J. reine angew. Math. 473 (1996), 69—85

Journal für die reine und angewandte Mathematik

An example of an arithmetic Fuchsian group

By *Daan Krammer* at Basel

1. Introduction

Let $G \subset \mathrm{PSL}_2(\mathbb{R})$ be an arithmetic Fuchsian lattice. The upper half plane modulo G is a complex curve S. It is likely that by studying some intersections of the form $G \cap g_i G g_i^{-1}$, with g_i in the commensurator of G, one can compute an algebraic equation for S, as well as the images in S of points in U of non-trivial stabilizer, and a so-called uniformizing differential equation. The idea is to find the combinatorics of $G \cap g_i G g_i^{-1}$ first. For an appropriate set of g_i this should determine the involved algebraic equations.

In this note, we illustrate the above idea by performing the computations for one particular arithmetic Fuchsian lattice W^+. The associated quaternion algebra is defined over $\mathbb{Q}$, with discriminant (3) (5). The curve S has genus zero, and the images of points with non-trivial stabilizer turn out to be $\{0, 1, \infty, 81\}$. In 9.1, we give a uniformizing differential equation. This differential equation appeared without proof in [C1], p. 193, (1; 3)-case, (4).

The group W^+ is not commensurable to a triangle group. (Otherwise, our computation would be trivial.) Since the quotient curve S has genus zero, the uniformizing differential equation is a counterexample to a conjecture by Dwork. This conjecture states that any globally nilpotent second order differential equation on $\mathbb{P}^1$ has either algebraic solutions, or has a correspondence to a Gauss hypergeometric one. We note that the explicit computation 9.1 is not necessary for our disproof.

This note is built up as follows. Sections 2 to 6 are of preliminary nature. In section 7 we define an arithmetic lattice $W \subset \mathrm{PGL}_2(\mathbb{R})$, and we give a presentation for W. In section 8, we choose one s in the commensurator of W, and we study the combinatorics of $W^+ \cap s W^+ s^{-1}$, where $W^+ = W \cap \mathrm{PSL}_2(\mathbb{R})$. We use this in section 9 to calculate a uniformizing differential equation on the curve $W^+ \backslash H^2$. In section 10 we show that some index 4 subgroup of W is the group of units in an order of a quaternion algebra. In section 11 we show that the uniformizing differential equation is a counterexample to Dwork's conjecture. The last section contains some suggestions for future research.

2. The hyperbolic plane

In this section, we will define the hyperbolic plane, thereby setting our notations.

Let V be a three dimensional real vector space equipped with a symmetric bilinear form $(\cdot,\cdot)$. Let us write $Q(x)=\frac{1}{2}(x,x)$. We assume that Q is of signature $(2,1)$, i.e., with respect to some basis of V we have $Q(x,y,z)=x^2+y^2-z^2$. We define

$$K=\{x\in V\,|\,Q(x)<0\}.$$

Note that K has two connected components; we denote these by K_1, K_2. We define the hyperbolic plane H^2 to be K_1 modulo scalar multiplication of $\mathbb{R}_{>0}$. Its compactification $\bar{H}^2$ is by definition $\bar{K}_1\backslash\{0\}$ modulo scalar multiplication.

Let $\mathrm{O}=\mathrm{O}(2,1)=\mathrm{O}(V,Q)$ be the orthogonal group. We define the following subgroups of O: $\mathrm{O}^+=\{g\in\mathrm{O}\,|\,gK_1=K_1\}$, $\mathrm{SO}=\{g\in\mathrm{O}\,|\,\det g=1\}$, $\mathrm{SO}^+=\mathrm{SO}\cap\mathrm{O}^+$. Note $\mathrm{O}=\mathrm{O}^+\times\{\pm 1\}$. The group O^+ acts faithfully on H^2 and SO^+ is the subgroup of orientation preserving elements. For any non-isotropic $v\in V$, let s_v denote the orthogonal reflection in v. If $Q(v)>0$ then $s_v\in\mathrm{O}^+$, $s_v\notin\mathrm{SO}^+$, whereas if $Q(v)<0$ then $-s_v\in\mathrm{SO}^+$.

In our calculations, we will use the above model for H^2. Of course, H^2 is isomorphic to the upper half plane $U=\{z\in\mathbb{C}\,|\,\mathrm{Im}\,z>0\}$ with isometry group $\mathrm{PGL}_2(\mathbb{R})\cong\mathrm{O}^+$. The action of $\mathrm{PGL}_2(\mathbb{R})$ descends from that of $\mathrm{GL}_2(\mathbb{R})$, which is defined as follows:

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(z)=\frac{az+b}{cz+d}\quad\text{if}\quad ad-bc>0,$$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(z)=\frac{a\bar{z}+b}{c\bar{z}+d}\quad\text{if}\quad ad-bc<0.$$

3. Fuchsian lattices

A Fuchsian group is a discrete subgroup of $\mathrm{SO}^+(2,1)$ (or $\mathrm{O}^+(2,1)$, or $\mathrm{PGL}_2(\mathbb{R})$, etc.). By a lattice we mean a discrete subgroup of finite covolume. Let $G\subset\mathrm{SO}^+(2,1)$ be a Fuchsian lattice. Then $S=G\backslash H^2$ is isomorphic to a compact smooth Riemann surface $\bar{S}$ minus a finite number of points, called cusps. We will define the signature of G. Let $\{p_1,\dots,p_m\}$ be the points in S above which the map $H^2\to S$ ramifies, let d_i be the ramification degree above p_i, and let g be the genus of $\bar{S}$. Then the signature is $(g;d_1,\dots,d_{m+n})$, where $d_i=\infty$ for $i>m$, and n equals the number of cusps. The Euler characteristic of G is defined to be $\chi(G)=2(1-g)-\sum(1-1/d_i)$; it is negative and proportional to the volume of S [Be], th. 10.4.3. A triangle group is defined to be a lattice in $\mathrm{SO}^+(2,1)$ of some signature $(0;d_1,d_2,d_3)$.

In the above situation, S, being a compact smooth Riemann surface minus a finite number of points, is an algebraic curve over $\mathbb{C}$. Thus a natural question is, when G is given, how to find an equation for this curve. Also, one would like to calculate the uniformizing differential equation, which will be defined in the next section. In general, these are very hard questions. For triangle groups however, it is easy to determine S and the three ramification points. Indeed $\bar{S} = \mathbb{P}^1$, and any three points on $\bar{S}$ can be moved by an automorphism of $\mathbb{P}^1$ to $0, 1$ and ∞ say. Moreover, a uniformizing differential equation can be given explicitly as the well-known Gauss hypergeometric differential equation

$$x(1-x)f'' + \big(c - (a+b+1)x\big)f' - abf = 0\,,$$

where $|1-c| = 1/d_1$, $|c-a-b| = 1/d_2$, $|a+b| = 1/d_3$.

4. Uniformizing differential equations

Let S be an irreducible curve over a field K of characteristic zero. Let $K(S)$ denote the function field, and fix a non-constant function $x \in K(S)$. Let $R = K(S)[d/dx]$ be the ring of differential operators, and write $D = d/dx$ and $f' = df/dx$. A *differential equation* on S is an equation of the form $Lf = 0$, $L \in R$. It will simply be referred to as L.

An equivalent definition runs as follows. A differential equation is a pair (M, f) (sometimes M for short) of a left R-module M generated by an element f. The equivalence of the two definitions is as follows. If a differential operator L is given, put $M = R/RL$, $f = 1 + RL \in M$. Details are left to the reader. We will go forth and back between both definitions, which should cause no confusion. The tensor product of (M, f) and (N, g) is defined to be $(M \otimes_{K(S)} N,\ f \otimes g)$, on which $D = d/dx$ acts by $D(p \otimes q) = (Dp) \otimes q + p \otimes (Dq)$. Here, we suppose that $M \otimes N$ is generated by $f \otimes g$ – otherwise, the tensor product of (M, f) and (N, g) is not defined.

The differential equation $D^n + p_1 D^{n-1} + \cdots + p_n$ is called *Fuchsian* if the following hold.

(1) In points $s \in S$ which are neither a pole nor a critical point of x, the function p_i has at most a pole of order i.

(2) In other points s, the same holds after replacing x by a function which has neither a pole or a critical point at s.

The above definition does not depend on x. Fuchs proved that the class of Fuchsian differential equations is closed under taking tensor products. If M and $M \otimes N$ are Fuchsian, then so is N.

4.1. Definition. Two differential equations (M, f), (N, g) are called *projectively equivalent* if there exists a first order differential equation (A, a) such that

$$(M, f) \cong (N, g) \otimes (A, a)\,.$$

They are called *algebraically equivalent* if in addition, we have $a'/a = kb'/b$ for some $k \in \mathbb{Q}^*$, $b \in K(S)$. (The last condition is equivalent to A being Fuchsian and having finite monodromy.) Let $L = D^2 + pD + q$ be a second order differential equation. For every $r \in K(S)$, there is a unique differential equation of the form $D^2 + rD + s$ projectively equivalent to L. For $r = 0$, it reads $D^2 + (q - p'/2 - p^2/4)$. See [Y], page 42.

4.2. Theorem. *Let $G \subset \mathrm{PSL}_2(\mathbb{R})$ be a lattice, so that $S = G\backslash H^2$ is a curve over $\mathbb{C}$. Let $\pi: H^2 \to S$ be the projection map. For every $p \in \mathbb{C}(S)$ there is a unique $q \in \mathbb{C}(S)$ such that the (multivalued) inverse of π is the quotient of two solutions of the differential equation*

$$f'' + pf' + qf = 0. \tag{1}$$

This differential equation is Fuchsian if and only if $f' + pf = 0$ is Fuchsian.

Proof. [Y], Ch. 4. □

4.3. Definition. In the notation of 4.2, the differential equation (1) is called a *uniformizing differential equation* for G, provided it is Fuchsian, and has only rational local exponents.

4.4. Lemma. *Let (A, a) be a first order Fuchsian differential equation on $S = \mathbb{P}^1/K$ with only rational exponents. Write $K(S) = K(x)$. Then for some $k \in \mathbb{Q}^*$, $b \in K(x)$, we have $a'/a = kb'/b$.*

Proof. Let us write $a' = pa$, $p \in K(x)$. Since (A, a) is Fuchsian, we have

$$p = \sum_i \frac{a_i}{x - b_i}, \quad b_i \in \bar{K},$$

where $\bar{K}$ denotes the algebraic closure of K. Solving $f' = pf$ gives

$$f = c \cdot \prod_i (x - b_i)^{a_i}.$$

It is given that the local exponents a_i are in $\mathbb{Q}$. Since $p \in K(x)$, we have $a_i = a_j$ whenever b_i, b_j are algebraically conjugate over K. This shows that for some $n > 0$, $f^n \in K(x)$, which proves the lemma. □

4.5. Corollary. *Two Fuchsian differential equations on $S = \mathbb{P}^1$ which are projectively equivalent and which have only rational local exponents, are algebraic equivalent.*

Proof. Let (M, f), (N, g) be projectively equivalent Fuchsian differential equations on $\mathbb{P}^1$ with only rational local exponents. Let (A, a) be a first order Fuchsian differential equation such that $(M, f) \cong (N, g) \otimes (A, a)$. Since (M, f) and (N, g) are Fuchsian, so is (A, a). Since (M, f) and (N, g) have only rational local exponents, so has (A, a). Hence 4.4 implies the existence of $k \in \mathbb{Q}^*$, $b \in K(x)$, with $a'/a = kb'/b$. Hence (M, f) and (N, g) are algebraically equivalent. □

5. Arithmetic Fuchsian lattices

We will define a special class of arithmetic Fuchsian lattices, which we call *rational* arithmetic Fuchsian lattices. Let A be a quaternion algebra over $\mathbb{Q}$. Suppose A is indefinite, i.e., $A \otimes \mathbb{R} \cong M_2(\mathbb{R})$, the algebra of real 2×2 matrices. Let R be an order in A. Then $R^*/\{\pm 1\} \subset \mathrm{PGL}_2(\mathbb{R})$ is a rational arithmetic Fuchsian lattice. Moreover, any rational arithmetic Fuchsian lattice is commensurable in the wide sense to such a group. Here, two subgroups of a group are called commensurable if their intersection has finite index in both of them, and they are called commensurable in the wide sense if one is commensurable to some conjugate of the other.

Two rational arithmetic Fuchsian lattices associated to quaternion algebras A_1, A_2 are commensurable in the wide sense if and only if A_1 and A_2 are isomorphic [T1]. Hence wide commensurability classes of arithmetic Fuchsian lattices are in bijection with the isomorphism classes of quaternion algebras.

A quaternion algebra $A/\mathbb{Q}$ is said to ramify at the place v of $\mathbb{Q}$ if $A \otimes \mathbb{Q}_v \ncong M_2(\mathbb{Q}_v)$. Let us write $S(A)$ for the set of ramification places. Then $|S(A)|$ is finite and even. Conversely [Bl], for any set S of an even number of places, there is a unique quaternion algebra A such that $S(A) = S$. We define the *discriminant* of A to be the product of the prime numbers p such that A ramifies at p.

Takeuchi [T2], theorem 2.1, proved that for given g, t, there exist only finitely many arithmetic lattices in $\mathrm{PSL}_2(\mathbb{R})$ of some signature $(g; d_1, \ldots, d_t)$, up to conjugation. He enumerated all arithmetic triangle groups [T1].

5.1. Lemma. *For any rational arithmetic triangle group, the associated quaternion algebra has discriminant* (1) *or* (2) (3).

Proof. This follows immediately from Takeuchi's list of arithmetic triangle groups [T1]. □

Another way of describing rational arithmetic Fuchsian lattices is by ternary quadratic forms. Let V be a $\mathbb{Z}$-module isomorphic to $\mathbb{Z}^3$, and let Q be a quadratic form on V of signature $(2,1)$. Then $\mathrm{O}^+(V,Q) \subset \mathrm{O}^+(2,1)$ is a rational arithmetic Fuchsian lattice, and all arithmetic Fuchsian lattices are commensurable with such a group. The equivalence of the two definitions of arithmetic Fuchsian lattices is established by noting that if A is a quaternion algebra with canonical anti-involution $x \mapsto \bar{x}$, and $V = \{x \in A \mid x + \bar{x} = 0\}$, then A^* acts on V by $a(x) = ax\bar{a}$, preserving the quadratic form $x\bar{x}$ on V if $a\bar{a} = 1$. A quadratic form $ax^2 + by^2 - z^2$ over $\mathbb{Q}$ ramifies at v if and only if the Hilbert symbol $(a,b)_v$ equals -1 [Bl].

6. Fuchsian reflection groups

Let $G \subset \mathrm{O}^+(2,1)$ be a lattice. We define a *fundamental polygon* for G to be an intersection P of finitely many closed half-spaces in H^2, such that

(1) H^2 is the union of all gP, $g \in G$,

(2) for all $g \in G$, $gP^0 \cap P^0 \neq \emptyset \Rightarrow g = 1$, where P^0 denotes the interior of P.

Let us call (P, X) a *fundamental pair* if in addition, $X = \{g \in G \mid P \cap gP$ has codimension $1\}$, and $gP \cap P$ is a facet of P for all $g \in X$.

Poincarés theorem [Be], Th. 9.8.4, gives a necessary and sufficient condition on a pair (P, X) to be a fundamental pair for some G. For the case of reflection groups, this theorem takes a particular comfortable form, which will be stated below. Since the example to be studied in this note will be a reflection group, this will suffice for us.

We retain the notation of sections 2 and 3.

6.1. Theorem. *Let $f_1, \ldots, f_n \in V$ (indices* mod n*) be such that*

(1) $(f_i, f_i) = 2$ *for all* i,

(2) $(f_i, f_{i+1}) = -2\cos(\pi/d_i)$ *for certain* $d_i \in \{2, 3, \ldots\} \cup \{\infty\}$,

(3) $(f_i, f_j) < -2$ *if* $i - j \notin \{-1, 0, 1\}$,

(4) *there exists* $x \in K_1$ *such that* $(f_i, x) > 0$ *for all* i.

Then (P, X) is a fundamental pair, where

$$P = \{x\mathbb{R}_{>0} \mid x \in K_1, (x, f_i) \geqq 0 \text{ for all } i\} \subset H^2,$$
$$X = \{s_i : x \mapsto x - (x, f_i) f_i\} \subset \mathrm{O}^+(2,1).$$

Let $W = \langle X \rangle$, $W^+ = W \cap \mathrm{SO}^+(2,1)$. The group W^+ is a Fuchsian lattice of signature $(0; d_1, \ldots, d_n)$. The stabilizer in W^+ of

$$p_i := \{x \in \bar{K}_1 \backslash \{0\} \mid (x, f_i) = (x, f_{i+1}) = 0\} \in \bar{H}^2$$

is generated by $s_i s_{i+1}$ and has order d_i. The group W has a presentation with generators r_i and relations r_i^2 and $(r_i r_{i+1})^{d_i}$. (Thus, it is a Coxeter group.)

Proof. [Be], Th. 9.8.4; [H], th. 5.13. □

7. An arithmetic Fuchsian lattice

In this section we will construct an arithmetic lattice $G \in \mathrm{O}^+(2,1)$, which is in fact generated by reflections.

Let us define a symmetric bilinear form on $\mathbb{Z}^3$ by $(x, y) = x^{\mathrm{T}} A y$, where A is the following matrix:

$$\begin{pmatrix} 2 & 0 & -5 \\ 0 & 6 & 0 \\ -5 & 0 & 10 \end{pmatrix}.$$

We write $Q(v) = \frac{1}{2}(v, v)$. Thus $Q(x, y, z) = 3y^2 + x^2 - 5xz + 5z^2$. Let

$$G = \mathrm{O}^+(\mathbb{Z}^3, Q) = \{g \in \mathrm{O}(\mathbb{Z}^3, Q) | gK_1 = K_1\}, \quad G^+ = \mathrm{SO}^+(\mathbb{Z}^3, Q) = \{g \in G | \det g = 1\}.$$

Note that a plus attached to G has a different meaning from one attached to SO. Let e_1, e_2, e_3 be the standard basis of $\mathbb{Z}^3$ and let $e_4 = (6, -2, 3)$. Write $f_i = e_i / Q(e_i)^{1/2}$. We give the matrices of inner products (e_i, e_j) and (f_i, f_j):

$$(2) \qquad \begin{pmatrix} 2 & 0 & -5 & -3 \\ 0 & 6 & 0 & -12 \\ -5 & 0 & 10 & 0 \\ -3 & -12 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & -\sqrt{5} & -\sqrt{3} \\ 0 & 2 & 0 & -4 \\ -\sqrt{5} & 0 & 2 & 0 \\ -\sqrt{3} & -4 & 0 & 2 \end{pmatrix}.$$

Let $s_i \in G$ be the reflection in e_i. In order to see that indeed $s_i \in G$, first note $Q(e_i) > 0$ whence $s_i \in \mathrm{O}^+(\mathbb{Q}^3, Q)$. Further, by definition,

$$s_i(e_j) = e_j - 2\frac{(e_i, e_j)}{(e_i, e_i)} e_i,$$

and for all j, we have $2(e_i, e_j)/(e_i, e_i) \in \mathbb{Z}$, which shows $s_i \in \mathrm{O}^+(\mathbb{Z}^3, Q) = G$.

The vectors $f_1, \ldots, f_4$ satisfy the conditions of 6.1. As to (4), note that for all i, on writing $e_i = (x, y, z)$, we have $x + y + z > 0$. Conditions (1), (2) and (3) are left to the reader. We only mention the values of d_i: $d_1 = d_2 = d_3 = 2$, $d_4 = 6$. Consequently, $s_1, \ldots, s_4$ generate a Fuchsian lattice W.

7.1. Proposition. (a) *The quadratic module* $(\mathbb{Q}^3, Q)$ *has discriminant* (3)(5).

(b) *The group G is not commensurable to any triangle group.*

(c) *We have $G = W$.*

Proof. (a) We have

$$Q(x, y, z) = 3y^2 + x^2 - 5xz + 5z^2 = 3y^2 + \left(x - \frac{5}{2}z\right)^2 - 5\left(\frac{1}{2}z\right)^2.$$

Hence we must calculate the Hilbert symbols $(5, -3)_p$, which is easy using [Se], Theorem III.1, page 20. We find

$$(5, -3)_p = \begin{cases} -1, & p = 3, 5, \\ 1, & \text{otherwise}, \end{cases}$$

whence the discriminant of $(\mathbb{Q}^3, Q)$ equals (3)(5).

(b) This follows immediately from (a) and 5.1.

(c) It is enough to prove $W^+ = G^+$, since $W \subset G$, $[G : G^+] = 2$ and $[W: W^+] = 2$. We have $W^+ \subset G^+$. By 6.1, W^+ is a lattice in $O^+(2,1)$ of signature (0; 2, 2, 2, 6). We calculate the Euler characteristic of W^+:

$$-\chi(W^+) = -2 + \sum (1 - 1/d_i) = 2 - (1/2 + 1/2 + 1/2 + 1/6) = 1/3 .$$

Now suppose G^+ has signature $(g;\ d_1, \ldots, d_t)$. Clearly $g = 0$, and by (b) $t \geqq 4$. Since $1 \leqq [G^+ : W^+] = \chi(W^+)/\chi(G^+)$, we find $-\chi(G^+) \leqq 1/3$. It is easily checked that for signature $(0;\ d_1, \ldots, d_t)$ with $t \geqq 4$, we have $-\chi(G^+) \geqq 1/6$, with equality only for signature (0; 2, 2, 2, 3). Supposing $G^+ \neq W^+$, it follows that G has signature (0; 2, 2, 2, 3). But then G^+ does not contain an element of exact order 6, whereas W^+ does. This contradicts the fact $W^+ \subset G^+$, which finishes the proof. □

8. A correspondence

8.1. Definition. Let us define $v = (5, -1, 2) \in \mathbb{Z}^3$, and let $s \in \mathrm{SO}^+(\mathbb{Q}^3, Q)$ be minus the reflection in v. We define $F = W \cap sWs$, $F^+ = W^+ \cap sW^+s$. We denote $X = F^+ \backslash H^2$, $Y = W^+ \backslash H^2$. Let $f : X \to Y$ be the natural map.

Since s normalizes F^+, s descends to an involution of X, which we also denote by s. The clue to the computation of the uniformizing differential equation of W^+ is the correspondence $(f, fs) : X \to Y \times Y$. Let us give a motivation for our choice of s. In general, one would study $W^+ \cap sW^+s^{-1}$ for some $s \in \mathrm{SO}^+(\mathbb{Q}^3, Q)$, but $s \notin W$. The most obvious choice is an involution $s = -s_v$ for some vector $v \in \mathbb{Z}^3$ with $Q(v) < 0$. In this case, the condition $s \notin W$ boils down to $Q(v) < -1$. Calculations will be simplest when we choose $Q(v) = -2$, which holds in our case.

The aim of this section is the following proposition, which gives the combinatorial data of the above mentioned correspondence.

8.2. Proposition. *The map f has degree* 3. *The curve X has genus* 0 *(as has Y). There exist $x_1, \ldots, x_8 \in X$, $y_1, \ldots, y_4 \in Y$ such that f ramifies only above the y_i, and s permutes the x_i except x_2 and x_4 by the following data. The ramification degree of f at x_i is denoted n_i.*

(3)

i	1	2	3	4	5	6	7	8
$s(x_i)$	x_8		x_5		x_3	x_7	x_6	x_1
n_i	1	2	1	2	1	1	1	3
$f(x_i)$	y_1	y_1	y_2	y_2	y_3	y_3	y_3	y_4

Figure 1 *shows the combinatorics of the maps f and s by an easy to understand convention.*

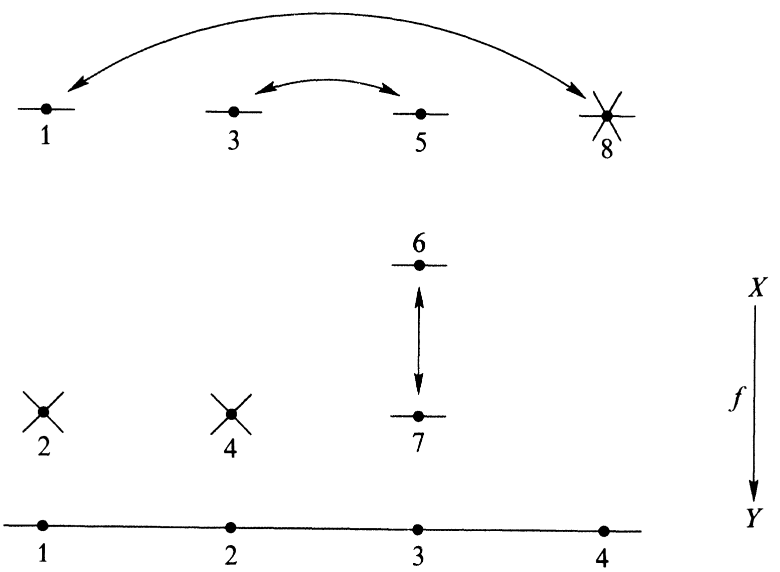

Figure 1. The combinatorics of the maps f and s

Note that over $\mathbb{F}_2$, Q is equivalent to $b^2 - ac$, since

$$Q(a, a+b+c, c) \equiv (a+b+c)^2 + a^2 + ac + c^2 \equiv b^2 + ac \pmod 2.$$

We will denote the quadratic form Q and the bilinear form $(\cdot,\cdot)$ over $\mathbb{F}_2$ by the same symbols as over $\mathbb{Z}$. This should cause no confusion. The quadric $\{x \in \mathbb{F}_2^3 \mid Q(x) = 0\}$ consists of three elements $v_1 = (1,1,0)$, $v_2 = (1,1,1)$, $v_3 = (0,1,1)$. The permutation action of $\mathrm{O}(\mathbb{F}_2^3, Q)$ on $\{v_1, v_2, v_3\}$ establishes an isomorphism between $\mathrm{O}(\mathbb{F}_2^3, Q)$ and the symmetric group S_3. We have a natural map $\pi: W \to \mathrm{O}(\mathbb{F}_2^3, Q) \cong S_3$. The following proposition shows that F is the pre-image under π of a certain subgroup of S_3. Note $v_1 = v \bmod 2$.

8.3. Proposition. $F = \{g \in W \mid gv \in v + 2\mathbb{Z}^3\}$.

Proof. We have $s(x) = x - 2\dfrac{(x,v)}{(v,v)}v = x + \dfrac{1}{2}(x,v)v$ and

$$(4)\qquad \begin{aligned} sgs(x) &= sg\left(x + \frac{1}{2}(x,v)v\right) = s\left(gx + \frac{1}{2}(x,v)gv\right) \\ &= gx + \frac{1}{2}(x,v)gv + \frac{1}{2}\left(gx + \frac{1}{2}(x,v)gv, v\right)v \\ &= gx + \frac{1}{2}(x,v)gv + \frac{1}{2}(gx,v)v + \frac{1}{4}(x,v)(gv,v)v. \end{aligned}$$

In order to prove $\supset$, let $g \in W$, $gv = v + 2w$, $w \in \mathbb{Z}^3$. We must show $sgs \in W$ or, equivalently, $sgs(x) \in \mathbb{Z}^3$ for any $x \in \mathbb{Z}^3$. Note $g^{-1}v = v - 2g^{-1}w$. We have

(5) $$\frac{1}{2}(x,v)gv+\frac{1}{2}(gx,v)v=\frac{1}{2}(x,v)(v+2w)+\frac{1}{2}(x,v-2g^{-1}w)v$$
$$=(x,v)(v+w)-(gx,w)v\in\mathbb{Z}^3,$$

and, inserting in (4),

$$sgs(x)\in gx+\frac{1}{4}(x,v)(gv,v)v+\mathbb{Z}^3.$$

Hence we are left to prove $(gv,v)\in 4\mathbb{Z}$. We have

$$(v,\ v)=(gv,gv)=(v+2w,v+2w)=(v,v)+4(v,w)+4(w,w),$$

whence $(v,w)=-(w,w)$. Since $(w,w)=2Q(w)$ is even, so is (v,w), and we find

$$(gv,v)=(v+2w,v)=(v,v)+2(v,w)=-4+2(v,w)\in 4\mathbb{Z}.$$

Now we will prove $\subset$. Let $g\in F$. For any $x\in\mathbb{Z}^3$, we have $sgs(x)\in\mathbb{Z}^3$ and a fortiori $4sgs(x)\in 2\mathbb{Z}^3$. By (4) we find $(x,v)(gv,v)\in 2\mathbb{Z}$. Choosing x such that (x,v) is odd (for example $x=(0,0,1)$) we find $(gv,v)\in 2\mathbb{Z}$. Let $i\in\{1,2,3\}$ be such that $gv_1=v_i$. Since $(gv,v)\in 2\mathbb{Z}$, we have $(v_i,v_1)=0$. A straightforward check shows that this implies $i=1$. Hence $gv_1=v_1$, or $gv\in v+2\mathbb{Z}^3$. □

We exhibit the matrices of $s_1,\dots,s_4$ with respect to the basis e_1,e_2,e_3.

$$s_1=\begin{pmatrix}-1&0&5\\0&1&0\\0&0&1\end{pmatrix},\quad s_2=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix},$$

$$s_3=\begin{pmatrix}1&0&0\\0&1&0\\1&0&-1\end{pmatrix},\quad s_4=\begin{pmatrix}7&24&0\\-2&-7&0\\3&12&1\end{pmatrix}\equiv s_3\ (\mathrm{mod}\,2).$$

Hence we find

$$\pi(s_1)=(23),\quad \pi(s_2)=(1),\quad \pi(s_3)=(12),\quad \pi(s_4)=(12).$$

Thus π is surjective. Since F is the pre-image under π of $\langle(23)\rangle$, we find

$$[W:F]=[W^+:F^+]=3,$$

and f has mapping degree 3. Put $r=s_4s_1$. Since $\pi(r)=(123)$ has order 3 and $r\notin F^+$, we have $W^+=F^+\amalg F^+r\amalg F^+r^{-1}$.

Recall that $p_i\in H^2$ is defined to be the fixed point of s_is_{i+1}. By 6.1, $\mathrm{Stab}_{W^+}(p_i)=\langle s_is_{i+1}\rangle$. Hence the stabilizer in W^+ of r^kp_i is generated by $g_{ki}=r^ks_is_{i+1}r^{-k}$. Let $\phi:H^2\to X$, $\psi:H^2\to Y$ denote the natural maps, and let d_{ki} be the degree of ramification of f in $\phi(r^kp_i)$. We have

(6) $$d_{ki} = [\mathrm{Stab}_{W^+}(r^k p_i) : \mathrm{Stab}_{F^+}(r^k p_i)]\,.$$

In other words, d_{ki} is the smallest positive integer d such that $\pi(g_{ki})^d \in \langle(23)\rangle$. The following table gives the $\pi(g_{ki})$ and d_{ki}. Note $\pi(r) = (123)$.

(7)

$\pi(g_{ki})$ and d_{ki}				
	$i=1$	$i=2$	$i=3$	$i=4$
$k=-1$	(12), 2	(31), 2	(1), 1	(123), 3
$k=0$	(23), 1	(12), 2	(1), 1	(123), 3
$k=1$	(31), 2	(23), 1	(1), 1	(123), 3

Note $f^{-1}(\psi(p_i)) = \{\phi p_i, \phi r p_i, \phi r^{-1} p_i\}$. Since $\psi(p_1)$ should have 3 pre-images in X (with multiplicities), we find $f^{-1}(\psi(p_1)) = \{\phi p_1, \phi r p_1\}$, with multiplicities 1, 2, respectively. By similar arguments, the two lower rows of (3) are proved, on writing $y_i = \psi p_i$, and x_i defined by the following table.

i	1	2	3	4	5	6	7	8
pre-image in H^2 of x_i	p_1	rp_1	rp_2	p_2	rp_3	p_3	$r^{-1}p_3$	p_4

Now we are able to compute the genus of X. Call it g. Note that Y has genus zero, by 6.1. By Riemann-Hurwitz, $2(g-1) = 3(-2) + \sum(d_i - 1) = -6 + 4$, where d_i are the degrees of ramification 2, 2, 3. Hence X has genus 0.

Note that the ramification degree of ϕ above $r^k p_i$ equals d_i/d_{ki}. Let us denote by $T \subset X$ the set of points above which ϕ ramifies. By (6) and (7), T consists of six points:

$$T = \phi\{p_1, rp_2, r^{-1}p_3, p_3, rp_3, p_4\}\,,$$

all ramification degrees being 2. The involution s of X permutes T. Our next goal is to determine this permutation.

First, we calculate $p_1, \ldots, p_4$. We find that p_i is the image in H^2 of q_i where

$$q_1 = (5, 0, 2),\quad q_2 = (2, 0, 1),\quad q_3 = (4, -1, 2),\quad q_4 = (20, -5, 8)\,.$$

Further,

$$r^{-1}q_3 = s_1 s_4 q_3 = s_1 q_3 = (6, -1, 2)\,.$$

We have $Q(q_1) = Q(q_4) = -5$, $Q(q_2) = Q(q_3) = -1$. Since

$$q_1 + q_4 = 5v \quad\text{and}\quad q_3 + r^{-1}q_3 = 2v,$$

we find

$$sp_1 = p_4,\quad sp_3 = r^{-1}p_3\,.$$

Here we use the easy fact that if $x + y = z$, $Q(x) = Q(y)$ then $s_z(x) = -y$.

We are left with rp_2 and rp_3. We leave it to the reader to check that

$$s(rp_2) = s_1 s_3 s_1 s_4 (p_3) = s_1 s_3 s_1 s_4 s_1 s_4 (rp_3)\,. \tag{8}$$

Since $\pi(s_1 s_3 s_1 s_4 s_1 s_4) = \pi((s_1 s_4)^3) = (1)$, we have $s_1 s_3 s_1 s_4 s_1 s_4 \in F^+$ and we have shown $s(\phi r p_2) = \phi r p_3$. This concludes the proof of 8.2.

The reader may wonder how we found identity (8). We used an algorithm that, in the situation of 6.1, given a vector $x \in H^2$, finds $w \in W$ such that $wx \in P$. The algorithm simply replaces x by $s_i x$ whenever $(x, f_i) < 0$, as long as it goes. It is known that it terminates, i.e., $x \in P$ after some time. For a proof we refer to [H], prop. 5.7.

9. The algebraic equations

From 8.2 we will deduce algebraic equations for the maps $f: X \to Y$ and $s: Y \to Y$, and a uniformizing differential equation on Y.

We may suppose $X = Y = \mathbb{P}^1$, $x_2 = y_1 = 0$, $x_4 = y_2 = 1$, $x_8 = y_4 = \infty$. This determines f, namely, $f(x) = 3x^2 - 2x^3$. This can be proved as follows. Since $f: \mathbb{P}^1 \to \mathbb{P}^1$ is a degree 3 map that fixes 0 and ∞ and ramifies there with degrees 2 and 3 respectively, we have $f = ax^2 + bx^3$. Now from $f(1) = 1, f'(1) = 0$ one finds the values of a and b.

We have $x_1 = \frac{3}{2}$, $x_3 = -\frac{1}{2}$. Let us write $u = x_5$, $v = x_6$, $w = x_7$. Now s exchanges $\frac{3}{2}$ with ∞, $-\frac{1}{2}$ with u and v with w. After translation over $\frac{3}{2}$, we find that some involution of $\mathbb{P}^1$ exchanges 0 with ∞, -2 with $u - \frac{3}{2}$ and $v - \frac{3}{2}$ with $w - \frac{3}{2}$. This involution has the form $x \mapsto p/x$ and we find

$$-2\left(u - \frac{3}{2}\right) = \left(v - \frac{3}{2}\right)\left(w - \frac{3}{2}\right). \tag{9}$$

It is possible to express the right-hand side of (9) in terms of u only, as follows. The numbers v and w are the solutions in t, different from u, of

$$f(t) = f(u), \quad 3t^2 - 2t^3 = 3u^2 - 2u^3,$$
$$(t - u)(3(t + u) - 2(t^2 + tu + u^2)) = 0, \quad t^2 + (t + u)\left(u - \frac{3}{2}\right) = 0\,.$$

Hence $(t - v)(t - w) = t^2 + (t + u)\left(u - \frac{3}{2}\right)$ for all t. Choosing $t = \frac{3}{2}$ and combining with (9) we find

$$-2\left(u - \frac{3}{2}\right) = \left(\frac{3}{2} - v\right)\left(\frac{3}{2} - w\right) = \left(\frac{3}{2}\right)^2 + \left(u + \frac{3}{2}\right)\left(u - \frac{3}{2}\right) = u^2,$$
$$u^2 + 2u - 3 = 0, \quad (u + 3)(u - 1) = 0\,.$$

Since $u = x_5 + x_4 = 1$, we find $x_5 = u = -3$, $y_3 = f(-3) = 81$, $p = -2\left(u - \frac{3}{2}\right) = 9$, and x_6, x_7 are roots of $2t^2 - 9t + 27 = 0$.

Proposition 8.2 also enables us to calculate a uniformizing differential equation on Y. For signature $(0; 2, 2, 2, e)$, it can generally be written as the Lamé equation

$$P(x)f'' + \frac{1}{2}P'(x)f' + \left(C - \frac{n(n+1)}{4}x\right)f = 0,$$

where $P(x) = x(x-1)(x-a)$ and $n = \frac{1}{e} - \frac{1}{2}$. The singularities are $0, 1, \infty, a$ with ramification degrees $2, 2, 2, e$, respectively. In our case $e = 6$ and $a = 81$. By 8.2, the pull-backs of this equation by the maps f and sf are projectively equivalent. Using a symbolic calculation package, we found that this implies $C = -1/2$.

We summarize in the following theorem.

9.1. Theorem. *We may take* $X = Y = \mathbb{P}^1$, $f(x) = 3x^2 - 2x^3$, $s(x) = \frac{9}{x - 3/2} + \frac{3}{2}$, *and the* x_i *and* y_i *have the following values*:

i	1	2	3	4	5	6, 7	8
x_i	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	-3	$\frac{3}{4}(3 \pm \sqrt{-15})$	∞
$f(x_i)$	0	0	1	1	81	81	∞

The uniformizing differential equation on Y *is*

$$P(x)f'' + \frac{1}{2}P'(x)f' + \frac{x-9}{18}f = 0, \tag{10}$$

where $P(x) = x(x-1)(x-81)$. □

The above differential equation is an entry in a list by D.V. and G.V. Chudnovsky of uniformizing differential equations of arithmetic groups of signature $(1; e)$ – see [C1], p. 193, (1; 3)-case, entry (4) or [C2], p. 23. These authors seem to have found their differential equations by high precision computer calculations, and gave no proofs.

10. The quaternion algebra

Consider the group homomorphism $L := \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g = \pm 1\} \to \mathrm{O}(2,1)$ defined by the symmetric square of the standard representation of $\mathrm{GL}_2(\mathbb{R})$. We have calculated pre-images S_i in L of the s_i. We found

$$S_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad S_2=\begin{pmatrix}-1&0\\0&1\end{pmatrix},$$

$$S_3=\begin{pmatrix}0&\alpha\\-\bar\alpha&0\end{pmatrix},\quad S_4=\begin{pmatrix}-2&\alpha\sqrt{3}\\\bar\alpha\sqrt{3}&2\end{pmatrix},$$

where $\alpha=\frac{1}{2}(1+\sqrt{5})$ and $x\mapsto\bar x$ is the non-trivial automorphism of $\mathbb{Q}(\sqrt{5})$. Let V be the pre-image of W in L: $V=\langle S_1,\dots,S_4,-1\rangle$. Let $\mathbb{Z}[V]$ denote the subalgebra of $M_2(\mathbb{R})$ generated by $\mathbb{Z}$ and V. Since $S_1,\dots,S_4,1$ are linearly independent over $\mathbb{Q}$, $\mathbb{Z}[V]$ is not an order in any quaternion algebra over $\mathbb{Q}$. Let us define

$$R=\left\{\begin{pmatrix}p&-q\sqrt{3}\\\bar q\sqrt{3}&\bar p\end{pmatrix}\,\middle|\,p,q\in\mathbb{Z}[\alpha]\right\}. \tag{11}$$

Now R is an order in a quaternion algebra.

10.1. Proposition. *We have $R^*\subset V$. More precisely, R^* is the kernel of the group homomorphism $\phi: V\to\mathbb{Z}/2\times\mathbb{Z}/2$ defined by*

$$-1\mapsto(0,0),\quad S_1\mapsto(1,0),\quad S_2\mapsto(0,1),\quad S_3\mapsto(1,1),\quad S_4\mapsto(0,1).$$

Proof. Note that S_1 and S_2 normalize R since

$$S_1\begin{pmatrix}a&b\\c&d\end{pmatrix}S_1^{-1}=\begin{pmatrix}d&c\\b&a\end{pmatrix},\quad S_2\begin{pmatrix}a&b\\c&d\end{pmatrix}S_2^{-1}=\begin{pmatrix}a&-b\\-c&d\end{pmatrix}.$$

Thus, $\tilde V:=\langle R^*, S_1, S_2\rangle$ is a normal extension of R^*. We have $\tilde V/R^*\cong\mathbb{Z}/2\times\mathbb{Z}/2$ since $S_1^2=S_2^2=-(S_1S_2)^2=1\in R^*$ and $S_1,S_2,S_1S_2\notin R^*$. Let $\phi:\tilde V\to\mathbb{Z}/2\times\mathbb{Z}/2$ be the homomorphism $R^*\to(0,0)$, $S_1\mapsto(1,0)$, $S_2\mapsto(0,1)$. We shall prove $V=\tilde V$. First, $V\subset\tilde V$ since

$$S_1S_2S_3=\begin{pmatrix}-\bar\alpha&0\\0&-\alpha\end{pmatrix}\in R^*,\quad S_2S_4=\begin{pmatrix}2&-\alpha\sqrt{3}\\\bar\alpha\sqrt{3}&2\end{pmatrix}\in R^*.$$

We note $\phi(S_3)=(1,1)$, $\phi(S_4)=(0,1)$. The other inclusion $V\supset\tilde V$ can be proved in the same way as we did 7.1 (c). This concludes the proof. □

11. A disproof of Dwork's conjecture

In this section, we will show that the arithmetic group of this note is a counterexample to a conjecture of Dwork [D], 11.5. This conjecture states that any globally nilpotent second order differential equation on $\mathbb{P}^1/\bar{\mathbb{Q}}$ has either algebraic solutions, or has a correspondence to a Gauss hypergeometric one. Here, a *correspondence* of two differential equations M_1, M_2 on curves S_1, S_2, respectively, is a correspondence $C\subset S_1\times S_2$ such that the pull-backs of M_1, M_2 to C are algebraically equivalent. For a definition of global nilpotency, we refer to [K], p. 177. Katz proved [K], Th. 13.0, that any singularity of a global nilpotent differential equation is regular with rational local exponents, and [K], Th. 10.0, that differential equations coming from geometry are global nilpotent. Here, the class of differential equations coming from geometry is the smallest class of differential equations on algebraic varieties over $\bar{\mathbb{Q}}$ containing Gauss-Manin connections, which is closed under taking submodules, direct products, tensor products, pull-backs under algebraic maps. (The Dwork-Siegel conjecture – not the one we will disprove – asserts the converse, i.e., globally nilpotent differential equations come from geometry.)

11.1. Theorem. *Let $G \subset \mathrm{PSL}_2(\mathbb{R})$ be a rational arithmetic Fuchsian lattice. Then $S := G\backslash H^2$ is defined over $\bar{\mathbb{Q}}$. Moreover, there exists a uniformizing differential equation on S for G which comes from geometry (and hence is globally nilpotent).*

Proof. Let $R \subset M_2(\mathbb{R})$ be a maximal order in the quaternion algebra associated to G. Let R^{*+} denote the group of invertible elements in R with determinant 1. It is known [Sh] that $R^{*+}\backslash H^2$ is the moduli space of abelian surfaces with multiplication by R. In particular, it is defined over $\mathbb{Q}$. Hence S is defined over $\bar{\mathbb{Q}}$. Let $X \to S$ be a family of abelian surfaces in which the general fibre over Gz is of the shape $\mathbb{C}^2/R\begin{pmatrix} z \\ 1 \end{pmatrix}$. Let ω be a 1-form on X which is holomorphic on the generic fibre. Let f denote the multivalued function on S of the form $f(s) = \int_{\gamma_s} \omega$, where γ_s is a closed non-contractible path in the fibre X_s, varying continuously in s. A straightforward computation shows that f satisfies a second order differential equation. Moreover, this is a uniformizing differential equation for G, and it clearly comes from geometry. □

11.2. Corollary. *In the notation of* 11.1, *suppose that S has genus zero. Then any uniformizing differential equation on S for G comes from geometry.*

Proof. Let M denote any uniformizing differential equation on S (defined over $\bar{\mathbb{Q}}$). Let N denote the uniformizing differential equation of 11.1. Then M and N are projectively equivalent, and have only rational local exponents. By 4.5, they are algebraically equivalent, i.e., $M \cong N \otimes A$, where A has algebraic solutions. Note that A is Fuchsian with only rational exponents, since the same holds for M and N. Hence A comes from geometry. Hence so does M. □

11.3. Lemma. *Let $p: S_1 \to S_2$ be a non-constant map of irreducible curves over $\mathbb{C}$ and let (M_i, f_i) be differential equations on S_i such that $(M_1, f_1) = p^*(M_2, f_2)$. Let $s_1 \in S_1$, $s_2 = p(s_1)$. Suppose that s_i is not a singularity of S_i or M_i and that p does not ramify in s_1. Let G_i be the monodromy groups, i.e., the images of $\pi_1(S_i^0, s_i)$ in $\mathrm{GL}_n(\mathbb{C})$, where S_i^0 is the set of points in S_i where M_i is regular. Then G_1 is a finite index subgroup of G_2.*

Proof. We may suppose that M_2 has no singularities, and that p is a non-ramifying proper map. (First remove in S_2 the singularities of M_2 and the points above which p ramifies or is not proper. Then replace S_1 by the pre-image of what is left of S_2.) The natural map $\pi_1(S_1, s_1) \to \pi_1(S_2, s_2)$ is an injection of finite index. The result follows immediately. □

We are now able to disprove Dwork's conjecture. Let $M = (M, f)$ be the differential equation on $Y \cong \mathbb{P}^1$ defined by (10). By 11.2, M is globally nilpotent.

We will prove that there is no correspondence between M and any hypergeometric connection P. Suppose we have such a correspondence $C \subset Y \times \mathbb{P}^1$, and let $p: C \to Y$ be the projection. By 11.3, the monodromy group H of the pull-back $p^*(M)$ is a finite index subgroup of the monodromy group W^+ of M. Hence H is a Fuchsian lattice. Applying the lemma to the other projection $C \to \mathbb{P}^1$, the monodromy group of P is a Fuchsian lattice, hence a triangle group. But W^+ is not commensurable to any triangle group (7.1 (b)), a contradiction.

The proof is finished by noting that Y has genus zero, and that the non-zero solutions of (M, f) are certainly not algebraic.

12. Suggestions for future research

In the proof of 11.1 we described a Shimura family of abelian surfaces over $R^{*+} \backslash H^2$. Choosing a principal polarization leads to a family of genus two curves. One question is how to calculate an algebraic equation for this family, or equivalently, an 'integral formula' for the solutions of the differential equation (10). A direct consequence of such a formula would be the global nilpotency.

Another question is to find an algorithm computing the uniformizing equation for any given arithmetic Fuchsian lattice.

The following conjecture looks a little like Dwork's one, and can be found in [C3], p. 141.

12.1. Conjecture. *Let (M, f) be a uniformizing differential equation on $S = G \backslash H^2$ for a Fuchsian lattice G. If M is globally nilpotent, then G is either arithmetic or commensurable to a triangle group.*

A stronger (not obviously equivalent) conjecture is obtained by replacing 'M is globally nilpotent' by 'S and (M, f) can be defined over $\bar{\mathbb{Q}}$'. A motivation for our conjecture is as follows. Let $G \subset \mathrm{PSL}_2(\mathbb{R})$ be a non-arithmetic Fuchsian lattice. A special case of a theorem by Margulis [M], Theorem (B), p. 298, states that a non-arithmetic Fuchsian lattice has finite index in its commensurator group. Thus, assume that G equals its commensurator group. The trick of studying some intersecting $G \cap gGg^{-1}$ cannot be applied now. The only case where the uniformizing differential equation can be calculated seems to be where G is a triangle group.

Takeuchi [T2], theorem 2.1, proved that for given g, t, there exist only finitely many arithmetic lattices in $\mathrm{PSL}_2(\mathbb{R})$ of some signature $(g; d_1, \ldots, d_t)$, up to conjugation. Hence 12.1 would imply that (up to automorphisms of $\mathbb{P}^1$ and algebraic equivalence) there are only finitely many counterexamples to Dwork's conjecture with a given number of singularities, and which uniformize some Fuchsian lattice. I do not know whether there exist infinitely many counterexamples.

Acknowledgement. I would like to thank F. Oort and F. Beukers for discussion and encouragement.

References

[Be] *A.F. Beardon*, The Geometry of Discrete Groups, Springer, New York 1983.

[Bl] *A. Blanchard*, Les corps non commutatifs, collection Sup, 1972.

[C1] *D.V. Chudnovsky* and *G.V. Chudnovsky*, Transcendental Methods and Theta-Functions, Proc. Symp. Pure Math. **49** (1989), part 2, 167–232.

[C2] *D.V. Chudnovsky* and *G.V. Chudnovsky*, Computational Problems in Arithmetic of Linear Differential Equations. Some Diophantine Applications, in: Number Theory, ed. D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn, M.B. Nathanson, Springer Lect. Notes Math. **1383** (1989), 12–49.

[C3] *D.V. Chudnovsky* and *G.V. Chudnovsky*, Computer Algebra in the Service of Mathematical Physics and Number Theory, in: Computers in Mathematics, ed. D.V. Chudnovsky, R.D. Jenks, Lect. Notes Pure Appl. Math. **125**, Dekker, New York (1990), 109–232.

[D] *B. Dwork*, Nilpotent second order Fuchsian differential equations, preprint.

[H] *J.E. Humphreys*, Reflection groups and Coxeter groups, Cambridge University Press, Cambridge 1990, 1992.

[K] *N. Katz*, Nilpotent connections and the monodromy theorem: applications of a result of Turrittin, Inst. Hautes Études Sci. Publ. Math. **32** (1970), 232–355.

[M] *G.A. Margulis*, Discrete Subgroups of Semi-simple Liegroups, Springer, Berlin 1991.

[Se] *J.-P. Serre*, A Course in Arithmetic, Springer, New York 1973.

[Sh] *G. Shimura*, On the theory of automorphic functions, Ann. Math. **2** (1959), 101–144.

[T1] *K. Takeuchi*, Commensurability classes of arithmetic triangle groups, J. Fac. Sc. Univ. Tokyo **24** (1977), 201–212.

[T2] *K. Takeuchi*, Arithmetic Fuchsian groups with signature (1; *e*), J. Math. Soc. Japan **35** (1983), 381–407.

[Y] *M. Yoshida*, Fuchsian Differential Equations, Aspects of Math. E11, Vieweg, Braunschweig 1987.

Mathcmatisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel

e-mail: krammer@math.unibas.ch

Eingegangen 30. Mai 1995