Non-Integrality of Binomial Sums and Fermat's Little Theorem

DANIEL LÓPEZ-AGUAYO
Centro de Ciencias Matemáticas UNAM
Morelia, Mexico
dlopez@matmor.unam.mx

In the April 2014 issue of THIS MAGAZINE, Marcel Chiriță proposed Problem 1942 which asked to prove that

$$\sum_{k=0}^{n} \frac{k}{k+r}\binom{n}{k} \tag{1}$$

is not an integer for $n \geq 2$ and $r = 1$. Two solutions to this problem appear in this issue (on page 238), and both apply Fermat's little theorem. A natural question is to ask if there are infinitely many integers r for which Equation (1) is not an integer. In this note we show that Equation (1) is nonintegral for $r = 2$, 3, and 4. Our results also hinge on Fermat's little theorem, as it may be used to prove the following useful lemma.

Lemma. *For every integer $r \geq 2$ we have that $2^r \not\equiv 1 \mod r$.*

All proofs below follow the same methodology. Rearrange the sum and apply the binomial theorem repeatedly to yield an expression that is shown to be nonintegral via the lemma.

Proposition 1. *Let $n \geq 2$. Then $\displaystyle\sum_{k=0}^{n} \frac{k}{k+2}\binom{n}{k}$ is not an integer.*

Proof. The summation may be rearranged so that

$$\begin{aligned}\sum_{k=0}^{n} \frac{k}{k+2}\binom{n}{k} &= \sum_{k=0}^{n}\left(1-\frac{2}{k+2}\right)\binom{n}{k} = 2^n - 2\sum_{k=0}^{n}\frac{1}{k+2}\binom{n}{k} \\ &= 2^n - \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2} \\ &= 2^n - \frac{2}{(n+1)(n+2)}\left[\sum_{k=0}^{n} k\binom{n+2}{k+2} + \sum_{k=0}^{n}\binom{n+2}{k+2}\right].\end{aligned}$$

To simplify the above expression, we first evaluate $\sum_{k=0}^{n} k\binom{n+2}{k+2}$. Differentiating $\sum_{k=0}^{n+2}\binom{n}{k}x^k = (1+x)^{n+2}$ and letting $x = 1$ yields

$$\sum_{k=0}^{n+2} k\binom{n}{k} = (n+2)2^{n+1}. \tag{2}$$

Math. Mag. **88** (2015) 231–234. doi:10.4169/math.mag.88.3.231.
MSC: Primary 11B65

By a change of variables,

$$\sum_{k=0}^{n} k\binom{n+2}{k+2} = \sum_{w=2}^{n+2}(w-2)\binom{n+2}{w} = \sum_{w=2}^{n+2} w\binom{n+2}{w} - \sum_{w=2}^{n+2} 2\binom{n+2}{w}. \tag{3}$$

Using Equation (2) and the binomial theorem, Equation (3) simplifies to

$$2^{n+1}(n+2) - (n+2) - 2\left[2^{n+2} - 1 - (n+2)\right] = 2^{n+1}n - 2^{n+2} + n + 4. \tag{4}$$

Because $\sum_{k=0}^{n}\binom{n+2}{k+2} = 2^{n+2} - 1 - (n+2)$, then

$$\sum_{k=0}^{n} \frac{k}{k+2}\binom{n}{k} = 2^n - \frac{2(2^{n+1}n+1)}{(n+1)(n+2)}.$$

It suffices to show that $\frac{2(2^{n+1}n+1)}{(n+1)(n+2)}$ is not an integer. Suppose otherwise. If $n+1$ is odd, then $2^{n+1}n+1 \equiv 0 \mod (n+1)$. This implies that $2^{n+1}n+1 \equiv 1 - 2^{n+1} \mod (n+1)$, which contradicts the lemma. If $n+1$ is even, then $n+2$ is odd and $2^{n+1}n+1 \equiv 0 \mod (n+2)$. This implies that $2^{n+1}n+1 \equiv 1-2^{n+2} \mod (n+2)$, which contradicts lemma. Therefore, the requisite sum is nonintegral. ∎

Proposition 2. *Let $n \geq 2$. Then $\sum_{k=0}^{n} \frac{k}{k+3}\binom{n}{k}$ is not an integer.*

Proof. Similar applications of the binomial theorem as in the proof of Proposition 1 gives

$$\sum_{k=0}^{n} \frac{k}{k+3}\binom{n}{k} = 2^n - \frac{6(2^n n^2 + 2^n n + 2^{n+1} - 1)}{(n+1)(n+2)(n+3)}.$$

The calculations may be found on the online supplement to this article [**1**].

We need to show that $\frac{6(2^n n^2+2^n n+2^{n+1}-1)}{(n+1)(n+2)(n+3)}$ is not an integer. Suppose first that $n+1$ is even, then $n+3$ is even, too, which implies that $(n+1)(n+3)$ is evenly divisible by 4. But $6(2^n n^2 + 2^n n + 2^{n+1} - 1)$ is not evenly divisible by 4, because

$$\begin{aligned} 6(2^n n^2 + 2^n n + 2^{n+1} - 1) &\equiv 2(2^n n^2 + 2^n n + 2^{n+1} - 1) \mod 4 \\ &\equiv 2^{n+1}[n(n+1)+2] - 2 \mod 4 \\ &\equiv -2 \mod 4. \end{aligned}$$

Hence, in this case, $\frac{6(2^n n^2+2^n n+2^{n+1}-1)}{(n+1)(n+2)(n+3)}$ is not an integer.

Suppose that $n+1$ is odd, but is not a multiple of 3. It must be the case that $n+1$ and 6 are relatively prime. Notice that

$$\begin{aligned} 2^n n^2 + 2^n n + 2^{n+1} - 1 &= 2^n n(n+1) + 2^{n+1} - 1 \\ &\equiv 2^{n+1} - 1 \mod (n+1). \end{aligned}$$

For $\frac{6(2^n n^2+2^n n+2^{n+1}-1)}{(n+1)(n+2)(n+3)}$ to be an integer, then $n+1$ must evenly divide $2^n n^2 + 2^n n + 2^{n+1} - 1$ or, equivalently, $2^{n+1} \equiv 1 \mod (n+1)$, which contradicts the lemma.

For $n+1$ odd and a multiple of 3, then $n+3$ is odd and not divisible by 3. This time $n+3$ and 6 are relatively prime. In a similar approach to the above case,

$$\begin{aligned}2^n n^2 + 2^n n + 2^{n+1} - 1 &\equiv 2^n(n^2+n+2) - 1 \mod (n+3)\\ &\equiv 2^n\left[(n-2)(n+3)+8\right] - 1 \mod (n+3)\\ &\equiv 2^{n+3} - 1 \mod (n+3).\end{aligned}$$

For $\frac{6(2^n n^2+2^n n+2^{n+1}-1)}{(n+1)(n+2)(n+3)}$ to be an integer, then $n+3$ must evenly divide $2^n n^2 + 2^n n + 2^{n+1} - 1$ or, equivalently, $2^{n+3} \equiv 1 \mod (n+3)$, which contradicts the lemma.

Therefore, $\frac{6(2^n n^2+2^n n+2^{n+1}-1)}{(n+1)(n+2)(n+3)}$ is not an integer, which completes the proof. ■

Proposition 3. *Let $n \geq 2$. Then $\sum_{k=0}^{n} \frac{k}{k+4}\binom{n}{k}$ is not an integer.*

Proof. Using a similar method as before, it follows that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{k}{k+4} = 2^n - \frac{8(2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3)}{(n+1)(n+2)(n+3)(n+4)}. \tag{5}$$

Suppose first that $n+3$ is odd but not divisible by 3. Because $n+3$ and 8 are relatively prime, if $\frac{8(2^n n^3+3\cdot 2^n n^2+2^{n+3}n+3)}{(n+1)(n+2)(n+3)(n+4)}$ is an integer, then $n+3$ must divide $2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3$. Notice that $n^3 + 3n^2 + 8n = (n+3)(n^2+8) - 24$ and therefore $n^3 + 3n^2 + 8n \equiv -24 \mod (n+3)$. Consequently,

$$\begin{aligned}2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3 &\equiv 2^n(-24) + 3 \mod (n+3)\\ &\equiv 3 - 3\cdot 2^{n+3} \mod (n+3).\end{aligned}$$

Thus $n+3$ divides $3(2^{n+3}-1)$ and because $n+3$ and 3 are relatively prime, then $2^{n+3} - 1 \equiv 0 \mod (n+3)$, which contradicts the lemma.

Suppose now that $n+3$ is odd but divisible by 3. Then $n+1$ is odd but not divisible by 3. As before, because $n+1$ and 8 are relatively prime, then for the expression in Equation (5) to be an integer, then $n+1$ must divide $2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3$. Now note that $n^3 + 3n^2 + 8n = (n+1)(n^2+2n+6) - 6$ and thus $n^3 + 3n^2 + 8n \equiv -6 \mod (n+1)$. Therefore,

$$\begin{aligned}2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3 &\equiv 2^n(-6) + 3 \mod (n+1)\\ &\equiv 3 - 3\cdot 2^{n+1} \mod (n+1).\end{aligned}$$

Therefore, $n+1$ divides $3(2^{n+1}-1)$. Because $n+1$ and 3 are relatively prime, then $2^{n+1} - 1 \equiv 0 \mod (n+1)$, which once again contradicts the lemma.

Suppose now that $n+3$ is even and that $n+2$ is not divisible by 3. For the expression in Equation (5) to be an integer, then $n+2$ must divide $2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3$. Since $n^3 + 3n^2 + 8n = (n+2)(n^2+n+6) - 12$, then $n^3 + 3n^2 + 8n \equiv -12 \mod (n+2)$. It follows that

$$\begin{aligned}2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3 &\equiv 2^n(-12) + 3 \mod (n+2)\\ &\equiv 3 - 3\cdot 2^{n+2} \mod (n+2).\end{aligned}$$

Therefore, $n+2$ divides $3(2^{n+2}-1)$. Because $n+2$ and 3 are relatively prime, then $2^{n+2} - 1 \equiv 0 \mod (n+2)$—another contradiction to the lemma.

Finally, assume that $n+3$ is even but $n+2$ is divisible by 3. In this case $n+4$ is odd and not divisible by 3. For the expression in Equation (5) to be an integer,

$n+4$ must divide $2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3$. Note that $n^3+3n^2+8n = (n^2-n+12)(n+4)-48$ and thus $n^3+n^2+8n \equiv -48 \mod (n+4)$. Therefore,

$$\begin{aligned}2^n n^3 + 3\cdot 2^n n^2 + 2^{n+3}n + 3 &\equiv 2^n(-48)+3 \mod (n+4)\\ &\equiv -2^n 2^4\cdot 3+3 \mod (n+4)\\ &\equiv 3-3\cdot 2^{n+4} \mod (n+4).\end{aligned}$$

It follows that $n+4$ divides $3(2^{n+4}-1)$. Since $n+4$ and 3 are relatively prime, then $n+4$ divides $2^{n+4}-1$; this is the final contradiction of the lemma and it completes the proof. ∎

Acknowledgment The author wishes to thank Moubariz Garaev for his helpful advice on this note.

REFERENCES

1. D. López-Aguayo, Online supplement to "Nonintegrality of Binomial Sums and Fermat's Little Theorem," www.maa.org/mathmag/supplements.

Summary. Problem 1942 from the April 2014 issue of the Magazine asks whether a particular binomial sum is nonintegral. We pose an open question whether there are infinitely many integers r for which the requisite sum is nonintegral when $k+1$ is replaced by $k+r$. We prove that the sum is nonintegral for $r=2$, 3, and 4 by an application of Fermat's little theorem.

DANIEL LÓPEZ-AGUAYO (MR Author ID 1105399) is a Ph.D. candidate at the Center of Mathematical Sciences UNAM working under the supervision of Raymundo Bautista. His field of research is the representation theory of finite dimensional algebras. He also has a keen interest in homological algebra, module theory, and the history of mathematics.