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DEGENERATE GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper we study terminating and ill-defined Gauss hypergeometric functions. For their hypergeometric equations, the set of 24 Kummer's solutions degenerates. We describe those solutions and relations between them.

1. Introduction

Throughout the paper, let $\mathbb{Z}_{\leq 0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of non-positive and non-negative integers, respectively. The *Gauss hypergeometric function* is defined by the series

$$ {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) = 1 + \frac{ab}{c \cdot 1!} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 2!} z^2 + \cdots . \tag{1} $$

The hypergeometric series terminate when $a \in \mathbb{Z}_{\leq 0}$ or $b \in \mathbb{Z}_{\leq 0}$; the function is usually undefined when $c \in \mathbb{Z}_{\leq 0}$. If a, c are non-positive integers and $a \geq c$, then we interpret (1) as a terminating hypergeometric series. The Gauss hypergeometric function can be analytically continued onto $\mathbb{C} \setminus [1, \infty)$. It satisfies the *hypergeometric differential equation*

$$ z(1-z)\frac{d^2y(z)}{dz^2} + (c - (a+b+1)z)\frac{dy(z)}{dz} - ab\,y(z) = 0. \tag{2} $$

Throughout the paper, we denote this equation by $E(a, b, c)$.

Kummer [**Kum36**] found that, in general, there are 24 hypergeometric series which express, up to simple power factors, solutions of the hypergeometric equation (2).

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Specifically, $E(a, b, c)$ is a Fuchsian equation with three regular singular points: $z = 0$, $z = 1$ and $z = \infty$. In general, for each singular point there is a basis of solutions expressible by hypergeometric series, so that we have six distinct hypergeometric solutions. Each of the six solutions can be represented in four ways as a hypergeometric series times a power factor.

This pattern degenerates if some or all of the numbers a, b, c, $c - a$, $c - b$, $a - b$, $c - a - b$ are integers. The subject of this paper is to investigate these degenerations. Note that in classical manuals [**AS64, Erd53**], a hypergeometric equation is called degenerate only when a, b, $c - a$ or $c - b$ are integers. This is precisely the case when the monodromy group is reducible or, equivalently, when there are terminating hypergeometric solutions. Without insisting on a change in terminology, in this paper we call the hypergeometric equation (2) *degenerate* if some or all of the numbers a, b, c, $c - a$, $c - b$, $a - b$, $c - a - b$ are integers. As mentioned, this is the case when the structure of 24 Kummer's hypergeometric solutions degenerates: some hypergeometric series are terminating or undefined, relations between them degenerate. Within this setting, it looks most practical to use the term 'degenerate' broadly.

In this paper, hypergeometric solutions of degenerate hypergeometric equations and relations between them are presented very explicitly and systematically. Theorem 2.2 and Corollary 2.3 give a convenient characterization of various degenerate cases. In particular, we describe the monodromy group in the degenerate cases. Table 1 gives the degenerate patterns of hypergeometric solutions. Sections 4–9 are devoted to separate degenerate cases. In each case, we give explicit expansions for hypergeometric and logarithmic solutions, and relations between them.

Previously, bases of solutions for degenerate hypergeometric equations have been presented in [**Erd53**, Sections 2.2 and 2.3]. Some of those solutions were reproduced in [**AS64**, Section 15.5]. Logarithmic solutions have been derived in several texts, for example in [**AAR99**, pp. 82–84], although not always correctly. Generators for the monodromy group in all degenerate cases are given in [**IKSY91**, Section 4.3]. Transformations of terminating Gauss hypergeometric series are widely used (see [**KS94**, Section 0.6] for example), but they are seldom fully exhibited.

2. General observations

The local exponents of $E(a, b, c)$ at the singular points are

$$0, 1 - c \quad \text{at } z = 0; \quad 0, c - a - b \quad \text{at } z = 1; \quad \text{and} \quad a, b \quad \text{at } z = \infty.$$

Permutations of the three singular points and of their local exponents are realized by transformations $y(z) \mapsto z^{-\alpha}(1-z)^{-\beta} y(\varphi(z))$ of $E(a,b,c)$ to other hypergeometric equations, where

$$\varphi(z) \in \left\{ z, \frac{z}{z-1}, 1-z, 1-\frac{1}{z}, \frac{1}{z}, \frac{1}{1-z} \right\} \tag{3}$$

and α, β are suitably chosen from the set $\{0, a, b, 1-c, c-a-b\}$. We refer to these transformations of hypergeometric equations as the *fractional-linear transformations*.

The 24 Kummer solutions and general relations between them were fully presented in [**Erd53**, Section 2.9]. They generally represent six distinct functions, since Euler's and Pfaff's formulas [**AAR99**, Theorem 2.2.5] identify four Kummer's series with each other. We refer to those identities as the *Euler–Pfaff transformations*; we recall them in formulas (4)–(6) below.

If we consider permutation of the upper parameters a, b as a non-trivial transformation, the fractional-linear transformations form a group of 48 elements acting on hypergeometric equations. This group acts on the parameters of $E(a,b,c)$ as follows.

LEMMA 2.1. *We have the following.*

(1) *Fractional-linear transformations can permute the three numbers* $1-c$, $c-a-b$, $b-a$ *and change their signs in any way.*

(2) *Fractional-linear transformations can permute the four numbers*

$$-\tfrac{1}{2}+a, \quad \tfrac{1}{2}-b, \quad -\tfrac{1}{2}+c-a, \quad \tfrac{1}{2}+b-c,$$

in any way, and can change their signs simultaneously.

Proof. We define $e_0 = 1-c$, $e_1 = c-a-b$, $e_\infty = b-a$. Note that e_0, e_1, e_∞ are the local exponent differences of $E(a,b,c)$ at the singular points. The first statement is clear once one accepts characterization of fractional-linear transformations as the transformations of hypergeometric equations which permute the three singular points and their local exponents. (Note that interchanging local exponents at a singular point changes the sign of the local exponent difference.)

For the second statement, note that the four listed numbers are equal to

$$\frac{-e_0-e_1-e_\infty}{2}, \quad \frac{e_0+e_1-e_\infty}{2}, \quad \frac{-e_0+e_1+e_\infty}{2}, \quad \frac{e_0-e_1+e_\infty}{2}.$$

By the first statement, we can permute e_i and change their signs. If we change the signs of even number of e_i, the four numbers are permuted; otherwise, they are multiplied by -1 and permuted.

Alternatively, one can show both statements by checking the list of 24 related hypergeometric equations:

$$\begin{aligned}
E(A,B,c) &\quad \text{with } A\in\{a,c-a\},\ B\in\{b,c-b\},\\
E(A,B,2-c) &\quad \text{with } A\in\{1-a,1+a-c\},\ B\in\{1-b,1+b-c\},\\
E(A,B,1+a+b-c) &\quad \text{with } A\in\{a,1+b-c\},\ B\in\{b,1+a-c\},\\
E(A,B,1+c-a-b) &\quad \text{with } A\in\{1-a,c-b\},\ B\in\{1-b,c-a\},\\
E(A,B,1+a-b) &\quad \text{with } A\in\{a,1-b\},\ B\in\{1+a-c,c-b\},\\
E(A,B,1+b-a) &\quad \text{with } A\in\{1-a,b\},\ B\in\{c-a,1+b-c\}. \qquad \square
\end{aligned}$$

The purpose of this paper is to present solutions (and relations between them) of degenerate hypergeometric equations. These equations usually have terminating or undefined hypergeometric solutions. In particular, if the local exponent difference at a singular point is an integer, then that point is either logarithmic or there is a terminating local solution at that point. At a *logarithmic point* there is only one local solution of the form $x^{\lambda}(1+\alpha_1 x+\alpha_2 x^2+\cdots)$, where x is a local parameter there. To get a basis of local solutions at a logarithmic point, one has to use the function $\log x$.

Terminating solutions occur if a, b, $c-a$ or $c-b$ is an integer. Then the hypergeometric equation has reducible monodromy group. Conversely, if the monodromy group is reducible, the hypergeometric equation has a solution which changes by a constant multiple under any monodromy action. The logarithmic derivative of such a solution is a rational function of z, which eventually means that the solution has a terminating series expression. This is clear from the Kovacic algorithm in differential Galois theory [**Kov86; VS03**, Section 4.3.4].

We recall briefly the role of the monodromy group. This group characterizes analytic continuation of solutions of the hypergeometric equation along paths in $\mathbb{C}\setminus\{0,1\}$; see [**Beu02**, Section 3.9]. Once a basis of local solutions at a non-singular point is chosen, we get a two-dimensional representation of the monodromy group. In general, the monodromy group is generated by two elements, say, those corresponding to paths circling $z=0$ or $z=1$ once. Fractional-linear transformations of $E(a,b,c)$ do not change the monodromy group. We are especially interested in the cases when a monodromy representation is a subgroup (up to conjugation) of the following subgroups of $\mathrm{GL}(2,\mathbb{C})$:

$$\mathbb{G}_m=\left\{\begin{pmatrix}1&0\\0&u\end{pmatrix}\,\middle|\,u\in\mathbb{C}^*\right\},\quad \mathbb{G}_a=\left\{\begin{pmatrix}1&u\\0&1\end{pmatrix}\,\middle|\,u\in\mathbb{C}\right\}.$$

The groups $\mathbb{G}_m$ and $\mathbb{G}_a$ are isomorphic to the multiplicative group $\mathbb{C}^*$ and the additive group $\mathbb{C}$, respectively. For Fuchsian equations, the monodromy group is closely related to the differential Galois group [**VS03**, Theorem 5.3]. The groups $\mathbb{G}_m$, $\mathbb{G}_a$ are examples of possible differential Galois groups for hypergeometric equations.

We now give a convenient characterization of hypergeometric equations with various types of monodromy groups.

THEOREM 2.2. *Consider a hypergeometric equation $E(a, b, c)$.*

(1) *The monodromy group is irreducible if and only if none of the numbers a, b, $c - a$, $c - b$ is an integer.*

(2) *If the monodromy group is abelian, then the sequence a, b, $c - a$, $c - b$ contains at least two integers.*

(3) *Suppose that the sequence a, $1 - b$, $c - a$, $1 + b - c$ contains precisely two integers. If those two integers are either both positive or both non-positive, then a monodromy representation is a non-trivial subgroup of $\mathbb{G}_m$. Otherwise the monodromy group is not abelian.*

(4) *Suppose that the numbers a, b, c are integers. If the sequence a, b, $c - a$, $c - b$ contains odd number of positive integers, then the monodromy group is trivial. Otherwise a monodromy representation is a non-trivial subgroup of $\mathbb{G}_a$.*

Proof. The first statement is a direct consequence of [**IKSY91**, Theorem 4.3.2]. Alternatively, this is [**Beu02**, Corollary 3.11].

The second statement is [**Beu02**, Lemma 3.13]. Alternatively, one can go through cases (B.2), (B.2)$'$, (B.2)$''$ and (C) of [**IKSY91**, Theorem 4.3.3].

For the third statement, we may assume, due to fractional-linear transformations, that $a, c - a$ are integers, and that $c \geq 1$. Then we are in part (B) of [**IKSY91**, Theorem 4.3.3], with $\rho_1 - \rho_2 = c - 1$ there. We have to check whether $0 < \rho_1 + \sigma_i + \tau_j \leq \rho_1 - \rho_2$ for those $i, j \in \{1, 2\}$ with $\rho_1 + \sigma_i + \tau_j \in \mathbb{Z}$. We have $\rho_1 + \sigma_i + \tau_j \in \{a, c - a, b, c - b\}$, therefore we are in case (B.2) if the two integers are both positive, and we are in case (B.1) if $a \leq 0$ and $c - a > 0$. The monodromy generators of case (B.1) do not commute, and they are (up to conjugation) inside $\mathbb{G}_m$ in case (B.2). Fractional-linear transformations preserve the property that the two integers in the sequence $a, 1 - b, c - a, 1 + b - c$ are either both positive, or both non-positive.

For the last statement, we use part (C) of [**IKSY91**, Theorem 4.3.3]. It is enough to show that if the sequence $a, b, c - a, c - b$ contains an odd number of positive integers, then we are in case (C.2). Up to fractional-linear transformations, we may assume that $a \leq 0$ and that the other three integers are positive. Then $a \leq 0 < b < c$.

We have $\rho_1 - \rho_2 = c - 1$, $\sigma_1 - \sigma_2 = c - a - b$, and for $i, j \in \{1, 2\}$ we have

$$\rho_1 + \sigma_i + \tau_j \in \{a, c - a, b, c - b\}, \quad \rho_i + \sigma_1 + \tau_j \in \{1 - a, c - b, c - a, 1 - b\}.$$

We check that $0 < \rho_1 + \sigma_i + \tau_j \le \rho_1 - \rho_2$ for any $i, j \in \{1, 2\}$ in the context of part (B) in [**IKSY91**], and $0 < \rho_i + \sigma_1 + \tau_j \le \sigma_1 - \sigma_2$ for any $i, j \in \{1, 2\}$ in the context of part (B)$'$. This means that we are indeed in case (C.2). Fractional-linear transformations preserve the property that the sequence $a, b, c - a, c - b$ contains an odd number of positive integers. □

COROLLARY 2.3. *We have the following.*

(1) *A monodromy representation of hypergeometric equation $E(a, b, c)$ is (up to conjugation) a non-trivial subgroup of $\mathbb{G}_m$ if and only if the sequence a, $1 - b$, $c - a$, $1 + b - c$ contains precisely two integers, and those two integers are either both positive or both non-positive.*

(2) *The monodromy group of $E(a, b, c)$ is trivial if and only if $a, b, c \in \mathbb{Z}$ and the sequence a, b, $c - a$, $c - b$ contains an odd number of positive integers.*

(3) *The monodromy representation of $E(a, b, c)$ is (up to conjugation) a non-trivial subgroup of $\mathbb{G}_a$ if and only if $a, b, c \in \mathbb{Z}$ and the sequence a, b, $c - a$, $c - b$ contains an even number of positive integers.*

Proof. Parts (2), (3) and (4) of Theorem 2.2 describe mutually exclusive cases for possible abelian monodromy groups of $E(a, b, c)$. □

In Sections 4–9 below we study various degenerate cases of Gauss hypergeometric functions. We concentrate on relations between their hypergeometric and logarithmic solutions. Here we present general forms of hypergeometric equations for each degeneration type.

THEOREM 2.4. *We have the following.*

(1) *Suppose that a hypergeometric equation has terminating hypergeometric solutions, non-abelian monodromy, and does not have logarithmic points. Up to fractional-linear transformations, the hypergeometric equation has the form $E(-n, a, c)$, where $n \in \mathbb{Z}_{\ge 0}$ and $a, c, c - a \notin \mathbb{Z}$.*

(2) *Suppose that a hypergeometric equation has logarithmic points, but does not have terminating hypergeometric solutions. Up to fractional-linear transformations, the hypergeometric equation has the form $E(a, b, m + 1)$, with $m \in \mathbb{Z}_{\ge 0}$ and $a, b \notin \mathbb{Z}$.*

(3) *Suppose that a hypergeometric equation has terminating solutions and logarithmic points, and that the monodromy group is not abelian. Up to fractional-linear transformations, the hypergeometric equation has the form* $E(a, -n, m+1)$, *with* $n, m \in \mathbb{Z}_{\geq 0}$ *and* $a \notin \mathbb{Z}$.

(4) *Suppose that a monodromy representation of a hypergeometric equation is a non-trivial subgroup of* $\mathbb{G}_m$. *Up to fractional-linear transformations, the hypergeometric equation has the form* $E(-n, a-m, -n-m)$, *with* $n, m \in \mathbb{Z}_{\geq 0}$ *and* $a \notin \mathbb{Z}$.

(5) *Up to fractional-linear transformations, a hypergeometric equation with the trivial monodromy group has the form* $E(-n, \ell+1, -n-m)$, *with* $n, m, \ell \in \mathbb{Z}_{\geq 0}$.

(6) *Suppose that a monodromy representation of a hypergeometric equation is a non-trivial subgroup of* $\mathbb{G}_a$. *Up to fractional-linear transformations, the hypergeometric equation has the form* $E(-\ell, -n-\ell, -n-m-2\ell)$, *with* $n, m, \ell \in \mathbb{Z}_{\geq 0}$.

Proof. Consider a hypergeometric equation $E(a, b, c)$ with no restrictions on the parameters a, b, c. In the first case, we may assume $b = -n \in \mathbb{Z}_{\leq 0}$. Since the monodromy group is not abelian and there are no logarithmic points, the local exponent differences $1-c, a+n, c-a+n$ are not integers.

In the second case, we may choose $z = 0$ as a logarithmic point. Then c is an integer, and we can choose it to be positive. Since there are no terminating solutions, a and b are not integers by part (1) of Theorem 2.2.

In the third case, we choose $z = 0$ as a logarithmic point as well, so c is a positive integer. Terminating solutions occur if a or b is an integer. Assume that b is an integer. Then $a \notin \mathbb{Z}$ by part (4) of Theorem 2.2, and either $b \leq 0$ or $c - b \leq 0$ by part (3) of the same theorem. While keeping c positive, we can permute b and $c - b$ by part (2) of Lemma 2.1, so we may assume that $b \leq 0$.

In the fourth case, we may assume $a = -n$ and use part (1) of Corollary 2.3, so one of the numbers $1-b, c-a, 1+b-c$ is a non-positive integer $-m$. Due to fractional-linear transformations we may assume that $c - a = -m$. We are allowed to rename b to $a - m$, for the purpose of symmetric presentation in Section 7.

In the fifth case, the numbers a, b, c are integers by part (2) of Corollary 2.3. One of the numbers in the sequence $a, b, c-a, c-b$ has different positivity than the others. Up to fractional-linear transformations, we may assume that $b = \ell + 1$ is a positive integer, and $a = -n$, $c - a = -m$ are non-positive integers.

TABLE 1. Kummer's solutions in degenerate cases.

Case	Kummer's series	Terminating solutions	Non-terminating solutions
(1)	24	6+6	4, 4, 4
(2)	12, 16 or 20	—	4, 4, 4 (and possibly 4, 4)
	10, 13 or 16	—	3, 3, 4 (and possibly 3, 3)
	6, 8 or 10	—	2, 2, 2; or 2, 3, 3; or 2, 2, 3, 3
(3)	16 or 20	6+4 or 8+4	3, 3 or 4, 4
(4)	24	6+4, 6+4	4
(5)	24	6+2, 6+2, 6+2	—
(6)	10, 13 or 16	6+2; or 8+2; or 10+2	2; or 3; or 4

In the last case, the numbers a, b, c are integers by part (3) of Corollary 2.3. The four numbers in part (2) of Lemma 2.1 cannot all be negative, since their sum is zero. We may assume that $a, b, c-a, c-b$ are all non-negative, and that $c-a \le b \le a \le 0$. Therefore we may set $a = -\ell$, $b-a = -n$ and $c-a-b = -m$. $\Box$

Table 1 describes concisely the set of Kummer's solutions in each degenerate case. The case numbers refer to Theorem 2.4. The second column gives the total number of distinct well-defined Kummer's series. In the third column, each terminating solution is represented by an additive expression, where the first integer gives the number of terminating hypergeometric expressions for the solution, and the second integer gives the number of non-terminating expressions for the same solution. The last column specifies solutions which have only non-terminating hypergeometric expressions; these solutions usually have four hypergeometric expressions due to Euler–Pfaff transformations. Multiple subcases are commented promptly below in the following section. We consider only relations between the 24 Kummer's solutions, so we do not take into account quadratic or higher degree transformations, nor do we consider artificial identities with terminating series such as

$$ {}_2\mathrm{F}_1\left(\begin{matrix} a, -1 \\ b \end{matrix} \,\middle|\, z\right) = {}_2\mathrm{F}_1\left(\begin{matrix} c, -1 \\ d \end{matrix} \,\middle|\, \frac{ad}{bc}z\right). $$

In particular, we consider two constant terminating ${}_2\mathrm{F}_1$ series (that is, series with a zero upper parameter) as distinct if other parameters or the argument are not equal. The correctness of Table 1 is evident from the detailed considerations in Sections 4–9.

3. Some explicit facts

Here we present some formulas which are useful in the following sections. In particular, we discuss degenerations of Euler–Pfaff transformations, and introduce an alternative normalization of Gauss hypergeometric function.

Recall that Euler–Pfaff transformations [**AAR99**, Theorem 2.2.5] identify the following hypergeometric series:

$$ {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) = (1-z)^{c-a-b}\,{}_2\mathrm{F}_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix} \,\middle|\, z\right) \tag{4} $$

$$ = (1-z)^{-a}\,{}_2\mathrm{F}_1\left(\begin{matrix} a, c-b \\ c \end{matrix} \,\middle|\, \frac{z}{z-1}\right) \tag{5} $$

$$ = (1-z)^{-b}\,{}_2\mathrm{F}_1\left(\begin{matrix} c-a, b \\ c \end{matrix} \,\middle|\, \frac{z}{z-1}\right). \tag{6} $$

These transformations hold when $c \notin \mathbb{Z}_{\le 0}$. If the local exponent differences $c-a-b$ and $a-b$ are non-zero, then these four series are distinct. If one of these local exponent differences is zero, we have three distinct series here. If $b = a$ and $c = 2a$, we have only two distinct series in (4)–(6).

If none of the four series (4)–(6) is terminating, i.e. $a, b, c-a, c-b \notin \mathbb{Z}_{\le 0}$, then no Kummer's series at $z = 1$, $z = \infty$ are equal to the left-hand side of (4). To see this, one may consider a solution basis and connection formulas in the general case [**Erd53**, Section 2.9] and in case (2) of Table 1 (see Section 5).

We formulate a general conclusion as follows. Let F denote a Gauss hypergeometric function with the argument $\varphi(z)$ as in (3), so that $\varphi(z)$ is a local parameter at a point $P \in \{0, 1, \infty\}$. Let $n(F) \in \{0, 1, 2\}$ denote the number of points in the set $\{0, 1, \infty\} \setminus \{P\}$ where the local exponent difference for the corresponding hypergeometric equation is equal to zero. Then we have exactly $4 - n(F)$ distinct hypergeometric series expressions for F, unless F can be expressed as a terminating series.

Now multiple cases of Table 1 can be better clarified. Recall that points with the zero local exponent difference are always logarithmic. In case (2) we can have one, two or three logarithmic points, and above that we may have no singular points with the zero local exponent difference (the first line), one such point (the second line), or two or three such points (the third line). In case (3) we have one logarithmic point where the local exponent difference can be zero. In case (6) we have two logarithmic points.

On a few occasions we use the following normalization of Gauss hypergeometric series:

$$\mathbf{F}\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) := \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(1+k)}. \tag{7}$$

This series is well defined when $a, b, c \notin \mathbb{Z}_{\le 0}$; in this case

$$\mathbf{F}\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right).$$

If $c = -N \in \mathbb{Z}_{\le 0}$ and $a, b \notin \mathbb{Z}_{\le 0}$, then we should take limits of the summands with singular gamma values:

$$\mathbf{F}\left(\begin{matrix} a, b \\ -N \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(a+N+1)\Gamma(b+N+1)}{(N+1)!} {}_2\mathrm{F}_1\left(\begin{matrix} a+N+1, b+N+1 \\ N+2 \end{matrix} \,\middle|\, z\right).$$

If one of the parameters a, b is a non-positive integer, but c is a smaller or equal integer, then the function in (7) is also well defined if we agree to evaluate quotients of singular gamma values by taking residues there. Then we have the following formulas.

LEMMA 3.1. *Suppose that $n, N \in \mathbb{Z}_{\ge 0}$, and that $n \le N$. Then*

$$\begin{aligned} \mathbf{F}\left(\begin{matrix} -n, a \\ -N \end{matrix} \,\middle|\, z\right) &= (-1)^{N-n} \frac{\Gamma(a)N!}{n!} {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ -N \end{matrix} \,\middle|\, z\right) \\ &\quad + \frac{\Gamma(a+N+1)(N-n)!}{(N+1)!} z^{N+1} {}_2\mathrm{F}_1\left(\begin{matrix} N-n+1, a+N+1 \\ N+2 \end{matrix} \,\middle|\, z\right). \end{aligned} \tag{8}$$

Correct versions of Euler–Pfaff transformations are

$$\begin{aligned} {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ -N \end{matrix} \,\middle|\, z\right) &= (1-z)^{-a+n-N} \frac{(-1)^n (N-n)!}{N!\Gamma(-a-N)} \mathbf{F}\left(\begin{matrix} n-N, -a-N \\ -N \end{matrix} \,\middle|\, z\right) && (9) \\ &= (1-z)^{-a} \frac{(-1)^n (N-n)!}{N!\Gamma(a)} \mathbf{F}\left(\begin{matrix} n-N, a \\ -N \end{matrix} \,\middle|\, \frac{z}{z-1}\right) && (10) \\ &= (1-z)^n {}_2\mathrm{F}_1\left(\begin{matrix} -n, -a-N \\ -N \end{matrix} \,\middle|\, \frac{z}{z-1}\right). && (11) \end{aligned}$$

Proof. Formula (8) is straightforward. For formula (9), we put $b = -n$, $c = -\nu - n$ in general Euler's formula (4) and take the limit $\nu \to N - n$. For the other two formulas, we take the same specialization and the same limit in Pfaff's formulas (5) and (6). □

For further convenience, we set forth the following two functions:

$$H_1 = \mathbf{F}\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right), \quad H_2 = z^{1-c} \mathbf{F}\left(\begin{matrix} 1+a-c, 1+b-c \\ 2-c \end{matrix} \,\middle|\, z\right). \tag{12}$$

In general, they form a basis of solutions for $E(a,b,c)$. Connection formulas for other hypergeometric solutions of $E(a,b,c)$ can be written as

$$\mathbf{F}\left(\begin{matrix} a, b \\ 1+a+b-c \end{matrix} \,\middle|\, 1-z\right) = \frac{H_1 - H_2}{K}, \tag{13}$$

$$(1-z)^{c-a-b}\mathbf{F}\left(\begin{matrix} c-a, c-b \\ 1+c-a-b \end{matrix} \,\middle|\, 1-z\right) = \frac{1}{K}\left(H_1 \frac{\sin \pi a \sin \pi b}{\sin \pi(c-a) \sin \pi(c-b)} - H_2\right), \tag{14}$$

$$(-z)^{-a}\mathbf{F}\left(\begin{matrix} a, 1+a-c \\ 1+a-b \end{matrix} \,\middle|\, \frac{1}{z}\right) = \frac{1}{\sin \pi c}(H_1 \sin \pi b + H_2 e^{i\pi c} \sin \pi(c-b)), \tag{15}$$

$$(-z)^{-b}\mathbf{F}\left(\begin{matrix} 1+b-c, b \\ 1+b-a \end{matrix} \,\middle|\, \frac{1}{z}\right) = \frac{1}{\sin \pi c}(H_1 \sin \pi a + H_2 e^{i\pi c} \sin \pi(c-a)), \tag{16}$$

where

$$K = \frac{\sin \pi c}{\pi} \Gamma(1+a-c)\Gamma(1+b-c).$$

These formulas hold for analytic continuations of the **F**-functions onto the upper half-plane, like in [**Erd53**, Section 2.9].

We shall use the following consequences of the reflection formula [**AAR99**, Theorem 1.2.1] for the gamma function:

$$\psi(x) - \psi(1-x) = -\frac{\pi}{\tan \pi x}, \tag{17}$$

$$\frac{\Gamma'(x)}{\Gamma(x)^2} = \frac{\Gamma'(1-x)}{\Gamma(x)\Gamma(1-x)} - \cos \pi x \,\Gamma(1-x). \tag{18}$$

Recall that $\psi(x) = \Gamma'(x)/\Gamma(x)$.

LEMMA 3.2. *Suppose that $b \notin \mathbb{Z}$ and $c \notin \mathbb{Z}_{\le 0}$. Then for $|z| < 1$ (and, eventually, after analytic continuation) we have*

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \psi(b+k) z^k = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \psi(1-b-k) z^k - \frac{\pi}{\tan \pi b}\, {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right).$$

Proof. The formula follows by applying (17) termwise. □

4. Terminating hypergeometric series

Here we consider hypergeometric equations which have terminating hypergeometric solutions and non-abelian monodromy group, but do not have logarithmic points. We are in case (1) of Theorem 2.4. A general equation is $E(-n, a, c)$, where n is a non-negative integer, and $a, c, c-a$ are not integers. All 24 Kummer's solutions are well defined. The monodromy group of this equation is reducible, because a terminating hypergeometric solution spans an invariant subspace.

It turns out that there are terminating hypergeometric solutions at each singular point of $E(-n, a, c)$. All terminating solutions lie in the one-dimensional invariant subspace. Here is their identification:

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, a\\ c\end{matrix}\,\right|\, z\right) &= (1-z)^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, c-a\\ c\end{matrix}\,\right|\, \frac{z}{z-1}\right) && (19)\\
&= \frac{(a)_n}{(c)_n}(-z)^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, 1-n-c\\ 1-n-a\end{matrix}\,\right|\, \frac{1}{z}\right) && (20)\\
&= \frac{(a)_n}{(c)_n}(1-z)^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, c-a\\ 1-n-a\end{matrix}\,\right|\, \frac{1}{1-z}\right) && (21)\\
&= \frac{(c-a)_n}{(c)_n}z^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, 1-n-c\\ 1-n+a-c\end{matrix}\,\right|\, 1-\frac{1}{z}\right) && (22)\\
&= \frac{(c-a)_n}{(c)_n}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n, a\\ 1-n+a-c\end{matrix}\,\right|\, 1-z\right). && (23)
\end{aligned}
$$

These formulas can be proved using the following two transformations a few times: rewriting a terminating hypergeometric sum in the opposite direction, and Pfaff's formula (5). Application of Euler's formula (4) to the above series gives non-terminating hypergeometric expressions for the same function. We have six terminating and six non-terminating hypergeometric expressions for this solution.

The six expressions (19)–(23) are valid for any terminating Gauss hypergeometric series, if only the numbers a, c, $c-a$ are not integers in the interval $[1-n, 0]$. Non-terminating solutions of $E(-n, a, c)$ are

$$
\begin{gathered}
z^{1-c}(1-z)^{c-a+n}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1+n, 1-a\\ 2-c\end{matrix}\,\right|\, z\right),\\
z^{1-c}(1-z)^{c-a+n}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1+n, 1-a\\ 1+n+c-a\end{matrix}\,\right|\, 1-z\right),\\
(-z)^{-c-n}(1-z)^{c-a+n}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1+n, c+n\\ 1+n+a\end{matrix}\,\right|\, \frac{1}{z}\right).
\end{gathered}
$$

For each of these functions, there are four non-terminating hypergeometric expressions by Euler–Pfaff transformations. This exhausts the 24 Kummer's solutions. Connection relations are evident from the following formulas:

$$
\begin{aligned}
&{}_2\mathrm{F}_1\left(\begin{matrix}1+n,1-a\\1+n+c-a\end{matrix}\,\middle|\,1-z\right)\\
&\quad=\frac{(c-a)_{n+1}}{(c-1)_{n+1}}{}_2\mathrm{F}_1\left(\begin{matrix}1+n,1-a\\2-c\end{matrix}\,\middle|\,z\right)\\
&\qquad+\frac{\Gamma(1+n+c-a)\Gamma(1-c)}{\Gamma(1-a)n!}z^{c-1}(1-z)^{a-c-n}{}_2\mathrm{F}_1\left(\begin{matrix}-n,a\\c\end{matrix}\,\middle|\,z\right),\\
&{}_2\mathrm{F}_1\left(\begin{matrix}1+n,c+n\\1+n+a\end{matrix}\,\middle|\,\frac{1}{z}\right)\\
&\quad=\frac{(a)_{n+1}}{(c-1)_{n+1}}(-z)^{n+1}{}_2\mathrm{F}_1\left(\begin{matrix}1+n,1-a\\2-c\end{matrix}\,\middle|\,z\right)\\
&\qquad+\frac{\Gamma(1+n+a)\Gamma(1-c)}{\Gamma(1+a-c)n!}(-z)^{c+n}(1-z)^{a-c+n}{}_2\mathrm{F}_1\left(\begin{matrix}-n,a\\c\end{matrix}\,\middle|\,z\right).
\end{aligned}
$$

We remark that any terminating series can be interpreted as an isolated Jacobi, Meixner or Meixner–Pollaczek polynomial [**KS94**, Sections 1.7, 1.8 and 1.9]:

$$
{}_2\mathrm{F}_1\left(\begin{matrix}-n,a\\c\end{matrix}\,\middle|\,z\right)=\frac{n!}{(c)_n}P_n^{(c-1,a-c-n)}(1-2z) \tag{24}
$$

$$
=M_n\left(-a;c,\frac{1}{1-z}\right) \tag{25}
$$

$$
=\frac{n!}{(c)_n(1-z)^{n/2}}P_n^{(c/2)}\left(\frac{ic}{2}-ia;\frac{i}{2}\log(1-z)\right). \tag{26}
$$

5. General logarithmic solutions

Here we consider hypergeometric equations which have logarithmic points, but do not have terminating hypergeometric solutions. We are in case (2) of Theorem 2.4. A general hypergeometric equation of this kind is $E(a,b,m+1)$, where m is a non-negative integer, and a,b are not integers.

The functions H_1, H_2 in (12) coincide in this case. The corresponding ${}_2\mathrm{F}_1$ series either coincide (if $m=0$), or only one of them is well defined (if $m\geq 1$). Formulas for a second independent local solution at $z=0$ are not pretty, but it must be important to have them. We choose to identify logarithmic solutions with

$$U_1 = (-1)^{m+1} m! \frac{\Gamma(a-m)\Gamma(b-m)}{\Gamma(a+b-m)} {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ a+b-m \end{matrix} \,\right|\, 1-z \right). \tag{27}$$

THEOREM 5.1. *The function U_1 has the following expressions:*

$$\begin{aligned} U_1 &= (-1)^{m+1} m! \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(m+2-a-b)} \\ &\quad \times (1-z)^{m+1-a-b} {}_2\mathrm{F}_1\left(\left. \begin{matrix} m+1-a, m+1-b \\ m+2-a-b \end{matrix} \,\right|\, 1-z \right) \\ &\quad - \frac{\pi \sin \pi(a+b)}{\sin \pi a \sin \pi b} {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ m+1 \end{matrix} \,\right|\, z \right) \qquad (28) \\ &= (-1)^{m+1} m! \frac{\Gamma(a-m)\Gamma(1-b)}{\Gamma(1+a-b)} (-z)^{-a} {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, a-m \\ 1+a-b \end{matrix} \,\right|\, \frac{1}{z} \right) \\ &\quad - \frac{\pi e^{-i\pi b}}{\sin \pi b} {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ m+1 \end{matrix} \,\right|\, z \right) \qquad (29) \\ &= {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ m+1 \end{matrix} \,\right|\, z \right) \log z + \frac{(-1)^{m+1} m! (m-1)!}{(1-a)_m (1-b)_m} z^{-m} \sum_{k=0}^{m-1} \frac{(a-m)_k (b-m)_k}{(1-m)_k k!} z^k \\ &\quad + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(m+1)_k k!} (\psi(a+k) + \psi(b+k) - \psi(m+k+1) - \psi(k+1)) z^k \\ &\qquad (30) \\ &= {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ m+1 \end{matrix} \,\right|\, z \right) \log z - \frac{\pi \sin \pi(a+b)}{\sin \pi a \sin \pi b} {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ m+1 \end{matrix} \,\right|\, z \right) \\ &\quad + \frac{(-1)^{m+1} m! (m-1)!}{(1-a)_m (1-b)_m} z^{-m} (1-z)^{m+1-a-b} \sum_{k=0}^{m-1} \frac{(1-a)_k (1-b)_k}{(1-m)_k k!} z^k \\ &\quad + (1-z)^{m+1-a-b} \sum_{k=0}^{\infty} \frac{(m+1-a)_k (m+1-b)_k}{(m+1)_k k!} \\ &\quad \times (\psi(m+k+1-a) + \psi(m+k+1-b) - \psi(m+k+1) - \psi(k+1)) z^k \\ &\qquad (31) \end{aligned}$$

$$
\begin{aligned}
&= {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ m+1 \end{matrix} \,\middle|\, z\right) \log \frac{z}{1-z} - \frac{\pi}{\tan \pi b}\, {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ m+1 \end{matrix} \,\middle|\, z\right) \\
&\quad + \frac{(-1)^{m+1} m!(m-1)!}{(1-a)_m (1-b)_m} z^{-m} (1-z)^{m-a} \sum_{k=0}^{m-1} \frac{(a-m)_k (1-b)_k}{(1-m)_k k!} \frac{z^k}{(z-1)^k} \\
&\quad + (1-z)^{-a} \sum_{k=0}^{\infty} \frac{(a)_k (m+1-b)_k}{(m+1)_k k!} \\
&\quad \times (\psi(a+k) + \psi(m+k+1-b) - \psi(m+k+1) - \psi(k+1)) \frac{z^k}{(z-1)^k}.
\end{aligned}
\tag{32}
$$

Proof. Formulas (28) and (29) are special cases of connection formulas 2.9.(33) and 2.9.(25) in [**Erd53**], respectively.

To prove formula (30), we apply formula (13) to the equation $E(a, b, m+1)$ and use expression (27). We take the limit $c \to m+1$ on the right-hand side of (13) by l'Hospital's rule and arrive at

$$
\begin{aligned}
U_1 &= -\frac{m!}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma'(m+k+1)}{k!\Gamma(m+k+1)^2} z^k \\
&\quad + \frac{m! z^{-m} \log z}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a-m+k)\Gamma(b-m+k)}{k!\Gamma(1-m+k)} z^k \\
&\quad + \frac{m! z^{-m}}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma'(a-m+k)\Gamma(b-m+k) + \Gamma(a-m+k)\Gamma'(b-m+k)}{k!\Gamma(1-m+k)} z^k \\
&\quad - \frac{m! z^{-m}}{\Gamma(a)\Gamma(b)} \lim_{c \to m+1} \sum_{k=0}^{\infty} \frac{\Gamma(1+a-c+k)\Gamma(1+b-c+k)\Gamma'(2-c+k)}{k!\Gamma(2-c+k)^2} z^k.
\end{aligned}
\tag{33}
$$

The first m terms of the second and of the third series are zero. The second series becomes the first term of (30). To compute the first m terms of the fourth series, we set $x = 2-c+k$ in formula (18) and get

$$
\lim_{c \to m+1} \frac{\Gamma'(2-c+k)}{\Gamma(2-c+k)^2} = (-1)^{m-k}\Gamma(m-k) = (-1)^m \frac{(m-1)!}{(1-m)_k}.
$$

The first m terms of the fourth series in (33) form the second term of formula (30).

The last term in (30) is obtained by combining the remaining terms of the third and fourth series in (33), and all terms of the first series in (33). This proof of (30) follows closely the derivation in [**AAR99**, pp. 82–84]; there are a few misprints in the formulas there.

To prove (31), we express the first term in (28) as a logarithmic function using (30). Then we simplify the term with $\log z$ by Euler's formula.

To prove (32), we apply Pfaff's transformation to the first term of (29) and substitution $z \mapsto z/(z-1)$ in (27) and (30). Then we simplify the logarithmic term by Pfaff's formula, observing that (for $z \in \mathbb{C}$ in the upper half-plane)

$$\log \frac{z}{z-1} = \log \frac{z}{1-z} - i\pi, \qquad \frac{\pi e^{-i\pi b}}{\sin \pi b} = \frac{\pi}{\tan \pi b} - i\pi. \qquad \square$$

More expressions for U_1 can be obtained by interchanging a, b in (29) and (32), by applying Euler–Pfaff transformations to individual hypergeometric functions in (27)–(32), and by applying Lemma 3.2 to sums with the ψ-function. In this way logarithmic solutions at $z = 0$ can be related to any other well-defined hypergeometric series at $z = 1$ or $z = \infty$. One may check that well-defined hypergeometric series at different points are independent.

The points $z = 1$ and $z = \infty$ are not logarithmic if and only if $a + b, a - b \notin \mathbb{Z}$. We have either four undefined Kummer's series at $z = 0$ (if $m > 0$) or four pairs of coinciding hypergeometric series there (if $m = 0$). In the latter case, Euler's transformation (4) acts trivially on some hypergeometric series at $z = 1$ and $z = \infty$, for example on (27). Then we have three (rather than usual four) distinct hypergeometric expressions for each Gauss hypergeometric function representable by well-defined series at $z = 1$ and $z = \infty$. In any case (when $z = 1$, $z = \infty$ are not logarithmic), we have five different Gauss hypergeometric solutions represented by 16 or 20 distinct Kummer's series.

If either $a + b$ or $a - b$ is an integer, then the equation $E(a, b, m + 1)$ has one other logarithmic point. For example, if $\ell \in \mathbb{Z}_{\geq 0}$ and $a = b + \ell$, then the point $z = \infty$ is logarithmic. A power series solution there is

$$z^{-a} \mathbf{F}\left(\left. \begin{matrix} a, a-m \\ \ell+1 \end{matrix} \,\right|\, \frac{1}{z} \right) = z^{-b} \mathbf{F}\left(\left. \begin{matrix} b, b-m \\ 1-\ell \end{matrix} \,\right|\, \frac{1}{z} \right).$$

Just as we obtained solution (30) from coinciding solutions $H_1 = H_2$ in (7) with

$c = m + 1$, we have the following local solution at $z = \infty$:

$$z^{-a}{}_2\mathrm{F}_1\left(\begin{matrix} a, a-m \\ \ell+1 \end{matrix} \,\middle|\, \frac{1}{z}\right) \log\frac{1}{z} + \frac{(-1)^{\ell+1}\ell!(\ell-1)!}{(1-a)_\ell(m+1-a)_\ell} z^{-b} \sum_{k=0}^{\ell-1} \frac{(b)_k(b-m)_k}{(1-\ell)_k k!} z^{-k}$$
$$+ z^{-a} \sum_{k=0}^{\infty} \frac{(a)_k(a-m)_k}{(\ell+1)_k k!} (\psi(a+k) + \psi(a-m+k) - \psi(\ell+k+1) - \psi(k+1)) z^{-k}.$$

This function has to be identified with the following constant multiple of U_1:

$$(-1)^{\ell+1}\ell! \frac{\Gamma(b)\Gamma(b-m)}{\Gamma(a+b-m)} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b-m \end{matrix} \,\middle|\, 1-z\right). \tag{34}$$

To see this, use the connection formula [**Erd53**, 2.9.(36)] (there is a misprint in this reference: $\Gamma(c)$ must be replaced by $\Gamma(b)$) before taking the limit $b \to a - \ell$ in the analogue of (13). If $m \neq 0$ and $\ell \neq 0$, then eight Kummer's series are undefined; other 16 Kummer's series are distinct and represent four different Gauss hypergeometric functions. If $m \neq 0$, $\ell = 0$, then four Kummer's series at $z = 0$ are undefined, and there are four pairs of coinciding solutions at $z = \infty$. The other four Kummer's series at $z = \infty$ are distinct expressions of one Gauss hypergeometric function. There are three more Gauss hypergeometric solutions, each represented by three distinct Kummer's series at $z = 0$ or $z = 1$. If $m = 0$, $\ell \neq 0$, we have a similar situation. If $m = \ell = 0$, then there are three distinct Kummer's series at $z = 0$ and at $z = \infty$, and four distinct series at $z = 1$. The groups of series at $z = 0$ or at $z = \infty$ represent single Gauss hypergeometric functions; the series at $z = 1$ represent two different functions.

All three singular points of $E(a, b, m+1)$ are logarithmic if and only if a and b are rational numbers with the denominator 2. Then we have three distinct Gauss hypergeometric functions, one for each singular point. The number of their hypergeometric expressions depends on the presence of singular points with the zero local exponent difference; recall discussion of Table 1. If all three local exponent differences are zero, then each Gauss hypergeometric solution is represented by just two distinct Kummer's series. In this case, the equation is $E(\frac{1}{2}, \frac{1}{2}, 1)$; its hypergeometric solutions are related to the well-known complete elliptic integral $K(k)$; see [**AAR99**, (3.2.3)]. The other renowned complete elliptic integral is $E(k)$ [**AAR99**, (3.2.14)], which is expressible via solutions of $E(-\frac{1}{2}, \frac{1}{2}, 1)$. This equation has one local exponent difference equal to zero, so the number of distinct Kummer's series is $3 + 3 + 4 = 10$. An example of hypergeometric equation with 8 distinct Kummer's series is $E(-\frac{1}{2}, -\frac{1}{2}, 1)$; it has two local exponent differences equal to zero.

6. Logarithmic and terminating solutions

Here we consider hypergeometric equations which have logarithmic points and terminating hypergeometric solutions, but the monodromy group is non-abelian. By part (3) of Theorem 2.4, a general hypergeometric equation of this kind is $E(a, -n, m+1)$, where $n, m \in \mathbb{Z}_{\geq 0}$ and $a \notin \mathbb{Z}$. We are in the non-abelian case of part (3) of Theorem 2.2. The point $z = 0$ is logarithmic; the points $z = 1$, $z = \infty$ are not logarithmic.

The terminating solution ${}_2\mathrm{F}_1\left(\begin{smallmatrix}-n,\, a\\ m+1\end{smallmatrix}\,\middle|\, z\right)$ has six terminating expressions (19)–(23) with $n+1$ terms, but there are two extra terminating expressions with $m+n+1$ terms if $m \neq 0$:

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\begin{matrix}-n, a\\ m+1\end{matrix}\,\middle|\, z\right) &= (1-z)^n\, {}_2\mathrm{F}_1\left(\begin{matrix}-n, m+1-a\\ m+1\end{matrix}\,\middle|\, \frac{z}{z-1}\right)\\
&= \frac{m!(a)_n}{(m+n)!}(-z)^n\, {}_2\mathrm{F}_1\left(\begin{matrix}-n, -m-n\\ 1-n-a\end{matrix}\,\middle|\, \frac{1}{z}\right)\\
&= \frac{m!(a)_n}{(m+n)!}(1-z)^n\, {}_2\mathrm{F}_1\left(\begin{matrix}-n, m+1-a\\ 1-n-a\end{matrix}\,\middle|\, \frac{1}{1-z}\right)\\
&= \frac{m!(m+1-a)_n}{(m+n)!}z^n\, {}_2\mathrm{F}_1\left(\begin{matrix}-n, -m-n\\ a-m-n\end{matrix}\,\middle|\, 1-\frac{1}{z}\right)\\
&= \frac{m!(m+1-a)_n}{(m+n)!}\, {}_2\mathrm{F}_1\left(\begin{matrix}-n, a\\ a-m-n\end{matrix}\,\middle|\, 1-z\right)\\
&= \frac{m!(a)_n}{(m+n)!}(-z)^{-m}(1-z)^{m+n}\, {}_2\mathrm{F}_1\left(\begin{matrix}-m-n, 1-a\\ 1-n-a\end{matrix}\,\middle|\, \frac{1}{1-z}\right)\\
&= \frac{m!(m+1-a)_n}{(m+n)!}z^{-m}\, {}_2\mathrm{F}_1\left(\begin{matrix}-m-n, a-m\\ a-m-n\end{matrix}\,\middle|\, 1-z\right).
\end{aligned}
$$

This solution has four non-terminating hypergeometric expressions with the argument z, $z/(z-1)$, $1/z$ or $1-1/z$, due to Euler's formula (4). There are two other Gauss hypergeometric solutions, represented by six (if $m > 0$) or eight (if $m = 0$) distinct non-terminating Kummer's series at $z = 1$ and $z = \infty$. As in Section 5, we miss four Kummer's series at $z = 0$, which are either undefined or coincide with listed terminating series.

The logarithmic solution U_1 of Section 5 is not defined in this case, since some values of the ψ-function become infinite in formulas (27)–(32). We should either apply those formulas to the equation $E(m+1-a, m+n+1, m+1)$, or consider the following solution of $E(a, -n, m+1)$, well defined for $b = -n$ by expression (31):

$$U_2 := U_1 + \frac{\pi \sin \pi(a+b)}{\sin \pi a \sin \pi b} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ m+1 \end{matrix} \middle| z\right). \tag{35}$$

The following theorem presents various expressions for this function.

THEOREM 6.1. *The function U_2 with $b = -n$ has the following expressions:*

$$\begin{aligned}
U_2 &= \frac{(-1)^{m+1} m!n!}{(1-a)_{m+n+1}} (1-z)^{m+n+1-a} {}_2\mathrm{F}_1\left(\begin{matrix} m+1-a, m+n+1 \\ m+n+2-a \end{matrix} \middle| 1-z\right) && (36)\\
&= \frac{(-1)^{m+1} m!n!}{(a-m)_{m+n+1}} (-z)^{-a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a-m \\ a+n+1 \end{matrix} \middle| \frac{1}{z}\right) + \frac{\pi e^{i\pi a}}{\sin \pi a} {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ m+1 \end{matrix} \middle| z\right) && (37)\\
&= {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ m+1 \end{matrix} \middle| z\right) \log z + \frac{\pi}{\tan \pi a} {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ m+1 \end{matrix} \middle| z\right) \\
&\quad + \frac{(-1)^{m+1} m!n!}{(1-a)_m} z^{-m} \sum_{k=0}^{m-1} \frac{(a-m)_k (m-k-1)!}{(m+n-k)!k!} z^k \\
&\quad + m!n! \sum_{k=0}^{n} \frac{(a)_k}{(m+k)!(n-k)!k!} \\
&\quad \times (\psi(a+k) + \psi(n-k+1) - \psi(m+k+1) - \psi(k+1))(-z)^k \\
&\quad + (-1)^n m!n! \sum_{k=n+1}^{\infty} \frac{(a)_k (k-n-1)!}{(m+k)!k!} z^k && (38)\\
&= {}_2\mathrm{F}_1\left(\begin{matrix} -n, a \\ m+1 \end{matrix} \middle| z\right) \log z \\
&\quad + \frac{(-1)^{m+1} m!}{(1-a)_m (m+n)!} z^{-m} (1-z)^{m+n+1-a} \\
&\quad \times \sum_{k=0}^{m-1} \frac{(1-a)_k (m-k-1)!(n+k)!}{k!} (-z)^k \\
&\quad + \frac{m!}{(m+n)!} (1-z)^{m+n+1-a} \sum_{k=0}^{\infty} \frac{(m+1-a)_k (m+n+k)!}{(m+k)!k!} \\
&\quad \times (\psi(m+k+1-a) + \psi(m+n+k+1) - \psi(m+k+1) - \psi(k+1)) z^k && (39)
\end{aligned}$$

$$
\begin{aligned}
&= {}_2\mathrm{F}_1\left(\begin{matrix}-n, a\\ m+1\end{matrix}\,\middle|\, z\right) \log\frac{z}{1-z} + \frac{\pi}{\tan\pi a}\,{}_2\mathrm{F}_1\left(\begin{matrix}-n, a\\ m+1\end{matrix}\,\middle|\, z\right)\\
&\quad + \frac{(-1)^{m+1} m!}{(1-a)_m (m+n)!} z^{-m}(1-z)^{m-a}\\
&\quad \times \sum_{k=0}^{m-1} \frac{(a-m)_k (m-k-1)!(n+k)!}{k!} \frac{z^k}{(1-z)^k}\\
&\quad + \frac{m!}{(m+n)!}(1-z)^{-a} \sum_{k=0}^{\infty} \frac{(a)_k (m+n+k)!}{(m+k)!k!}\\
&\quad \times (\psi(a+k) + \psi(m+n+k+1) - \psi(m+k+1) - \psi(k+1)) \frac{z^k}{(z-1)^k}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&= {}_2\mathrm{F}_1\left(\begin{matrix}-n, a\\ m+1\end{matrix}\,\middle|\, z\right) \log\frac{z}{1-z}\\
&\quad + \frac{(-1)^{m+1} m!n!}{(1-a)_m} z^{-m}(1-z)^{n+m} \sum_{k=0}^{m-1} \frac{(1-a)_k (m-k-1)!}{(m+n-k)!k!} \frac{z^k}{(z-1)^k}\\
&\quad + m!n!(1-z)^n \sum_{k=0}^{n} \frac{(m+1-a)_k}{(m+k)!(n-k)!k!}\\
&\quad \times (\psi(m+k+1-a) + \psi(n-k+1) - \psi(m+k+1) - \psi(k+1)) \frac{z^k}{(1-z)^k}\\
&\quad + m!n!(z-1)^n \sum_{k=n+1}^{\infty} \frac{(m+1-a)_k (k-n-1)!}{(m+k)!k!} \frac{z^k}{(z-1)^k}.
\end{aligned} \tag{41}
$$

Proof. To show the first two formulas we apply, respectively, (28) or (29) to expression (35). Then we collect the two terms with ${}_2\mathrm{F}_1\left(\begin{smallmatrix}a, b\\ m+1\end{smallmatrix}\,\middle|\, z\right)$ and take the limit $b \to -n$.

To prove (38), we apply Lemma 3.2 to expression (30) and arrive at

$$
\begin{aligned}
U_2 &= {}_2\mathrm{F}_1\left(\begin{matrix}a, b\\ m+1\end{matrix}\,\middle|\, z\right) \log z + \left(\frac{\pi \sin\pi(a+b)}{\sin\pi a \sin\pi b} - \frac{\pi}{\tan\pi b}\right) {}_2\mathrm{F}_1\left(\begin{matrix}a, b\\ m+1\end{matrix}\,\middle|\, z\right)\\
&\quad + \frac{(-1)^{m+1} m!(m-1)!}{(1-a)_m (1-b)_m} z^{-m} \sum_{k=0}^{m-1} \frac{(a-m)_k (b-m)_k}{(1-m)_k k!} z^k\\
&\quad + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(m+1)_k k!} (\psi(a+k) + \psi(1-b-k) - \psi(m+k+1) - \psi(k+1)) z^k.
\end{aligned}
$$

Now we take the limit $b \to -n$ of each term. In particular, for $k \geq n+1$ we apply formula (17) with $x = b + k$, and then formula (18) to obtain

$$\begin{aligned}\lim_{b\to -n}(b)_k\psi(1-b-k) &= \lim_{b\to -n}\frac{\Gamma(b+k)}{\Gamma(b)}\left(\psi'(b+k)+\frac{\pi\cos\pi b}{\sin\pi b}\right)\\ &= \lim_{b\to -n}\left(\frac{\Gamma'(b+k)}{\Gamma(b)}+\cos\pi b\,\Gamma(1-b)\Gamma(b+k)\right)\\ &= (-1)^n n!(k-n-1)!.\end{aligned}$$

Note that the last sum in (38) can be easily missed out; see [**AAR99**, p. 84].

Formulas (39) and (40) are direct consequences of (31) and (32) applied to (35). To obtain formula (41), consider (32) with interchanged roles of the parameters a and b, so that a approaches the integer $-n$ (and b can be renamed to a at some point). Then we apply Lemma 3.2 similarly as in the proof of (38). □

7. Completely reducible monodromy group

Here we consider hypergeometric equations with completely reducible but non-trivial monodromy group. Up to conjugation, the monodromy representation is a subgroup of $\mathbb{G}_m$. By part (4) of Theorem 2.4, a general hypergeometric equation of this type is $E(-n, a-m, -n-m)$, where n, m are non-negative integers, and $a \notin \mathbb{Z}$.

Since the monodromy group is completely reducible, there is a basis of terminating solutions of $E(-n, a-m, -n-m)$. Such a basis is

$$ {}_2\mathrm{F}_1\left(\begin{matrix}-n,\, a-m\\ -n-m\end{matrix}\,\middle|\, z\right), \quad (1-z)^{-a}\,{}_2\mathrm{F}_1\left(\begin{matrix}-m,\, -a-n\\ -n-m\end{matrix}\,\middle|\, z\right). \tag{42}$$

Although these two hypergeometric functions seem to be equal by standard Euler's formula (4), the correct Euler's transformation in this situation is formula (9) of Lemma 3.1. Alternative terminating expressions of the basis solutions (42) are obtained by using formulas (19)–(23). For example,

$$\begin{aligned}{}_2\mathrm{F}_1\left(\begin{matrix}-n,\, a-m\\ -n-m\end{matrix}\,\middle|\, z\right) &= (1-z)^n\,{}_2\mathrm{F}_1\left(\begin{matrix}-n,\, -a-n\\ -n-m\end{matrix}\,\middle|\, \frac{z}{z-1}\right)\\ &= \frac{m!(a-m)_n}{(n+m)!}z^n\,{}_2\mathrm{F}_1\left(\begin{matrix}-n,\, m+1\\ 1-a+m-n\end{matrix}\,\middle|\, \frac{1}{z}\right)\end{aligned}$$

$$= \frac{m!(a-m)_n}{(n+m)!}(z-1)^n {}_2\mathrm{F}_1\left(\begin{matrix}-n,\,-a-n\\1-a+m-n\end{matrix}\,\middle|\,\frac{1}{1-z}\right)$$

$$= \frac{m!(a+1)_n}{(n+m)!} z^n {}_2\mathrm{F}_1\left(\begin{matrix}-n,\,m+1\\a+1\end{matrix}\,\middle|\,1-\frac{1}{z}\right)$$

$$= \frac{m!(a+1)_n}{(n+m)!} {}_2\mathrm{F}_1\left(\begin{matrix}-n,\,a-m\\a+1\end{matrix}\,\middle|\,1-z\right).$$

For the last four expressions, standard Euler's formula (4) can be applied; this gives us four non-terminating hypergeometric expressions. In total, we have six terminating and four non-terminating hypergeometric expressions for each basis solution in (42). The remaining four Kummer's solutions are related by Euler–Pfaff transformations; they represent one Gauss hypergeometric function. The relation between this non-terminating and the two terminating solutions is a consequence of formula (8) in Lemma 3.1:

$$\begin{aligned}(1-z)^{-a} {}_2\mathrm{F}_1\left(\begin{matrix}-m,\,-a-n\\-n-m\end{matrix}\,\middle|\,z\right)\\ = {}_2\mathrm{F}_1\left(\begin{matrix}-n,\,a-m\\-n-m\end{matrix}\,\middle|\,z\right)\\ + (-1)^m \frac{n!m!(a-m)_{n+m+1}}{(n+m)!(n+m+1)!} z^{n+m+1} {}_2\mathrm{F}_1\left(\begin{matrix}m+1,\,a+n+1\\n+m+2\end{matrix}\,\middle|\,z\right).\end{aligned} \tag{43}$$

8. The trivial monodromy group

Here we consider hypergeometric equations with the trivial monodromy group. That means that solutions can be meromorphically continued to the entire projective line $\mathbb{P}^1$, so they are rational functions. By part (5) of Theorem 2.4, general hypergeometric equation with trivial monodromy group is $E(-n, \ell+1, -m-n)$. We have the following three terminating solutions:

$$\begin{gathered}{}_2\mathrm{F}_1\left(\begin{matrix}-n,\,\ell+1\\-n-m\end{matrix}\,\middle|\,z\right), \quad (1-z)^{-\ell-1} {}_2\mathrm{F}_1\left(\begin{matrix}-m,\,\ell+1\\-n-m\end{matrix}\,\middle|\,\frac{z}{z-1}\right),\\ z^{n+m+1}(1-z)^{-m-\ell-1} {}_2\mathrm{F}_1\left(\begin{matrix}-\ell,\,n+1\\-m-\ell\end{matrix}\,\middle|\,1-z\right).\end{gathered} \tag{44}$$

Each of them can be transformed by formulas (19)–(23). For example,

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, \ell+1\\ -n-m\end{matrix}\,\right|\, z\right) &= (1-z)^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, -n-m-\ell-1\\ -n-m\end{matrix}\,\right|\, \frac{z}{z-1}\right)\\
&= \frac{m!(n+\ell)!}{\ell!(n+m)!} z^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, m+1\\ -n-\ell\end{matrix}\,\right|\, \frac{1}{z}\right)\\
&= \frac{m!(n+\ell)!}{\ell!(n+m)!}(z-1)^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, -n-m-\ell-1\\ -n-\ell\end{matrix}\,\right|\, \frac{1}{1-z}\right)\\
&= \frac{m!(n+m+\ell+1)!}{(n+m)!(m+\ell+1)!} z^n\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, m+1\\ 2+m+\ell\end{matrix}\,\right|\, 1-\frac{1}{z}\right)\\
&= \frac{m!(n+m+\ell+1)!}{(n+m)!(m+\ell+1)!}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, \ell+1\\ 2+m+\ell\end{matrix}\,\right|\, 1-z\right).
\end{aligned}
$$

Note how permutation of the numbers m, n, ℓ permutes the three sets of hypergeometric representations for solutions in (8), if we ignore the front factors and change of the argument. The last two identities can be transformed to non-terminating series by Euler's formula, so in total we have six terminating and two non-terminating hypergeometric expressions for each of the three solutions. This exhausts the 24 Kummer's series. Relation between the three solutions is a consequence of (43):

$$
\begin{aligned}
&(1-z)^{-\ell-1}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-m,\, \ell+1\\ -n-m\end{matrix}\,\right|\, \frac{z}{z-1}\right)\\
&\quad = {}_2\mathrm{F}_1\left(\left.\begin{matrix}-n,\, \ell+1\\ -n-m\end{matrix}\,\right|\, z\right)\\
&\qquad + (-1)^m \frac{n!(m+\ell)!}{\ell!(n+m)!} z^{n+m+1}(1-z)^{-m-\ell-1}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}-\ell,\, n+1\\ -m-\ell\end{matrix}\,\right|\, 1-z\right).
\end{aligned}
$$

9. Additive monodromy group

Here we consider hypergeometric equations whose monodromy group is (up to conjugation) a non-trivial subgroup of $\mathbb{G}_a$. By part (6) of Theorem 2.4, general hypergeometric equation of this type is $E(-\ell, -n-\ell, -m-n-2\ell)$. This is the same equation as $E(-\ell, -n-\ell, m+1)$ with z replaced by $1-z$. The point $z=0$ is not logarithmic, there is a basis of power series solutions there:

$$
{}_2\mathrm{F}_1\left(\left.\begin{matrix}-\ell,\, -n-\ell\\ -m-n-2\ell\end{matrix}\,\right|\, z\right), \qquad z^{m+n+2\ell+1}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}m+\ell+1,\, m+n+\ell+1\\ m+n+2\ell+2\end{matrix}\,\right|\, z\right). \tag{45}
$$

The first solution has the following terminating expressions:

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-n-\ell\\-n-m-2\ell\end{array}\right|z\right) &= (1-z)^{\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-m-\ell\\-n-m-2\ell\end{array}\right|\frac{z}{z-1}\right)\\
&= C_1(-z)^{\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,n+m+\ell+1\\n+1\end{array}\right|\frac{1}{z}\right)\\
&= C_1(1-z)^{\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-m-\ell\\n+1\end{array}\right|\frac{1}{1-z}\right)\\
&= C_2 z^{\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,n+m+\ell+1\\m+1\end{array}\right|1-\frac{1}{z}\right)\\
&= C_2\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-n-\ell\\m+1\end{array}\right|1-z\right)\\
&= (1-z)^{-m}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-m-\ell,-n-m-\ell\\-n-m-2\ell\end{array}\right|z\right)\\
&= (1-z)^{n+\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-n-\ell,-n-m-\ell\\-n-m-2\ell\end{array}\right|\frac{z}{z-1}\right)\\
&= C_1(-z)^{m+\ell}(1-z)^{-m}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-m-\ell,n+\ell+1\\n+1\end{array}\right|\frac{1}{z}\right)\\
&= C_2 z^{n+\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-n-\ell,m+\ell+1\\m+1\end{array}\right|1-\frac{1}{z}\right),
\end{aligned}
$$

where

$$
C_1=\frac{(n+\ell)!(n+m+\ell)!}{n!(n+m+2\ell)!},\quad C_2=\frac{(m+\ell)!(n+m+\ell)!}{m!(n+m+2\ell)!}.
$$

This solution has also non-terminating hypergeometric expressions, with the argument $1-z$ or $1/(1-z)$ by Euler's formula. In total we have 10 terminating and two non-terminating hypergeometric expressions for this solution. The number of distinct terminating expressions may drop to eight (if $m=0$ or $n=0$) or to six (if $m=n=0$). The second solution in (45) has two, three or four distinct hypergeometric expressions due to Euler–Pfaff transformations.

Other Kummer's series at $z=1$ and $z=\infty$ are undefined (or coincide with terminating expressions, if $m=0$ or $n=0$). Consequently, there is no basis of power series solutions at these points $z=1$, $z=\infty$; they are logarithmic. In the following theorem, we present logarithmic expressions for the function

$$
U_3=\frac{(-1)^{m+1}}{(m+n+2\ell+1)!}z^{m+n+2\ell+1}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}m+\ell+1,m+n+\ell+1\\m+n+2\ell+2\end{array}\right|z\right). \tag{46}
$$

Note that all terms with the ψ-function can be written as terminating sums of rational numbers, and that all sums in expression (47) are terminating.

THEOREM 9.1. *Set* $C_3 = 1/\ell!(m+\ell)!(n+\ell)!(m+n+\ell)!$. *The following formulas hold:*

$$
\begin{aligned}
U_3 &= C_3(m+n+2\ell)!\,{}_2\mathrm{F}_1\left(\begin{matrix}-\ell, -n-\ell\\ -m-n-2\ell\end{matrix}\,\middle|\, z\right)\log(1-z)\\
&\quad+\sum_{k=0}^{\ell}\frac{\psi(n+\ell-k+1)+\psi(\ell-k+1)-\psi(m+k+1)-\psi(k+1)}{(m+k)!(n+\ell-k)!(\ell-k)!k!}(1-z)^k\\
&\quad-(z-1)^{-m}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{(m+n+\ell-k)!(m+\ell-k)!k!}(z-1)^k\\
&\quad+(-1)^{\ell}(z-1)^{n+\ell}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{(m+n+\ell-k)!(n+\ell-k)!k!}\frac{1}{(z-1)^k} \qquad (47)\\
&= C_3(m+n+2\ell)!\,{}_2\mathrm{F}_1\left(\begin{matrix}-\ell, -n-\ell\\ -m-n-2\ell\end{matrix}\,\middle|\, z\right)\log(1-z)\\
&\quad-C_3 z^{m+n+2\ell+1}(z-1)^{-m}\sum_{k=0}^{m-1}\frac{(n+\ell+k)!(\ell+k)!(m-k-1)!}{k!}(z-1)^k\\
&\quad+C_3 z^{m+n+2\ell+1}\sum_{k=0}^{\infty}\frac{(m+\ell+k)!(m+n+\ell+k)!}{(m+k)!k!}(1-z)^k\\
&\quad\times(\psi(m+n+\ell+k+1)+\psi(m+\ell+k+1)-\psi(m+k+1)-\psi(k+1)) \qquad (48)\\
&= C_3(m+n+2\ell)!\,{}_2\mathrm{F}_1\left(\begin{matrix}-\ell, -n-\ell\\ -m-n-2\ell\end{matrix}\,\middle|\, z\right)\log\frac{1-z}{z}\\
&\quad-\frac{z^{m+\ell}(z-1)^{-m}}{(n+\ell)!(m+n+\ell)!}\sum_{k=0}^{m-1}\frac{(n+\ell+k)!(m-k-1)!}{(m+\ell-k)!k!}\frac{(z-1)^k}{z^k}\\
&\quad+\frac{z^{\ell}}{(n+\ell)!(m+n+\ell)!}\sum_{k=0}^{\ell}\frac{(m+n+\ell+k)!}{(m+k)!(\ell-k)!k!}\\
&\quad\times(\psi(m+n+\ell+k+1)+\psi(\ell-k+1)-\psi(m+k+1)-\psi(k+1))\frac{(1-z)^k}{z^k}\\
&\quad+\frac{(-z)^{\ell}}{(n+\ell)!(m+n+\ell)!}\sum_{k=\ell+1}^{\infty}\frac{(m+n+\ell+k)!(k-\ell-1)!}{(m+k)!k!}\frac{(z-1)^k}{z^k} \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
&= C_3(m+n+2\ell)!\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-n-\ell\\ -m-n-2\ell\end{array}\right|\,z\right)\log\frac{1-z}{z}\\
&\quad-\frac{z^{m+n+\ell}(z-1)^{-m}}{\ell!(m+\ell)!}\sum_{k=0}^{m-1}\frac{(\ell+k)!(m-k-1)!}{(m+n+\ell-k)!k!}\frac{(z-1)^k}{z^k}\\
&\quad+\frac{z^{n+\ell}}{\ell!(m+\ell)!}\sum_{k=0}^{n+\ell}\frac{(m+\ell+k)!}{(m+k)!(n+\ell-k)!k!}\\
&\quad\times(\psi(m+\ell+k+1)+\psi(n+\ell-k+1)-\psi(m+k+1)-\psi(k+1))\frac{(1-z)^k}{z^k}\\
&\quad+\frac{(-z)^{n+\ell}}{\ell!(m+\ell)!}\sum_{k=n+\ell+1}^{\infty}\frac{(m+\ell+k)!(k-n-\ell-1)!}{(m+k)!k!}\frac{(z-1)^k}{z^k}. \qquad (50)
\end{aligned}
$$

In addition, in each expression one can interchange m and n, provided that z is replaced by $z/(z-1)$ and the whole expression is multiplied by $(-1)(1-z)^{\ell}$.

Proof. We apply Theorem 6.1 with n replaced by ℓ, z replaced by $1-z$, and a approaching $-n-\ell$. By formula (36), we may identify $U_3 = U_2\big/m!(n+\ell)!\ell!$. In formula (38), we get rid of the singular ψ-values and the tangent term by using Lemma 3.2. The result is

$$
\begin{aligned}
U_3 &= \frac{1}{m!(n+\ell)!\ell!}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}-\ell,-n-\ell\\ m+1\end{array}\right|\,1-z\right)\log(1-z)\\
&\quad+(-1)^{m+1}(1-z)^{-m}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{(m+n+\ell-k)!(m+\ell-k)!k!}(z-1)^k\\
&\quad+\sum_{k=0}^{\ell}\frac{\psi(n+\ell-k+1)+\psi(\ell-k+1)-\psi(m+k+1)-\psi(k+1)}{(m+k)!(n+\ell-k)!(\ell-k)!k!}(1-z)^k\\
&\quad+(-1)^{\ell}\sum_{k=\ell+1}^{\ell+n}\frac{(k-\ell-1)!}{(n+\ell-k)!(m+k)!k!}(z-1)^k.
\end{aligned}
$$

We apply Euler's transformation (4) to the first hypergeometric sum, rewrite the last sum in the opposite direction, and obtain (47). Formulas (48) and (49) are just rewritten versions of expressions (39) and (41), respectively. Formula (50) can be obtained from (41) after interchanging the first two parameters of $E(-\ell, -n-\ell, m+1)$; the same formula can be obtained by carefully applying Lemma 3.2 to expression (40).

To see the last statement, one can check the described transformation on formula (46) and compare it with Pfaff's transformation. (Formula (47) is invariant under this transformation as well, up to Euler's transformation of the first term and summing the second term in the opposite direction. Interchanging the singular points $z = 1$, $z = \infty$ produces the same transformation.) $\square$

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