INTEGRAL RATIOS OF FACTORIALS AND ALGEBRAIC HYPERGEOMETRIC FUNCTIONS

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Chebychev in his work on the distribution of primes numbers used the following fact

$$u_n := \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z}, \qquad n = 0, 1, 2, \ldots$$

This is not immediately obvious (for example, this ratio of factorials is not a product of multinomial coefficients) but it is not hard to prove. The only proof I know proceeds by checking that the valuations $v_p(u_n)$ are non-negative for every prime p; an interpretation of u_n as counting natural objects or being dimensions of natural vector spaces is far from clear.

As it turns out, the generating function

$$u := \sum_{\nu \geq 1} u_n \lambda^n$$

is algebraic over $\mathbb{Q}(\lambda)$; i.e. there is a polynomial $F \in \mathbb{Z}[x, y]$ such that

$$F(\lambda, u(\lambda)) = 0.$$

However, we are not likely to see this polynomial explicitly any time soon as its degree is $483,840$ (!)

What is the connection between u_n being an integer for all n and u being algebraic? Consider the more general situation

$$u_n := \prod_{\nu \geq 1} (\nu n)!^{\gamma_\nu},$$

where the sequence $\gamma = (\gamma_\nu)$ for $\nu \in \mathbb{N}$ consists of integers which are zero except for finitely many.

We assume throughout that γ is *regular*, i.e.,

$$\sum_{\nu \geq 1} \nu \gamma_\nu = 0,$$

which, by Stirling's formula, is equivalent to the generating series $u := \sum_{\nu \geq 1} u_n \lambda^n$ having finite non-zero radius of convergence. We define the *dimension* of γ to be

$$d := -\sum_{\nu \geq 1} \gamma_\nu.$$

To abbreviate, we will say that γ is *integral* if $u_n \in \mathbb{Z}$ for every $n = 0, 1, 2, \ldots$.

We can now state the main theorem of the talk.

Theorem 1. *Let $\gamma \neq 0$ be regular; then u is algebraic if and only if γ is integral and $d = 1$.*

One direction is fairly straightforward. If u is algebraic, by a theorem of Eisenstein, there exists an $N \in \mathbb{N}$ such that $N^n u_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. It is not hard to see that in our case if such an N exists then it must equal 1. To see that $d = 1$ we need to introduce the *monodromy representation*.

The power series u satisfies a linear differential equation $Lu = 0$. After possibly scaling λ this equation has singularities only at $0, 1$ and ∞. Indeed, u is a hypergeometric series. Moreover, these singularities are regular singularities precisely because we assumed γ to be regular.

If we let V be the space of local solutions to $Lu = 0$ at some base point not $0, 1$ or ∞ then analytic continuation gives a representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \longrightarrow GL(V).$$

We let the *monodromy group* Γ be the image of ρ and let B, A, σ be the monodromies around $0, \infty, 1$, respectively, with orientations chosen so that $A = B\sigma$. The main use of the monodromy group for us is the fact that u is algebraic if and only if Γ is finite.

As it happens the multiplicity of the eigenvalue 1 for B is d and it is also true that the corresponding Jordan block of B is of size d. Hence, Γ is not finite if $d > 1$.

To prove the converse we appeal to the work of Beukers and Heckman [1] who extended Schwartz work and described all algebraic hypergeometric functions. Let p and q be the characteristic polynomials of A and B respectively. In our situation p and q are relatively prime polinomials in $\mathbb{Z}[x]$ (which are products of cyclotomic polynomials). Their work tells us that Γ is finite if and only if the roots of p and q interlace in the unit circle.

The key step in the proof of this beautiful fact is to determine when Γ fixes a non-trivial positive definite Hermitian form H on V (which guarantees that Γ is compact). I explained in my talk how H can be defined using a variant of a construction going back to Bezout. Consider the two variable polynomial

$$\frac{p(x)q(y) - p(y)q(x)}{x - y} = \sum_{i,j} B_{i,j} x^i y^k$$

and define the *Bezoutian* of p and q as

$$\mathrm{Bez}(p, q) = (B_{i,j}).$$

We need two facts about this matrix. First, the determinant of $\mathrm{Bez}(p, q)$ equals the resultant of p and q (in passing I should mention that this is a useful fact computationally since the matrix is of smaller size than the usual Sylvester matrix). Second, note that $\mathrm{Bez}(p, q)$ is symmetric. Hence it carries more information than just its determinant as it defines a quadratic form H. It is a classical fact (due to Hermite and Hurwitz) that the signature of H has a topological interpretation.

Consider the continuous map $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ given by the rational function p/q. Since $\mathbb{P}^1(\mathbb{R})$ is topologically a circle we have $H^1(\mathbb{P}^1(\mathbb{R}), \mathbb{Z}) \simeq \mathbb{Z}$ and the induced map $H^1(\mathbb{P}^1(\mathbb{R}), \mathbb{Z}) \to H^1(\mathbb{P}^1(\mathbb{R}), \mathbb{Z})$ is multiplication by some integer s, which is none other than the signature of H. In particular, H is definite if and only if the roots of p and q interlace on $\mathbb{R}$. A twisted form of this construction and analogous signature result can be applied to the hypergeometric situation; in this way we recover the facts about the Hermitian form fixed by Γ proved by Beukers and Heckman.

Finally, to make the connection with the integrality of γ we define the *Landau function*

$$\mathcal{L}(x) := -\sum_{\nu \geq 1} \gamma_\nu \{\nu x\}, \qquad x \in \mathbb{R}$$

where $\{x\}$ denotes fractional part. It is simple to verify that

$$v_p(u_n) = \sum_{k \geq 1} \mathcal{L}\left(\frac{n}{p^k}\right).$$

Landau [2] proved a nice criterion for integrality: γ is integral if and only if $\mathcal{L}(x) \geq 0$ for all $x \in \mathbb{R}$.

Write

$$p(t) = \prod_{j=1}^{r} (t - e^{2\pi i \alpha_j}), \qquad q(t) = \prod_{j=1}^{r} (t - e^{2\pi i \beta_j}),$$

where $r = \dim V$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r < 1$ and $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_r < 1$ are rational.

The function $\mathcal{L}$ satisfies a number of simple properties: it is locally constant (by regularity), periodic modulo 1, right continuous with discontinuity points exactly at $x \equiv \alpha_j \bmod 1$ or $x \equiv \beta_j \bmod 1$ for some $j = 1, \ldots, r$ and takes only integer values. More precisely,

$$\mathcal{L}(x) = \#\{j \mid \alpha_j \leq x\} - \#\{j \mid 0 < \beta_j \leq x\}.$$

Away from the discontinuity points of $\mathcal{L}$ we have

$$\mathcal{L}(-x) = d - \mathcal{L}(x).$$

In particular, $\mathcal{L}(x) \geq 0$ if and only if $\mathcal{L}(x) \leq d$.

It is now easy to verify that if $d = 1$ and $\mathcal{L}(x) \geq 0$ then the roots of p and q must necessarily interlace on the unit circle finishing the proof. (Some further elaboration would also yield the other implication in the theorem independently of our previous argument.)

As a final note, let me mention that the examples in the theorem are a case of the ADE phenomenon; up to the obvious scaling $n \mapsto dn$ for some $d \in \mathbb{N}$, they come in two infinite families A and D, which are easy to describe, and some sporadic ones (10 of type E_6, 10 of type E_7 and 30 of type E_8).

References

[1] F. Beukers and G. Heckman *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95** (1989), 325–354.

[2] E. Landau *Sur les conditions de divisibilité d'un produit de factorielles par un autre.* Collected works, I, p. 116, Thales-Verlag, Essen, 1985.

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