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A remark on Apéry's numbers

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Abstract

The Apéry numbers, introduced in Apéry's celebrated proof of the irrationality of $\zeta(3)$, are defined by $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. They have the following nice property: if p is a prime number, and $n = \sum n_j p^j$ is the base p expansion of n, then $a_n \equiv \prod a_{n_j} \bmod p$. In a paper which appeared in this journal (64 (1995)11–19), C. Radoux asserted that the same property holds, provided $p \geqslant 5$, if p is replaced by p^2 both for the base and for the congruence, and if p is replaced by p^3 both for the base and for the congruence. We show that these two statements are not correct.

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The Apéry numbers are defined by

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and have been introduced by Apéry in his celebrated proof of the irrationality of $\zeta(3)$. Many authors have studied the values of these numbers modulo a prime or a prime power. In particular, several conjectures were made in [2]. These conjectures were proved by Gessel [3]; some of them proved by Radoux [5] appeared earlier, but considering the submission dates, priority should be given to [3].

An interesting relation satisfied by the numbers a_n (see [3, 5]) is that, if the expansion of the integer n in base p is given by $n = \sum_k n_k p^k$, where $0 \leqslant n_k \leqslant p-1$ (and of course only a finite number of the n_k's are non-zero), then

$$a_n \equiv \prod_k a_{n_k} \bmod p. \tag{1}$$

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This relation is linked to the famous Lucas property: if $n=\sum_k n_k p^k$ and $m=\sum_k m_k p^k$ are the expansions of the integers n and m in base p, where p is a prime number, then

$$\binom{n}{m} \equiv \prod_k \binom{n_k}{m_k} \bmod p.$$

This relation has been generalized to many unidimensional or bidimensional sequences, see [4, 1], and a natural question is whether such congruences hold when replacing the prime number p by, e.g., a prime power. In particular, it is stated in [6] that Eq. (1) is true when the prime number p is replaced both for the base and for the congruence by p^2 or p^3, where p is a prime number $\geqslant 5$. We prove in this note that this result is not correct.

1. The idea of a counterexample

In view of the claim in [5] that, for $n=\sum_k n_k(p^2)^k$, with $0 \leqslant n_k \leqslant p^2-1$, one would have $a_n \equiv \prod [a_{n_k} \bmod p^2$ [6, Theorem, p. 13] and [3, Theorem 4] that gives also a congruence modulo p^2, one wants to compare both results. Let us recall Theorem 4 of [3]: let p be a prime number and $0 \leqslant k < p$, then

$$a_{k+pn} \equiv (a_k + pnb_k)a_n \bmod p^2,$$

where the sequence (b_n) is defined as the solution of a recurrence equation of order 2 with polynomial coefficients resembling the one satisfied by the sequence (a_n), or by the relation

$$b_n = 2\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (H_{n+k} - H_{n-k}),$$

where the H_k's are given by $H_0=0$ and $H_k = 1+\frac{1}{2}+\cdots+1/k$. In particular, $b_0=0$, $b_1=12$, $b_2=210$, $b_3=4438$.

Let us take $n=xp^2+yp+z$, with $1 \leqslant x \leqslant p-1$, $0 \leqslant y$, $z \leqslant p-1$. Then, applying the result of [3] twice, one obtains

$$\begin{aligned} a_{xp^2+yp+z} &= a_{z+p(y+xp)} \\ &\equiv (a_z + p(y+xp)b_z)a_{y+xp} \bmod p^2 \\ &\equiv (a_z + pyb_z)a_{y+xp} \bmod p^2 \\ &\equiv (a_z + pyb_z)(a_y + pxb_y)a_x \bmod p^2 \\ &\equiv (a_z a_y + p(yb_z a_y + xa_z b_y))a_x \bmod p^2. \end{aligned}$$

Now, one notices that $n=xp^2+yp+z$ is a two-digit number in base p^2 (the rightmost digit being $yp+z$ and the leftmost digit being x). Hence, applying the claim in [6] and again the result of [3], one can write

$$\begin{aligned} a_{xp^2+(yp+z)} &\equiv a_x a_{yp+z} \bmod p^2 \\ &\equiv a_x(a_z + pyb_z)a_y \bmod p^2. \end{aligned}$$

Comparing the two expressions for a_{xp^2+yp+z} one should have

$$pxa_zb_ya_x \equiv 0 \bmod p^2,$$

i.e.,

$$xa_zb_ya_x \equiv 0 \bmod p.$$

This permits to construct infinitely many counterexamples, namely, take $x=y=z=1$, hence $a_x=a_z=a_1=5$, and $b_y=b_1=12$, which gives $xa_zb_ya_x=300$. The contradiction results in taking any prime number not dividing 300, e.g. $p=7$.

Proposition. *If* $n=57$, *i.e.*, $n=\overline{18}$ *in base* 49, *then* $a_n=a_{57}\equiv 13 \bmod 49$ *and* $a_1a_8\equiv 20 \bmod 49$.

Remarks.

- The result above can be checked using `Maple` for example.
- In the same spirit one can find counterexamples to the Theorem in [6] where the base and the congruence are equal to p^3. For example, take $n=400$ and $p=7$. In base $7^3=343$, this number has two digits (the rightmost being 57, the leftmost being 1). Hence, if the statement of [6] were true, one would have $a_{400}\equiv a_1a_{57} \bmod 7^3$ hence, a fortiori, $a_{400}\equiv a_1a_{57} \bmod 49$.

  On the other hand, using Theorem 4 of [3], one has

$$a_{400}=a_{1+7\times 57}\equiv (a_1+7\times 57b_1)a_{57} \bmod 49.$$

Both relations would imply that $7\times 57b_1a_{57}\equiv 0 \bmod 49$, i.e., $57b_1a_{57}\equiv 0 \bmod 7$. But we saw that $a_{57}\equiv 13 \bmod 49$ and we have that $b_1=12$, hence the desired contradiction.

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