results concerning Galois groups. In addition we give the following results which are concerned exclusively with the Galois theory of linear differential equations of order four. One can find the definitions of **solvable in terms of lower order** and **liouvillian** in [18].

Theorem 4.1.6 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ is* **not** *solvable in terms of solutions of lower order equations if and only if $L_{Sym^4}(y)$ is irreducible.*

Proof. Theorem 5.1 from [18] implies that $L(y) = 0$ is not solvable in terms of solutions to equations of lower order if and only if the Lie algebra associated with the connected component G^0 if $Gal(L)$ is both simple and has no faithful representation of degree less than 4. The only algebraic subgroups of SL(4, C) satisfying both of these conditions are the infinite primitive groups G where G^0 is either SL(4, C) or SP(4, C). By observing the tables one sees that these are the only subgroups of SL(4, C) which act irreducibly on Sym^4V.

Theorem 4.1.7 *Let $L(y) = 0$ be an irreducible order four linear homogeneous differential equation with coefficients in k and unimodular differential Galois group $Gal(L)$. Then $L(y) = 0$ has only liouvillian solutions if and only if one of the following conditions is satisfied.*

1. *$L_{Sym^4}(y)$ has an irreducible factor of even order k where $k \neq$ 6,10, or 16.*
2. *$L_{Sym^4}(y)$ has more than one order 1 factor.*

3. *For some* $k \neq 5, 10$, $L_{Sym^4}(y)$ *has an even number of irreducible factors of order* k*.*
4. *For each* k, $L_{Sym^4}(y)$ *has at most one factor of order* k *and there is an* $m \neq 7$ *such that* $L_{Sym^6}(y)$ *has more than one irreducible factor of order* m*.*
5. *None of the above hold and* $L_{Sym^6}(y)$ *has no irreducible factor of order equal to a multiple of 7.*

Proof. A theorem of Kolchin (Theorem 3.2 in [20] pp.18) implies that $L(y) = 0$ has only liouvillian solutions if and only if the connected component of the identity $Gal(L)^0$ is a solvable group. In the order four case $Gal(L)^0$ is solvable if and only if $Gal(L)$ is a finite group, a monomial group, or an imprimitive nonmonomial group $\mathrm{H} \cup \gamma\mathrm{H}$ where $\mathrm{H} \subseteq \mathrm{T} \times \mathrm{T}$ and T is projectively finite. The result then follows from the information found in the tables from the Appendix.

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Appendix

Summary of Group Notation

We provide a quick reference for the groups constructed in section 3. This summary contains all of the notation for the groups and can be used to make the representation tables easier to read.

Imprimitive Monomial Groups

A monomial group G contains a maximal diagonal subgroup H such that $\mathrm{G}/\mathrm{H} \cong \mathscr{P}_{\mathrm{G}}$ where $\mathscr{P}_{\mathrm{G}}$ is a transitive subgroup of the permutation group S_4. We refer to this group $\mathscr{P}_{\mathrm{G}}$ as the permutation action of G. If G is an infinite monomial group then H must be conjugate to one of the following groups:

- $\mathbf{H_1^n} = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & \xi^{-1} a & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix} : \ a \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$
- $\mathbf{H_2^n} = \left\{ \begin{bmatrix} \xi a & 0 & 0 & 0 \\ 0 & \xi^{-1} b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} : \ a, b \in \mathrm{C}^*, \xi \in \mathrm{U}_n \right\}$
- $\mathbf{H_3} = \mathrm{diag}(\mathrm{SL}(4, \mathrm{C}))$

Suppose one of the above is the maximal diagonal subgroup of a monomial group G with permutation action $\mathscr{P}_{\mathrm{G}}$. We denote such a group G by

$$\mathrm{G}^{\mathbf{i,n}}_{\mathscr{P}_{\mathrm{G}}}$$

where we delete the integer n when $i = 3$.

Imprimitive Nonmonomial Groups

We may write such a group G as the union $\mathrm{H} \cup \gamma \mathrm{H}$ where H is in block diagonal form (so H can be thought of as a subgroup of $\mathrm{GL}(2, \mathrm{C}) \times \mathrm{GL}(2, \mathrm{C})$) and γ is in block off-diagonal form. Since G is imprimitive and nonmonomial, the set of elements that occur in either the upper left hand corner of H or the lower right hand corner of H is equal to a primitive subgroup T of GL(2, C).

Suppose T is projectively finite. Then H has the following properties.

- T projects onto a degree two projective representation of A_4, S_4 or A_5. This permutation group will be denoted by P.
- There exists a normal subgroup N of P such that if $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in H$ then $B \in A\tilde{N}$ where $\tilde{N}$ is a subgroup of T which projects onto P. This can be shown using a generalization of Goursat's Lemma.

Given such a subgroup H there are two possible types of groups G. In particular,

- if T projects onto either S_4 or A_5, then G is conjugate to $H \cup \gamma_1 H$ where $\gamma_1 = \begin{bmatrix} 0 & I \\ \xi & 0 \end{bmatrix}$, $\xi \in C^*$.
- if T projects onto A_4 then there are basically two possibilities for G. Either $H \cup \gamma_1 H$ where γ_1 is as above or $H \cup \gamma_2 H$ where $\gamma_2 = \begin{bmatrix} 0 & O_{S_4} \\ \xi O_{S_4} & 0 \end{bmatrix}$.

So given an imprimitive nonmonomial group $G = H \cup \gamma_i H$ where T projects onto the permutation group P and where $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in H$ implies $B \in A\tilde{N}$ we denote G by

$$G_{\mathbf{i}}(P, N).$$

Suppose T is projectively infinite. Then there exists a nonnegative integer n such that $G = H \cup \gamma H$ where H is one of the following groups and $\gamma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Recall that $SL(2, C)_n = \{A \in GL(2, C) : \det A^n = 1\}$.

- $\mathbf{H_1} = \left\{ \begin{bmatrix} B & 0 \\ 0 & d_B B \end{bmatrix} : B \in SL(2, C)_n,\ d_B = \frac{1}{\det B} \right\}$
- $\mathbf{H_2} = \left\{ \begin{bmatrix} B & 0 \\ 0 & \pm d_B B \end{bmatrix} : B \in SL(2, C)_n,\ d_B = \frac{1}{\det B} \right\}$
- $\mathbf{H_3} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in SL(2, C)_n,\ \det AB = 1 \right\}$

Let n be a nonnegative integer such that $G = H_i \cup \gamma H_i$. Then we denote G by

$$G^{\mathbf{i}}_{\mathbf{n}}.$$

Primitive Groups

Suppose G is a finite primitive group. Then G projects onto one of the 30 primitive projective groups listed in [3]. For each of these projective groups there is a largest subgroup of SL(4, C) which projects onto this group. These subgroups of SL(4, C) are denoted by

$$\mathbf{FP}1, \dots, \mathbf{FP30}.$$

Suppose G is an infinite primitive group. Then G is a finite extension of its connected component G^0. In this case either G^0 is one of the following semisimple groups or G is a homomorphic image of $\tilde{H} \times SL(2, C)$ acting on $V(1) \otimes V(1)$ where $\tilde{H} \in \{A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2}, (A_4^{SL_2})_2, (S_4^{SL_2})_2, (A_5^{SL_2})_2\}$. A group of this last type will be denoted by $\tilde{H} \otimes SL(2, C)$.

- **SL(4,C)** acting on its standard representation, V.
- **SL(4,C)** acting on the dual, V^*.
- **SL(2,C)** acting on $\mathrm{Sym}^3 V$ where V is its standard representation.
- **SP(4,C)** acting on its standard representation, V.
- **SO(4,C)** $\approx$ **SL(2,C)** $\times$ **SL(2,C)** acting on $V \otimes V$ where V is the standard representation for SL(2, C).

Note. We observed that the dimensions of the irreducible components in the decomposition of the G− modules from Section 3 where G is either of the above representations of SL(4, C) are identical. Hence we do not distinguish between them in the tables.

Table 1. Module Decompositions for the Infinite Monomial Groups

	$\wedge^2$V	Sym^2V	Sym^3V	Sym^4V
$G_{C_4}^{1,n},\ n = 1, 2$	$2, 1^4$	$2^3, 1^4$	2^{10}	$2^{13}, 1^9$
$G_{C_4}^{2,n},\ n = 1, 2$	$4, 1^2$	$4^2, 1^2$	4^5	$4^8, 1^3$
$G_{C_4}^{2,n},\ n > 2, G_{C_4}^3$	$4, 2$	$4^2, 2$	4^5	$4^8, 2, 1$
$G_K^{1,n},\ n = 1, 2$	$2, 1^4$	$2^3, 1^4$	2^{10}	$2^{13}, 1^9$
$G_K^{2,n},\ n = 1, 2$	$2^2, 1^2$	$4, 2^2, 1^2$	4^5	$4^5, 2^6, 1^3$
$G_K^{2,n},\ n > 2, G_K^3$	2^3	$4, 2^3$	4^5	$4^7, 2^3, 1$
$G_{P_8}^{1,n},\ n = 1, 2$	$2, 1^4$	$2^3, 1^4$	2^{10}	$2^{13}, 1^9$
$G_{P_8}^{2,n},\ n = 1, 2$	$4, 1^2$	$4^2, 1^2$	$8, 4^3$	$8, 4^6, 1^3$
$G_{P_8}^{2,n},\ n > 2, G_{P_8}^3$	$4, 2$	$4^2, 2$	$8, 4^3$	$8^2, 4^4, 2, 1$
$G_{A_4}^3$	6	6, 4	$12, 4^2$	$12^2, 6, 4, 1$
$G_{S_4}^3$	6	6, 4	$12, 4^2$	$12^2, 6, 4, 1$

• V is the usual representation of the monomial group, G
• For convenience of notation in the table the group $< (1234) >$ will be denoted by C_4 and the group $< (1234), (12)(34) >$ will be denoted by P_8

Table 2. Decompositions for Nonmonomial G where T is Projectively Infinite

	G	$\wedge^2$V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
G_n^1	$n = 1$	$3, 1^3$	$3^3, 1$	$4^4, 2^2$	$5^5, 3^3, 1$	$7^7, 5^5, 3^3, 1$
	$n = 2$	*	*	$8^2, 4$	*	*
	$n = 3$	3, 2, 1	6, 3, 1	$8, 4^3$	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n = 4$	*	*	$8^2, 4$	$5^5, 3^3, 1$	$14^2, 10, 7^3, 6, 5^3, 3, 1$
	$n = 5, 6$	*	*	*	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n \geq 7$	*	*	*	*	$14^3, 10^2, 7, 6, 5, 3, 1$
G_n^2	$n = 1$	$3, 1^3$	$3^3, 1$	$4^4, 2^2$	$5^5, 3^3, 1$	$7^7, 5^5, 3^3, 1$
	$n = 2$	*	*	$8^2, 4$	*	*
	$n = 3$	3, 2, 1	6, 3, 1	$8, 4^3$	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n = 4$	*	*	$8^2, 4$	$5^5, 3^3, 1$	$14^2, 10, 7^3, 6, 5^3, 3, 1$
	$n = 5, 6$	*	*	*	$10^2, 6, 5, 3, 1$	$14^2, 10^2, 7^3, 6, 5, 3, 1$
	$n \geq 7$	*	*	*	*	$14^3, 10^2, 7, 6, 5, 3, 1$
G_n^3	any n	$4, 1^2$	6, 4	12, 8	16, 10, 9	30, 24, 16, 14

• A star appears in the table below wherever the decomposition is exactly as that immediately above

Table 3. Decompositions of Nonmonomial G where T is projectively Finite

	$\wedge^2$V	Sym^2V	Sym^3V	Sym^4V	Sym^6V
$G_i(A_4, I)$	$1^3, 3$	$1, 3^3$	4^5	$1^{11}, 3^8$	$1^{18}, 3^{22}$
	1, 2, 3	1, 3, 6	$4^3, 8$	#	#
			$4, 8^2$	#	#
$G_i(A_4, K)$	$1^2, 4$	4, 6	$4^2, 12$	$1^4, 3, 4^4, 6^2$	$1^2, 3^2, 4^{10}, 6^6$
	2, 4		8, 12	$1^2, 2, 3^3, 4^2, 6, 8$	$1^2, 3^6, 4^4, 6^4, 8^3$
				#	#
$G_i(A_4, A_4)$	$1^2, 4$	4, 6	$4^2, 12$	$1^2, 2, 4^2, 6, 8, 9$	$1^2, 4^4, 6^4, 8^3, 9^2$
	2, 4		8, 12	#	
$G_1(S_4, I)$	1, 2, 3	1, 3, 6	$4, 8^2$	$1, 2^5, 3^4, 6^2$	$1^4, 2^7, 3^{10}, 6^6$
			4,16	#	#
$G_1(S_4, K)$	2, 4	4, 6	8, 12	$2^2, 3, 6^2, 8^2$	$2, 4^4, 6^7, 8^3$
				#	#
$G_1(S_4, A_4)$	2, 4	4, 6	8, 12	4, 6, 9, 16	$2, 4^2, 6^2, 8^2, 12, 16, 18$
					#
$G_1(S_4, S_4)$	1, 2, 3	4, 6	8, 12	4, 6, 9, 16	$2, 6^2, 8, 12, 16^2, 18$
					#
$G_1(A_5, I)$	$1^3, 3$	$1, 3^3$	$4, 8^2$	$1, 3^3, 5^5$	$1, 3^{10}, 4^7, 5^5$
	1, 2, 3	1, 3, 6	4, 16	#	#
$G_1(A_5, A_5)$	$1^2, 4$	4,6	8, 12	9, 10, 16	6, 8, 16, 24, 30
	2, 4				

• A # indicates a list of all possible lists of dimensions that can be constructed from the given list of dimensions via combining two irreducible spaces of dimension d into one irreducible space of dimenison $2d$

Table 4. Decompositions for the Infinite Primitive Groups

	$\wedge^2 V$	$Sym^2 V$	$Sym^3 V$	$Sym^4 V$	$Sym^6 V$
$G^0 = SL(2, C)$	1, 5	3, 7	4, 6, 10	1, 5, 7, 9, 13	$3, 7^2, 9, 11, 13, 15, 19$
$G^0 = SL(4, C)$	6	10	20	35	84
$G^0 = SP(4, C)$	1, 5	10	20	35	84
$G^0 = SO(4, C)$	3^2	1, 9	4, 16	1, 9, 25	1, 9, 25, 49
$A_4^{SL_2} \otimes SL(2, C)$	3^2	1, 9	$4, 8^2$	$1, 5^2, 9, 15$	$1, 5^2, 7, 9, 15, 21^2$
$S_4^{SL_2} \otimes SL(2, C)$	3^2	1, 9	4, 16	1, 9, 10, 15	$1, 7, 9, 10, 15, 21^2$
$A_5^{SL_2} \otimes SL(2, C)$	3^2	1, 9	4, 16	1, 9, 25	1, 9, 21, 25, 28

Table 5. Decompositions for the Finite Primitive Groups

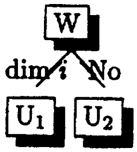
	$\wedge^2 V$	$Sym^2 V$	$Sym^3 V$	$Sym^4 V$	$Sym^6 V$
FP1	3^2	1, 9	4^5	$1^5, 3^4, 9^2$	$1^6, 3^8, 9^6$
FP2	3^2	1, 9	$4, 8^2$	$1, 2^2, 3^2, 6, 9^2$	$1^2, 2^2, 3^6, 6, 9^6$
FP3	3^2	1,9	$4, 8^2$	$1, 5^2, 9, 15$	$1, 3, 4, 5^2, 9^3, 12^2, 15$
FP4	3^2	1, 9	4, 16	1, 9, 10, 15	$1, 3, 4, 9^3, 10, 12^2, 15$
FP5	3^2	1, 9	4, 16	$1, 4, 6^2, 9^2$	$1^2, 3^4, 4, 6^2, 9^6$
FP6	3^2	1, 9	4, 16	1, 9, 25	$1, 9^2, 12^2, 16, 25$
FP7	3^2	1, 9	$4, 8^2$	$1, 2^2, 6^2, 9^2$	$1^2, 2^2, 3^4, 6^2, 9^6$
FP8	6	1, 9	$4^3, 8$	$1^3, 2, 6^2, 9^2$	$1^4, 2, 6^4, 9^6$
FP9	6	1, 9	$4, 8^2$	$1, 2^2, 9^2, 12$	$1^2, 2^2, 6^2, 9^6, 12$
FP10	6	1, 9	4, 16	$1, 4, 9^2, 12$	$1^2, 4, 6^2, 9^6, 12$
FP11	6	1, 9	4, 16	1, 9, 25	$1, 9^2, 16, 24, 25$
FP12	6	1, 9	4, 16	$1, 4, 9^2, 12$	$1^2, 4, 6^2, 9^4, 12, 18$
FP13	1, 5	5^2	4^5	$1^5, 5^6$	$1^4, 5^{16}$
FP14	1, 5	5^2	$4, 8^2$	$1, 2^2, 5^6$	$2^2, 5^{16}$
FP15	1, 5	10	4, 16	$1, 4, 5^2, 10^2$	$4, 5^4, 10^6$
FP16	1, 5	10	20	$5^3, 20$	$4, 5, 10^2, 15, 20^2$
FP17	6	10	4, 16	$1, 4, 15^2$	$10^4, 12^2, 20$
FP18	1, 5	10	20	5, 10, 20	$4, 5, 10^2, 15, 20^2$
FP19	6	10	4, 16	$1, 4, 15^2$	$10^4, 20, 24$
FP20	6	10	20	5, 30	$10^2, 24, 40$
FP21	6	10	20	5, 30	$10^2, 24, 40$
FP22	3^2	1, 4, 5	$1, 3^2, 4^2, 5$	$1^2, 3^2, 4^3, 5^3$	$1^3, 3^6, 4^7, 5^7$
FP23	1, 5	$3^2, 4$	$4^2, 6^2$	$1^2, 3^2, 4^3, 5^3$	$1, 3^{10}, 4^7, 5^5$
FP24	1, 5	10	10^2	$5^2, 8^2, 9$	$5^2, 8^2, 9^2, 10^4$
FP25	6	10	20	14, 21	$10^2, 14, 15, 35$
FP26	6	3,7	$4^2, 6^2$	$1, 6^2, 7^2, 8$	$1, 3^4, 6^2, 7^5, 8^3$
FP27	6	10	20	5, 30	15, 24, 45
FP28	6	1,4,5	$1, 4^2, 5, 6$	$1^2, 4^3, 5^3, 6$	$1^3, 4^7, 5^7, 6^3$
FP29	$1^2, 4$	4, 6	$4^2, 6^2$	$4, 5^5, 6$	$1, 4^7, 5^5, 6^5$
FP29B	1, 5	4, 6	$4^2, 6^2$	$1^2, 4^3, 5^3, 6$	$1, 4^7, 5^5, 6^5$
FP30	1, 5	10	20	$5^2, 9, 16$	$5^2, 9^2, 10^4, 16$

Table 6. Decompositions for the Finite Primitive Subgroups of SL(2, C) (from [20])

	Sym^2V	Sym^3V	Sym^4V	Sym^5V	Sym^6V
$A_5^{SL_2}$	3	4	5	6	3, 4
$S_4^{SL_2}$	3	4	2, 3	2, 4	$1, 3^2$
$A_4^{SL_2}$	3	2^2	$1^2, 3$	2^3	$1, 3^2$

Explanation of the flow charts.

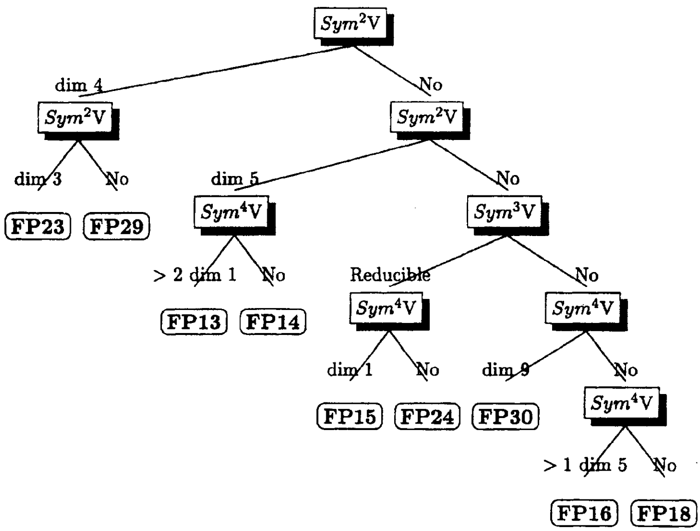
A branching of the following type



means that if W has an invariant subspace of dimension i then consider the space U_1, if not consider the space U_2. In general W, U_1, and U_2 are constructions of $V \cong C^4$. Sometimes type phrase "dim i" is replaced by "irr dim i" (respectively "$> k$ dim i"). These apply when W has an irreducible invariant subspace of dimension i (respectively, when W has an invariant subspace that is the sum of at least k invariant subspace of dimension i.)

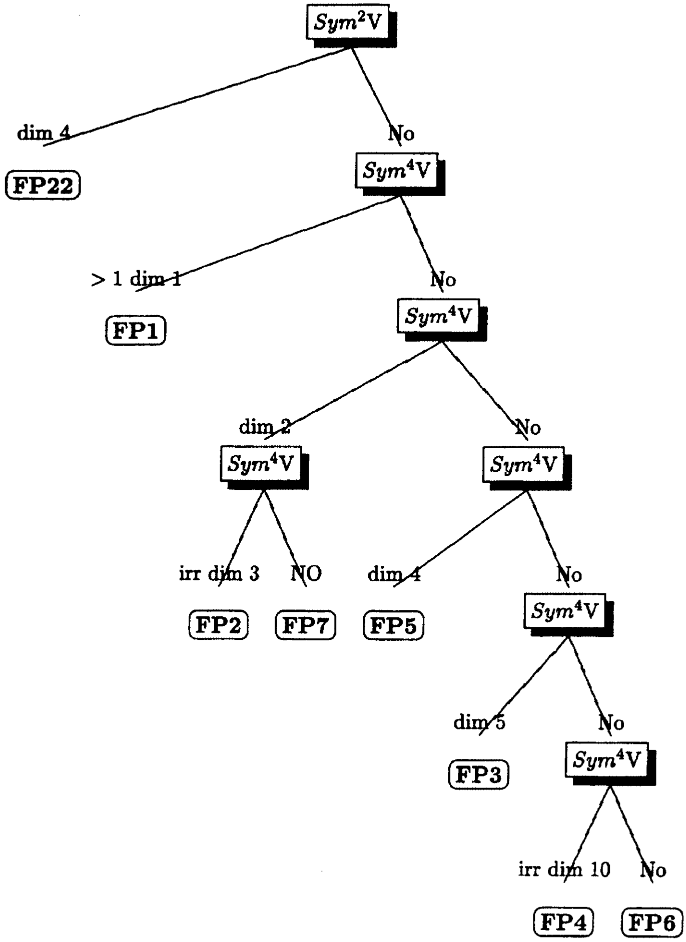
Flow Chart A

$\wedge^2$ V has dim 1 invariant.



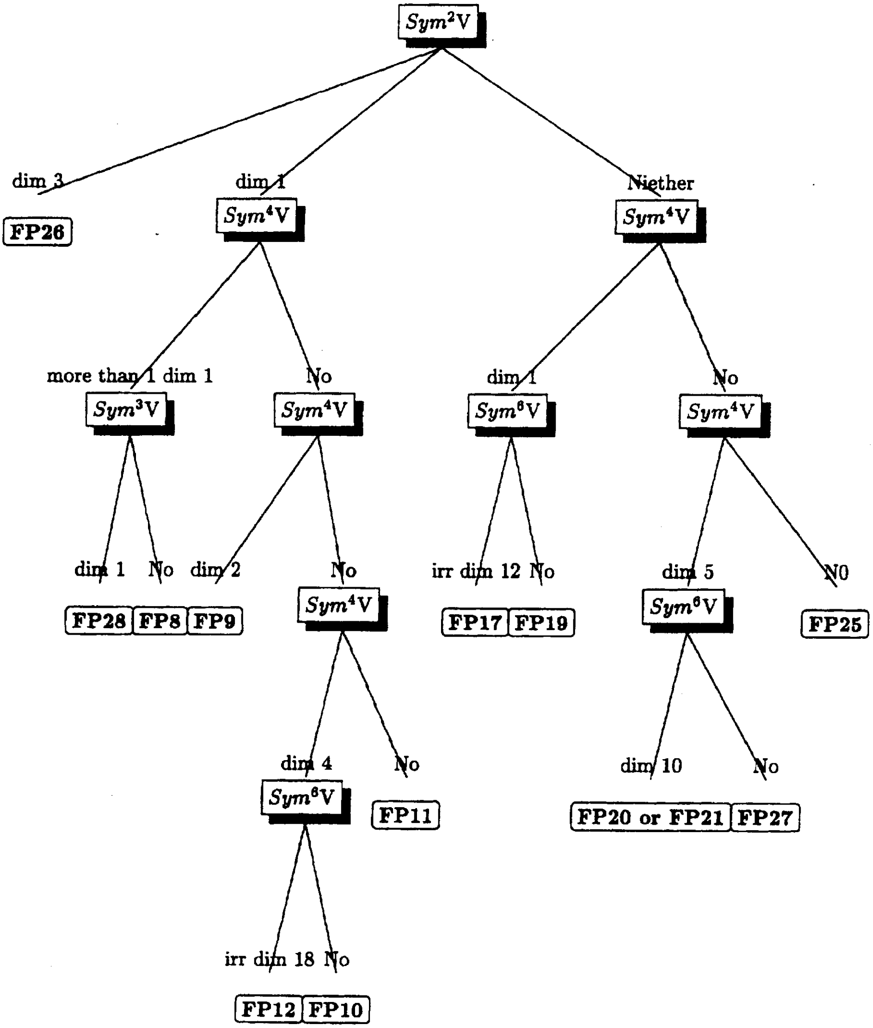
Flow Chart B

$\wedge^2$ *V* has dim 3 invariant.

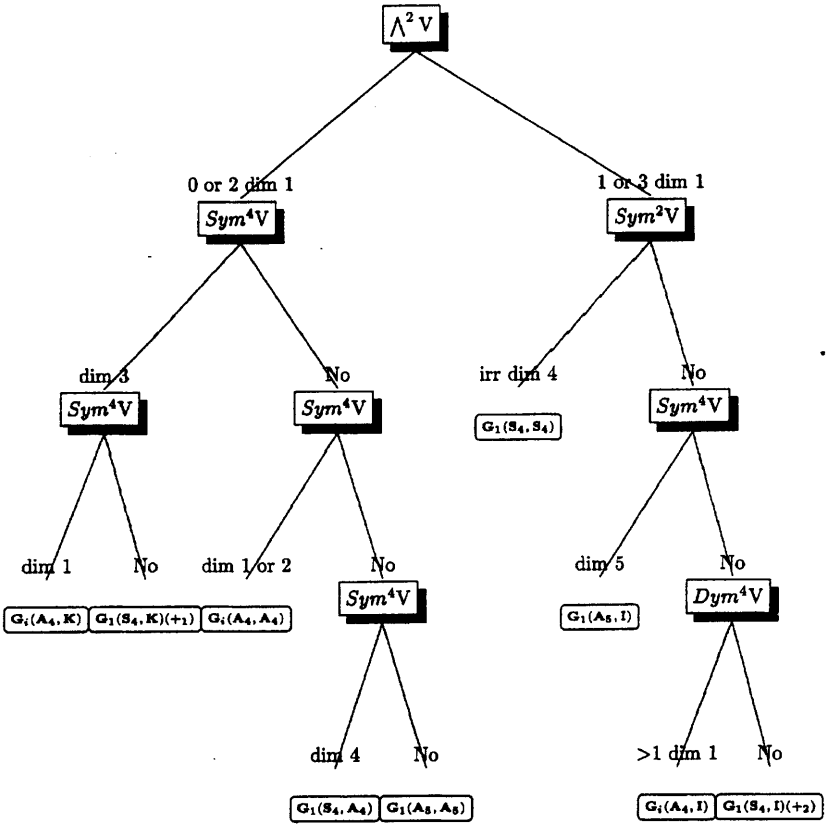


Flow Chart C

$\wedge^2 V$ is irreducible.



Flow Chart D



Notes:

($+_1$) It is most likely that the group is $G_1(S_4, K)$ but it may also be $G_i(A_4, K)$
($+_2$) It is most likely that the group is $G_1(S_4, I)$ but it may also be $G_i(A_4, I)$