ON p-INTEGRALITY OF INSTANTON NUMBERS

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1. Introduction

The motivation for this paper comes from the striking work [7] of Candelas, de la Ossa, Green and Parkes in the study of mirror symmetry of quintic threefolds from 1991. The story has been told many times, so we will give only a very brief description. For more details we like to refer to Duco van Straten's excellent [18] and the many references therein. Our short story starts with the differential operator

$$L = \theta^4 - 5t(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4),$$

where θ denotes $t\frac{d}{dt}$. The unique holomorphic solution to L(y) = 0 is of hypergeometric type and given by

$$F_0 := \sum_{n>0} \frac{(5n)!}{(n!)^5} t^n.$$

The equation L(y) = 0 has a unique basis of solutions of the form

$$y_0 = F_0, \ y_1 = F_0 \log t + F_1, \ y_2 = \frac{1}{2} F_0 \log^2 t + F_1 \log t + F_2,$$

$$y_3 = \frac{1}{6} F_0 \log^3 t + \frac{1}{2} F_1 \log^2 t + F_2 \log t + F_3,$$

where F_0 is given above and $F_1, F_2, F_3 \in t\mathbb{Q}[\![t]\!]$. Straightforward computation shows that the coefficients of F_1 are certainly not integral. The surprise is that $q(t) := t \exp(F_1/F_0)$, expanded as power series in t, does have integer coefficients. The function q(t) is called the *canonical coordinate*. The inverse power series t(q) is called the *mirror map*. Using this inverse series one can rewrite the solutions y_i as functions of q. In particular,

$$y_1/y_0 = \log q$$
, $y_2/y_0 = \frac{1}{2}\log^2 q + V(q)$

for some $V(q) \in q\mathbb{Q}[\![q]\!]$. Let $\theta_q = q\frac{d}{dq}$. Then $K(q) := 1 + \theta_q^2 V$ is called the Yukawa coupling for reasons coming from theoretical physics. In [7] the Yukawa coupling is expanded as a so-called Lambert series in the form

$$K(q) = 1 + \sum_{n>1} \frac{A_n q^n}{1 - q^n}.$$

Candelas et al conjectured that the numbers $a_n = 5A_n/n^3$ are integers which are equal to the (virtual) number of degree n rational curves on a generic quintic threefold in \mathbb{P}^3 . This counting relation is now a part of the mirror symmetry theory from theoretical physics. The numbers a_n are called *instanton numbers*. In this paper we shall concentrate on their integrality.

It turns out that there are many other examples of differential equations, similar to the example above, which display integrality properties of the associated instanton numbers. Motivated by this, Almkvist, van Enckevort, van Straten and Zudilin [1] compiled an extensive list of so-called Calabi-Yau operators. For each entry in the list the authors give experimental evidence that the instanton numbers are p-adically integral for almost all primes p, with some obvious exceptions. The quintic example that we started with belongs to this list. Presumably, each equation in the

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list is a so-called Picard-Fuchs equation corresponding to a one parameter family of Calabi-Yau threefolds.

Around 1995 Giventhal [9] showed that instanton numbers can be related to the geometrical Gromov-Witten invariants of Calabi-Yau threefolds. However, the integrality of these invariants is not a priori clear. Around 1998 Gopakumar and Vafa introduced the so-called BPS-numbers for Calabi-Yau threefolds, which include the instanton numbers as g=0 case, see [6, Def 1.1]. In 2018 Ionel and Parker [10] proved that these BPS-numbers are integers by using methods from symplectic topology. So this opens up a path towards proving integrality of instanton numbers.

Another approach to the integrality of instanton numbers was the use of Dwork's ideas in p-adic cohomology. This was started by Jan Stienstra [17] in 2003 with partial success and later Kontsevich, Schwarz and Vologodsky [13], [19] laid out the ideas for a more complete approach. Unfortunately it is hard to find a satisfactory account of their approach.

One of the main goals of the present paper is to show by elementary means that for the quintic example and a few related examples the a_n are p-adic integers for almost all p. The method showed up as a by-product of our prior work in the paper Dwork crystals III, [5], which is a continuation of the preceding papers Dwork crystals I,II ([3],[4]). By the time this paper was finished we realized that our methods also enable us to prove p-integrality for g = 0 BPS-numbers associated to at least one family of Calabi-Yau fourfolds, see Theorem 5.11.

We begin by sketching an outline of our approach. To study p-integrality we let p be an odd prime. Let $\sigma: \mathbb{Z}_p[\![t]\!] \to \mathbb{Z}_p[\![t]\!]$ be a p-th power Frobenius lift, which is a ring endomorphism such that $h(t)^{\sigma} := \sigma(h(t)) \equiv h(t^p) \pmod{p}$ for all $h \in \mathbb{Z}_p[\![t]\!]$. In this paper we shall take it to be of the form $t^{\sigma} = t^p(1 + ptu(t))$ with $u(t) \in \mathbb{Z}_p[\![t]\!]$. We start with a linear differential equation L(y) = 0 where

$$L = \theta^r + a_{r-1}(t)\theta^{r-1} + \dots + a_1(t)\theta + a_0(t) \in \mathbb{Z}_p[\![t]\!][\theta], \quad \theta = t\frac{d}{dt}.$$

We also assume that it is of MUM-type (maximal unipotent monodromy) at the origin, i.e $a_i(0) = 0$ for i = 0, ..., r - 1. The main examples of such equations are the Picard-Fuchs equations which are associated to certain one parameter families of algebraic varieties. We shall give a more precise definition later on. For such an equation there is a unique basis of local solutions $y_0, y_1, ..., y_{r-1}$ of the form

(1)
$$y_i = F_0 \frac{\log^i t}{i!} + F_1 \frac{\log^{i-1} t}{(i-1)!} + \dots + F_{i-1} \log t + F_i, \ i = 0, \dots, r-1,$$

and $F_i \in \mathbb{Q}_p[\![t]\!]$, $F_0(0) = 1$ and $F_i(0) = 0$ if i > 0. We extend the Frobenius lift σ to the solutions y_i^{σ} in the obvious way, i.e. by taking $F_i^{\sigma} = F_i(t^{\sigma})$ and

$$\log t^{\sigma} = p \log t - \sum_{m \ge 1} \frac{1}{m} (-p t u(t))^{m}.$$

A familiar property of Picard-Fuchs equations is the presence of a Frobenius structure for almost all p. We give a very concrete definition in terms of its solutions.

Definition 1.1 (Frobenius structure). Let $1 \leq s \leq r$. The solutions $y_0, y_1, \ldots, y_{s-1}$ are said to have a (p-th power) Frobenius structure if there exists an operator $A \in \mathbb{Z}_p[\![t]\!][\theta]$ and $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{s-1} \in \mathbb{Z}_p$ such that

$$\mathcal{A}(y_i^{\sigma}) = p^i \sum_{j=0}^i \alpha_j y_{i-j}, \ i = 0, \dots, s-1.$$

Notice there are no restrictions on the order of \mathcal{A} . If we write $\mathcal{A} = \sum_{j\geq 0} A_j(t)\theta^j$, one can easily check that $\alpha_j = A_j(0)$ for $j = 0, 1, \ldots, s-1$.

Recall the canonical coordinate $q(t) = t \exp(F_1/F_0)$ and the inverse series t(q) which we called the mirror map. It is expected that for Picard-Fuchs equations of MUM-type these series are

in $\mathbb{Z}_p[\![t]\!]$ for all but finitely many primes p. So far this has been shown for a large number of special cases, e.g [15], [14], [16], [8]. In this paper we prove the following.

Theorem 1.2 (p-integrality of the mirror map). Suppose that $r \geq 2$ and y_0, y_1 have a Frobenius structure. Then $\exp(F_1/F_0) \in \mathbb{Z}_p[\![t]\!]$.

Let us now suppose that $\exp(F_1/F_0) \in \mathbb{Z}_p[\![t]\!]$ and $r \geq 3$. Take $q = t \exp(F_1/F_0) \in t + t^2\mathbb{Z}_p[\![t]\!]$. We choose the special Frobenius lift given by $q^{\sigma} = q^p$, called the excellent lift in [5]. Define $\theta_q = q \frac{d}{dq}$. Then the operator θ_q^2 annihilates the functions $y_1/y_0 = \log q$ and $y_0/y_0 = 1$.

Definition 1.3. The function $K(q) = \theta_q^2(y_2/y_0)$ is called the Yukawa coupling associated to the operator L. It is equal to $1 + \theta_q^2 V$, where we consider $V = F_2/F_0 - \frac{1}{2}(F_1/F_0)^2$ as function of q.

As in the quintic example we expand

$$K(q) = 1 + \sum_{n>1} \frac{A_n q^n}{1 - q^n}.$$

The Frobenius structure constants α_i are notoriously difficult to compute, but for the proof of the following theorems we only need the requirement $\alpha_1 = 0$. In Proposition 1.7 we give a sufficient criterion for the vanishing of α_1 .

Theorem 1.4. Suppose $p \ge r \ge 3$ and suppose that y_0, y_1, y_2 have a Frobenius structure with $\alpha_1 = 0$. Then $A_n/n^2 \in \mathbb{Z}_p$ for all $n \ge 1$.

The p-integrality of A_n/n^3 follows if the differential equation has order 4 and is self-dual. This is precisely the type of operators that is studied in [1] and which are named Calabi-Yau operators. There we find that an operator $L = \theta^4 + a_3\theta^3 + a_2\theta^2 + a_1\theta + a_0$ is called self-dual if

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + \theta(a_2) - \frac{3}{4}a_3\theta(a_3) - \frac{1}{2}\theta^2(a_3).$$

There are more intrinsic characterizations of self-duality but what matters for us is that in the case of self-duality we have

$$\begin{vmatrix} y_0 & y_3 \\ \theta y_0 & \theta y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{vmatrix}.$$

Theorem 1.5 (p-integrality of instanton numbers). Suppose r=4, $p\geq 5$ and suppose that y_0, y_1, y_2, y_3 have a Frobenius structure. Suppose also that the differential operator L is self-dual and $\alpha_1=0$. Then $A_n/n^3\in\mathbb{Z}_p$ for all $n\geq 1$.

We remark that the theorem is sensitive to the normalization of t. In other words, if we replace t by 2t for example, the theorem is not true anymore. The value α_1 has become non-zero. It is worth mentioning that the instanton numbers and the Frobenius structure behave nicely under the change of variables $t \to t^w$. One can easily check that the new Yukawa coupling is given by $K(q^w)$ and the new constants α_i of the Frobenius structure are given by α^i/w^i , $i=0,1,\ldots$

Theorems 1.2, 1.4 and 1.5 are proven in Section 3. Proposition 1.7 below gives a sufficient criterion for $\alpha_1 = 0$.

Let us now sketch the set up which we use to construct the Frobenius structure for differential operators. We start with a Laurent polynomial $f(\mathbf{x})$ in $\mathbf{x} = (x_1, \dots, x_n)$ with coefficients in $\mathbb{Z}[t]$. Let Δ be the Newton polytope of f and Δ° its interior. Let R be a ring containing $\mathbb{Z}[t]$. By Ω_f we denote the R-module generated by the functions $(k-1)!A/f^k$, where A is a Laurent polynomial with coefficients in R and support in $k\Delta$ and $k \geq 1$. We define Ω_f° similarly, consisting of rational functions as above, but with support of A in $k\Delta^{\circ}$. In the notation of Dwork crystals I,II we use $\Omega_f := \Omega_f(\Delta)$ and $\Omega_f^{\circ} := \Omega_f(\Delta^{\circ})$. By $d\Omega_f$ we define the module spanned by the partial derivatives of the form $x_i \frac{\partial}{\partial x_i}((k-1)!A/f^k)$ with $A/f^k \in \Omega_f$ and $i=1,\ldots,n$. We will denote these operations of partial derivation as $\theta_i = x_i \frac{\partial}{\partial x_i}$.

When f has suitable regularity properties, there exists a polynomial $D_f(t) \in \mathbb{Z}[t]$ such that the quotient $\Omega_f^{\circ}/d\Omega_f$ is a module over $R = \mathbb{Z}[t, 1/D_f]$ of finite rank. It can roughly be identified with the (n-1)-st De Rham cohomology of the zero set of f (although we shall not use this).

The derivation $\theta = t \frac{d}{dt}$ on $\mathbb{Z}[t]$ can be extended to Ω_f in the obvious way. On easily checks that θ sends Ω_f° to itself as well as $d\Omega_f$. Hence θ maps $\Omega_f^{\circ}/d\Omega_f$ to itself. This gives us the so-called Gauss-Manin connection on $\Omega_f^{\circ}/d\Omega_f$.

Our basic example is

$$f = 1 - t \left(x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4} \right).$$

Then $\Omega_f^{\circ}/d\Omega_f$ is a free $\mathbb{Z}[t,1/((5t)^6-5t)]$ -module of rank 4 with a basis given by $\theta^i(1/f)$ with i=0,1,2,3. Clearly $\theta^4(1/f)$ depends on these and the relation is given by $L(1/f) \in d\Omega_f$, where L is the linear differential operator

$$L = \theta^4 - (5t)^5(\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4).$$

This operator is related to the hypergeometric operator with which we began our introduction via the change of variable $t \to t^5$. We call it the *quintic example*.

Our second main example illustrates the use of symmetries of f. Let \mathcal{G} be a finite group of monomial substitutions under which f is invariant. A monomial substitution has the form $x_i \to \mathbf{x}^{\mathbf{a}_i}, i = 1, \ldots, n$ where \mathbf{a}_i are vectors of integers which form a basis of \mathbb{Z}^n . We denote the submodule of invariant rational functions by $(\Omega_f^{\circ})^{\mathcal{G}}$. Consider

$$f = 1 - t \left(x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} + x_4 + \frac{1}{x_4} \right).$$

It turns out that $\Omega_f^{\circ}/d\Omega_f$ has rank 10 as module over $\mathbb{Z}[t, (2t(1-80t^2+1024t^4))^{-1}]$. However, f is invariant under the group \mathcal{G} of monomial transformations generated by $x_1 \to 1/x_1, x_2 \to x_2, x_3 \to x_3, x_4 \to x_4$ and the permutations of x_1, x_2, x_3, x_4 . The quotient module $(\Omega_f^{\circ})^{\mathcal{G}}/d\Omega_f$ turns out to have rank 4 and is generated by $\theta^i(1/f)$ for i = 0, 1, 2, 3. Then $L(1/f) \in d\Omega_f$ with

$$L = (1024t^4 - 80t^2 + 1)\theta^4 + 64(128t^4 - 5t^2)\theta^3 +16(1472t^4 - 33t^2)\theta^2 + 32(896t^4 - 13t^2)\theta + 128(96t^4 - t^2).$$

This is #16 in the database of Calabi-Yau equations [1] with $z=t^2$. We call it the diagonal example.

Prompted by these examples we now restrict the Laurent polynomial $f(\mathbf{x})$ to a polynomial of the form $f = 1 - tg(\mathbf{x})$, where g is a Laurent polynomial in the variables $\mathbf{x} = (x_1, \dots, x_n)$ with coefficients in \mathbb{Z} . We also assume that its Newton polytope $\Delta \subset \mathbb{R}^n$ is reflexive.

We remind the reader that a lattice polytope $\Delta \subset \mathbb{R}^n$ is reflexive if it is of maximal dimension, contains $\mathbf{0}$ and each of its codimension 1 faces can be given by an equation $\sum_{i=1}^n a_i x_i = 1$ with coefficients $a_i \in \mathbb{Z}$. It follows from this definition that $\mathbf{0}$ is the unique lattice point in Δ° . Reflexivity is a requirement needed for the p-adic considerations later on. Unfortunately it gives a strong restriction on the cases that we can deal with.

Let us remark that in Dwork crystals II [4] we prove congruences of Dwork type for the power series $y_0(t) = \sum_{m\geq 0} g_m t^m$, with $g_m = \text{constant term of } g(\mathbf{x})^m$, which are associated to Ω_f for these particular f. The above relation $L(1/f) \in d\Omega_f$ implies that $y_0(t)$ is a power series solution to the differential equation L(y) = 0. This follows from the properties of the period map constructed in [4, §2]. See also the Dwork type families in [5].

Another property of reflexive polytopes which we will use is that to every lattice point $\mathbf{u} \in \mathbb{Z}^n$ there is a unique integer $d \geq 0$ such that \mathbf{u} lies on the boundary of $d\Delta$. We call this integer d the degree of both \mathbf{u} and its respective monomial $\mathbf{x}^{\mathbf{u}}$. For a Laurent polynomial $A \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ its degree $\deg(A)$ is the maximum degree of the monomials it contains. Suppose that R is a localization of $\mathbb{Z}[t]$ as above and t is not invertible in R. A Laurent polynomial $A(x) = \sum_{\mathbf{u}} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ with coefficients in R is called admissible if for every monomial $\mathbf{x}^{\mathbf{u}}$ in A the t-adic order of its coefficient $a_{\mathbf{u}}$ is greater or equal to $\deg(\mathbf{x}^{\mathbf{u}})$. For example, $f(\mathbf{x}) = 1 - tg(\mathbf{x})$ is admissible. Throughout the paper we will work with the submodule

$$\mathcal{O}_f = \left\{ (k-1)! \frac{A(\mathbf{x})}{f(\mathbf{x})^k} \mid k \geq 1, A \text{ is admissible, } Supp(A) \subset k\Delta \right\} \subset \Omega_f.$$

We will call elements of \mathcal{O}_f admissible rational functions. The submodule $\mathcal{O}_f^{\circ} \subset \mathcal{O}_f$ is defined by the stronger condition $Supp(A) \subset k\Delta^{\circ}$. The submodule of derivatives $d\mathcal{O}_f \subset \mathcal{O}_f$ is the R-module generated by $x_i \frac{\partial}{\partial x_i}(\omega)$ for $\omega \in \mathcal{O}_f$ and $i = 1, \ldots, n$.

The main motivation for choosing \mathcal{O}_f is that in our examples the quotient module $\mathcal{O}_f^{\circ}/d\mathcal{O}_f$ is of finite rank over the ring $R = \mathbb{Z}[t, 1/D_f(t)]$, where now $D_f(0) \neq 0$. So when p does not divide $D_f(0)$, this ring can be embedded in $\mathbb{Z}_p[\![t]\!]$, which is crucial for our p-adic considerations.

Letting \mathcal{G} be a group of monomial automorphisms of $f(\mathbf{x})$, we denote the submodule of \mathcal{G} -invariant admissible rational functions by $(\mathcal{O}_f^{\circ})^{\mathcal{G}}$. It is not hard to verify that $\gamma \circ \theta = \theta \circ \gamma$ for all $\gamma \in \mathcal{G}$. In particular this implies that θ maps $(\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ to itself. We now assume that $M := (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ is free of finite rank r and is generated by the elements $\theta^i(1/f)$ for $i = 0, 1, \ldots, r-1$. We actually need a more refined assumptions to be satisfied, in which case we call M a cyclic θ -module of MUM-type. See Section 3 for the exact definition. Just as in the above two examples we can associate to a cyclic θ -module M of MUM-type an r-th order linear differential equation of MUM-type. We call this the Picard-Fuchs equation associated to M. This is precisely the type of differential equations that we want to consider.

To introduce the Frobenius structure on the Picard-Fuchs equations we choose an odd prime p and assume that $\mathbb{Z}[t, 1/D_f(t)]$ can be embedded in $\mathbb{Z}_p[\![t]\!]$ so that we are in the situation from the beginning of this introduction, including a Frobenius action σ on $\mathbb{Z}_p[\![t]\!]$.

We recall the Cartier operator \mathscr{C}_p which was introduced in Dwork crystals I, [3]. Using a vertex **b** of Δ we can expand any element of Ω_f as a formal Laurent series with support in the cone spanned by $\Delta - \mathbf{b}$. On such Laurent series the Cartier operator acts as

$$\mathscr{C}_p: \sum_{\mathbf{k} \in C(\Delta - \mathbf{b})} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \mapsto \sum_{\mathbf{k} \in C(\Delta - \mathbf{b})} a_{p\mathbf{k}} \mathbf{x}^{\mathbf{k}}.$$

In [3, Prop 3.3] we see that \mathscr{C}_p maps $\widehat{\Omega}_f$ to $\widehat{\Omega}_{f^{\sigma}}$, where $\widehat{\Omega}_f$ denotes the p-adic completion of Ω_f and f^{σ} is simply f with σ applied to its coefficients. This map is defined over the p-adic completion of $\mathbb{Z}[t,1/D_f(t)]$, the so-called ring of analytic elements. However, we enlarge this ring to $R = \mathbb{Z}_p[\![t]\!]$. Similarly \mathscr{C}_p maps $\widehat{\Omega}_f^{\circ}$ and $d\widehat{\Omega}_f$ to $\widehat{\Omega}_{f^{\sigma}}^{\circ}$ respectively $d\widehat{\Omega}_{f^{\sigma}}$. This means that

$$\mathscr{C}_p:\widehat{\Omega}_f^{\circ}/d\widehat{\Omega}_f\to\widehat{\Omega}_{f^{\sigma}}^{\circ}/d\widehat{\Omega}_{f^{\sigma}}.$$

When $\Omega_f^{\circ}/d\Omega_f$ has finite rank, then $\widehat{\Omega}_f^{\circ}/d\widehat{\Omega}_f \cong \Omega_f^{\circ}/d\Omega_f$ and this yields an action of \mathscr{C}_p on $\Omega_f^{\circ}/d\Omega_f$.

The action of \mathscr{C}_p on $(\mathcal{O}_f^{\circ})^{\mathcal{G}}$ will be crucial for our construction of the Frobenius structure. Throughout the paper we assume that

$$p \nmid \#\mathcal{G}$$
.

It was shown in [5, §2] that under this condition our Cartier operator descends to the *p*-adic completions $\mathscr{C}_p: (\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}} \to (\widehat{\mathcal{O}}_{f^{\sigma}}^{\circ})^{\mathcal{G}}$. Using the commutation relation $\theta \circ \mathscr{C}_p = \mathscr{C}_p \circ \theta$, we obtain the following

Proposition 1.6. Let $M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ be a cyclic θ -module of MUM-type and rank r. Suppose $p \geq r$. Let $y_0, y_1, \ldots, y_{r-1}$ be the standard basis of solutions of the associated differential equation L(y) = 0. Then $y_0, y_1, \ldots, y_{r-1}$ have a Frobenius structure in the sense of Definition 1.1. A fortiori, each s-tuple y_0, \ldots, y_{s-1} with $s \leq r$ has a Frobenius structure.

We give a proof of this proposition in Section 3 just after the proof of Proposition 3.2.

We now turn to the sufficient criterion for $\alpha_1 = 0$. If F is a simplicial face of Δ spanned by the vertices $\mathbf{b}_1, \ldots, \mathbf{b}_s$, we define the volume of F as the index of the \mathbb{Z} -lattice spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_s$ inside the lattice points in the \mathbb{Q} -vector space generated by $\mathbf{b}_1, \ldots, \mathbf{b}_s$.

Proposition 1.7. Let M be a cyclic θ -module of MUM-type and rank $r \geq 2$. Let p be a prime with $p \geq r$. Suppose the following conditions hold:

- (a1) The set $\Delta_{\mathbb{Z}} = \Delta \cap \mathbb{Z}^n$ consists of **0** and the vertices of Δ .
- (a2) The faces of Δ are simplices of volume ≤ 2 .

- (a3) The \mathcal{G} -invariant Laurent polynomials with support in Δ lie in the module generated by $1, g, \text{ and } x_i \frac{\partial}{\partial x_i}(g) \text{ for } i = 1, \ldots, n.$
- (a4) The non-constant terms of g have coefficient 1.

Then $\alpha_1 = 0$.

The proof of this proposition is given in Section 4. In it we shall rely on some results from Dwork crystals III [5], where we defined for any $k \ge 1$ the submodule

$$\mathscr{F}_k := \{ \omega \in \widehat{\mathcal{O}}_f \mid \mathscr{C}_p^s \omega \equiv 0 (\text{mod } p^{sk} \widehat{\mathcal{O}}_{f^{\sigma^s}}) \text{ for all } s \geq 1 \}.$$

In Dwork crystals III it is shown that the formal Laurent series expansions of elements of \mathscr{F}_k are linear combinations of the k-th partial derivatives of formal Laurent series and vice-versa. In the case k=1 Katz [11, p.258] called them 'forms that die under formal expansion'. In particular, \mathscr{F}_1 contains $d\widehat{\mathcal{O}}_f$. By definition we see that \mathscr{C}_p maps \mathscr{F}_k to $p^k\mathscr{F}_k$. Hence $\widehat{\mathcal{O}}_f/\mathscr{F}_k$ and $\widehat{\mathcal{O}}_f^{\circ}/\mathscr{F}_k$ are $\mathbb{Z}_p[\![t]\!]$ -modules with Cartier action. In Dwork crystals III we show that under the so-called k-th Hasse-Witt condition

$$\widehat{\mathcal{O}}_f^{\circ}/\mathscr{F}_k \cong (\mathcal{O}_f^{\circ})(k).$$

Here $(\mathcal{O}_f^{\circ})(k)$ is the $\mathbb{Z}_p[\![t]\!]$ -span of the functions $t^{\deg(\mathbf{u})}\mathbf{x}^{\mathbf{u}}/f^k$ with $\mathbf{u} \in k\Delta^{\circ}$. The k-th Hasse-Witt condition will be defined in Section 4. To prove Proposition 1.7 we will consider our cyclic θ -module M modulo \mathscr{F}_2 . We mentioned earlier that the Frobenius structure in Proposition 1.6 is constructed using the Cartier matrix on M. Modulo \mathscr{F}_2 we obtain certain congruences for expansion coefficients of rational functions which appear to be sufficient to recover the value of α_1 . Our computations indicate that to access the next constant α_2 one should work modulo \mathscr{F}_3 , and so on. Higher α_i 's seem to be interesting p-adic constants which we plan to consider in a future paper.

Corollary 1.8. In the 'quintic example' and the 'diagonal example' above the instanton numbers are p-integral for every prime p > 5 in the quintic case and $p \ge 5$ in the diagonal case.

Proof. This follows immediately from Theorem 5.6, which is proven in Section 5. \Box

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2. Proof of Theorems 1.2, 1.4, 1.5

We start with a classical lemma.

Lemma 2.1 (Dieudonné-Dwork lemma). Let $g \in t\mathbb{Q}_p[\![t]\!]$. Then $g(t) - \frac{1}{p}g(t^{\sigma}) \in \mathbb{Z}_p[\![t]\!]$ if and only if $\exp(g) \in \mathbb{Z}_p[\![t]\!]$.

Lemma 2.2. Let $G(q) \in 1 + q\mathbb{Q}_p[\![q]\!]$. Consider its Lambert series expansion

$$G(q) = 1 + \sum_{n \ge 1} \frac{h_n q^n}{1 - q^n}.$$

Suppose there exists $s \ge 1$ and $\phi \in \mathbb{Z}_p[\![q]\!]$ such that $G(q^p) - G(q) = \theta_q^s \phi$. Then $h_n/n^s \in \mathbb{Z}_p$ for all $n \ge 1$.

Proof. Write $G(q) = 1 + \sum_{n \geq 1} g_n q^n$. Then $G(q^p) - G(q) = \theta_q^s \phi$ implies that $g_n \equiv g_{n/p} \pmod{n^s \mathbb{Z}_p}$ for all $n \geq 1$ (use the convention $g_{n/p} = 0$ if p does not divide n). One has $g_n = \sum_{d|n} h_d$ and its Möbius inversion reads $h_n = \sum_{d|n} \mu(d) g_{n/d}$, where $\mu(n)$ is the Möbius function. Write

$$h_n = \sum_{d|n} \mu(d)g_{n/d}$$

$$= \sum_{d|(n/p), d \neq 0(p)} \mu(dp)g_{n/pd} + \mu(d)g_{n/d}$$

$$= \sum_{d|(n/p), d \neq 0(p)} \mu(d)(g_{n/d} - g_{n/pd})$$

The latter sum is in $n^s\mathbb{Z}_p$ because of the congruences for g_n .

The following lemma gives a criterion to recognize elements of $\mathbb{Z}_p[\![t]\!]$.

Lemma 2.3. Let $u(t) \in 1 + t\mathbb{Q}_p[\![t]\!]$ and suppose there exists a linear operator $A \in \mathbb{Z}_p[\![t]\!][\theta]$ such that $A(u^{\sigma}) = u$. Then $u \in 1 + t\mathbb{Z}_p[\![t]\!]$.

Proof. Write $\mathcal{A} = \sum_{i \geq 0} A_i(t)\theta^i$. Then, by setting t = 0 in $\mathcal{A}(u^{\sigma}) = u$ we see that $A_0(0) = 1$. Consider the recursion $u_{i+1} = \mathcal{A}(u_i^{\sigma}), i \geq 0$ and $u_0 = 1$. Since $A_0(0) = 1$ we get that $u_i \in 1 + t\mathbb{Z}_p[\![t]\!]$ for all $i \geq 0$.

By induction on i we now prove that $u_i \equiv u \pmod{t^{p^i}}$ for all $i \geq 0$. For i = 0 this is trivial. For $i \geq 0$ we see that $u_i \equiv u \pmod{t^{p^i}}$ implies $u_i^{\sigma} \equiv u^{\sigma} \pmod{t^{p^{i+1}}}$. Application of \mathcal{A} on both sides gives $u_{i+1} \equiv u \pmod{t^{p^{i+1}}}$. We conclude that $u \in 1 + t\mathbb{Z}_p[\![t]\!]$.

Proof of Theorem 1.2. Since the pair y_0, y_1 has a Frobenius structure there exists $\mathcal{A} \in \mathbb{Z}_p[\![t]\!][\theta]$ such that $\mathcal{A}(y_0^{\sigma}) = y_0$. By Lemma 2.3 we have $y_0 \in 1 + t\mathbb{Z}_p[\![t]\!]$. We also have that $\mathcal{A}(y_1^{\sigma}) = p(y_1 + \alpha_1 y_0)$. Let us define $v = y_1/y_0$ and rewrite $\mathcal{A}(y_1^{\sigma})$ as

$$\mathcal{A}(v^{\sigma}y_0^{\sigma}) = v^{\sigma}\mathcal{A}(y_0^{\sigma}) + \mathcal{A}_1(\theta(v^{\sigma}))$$

for some operator $\mathcal{A}_1 \in \mathbb{Z}_p[\![t]\!][\theta]$. Since $\mathcal{A}(y_0^{\sigma}) = y_0$ we get

(2)
$$v^{\sigma}y_0 + \mathcal{A}_1(\theta(v^{\sigma})) = p(v + \alpha_1)y_0.$$

Divide on both sides by y_0 and apply θ . We get, after division by p,

$$\mathcal{A}_2(\frac{1}{p}\theta(v^{\sigma})) = \theta v$$

for some operator $\mathcal{A}_2 \in \mathbb{Z}_p[\![t]\!][\theta]$. Write $\frac{1}{p}\theta(v^{\sigma}) = \frac{1}{p}\frac{\theta(t^{\sigma})}{t^{\sigma}}(\theta v)^{\sigma}$ and note that $z(t) := \frac{1}{p}\frac{\theta(t^{\sigma})}{t^{\sigma}} \in \mathbb{Z}_p[\![t]\!]$. Hence $\mathcal{A}_2(z(t)(\theta v)^{\sigma}) = \theta v$. We conclude that θv has a Frobenius structure. Moreover, $(\theta v)(0) = 1$. Hence $\theta v \in 1 + t\mathbb{Z}_p[\![t]\!]$ by Lemma 2.3. This also implies $\theta(v^{\sigma}) = pz(t)(\theta v)^{\sigma} \in p\mathbb{Z}_p[\![t]\!]$. Using this in (2) and division by py_0 yields

$$\frac{1}{p}v^{\sigma} - v \in \mathbb{Z}_p[\![t]\!],$$

hence

$$\frac{1}{p}\frac{F_1^{\sigma}}{F_0^{\sigma}} - \frac{F_1}{F_0} \in \mathbb{Z}_p[\![t]\!].$$

The Dwork-Dieudonné lemma then implies that $\exp(F_1/F_0) \in \mathbb{Z}_p[\![t]\!]$.

Proof of Theorem 1.4. By Theorem 1.2 we have $q = \exp(y_1/y_0) \in t + t^2 \mathbb{Z}_p[\![t]\!]$. We then have $\mathbb{Z}_p[\![q]\!] = \mathbb{Z}_p[\![t]\!]$. Consider the special Frobenius lift given by $q^{\sigma} = q^p$. Since y_0, y_1, y_2 have a Frobenius structure, the same holds for $1 = y_0/y_0, y_1/y_0, y_2/y_0$. Simply replace the operator \mathcal{A} in Definition 1.1 by $\frac{1}{y_0} \circ \mathcal{A} \circ y_0^{\sigma}$. This change does not affect the value of α_1 , which is still 0. We write these functions in terms of q. We have $y_0/y_0 = 1, y_1/y_0 = \log q, y_2/y_0 = \frac{1}{2}\log^2 q + V_2(q)$. Frobenius structure with $\alpha_1 = 0$ for these functions implies that there is an operator $\mathcal{A} \in \mathbb{Z}_p[\![q]\!][\theta_q]$ such that

(3)
$$\mathcal{A}(1) = 1, \quad \mathcal{A}(\log q) = \log q,$$
$$\mathcal{A}\left(\frac{1}{2}\log^2(q^p) + V_2(q^p)\right) = p^2\left(\frac{1}{2}\log^2 q + V_2(q) + \alpha_2\right).$$

It follows from the first two identities that $\mathcal{A} = 1 + \mathcal{A}_2 \theta_q^2$ with some operator $\mathcal{A}_2 \in \mathbb{Z}_p[\![q]\!][\theta_q]$. The second identity above then turns into

(4)
$$V_2(q^p) + \mathcal{A}_2(p^2 + p^2(\theta_q^2 V_2)(q^p)) = p^2(V_2(q) + \alpha_2).$$

Apply θ_q^2 , divide by p^2 and add 1 on both sides. We obtain

$$(1 + \theta_q^2 \mathcal{A}_2) (1 + (\theta_q^2 V_2)(q^p)) = 1 + (\theta_q^2 V_2)(q).$$

Using Lemma 2.3 we find that $K(q) = 1 + (\theta_q^2 V_2)(q) \in \mathbb{Z}_p[\![q]\!]$. Using this in (4) we obtain

$$\frac{1}{n^2}V_2(q^p) - V_2(q) \in \mathbb{Z}_p[\![q]\!].$$

Denote this function by $\phi(q)$. Apply θ_q^2 to get $K(q^p) - K(q) = \theta_q^2 \phi(q)$. Lemma 2.2 with G(q) = K(q) and s = 2 then implies our theorem.

Proof of Theorem 1.5. We again work over the ring $\mathbb{Z}_p[\![q]\!]$ and Frobenius lift given by $q^{\sigma} = q^p$. Again $y_i/y_0, i = 0, 1, 2, 3$ have a Frobenius structure with $\alpha_1 = 0$. As functions of q we have

$$y_0/y_0 = 1$$
, $y_1/y_0 = \log q$, $y_2/y_0 = \frac{1}{2}\log^2 q + V_2(q)$, $y_3/y_0 = \frac{1}{6}\log^3 q + V_2(q)\log q + V_3(q)$.

The self-duality relation

$$\begin{vmatrix} y_0 & y_3 \\ \theta_q y_0 & \theta_q y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta_q y_1 & \theta_q y_2 \end{vmatrix}$$

evaluates to $2V_2 + \theta_q V_3 = 0$. By the Frobenius structure of the y_i/y_0 with $\alpha_1 = 0$ there is an operator $\mathcal{A} \in \mathbb{Z}_p[\![q]\!][\theta_q]$ such that the identities (3) hold along with the additional identity

$$\mathcal{A}\left(\frac{1}{6}\log^3(q^p) + V_2(q^p)\log(q^p) + V_3(q^p)\right) = p^3\left(\frac{1}{6}\log^3 q + V_2(q)\log q + V_3(q) + \alpha_2\log q + \alpha_3\right).$$

Here $\alpha_2, \alpha_3 \in \mathbb{Z}_p$. Following the arguments from the proof of Theorem 1.4 we can write $\mathcal{A} = 1 + \mathcal{A}_2 \theta_q^2$ with some differential operator \mathcal{A}_2 satisfying (4). Our additional identity now becomes

$$pV_2(q^p)\log q + V_3(q^p) + \mathcal{A}_2\left[p^3\log q + 2p^2(\theta V_2)(q^p) + p^3(\theta^2 V_2)(q^p)\log q + p^2(\theta^2 V_3)(q^p)\right] = p^3\left(V_2(q)\log q + V_3(q) + \alpha_2\log q + \alpha_3\right).$$

Use the relation $2\theta_q V_2 + \theta_q^2 V_3 = 0$, which follows from the duality relation $2V_2 + \theta_q V_3 = 0$. We get after division by p^3 ,

$$\frac{1}{p^3}V_3(q^p) + \frac{1}{p^2}V_2(q^p)\log q + \mathcal{A}_2(K(q^p)\log q) = (V_2(q) + \alpha_2)\log q + V_3(q) + \alpha_3.$$

We know from Theorem 1.4 that $K(q) \in \mathbb{Z}_p[\![q]\!]$. Terms with $\log q$ cancel due to the identity (4) and the remaining terms yield

$$\frac{1}{n^3} V_3(q^p) - V_3(q) \in \mathbb{Z}_p[\![q]\!].$$

We denote this function by $\psi(q)$ and apply θ_q^3 to get $(\theta_q^3 V_3)(q^p) - (\theta_q V_3)(q) = \theta_q^3 \psi$. Hence $K(q^p) - K(q) = -\frac{1}{2}\theta_q^3 \psi$. Lemma 2.2 with G(q) = K(q) and s = 3 then implies our theorem. \square

3. Construction of the Frobenius Structure

Now we turn to the proof of Proposition 1.6. Let us recall our basic setup. We work with a Laurent polynomial $f(\mathbf{x}) = 1 - tg(\mathbf{x})$ where $g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The Newton polytope of $f(\mathbf{x})$ is denoted by Δ , it is assumed to be reflexive. The degree of a monomial $\mathbf{x}^{\mathbf{u}}$, $\mathbf{u} \in \mathbb{Z}^n$ is the integer $d \geq 0$ such that \mathbf{u} lies on the boundary of $d\Delta$. An admissible polynomial is a Laurent polynomial $A = \sum_{\mathbf{u}} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ with coefficients in $\mathbb{Z}_p[\![t]\!]$ such that the t-adic order of each $a_{\mathbf{u}}$ is greater or equal to the degree of \mathbf{u} . We denote by \mathcal{O}_f the $\mathbb{Z}_p[\![t]\!]$ -module generated by rational functions $(k-1)!A(\mathbf{x})/f(\mathbf{x})^k$ with $k \geq 1$ and admissible polynomial A supported in $k\Delta$. The submodule $\mathcal{O}_f^{\circ} \subset \mathcal{O}_f$ consists of admissible rational functions as above with denominators $A(\mathbf{x})$ supported in $(k-1)\Delta$. We denote by $d\mathcal{O}_f \subset \mathcal{O}_f$ the submodule generated by partial derivatives $\theta_f = x_j \partial/\partial x_j$, $1 \leq j \leq n$ of elements of \mathcal{O}_f .

Fix a group $\mathcal{G} \subset GL(n,\mathbb{Z})$ of monomial substitutions which preserve $f(\mathbf{x})$ and assume that $p \nmid \#\mathcal{G}$. We note that \mathcal{O}_f , \mathcal{O}_f° and its submodule of \mathcal{G} -invariant rational functions $(\mathcal{O}_f^{\circ})^{\mathcal{G}}$ are Dwork crystals in the sense of [5, §2]. In the notation of [5] we have $(\mathcal{O}_f^{\circ})^{\mathcal{G}} = \Omega_{\mathcal{L},f}(\Delta^{\circ})$, where $\mathcal{L} \subset \mathbb{Z}_p[\![t]\!][x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ is the submodule of admissible Laurent polynomials which are invariant under \mathcal{G} .

Definition 3.1. We shall say that $M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ is a cyclic θ -module of rank r if

(i) For every $m \geq 0$ and for every \mathcal{G} -invariant admissible Laurent polynomial $A(\mathbf{x})$ supported in $m\Delta$ we have

$$m! \frac{A(\mathbf{x})}{f(\mathbf{x})^{m+1}} \equiv \sum_{j=0}^{\min(m,r-1)} b_j(t)\theta^j(1/f) \pmod{d\mathcal{O}_f}$$

with $b_j(t) \in \mathbb{Z}_p[\![t]\!]$ for all j.

(ii) The functions $\theta^i(1/f)$, i = 0, 1, ..., r-1 are independent modulo $d\mathcal{O}_f$.

Let $L = \theta^r + a_{r-1}\theta^{r-1} + \cdots + a_1\theta + a_0 \in \mathbb{Z}_p[\![t]\!][\theta]$ be the linear operator such that L(1/f) = 0 in M. We shall say that a cyclic θ -module M is of MUM-type (maximal unipotent local monodromy) if $a_i \in t\mathbb{Z}_p[\![t]\!]$ for $i = 0, 1, \ldots, r-1$.

In particular, a cyclic θ -module M is a free $\mathbb{Z}_p[\![t]\!]$ -module with the basis $\theta^i(1/f)$, $i=0,1,\ldots,r-1$. The differential equation L(y)=0 is called the associated *Picard-Fuchs equation*. In Proposition 5.4 we give a sufficient criterion which allows us to check the requirements in a finite number of steps.

We fix a Frobenius lift σ on $\mathbb{Z}_p[\![t]\!]$ and assume it is given by $t^{\sigma} = t^p(1 + p t u(t))$ for some $u \in \mathbb{Z}_p[\![t]\!]$. In what follows we define $\mathcal{O}_{f^{\sigma}}$ to be the $\mathbb{Z}_p[\![t]\!]$ -module generated by the image of \mathcal{O}_f under σ . Similarly we define $\mathcal{O}_{f^{\sigma}}^{\circ}$.

Under the assumption $p \nmid \#\mathcal{G}$ the Cartier operation $\mathscr{C}_p : \widehat{\Omega}_f \to \widehat{\Omega}_{f^{\sigma}}$ restricts to $\mathscr{C}_p : \widehat{\mathcal{O}}_f \to \widehat{\mathcal{O}}_{f^{\sigma}}$ and sends $(\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}}$ to $(\widehat{\mathcal{O}}_{f^{\sigma}}^{\circ})^{\mathcal{G}}$, see [5, Proposition 2.10].

Proposition 3.2. Let $M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ be a cyclic θ -module of rank r. Then there exist $\lambda_i(t) \in \mathbb{Z}_p[\![t]\!]$ such that

(5)
$$\mathscr{C}_p(1/f) \equiv \sum_{i=0}^{r-1} \lambda_i(t) (\theta^i(1/f))^{\sigma} \pmod{d\widehat{\mathcal{O}}_{f^{\sigma}}}.$$

Suppose in addition that $p \ge r$. Then $\lambda_i(t)$ is divisible by p^i for all i.

Proof. Let $M^{\sigma} = (\mathcal{O}_{f^{\sigma}}^{\circ})^{\mathcal{G}}/d\mathcal{O}_{f^{\sigma}}$ be the $\mathbb{Z}_p[\![t]\!]$ -module generated by the images of elements of M under σ . It is a free module with a basis given by $(\theta^i(1/f))^{\sigma}$, $i = 0, \ldots, r-1$. Since M and M^{σ} are free $\mathbb{Z}_p[\![t]\!]$ -modules, it is clear that $(\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}}/d\widehat{\mathcal{O}}_f = M$ and $(\widehat{\mathcal{O}}_{f^{\sigma}}^{\circ})^{\mathcal{G}}/d\widehat{\mathcal{O}}_{f^{\sigma}} = M^{\sigma}$. Therefore we have a well defined Cartier action

$$\mathscr{C}_n: M \to M^{\sigma}$$
.

It follows from this that $\lambda_0, \ldots, \lambda_{r-1} \in \mathbb{Z}_p[\![t]\!]$ as in (5) exist.

To prove the second claim, consider the expansion formula for $\mathcal{C}_p(1/f)$ from the proof of Proposition 3.3 in Dwork crystals I. It has the form

$$\mathscr{C}_p(1/f) = \sum_{k>0} \frac{p^k}{k!} \times k! \frac{Q_k(\mathbf{x})}{(f^{\sigma})^{k+1}},$$

where $Q_k(\mathbf{x}) = \mathscr{C}_p(G(\mathbf{x})^k f(\mathbf{x})^{p-1})$ and

$$G(\mathbf{x}) = \frac{1}{p} (f^{\sigma}(\mathbf{x}^p) - f(\mathbf{x})^p).$$

Notice that $f(\mathbf{x})$ and $G(\mathbf{x})$ are admissible and supported in $\Delta, p\Delta$ respectively. It follows that $G(\mathbf{x})^k f(\mathbf{x})^{p-1}$ is admissible and supported in $(kp+p-1)\Delta$. By application of Lemma 2.5 of Dwork crystals III one finds that $Q_k = \mathscr{C}_p(G(\mathbf{x})^k f(\mathbf{x})^{p-1})$ lies in the $\mathbb{Z}_p[\![t]\!]$ span of the σ -image of admissible Laurent polynomials. Moreover it is supported in $k\Delta$. So we have $k!Q_k(\mathbf{x})/f^{\sigma}(\mathbf{x})^{k+1} \in (\mathcal{O}_{f^{\sigma}}^{\circ})^{\mathcal{G}}$.

Let s be an integer with s < r. Because $p \ge r > s$ we see that $\operatorname{ord}_p(p^k/k!) \ge s$ when $k \ge s$. So we get

$$\mathscr{C}_p(1/f) \equiv \sum_{k=0}^{s-1} \frac{p^k}{k!} \times k! \frac{Q_k(\mathbf{x})}{(f^{\sigma})^{k+1}} \pmod{p^s \widehat{\mathcal{O}}_{f^{\sigma}}}.$$

By Definition 3.1(i) the terms $k!Q_k(\mathbf{x})/(f^{\sigma})^{k+1}$ with k < s are equivalent modulo $d\mathcal{O}_{f^{\sigma}}$ to a $\mathbb{Z}_p[\![t]\!]$ -linear combination of $(\theta^i(1/f))^{\sigma}$, $i = 0, 1, \ldots, s-1$. Hence

$$\mathscr{C}_p(1/f) \equiv \sum_{i=0}^{s-1} \lambda_i(t) (\theta^i(1/f))^{\sigma} (\text{mod } p^s \widehat{\mathcal{O}}_{f^{\sigma}}, d\widehat{\mathcal{O}}_{f^{\sigma}}).$$

In particular this implies that $\lambda_i(t)$ is divisible by p^s for all $i \geq s$. This proves our divisibility statement.

Proof of Proposition 1.6. Let L be the differential operator associated to M. Let L^{σ} be the operator obtained from L by applying σ to its coefficients and replacing θ by $\theta^{\sigma} = \frac{t^{\sigma}}{\theta(t^{\sigma})}\theta$. We then normalize so that the coefficient of θ^r becomes 1. Clearly the operator L^{σ} annihilates y_i^{σ} for $i = 0, \ldots, r-1$ and we have $L^{\sigma}(1/f^{\sigma}) \in d\mathcal{O}_{f^{\sigma}}$.

Consider (5) and replace $(\theta^i(1/f))^{\sigma} = (\theta^{\sigma})^i(1/f^{\sigma})$ by $(\frac{t^{\sigma}}{\theta t^{\sigma}}\theta)^i(1/f^{\sigma})$. Note that $\frac{\theta t^{\sigma}}{t^{\sigma}} = p(1+u(t))$ for some $u(t) \in t\mathbb{Z}_p[\![t]\!]$. Since $\lambda_j(t)$ is divisible by p^j for all j we can rewrite (5) as

$$\mathscr{C}_p(1/f) \equiv \mathcal{A}(1/f^{\sigma}) \pmod{d\widehat{\mathcal{O}}_{f^{\sigma}}},$$

where

$$A = A_0(t) + A_1(t)\theta + \dots + A_{r-1}(t)\theta^{r-1} \in \mathbb{Z}_p[\![t]\!][\theta].$$

Apply L from the left and observe that

$$L(\mathscr{C}_p(1/f)) = \mathscr{C}_p(L(1/f)) \subset \mathscr{C}_p(d\widehat{\mathcal{O}}_f) \subset d\widehat{\mathcal{O}}_{f^{\sigma}}.$$

Hence $L \circ \mathcal{A}(1/f^{\sigma}) \in d\widehat{\mathcal{O}}_{f^{\sigma}}$. Using the right Euclidean algorithm for differential operators we can find \mathcal{B} and \mathcal{N} of order less than r such that $L \circ \mathcal{A} = \mathcal{B} \circ L^{\sigma} + \mathcal{N}$. Since $L^{\sigma}(1/f^{\sigma}) \in d\mathcal{O}_{f^{\sigma}}$ and \mathcal{B} sends $d\mathcal{O}_{f^{\sigma}}$ to itself, it follows that $\mathcal{N}(1/f^{\sigma}) \in d\widehat{\mathcal{O}}_{f^{\sigma}}$. Using the independence assumption (ii) in Definition 3.1 we conclude that $\mathcal{N} = 0$. Hence L^{σ} is a right divisor of $L \circ \mathcal{A}$. In particular this implies that $\mathcal{A}(y_i^{\sigma})$ lies in the kernel of L for $i = 0, 1, \ldots, r - 1$. Choose i. Then $\mathcal{A}(y_i^{\sigma})$ is a solution of L(y) = 0. So there exist constants b_0, \ldots, b_{r-1} such that

(6)
$$\mathcal{A}(y_i^{\sigma}) = b_0 y_0 + b_1 y_1 + \dots + b_{r-1} y_{r-1}.$$

This is an equality in the ring $\mathbb{Q}_p[\![t]\!][\log t]$. When we say that we take the constant term of an element of $\mathbb{Q}_p[\![t]\!][\log t]$ we simply mean that we drop all powers of t and keep de powers of $\log t$. In particular the constant term of y_i is $\frac{1}{i!}\log^i t$ and the constant term of y_i^{σ} is $\frac{p^i}{i!}\log^i t$. It is not hard to see that the operation of taking the constant terms commutes with θ . Hence if we take constant terms on both sides of (6) we get

$$\sum_{j=0}^{r-1} A_j(0)\theta^j \left(p^i \frac{\log^i t}{i!} \right) = \sum_{l=0}^{r-1} b_l \frac{\log^l t}{l!}.$$

Elaboration of the left hand side gives us

$$p^{i} \sum_{j=0}^{i} A_{j}(0) \frac{\log^{i-j} t}{(i-j)!} = \sum_{l=0}^{r-1} b_{l} \frac{\log^{l} t}{l!}.$$

We conclude that $b_l = p^i A_{i-l}(0)$ if $l \le i$ and $b_l = 0$ if l > i. We put $\alpha_i = A_i(0)$.

Finally we must show that $\alpha_0 = A_0(0) = 1$. For this we will use the *period map* p_0 which was defined in §2 of Dwork crystals II. To define the period p_0 of a rational function $h(\mathbf{x})/f(\mathbf{x})^k$ we write it as

$$h(\mathbf{x}) \left(\sum_{n \ge 0} t^n g(\mathbf{x})^n \right)^k$$

and take the coefficient at $\mathbf{x}^{\mathbf{0}}$. This procedure yields a map $p_{\mathbf{0}}: \mathcal{O}_f \to \mathbb{Z}_p[\![t]\!]$ which vanishes on $d\mathcal{O}_f$ and commutes with θ . In the introduction we mentioned that $p_{\mathbf{0}}(1/f) = F_0(t)$. This period map clearly extends to $\widehat{\mathcal{O}}_f$ and it was shown in Dwork crystals II that $p_{\mathbf{0}}$ is \mathscr{C}_p -invariant. Applying the period map $p_{\mathbf{0}}$ to the identity $\mathscr{C}_p(1/f) \equiv \mathcal{A}(1/f^{\sigma}) \pmod{d\widehat{\mathcal{O}}_{f^{\sigma}}}$ we get $F_0(t) = \mathcal{A}(F_0^{\sigma})$. Setting t = 0 on both sides gives us $1 = A_0(0)$, as desired.

We remark that the Frobenius structure constructed in Proposition 1.6 has $\alpha_i = p^{-i}\lambda_i(0)$, where $\lambda_i(t) \in p^i \mathbb{Z}[t]$ are the Cartier entries in (5).

4. Proof of Proposition 1.7

We continue working in the setting of the previous section. Let us start by recalling the main result from Dwork crystals III [5], which will be used in our proofs. For a subset $S \subset \Delta$ consider the submodule

$$\mathcal{O}_f(S) = \left\{ (m-1)! \frac{A(\mathbf{x})}{f(\mathbf{x})^m} \mid m \ge 1, A \text{ is admissible, } Supp(A) \subset mS \right\} \subset \mathcal{O}_f.$$

With this notation we have $\mathcal{O}_f^{\circ} = \mathcal{O}_f(\Delta^{\circ})$. We denote by $\widehat{\mathcal{O}}_f(S)$ the p-adic completion of this module. Fix a number $1 \leq k < p$ and denote by $HW^{(k)}(S)$ the matrix labelled by the integer points in kS such that the matrix element with indices $\mathbf{u}, \mathbf{v} \in (kS)_{\mathbb{Z}}$ is equal to $t^{\deg(\mathbf{u})-p\deg(\mathbf{v})}$ times the coefficient of $\mathbf{x}^{p\mathbf{v}-\mathbf{u}}$ in the polynomial

(7)
$$\mathcal{F}^{(k)}(\mathbf{x}) := f(\mathbf{x})^{p-k} \sum_{r=0}^{k-1} (f^{\sigma}(\mathbf{x}^p) - f(\mathbf{x})^p)^r f^{\sigma}(\mathbf{x}^p)^{k-1-r}.$$

Since $\mathcal{F}^{(k)}(\mathbf{x})$ is admissible and $\deg(p\mathbf{v}-\mathbf{u}) \geq p \deg(\mathbf{v}) - \deg(\mathbf{u})$, the entries of $HW^{(k)}(S)$ are in $\mathbb{Z}_p[\![t]\!]$. A set $S \subseteq \Delta$ is called *open* if $\Delta \setminus S$ is a union of faces of any dimension. The respective module $\mathcal{O}_f(S)$ was denoted by $\Omega_f^{adm}(S)$ in [5, §2]. The matrix $HW^{(k)}(S)$ is the kth Hasse-Witt matrix corresponding to the Dwork crystal $\widehat{\mathcal{O}}_f(S)$, it was denoted by $HW^{(k)}_{adm}(S)$ in [5, §5]. In [5, Proposition 5.8] we showed that for any open S the p-adic order of the determinant of $HW^{(k)}(S)$ is at least

$$L(k,S) := (k-1)\#(kS)_{\mathbb{Z}} - \sum_{\ell=1}^{k-1} \#(\ell S)_{\mathbb{Z}}.$$

The series

$$hw^{(k)}(S) := p^{-L(k,S)} \det HW^{(k)}(S) \in \mathbb{Z}_p[\![t]\!]$$

is called the kth Hasse-Witt determinant. We say that the kth Hasse-Witt condition holds for S if the first k Hasse-Witt determinants are invertible:

$$hw^{(1)}(S), \dots, hw^{(k)}(S) \in \mathbb{Z}_p[\![t]\!]^{\times}.$$

By the main result of Dwork crystals III, see [5, Theorem 4.3 and Remark 5.11], if the kth Hasse–Witt condition holds for S and k < p then we have a direct sum decomposition of $\mathbb{Z}_p[\![t]\!]$ -modules

(8)
$$\widehat{\mathcal{O}}_f(S) = \mathcal{O}_f(S)(k) \oplus \mathscr{F}_k.$$

Here $\mathcal{O}_f(S)(k)$ is the free $\mathbb{Z}_p[\![t]\!]$ -module consisting of the functions $A(\mathbf{x})/f(\mathbf{x})^k$ with admissible $A(\mathbf{x})$ supported in kS and \mathscr{F}_k is to be read as $\mathscr{F}_k \cap \widehat{\mathcal{O}}_f(S)$. We remind the reader that this latter module was defined in the introduction as

$$\mathscr{F}_k = \{ \omega \in \widehat{\mathcal{O}}_f \mid \mathscr{C}_p^s \omega \in p^{ks} \widehat{\mathcal{O}}_{f^{\sigma^s}} \text{ for all } s \ge 1 \}.$$

For our proof of Proposition 1.7 we will use decompositions (8) for $S = \Delta^{\circ}$ and Δ with k = 1 and 2. The reader may immediately observe that

(9)
$$hw^{(1)}(\Delta^{\circ}) = \text{ coefficient of } \mathbf{x}^{\mathbf{0}} \text{ in } f(\mathbf{x})^{p-1} \in 1 + O(t) \subset \mathbb{Z}_p[\![t]\!]^{\times}.$$

We shall now verify the remaining Hasse-Witt conditions

(10)
$$hw^{(1)}(\Delta), hw^{(2)}(\Delta^{\circ}), hw^{(2)}(\Delta) \in \mathbb{Z}_p[\![t]\!]^{\times}.$$

The following two lemmas will serve as a preparation.

Lemma 4.1. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_M\}$ be a set of points in Δ and let \mathbf{b} be a point on a proper face $F \subseteq \Delta$ of arbitrary dimension. Suppose there exist $R, r_1, \ldots, r_M \geq 0$ such that $r_1 + \cdots + r_M = R$ and $r_1 \mathbf{b}_1 + \cdots + r_M \mathbf{b}_M = R \mathbf{b}$. Then $r_i > 0$ implies that $\mathbf{b}_i \in F$.

Proof. Let ℓ be a linear form such that Δ lies in the half space $\ell(\mathbf{y}) \geq 1$ and F is the intersection of Δ and the plane $\ell(\mathbf{y}) = 1$. From $\sum_{i=1}^{M} r_i \mathbf{b}_i = R \mathbf{b}$ it follows that $\sum_{i=1}^{M} r_i \ell(\mathbf{b}_i) = R$. Since $R = \sum_{i=1}^{M} r_i$ we get $\sum_{i=1}^{M} r_i (\ell(\mathbf{b}_i) - 1) = 0$. Since $r_i \geq 0$ for all i this implies that $\ell(\mathbf{b}_i) = 1$ whenever $r_i > 0$. Since F is the intersection of Δ and $\ell(\mathbf{y}) \geq 1$, we conclude that $\mathbf{b}_i \in F$ whenever $r_i > 0$.

Lemma 4.2. For any $k \ge 1$ we have

(11)
$$\det(HW^{(k)}(\Delta)) = \prod_{F \subset \Delta} \det(HW^{(k)}(F^{\circ})),$$

where the product is over all faces of Δ . By F° we denote the interior of a face F, except when F is a vertex. Then we take $F^{\circ} = F$. If F has positive dimension and F has no lattice points in its interior we take $\det(HW^{(k)}(F^{\circ})) = 1$.

Moreover, for every proper face F the matrix $HW^{(k)}(F^{\circ})$ will not change if in (7) we substitute $f(\mathbf{x})$ with the restriction $f|_F(\mathbf{x})$. Here $f|_F$ denotes the sum of all terms of f whose support is in F.

Let us mention one consequence of the second statement. Note that F° is an *open* subset of F in our rudimentary topology, even though it is not an *open* subset of F. Since F is the Newton polytope of $f|_F$, it then follows from [5, Proposition 5.8] that the F-adic order of $\det(HW^{(k)}(F^{\circ}))$ is at least $L(k,F^{\circ})$. Therefore we have $hw^{(k)}(F^{\circ}) \in \mathbb{Z}_p[\![t]\!]$. Dividing (11) by the respective power of F we obtain the following useful formula:

(12)
$$hw^{(k)}(\Delta) = \prod_{F \subset \Delta} hw^{(k)}(F^{\circ}).$$

Proof of Lemma 4.2. Let us fix some order of faces $F \subset \Delta$ in which their dimension is non-decreasing. We now list integral points in $(k\Delta)_{\mathbb{Z}}$ by taking the faces F in the above order and listing the points in $(kF^{\circ})_{\mathbb{Z}}$. We will show that $HW^{(k)}(\Delta)$ is a block upper-triangular matrix where the blocks are given by the points in $(kF^{\circ})_{\mathbb{Z}}$. Our first claim (11) immediately follows from this

Let $\mathbf{v} \in kF^{\circ}$, where F is a proper face of Δ . Take any $\mathbf{u} \in k\Delta$. The entry of $HW^{(k)}(\Delta)$ at position \mathbf{u}, \mathbf{v} is given by the coefficient of $\mathbf{x}^{p\mathbf{v}}$ in $\mathbf{x}^{\mathbf{u}}\mathcal{F}^{(k)}(\mathbf{x})$. In (7) we can rewrite

$$\mathcal{F}^{(k)} = \sum_{0 \le j \le r \le k-1} (-1)^j \binom{r}{j} f(\mathbf{x})^{p(j+1)-k} f^{\sigma}(\mathbf{x}^p)^{k-(j+1)}.$$

Hence the monomials in $\mathbf{x}^{\mathbf{u}} \mathcal{F}^{(k)}(\mathbf{x})$ are products of \mathbf{u} and precisely k(p-1) monomials $\mathbf{x}^{\mathbf{w}}$ with $\mathbf{w} \in \Delta_{\mathbb{Z}}$. For non-zero entries we thus have

$$p\mathbf{v} = \sum_{\mathbf{w} \in \Delta_{\mathbb{Z}}} \beta_{\mathbf{w}} \mathbf{w} + \mathbf{u}$$

with some non-negative integers $\beta_{\mathbf{w}}$ satisfying $\sum_{\mathbf{w}} \beta_{\mathbf{w}} = k(p-1)$. We divide by k to get

$$p(\mathbf{v}/k) = \sum_{\mathbf{w} \in \Delta_{\mathbb{Z}}} (\beta_{\mathbf{w}}/k)\mathbf{w} + \mathbf{u}/k$$

and apply Lemma 4.1 with $\mathbf{b} = \mathbf{v}/k \in F^{\circ}$ and the set $\{\mathbf{b}_i\} = \Delta_{\mathbb{Z}} \cup \{\mathbf{u}/k\}$. It follows that $\mathbf{u}/k \in F$. So \mathbf{u} is either in kF° or in kG° , where G is a proper subface of F. This gives us the

upper-triangular block structure for the \mathbf{v} on the boundary of $k\Delta$. The \mathbf{v} in the interior of $k\Delta$ form the last block.

For the second claim we observe that in the above argument we also have $\beta_{\mathbf{w}} = 0$ when $\mathbf{w} \notin F$ as a consequence of Lemma 4.1.

Now we are in a position to verify the Hasse–Witt conditions (10) efficiently.

Lemma 4.3. Suppose

- (a1) $\Delta_{\mathbb{Z}}$ consists of **0** and the vertices of Δ .
- (a4') The vertex coefficients of $g(\mathbf{x}) = \sum_{\mathbf{w}}^{\bullet} g_{\mathbf{w}} \mathbf{x}^{\mathbf{w}}$ are in \mathbb{Z}_{p}^{\times} .

Then $hw^{(1)}(\Delta)$ and $hw^{(2)}(\Delta^{\circ})$ are in $\mathbb{Z}_p[\![t]\!]^{\times}$.

Proof. For the first claim we use (12) with k = 1. In (9)we checked that $hw^{(1)}(\Delta^{\circ}) \in \mathbb{Z}_p[\![t]\!]^{\times}$. By (a1) the sets $(F^{\circ})_{\mathbb{Z}}$ are empty for all proper faces of positive dimension. It remains to consider $F = \{\mathbf{v}\}$ for a vertex $\mathbf{v} \in \Delta$. Note that the restriction of f to the vertex is $f|_F(\mathbf{x}) = -tg_{\mathbf{v}}\mathbf{x}^{\mathbf{v}}$. Therefore $hw^{(1)}(\mathbf{v}) = (-g_{\mathbf{v}})^{p-1} \in \mathbb{Z}_p^{\times}$ due to (a4').

For the second claim we first note that $(2\Delta^{\circ})_{\mathbb{Z}} = \Delta_{\mathbb{Z}}$ because Δ is a reflexive polytope. To slightly simplify our notation we denote $H := HW^{(2)}(\Delta^{\circ})$. We consider H(mod O(t)). Let \mathbf{v} be a vertex of Δ . When \mathbf{u} is another vertex, then $H_{\mathbf{u},\mathbf{v}}$ is t^{1-p} times the the coefficient at $\mathbf{x}^{p\mathbf{v}-\mathbf{u}}$ in

$$\mathcal{F}^{(2)}(\mathbf{x}) = 2f(\mathbf{x})^{p-2} f^{\sigma}(\mathbf{x}^p) - f(\mathbf{x})^{p-2}.$$

We have $f^{\sigma}(\mathbf{x}^p) \equiv 1 \pmod{t^p}$ and hence

$$\mathcal{F}^{(2)}(\mathbf{x}) \equiv 2f(\mathbf{x})^{p-2} - f(\mathbf{x})^{2p-2} \pmod{t^p}$$

$$= \sum_{k=0}^{p-2} \left(2\binom{p-1}{k} - \binom{2p-2}{k} \right) (-tg(\mathbf{x}))^k - \binom{2p-2}{p-1} (-tg(\mathbf{x}))^{p-1} \pmod{t^p}.$$

Then we see that the contributions to $H_{\mathbf{u},\mathbf{v}}(\text{mod }t)$ can only come from $g(\mathbf{x})^{p-1}$, in which case we should also have $\mathbf{u} = \mathbf{v}$ as a result of Lemma 4.1. This same lemma also tells us that there is precisely one contribution, namely $-\binom{2p-2}{p-1}(-tg_{\mathbf{v}}\mathbf{x}^{\mathbf{v}})^{p-1}$. Thus we have $H_{\mathbf{v},\mathbf{v}}(0) = -\binom{2p-2}{p-1}(-g_{\mathbf{v}})^{p-1} \in p\mathbb{Z}_p^{\times}$ and in the column of H(mod t) indexed by \mathbf{v} all non-diagonal entries are zero except for possibly $H_{\mathbf{0},\mathbf{v}}$.

To see what happens when $\mathbf{v} = \mathbf{0}$ note that $\mathcal{F}^{(2)}(\mathbf{x}) \equiv 1 \pmod{t}$. Hence we see that all $H_{\mathbf{u},\mathbf{0}}$ are divisible by t, except $H_{\mathbf{0},\mathbf{0}} \equiv 1 \pmod{t}$.

As a result we get that $\det(H(0)) = \prod_{\mathbf{w} \in \Delta_{\mathbb{Z}}} H_{\mathbf{w},\mathbf{w}}(0) \in p^{\#\Delta_{\mathbb{Z}}-1}\mathbb{Z}_p^{\times}$. Since $L(2,\Delta^{\circ}) = \#\Delta_{\mathbb{Z}} - 1$, one immediately concludes that $hw^{(2)}(\Delta^{\circ}) \in \mathbb{Z}_p[\![t]\!]^{\times}$.

In the following lemma, if F is a simplicial face of Δ spanned by the vertices $\mathbf{b}_1, \ldots, \mathbf{b}_s$, we define the volume of F as the index of the \mathbb{Z} -lattice spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_s$ inside the lattice points in the \mathbb{Q} -vector space generated by $\mathbf{b}_1, \ldots, \mathbf{b}_s$. Notation: Vol(F).

Lemma 4.4. Suppose

- (a1) $\Delta_{\mathbb{Z}}$ consists of **0** and the vertices of Δ .
- (a2) The faces of Δ are simplices of volume ≤ 2 .
- (a4') The non-constant coefficients of g are in \mathbb{Z}_p^{\times} .

Then $hw^{(2)}(\Delta) \in \mathbb{Z}_p[\![t]\!]^{\times}$.

Proof. We shall use the factorization in Lemma 4.2 and its consequence (12). By Lemma 4.3 we have $hw^{(2)}(\Delta^{\circ}) \in \mathbb{Z}_p[\![t]\!]^{\times}$. It remains to determine $hw^{(2)}(F^{\circ})$ for all proper faces F of Δ . First we determine the possible lattice points in $2F^{\circ}$ for any proper face F of Δ . Suppose F is spanned by the vertices $\mathbf{b}_1, \ldots, \mathbf{b}_s$. Let $\mathbf{q} \in 2F^{\circ}$. We distinguish three cases according to the value of $\operatorname{Vol}(F)$.

Vol(F) = 1. Then there exist positive integers λ_i such that $\mathbf{q} = \sum_{i=1}^s \lambda_i \mathbf{b}_i$ and $\sum_{i=1}^s \lambda_i = 2$. Hence either s = 1 and $\lambda_1 = 2$, in which case \mathbf{q} is twice a vertex, or s = 2 and $\lambda_1 = \lambda_2 = 1$ in which case \mathbf{q} is a sum of two distinct vertices. Vol(F) = 2. Then there exist positive integers λ_i such that $\mathbf{q} = \frac{1}{2} \sum_{i=1}^{s} \lambda_i \mathbf{b}_i$ and $\sum_{i=1}^{s} \lambda_i = 4$. Suppose that $\lambda_i \geq 2$ for some i. Then $\mathbf{q} - \mathbf{b}_i$ lies on the face F, hence it equals \mathbf{b}_j for some j. So we get $\mathbf{q} = \mathbf{b}_i + \mathbf{b}_j$ and s = 2. In the remaining case we have $\lambda_i = 1$ for all i. Hence s = 4 and we see that $\mathbf{q} = (\mathbf{b}_1 + \dots + \mathbf{b}_4)/2$. It is the unique lattice point in $2F^{\circ}$.

We now use the second claim of Lemma 4.2 to compute $hw^{(2)}(F^{\circ})$ in each of the above cases. First we take for $F = F^{\circ}$ a vertex **b** of Δ and $\mathbf{v} = 2\mathbf{b}$. Then $f|_F = -tg_{\mathbf{b}}\mathbf{x}^{\mathbf{b}}$ and $HW^{(2)}(F^{\circ})$ is a 1×1 -matrix with entry t^{2-2p} times the coefficient of $\mathbf{x}^{2(p-1)\mathbf{b}}$ in

$$2f|_{F}(\mathbf{x})^{p-2}f|_{F}^{\sigma}(\mathbf{x}^{p}) - f|_{F}(\mathbf{x})^{2p-2} = (2g_{\mathbf{b}}^{p-1}t^{p-2}t^{\sigma} - g_{\mathbf{b}}^{p-1}t^{2p-2})\mathbf{x}^{2(p-1)\mathbf{b}}.$$

It follows that $hw^{(2)}(F^{\circ}) = HW^{(2)}(F^{\circ}) \in 1 + O(t)$.

Now let \mathbf{v} be the sum of two vertices $\mathbf{b}_1, \mathbf{b}_2$, say. It is the unique lattice point in the interior of the face F spanned by $\mathbf{b}_1, \mathbf{b}_2$. Then $f|_F = -t(g_{\mathbf{b}_1}\mathbf{x}^{\mathbf{b}_1} + g_{\mathbf{b}_2}\mathbf{x}^{\mathbf{b}_2})$ and $HW^{(2)}(F^{\circ})$ is again a 1×1 -matrix with entry t^{2-2p} times the coefficient of $\mathbf{x}^{(p-1)\mathbf{v}}$ in

$$2f|_F(\mathbf{x})^{p-2}f|_F^{\sigma}(\mathbf{x}^p) - f|_F(\mathbf{x})^{2p-2}.$$

Clearly the contribution to this coefficient can only come from the second term. Therefore we have

$$hw^{(2)}(F^{\circ}) = \frac{1}{p}HW^{(2)}(F^{\circ}) = -\frac{1}{p}\binom{2p-2}{p-1}(g_{\mathbf{b}_1}g_{\mathbf{b}_2})^{p-1} \in \mathbb{Z}_p^{\times}.$$

When $\mathbf{v} = \frac{1}{2} \sum_{i=1}^{4} \mathbf{b}_i$ we can use a similar argument. Letting F be the face spanned by $\mathbf{b}_1, \dots, \mathbf{b}_4$ the matrix $HW^{(2)}(F^{\circ})$ is again 1×1 with entry the coefficient of $\mathbf{x}^{(p-1)\mathbf{v}}$ in $-\left(\sum_{i=1}^{4} g_{\mathbf{b}_i} \mathbf{x}^{\mathbf{b}_i}\right)^{2p-2}$. Therefore

$$hw^{(2)}(F^{\circ}) = \frac{1}{p}HW^{(2)}(F^{\circ}) = -\frac{1}{p}\frac{(2p-2)!}{(\frac{p-1}{2}!)^4}(g_{\mathbf{b}_1}\cdots g_{\mathbf{b}_4})^{\frac{p-1}{2}} \in \mathbb{Z}_p^{\times}.$$

This finishes our consideration of proper faces which contribute to $hw^{(2)}(\Delta)$. The claim is now proved.

From the last two Lemmas we conclude that under assumptions (a1),(a2) and (a4') the 2nd Hasse–Witt condition holds for $S = \Delta$. We now derive an important consequence.

Lemma 4.5. Let p be an odd prime. Let $M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ be a cyclic θ -module of rank ≥ 2 . Suppose that the 2nd Hasse-Witt condition holds for Δ . Then $(\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}}/(d\mathcal{O}_f + \mathscr{F}_2)$ is a free $\mathbb{Z}_p[\![t]\!]$ -module with generators 1/f, $\theta(1/f)$. If in addition

(a3) all \mathcal{G} -invariant Laurent polynomials supported in Δ lie in the module generated by $1, g(\mathbf{x})$ and $(\theta_i g)(\mathbf{x}), i = 1, \ldots, n$,

then every element $d\omega \in (d\mathcal{O}_f + \mathscr{F}_2) \cap (\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}}$ is equivalent modulo \mathscr{F}_2 to a $\mathbb{Z}_p[\![t]\!]$ -linear combination of partial derivatives $\theta_i(1/f)$, $i=1,\ldots,n$.

Proof. Note that decomposition formula (12) implies that if the kth Hasse–Witt condition holds for Δ then it also holds for Δ° . In our case k=2 and [5, Theorem 4.3 and Remark 5.11] implies that $\widehat{\mathcal{O}}_{f}^{\circ}/\mathscr{F}_{2} \cong \mathcal{O}_{f}^{\circ}(2)$. Taking \mathcal{G} -invariants we conclude that $(\widehat{\mathcal{O}}_{f}^{\circ})^{\mathcal{G}}/\mathscr{F}_{2}$ is a free $\mathbb{Z}_{p}[\![t]\!]$ -module consisting of rational functions $A/f^{2} \in (\mathcal{O}_{f}^{\circ})^{\mathcal{G}}$ with \mathcal{G} -invariant admissible polynomials A supported in Δ . Since each of these elements is equivalent modulo $d\mathcal{O}_{f}$ to a $\mathbb{Z}_{p}[\![t]\!]$ -linear combination of 1/f, $\theta(1/f)$ (Definition 3.1(i) of cyclic θ -modules) we find that $(\widehat{\mathcal{O}}_{f}^{\circ})^{\mathcal{G}}/(d\mathcal{O}_{f}+\mathscr{F}_{2})$ is generated by 1/f, $\theta(1/f)$. It remains to show that these generators are independent modulo $d\mathcal{O}_{f} + \mathscr{F}_{2}$.

Suppose that there exist $\lambda, \mu \in \mathbb{Z}_p[\![t]\!]$ and $d\omega \in d\mathcal{O}_f$ and $\nu_2 \in \mathscr{F}_2$ such that $\lambda(1/f) + \mu\theta(1/f) = d\omega + \nu_2$. Since the 1st Hasse–Witt condition holds for Δ , we know that any element in \mathcal{O}_f is equivalent modulo \mathscr{F}_1 to an element in $\mathcal{O}_f(1)$. Any derivative $x_i \frac{\partial}{\partial x_i}$ maps \mathscr{F}_1 to \mathscr{F}_2 . Hence any element $d\omega \in d\mathcal{O}_f$ is equivalent modulo \mathscr{F}_2 to an element $d\omega'$ in $d\mathcal{O}_f(1)$. Here $d\mathcal{O}_f(1)$ is generated by the derivatives of elements A/f with support of A in Δ . Since the 2nd Hasse–Witt condition holds for Δ , we have $\mathscr{F}_2 \cap \mathcal{O}_f(2) = \{0\}$. We conclude that $\lambda(1/f) + \mu\theta(1/f) = d\omega'$.

However, 1/f, $\theta(1/f)$ are known to be independent modulo $d\mathcal{O}_f$. Hence $\lambda = \mu = 0$, which proves the independence of 1/f, $\theta(1/f)$ modulo $d\mathcal{O}_f + \mathscr{F}_2$. This concludes the proof of our first assertion

To prove the second statement, consider an element $d\omega \in (d\mathcal{O}_f + \mathscr{F}_2) \cap (\widehat{\mathcal{O}}_f^{\circ})^{\mathcal{G}}$. By the argument given in the preceding paragraph, this $d\omega$ is equivalent modulo \mathscr{F}_2 to an element $d\omega' = A/f^2 \in d\mathcal{O}_f(1)$. Moreover, since $\mathcal{O}_f^{\circ}/\mathscr{F}_2 \cong \mathcal{O}_f^{\circ}(2)$ and $d\omega \in \widehat{\mathcal{O}}_f^{\circ}$, the element A/f^2 is uniquely determined and we have $A/f^2 \in \mathcal{O}_f^{\circ}(2)$. Therefore A is an admissible Laurent polynomial supported in Δ . Since $d\omega$ is \mathcal{G} -invariant and the respective $d\omega'$ is unique, it follows that $d\omega'$, hence A, is also \mathcal{G} -invariant. Under condition (a3) we can write it as $A(\mathbf{x}) = \lambda + \mu f(\mathbf{x}) + \sum_{i=1}^n \nu_i(\theta_i f)(\mathbf{x})$ with some $\lambda, \mu, \nu_i \in \mathbb{Z}_p[\![t]\!]$. Since

$$d\omega' = \frac{A}{f^2} = (\lambda + \mu)\frac{1}{f} + \lambda\theta(1/f) - \sum_{i=1}^{n} \nu_i \theta_i(1/f)$$

and the rightmost sum also belongs to $d\mathcal{O}_f$, it follows that $(\lambda + \mu)(1/f) + \lambda\theta(1/f) \in d\mathcal{O}_f$. Hence this linear combination is trivial and we find that $\lambda = \mu = 0$. We thus obtain that $d\omega$ is equivalent modulo \mathscr{F}_2 to the linear combination $-\sum_i \nu_i \theta_i(1/f)$.

Let us fix a vertex **b** of Δ such that the coefficient $g_{\mathbf{b}}$ of $\mathbf{x}^{\mathbf{b}}$ in $g(\mathbf{x})$ is a unit in \mathbb{Z}_p . As we mentioned in the introduction, in this case any element $\omega = A/f^m \in \mathcal{O}_f$ can be expanded as a formal Laurent series $\omega = \sum_{\mathbf{k} \in C(\Delta - \mathbf{b})} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ with coefficients $a_{\mathbf{k}} \in \mathbb{Z}_p[\![t]\!][t^{-1}]$. Elements $\omega \in \mathscr{F}_s$ are then characterized by the following property: for every $\mathbf{k} \in C(\Delta - \mathbf{b})$ one has $\operatorname{ord}_p(a_{\mathbf{k}}) \geq \operatorname{ord}_p(\mathbf{k}) s$ where $\operatorname{ord}_p(\mathbf{k})$ is the highest power of p dividing every coordinate of \mathbf{k} . In our proof of Proposition 1.7 we will need the following property of expansion coefficients of $\omega = 1/f$:

Lemma 4.6. Consider the formal expansion

$$\frac{1}{f} = \sum_{\mathbf{k} \in C(\Delta - \mathbf{b})} \alpha_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}.$$

Then for all $N \geq 1$ we have

(13)
$$\alpha_{-N\mathbf{b}} = -(tg_{\mathbf{b}})^{-N}a_N(t),$$

where $a_N \in \mathbb{Z}_p[t]$ is a polynomial of degree < N with $a_N(0) = 1$.

Proof.

$$\frac{1}{f(\mathbf{x})} = \frac{1}{-tg_{\mathbf{b}}\mathbf{x}^{\mathbf{b}}(1+\ell(\mathbf{x}))} = -\frac{\mathbf{x}^{-\mathbf{b}}}{tg_{\mathbf{b}}\mathbf{x}^{\mathbf{b}}} \sum_{m=0}^{\infty} (-1)^m \ell(\mathbf{x})^m,$$

where

$$\ell(\mathbf{x}) = -\frac{\mathbf{x}^{-\mathbf{b}}}{tq_{\mathbf{b}}} + rest.$$

The contribution from each m to $\alpha_{-N\mathbf{b}}$ is given by $-(tg_{\mathbf{b}})^{-1}$ times

$$(-1)^m \left[\ell(\mathbf{x})^m\right]_{\mathbf{x}^{-(N-1)\mathbf{b}}} = (-1)^m \sum_{s=0}^m \binom{m}{s} \left(-\frac{\mathbf{x}^{-\mathbf{b}}}{tg_{\mathbf{b}}}\right)^s \left[(rest)^{m-s}\right]_{\mathbf{x}^{-(N-1-s)\mathbf{b}}}.$$

Since rest is supported in $\Delta - \mathbf{b}$, which is a convex cone, the terms with $s \geq N$ in the above sum vanish. Since rest is independent of t, the above sum is a polynomial in (1/t) of degree at most N-1. We conclude that (13) is true with some polynomial $a_N \in \mathbb{Z}_p[t]$ of degree at most N. We now compute its value at t=0 as the sum of contributions from s=N-1 in the previous formula:

$$a_N(0) = \sum_{m>N-1} (-1)^{m-N+1} \binom{m}{N-1} \left[(rest)^{m-N+1} \right]_{\mathbf{x}^0}.$$

Now note that *rest* is supported in $\Delta - \mathbf{b}$ and has no constant term, thus the only non-zero term in this sum is for m = N - 1 and we obtain $a_N(0) = 1$.

The last ingredient we need for the proof of Proposition 1.7 is the following.

Lemma 4.7. Let $\ell_i(\mathbf{y})$, i = 1, ..., m in $\mathbf{y} = y_1, ..., y_n$ be a \mathbb{Z} -basis of all linear forms $\ell \in \mathbb{Z}[\mathbf{y}]$ such that $\ell(\theta_1 g, ..., \theta_n g)$ is \mathcal{G} -invariant. Then there exist vertices $\mathbf{b}_1, ..., \mathbf{b}_{m+1}$ of Δ such that the determinant of

(14)
$$\begin{pmatrix} 1 & \ell_1(\mathbf{b}_1) & \cdots & \ell_m(\mathbf{b}_1) \\ 1 & \ell_1(\mathbf{b}_2) & \cdots & \ell_m(\mathbf{b}_2) \\ \vdots & \vdots & & \vdots \\ 1 & \ell_1(\mathbf{b}_{m+1}) & \cdots & \ell_m(\mathbf{b}_{m+1}) \end{pmatrix}$$

is non-zero. This lemma can be considered trivial if m = 0.

Proof. Let $\mathbf{b}_1, \ldots, \mathbf{b}_M$ be the set of vertices of Δ . This set of vectors has rank n. Hence the vectors $(\ell_1(\mathbf{b}_i), \ldots, \ell_m(\mathbf{b}_i))$, $i = 1, \ldots, M$ have rank m. We claim that the set of augmented vectors $(1, \ell_1(\mathbf{b}_i), \ldots, \ell_m(\mathbf{b}_i))$, $i = 1, \ldots, M$ has rank m + 1. To see this, choose $\lambda_1, \ldots, \lambda_M > 0$ such that $\sum_{i=1}^{M} \lambda_i \mathbf{b}_i = \mathbf{0}$. This is possible because $\mathbf{0}$ lies in the interior of Δ . Hence we get that

$$\sum_{i=1}^{M} \lambda_i(1, \ell_1(\mathbf{b}_i), \dots, \ell_m(\mathbf{b}_i)) = \left(\sum_{i=1}^{M} \lambda_M, 0, \dots, 0\right)$$

lies in the span of the augmented vectors. Since $\sum_i \lambda_i > 0$, hence non-zero, this gives us the desired rank m+1.

Proof of Proposition 1.7. Due to Lemma 4.4 and Lemma 4.3 assumptions (a1),(a2) and (a4) imply that the 2nd Hasse-Witt condition holds for Δ . Then according to Lemma 4.5 with assumption (a3) the quotient $(\widehat{\mathcal{O}}_f^{\sigma})^{\mathcal{G}}/(d\mathcal{O}_f + \mathscr{F}_2)$ is a free $\mathbb{Z}_p[\![t]\!]$ -module of rank 2 generated by 1/f, $\theta(1/f)$. The same is true for f^{σ} in place of f, in which case the basis in the respective quotient module is given by $1/f^{\sigma}$, $\theta(1/f)^{\sigma}$.

Consider $A, B \in \mathbb{Z}_p[\![t]\!]$ such that

$$\theta^2(1/f) \equiv A(1/f) + B\theta(1/f) \pmod{d\mathcal{O}_f + \mathscr{F}_2}.$$

Note that the differential operator $P=\theta^2-B\theta-A$ is a right divisor of the Picard–Fuchs differential operator $L=\theta^r+u_{r-1}\theta^{r-1}+\ldots+u_0$ attached to the cyclic theta module M. Indeed, using the right-division algorithm in the ring $\mathbb{Z}_p[\![t]\!][\theta]$ we can write $L=QP+C\theta+D$ with some differential operator Q of order r-2 and $C,D\in\mathbb{Z}_p[\![t]\!]$. Since $L(1/f)\in d\mathcal{O}_f$ and $P(1/f)\in d\mathcal{O}_f+\mathscr{F}_2$, we have $C\theta(1/f)+D/f\in d\mathcal{O}_f+\mathscr{F}_2$. Since 1/f and $\theta(1/f)$ are linearly independent modulo $d\mathcal{O}_f+\mathscr{F}_2$, we conclude that C=D=0.

Since L is of MUM type, it follows that P is also of MUM type. It follows that $A, B \in t\mathbb{Z}_p[\![t]\!]$ and also for every $s \geq 2$ there exist $A_s, B_s \in t\mathbb{Z}_p[\![t]\!]$ such that

(15)
$$\theta^{s}(1/f) \equiv A_{s}(t)(1/f) + B_{s}(t)\theta(1/f) \pmod{d\mathcal{O}_{f} + \mathscr{F}_{2}}.$$

By Proposition 3.2 there exist $\lambda_i(t) \in \mathbb{Z}_p[\![t]\!]$ such that

$$\mathscr{C}_p(1/f) \equiv \lambda_0(t)(1/f)^{\sigma} + \lambda_1(t)(\theta(1/f))^{\sigma} + \dots + \lambda_{r-1}(t)(\theta^{r-1}(1/f))^{\sigma} \pmod{d\widehat{\mathcal{O}}_{f^{\sigma}}}.$$

Using the relations (15) with $1/f^{\sigma}$ and $\theta(1/f)^{\sigma}$ instead of $1/f, \theta(1/f)$ we find that there exist $\mu_0(t), \mu_1(t) \in \mathbb{Z}_p[\![t]\!]$ such that

$$\mathscr{C}_p(1/f) \equiv \mu_0(t)(1/f)^{\sigma} + \mu_1(t)\theta(1/f)^{\sigma} \pmod{d\mathcal{O}_{f^{\sigma}} + \mathscr{F}_2}.$$

Because the reduction coefficients A_s^{σ} , B_s^{σ} all have the property that $A_s^{\sigma}(0) = B_s^{\sigma}(0) = 0$ we find that $\mu_0(0) = \lambda_0(0) = 1$ and $\mu_1(0) = \lambda_1(0) = p\alpha_1$.

We now proceed to determine $\mu_1(0)$. Note that

$$\mathscr{C}_p(1/f) - \mu_0(1/f)^{\sigma} - \mu_1 \theta(1/f)^{\sigma} \in (\widehat{\mathcal{O}}_{f^{\sigma}}^{\circ})^{\mathcal{G}} \cap (d\mathcal{O}_{f^{\sigma}} + \mathscr{F}_2).$$

We now use assumption (a3). Let us apply to the above element the second assertion of Lemma 4.5 for f^{σ} in place of f. Using a basis of linear functionals as in Lemma 4.7, we can write

$$\mathscr{C}_p(1/f) = \mu_0(t)(1/f^{\sigma}) + \mu_1(t)\theta(1/f)^{\sigma} + \sum_{i=1}^m \nu_i(t)\ell_i(\mathbf{x}\frac{\partial}{\partial \mathbf{x}})(1/f)^{\sigma} \pmod{\mathscr{F}_2}$$

with some $\nu_1, \ldots, \nu_m \in \mathbb{Z}_p[\![t]\!]$. Expanding both sides at a vertex **b** of Δ and reading the coefficient at $\mathbf{x}^{-p^s\mathbf{b}}$ we find that

$$\alpha_{-p^{s+1}\mathbf{b}} \equiv \mu_0 \alpha_{-p^s\mathbf{b}}^{\sigma} + \mu_1 (\theta \alpha_{-p^s\mathbf{b}})^{\sigma} - p^s \left(\sum_{i=1}^m \nu_i \ell_i(\mathbf{b}) \right) \alpha_{-p^s\mathbf{b}}^{\sigma} \pmod{p^{2s} \mathbb{Z}_p[\![t]\!][t^{-1}]}.$$

Here $\alpha_{\mathbf{k}}$ for $\mathbf{k} \in C(\Delta - \mathbf{b})$ are the expansion coefficients of 1/f at the vertex \mathbf{b} , they were considered in Lemma 4.6. Multiplying by $-(g_{\mathbf{b}}t^{\sigma})^{p^s}$ we obtain

$$\left(\frac{t^{\sigma}}{t^{p}}\right)^{p^{s}} (g_{\mathbf{b}})^{(1-p)p^{s}} a_{p^{s+1}}(t) \equiv \mu_{0}(t) a_{p^{s}}(t^{\sigma}) + \mu_{1}(t) \left((\theta - p^{s}) a_{p^{s}}\right) (t^{\sigma}) - p^{s} \left(\sum_{i=1}^{m} \nu_{i}(t) \ell_{i}(\mathbf{b})\right) a_{p^{s}}(t^{\sigma}) \pmod{p^{2s} \mathbb{Z}_{p}[\![t]\!]}.$$

The above congruence evaluates at t = 0 to

$$(g_{\mathbf{b}})^{(1-p)p^s} \equiv \mu_0(0) - p^s \left(\mu_1(0) + \sum_{i=1}^m \nu_i(0)\ell_i(\mathbf{b})\right) \pmod{p^{2s}}.$$

By looking at this congruence modulo p^s we find that $\mu_0(0) = 1$. We substitute this value back in the above congruence and divide by p^s . Since s here is arbitrary and for $a \in 1 + p\mathbb{Z}_p$ one has $\log_p(a) \equiv p^{-s}(a^{p^s} - 1)$, we find that

$$\mu_1(0) + \sum_{i=1}^m \nu_i(0)\ell_i(\mathbf{b}) = \log_p(g_{\mathbf{b}}^{(p-1)}).$$

Using these relations for $\mathbf{b} = \mathbf{b}_1, \dots, \mathbf{b}_{m+1}$ from Lemma 4.7, we can recover the value of $\mu_1(0)$. Under assumption (a4) all $g_{\mathbf{b}_i} = 1$ and we obtain $\mu_1(0) = 0$.

5. Examples

In this section we list a number of examples of Theorem 1.5.

Let us consider $f(\mathbf{x}) = 1 - tg(\mathbf{x})$, where $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a Laurent polynomial whose Newton polytope Δ is reflexive. As was mentioned in the introduction, under this condition every point $\mathbf{u} \in \mathbb{Z}^n$ belongs to an integral dilation of the boundary of Δ : we have $\mathbf{u} \in m\partial \Delta$ for a unique non-negative integer m. In our notation $\deg(\mathbf{x}^{\mathbf{u}}) = m$.

Let \mathcal{G} be a finite group of monomial substitutions preserving Δ and $g(\mathbf{x})$. The elements of \mathcal{G} can be described by $n \times n$ -matrices in $GL(n,\mathbb{Z})$ and an element $\gamma \in \mathcal{G}$ acts on a monomial $\mathbf{x}^{\mathbf{u}}$ by sending it to $\mathbf{x}^{\mathbf{u}\gamma}$. This action can be extended to all Laurent polynomials. The image of a Laurent polynomial h under $\gamma \in \mathcal{G}$ is denoted by h^{γ} . Finally, given any Laurent polynomial h we define its symmetrization by

$$h^{\mathcal{G}} := \frac{1}{\# \mathcal{G}} \sum_{\gamma \in \mathcal{G}} h^{\gamma}.$$

Clearly we must assume that $p \nmid \#\mathcal{G}$. In particular, if h is \mathcal{G} -symmetric then $h^{\mathcal{G}} = h$. Before giving the examples we describe conditions which will ensure that

$$M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$$

is a cyclic θ -module in the sense of Definition 3.1. It will turn out that in all our examples the rank r is equal to n, the number of variables x_1, \ldots, x_n . Later we will also take n = 4. To

satisfy Definition 3.1(i) we must apply Dwork-Griffiths reduction to each element of $(\mathcal{O}_f^{\circ})^{\mathcal{G}}$. We shall reduce this infinite amount of work to a finite number of steps.

Step 1: Show that every $t^l \mathbf{x}^{\mathbf{u}}$ with $l = \deg(\mathbf{x}^{\mathbf{u}})$ and $\mathbf{u} \in (n+1)\Delta$ can be written in the form

$$t^{l}\mathbf{x}^{\mathbf{u}} = \sum_{i=1}^{n} A_{i}(\mathbf{x})\theta_{i}(f) + A_{0}(\mathbf{x})f,$$

where the A_i are admissible Laurent polynomials with support in $n\Delta$.

We note that [2, Theorem 4.8 (vii)] suggests that if f is Δ -regular then this property holds with some, non necessarily admissible, Laurent polynomials A_i . But since we work with admissible polynomials it is wise to verify it explicitly.

Step 2: For every $1 \le k \le n$ determine a finite set of admissible polynomials H_k , supported in $k\Delta$, such that every $t^l\mathbf{x}^\mathbf{u}$ with $l = \deg(\mathbf{x}^\mathbf{u})$ and $\mathbf{u} \in k\Delta$ can be written in the form

$$t^{l}\mathbf{x}^{\mathbf{u}} = \sum_{i=1}^{n} A_{i}(\mathbf{x})\theta_{i}(f) + A_{0}(\mathbf{x})f + \sum_{h \in H_{k}} \delta_{h}h(\mathbf{x}),$$

where the A_i are admissible Laurent polynomials with support in $(k-1)\Delta$ and $\delta_h \in \mathbb{Z}_p[\![t]\!]$ for all $h \in H_k$.

It is clear that such a set H_k always exists, the problem is to find it efficiently. In practice a quick trial and error search does the job. In all examples the sets H_k will be chosen independently of the prime number p. For any choice of H_k , we will give an effective method to determine the set of those primes p for which Step 2 can be performed.

Again, when f is Δ -regular, then by [2, Remark 2.13 and Theorem 4.8 (i)] there should be sets H_k of size $\#H_k = \psi_k(\Delta)$, such that this property holds with some, not necessarily admissible A_i and δ_h . These numbers are defined by the generating series $\sum_{k=0}^n \psi_k(\Delta) T^k = (1-T)^{n+1} \sum_{m\geq 0} \#(m\Delta)_{\mathbb{Z}} T^m$. For reflexive polytopes they satisfy $\psi_{n-k}(\Delta) = \psi_k(\Delta)$; in particular $\psi_n(\Delta) = 1$. We would like to remark that in all our examples $\#H_k = \psi_k(\Delta)$, $k = 1, \ldots, n$ and we can choose $H_n = \{1\}$.

In the next step we verify property (i) of Definition 3.1 only for \mathcal{G} -symmetrizations of the polynomials $h \in H_k$. This will turn out to be sufficient (Proposition 5.4).

Step 3: For each $1 \le k \le n$ and each $h \in H_k$ we check that there exist $b_i(t) \in \mathbb{Z}_p[\![t]\!]$ for $i = 0, 1, \ldots, \min(k, n - 1)$ such that

(16)
$$k! \frac{h^{\mathcal{G}}}{f^{k+1}} = \sum_{i=0}^{\min(k, n-1)} b_i(t) \theta^i(1/f) \pmod{d\mathcal{O}_f}.$$

We notice that $\theta^i(1/f)$ for $i=0,1,\ldots,k$ generate the same module as $i!/f^{i+1}$ for $i=0,1,\ldots,k$. This is because of the straightforward relation

(17)
$$\frac{i!}{f^{i+1}} = (\theta + i) \cdots (\theta + 2)(\theta + 1)(1/f)$$

So in the verification of (16) we can also take $i!/f^{i+1}$ instead of $\theta^i(1/f)$. With other coefficients $b_i(t)$ of course.

At the final step we will check property (ii) of of Definition 3.1. Earlier we mentioned that in our examples we always have $H_n = \{1\}$. Therefore at Step 3 we are urged to find a representation of $n!/f^{n+1}$ as a $\mathbb{Z}_p[\![t]\!]$ -linear combination of $\theta^i(1/f)$, $i=1,\ldots,n-1$ modulo $d\mathcal{O}_f$. Thus at Step 3 we find the monic nth order differential operator $L \in \mathbb{Z}_p[\![t]\!][\theta]$ such that $L(1/f) \in d\mathcal{O}_f$. The coefficients of L are in $\mathbb{Q}(t)$ because they can be determined by solving a finite set of linear equations with coefficients in $\mathbb{Q}(t)$.

Step 4: Check that L is irreducible in $\mathbb{Q}[t][\theta]$.

Lemma 5.1. Suppose that Step 4 has been verified. Then $\theta^i(1/f)$, i = 0, ..., n-1 are $\mathbb{Z}_p[\![t]\!]$ -linearly independent modulo $d\mathcal{O}_f$.

Proof. Suppose $\theta^i(1/f)$, i = 0, ..., n-1 are dependent. Then there exist $b_i(t) \in \mathbb{Z}_p[\![t]\!]$ and Laurent polynomials A_i with support in $k\Delta$ for some $k \geq 1$ and coefficients in $\mathbb{Z}_p[\![t]\!]$ such that

$$\sum_{i=0}^{n-1} b_i(t)\theta^i(1/f) = \sum_{j=1}^n \theta_j\left(\frac{A_j}{f^k}\right).$$

Similarly to the coefficients of L, the coefficients involved can be determined by solution of a finite set of linear equations with coefficients in $\mathbb{Q}(t)$. Hence we can assume that $b_i(t) \in \mathbb{Q}(t)$ for all i. Taking the period map we find that $L_1(F_0) = 0$, where $L_1 = \sum_{i=0}^{n-1} b_i(t)\theta^i$. Hence F_0 satisfies a lower order differential equation over $\mathbb{Q}(t)$. Then the right greatest common divisor of L, L_1 is a non-trivial right divisor of L, which contradicts the irreducibility of L.

Lemma 5.2. For every integral point $\mathbf{u} \in m \partial \Delta$ with $m \geq n$ there is a vertex \mathbf{v} of Δ such that $\mathbf{u} - \mathbf{v} \in (m-1)\Delta$.

Proof. There exist n vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that \mathbf{u} is in the positive cone spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n$. Write $\mathbf{u} = \sum_{i=1}^n \beta_i \mathbf{v}_i$ with each $\beta_i \geq 0$. Then $\sum_{i=1}^n \beta_i \geq m$. Since $m \geq n$ there is at least one index i_0 such that $\beta_{i_0} \geq 1$. It then follows that $\mathbf{u} - \mathbf{v}_{i_0} \in (m-1)\Delta$.

Corollary 5.3. Suppose the conditions of Steps 1-2 are fulfilled. Then for every $k \geq 1$ there exists a finite set of admissible polynomials H_k , supported in $k\Delta$, such that every $t^l\mathbf{x}^{\mathbf{u}}$ with $l = \deg(\mathbf{x}^{\mathbf{u}})$ and $\mathbf{u} \in k\Delta$ can be written in the form

(18)
$$t^{l}\mathbf{x}^{\mathbf{u}} = \sum_{i=1}^{n} A_{i}(\mathbf{x})\theta_{i}(f) + A_{0}(\mathbf{x})f + \sum_{h \in H_{k}} \delta_{h}h(\mathbf{x}),$$

where the A_i are admissible Laurent polynomials with support in $(k-1)\Delta$ and $\delta_h \in \mathbb{Z}_p[\![t]\!]$ for all $h \in H_k$. When k > n we can take $H_k = \emptyset$.

Proof. From Step 2 we see that (18) holds for $k \leq n$. For k = n + 1 it follows from Step 1. We can take $H_{n+1} = \emptyset$. For k > n + 1 we apply induction on k. Using Lemma 5.2 we can find find a vertex \mathbf{v} such that $\mathbf{u} - \mathbf{v} \in (k-1)\Delta$. Then multiply the relation (18) for $\mathbf{x}^{\mathbf{u}-\mathbf{v}}$ by $t\mathbf{x}^{\mathbf{v}}$ to get the desired relation.

Proposition 5.4. Suppose the conditions in Steps 1-4 are fulfilled. Then $(\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ is a cyclic θ -module of rank n.

Proof. Condition (ii) of Definition 3.1 follows from Lemma 5.1. We verify condition (i) by induction on m. For m=0 we can only have A=1 and (i) clearly holds. Suppose that m>0 and (i) holds for m-1. Without loss of generality we can assume that A is the symmetrization of a monomial $t^l\mathbf{x}^{\mathbf{u}}$ of degree l and support in $m\Delta$. We write $t^l\mathbf{x}^{\mathbf{u}}$ in the form (18) and symmetrize. Then we find new A_i such that

$$t^{l}(\mathbf{x}^{\mathbf{u}})^{\mathcal{G}} = \sum_{i=1}^{n} A_{i}(\mathbf{x})\theta_{i}(f) + A_{0}(\mathbf{x})^{\mathcal{G}}f + \sum_{h \in H_{-}} \delta_{h}h^{\mathcal{G}}(\mathbf{x}).$$

Hence, after multiplication with $m!/f^{m+1}$.

$$m! t^{l} \frac{(\mathbf{x}^{\mathbf{u}})^{\mathcal{G}}}{f^{m+1}} = \sum_{i=1}^{n} m! \frac{A_{i}\theta_{i}(f)}{f^{m+1}} + m! \frac{A_{0}^{\mathcal{G}}}{f^{m}} + \sum_{h \in H_{m}} \delta_{h} m! \frac{h^{\mathcal{G}}}{f^{m+1}}.$$

We apply Step 3 to the summation involving $h \in H_m$. We apply our induction hypothesis to the term with $A_0^{\mathcal{G}}$. The first summation on the right can be rewritten as

$$-\sum_{i=1}^{n} \theta_{i} \left((m-1)! \frac{A_{i}}{f^{m}} \right) + \sum_{i=1}^{n} (m-1)! \frac{\theta_{i}(A_{i})}{f^{m}}.$$

Clearly the first summation is in $d\mathcal{O}_f$. After symmetrization we can apply our induction hypothesis to the second summation with terms $(m-1)!\theta_i(A_i)^{\mathcal{G}}/f^m$.

We now present a practical way to perform Steps 1 and 2. We start with Step 1. For any \mathbf{u} we define $w_{\mathbf{u}} := t^l \mathbf{x}^{\mathbf{u}}$ where $l = \deg(\mathbf{x}^{\mathbf{u}})$. Consider the $\mathbb{Z}_p[\![t]\!]$ -module S generated by $w_{\mathbf{u}}\theta_i(f), i = 1, \ldots, n$ and $w_{\mathbf{u}}f$ for $\mathbf{u} \in n\Delta$. It is a submodule of the $\mathbb{Z}_p[\![t]\!]$ -module T of admissible Laurent polynomials with support in $(n+1)\Delta$. To verify Step 1 we need to verify that these modules coincide. To do so we write each generator of the submodule S in a coordinate vector with respect to the basis $w_{\mathbf{v}}, \mathbf{v} \in (n+1)\Delta$ of T. One easily verifies that the coordinates of these vectors are in $\mathbb{Z}_p[\![t]\!] \subset \mathbb{Z}_p[\![t]\!]$. We now use the following observation.

Lemma 5.5. Let $v_1(t), \ldots, v_K(t)$ be a finite set of vectors of length N with entries in $\mathbb{Z}_p[\![t]\!]$. Then they generate $(\mathbb{Z}_p[\![t]\!])^N$ if and only if the vectors $v_1(0), \ldots, v_K(0)$ generate \mathbb{Z}_p^N .

Proof. Suppose $\mathbf{v}_1(t), \dots, \mathbf{v}_K(t)$ generate $(\mathbb{Z}_p[\![t]\!])^N$. Take any vector \mathbf{a} in \mathbb{Z}_p^N . This can be written as $\mathbb{Z}_p[\![t]\!]$ linear combination of $\mathbf{v}_1(t), \dots, \mathbf{v}_K(t)$. Set t=0 in this linear combination and we find that \mathbf{a} is a \mathbb{Z}_p -linear combination of $\mathbf{v}_1(0), \dots, \mathbf{v}_K(0)$.

Suppose $\mathbf{v}_1(0), \dots, \mathbf{v}_K(0)$ generate \mathbb{Z}_p^N . Every standard basis vector \mathbf{e}_k (i.e k-th coordinate is 1, the others are 0) in \mathbb{Z}_p^N can be written in the form $\mathbf{e}_k = \sum_{j=1}^K \lambda_{kj} \mathbf{v}_k(0)$ with $\lambda_{kj} \in \mathbb{Z}_p$. Define $\mathbf{e}_k(t) := \sum_{j=1}^K \lambda_{kj} \mathbf{v}_k(t)$. Then $\mathbf{e}_1(t), \dots, \mathbf{e}_N(t)$ have a determinant in $\mathbb{Z}_p[\![t]\!]$ whose constant term is 1. This implies that every vector in $(\mathbb{Z}_p[\![t]\!])^N$ is a $\mathbb{Z}_p[\![t]\!]$ -linear combination of the $\mathbf{e}_j(t)$, hence the $\mathbf{v}_j(t)$.

In practice we consider the \mathbb{Z} -module generated by the generators of S and then set t=0 in the coordinate vectors with respect to the generators of T. By a reduction procedure (for example LLL) we determine the index of the lattice generated by these integer vectors in $\mathbb{Z}^{|((n+1)\Delta)_{\mathbb{Z}}|}$, which we denote by DG(n+1,g), DG being the initials of Dwork-Griffiths. If DG(n+1,g) is infinite, Step 1 fails for any p. If it is finite then Step 1 is successful for every prime p that does not divide DG(n+1,g).

For Step 2 we use an extension of the above idea. Let S_k be the set of $w_{\mathbf{u}}\theta_i(f)$, $i=1,\ldots,n$ and $w_{\mathbf{u}}f$ for $\mathbf{u} \in (k-1)\Delta$. Usually, when $k \leq n$, the $\mathbb{Z}_p[\![t]\!]$ -module generated by S_k will not be equal to the module generated by $T_k = \{w_{\mathbf{v}} | \mathbf{v} \in k\Delta\}$. The idea is to supplement S_k with a finite set H_k of elements in T_k . until we have equality. The choice of H_k is by a simple trial and error process. To that end we write each element in $S_k \cup H_k$ with respect to T_k and set t=0. The index of the \mathbb{Z} -module generated by these vectors in $\mathbb{Z}^{\lfloor (k\Delta)_{\mathbb{Z}} \rfloor}$ is denoted by $DG(k,g;H_k)$. We choose H_k such that $DG(k,g;H_k)$ is finite. Then Step 2 succeeds for every prime p that does not divide $DG(k,g;H_k)$.

Theorem 5.6. In the four examples below the instanton numbers are p-integral for all p > 5 and $p \ge 5$ in the non-quintic cases.

Proof. We first use Proposition 5.4 to check that $M = (\mathcal{O}_f^{\circ})^{\mathcal{G}}/d\mathcal{O}_f$ is a cyclic θ-module of MUM-type. In each of the examples we shall explicitly verify Steps 1-4 below. We then use Proposition 1.6 to conclude that the respective Picard–Fuchs differential operator has a Frobenius structure. One easily verifies that all four examples satisfy conditions (a1)-(a4) of Proposition 1.7. Having done this we can conclude that $\alpha_1 = 0$. Theorem 1.5 then implies our theorem.

Example 5.7. Let

$$g = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4},$$

this is our quintic case. Here we can take the trivial group $\mathcal{G} = \{1\}$. We compute

$$DG(g,5) = 1$$
, $DG(g,4;1) = 5^4$, $DG(g,3;1) = 5^3$, $DG(g,2;1) = 5^2$, $DG(g,1;1) = 5$.

So Steps 1,2 can be taken for p > 5. Here we have $H_k = \{1\}$ for k = 1, 2, 3, 4. Note that Step 3 with k = 1, 2, 3 is fulfilled trivially since functions $k!/f^{k+1}$ are already in the desired form. For

Step 3 with k = 4 we verify that

$$(1 - (5t)^5) \frac{4!}{f^5} \equiv 10 \frac{3!}{f^4} - 25 \frac{2!}{f^3} + 15 \frac{1}{f^2} - \frac{1}{f} \pmod{d\mathcal{O}_f}.$$

Using (17) yields the Picard-Fuchs equation

$$L(1/f) \equiv 0 \pmod{d\mathcal{O}_f}, \quad L = -\theta^4 + (5t)^5(\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4).$$

Note that L is of MUM-type and hypergeometric with solution

$$F_0(t) = {}_{4}F_3(1/5, 2/5, 3/5, 4/5; 1, 1, 1|(5t)^5).$$

As this differential operator is irreducible, Step 4 is fulfilled and we conclude that $M = \mathcal{O}_f^{\circ}/d\mathcal{O}_f$ is a cyclic θ -module of rank 4.

Example 5.8. Consider

$$g(\mathbf{x}) = x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} + x_4 + \frac{1}{x_4}.$$

As symmetry group \mathcal{G} we take the group consisting of $x_i \to (x_{s(i)})^{\pm 1}$, where s is a permutation of 1, 2, 3, 4. We have $\#\mathcal{G} = 48$ and take $p \geq 5$. This is our diagonal case from the introduction. We compute

$$DG(g,5) = 1$$
, $DG(g,4;1) = 2^7 \cdot 3$, $DG(g,3;1,x_1,x_2,x_3) = 2^{13} \cdot 3$, $DG(g,2;1,x_1,x_2,x_3,x_1x_2,x_1x_3) = 2^6$, $DG(g,1;1,x_1,x_2,x_3) = 2$.

For Step 3 with k = 4 we verify that

$$(1 - 80t^{2} + 1024t^{4})\frac{4!}{f^{5}} \equiv (10 - 480t^{2} + 2048t^{4})\frac{3!}{f^{4}} + (-25 + 608t^{2})\frac{2!}{f^{3}} + (15 - 128t^{2})\frac{1}{f^{2}} - \frac{1}{f} \pmod{d\mathcal{O}_{f}}.$$

Using (17) we find the Picard-Fuchs equation $L(1/f) \equiv 0 \pmod{d\mathcal{O}_f}$ with

$$L = (1024t^4 - 80t^2 + 1)\theta^4 + 64(128t^4 - 5t^2)\theta^3 + 16(1472t^4 - 33t^2)\theta^2 + 32(896t^4 - 13t^2)\theta + 128(96t^4 - t^2).$$

Observe that it has MUM-type. Furthermore it is non-hypergeometric and listed as #16 in [1] with $z=t^2$. Step 3 for the elements $1 \in H_k$ with k=3,2,1 holds trivially, just as in Example 5.7. The symmetrization of x_1, x_2, x_3 is simply the Laurent polynomial g. Observe that

(19)
$$\frac{tg}{f^{k+1}} = -\frac{1}{f^{k+1}} + \frac{1}{f^k}$$

for k = 3, 2, 1. Finally, a partial symmetrization of x_1x_2 is $\sum_{i < j} x_ix_j$. We compute

$$2! \frac{t^2 \sum_{i < j} x_i x_j}{f^3} \equiv \frac{1}{8} \left(2! \frac{1 - 16t^2}{f^3} - \frac{3}{f^2} + \frac{1}{f} \right) \pmod{d\mathcal{O}_f}.$$

After symmetrization, this completes our verification of Step 3. A Maple check shows that L is irreducible.

Example 5.9. Consider

$$g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^2 x_2 x_3 x_4}$$

with the trivial group $\mathcal{G} = \{1\}$. We take $p \geq 5$ and compute

$$D(g,5) = 4$$
, $D(g,4;1) = 2^5 \cdot 3^4$, $D(g,3;1) = 2^4 \cdot 3^3$, $D(g,2;1,1/x_1) = 2^2 \cdot 3^2$, $D(g,1;1) = 2 \cdot 3$.

This was Step 1 and 2. The check of Step 3 with k = 4 reads

$$(1 - 108^2 t^6) \frac{4!}{f^5} \equiv (10 + 23328t^6) \frac{3!}{f^4} - (25 - 23328t^6) \frac{2!}{f^3} + 15 \frac{1}{f^2} - \frac{1}{f} \pmod{d\mathcal{O}_f}.$$

Using (17) we derive the Picard-Fuchs equation $L(1/f) \equiv 0 \pmod{d\mathcal{O}_f}$ with

$$L = -\theta^4 + 108^2 t^6 (\theta + 1)(\theta + 2)(\theta + 4)(\theta + 5).$$

The equation is hypergeometric with

$$F_0(t) = {}_{4}F_3(1/6, 1/3, 2/3, 5/6; 1, 1, 1|108^2t^6).$$

Step 3 for $1 \in H_k, k = 3, 2, 1$ is immediate. Finally we verify that

$$2! \frac{t^2/x_1}{f^3} \equiv 2! \frac{3t^3}{f^3} \pmod{d\mathcal{O}_f},$$

which finishes our check of Step 3. Similarly to Example 5.7, Step 4 follows from the known criterion of irreducibility for hypergeometric differential operators.

Example 5.10. Consider

$$g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2} + \frac{1}{x_3 x_4}.$$

The symmetry group \mathcal{G} is generated by $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$ and $x_1 \to x_3, x_2 \to x_4$ and $x_1 \to 1/x_1x_2$. Note that $\#\mathcal{G} = 18$. We take $p \geq 5$ and compute

$$DG(g,5) = 1$$
, $DG(g,4;1) = 2 \cdot 3^5$, $DG(g,3;1,x_1) = 3^6$, $DG(g,2;1,x_1,x_1^2) = 3^3$, $DG(g,1;1,x_1) = 3$.

Step 1 and 2 have been taken. Step 3 for k = 4 reads

$$(-1+189t^{3}+5832t^{6})\frac{4!}{f^{5}} = (-10+756t^{3}-11664t^{6})\frac{3!}{f^{4}} + (25-486t^{3}-11664t^{6})\frac{2!}{f^{3}}-15\frac{1}{f^{2}}+\frac{1}{f} \pmod{d\mathcal{O}_{f}}$$

Using (17) we get the Picard-Fuchs equation $L(1/f) \equiv 0 \pmod{d\mathcal{O}_f}$ with

$$L = -\theta^4 + 3^3 t^3 (1+\theta)(2+\theta)(18+21\theta+7\theta^2) + 18^3 t^6 (1+\theta)(2+\theta)(4+\theta)(5+\theta)$$

This is #15 in [1] with $z = t^3$. For $1 \in H_k$ for k = 3, 2, 1 we proceed as before. The symmetrization of x_1 is simply g itself. We can deal with this using the observation (19). Finally we compute

$$2!\frac{3t^2(x_1^2)^{\mathcal{G}}}{f^3} \equiv (2t^3 + \frac{1}{54})\frac{2!}{f^3} - \frac{2/9}{f^2} + \frac{5/27}{f} \pmod{d\mathcal{O}_f}.$$

Beyond these examples there are a number of other cases for which Steps 1-4 apply. For example

$$g = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^4 x_2 x_3 x_4}$$

corresponding to the hypergeometric equation with holomorphic solution

$$F_0(t) = {}_{4}F_3(1/8, 3/8, 5/8, 7/8; 1, 1, 1|(4t)^8)$$

and

$$g = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^5 x_2^2 x_3 x_4}$$

corresponding to the hypergeometric equation with solution

$$F_0(t) = {}_{4}F_3(1/10, 3/10, 7/10, 9/10; 1, 1, 1|2^8 5^5 t^{10}).$$

Unfortunately, conditions (a1)-(a4) of Proposition 1.7 are not all satisfied and we would have to do some extra work.

We end with with an example of a fifth order Picard-Fuchs equation. Consider

$$g = x_1 + \dots + x_5 + \frac{1}{x_1 \cdots x_5}$$

corresponding to the hypergeometric equation with holomorphic solution

$$F_0(t) = {}_{5}F_4(1/6, 1/3, 1/2, 2/3, 5/6; 1, 1, 1, 1, 1/6t)^6$$
.

Theorem 5.11. In the above example let

$$6K(q) = 6 + \sum_{n>1} A_n \frac{q^{6n}}{1 - q^{6n}}$$

be 6 times the Yukawa coupling. Then A_n/n^2 is p-adic integer for all $n \ge 1$ and all primes $p \ge 7$.

Proof. This is an application of Theorem 1.4 and the conditions are checked in a similar way as in the quintic example. \Box

We like to point out that the numbers A_n/n^2 are precisely the so-called BPS-numbers for g=0 corresponding to a family of Calabi-Yau fourfolds, as treated in [12, §6.1]. This answers a question posed to us by Martijn Kool about BPS-numbers for Calabi-Yau fourfolds in one instance.

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