Differential Galois Theory and Integration

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Abstract In this chapter, we present methods to simplify reducible linear differential systems before solving. Classical integrals appear naturally as solutions of such systems. We will illustrate the methods developed in [DW19] on several examples to reduce the differential system. This will give information on potential algebraic relations between integrals.

Key words: Ordinary Differential Equations, Differential Galois Theory, Computer Algebra, Integrals, Lie Algebras, D-finite functions.

Introduction

In this chapter, we will review properties of block triangular linear differential systems and their use to compute properties of integrals.

Let $\mathbf{k} = \mathbb{C}(x)$ and $A \in \mathrm{Mat}(n, \mathbf{k})$. We will study the corresponding *linear differential system* $[A] : \partial_x Y = AY$. More generally, we might consider linear differential system over a differential field $(\mathbf{k}, \partial)$ of characteristic zero, that is a field $\mathbf{k}$ equipped with an additive morphism ∂ satisfying the Leibniz rule $\partial(ab) = a\partial(b) + \partial(a)b$.

The Galois theory of linear differential equation aims at understanding what are the algebraic relations between the solutions of $[A]$. We attach to $[A]$ a group that measures the relations. The computation of this differential Galois group is a hard task in full generality. The goal of this chapter is to illustrate on examples the

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method described in [DW19] that focuses on the reduction of block-triangular linear differential systems. This approach is powerful enough to understand the desired relations on the solutions.

Given an invertible matrix $P \in GL(n, \mathbf{k})$, the linear change of variables

$$Y = PZ$$

produces a new differential systems denoted $Z' = P[A]Z$ where

$$P[A] := P^{-1}AP - P^{-1}P'.$$

Two linear differential systems $[A]$ and $[B]$ are called *(gauge) equivalent* over $\mathbf{k}$ if there exists a gauge transformation, an invertible matrix $P \in GL(n, \mathbf{k})$, such that $B = P[A]$. A linear differential system is called *reducible* (over $\mathbf{k}$) when it is gauge equivalent to a linear differential system in block triangular form:

$$[A] : \ \partial Y = AY, \ \text{ with } A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline S & A_2 \end{array} \right) \in \mathrm{Mat}(\mathbf{k}). \tag{1}$$

It turns out that computing properties of integrals of D-finite functions or of some types of iterated integrals may be reduced to computing solutions of such reducible linear differential systems. Computing the differential Galois groups of block triangular systems gives, in turn, information on properties of their solutions. This idea was promoted by Bertrand [Ber01] or Berman and Singer [BS99] who showed how to compute Galois groups of some reducible systems and how this would reveal algebraic properties of integrals.

Our aim, in this chapter, is to show how to compute such algebraic properties. The underlying theory is developed in [DW19] and [CW18]; general references for differential Galois theory are for example [PS03, CH11, Sin09]; general references for constructive theory of reduced forms of differential systems are [AMCW13, MS02, AMDW16, BCDVW20, BCDVW16]. We will rely on many examples rather than on cumbersome theory and provide references for interested readers. For example, we will consider the dilogarithm function Li_2 defined by $\mathrm{Li}_2'(x) = -\frac{\ln(1-x)}{x}$ and our 4-dimensional example (in the next sections) will provide a simple algorithmic proof that it is not only transcendent but algebraically independent of e^x, $\ln(x)$ and $\ln(1 - x)$; the proof will only use rational solutions of linear first order differential equations.

The chapter is organized as follows. We begin by some examples to illustrate what the method will provide. Then we propose a review of differential Galois theory and reduced forms of differential systems. We finish by explaining the strategy of [DW19] on two examples that are chosen so that almost all calculations can be reproduced easily.

1 Examples

1.1 A first toy example

We consider two confluent Heun functions

$$f_1(x) = \exp\left(\frac{\sqrt{3}}{12}x\right) \text{HeunC}\left(\frac{\sqrt{3}}{6}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{48}, \frac{11}{48}, x\right)$$

$$= 1 - \frac{7}{96}x - \frac{719}{46080}x^2 - \frac{127307}{10616832}x^3 - \frac{82319293}{10192158720}x^4 + O\left(x^5\right)$$

and

$$f_2(x) = \exp\left(\frac{\sqrt{3}}{12}x\right) \sqrt[3]{x}\,\text{HeunC}\left(1/6\,\sqrt{3}, 1/3, -1/3, 1/48, \frac{11}{48}, x\right)$$

$$= x^{\frac{1}{3}} e^{\frac{\sqrt{3}}{12}x}\left(1 + \left(\frac{25}{192} - \frac{\sqrt{3}}{12}\right)x + \left(\frac{10321}{129024} - \frac{25\sqrt{3}}{2304}\right)x^2 + O\left(x^3\right)\right).$$

They form a basis of solutions of the second order linear differential equation $L(y) = 0$

$$L(y) := \frac{d^2}{dx^2}y(x) + \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d}{dx}y(x) - \frac{(3x^2 - 6x + 7)}{144x(x-1)}y(x) = 0$$

The Wronskian relation gives us the algebraic relation $(f_1 f_2' - f_1' f_2)^3 = x^2(x-1)^2$. The equation has order two so the Kovacic algorithm [Kov86, UW96, vHW05] can be used to compute the differential Galois group and we find that no other algebraic relations exist between f_1, f_2, f_1' and f_2'. Now let $F_i(x) := \int^x f_i(t)dt$ be a primitive. We want to determine whether F_1 and F_2 are algebraically independent of the f_i and f_i' or not. The techniques explained below will show that this question reduces to asking whether there is a *rational* solution to the linear differential system

$$Z' = \begin{pmatrix} 0 & \frac{-3x^2 + 6x - 7}{144x(x-1)} \\ -1 & \frac{1}{3}\frac{4x-2}{x(x-1)} \end{pmatrix} \cdot Z + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or, equivalently, whether the following linear differential equation (the left hand side turns out to be the adjoint operator of L) has a rational solution:

$$-g''(x) + \frac{2}{3}\frac{(2x-1)}{x(x-1)}g'(x) + \frac{(3x^4 - 9x^3 - 179x^2 + 185x - 96)}{144x^2(x-1)^2}g(x) = 1.$$

It can be seen directly or with a computer algebra system that this equation has no rational solution. The underlying theoretical tools are from the constructive differ-

ential Galois theory. However, in operational terms, it is rather easy to compute and check : no hard theory is required for the calculation. Let us unveil a corner of the underlying tools.

We have in fact studied the differential Galois group of the differential system $[A]$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3x^2-6x+7}{144(x-1)^3x^2} & -\frac{2}{3}\left(\frac{1}{x-1}+\frac{1}{x}\right) & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which admits the fundamental solution matrix

$$U := \begin{pmatrix} f_1(x) & f_2(x) & 0 \\ f_1'(x) & f_2'(x) & 0 \\ \int^x f_1(t)\mathrm{dt} & \int^x f_2(t)\mathrm{dt} & 1 \end{pmatrix}.$$

Our calculation of rational solution above shows (with the tools displayed below) that the differential Galois group of the system $[A]$ will have dimension 5. This in turn shows that the integrals $\int^x f_j(t)\mathrm{dt}$ are algebraically independent of f_1, f_2 and their derivatives. In fact, it even shows that both integrals are algebraically independent.

1.2 A second toy example

Remind that the hypergeometric function is given by the formula

$$_2F_1([a,b],[c])(x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \quad \text{with} \quad (a)_0 = 1, \ (a)_n = a(a+1)\ldots(a+n-1).$$

We now take two hypergeometric functions (with nice modular properties)

$$f_1(x) = {}_2F_1\left(\left[-\frac{1}{3},\frac{1}{12}\right],\left[\frac{7}{12}\right]\right)(x)$$

and

$$f_2(x) = x^{5/12}{}_2F_1\left(\left[\frac{1}{12},\frac{1}{2}\right],\left[\frac{17}{12}\right]\right)(x).$$

The vectors $Y_i := (f_i, f_i')^T$ are solutions of $Y' = A_1Y$ below. To study properties of their integrals, we set $Y := (f, f', \int^x f(t)dt)^T$ and we have $Y' = AY$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ \frac{1}{36}\frac{1}{x(x-1)} & -\frac{7}{12x} - \frac{1}{6(x-1)} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & 0 \\ 1 0 & 0 \end{pmatrix}.$$

Proceeding as on the first example, we find a matrix

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{15}{44} + \frac{45\,x}{44} & -\frac{9\,x(x-1)}{11} & 1 \end{pmatrix}$$

such that

$$Q[A] = \begin{pmatrix} A_1 & 0 \\ 0\,0 & 0 \end{pmatrix}.$$

This shows that

$$\int^x f_i(t)\,\mathrm{dt} = -\frac{9}{11}\,x\,(x-1)\,f_i'(x) + \frac{15}{44}\,(3\,x-1)\,f_i(x) + \frac{9}{11}\,c_i, \quad c_i \in \mathbb{C},$$

and the differential Galois group of $[A]$ has dimension 3. We note that none of these two examples is new and that efficient methods to handle these questions have been developed by Abramov and van Hoeij in [AvH99, AvH97].

1.3 Integrals via reducible systems

These two examples have shown how, by augmenting the dimension of a linear differential system, we can study integrals of its solutions. Conversely, block triangular systems give rise to integrals via variation of constants; we review this for clarification. For a factorized reducible system $[A]$ of the form

$$A(x) = \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline S(x) & A_2(x) \end{array} \right) = A_{diag}(x) + A_{sub}(x),$$

with $A_{diag}(x) := \left(\begin{array}{c|c} A_1(x) & 0 \\ \hline 0 & A_2(x) \end{array} \right)$ and $A_{sub}(x) := \left(\begin{array}{c|c} 0 & 0 \\ \hline S(x) & 0 \end{array} \right)$, we have a fundamental solution matrix of the form

$$U = \left(\begin{array}{c|c} U_1 & 0 \\ \hline U_2V & U_2 \end{array} \right) = \left(\begin{array}{c|c} U_1 & 0 \\ \hline 0 & U_2 \end{array} \right) \left(\begin{array}{c|c} \mathrm{Id}_{n_1} & 0 \\ \hline V & \mathrm{Id}_{n_2} \end{array} \right).$$

Once U_1 and U_2 are known, V is given by integrals : $\partial V = U_2^{-1} S U_1$. So it may seem that no further theory is required. However, U_1 and U_2 are fundamental solution matrices for systems $\partial U_i = A_i U_i$ so the relation $\partial V = U_2^{-1} S U_1$ may involve integrals of complicated D-finite functions.

Our approach will be to first "reduce" S as much as possible, using manipulations with rational functions, to prepare the system for easier solving. In return, we will obtain algebraic information on all (possibly iterated) integrals that may occur in the relation $\partial V = U_2^{-1} S U_1$.

We note, for the record, that factorized differential operators also correspond to block triangular differential systems.

1.4 Example of situations involving reducible linear differential systems

1.4.1 Operators from statistical physics and combinatorics

In many models attached to statistical mechanics, quantities are expressed as multiple integrals depending on a parameter. They are generally holonomic in this parameter, meaning that they are D-finite, i.e they are solutions to linear differential operators (for example, the so-called Ising operators). A description of this setting may be found in the books of Baxter [Bax82] and McCoy [McC10] (notably Chapters 10 and 12) or in the surveys [MM12, MAB$^+$11] (and references therein). Similarly, in combinatorics, sequences that satisfy recurrence relation may be studied through their generating series, which are often D-finite.

Experimentally (see, for example, [BHM$^+$07, BBH$^+$09, BBH$^+$10, BBH$^+$11, BHMW14, BHMW15] or [HKMZ16, Kou13, ABKM20, ABKM18]), it turns out that many differential operators coming from these processes are factored into a product of smaller order factors - and the corresponding companion systems are reducible.

Other cases of reducible systems are the ones admitting reducible monodromies such as [ST16] or in the works of Kalmykov on Feynman integrals via Mellin-Barnes integrals [BKK13, Kal06, BKK10, KK12, KK17, KK11]; about differential equations for Feynman integrals, one may consult the works of Smirnov [Smi12, LSS18].

Finally, we mention the paper [ABRS14] with other examples of integrals where the techniques presented below may offer alternative approaches for some of the computations.

1.4.2 Variational equations of nonlinear differential systems

Another natural source of reducible systems is the old method of *variational equations* of nonlinear differential systems along a particular solution. Given a system of nonlinear differential equations, we assume the a particular solution ϕ is known; one can form the linear differential equation describing perturbations along this solution ϕ. The general principle is that obstructions to integrability of the nonlinear system can be read on this linear differential system. This is the theory of Morales-Ramis and Simó [MRRS07, MRR10, MRR01] for Hamiltonian systems; extensions to non-hamiltonian differential equations can be found in [AZ10] or [Cas09, CW18]

These variational equation can be written in the form of reducible linear differential systems of big dimension. The simplification techniques outlined below it hence particularly relevant to make computation on such systems practical (see [AMDW16]).

We note that the Morales-Ramis theory has had a spectacular recent development, initiated in [MR20] where this variational approach is applied to path integrals thus

establishing a beautiful and unexpected bridge with the previous subsection.

Reducible linear differential systems appear naturally in another type of perturbative approach: in ϵ-expansions of solutions for perturbed systems $\frac{d}{dx}Y = B(x, \epsilon)Y$ like the ones that appear in works of J. Blümlein, C. Raab, C. Schneider, J. Henn and others for example.

2 Reduced forms of linear differential systems

2.1 Ingredient #1 : Differential Galois-Lie Algebra

In what follows, $(\mathbf{k}, \partial)$ is a differential field of characteristic 0. We outline a brief exposition of the Galois theory of linear differential equations, see [PS03, CH11, Sin09] for expositions with proofs. We consider a linear differential system

$$[A]: \quad \partial Y = AY, \quad A \in \text{Mat}(n, \mathbf{k}). \tag{2}$$

Let C be the field of constants of the differential field $\mathbf{k}$, that is $C := \{a \in \mathbf{k} | \partial(a) = 0\}$. We will assume that C is algebraically closed.

A *Picard-Vessiot extension* is a field $K = \mathbf{k}(U)$, where U is a fundamental solution matrix[1] of $[A]$, such that the field of constants of K is still C. This can be constructed algebraically; alternatively, when $\mathbf{k}$ is a field of meromorphic functions, one may consider a local matrix of power series solutions at a regular point and this gives a Picard-Vessiot extension. A Picard-Vessiot extension is unique modulo differential field isomorphisms.

The *differential Galois group* $G := \text{Aut}_\partial(K/\mathbf{k})$ is the set of automorphisms of K which leave the base field $\mathbf{k}$ fixed and commute with the derivation. Let $\sigma \in G$. By construction, $\sigma(U)$ is also a fundamental solution matrix in K and we find that there exists a matrix $[\sigma] \in \text{GL}(n, C)$ such that $\sigma(U) = U \cdot [\sigma]$. The map $\sigma \mapsto [\sigma]$ provides a faithful representation of G as a subgroup of $\text{GL}(n, C)$, actually a linear algebraic group. If we change the fundamental solution, we obtain a conjugate representation.

We recall that, given an invertible matrix $P \in \text{GL}(n, \mathbf{k})$, the linear change of variables $Y = PZ$ produces a new differential systems denoted $\partial Z = P[A]Z$ where $P[A] := P^{-1}AP - P^{-1}\partial P$. We note that such a gauge transformation $P[A]$, with $P \in \text{GL}(n, \mathbf{k})$, does not change the Galois group.

Given a Picard-Vessiot extension $K = \mathbf{k}(U)$, the polynomial relations among all entries of U (over $\mathbf{k}$) form an ideal I. The Galois group G can then be viewed as the set of matrices which stabilize this ideal I of relations. Thus the computation of G is strongly related to the understanding of the algebraic relations among the solutions.

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[1] An invertible $n \times n$ matrix U such that $U' = AU$.

The *Galois-Lie algebra of* $[A]$ is the Lie algebra $\mathfrak{g}$ of the differential Galois group G. It is defined as the tangent space of G at the identity Id. The dimension of $\mathfrak{g}$ measures the transcendence degree of K over $\mathbf{k}$, that is $\dim_C \mathfrak{g} = \operatorname{trdeg}(K/\mathbf{k})$.

One way of computing the Lie algebra ([PS03]) is the following : $\mathfrak{g}$ is the set of matrices N such that $\operatorname{Id} + \epsilon N \in G(C[\epsilon])$ with $\epsilon^2 = 0$. In other terms, $\operatorname{Id} + \epsilon N$ satisfies the defining equations of the group modulo ϵ^2. For example, $\operatorname{SL}_n(C)$ (set of M such that $\det(M) = 1$) gives the Lie algebra $\mathfrak{sl}_n(C)$ of matrices N such that $\operatorname{Tr}(N) = 0$. The symplectic group $\operatorname{Sp}_{2n}(C)$ is the set of M such that $M^T \cdot J \cdot M = J$; its Lie algebra $\mathfrak{sp}_n(C)$ is found to the set of N such that $N^T J + JN = 0$, with $J = \begin{pmatrix} 0 & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{pmatrix}$. The additive group $\mathbb{G}_a := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in C \right\}$ admits the Lie algebra $\mathfrak{g}_a = Span_C \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$; the multiplicative group $\mathbb{G}_m = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in C^* \right\}$ admits the Lie algebra $\mathfrak{g}_m = Span_C \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

The reduction technique exposed in this chapter aims at computing directly the Galois-Lie algebra $\mathfrak{g}$ before computing G itself. Although the theory is not obvious, the resulting calculations are reasonably simple.

2.2 Ingredient #2 : Lie algebra $\operatorname{Lie}(A(x))$ associated to $A(x)$

The Lie algebra $\operatorname{Lie}(A)$ associated to the matrix A is defined as follows. Let $a_1(x), \ldots, a_s(x) \in \mathbf{k}$ be a basis of the C-vector space generated by the coefficients of A. We can then decompose A as $A = \sum_{i=1}^s a_i(x)M_i$ where the M_i are constant matrices. As we took the coefficients $a_i(x)$ to be a basis of the C-vector space of coefficients of A, this decomposition is well defined. It is not unique but the vector space generated by the M_i is unique and well defined. Now, we consider the smallest Lie algebra containing all the M_i: this is the vector space generated by the M_i and all their iterated Lie brackets ($[M, N] := MN - NM$); then we take its algebraic envelope:

Definition 1 The Lie algebra $\operatorname{Lie}(A)$ associated to the matrix A is the smallest *algebraic*[2] Lie algebra containing all the M_i.

This Lie algebra $\operatorname{Lie}(A)$ appears in works of Magnus [Mag54] or Feynman who use the Baker-Campbell-Hausdorf formula to write solutions of $\partial Y = AY$ as (infinite) products of exponentials constructed with Lie brackets. Wei and Norman give in [WN63, WN64] a finite formula to solve the system when $\operatorname{Lie}(A)$ is solvable. This formula is well-known in physics and control theory but not as well among mathematicians. The terminology of Lie algebra associated to A appears[3] in

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[2] A Lie algebra is called algebraic when it is the Lie algebra of a linear algebraic group. When the M_i are given, this can be computed, see [FdG07], Section 3, or [DW19], Section 6.

[3] For this reason, some authors, including ourselves, call the decomposition $A = \sum_{i=1}^s a_i(x)M_i$ a *Wei-Norman decomposition of A.*

[WN63, WN64] (in there, it is defined as the Lie algebra generated by all values of $A(z_0)$ for z_0 spanning all constants minus singularities and the algebraic envelope is missing). In the sequel, we will study a 4-dimensional example and an 8-dimensional example where our technique will have some relations to the Wei-Norman approach; namely, we will change A to obtain an associated Lie algebra Lie(A) of minimal dimension so that solving formulas become optimal in some sense.

Example 1 (A 4-dimensional example) Let

$$A := \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ 0 & 0 & 1 & -\frac{1}{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that this system is upper triangular, contrary to (1). This will illustrate that our method can be equivalently applied to upper and lower triangular systems. We obtain a Wei-Norman decomposition $A = M_1 + \frac{1}{x} M_2 + \frac{1}{x-1} M_3$, where

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ M_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ M_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have

$$M_4 := [M_2, M_3] = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

All the other brackets in $\langle M_1, M_2, M_3, M_4 \rangle$ are zero and we find that Lie(A) = $\langle M_1, M_2, M_3, M_4 \rangle$. It is solvable of depth 2: the first derived algebra (the set of all matrices in Lie(A) which can be written as a Lie bracket) is $\langle M_4 \rangle$ and the second derived algebra is $\{0\}$. We'll continue below with this example. ◇

2.3 Linear differential systems in reduced form

We now turn to the link between Lie(A) and differential Galois theory, based on two important results of Kolchin and Kovacic. Proofs can be found in [PS03], Proposition 1.31 and Corollary 1.32; see also [BSMR10], Theorem 5.8, and [AMCW13], § 5.3 after Remark 31. Let $A \in \text{Mat}(n, \mathbf{k})$; let G be the differential Galois group of [A] and $\mathfrak{g}$ its Lie algebra, the Galois-Lie algebra of the system [A]. The first result is

$$\mathfrak{g} \subset \mathrm{Lie}(A).$$

So, the Lie algebra associated to A, an object which is very easy to compute, provides an "upper bound" on $\mathfrak{g}$. When we perform a gauge transformation $P \in \mathrm{GL}(n, \mathbf{k})$ to obtain a new system $P[A]$, G and $\mathfrak{g}$ are invariant while $\mathrm{Lie}(P[A])$ may vary. This shows that $\mathfrak{g}$ is a lower bound on all $\mathrm{Lie}(P[A])$, for all gauge transformations $P \in \mathrm{GL}(n, \mathbf{k})$. The second result of Kolchin and Kovacic is that this lower bound is reached. Let $\mathfrak{h} := \mathrm{Lie}(A)$. It is the Lie algebra of an algebraic connected group H. Then there exists a gauge transformation $P \in H(\overline{\mathbf{k}})$ such that $\mathfrak{g} = \mathrm{Lie}(P[A])$. Furthermore, if G is connected and under the very mild additional condition that $\mathbf{k}$ is a C^1-field (it will be the case in the situations handled in this chapter), then we may choose $P \in H(\mathbf{k})$ (no algebraic extension).

Definition 2 A system $\partial Y = AY$ is in *reduced form* when the Lie algebra $\mathrm{Lie}(A)$ associated to A is equal to the Lie algebra of the differential Galois group of $\partial Y = AY$.

The results of Kolchin and Kovacic show that a reduced form always exists. We provide, in the sequel, constructive methods to obtain them when the system is reduced. They will be illustrated on our 4-dimensional example in the next section.

Example 2 (4-dimensional example, continued) We will show, in the next section, that $[A]$ is in reduced form. This system is easily integrated step by step and we find a fundamental solution matrix

$$U = e^x \begin{pmatrix} 1 & \ln(x) & \ln(x-1) & 2\,\mathrm{dilog}(x) + \ln(x-1)\ln(x) \\ 0 & 1 & 0 & \ln(x-1) \\ 0 & 0 & 1 & -\ln(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where dilog is defined by $\mathrm{dilog}'(x) = \frac{\ln(x)}{1-x}$. We may take $\mathrm{dilog}(x) = \mathrm{Li}_2(1-x)$. When computing U, the terms requiring one integration ($\ln(x)$ and $\ln(x-1)$) correspond to the terms in M_2 and M_3 in the Wei-Norman decomposition of A. The term dilog comes from the existence of M_4, the Lie bracket of M_2 and M_3 in $\mathrm{Lie}(A)$.

The corresponding Galois group is a semi-direct product of a 1-dimensional torus (giving rise to the $\exp(x)$ in the solution) and of a vector group generated by M_2, M_3 (giving rise to the terms in ln) and M_4 (giving rise to the dilog). Its Lie algebra is $\mathrm{Lie}(A)$. $\diamond$

The ideas behind this notion of reduced form have been used for inverse problems in differential Galois theory: given an algebraic group G, construct a differential system $[A]$ having G as its differential Galois group. It is also a technique known in differential geometry. Its use for direct problems in differential Galois theory is more recent. A remark in [PS03] suggests that this would be a good idea. In the context of Lie-Vessiot systems, Blazquez and Morales exploit this idea in [BSMR10]. It is developed in [AMDW16, AMW12, AMW11] in order to study variational equations

in the context of integrability of Hamiltonian systems and the Morales-Ramis-Simó theory (and later in [CW18] to study algebraic properties of Painlevé equations). For irreducible systems (or systems in block diagonal form), a criterion for reduced forms is established in [AMCW13] with a decision procedure. Another, much more efficient approach is given in [BCDVW16, BCDVW20] together with generalizations of the criterion of [AMCW13].

The approach described below allows, given the above results, to compute a reduced form of a block triangular linear differential system (the last case remaining after all the above contributions). It is based upon these works, notably [CW18], and is constructed in [DW19].

3 How to Compute a Reduced Form of a Reducible System

Assume now that $\mathbf{k} = C(x)$, where C is algebraically closed of characteristic zero where the derivation ∂ acts trivially. It is in particular a C^1-field. We consider a *block triangular* system over the differential field $C(x)$ in the same form as (1), that is

$$[A] : \partial Y = AY, \quad \text{with } A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline S & A_2 \end{array} \right) \in \mathrm{Mat}(n, \mathbf{k}).$$

Let $A_{\mathrm{diag}} := \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right)$. In what follows, we will assume that the block diagonal part A_{diag} is in *reduced form* and we will show how to find a gauge transformation P such that $P[A]$ is in reduced form. By [DW19], Lemma 2.7, the differential Galois group is connected and the reduction matrix we are looking for has coefficients in $\mathbf{k}$. Instead of reproving all the theory (which, in this case, can be mostly found in [DW19]), we will work out in details a simple example where most of the required algorithmic elements appear. This may help convince the reader of how the method works (the details in [DW19] may be technical, at least in a first approach).

3.1 Shape of the gauge transformation

Let $\mathfrak{h}_{sub}$ be the set of *off-diagonal* constant matrices of the form $\left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right)$ (same sizes as in relation (1)). Our first step is that we may find a reduction matrix in a very particular shape.

Lemma 1 ([AMDW16], Lemma 3.4) *There exists a gauge transformation $P \in \left\{ \mathrm{Id} + B, B \in \mathfrak{h}_{sub}(\mathbf{k}) \right\}$ such that $\partial Y = P[A]Y$ is in reduced form.*

The following is is based on an observation from [AMW12, AMW11]. Let $P = \mathrm{Id}+B$, $B \in \mathfrak{h}_{sub}(\mathbf{k})$. Suppose that, for all $Q \in \{\mathrm{Id} + B, B \in \mathfrak{h}_{sub}(\mathbf{k})\}$, we have $\mathrm{Lie}(P[A]) \subseteq$

Lie($Q[P[A]]$); then, $P[A]$ is in reduced form. In other terms, no rational gauge transformation can turn it into a system with a smaller associated Lie algebra. In this case, Lie($P[A]$) will be the Lie algebra of the differential Galois group and this will give us transcendence relations and algebraic relations on the solutions; this will be seen on the main example of this section.

More generally, as we can see in [DW19], Section 5, if our method can reduce a system with two diagonal blocks then we can iterate this method to obtain a reduced form of a block-triangular system with an arbitrary number of blocks on the diagonal. Let us illustrate this iteration on a system with three diagonal blocks of the form

$$\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline S_{2,1} & A_2 & 0 \\ \hline S_{3,1} & S_{3,2} & A_3 \end{array} \right)$$

where the block diagonal part is in reduced form (see [BCDVW20, BCDVW16] for this). We will first reduce the south-east part (which is of the same form as (1)) into a form

$$\left(\begin{array}{c|c} A_2 & 0 \\ \hline S & A_3 \end{array} \right)$$

Let P_1 be the reduction matrix. By [DW19], Lemma 5.1, the following system is automatically in reduced form

$$A_d := \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & S & A_3 \end{array} \right).$$

Now we perform the gauge transformation $\left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & P_1 \end{array} \right)$ to obtain a system of the form

$$\left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline \tilde{S}_{2,1} & A_2 & 0 \\ \hline \tilde{S}_{3,1} & S & A_3 \end{array} \right)$$

(the $S_{i,j}$ may have changed after the first reduction step). We now see that this system in the same form as (1) with A_d as the block diagonal matrix. So a second reduction of a two-blocks triangular system allows to reduce the initial three-blocks triangular system.

This iteration is well seen in our 4-dimensional example below.

Example 3 (4-dimensional example, continued) Let

$$A := \left(\begin{array}{cccc} 1 & 0 & \frac{1}{x} & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 \end{array} \right).$$

A simple application of a factorization algorithm shows that it is reducible. Indeed, letting

$$P := \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

we have

$$P[A] = \left(\begin{array}{cc|c|c} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ \hline 0 & 0 & 1 & -\frac{1}{x} \\ \hline 0 & 0 & 0 & 1 \end{array} \right).$$

This example is, of course, particularly simple. We use it to show how to apply the iteration procedure and Lemma 1 to simplify the system or prove that it cannot be simplified further.

Since we consider an upper triangular system, we start with the "north-west" corner. We let

$$B := \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix}.$$

The diagonal part is in reduced form (solutions are e^x and cannot be simplified using rational functions). The associated Lie algebra $\mathrm{Lie}(B)$ has dimension 2. Reduction would imply to have dimension 1. By Lemma 1, a reduction matrix would have the form

$$P := \begin{pmatrix} 1 & f(x) \\ 0 & 1 \end{pmatrix}.$$

The north-east coefficient of $P[B]$ is $\frac{1}{x} - f'(x)$. The coefficient $\frac{1}{x} - f'(x)$ could never be constant (the equation $f'(x) = \frac{1}{x}$ has no rational solution, the simple pole $\frac{1}{x}$ cannot be canceled by the derivative of a rational function). For any choice of f, $\mathrm{Lie}(P[B])$ will have dimension 2. It follows that $[B]$ is in reduced form. So we iterate.

We now pick a bigger matrix B:

$$B := \left(\begin{array}{cc|c} 1 & \frac{1}{x} & \frac{1}{x-1} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \quad \text{and} \quad B_{\mathrm{diag}} := \left(\begin{array}{cc|c} 1 & \frac{1}{x} & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

By [DW19], Lemma 5.1, and by the above calculation we find that $[B_{\mathrm{diag}}]$ is in reduced form. Lemma 1 thus shows that a reduction matrix would have the simple form

$$P := \begin{pmatrix} 1 & 0 & f(x) \\ 0 & 1 & g(x) \\ 0 & 0 & 1 \end{pmatrix}.$$

Now $\mathrm{Lie}(B_{\mathrm{diag}})$ has dimension 2 and $\mathrm{Lie}(B)$ has dimension 3. A reduction matrix should therefore map B to a matrix with an associated Lie algebra of dimension 2. We have $P[B]_{2,3} = -g'(x)$ so there should exist constants g_1, g_2 and a rational function $g(x)$ such that $-g'(x) = g_1 + g_2 \frac{1}{x}$. A necessary condition is $g_2 = 0$ and $g(x) = g_0 + g_1 x$. Similarly, there should exist constants f_1, f_2 such that $P[B]_{1,3} = f_1 + f_2 \frac{1}{x}$. We now plug our condition on g into this relation and find that there should be a rational function f such that

$$f'(x) = f_1 - g_1 + (g_0 + f_2)\frac{1}{x} + \frac{1}{x-1}.$$

Now, because of the pole of order 1 at $x = 1$, this can never have a rational solution (whatever the values of the unknown constants). It follows that, for any choice of f, $\mathrm{Lie}(P[B])$ will have dimension 3. So P is in reduced form.

Note that our main ingredient here has been to look for a rational solution of an inhomogeneous linear differential equation whose right-hand side contains parameters. There exist algorithms to compute conditions on the parameters (from the right-hand side) so that such an equation has rational solutions and this will be the key to what follows.

We continue iterating the reduction process. Now we'll have

$$B := \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{1}{x-1} \\ 0 & 0 & 1 & -\frac{1}{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{\mathrm{diag}} := \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and } P := \begin{pmatrix} 1 & 0 & 0 & f_1(x) \\ 0 & 1 & 0 & f_2(x) \\ 0 & 0 & 1 & f_3(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using again [DW19], Lemma 5.1, we see that B_{diag} is in reduced form. Furthermore, $\mathrm{Lie}(B_{\mathrm{diag}})$ has dimension 3 and $\mathrm{Lie}(B)$ has dimension 4. We compute $P[B]$.

$$P[B] = \begin{pmatrix} 1 & \frac{1}{x} & \frac{1}{x-1} & \frac{f_2(x)}{x} + \frac{f_3(x)}{x-1} - \frac{\mathrm{d}}{\mathrm{d}x}f_1(x) \\ 0 & 1 & 0 & \frac{1}{x-1} - \frac{\mathrm{d}}{\mathrm{d}x}f_2(x) \\ 0 & 0 & 1 & -\frac{1}{x} - \frac{\mathrm{d}}{\mathrm{d}x}f_3(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The relation $P[B]_{3,4} = c_{1,3} + c_{2,3}\frac{1}{x} + c_{3,3}\frac{1}{x-1}$ gives conditions $c_{3,3} = 0$, $c_{2,3} = -1$ and $f_3(x) = -c_{1,3}x + c_{0,3}$. The same study on $P[B]_{2,4}$ gives us $f_2(x) = -c_{1,2}x + c_{0,2}$. Without finishing with the last coefficient, we see that $\mathrm{Lie}(P[B])$ contains matrices of the following forms (respectively because of terms in $\frac{1}{x}$ and $\frac{1}{x-1}$):

$$M_2 := \begin{pmatrix} 0 & 1 & 0 & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_3 := \begin{pmatrix} 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

whose Lie bracket is

$$M_4 := [M_2, M_3] = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we see that, whatever our future choices may be, $\mathrm{Lie}(P[B])$ will contain M_4 and hence have dimension 4. This shows that our system cannot be reduced so it is in reduced form. Furthermore, this suggests that our reduction conditions might have been stronger : requiring $P[B]_{3,4} = c_{1,3} + c_{2,3}\frac{1}{x} + c_{3,3}\frac{1}{x}$ was not enough. We could have imposed $P[B]_{3,4} = 0$ and $P[B]_{2,4} = 0$. Both these relations are easily seen to be impossible to fullfill with rational functions so our system is again seen to be in reduced form. ◇

To summarize what this example suggests: we need to "cancel" terms in the purely triangular part; this reduces to finding rational solutions of linear differential equations with parametrized right-hand sides. And the order of the computations matters: here, one needs to study the relations on f_2 and f_3 before studying relations on f_1. We will show, in the sequel, how to systematize these ideas, using an isotypical decomposition and an adapted flag structure, and how to make them algorithmic so that a computer algebra system may perform the calculations.

In this example, we had seen that a fundamental solution matrix could be written using $e^x, \ln(x), \ln(x-1)$ and dilog(x). As $[B]$ is in reduced form, $\mathrm{Lie}(B)$ is the Lie algebra of the Galois group and it has dimension 4. This shows that these four functions are transcendent and algebraically independent. So our calculation above (long but not hard) gave us a simple proof that dilog(x) is algebraically independent of $e^x, \ln(x), \ln(x-1)$.

3.2 The adjoint action of the diagonal

We recall our notations so far. We have $n_i \times n_i$ matrices A_i and

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline S & A_2 \end{array} \right), \quad A_{\mathrm{diag}} := \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right), \quad \mathfrak{h}_{sub} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline S & 0 \end{array} \right) \right\}.$$

If we take two matrices B_1 and B_2 in $\mathfrak{h}_{sub}$, we have $B_1.B_2 = 0$. This allows two simple calculations. First, let $P := \mathrm{Id} + \sum_i f_i B_i$, with $f_i \in \mathbf{k}$, $B_i \in \mathfrak{h}_{sub}$. Then

$$P[A] = A + \sum_i f_i [B_i, A_{\mathrm{diag}}] - \sum_i \partial(f_i) B_i. \tag{3}$$

Furthermore, $[B_i, A_{\mathrm{diag}}] \in \mathfrak{h}_{sub}(\mathbf{k})$. These two calculations show that reduction will be governed by the *adjoint action* $\Psi : X \mapsto [X, A_{\mathrm{diag}}]$ of the block diagonal part A_{diag} on $\mathfrak{h}_{sub}(\mathbf{k})$. This adjoint action Ψ is a linear map. Its matrix, on the canonical basis of $\mathfrak{h}_{sub}$, is

$$\Psi = A_2 \otimes \mathrm{Id}_{n_1} - \mathrm{Id}_{n_2} \otimes A_1^T.$$

When $\partial Y = A_{\mathrm{diag}} Y$ has an abelian Lie algebra we may easily compute a Jordan normal form of $\Psi : X \mapsto [X, A_{\mathrm{diag}}]$. Furthermore the eigenvalues of Ψ belong to $\mathbf{k}$. This is the idea behind [AMDW16]. In our case, we will need a more subtle structure, an isotypical decomposition into Ψ-invariant subspaces of $\mathfrak{h}_{sub}$.

Example 4 (An 8-dimensional example) We consider a matrix A given by

$$A(x) := \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & 1 & 0 & \frac{1}{x} & 0 \\ \star & \star & \star & \star & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\ \star & \star & \star & \star & 0 & 0 & 1 & 0 \\ \star & \star & \star & \star & 0 & 0 & \frac{1}{x-1} & 1 \end{array} \right).$$

The block diagonal part is given by two copies of our 4-dimensional example A_1 (here, $A_2 = A_1$) and we have shown that it was in reduced form. We will give the off-diagonal part later.

As $A_2 = A_1$, the matrix of the adjoint action of the diagonal on $\mathfrak{h}_{sub}$ is a (sparse) 16×16 matrix given by $\Psi = A_1 \otimes \mathrm{Id}_4 - \mathrm{Id}_4 \otimes A_1^T$. $\diamond$

3.2.1 Isotypical decomposition

Remind that $\mathfrak{h}_{sub}$ is the C-vector space of off-diagonal matrices. We now show how the adjoint action Ψ of the diagonal will govern the reduction strategy on $\mathfrak{h}_{sub}$.

Definition 3 A vector space $W \subset \mathfrak{h}_{sub}$ will be called a Ψ-*space* if $\Psi(W) \subset W \otimes_C \mathbf{k}$.

The importance of these Ψ-spaces is stated in the following lemma.

Lemma 2 ([DW19], Lemma 2.11) *Let* $A := \left(\begin{array}{c|c} A_1 & 0 \\ \hline S & A_2 \end{array} \right)$ *and assume that* $\partial Y = AY$ *is in reduced form. Then,* $\mathrm{Lie}(A) \cap \mathfrak{h}_{sub}$ *is a* Ψ-*space.*

So our reduction strategy will be to try to project onto the smallest possible Ψ-space using rational gauge transformations. In [DW19], we provide references to algorithms to decompose and factor into Ψ-spaces. This is obtained using an isotypical decomposition (eigenring methods) and a flag structure.

Lemma 3 (**Krull-Schmidt**) *The C-vector space $\mathfrak{h}_{sub}$ admits a unique isotypical decomposition*

$$\mathfrak{h}_{sub} = \bigoplus_{i=1}^{\kappa} W_i$$

where

- *each W_i is a Ψ-space;*
- *$W_i \simeq v_i V_i$, where V_i is an indecomposable Ψ-space which admits a* flag *decomposition*

$$V_i = V_i^{[\mu]} \supsetneq V_i^{[\mu-1]} \supsetneq \cdots \supsetneq V_i^{[1]} \supsetneq V_i^{[0]} = \{0\}$$

  and $V_i^{[j]}/V_i^{[j-1]}$ is a sum of isomorphic irreducible Ψ-spaces for $1 \le j \le \mu$;
- *For $i \neq j$, the Ψ-spaces $V_i \subset W_i$ and $V_j \subset W_j$ are not isomorphic.*

Once this decomposition and flag structure are computed, we perform, at each stage, a projection on a minimal Ψ-subspace in $V_{i,j}$. This reduces to computing a matrix $E_{i,j}(x)$ (linear algebra) and tuples $(\mathbf{F}(x), c_1, \ldots, c_s)$, with $\mathbf{F}(x) \in \mathbf{k}^N$ and c_i constant, such that

$$\mathbf{F}'(x) = E_{i,j}(x) \cdot \mathbf{F}(x) + \sum_i c_i \mathbf{b_i}.$$

The resulting system $P[A]$ will be "minimal": it will be in reduced form. The proof of this result is technical and can be found in [DW19]. We will illustrate the process on our main example.

Example 5 (8-dimensional example, continued) In this example, $\mathfrak{h}_{sub}$ decomposes as a direct sum $\mathfrak{h}_{sub} = \mathfrak{h}_1 \oplus \mathfrak{h}_5 \oplus \mathfrak{h}_{10}$ of three indecomposable Ψ-spaces. We may compute the structure on $\mathfrak{h}_5$. We find that the adjoint action of $\Psi := [\bullet, A_{\mathrm{diag}}]$ on $\mathfrak{h}_5$ in a good basis $\tilde{N}_2, \ldots, \tilde{N}_6$ of $\mathfrak{h}_5$ is

$$\Psi_5 := \left(\begin{array}{cc|c|cc} 0 & 0 & -\frac{2}{x-1} & 0 & 0 \\ 0 & 0 & -\frac{2}{x} & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{2}{x} & -\frac{2}{x-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The flag structure on $\mathfrak{h}_5$ suggests the following reduction path. Try to remove elements in $\langle \tilde{N}_5, \tilde{N}_6 \rangle$ if possible; then in $\langle \tilde{N}_4 \rangle$; then in $\langle \tilde{N}_2, \tilde{N}_3 \rangle$. How to do this will be made clear in the next section; the flag structure guides the order in which computations should be handled.

Similarly, we have the following structure on $\mathfrak{h}_{10}$. The action of $[\bullet, A_{\mathrm{diag}}]$ on $\mathfrak{h}_{10}$ in an adapted basis $\tilde{N}_7, \ldots, \tilde{N}_{16}$ is:

$$\Psi_{10} := \left(\begin{array}{c|ccc|cccc|ccc}
0 & \frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{1}{x} & -\frac{1}{x-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{x} & \frac{1}{x-1} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

$\diamond$

3.2.2 Intermezzo : Reduction and Rational Solutions

Before we continue, let us make a quick excursion into our main algorithmic toolbox. we start with a simple case. We look for a condition on $P := \mathrm{Id} + \begin{pmatrix} 0 & 0 \\ \hline \beta & 0 \end{pmatrix}$ to have

$$A = \begin{pmatrix} A_1 & 0 \\ \hline S & A_2 \end{pmatrix} \longrightarrow P[A] = \begin{pmatrix} A_1 & 0 \\ \hline \textcolor{red}{0} & A_2 \end{pmatrix}.$$

A simple calculation shows that β should be a rational solution of the matrix linear differential system $\beta' = A_1\beta - \beta A_2 - S$. If we let vec denote the operator transforming a matrix into a vector by stacking its columns, we find (see [DW19]) that $\mathrm{vec}(\beta)' = \psi \cdot \mathrm{vec}(\beta) - \mathrm{vec}(S)$, where Ψ is again the adjoint action of the diagonal defined above. So reduction will be governed by computing rational solutions of linear differential systems. A computer algebra algorithm for this task has been given by Barkatou in [Bar99], see [BCEBW12] for a generalisation to linear partial differential systems and a Maple implementation.

Now, our general tool (also found in the above references) will be an apparently more complicated problem. Given a matrix Ψ and vectors $\mathbf{b_1}, \ldots, \mathbf{b_s}$, we will look for tuples $(\mathbf{F}(x), c_1, \ldots, c_s)$, with $\mathbf{F}(x) \in \mathbf{k}^N$ and c_i constant, such that $\mathbf{F}'(x) = \Psi(x) \cdot \mathbf{F}(x) + \sum_i c_i \mathbf{b_i}$. This forms a *computable* vector space and the algorithms in [Bar99, BCEBW12] provide this.

We now pick concrete coefficients to show how to perform the reduction on our 8-dimensional example. A Maple worksheet[4] with this example and the chosen coefficients may be found at [DW20].

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[4] The reader may also find a pdf version at
http://www.unilim.fr/pages_perso/jacques-arthur.weil/
DreyfusWeilReductionExamples.pdf

3.2.3 Reduction on $\mathfrak{h}_5$ (8-dimensional example)

To remove all of $\mathfrak{h}_5$, it would be enough to have a rational solution to the system

$$\mathbf{Y}' = \Psi_5.\mathbf{Y} + \mathbf{b} \text{ with } \mathbf{b} = \begin{pmatrix} \frac{2}{x} \\ \frac{2\frac{x}{x}+1}{x^2} \\ \frac{x+1}{x^2} \\ 0 \\ \frac{1}{x^2} \end{pmatrix}$$

and Ψ_5 is given in Example 5 (page 17). This gives us reduction equations

$$(W^{[3]}) : \begin{cases} f'_{3,1}(x) = \frac{1}{x^2} \\ f'_{3,2}(x) = 0 \end{cases}$$
$$(W^{[2]}) : \left\{ f'_{2,1}(x) = -\frac{2}{x-1}f_{3,1}(x) - \frac{2}{x}f_{3,2}(x) + \frac{x+1}{x^2} \right.$$
$$(W^{[1]}) : \begin{cases} f'_{1,1}(x) = -\frac{2}{x}f_{2,1}(x) + \frac{2x+1}{x^2} \\ f'_{1,2}(x) = -\frac{2}{x-1}f_{2,1}(x) + \frac{2}{x}. \end{cases}$$

The first two equations correspond to the highest level $W^{[3]}$ of the flag. To remove an element from $W^{[3]}$, there should be a rational solution to the equation $y' = c_1.\frac{1}{x^2} + c_2.0$. The C-vector space of triples (f, c_1, c_2) with $f \in \mathbf{k}$ and $c_i \in C$ such that $f' = c_1.\frac{1}{x^2} + c_2.0$ is found to be 2-dimensional; for $\underline{c} = (1, 0)$, we have $f_{3,1} := -\frac{1}{x} + c_{3,1}$; for $\underline{c} = (0, 1)$, we have $f_{3,2} := c_{3,2}$, where the $c_{3,i}$ are arbitrary constants (their importance will soon be visible). Our gauge transformation is $P^{[3]} = \text{Id} + f_{3,1}B_1^{[3]} + f_{3,2}B_2^{[3]}$ and $A^{[2]} := P^{[3]}[A]$ does not contain any terms from $W^{[3]}$.

Now $W^{[2]}$ is 1-dimensional. The equation for the reduction on $W_2^{[2]}$ is now

$$y' = -\frac{2}{x-1}f_{3,1} - \frac{2}{x}f_{3,2} + \frac{x+1}{x^2} = -\frac{2}{x-1}\left(-\frac{1}{x} + c_{3,1}\right) - \frac{2}{x}c_{3,2} + \frac{x+1}{x^2}$$
$$= \frac{-2c_{3,2}-1}{x} + \frac{-2c_{3,1}+2}{x-1} + \frac{1}{x^2}.$$

We have necessary and sufficient conditions on the parameters $c_{3,i}$ to have a rational solution, namely $c_{3,1} = 1$, $c_{3,2} = -\frac{1}{2}$ and then a general rational solution $f_{2,1} := \frac{-1}{x} + c_{2,1}$. Our new gauge transformation is $P^{[2]} = \text{Id} + (-\frac{1}{x} + c_{2,1})B_1^{[2]}$ and $A^{[1]} := P^{[2]}[A]$ does not contain any term from $W^{[2]}$ any more.

Finally, we look for triples (f, c_1, c_2) with c_i constants such that f is a rational solution of

$$
\begin{aligned}
y' &= c_1 \left(-\frac{2}{x} f_{2,1} + \frac{2x+1}{x^2} \right) + c_2 \left(-\frac{2}{x-1} f_{2,1} + \frac{2}{x} \right) \\
&= c_1 \left(-\frac{2}{x} \left(\frac{-1}{x} + c_{2,1} \right) + \frac{2x+1}{x^2} \right) + c_2 \left(-\frac{2}{x-1} \left(\frac{-1}{x} + c_{2,1} \right) + \frac{2}{x} \right) \\
&= \frac{-2c_1 c_{2,1} + 2 c_1}{x} + \frac{2c_2 \left(-c_{2,1} + 1 \right)}{x-1} + \frac{3c_1}{x^2}.
\end{aligned}
$$

This integral is rational if and only if both residues are zero. As the solution $c_1 = c_2 = 0$ is not admissible, we see that a necessary and sufficient condition is $c_{2,1} = 1$. The set of desired triples (f, c_1, c_2) is of dimension 2. For $\underline{c} = (1, 0)$, we have $f_{1,1} := -\frac{3}{x} + c_{1,1}$; for $\underline{c} = (0, 1)$, we have $f_{1,2} := c_{1,2}$, where the $c_{1,i}$ are constants and can be chosen arbitrarily. So our last gauge transformation matrix will be $P^{[1]} = \mathrm{Id} - \frac{3}{x} B_1^{[1]}$ and the reduction matrix on $\mathfrak{h}_5$ is

$$
P_5 := P^{[3]} P^{[2]} P^{[1]} = \mathrm{Id} + \left(-\frac{1}{x} + 1 \right) B_1^{[3]} - \frac{1}{2} B_2^{[3]} + \left(-\frac{1}{x} + 1 \right) B_1^{[2]} - \frac{3}{x} B_1^{[1]}.
$$

The resulting matrix $\tilde{A} := P_5[A]$ contains no terms from $\mathfrak{h}_5$.

### 3.2.4 Reduction on $\mathfrak{h}_{10}$ (8-dimensional example)

The matrix Ψ_{10}, in a good basis, is given in Example 5 page 17. The reduction equations are now

$$
\begin{aligned}
&(W^{[5]}) : \left\{ f_{5,1}'(x) = 0 \right. \\
&(W^{[4]}) : \begin{cases} f_{4,1}'(x) = \frac{1}{x} f_{5,1}(x) + \frac{1}{x} \\ f_{4,2}'(x) = \frac{1}{x-1} f_{5,1}(x) + \frac{1}{x-1} \end{cases} \\
&(W^{[3]}) : \begin{cases} f_{3,1}'(x) = \frac{1}{x} f_{4,1}(x) \\ f_{3,2}'(x) = \frac{1}{x-1} f_{4,1}(x) - \frac{1}{x-1} \\ f_{3,3}'(x) = \frac{1}{x} f_{4,2}(x) \\ f_{3,4}'(x) = \frac{1}{x-1} f_{4,2}(x) \end{cases} \\
&(W^{[2]}) : \begin{cases} f_{2,1}'(x) = \frac{1}{x-1} f_{3,1}(x) - \frac{1}{x} f_{3,3}(x) + \frac{1}{x^2} \\ f_{2,2}'(x) = -\frac{1}{x-1} f_{3,2}(x) + \frac{1}{x} f_{3,4}(x) - \frac{1}{x-1} + \frac{1}{x} \end{cases} \\
&(W^{[1]}) : \left\{ f_{1,1}'(x) = \frac{1}{x-1} f_{2,1}(x) + \frac{1}{x} f_{2,2}(x) + \frac{1}{x^2}. \right.
\end{aligned}
$$

We will let the reader solve this iteratively following the method from the previous section. This will give the following successive reductions
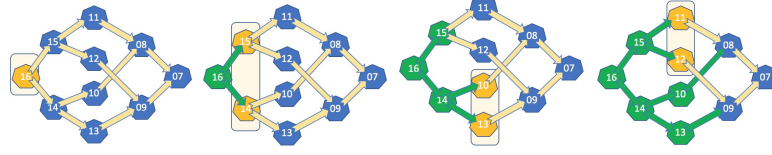
where the green denotes parts that have been successfully removed. However, we reach an obstruction when trying to remove N_{11} (once the equation for $f_{3,2}$ has a rational solution, the equation for $f_{3,1}(x)$ cannot have a rational solution).
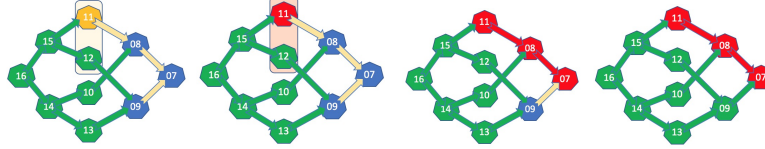
We obtain the reduced form

$$
A_{\mathrm{red}} := \left(\begin{array}{cccc|cccc}
1 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x-1} & 1 & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{x-1} & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -\frac{1}{x} & 1 & 0 & \frac{1}{x} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{x-1} & 1 & 0 & -\frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x-1} & 1
\end{array}\right).
$$

The associated Lie algebra is spanned by

$$
\left(\begin{array}{cccc|cccc}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{array}\right),
\left(\begin{array}{cccc|cccc}
0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0
\end{array}\right),
\left(\begin{array}{cccc|cccc}
0&0&1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
\hline
0&0&0&-1&0&0&1&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{array}\right),
\left(\begin{array}{cccc|cccc}
0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
\hline
0&0&1/2&0&0&0&0&0\\
0&0&0&-1/2&0&0&1&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{array}\right),
\left(\begin{array}{cccc|cccc}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{array}\right).
$$

This gives us the Lie algebra $\mathfrak{g} = \mathrm{Lie}(A_{\mathrm{red}})$ of the differential Galois group. We remark that two of the matrices that could not be removed from $\mathfrak{h}_{sub}$ are "absorbed" as lower triangular parts of matrices coming from A_{diag}. It is 5-dimensional, whereas the Lie algebra $\mathrm{Lie}(A)$ associated to the original matrix A had dimension 14. This shows that the Picard-Vessiot extension is obtained from the Picard-Vessiot extension K_{diag} for $[A_{\mathrm{diag}}]$ by performing only one integral and the system has indeed been transformed into a form where solving is much simpler than before - and we also have proofs of transcendence properties for the remaining objects.

References

ABKM18. Y. Abdelaziz, S. Boukraa, C. Koutschan, and J.-M. Maillard, *Diagonals of rational functions, pullbacked $_2F_1$ hypergeometric functions and modular forms*, J. Phys. A

51 (2018), no. 45, 455201, 30. MR 3863298

ABKM20. ———, *Heun functions and diagonals of rational functions*, J. Phys. A **53** (2020), no. 7, 075206, 24. MR 4071098

ABRS14. J. Ablinger, J. Blümlein, C. G. Raab, and C. Schneider, *Iterated binomial sums and their associated iterated integrals*, J. Math. Phys. **55** (2014), no. 11, 112301, 57. MR 3390498

AMCW13. Ainhoa Aparicio-Monforte, Élie Compoint, and Jacques-Arthur Weil, *A characterization of reduced forms of linear differential systems*, Journal of Pure and Applied Algebra **217** (2013), no. 8, 1504–1516.

AMDW16. A. Aparicio-Monforte, T. Dreyfus, and J.-A. Weil, *Liouville integrability: an effective Morales–Ramis–Simó theorem*, Journal of Symbolic Computation **74** (2016), 537 – 560.

AMW11. Ainhoa Aparicio-Monforte and Jacques-Arthur Weil, *A reduction method for higher order variational equations of Hamiltonian systems*, Symmetries and Related Topics in Differential and Difference Equations, Contemporary Mathematics, vol. 549, Amer. Math. Soc., Providence, RI, September 2011, pp. 1–15.

AMW12. Ainhoa Aparicio-Monforte and Jacques-Arthur Weil, *A reduced form for linear differential systems and its application to integrability of Hamiltonian systems*, Journal of Symbolic Computation **47** (2012), no. 2, 192 – 213.

AvH97. Sergei A. Abramov and Mark van Hoeij, *A method for the integration of solutions of Ore equations*, Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI), ACM, New York, 1997, pp. 172–175. MR 1809984

AvH99. ———, *Integration of solutions of linear functional equations*, Integral Transform. Spec. Funct. **8** (1999), no. 1-2, 3–12. MR 1730611

AZ10. Michaël Ayoul and Nguyen Tien Zung, *Galoisian obstructions to non-Hamiltonian integrability*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 23-24, 1323–1326.

Bar99. Moulay A. Barkatou, *On rational solutions of systems of linear differential equations*, J. Symbolic Comput. **28** (1999), no. 4-5, 547–567.

Bax82. Rodney J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982.

BBH+09. A. Bostan, S. Boukraa, S. Hassani, J.-M. Maillard, J.-A. Weil, and N. Zenine, *Globally nilpotent differential operators and the square Ising model*, J. Phys. A **42** (2009), no. 12, 125206, 50.

BBH+10. A. Bostan, S. Boukraa, S. Hassani, J.-M. Maillard, J.-A. Weil, N. Zenine, and N. Abarenkova, *Renormalization, isogenies, and rational symmetries of differential equations*, Adv. Math. Phys. (2010), 44p.

BBH+11. A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J.-M. Maillard, J.-A. Weil, and N. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, J. Phys. A: Math. Theor. **44** (2011), no. 4, 045204, 44.

BCDVW16. Moulay A. Barkatou, Thomas Cluzeau, Lucia Di Vizio, and Jacques-Arthur Weil, *Computing the lie algebra of the differential galois group of a linear differential system*, Proceedings of the 40th international symposium on Symbolic and algebraic computation (New York, NY, USA), ISSAC '16, ACM, 2016, p. To appear.

BCDVW20. Moulay Barkatou, Thomas Cluzeau, Lucia Di Vizio, and Jacques-Arthur Weil, *Reduced forms of linear differential systems and the intrinsic Galois-Lie algebra of Katz*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 054, 13. MR 4112733

BCEBW12. Moulay A. Barkatou, Thomas Cluzeau, Carole El Bacha, and Jacques-Arthur Weil, *Computing closed form solutions of integrable connections*, Proceedings of the 36th international symposium on Symbolic and algebraic computation (New York, NY, USA), ISSAC '12, ACM, 2012.

Ber01. D. Bertrand, *Unipotent radicals of differential Galois group and integrals of solutions of inhomogeneous equations*, Math. Ann. **321** (2001), no. 3, 645–666.

BHM$^+$07.   S. Boukraa, S. Hassani, J.-M. Maillard, B. M. McCoy, J.-A. Weil, and N. Zenine, *Fuchs versus Painlevé*, J. Phys. A **40** (2007), no. 42, 12589–12605.

BHMW14.   S. Boukraa, S. Hassani, J.-M. Maillard, and J.-A. Weil, *Differential algebra on lattice green and calabi-yau operators*, J. Phys. A: Math. Theor. **47** (2014), no. 9, 095203.

BHMW15.   S Boukraa, S Hassani, J-M Maillard, and J-A Weil, *Canonical decomposition of irreducible linear differential operators with symplectic or orthogonal differential galois groups*, Journal of Physics A: Mathematical and Theoretical **48** (2015), no. 10, 105202.

BKK10.   Vladimir V. Bytev, Mikhail Yu. Kalmykov, and Bernd A. Kniehl, *Differential reduction of generalized hypergeometric functions from Feynman diagrams: one-variable case*, Nuclear Phys. B **836** (2010), no. 3, 129–170. MR 2646503

BKK13.   _____ , *HYPERDIRE, HYPERgeometric functions DIfferential REduction: MATHEMATICA-based packages for differential reduction of generalized hypergeometric functions $_pF_{p-1}$, F_1, F_2, F_3, F_4*, Comput. Phys. Commun. **184** (2013), no. 10, 2332–2342. MR 3101372

BS99.   P. H. Berman and M. F. Singer, *Calculating the Galois group of $L_1(L_2(y)) = 0$, L_1, L_2 completely reducible operators*, J. Pure Appl. Algebra **139** (1999), no. 1-3, 3–23, Effective methods in algebraic geometry (Saint-Malo, 1998).

BSMR10.   David Blázquez-Sanz and Juan José Morales-Ruiz, *Differential Galois theory of algebraic Lie-Vessiot systems*, Differential algebra, complex analysis and orthogonal polynomials, Contemp. Math., vol. 509, Amer. Math. Soc., Providence, RI, 2010, pp. 1–58.

Cas09.   Guy Casale, *Morales-Ramis theorems via Malgrange pseudogroup*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 7, 2593–2610.

CH11.   Teresa Crespo and Zbigniew Hajto, *Algebraic groups and differential Galois theory*, Graduate Studies in Mathematics, vol. 122, American Mathematical Society, Providence, RI, 2011.

CW18.   Guy Casale and Jacques-Arthur Weil, *Galoisian methods for testing irreducibility of order two nonlinear differential equations*, Pacific J. Math. **297** (2018), no. 2, 299–337. MR 3893430

DW19.   Thomas Dreyfus and Jacques-Arthur Weil, *Computing the Lie algebra of the differential Galois group: the reducible case*, ArXiv **1904.07925** (2019).

DW20.   _____ , *Maple worksheet with the examples for this paper:* `http://www.unilim.fr/pages_perso/jacques-arthur.weil/ DreyfusWeilReductionExamples.mw`, 2020.

FdG07.   Claus Fieker and Willem A. de Graaf, *Finding integral linear dependencies of algebraic numbers and algebraic Lie algebras*, LMS J. Comput. Math. **10** (2007), 271–287.

HKMZ16.   S. Hassani, Ch. Koutschan, J.-M. Maillard, and N. Zenine, *Lattice Green functions: the d-dimensional face-centered cubic lattice, d = 8, 9, 10, 11, 12*, J. Phys. A **49** (2016), no. 16, 164003, 30. MR 3491321

Kal06.   Mikhail Yu. Kalmykov, *Gauss hypergeometric function: reduction, ϵ-expansion for integer/half-integer parameters and Feynman diagrams*, J. High Energy Phys. (2006), no. 4, 056, 21. MR 2219052

KK11.   Mikhail Yu. Kalmykov and Bernd A. Kniehl, *Counting master integrals: integration by parts vs. differential reduction*, Phys. Lett. B **702** (2011), no. 4, 268–271. MR 2823288

KK12.   _____ , *Mellin-Barnes representations of Feynman diagrams, linear systems of differential equations, and polynomial solutions*, Phys. Lett. B **714** (2012), no. 1, 103–109. MR 2954894

KK17.   _____ , *Counting the number of master integrals for sunrise diagrams via the Mellin-Barnes representation*, J. High Energy Phys. (2017), no. 7, 031, front matter+27. MR 3688369

Kou13.   Christoph Koutschan, *Lattice Green functions of the higher-dimensional face-centered cubic lattices*, J. Phys. A **46** (2013), no. 12, 125005, 14.

Kov86.   Jerald J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput. **2** (1986), no. 1, 3–43.

LSS18.   Roman N. Lee, Alexander V. Smirnov, and Vladimir A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, J. High Energy Phys. (2018), no. 3, 008, front matter+14. MR 3798577

MAB+11.   Barry M. McCoy, Michael Assis, Salah Boukraa, Saoud Hassani, Jean-Marie Maillard, William P. Orrick, and Nadjah Zenine, *The saga of the Ising susceptibility*, New trends in quantum integrable systems, World Sci. Publ., Hackensack, NJ, 2011, pp. 287–306.

Mag54.   Wilhelm Magnus, *On the exponential solution of differential equations for a linear operator*, Comm. Pure Appl. Math. **7** (1954), 649–673.

McC10.   Barry M. McCoy, *Advanced statistical mechanics*, International Series of Monographs on Physics, vol. 146, Oxford University Press, Oxford, 2010.

MM12.   B. M. McCoy and J-M. Maillard, *The importance of the ising model*, Prog. Theor. Phys. 127 (2012), 791-817, 2012.

MR20.   Juan J. Morales-Ruiz, *A differential Galois approach to path integrals*, J. Math. Phys. **61** (2020), no. 5, 052103, 12. MR 4097805

MRR01.   Juan J. Morales-Ruiz and Jean Pierre Ramis, *Galoisian obstructions to integrability of Hamiltonian systems. I, II*, Methods Appl. Anal. **8** (2001), no. 1, 33–95, 97–111.

MRR10.   Juan J. Morales-Ruiz and Jean-Pierre Ramis, *Integrability of dynamical systems through differential Galois theory: a practical guide*, Differential algebra, complex analysis and orthogonal polynomials, Contemp. Math., vol. 509, Amer. Math. Soc., Providence, RI, 2010, pp. 143–220.

MRRS07.   Juan J. Morales-Ruiz, Jean-Pierre Ramis, and Carles Simo, *Integrability of Hamiltonian systems and differential Galois groups of higher variational equations*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 6, 845–884.

MS02.   Claude Mitschi and Michael F. Singer, *Solvable-by-finite groups as differential Galois groups*, Ann. Fac. Sci. Toulouse Math. (6) **11** (2002), no. 3, 403–423.

PS03.   Marius van der Put and Michael F. Singer, *Galois theory of linear differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 328, Springer-Verlag, Berlin, 2003.

Sin09.   Michael F. Singer, *Introduction to the Galois theory of linear differential equations*, Algebraic theory of differential equations, London Math. Soc. Lecture Note Ser., vol. 357, Cambridge Univ. Press, Cambridge, 2009, pp. 1–82.

Smi12.   Vladimir A. Smirnov, *Analytic tools for Feynman integrals*, Springer Tracts in Modern Physics, vol. 250, Springer, Heidelberg, 2012. MR 3014599

ST16.   T. M. Sadykov and S. Tanabé, *Maximally reducible monodromy of bivariate hypergeometric systems*, Izv. Ross. Akad. Nauk Ser. Mat. **80** (2016), no. 1, 235–280. MR 3462681

UW96.   Felix Ulmer and Jacques-Arthur Weil, *Note on Kovacic's algorithm*, J. Symbolic Comput. **22** (1996), no. 2, 179–200.

vHW05.   Mark van Hoeij and Jacques-Arthur Weil, *Solving second order differential equations with Klein's theorem*, ISSAC 2005 (Beijing), ACM, New York, 2005.

WN63.   James Wei and Edward Norman, *Lie algebraic solution of linear differential equations*, J. Mathematical Phys. **4** (1963), 575–581.

WN64.   J. Wei and E. Norman, *On global representations of the solutions of linear differential equations as a product of exponentials*, Proc. Amer. Math. Soc. **15** (1964), 327–334.