COROLLARY. $\sum[\mu] = (\sum\langle\mu^*\rangle)^*$.

Example: $\langle 3\rangle + \langle 21\rangle + \langle 2\rangle = ([1^3] + [21] + [1^2])^*$.

Now the theorem stated by Murnaghan could be deduced from the theorem of conjugates of Littlewood,[3] i.e., if $[\lambda] \otimes \{\mu\} = \sum\{\nu\}$, where λ is a partition of an integer m, then $\langle\lambda^*\rangle \otimes \{\mu\} = \sum\{\nu^*\}$ if m is even and $\langle\lambda^*\rangle \otimes \{\mu^*\} = \sum\{\nu^*\}$ if m is odd. For the $2k$-dimensional group

$$\langle 2\rangle \otimes \{2\} = \langle 4\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle; \text{ then, } [1^2] \otimes \{2\} = [1^4] + [2^2] + [2] + [0]; \text{ also if } [3] \otimes \{2\} = [6] + [42] + [4] + [2^2] + [2] + [0], \text{ then } \langle 1^3\rangle \otimes \{1^2\} = \langle 1^6\rangle + \langle 2^21^2\rangle + \langle 1^4\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle.$$

COROLLARY.

(i) Since $\{\lambda\} \otimes (\{\mu_1\} + \{\mu_2\} + \ldots) = \{\lambda\} \otimes \{\mu_1\} + \{\lambda\} \otimes \{\mu_2\} + \ldots$, then, if $\{\lambda\} \otimes [\mu] = \sum[\nu]$, where λ is a partition of an integer m, $\{\lambda^*\} \otimes [\mu] = \sum\langle\nu^*\rangle$ if m is even, and $\{\lambda^*\} \otimes \langle\mu^*\rangle = \sum\langle\nu^*\rangle$ if m is odd.

(ii) If $[\lambda] \otimes [\mu] = \sum[\nu]$, λ is a partition of m, then $\langle\lambda^*\rangle \otimes [\mu] = \sum\langle\nu^*\rangle$ (m even) and $\langle\lambda^*\rangle \otimes \langle\mu^*\rangle = \sum\langle\nu^*\rangle$ (m odd).

(iii) If $\langle\lambda\rangle \otimes [\mu] = \sum\{\nu\}$, λ a partition of m, then $[\lambda^*] \otimes [\mu] = \sum\{\nu^*\}$ (m even) and $[\lambda^*] \otimes \langle\mu^*\rangle = \sum\{\nu^*\}$ (m odd).

1 F. D. Murnaghan, PROC. NATL. ACAD. SCI., **38**, 966–973 (1952).

2 D. E. Littlewood, *Phil. Trans. R. Soc.*, ser. A, No. 809, **239**, 387–417 (1944).

3 D. E. Littlewood, *ibid.*, No. 807, pp. 305–65.

ON THE ARITHMETIC NORMALITY OF THE GRASSMANN VARIETY*

BY JUN-ICHI IGUSA

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, AND KYOTO UNIVERSITY, JAPAN

Communicated by Oscar Zariski, March 1, 1954

In this paper we shall prove, by a method which is valid for a universal domain of arbitrary characteristic, a theorem of Severi to the effect that every positive divisor on a Grassmann variety, which is canonically imbedded in a projective space, is a complete intersection by a hypersurface of the ambient space. The proof is a simple combination of formal lemmas due to Hodge, the well-known characterization of normality due to Muhly-Zariski, and the rationality of the Grassmann variety. Actually, Severi gave two proofs[1] of this assertion, neither of which can be applied immediately to the case of prime characteristic. We note that Severi's theorem and the first main theorem for the unimodular group[2] are substantially equivalent. Therefore, we have also proved this latter theorem in the case of prime characteristic. As a consequence, we shall generalize to the case of arbitrary characteristic a remark of Hodge to the effect that any Chow form is a homogeneous polynomial in the Plücker co-ordinates of the indeterminates of the form.

The subject of this paper was called to my attention by Professor Zariski, and I should like to thank him for doing so, and also for his interest shown to my proof.

1. *Arithmetic Normality.*—The Grassmann variety V is defined in the following way. We fix a projective space L^n of n dimension over the universal domain $\mathbf{K}$.[3] If L^r is a linear variety in L^n which is spanned by a set of $r + 1$ independent generic points $x_0, x_1, \ldots, x_r$ of L^n over a field K, then L^r is defined over the purely transcendental extension $K(x_0, x_1, \ldots, x_r)$ of K. Therefore, its Chow point p has a locus V over K, which coincides with the variety of positive cycles of dimension r and of degree 1 in L^n. It is a simple matter to see that the homogeneous co-ordinates of the point p coincide with the Plücker co-ordinates

$$p_{i_0 i_1 \ldots i_r} = \begin{vmatrix} x_{0i_0} x_{0i_1} & \ldots & x_{0i_r} \\ x_{1i_0} x_{1i_1} & \ldots & x_{1i_r} \\ \ldots & & \ldots \\ x_{ri_0} x_{ri_1} & \ldots & x_{ri_r} \end{vmatrix}$$

of L^r, where $x_{\alpha 0}, x_{\alpha 1}, \ldots, x_{\alpha n}$ are the homogeneous co-ordinates of x_α for $\alpha = 0, 1, \ldots, r$. Since the group of projective transformations in L^n operates transitively on the totality of L^r, we conclude that V is a "homeogeneous variety." Therefore, V is nonsingular and a fortiori everywhere locally normal.

We take a set of letters X_{ij} and denote by capitals $P_{i_0 i_1 \ldots i_r}$ the expressions which result if in the above determinants $p_{i_0 i_1 \ldots i_r}$ the x_{ij} are replaced by X_{ij}. Our purpose is to prove the following:

THEOREM 1 (ARITHMETIC NORMALITY). *The ring of polynomials* $\mathbf{K}[P]$ *in* $P_{i_0 i_1 \ldots i_r}$ *with coefficients in* $\mathbf{K}$ *is integrally closed.*

This can be done by a combination of some lemmas due to Hodge and stated below.

Any monomial of degree m

$$P_{i_0 i_1 \ldots i_r} P_{j_0 j_1 \ldots j_r} \ldots P_{k_0 k_1 \ldots k_r}$$

can be described by the following diagram

$$\begin{matrix} i_0 \, j_0 \ldots k_0, \\ i_1 \, j_1 \ldots k_1, \\ \ldots \quad \ldots, \\ i_r \, j_r \ldots k_r. \end{matrix}$$

We shall assume that $i_0 < i_1 < \ldots < i_r$, $j_0 < j_1 < \ldots < j_r$, etc. If we can change the order of the factors in product in such a way that $i_\alpha \leq j_\alpha \leq \ldots \leq k_\alpha$ for $\alpha = 0, 1, \ldots, r$, we call it, after A. Young, a *standard product.*

LEMMA 1. *The set of standard products of degree m forms a* $\mathbf{K}$*-linear base of the module of homogeneous elements of degree m in* $\mathbf{K}[P]$.[4]

This follows readily from the quadratic relations between $P_{i_0 i_1 \ldots i_r}$ of the form

$$\textstyle\sum_\sigma \operatorname{sgn}(\sigma) P_{i_0 \ldots i_{\alpha-1} i_\alpha^\sigma \ldots i_r^\sigma} P_{j_0^\sigma \ldots j_\alpha^\sigma j_{\alpha+1} \ldots j_r} = 0,$$

where σ runs over the $\binom{r+2}{\alpha+1}$ permutations of $(i_\alpha \ldots i_r j_0 \ldots j_\alpha)$ such that $i_\alpha^\sigma < \ldots < i_r^\sigma$, $j_0^\sigma < \ldots < j_\alpha^\sigma$.

LEMMA 2. *Let*

$$F(P) = \textstyle\sum c_{ij \ldots k} P_{i_0 i_1 \ldots i_r} P_{j_0 j_1 \ldots j_r} \ldots P_{k_0 k_1 \ldots k_r}$$

be a linear form in the standard products of degree m with coefficients in $\mathbf{K}$, *which vanishes after the substitution*

$$X_{ij} = X_{rr} = 0 \ (i > j).$$

Then $c_{ij} \ldots {}_k = 0$ *for* $(i_0 i_1 \ldots i_r) \neq (01 \ldots r)$.[5]

This can be seen by examining the coefficient of the smallest power of T after the substitution $X_{ij} = Y_{ij}T^{(n+1-i)(n+1-i+j)}$. The same reasoning shows that the standard products of degree m are linearly independent over $\mathbf{K}$.

We are now in a position to prove our Theorem 1. Let G be a homogeneous element of the integral closure of $\mathbf{K}[P]$. Then G is necessarily contained in this ring when $\deg(G) \geq m$, where m does not depend on G.[6] We assume that m is the smallest integer of this nature, and we shall show that $m = 0$. Otherwise we can find a homogeneous element G of degree $m - 1$ in the integral closure of $\mathbf{K}[P]$, which is not contained in this ring. Since G is an element of $\mathbf{K}(P)$ which is integral over $\mathbf{K}[P]$, it is a fortiori an element of $\mathbf{K}(X)$ which is integral over $\mathbf{K}[X]$. Since $\mathbf{K}[X]$ is integrally closed, G can be written uniquely as a polynomial $G(X)$ in X_{ij} with coefficients in $\mathbf{K}$. On the other hand $P_{01} \ldots {}_r G$ is a homogeneous element of degree m in the integral closure of $\mathbf{K}[P]$, whence it can be written in the form

$$P_{01} \ldots {}_r G = \sum c_{ij} \ldots {}_k P_{i_0 i_1} \ldots {}_{i_r} \ldots$$

as a linear combination of standard products of degree m, by Lemma 1. Let $\sum = P_{01} \ldots {}_r \, G'(P) + \sum''$ be the decomposition of $\sum$ such that $\sum''$ is the part of $\sum$ no term of which contains $P_{01} \ldots {}_r$. Then $P_{01} \ldots {}_r \, [G - G'(P)] = \sum''$ is a polynomial in X_{ij}, which vanishes after the substitution in Lemma 2. Therefore, by the same lemma, each coefficient $c_{ij} \ldots {}_k$ of $\sum''$ must be zero; whence $G = G'(P)$, completing the proof.

THEOREM 2 (SEVERI'S THEOREM). *If W is a positive divisor on V, then W can be expressed as an intersection-product of V and a hypersurface in the ambient space of V.*

In fact, by the arithmetic normality of V, the intersection-products $V.H_m$ of V and hypersurfaces H_m of degree m in the ambient space of V not containing V form a complete linear system on V.[6] On the other hand, by a result of the author, the algebraic family of positive divisors of a given degree on V is linear,[7] whence our theorem follows.

2. *Invariant Theoretic Meaning.*—We shall now consider the relation of the arithmetic normality of the Grassmann variety and the first main theorem for the unimodular group. We prove the following theorem, which is stronger than Theorem 1 but in the proof of which we use Theorem 1:

THEOREM 3. *If* $\mathbf{K}[X]$ *is the ring of polynomials in* X_{ij} *with coefficients in* $\mathbf{K}$, *then* $\mathbf{K}[X] \cap \mathbf{K}(P) = \mathbf{K}[P]$.

We have only to show that every polynomial $F(X)$ contained in $\mathbf{K}(P)$ is integral over $\mathbf{K}[P]$. If $F(X) = \sum_\rho F_\rho(X)$ is the decomposition of $F(X)$ into homogeneous parts, then, for any quantity t, $F(tX) = \sum_\rho t^\rho F_\rho(X)$ is contained in $\mathbf{K}[X] \cap \mathbf{K}(P)$. Therefore, we may assume that $F(X)$ itself is a homogeneous polynomial in the $(r + 1)(n + 1)$ letters X_{ij}. Let V^* be the "representative cone" of V and let F^* be the function defined by $F(X)$ on V^*. If $F(X)$ is not integral over $\mathbf{K}[P]$, the function F^* has at least one polar variety W^*, which is also a cone, of dimension one less than the dimension of V^*.[8] Let K be a field of definition of F^* containing the coefficients of $F(X)$; let $p'_{i_0 i_1 \ldots i_r}$ be the co-ordinates of a generic point p'^* of W^* over $\bar{K}$ and t a variable over $K(p'^*)$. Without loss of generality, we may assume that $p'_{01 \ldots r} \neq 0$. Then the set of $r + 1$ points $x'_0, x'_0, \ldots, x'_r$ with homogeneous co-ordinates

$$x'_{\alpha j} = tp'_{01 \ldots \alpha-1, \alpha+1 \ldots r, j} \ (0 \leq j \leq n)$$

for $\alpha = 0, 1, \ldots, r$ also determines a generic point p''^* of W^* over $\bar{K}$. However, $F(X)$ is finite at x', but F^* is not so at p''^*—a contradiction.

We note that Theorem 1 and Theorem 3 hold actually for any reference field. The following theorem is the *first main theorem of covariant vector invariants for* $SL(r + 1)$:

THEOREM 4. *If $F(X)$ is a polynomial in X_{ij} with coefficients in a field K which is invariant under unimodular substitutions*

$$X'_{ij} = \sum_{k=o}^{r} a_{ik} X_{kj},$$

then $F(X)$ can be expressed as a polynomial in the $P_{i_0 i_1 \ldots i_i}$ with coefficients in K.

Proof: Since each homogeneous part of $F(X)$ is invariant, we may assume that $F(X)$ itself is homogeneous. In this case we conclude from the absolute irreducibility of a determinant that we have $F(\ldots, \sum_{k=0}^{r} a_{ik} X_{kj}, \ldots) = \det(a_{ik})^m F(X)$ with some nonnegative integer m. Therefore, $F(X)$ is an element of $K(P)$; whence our assertion follows by Theorem 4.

We note that Theorem 4 is apparently stronger than Theorem 3.

Finally, we note that $SL(r + 1)$ is generated by the following set of $r(r + 1)$ substitutions:

$$X'_i = X_i \quad (i \neq i_0),$$
$$X'_{i_0} = X_{i_0} + t_{i_0 j_0} X_{j_0} \ (i_0 \neq j_0).$$

In the case of characteristic zero the invariance of $F(X)$ under such substitutions is expressed simply by first "polarizations,"

$$\sum_{j=0}^{n} X_{j_0 j} . \partial F / \partial X_{i_0 j} = 0 \ (i_0 \neq j_0),$$

and this was the starting point of Severi's algebraic proof.

3. *Application to Chow Forms.*—We add here an application of the first main theorem for the unimodular group to a general theorem in algebraic geometry. Let A^r be a variety of r dimension in L^n, which is defined over a field K; let y be a generic point of A over K and let L^{n-1} be the hyperplane in the "dual space" of L^n, which corresponds to y. There exists one and only one subvariety T in the product of A and $(r + 1)$-ply projective space $L^n \times L^n \times \ldots \times L^n$, which is defined over K, such that $T.(y \times L^n \times \ldots \times L^n) = y \times L^{n-1} \times \ldots \times L^{n-1}$. This variety does not depend on K and y when A is given. It is a simple matter to see that $pr_{L^n \times \ldots \times L^n}(T)$ is a variety of $(r + 1)n - 1$ dimension. Therefore, if we denote by U_{ij} a set of letters to describe a variety in $L^n \times L^n \times \ldots \times L^n$, the above algebraic projection can be defined by a single absolutely irreducible $(r + 1)$-ply homogeneous polynomial $F(U)$. This polynomial is uniquely determined up to a constant by A and is called the *Chow form of A*.[9] The Chow form of any positive cycle of r dimension in L^n can be defined by the corresponding product of Chow forms of its irreducible components. We shall now prove the following assertion:

THEOREM 5. *Any Chow form $F(U)$ of a positive cycle of dimension r and of degree m in L^n can be expressed as a homogeneous polynomial of degree m in the determinants*

$$\begin{vmatrix} U_{0i_0}U_{0i_1} \dots U_{0i_r} \\ U_{1i_0}U_{1i_1} \dots U_{1i_r} \\ \dots \quad \dots \\ U_{ri_0}U_{ri_1} \dots U_{ri_r} \end{vmatrix}.$$

Proof: We may assume that the given cycle A^r is a variety. If $U'_{ij} = \sum_{k=0}^{n} a_{ik}U_{kj}$ is any unimodular substitution, $F'(U) = F(U')$ is also absolutely irreducible and defines the same variety A^r as $F(U)$. Therefore, $F(U)$ is at least a relative hence an absolute invariant, of $SL(r + 1)$. On the other hand, it is a simple matter to see that $F(U)$ is of degree m in each $U_{\alpha j}$ $(0 \leq j \leq n)$ for $\alpha = 0, 1, \dots, r$; whence our assertion follows by Theorem 4.

It follows from this theorem that we can define a one-to-one "birational" transformation from the set of positive divisors of dimension r and of degree m into the complete linear system of hypersurface sections of degree m of the Grassmann variety V.

* This work was supported by a research project at Harvard University, sponsored by the Office of Ordnance Research, U.S. Army.

1 F. Severi, "Sulla varietà che rappresenta gli spazi subordinati di data dimensione immersi in uno spazo lineare," *Ann. di Mat.*, Vol. **24** (1915).

2 H. Weyl, *Classical Groups* (Princeton: Princeton University Press, 1946), **1**, 45.

3 A. Weil, *Foundations of Algebraic Geometry* ("Am. Math. Soc. Colloquium Publications," Vol. **29** [1946]).

4 W. V. D. Hodge, "Some Enumerative Results in the Theory of Forms," *Proc. Cambridge Phil. Soc.*, **39**, 24–26 (1943).

5 *Ibid.*, pp. 25–27.

6 O. Zariski, "Complete Linear Systems on Normal Varieties and a Generalization of a Lemma of Enriques-Severi," *Ann. Math.*, **55**, 563 (1952).

7 The theory of Grassmann varieties over fields of arbitrary characteristic will be published as a preliminary part of a theory of canonical classes of nonsingular projective models.

8 Weil, *op. cit.*, p. 270.

9 W. L. Chow and B. L. van der Waerden, "Ueber zugeordnete Formen und algebraische Systeme von algebraischen Mannigfaltigkeiten," *Math. Ann.*, Vol. **113** (1937).

ON KÄHLER VARIETIES OF RESTRICTED TYPE*

BY K. KODAIRA

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by S. Bochner, February 23, 1954

1. This is a preliminary report of a paper[1] concerning Kähler varieties of restricted type. A compact complex analytic variety V of complex dimension $n \geqq 2$ is called a *Kähler variety of restricted type*[2] or a *Hodge variety*[3] if V carries a Kähler metric $ds^2 = 2 \sum g_{\alpha\bar{\beta}} (dz^\alpha \, d\bar{z}^\beta)$ such that the associated exterior form $\omega = i \sum g_{\alpha\bar{\beta}} dz^\alpha \, d\bar{z}^\beta$ belongs to the cohomology class of an *integral* 2-cocycle on V. In what follows, such a metric will be called a *Hodge metric*. It is well known that *every non-*