A PROOF OF THE SEVEN CUBE THEOREM

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It was proved by Landau in 1909 that every sufficiently large number is representable as the sum of eight cubes of positive integers†. This was one of the few results on Waring's problem which was not superseded when Hardy and Littlewood developed their powerful analytical method. It was not until 1942 that Linnik proved that every sufficiently large number is the sum of seven positive integral cubes‡ His proof is very difficult, and requires some deep results about the representation of numbers by ternary quadratic forms.

I give here a simple proof of the seven cube theorem, which uses nothing more recondite than the estimate, now well known, for the number of primes in an arithmetic progression. This estimate (Lemma 1 below) resulted from the work of Siegel§ on the magnitude of the class-number.

I am greatly indebted to Prof. Davenport for his help and encouragement in the publication of this paper, as well as for supplying references and putting it into a suitable form.

A further paper giving some generalizations of the seven cube theorem is in preparation.

LEMMA 1. *Let $\pi(X;\, k, l)$ denote the number of primes p satisfying $p \equiv l \pmod{k}$ and $p \leqslant X$, where k and l are coprime. Then for any $\epsilon > 0$ there is a positive number $C = C(\epsilon)$ such that*

$$(1) \qquad \pi(X;\, k, l) = \frac{1}{\phi(k)} \int_2^X \frac{dt}{\log t} + O\left(Xe^{-A(\log X)^{\frac{1}{2}}}\right) + O\left(\frac{X^{1-Ck^{-\epsilon}}}{\phi(k) \log X}\right),$$

where A is a certain positive absolute constant. The constants implied by the symbol O are independent of k and l.

LEMMA 2. *If X is sufficiently large, $k < (\log X)^{100}$ and $(k, l) = 1$, there is a prime p satisfying $p \equiv l \pmod{k}$ and*

$$(2) \qquad X < p < 1{\cdot}01X.$$

* Received 28 August, 1950; read 17 November, 1950.

† The proof is given in his *Vorlesungen über Zahlentheorie*, 2, 29–30.

‡ *Comptes Rendus* (*Doklady*) *Acad. Sci. U.S.S.R.*, 35 (1942), 162; and *Recueil Math.*, 12 (1943), 218–224. See also G. Pall, *American J. of Math.*, 64 (1942), 503–513.

§ *Acta Arithmetica*, 1 (1935), 83–86. A simple proof has been given recently by Estermann in this *Journal*, 23 (1948), 275–279. For the actual result (1), it is only necessary to substitute Siegel's result $\sigma_1 \leqslant 1 - Ck^{-\epsilon}$ in the theorem of Page, *Proc. London Math. Soc.* (2), 39 (1935), 116–141.

Proof. Subtracting (1) from the same formula with $1{\cdot}01X$ for X, the main term is

$$\frac{1}{\phi(k)}\int_X^{1\cdot01X}\frac{dt}{\log t} > \frac{0{\cdot}01X}{\phi(k)\log(1{\cdot}01X)}.$$

It suffices to prove that the two error terms are negligible in comparison with this. Now for large X,

$$Xe^{-A(\log X)^{\frac12}} < \frac{X}{(\log X)^{102}} < \frac{X}{\phi(k)(\log X)^2}.$$

Also, taking $\epsilon = 1/200$, we have $k^{\epsilon} < (\log X)^{\frac12}$, and

$$X^{-Ck^{-\epsilon}} < e^{-C(\log X)^{\frac12}}.$$

The result follows.

LEMMA 3. *Let N be a positive integer, and suppose there exist distinct primes p, q, r such that*

(3) $$p \equiv q \equiv r \equiv -1 \pmod 6,$$

(4) $$r < q < 1{\cdot}01r,$$

(5) $$\tfrac{3}{4}q^{18}p^3 < N < q^{18}p^3,$$

(6) $$N \equiv 3p \pmod{6p},$$

(7) $$4N \equiv r^{18}p^3 \pmod{q^6},$$

(8) $$2N \equiv q^{18}p^3 \pmod{r^6}.$$

Then N is representable as the sum of six positive integral cubes.

Proof. The conditions (4) and (5) imply that

(9) $$(4q^{18}+2r^{18})p^3 < 8N < (4q^{18}+8r^{18})p^3.$$

The congruences (6), (7) and (8) obviously imply that

$$8N \equiv (4q^{18}+2r^{18})p^3+18q^6r^6p \pmod{q^6r^6p}.$$

Now $8N \equiv 24 \pmod{48}$, and the expression on the right is $\equiv 0 \pmod 3$ and $\equiv 8 \pmod{16}$, and so is also $\equiv 24 \pmod{48}$. We can therefore write

$$8N = (4q^{18}+2r^{18})p^3+18q^6r^6p+48q^6r^6pu.$$

By (9) we have

$$0 < 6q^6r^6p(8u+3) < 6r^{18}p^3,$$

that is, $$0 < 8u+3 < q^{-6}r^{12}p^2.$$

By a classical result, $8u+3$ is representable as $x^2+y^2+z^2$, where x, y, z are odd positive integers, each obviously less than $q^{-3}r^6p$.

We now have

$$\begin{aligned} 8N &= (4q^{18}+2r^{18})p^3+6q^6r^6p(x^2+y^2+z^2) \\ &= (q^6p+r^3x)^3+(q^6p-r^3x)^3+(q^6p+r^3y)^3+(q^6p-r^3y)^3 \\ &\qquad +(r^6p+q^3z)^3+(r^6p-q^3z)^3. \end{aligned}$$

The numbers in brackets are all positive and even, which proves the result.

THEOREM. *Every sufficiently large positive integer is representable as the sum of seven positive integral cubes.*

Proof. If n is sufficiently large, we can find two primes q and r which satisfy (3) and (4) and are less than $(\log n)^2$, and neither of which is a factor of n. For by Lemma 1, the number of primes up to X that are $\equiv -1 \pmod 6$ is asymptotically $\frac{1}{2}X/(\log X)$. Taking X to be first $(\log n)^2$ and then $100\,(\log n)^2/101$, and subtracting, we have asymptotically

$$\frac{(\log n)^2}{404 \log\log n}$$

primes, any two of which satisfy (3) and (4) and are less than $(\log n)^2$. But the number of distinct prime factors of n is $O(\log n/\log\log n)$, which is of lower order.

Choose two such primes q and r, and write

$$X = n^{\frac{1}{3}}q^{-6}. \tag{10}$$

Since every number prime to qr is congruent to a cube to the modulus q^6 and also to the modulus r^6, we can find a number l such that

$$4n \equiv r^{18}l^3 \pmod{q^6}, \tag{11}$$

$$2n \equiv q^{18}l^3 \pmod{r^6}. \tag{12}$$

Take $k = 6q^6r^6$ in Lemma 2. The hypothesis of the lemma is satisfied, since $k < 6(\log n)^{24}$ and $\log X > \frac{1}{4}\log n$ if n is sufficiently large. Hence there exists a prime p satisfying

$$\begin{aligned} &p \equiv -1 \pmod 6, \\ &p \equiv l \pmod{q^6r^6}, \\ &X < p < 1{\cdot}01X. \end{aligned}$$

By (11) and (12) we have

$$4n \equiv r^{18}p^3 \pmod{q^6}, \qquad 2n \equiv q^{18}p^3 \pmod{r^6}.$$

Now choose an integer t so that

$$t^3 \equiv n-3p \pmod{6p},$$

$$t \equiv 0 \pmod{q^2 r^2},$$

$$0 < t \leqslant 6p q^2 r^2,$$

which is certainly possible since every number is congruent to a cube (mod $6p$). Now $N = n - t^3$ satisfies the congruences (6), (7) and (8) of Lemma 3. Also

$$n - t^3 < n = q^{18} X^3 < q^{18} p^3,$$

$$n - t^3 \geqslant n - 216 p^3 q^6 r^6 = q^{18} X^3 - 216 p^3 q^6 r^6$$

$$> (1{\cdot}01)^{-3} q^{18} p^3 - 216 q^6 r^6 p^3 > \tfrac{3}{4} q^{18} p^3.$$

All the conditions of Lemma 3 are satisfied, since obviously p is different from q and r by (2) and (10). Hence $n - t^3$ is representable as the sum of six positive integral cubes, which proves the theorem.

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George Henry Livens, Professor of Mathematics in the University College of South Wales and Monmouthshire, Cardiff, died on March 26, 1950.

Born in 1886, he received his early education at the Latymer Upper School, Hammersmith, before going up to Cambridge in 1906, having been elected to an open Scholarship at Jesus College. He took Part I, then the main part of the Tripos, in 1909, the last year in which the lists were published in order of merit, and was one of a group of five bracketed as fourth wranglers, the first three being P. J. Daniell, E. H. Neville and L. J. Mordell.

In 1910 Livens took Part II of the Tripos and was placed, together with all the men who took Part II in that year, in Class I, Division 2. In 1911 he was awarded the first Smith's prize for an essay entitled "The influence of density on the position of the emission and absorption lines in a gas-spectrum". In the same year he was elected a Fellow of Jesus College and published his first paper.

Meanwhile he had been appointed lecturer in Geometry at the University of Sheffield, where he stayed until 1919, when he became senior mathematical lecturer at the University of Manchester. He was elected to the Chair of Mathematics at Cardiff in 1922.