Liouvillian Solutions of Third Order Linear Differential Equations: New Bounds and Necessary Conditions

EXTENDED ABSTRACT

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Abstract

In this note, we report on recent work [5] where we reinterpret the Kovacic algorithm for solving second order linear differential equations and show how this can be extended to give a similar explicit algorithm for third order equations. We also give a set of powerful necessary conditions extending (and improving) those of Kovacic.

1 Introduction

A given differential equation $L(y) = y''' + a_2y'' + a_1y' + a_0y = 0$ whose coefficients belongs to the field k, of rational functions over $\mathbb{C}$ or $\overline{\mathbb{Q}}$, can be transformed in

$$\begin{aligned} L_{SL}(y) &= y''' + \left(a_1 - \frac{{a_2}^2}{3} + a_2'\right) y' \\ &\quad + \left(a_0 - \frac{a_1a_2}{3} - \frac{a_2''}{3} + \frac{2{a_2}^3}{27}\right) y, \end{aligned}$$

using the variable transformation

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$y = z \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$. We assume from now on that a third order differential equation $L(y) = 0$ is of the form

$$L(y) = y''' + Ay' + By = 0, \qquad A, B \in k.$$

A *liouvillian function* over a function field k is a function that can be written using functions of k, the symbol $\int$, the exponential function, algebraic functions and composition of these functions.

Example 1: The Function $\sqrt{x^2+1}\log(x^2+1)$ is a liouvillian function over $(\mathbb{Q}(x), \frac{\mathrm{d}}{\mathrm{d}x})$ and is a solution of the differential equation:

$$L(y) = \frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}\,y(x)}{\mathrm{d}x} + \frac{x^2}{x^4+2x^2+1} y(x) = 0$$

∎

In [1] J. Kovacic gives a very explicit algorithm which allows one to compute the liouvillian solutions of a second order differential equations. The algorithm is now available in most computer algebra systems. A general algorithm for differential equations of arbitrary order has been given by the first author (cf. [4]). This general algorithm, although constructive, is much less explicit than the Kovacic algorithm.

Kovacic's algorithm is divided into three cases. We interpret each of these as corresponding to certain behavior of the galois group. We also show that when a similar analysis is made for third order equations, one gets a much better algorithm (cf. [5]) than the general algorithm of [4]. A very useful feature of Kovacic's algorithm is that it gives very strong necessary conditions that allow one to frequently bypass some of the more involved calculations of the full algorithm. In this note, we shall describe the corresponding necessary conditions that our algorithm gives for third order equations.

2 Outline of Algorithm

We see Kovacic's algorithm to be based on the following facts. To any homogeneous linear differential equation $L(y) = 0$ of order n with coefficients in $\overline{\mathbb{Q}}(x)$, one can associate a group of $n \times n$ matrices $Gal(L) \subseteq GL(n, \overline{\mathbb{Q}})$ called the differential galois group of $L(y) = 0$. Differential and algebraic properties of the equation are mirrored by group theoretic properties of this group. Given $G \subseteq GL(n, \overline{\mathbb{Q}})$, one can make the following distinctions. The linear group G is said to be *reducible* if there exists a proper, non-trivial G-invariant subspace of $\overline{\mathbb{Q}}^n$; otherwise it is said to be *irreducible*. An irreducible linear group G is said to be *imprimitive* if there are non trivial subspaces $V_1, \ldots, V_k$ of $\overline{\mathbb{Q}}^n$ such that each element of G permutes these subspaces; otherwise G is said to be *primitive*. Kovacic's algorithm proceeds by testing if the differential galois group has one of these properties.

A differential equation $L(y)$ with coefficients in k is called *reducible* if $L(y)$ can be written as $L_1(L_2(y))$, where $L_1(y)$ and $L_2(y)$ are differential equations with coefficients in k of order > 0.

Example 2:

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} - \frac{8x^2-2x-3}{2x(4x+1)}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{12x+11}{4x(4x+1)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{4x+5}{4x(4x+1)} = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{4x^2+x-1}{x(4x+1)}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{4x}\right)$$

If k is the field of rational functions over $\overline{\mathbb{Q}}$, then an algorithm for the decomposition of $L(y)$ as a product of irreducible differential operators is given for example in [3]. A differential equation is reducible if and only if its differential galois group is reducible.

Case 1 of Kovacic's algorithm tests to see if a second order differential equation $L(y) = y'' - ry = 0$ ($r \in \overline{\mathbb{Q}}(x)$) is reducible. If it is, $L(y) = L_1(L_2(y))$ where $L_2(y) = y' + ay$ ($r \in \overline{\mathbb{Q}}(x)$) ($a \in \overline{\mathbb{Q}}(x)$) so $y = \exp\left(\int a\right)$ will be a liouvillian solution of $L(y) = 0$. Another liouvillian solution can be found using the reduction of order (d'Alembert's) method.

Assuming $Gal(L) \subseteq GL(2, \overline{\mathbb{Q}})$ (and therefore $y'' - ry = 0$) is irreducible, the algorithm proceeds to test if $Gal(L) \subseteq GL(2, \overline{\mathbb{Q}})$ is imprimitive. This will be the case if and only if $L(y) = 0$ has a solution y such that y'/y is algebraic of degree 2 . Case 2 of Kovacic's algorithm tests to see if this is the case by trying to construct the minimal polynomial of y'/y. If it succeds, a liouvillian solution is found. If it fails, one can assume that the galois group is a primitive linear group. One then shows that (under the assumption that there is a liouvillian solution) $Gal(L) \subseteq GL(2, \overline{\mathbb{Q}})$ must be finite and that $L(y) = 0$ will have a solution y such that y'/y is algebraic of degree 4, 6 or 12. Case 3 of Kovacic's algorithm attempts to construct the minimum polynomial of y'/y. If it succeeds, a liouvillian solution is found. If not, then $L(y) = 0$ has no liouvillian solutions.

Our algorithm now proceeds along the same lines. Case 1 tests to see if a third order differential equation $L(y) = 0$ is reducible. If it is, the question of finding all liouvillian solutions is reduced to the same question for a second order equation. If $L(y) = 0$ is irreducible, the algorithm proceeds to case 2 which tests if the differential galois group is an imprimitive linear group. This will happen if and only if $L(y) = 0$ has a solution y such that y'/y is algebraic of degree 3. The algorithm attempts to construct the minimal polynomial of y'/y. If it fails, $Gal(L) \subseteq GL(3, \overline{\mathbb{Q}})$ must be a primitive linear group. In this case (under the assumption that $L(y) = 0$ has a liouvillian solution) the differential galois group must be finite. The possible differential galois groups can be determined and using their character tables we can further determine that $L(y) = 0$ must have a solution y such that y'/y is algebraic of degree 6, 9, 21 or 36 (these numbers are sharp and improve the bounds of [6]). Case 3 of our algorithm attempts to construct the minimal polynomial of y'/y. If it succeds, exponentials of roots of this polynomial yield a full set of liouvillian solutions and if not the equation has no liouvillian solution.

In what follows, we present strong necessary conditions that are a consequence of this algorithm. To do this we need the following concepts. If $L(y) = a_n(x)y^{(n)} + \ldots + a_0(x)y$ has a solution of the form $y = x^\rho \sum_{j\geq 0} c_j x^j$, $c_0 \neq 0$, then formally substituting this expression into $L(y) = 0$ and examining the coefficient of the smallest power of x, one sees that ρ will satisfy the equation

$$P(\rho) = \sum a_{k_0}(\rho)_k = 0$$

where $(\rho)_j = \rho(\rho-1)\ldots(\rho-(j-1))$, $a_j(x) = a_{j_0}x^{j_0} + \ldots$ (higher order terms) and the sum is over all k with $k_0 - k = \min_{0\leq j\leq n}\{j_0 - j\}$. $P(\rho)$ is called the *indical equation of $L(y)$ (at 0)*. We shall refer to the roots of the indicial polynomial as *exponents of $L(y)$ (at 0)*.

Analogously, we can define the indicial equation at any c and exponents at c. Via the transformation $x = 1/t$, $\frac{d}{dx} = -t^2\frac{d}{dt}$ we can also define the indicial equation and exponents at infinity by considering the point $t = 0$ of the transformed equation. We say that $x = c$ is an *ordinary point* if, for $0 \leq i \leq n$, $a_i(x)/a_n(x)$ is analytic at $x = c$ (for $c = \infty$ this property is related to the point $t = 0$ of the transformed equation). If c is not an ordinary point it is called a singular point.

A third order differential equation has at most 3 exponents at each singular point. The number of exponents can also be less than 3:

Example 3: The only singular point of the differential equation

$$L(y) \;=\; \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y$$

is ∞ and the only exponent of $L(y) = 0$ at ∞ is $1/2$.

If we considere an "arbitrary" solution u of $L(y) = 0$, then differentiation of the function $Y = u^3$ gives:

$$\begin{aligned} Y &= u^3 \\ Y' &= 3u'u^3 \\ Y'' &= 3u''u^3 + 6u'^2u \\ &\ldots \end{aligned}$$

Using the differential equation $L(y) = 0$ we can always replace higher derivatives of u by derivatives of order 1 or 2 (e.g. $u''' = -Au' - Bu$). Thus on the right hand side there are only terms $u^i u'^j u''^k$, where i, j, k are positive integrers such that $i + j + k = 3$. Thus there are at most 10 unknowns, and the function Y must satisfy a differential equation of order at most 10. We denote the differential equation of lowest order which has Y as a solution by $L^{\circledS^3}(y)$.

Example 4: For the the differential equation equation $L(y) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y$ we get

$$\begin{aligned} L^{\circledS^3}(y) \;=\; & \frac{\mathrm{d}^7y}{\mathrm{d}x^7} - 56x\frac{\mathrm{d}^5y}{\mathrm{d}x^5} - 140\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + 784x^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \\ & +2352x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4(576x^3 - 295)\frac{\mathrm{d}y}{\mathrm{d}x} \\ & -3456x^2y. \end{aligned}$$

3 First Step

We begin by looking for solutions of $L(y) = 0$ which are of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i.

For a given equation $L(y) = 0$ of degree n, there are only a finite number of singularities and at each singularity c_i, there are at most n possible exponents a_i. If $L(y) = 0$ has a solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i, then there are at most a finite number of possibilities for $\prod_i(x-c_i)^{a_i}$. For each possible term $\prod_i(x-c_i)^{a_i}$, we considere the differential equation:

$$\tilde{L}(y) \;:=\; \frac{L\left(y\prod_i(x-c_i)^{a_i}\right)}{\prod_i(x-c_i)^{a_i}} = 0.$$

After simplification, the coefficients of $\tilde{L}(y)$ belong also to k. If $L(y) = 0$ has a solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, then $\tilde{L}(y) = 0$ has a solution $P(x) \in k$. We thus have to compute a basis $\{p_1(x), \cdots, p_k(x)\}$ ($k \leq n$) of the polynomial solutions of $\tilde{L}(y) = 0$ to get a basis of the solutions of the form $P(x)\prod_i(x-c_i)^{a_i}$. An algorithm to compute the rational solutions of a linear differential equation is well known (see [2]).

We note the following necessary condition for $L(y) = 0$ to have such a solution:

Lemma 3.1 *If a linear differential equation $L(y) = 0$ of degree n has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i, then at each finite singular point c_i there are exponents a_i such that for some exponent e_∞ at ∞, the sum $(\sum_i a_i) + e_\infty$ is a non-positive integer.*

4 Necessary conditions for the first case

If the necessary conditions stated in this section does not hold, then $L(y) = 0$ is an irreducible differential equation. If the differential equation $L(y) = 0$ is reducible, then the problem of finding the liouvillian solutions of $L(y) = 0$ can be reduced to the computation of the liouvillian solutions of linear differential equations of lower order (see [5]).

For the Laurent series at 0 or ∞ of A and B we introduce the following notation:

$$\begin{aligned} A &= \alpha x^a + \dots \quad \text{(higher order terms)} \\ B &= \beta x^b + \dots \quad \text{(higher order terms)} \\ A &= \alpha_\infty x^{a_\infty} + \dots \quad \text{(lower order terms)} \\ B &= \beta_\infty x^{b_\infty} + \dots \quad \text{(lower order terms)} \end{aligned}$$

For $A = A_1/A_2$ where $A_i \in \overline{\mathbb{Q}}[x]$, a_∞ denotes $deg_x(A_1) - deg_x(A_2)$.

The following necessary condition is a consequence of the fact that if $L(y) = 0$ is reducible then either $L(y) = 0$ or its adjoint, $L^*(y) = 0$, has a solution y such that y'/y is a rational function.

Necessary condition 1 *Let $L(y) = 0$ be a third order linear differential equation with coefficients in $\overline{\mathbb{Q}}(x)$ such that $L(y) = 0$ (resp. the adjoint $L^*(y) = 0$ of $L(y) = 0$) has no solutions of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $P(x) \in \overline{\mathbb{Q}}[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ (resp. $L^*(y) = 0$) and a_i are exponents of $L(y) = 0$ (resp. $L^*(y) = 0$) at c_i. If $L(y)$ is reducible over $\overline{\mathbb{Q}}(x)$, then one of the following holds:*

- *At some finite singular point of $L(y) = 0$ the coefficients A and B have exponents a and b such that ($a \in 2\mathbb{Z}$, $a < -4$ and $3a/2 \leq b$) or ($b \in 3\mathbb{Z}$, $b < -6$ and $2b/3 \leq a$) or ($b < -6$, $b < a - 1$ and $a < -4$).*
- *($a_\infty \in 2\mathbb{Z}$, $a_\infty \geq 0$ and $3a_\infty/2 \geq b_\infty$) or ($b_\infty \in 3\mathbb{Z}$, $b_\infty \geq 0$ and $2b_\infty/3 \geq a_\infty$) or ($b_\infty = a_\infty = 0$) or ($b_\infty \geq 3$, $a_\infty \geq 2$ and $b_\infty > a_\infty$).*

Example: We use the above necessary conditions to show that for the following equation (which is the second symmetric power of the Airy equation)

$$L(y) \quad = \quad \frac{d^3y}{dx^3} - 4x\frac{dy}{dx} - 2y,$$

case 1 (of a reducible eqaution $L(y) = 0$) cannot occur.
The point ∞ is the only singular point of $L(y)$, and the only exponent at ∞ is $1/2$. From Lemma 3.1 we get that $L(y) = 0$ has no solution of the form $P(x)\prod_i (x - c_i)^{a_i}$. Since $L(y) = 0$ has no finite singular point, the above first condition does not hold. Since $a_\infty = 1$ and $b_\infty = 0$ the second condition above does not hold. Thus $L(y)$ is an irreducible equation. ∎

5 Necessary conditions for the second case

In this section we assume that $L(y) = 0$ is an irreducible equation (which is the case if the previous necessary conditions for case 1 does not hold). The following necessary conditions correspond to the case where the differential galois group of $L(y) = 0$ is an imprimitive linear subgroup of $SL(3, \mathbb{C})$ (see e.g. [5]).This necessary condition is a consequence of the fact that if $Gal(L) \subseteq SL(3, \overline{\mathbb{Q}})$ is imprimitive, then there exist three solutions y_1, y_2, y_3 of $L(y) = 0$ such that $(y_1y_2y_3)^2$ is left fixed by $Gal(L)$ and so lies in $\overline{\mathbb{Q}}(x)$.

Necessary condition 2 *Let $L(y) = 0$ be an irreducible third order linear differential equation $L(y) = 0$ over $\overline{\mathbb{Q}}(x)$ with imprimitive differential galois group $\mathcal{G}(L) \subseteq SL(3, \mathbb{C})$. The third order symmetric power $L^{\circledS^3}(y) = 0$ of $L(y) = 0$ must have a non trivial solution of the form $P(x)\prod_i(x - c_i)^{a_i}$, where $P(x) \in \overline{\mathbb{Q}}[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and $a_i \in \frac{1}{2}\mathbb{Z}$.
In particular, at any singularity, $L^{\circledS^3}(y) = 0$ must have an exponent of the form $a/2$, where $a \in \mathbb{Z}$.*

Example: We now use the above necessary conditions to show that, for the second symmetric power of the Airy equation

$$L(y) \;=\; \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y,$$

the above necessary conditions does not hold. We have to compute $L^{\circledS^3}(y)$, which is given in example 4, and to test if $L^{\circledS^3}(y) = 0$ has a solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x) \in \overline{\mathbb{Q}}[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and $a_i \in \frac{1}{2}\mathbb{Z}$. In Section 3 we show how this problem can always be reduced to the problem of finding a rational solution of a differential equation with coefficients in $\overline{\mathbb{Q}}(x)$. In the above case, since 3/2 is the only exponent at the only singularity ∞, the sum of exponents can never be a non-positive integer. It followns from Lemma 3.1 that $L^{\circledS^3}(y) = 0$ has no non trivial solution of the form $P(x)\prod_i(x-c_i)^{a_i}$, where $P(x) \in \overline{\mathbb{Q}}[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and $a_i \in \frac{1}{2}\mathbb{Z}$. Thus the above necessary conditions does not hold.

6 Necessary conditions for the third case

In this section we assume that $L(y) = 0$ is an irreducible equation (which is the case if the previous necessary conditions for case 1 does not hold). The following necessary conditions correspond to the case where the differential galois group of $L(y) = 0$ is a primitive linear subgroup of $SL(3,\mathbb{C})$ (see e.g. [5]). In this case all solutions must be algebraic functions over k. These necessary conditions are a consequence of two facts in this case. The first is that all exponents must rational and the denominator of any exponent (in simplified form) must divide the order of an element in $Gal(L)$. The second is that for each of these groups we can show that some product $y_1 \cdot \ldots \cdot y_m$ of solutions (e.g. $m = 36$ in the first case) must lie in $\overline{\mathbb{Q}}(x)$.

Necessary condition 3 *Let $L(y)$ be an irreducible third order linear differential equation with galois group a primitive group $\mathcal{G}(L) \subset SL(3,\mathbb{C})$. If $L(y) = 0$ has a liouvillian solution then $L(y) = 0$ must have n distinct rational exponents at any singularity and one of the following must hold:*

1. *The exists positive integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 36$ and at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,4,5,6,12,15\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
2. *The exists positive integers n_1, n_2, n_3 such that $\sum_{i=1}^3 n_i = 6$ and at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,5\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
3. *The exists positive integers n_1, n_2, n_3 such that $\sum_{i=1}^3 n_i = 6$ and at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,5,6,15\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $3(\sum_{i=1}^3 n_i e_i) \in \mathbb{Z}$.*
4. *There exists positive integers n_1, n_2, n_3 such that $\sum_{i=1}^3 n_i = 21$ and at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,4,7\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $\sum_{i=1}^3 n_i e_i \in \mathbb{Z}$.*
5. *There exists positive integers n_1, n_2, n_3 such that $\sum_{i=1}^3 n_i = 21$ and at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,4,6,7,12,21\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $3(\sum_{i=1}^3 n_i e_i) \in \mathbb{Z}$.*
6. *There exists positive integers n_1, n_2, n_3, such that $\sum_{i=1}^3 n_i = 9$ and such that at each singular point of $L(y) = 0$:*
   - *(a) there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1,2,3,4,6,9,12,18\}$ and $(a,m) = 1$.*
   - *(b) there exists (possibly repeated) exponents e_1, e_2, e_3 such that $3(\sum_{i=1}^3 n_i e_i) \in \mathbb{Z}$.*

7. *There exists positive integers n_1, n_2, n_3, such that $\sum_{i=1}^{3} n_i = 9$ and such that at each singular point of $L(y) = 0$:*
   (a) *there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1, 2, 3, 4, 6, 12\}$ and $(a, m) = 1$.*
   (b) *there exists (possibly repeated) exponents e_1, e_2, e_3 such that $2(\sum_{i=1}^{3} n_i e_i) \in \mathbb{Z}$.*
8. *There exists positive integers n_1, n_2, n_3, such that $\sum_{i=1}^{3} n_i = 6$ and such that at each singular point of $L(y) = 0$:*
   (a) *there are 3 distinct exponents of the form a/m, where $a \in \mathbb{Z}$ and $m \in \{1, 2, 3, 4, 6, 12\}$ and $(a, m) = 1$.*
   (b) *there exists (possibly repeated) exponents e_1, e_2, e_3 such that $4(\sum_{i=1}^{3} n_i e_i) \in \mathbb{Z}$.*

In order to exclude this case it is enough to show that none of the conditions on the exponents is satisfied. Similar conditions can be tated for second order equations and strengthen the results of Kovacic in this case.

Example: We now show that the above necessary conditions does not hold for

$$L(y) = \frac{d^3y}{dx^3} - 4x\frac{dy}{dx} - 2y.$$

This immediately follows from the fact that $L(y) = 0$ has only one exponent at the singularity ∞.

7 Conclusion

In this paper we give a set of necessary conditions which allow to test if a third order differential equation has liouvillian solutions. In order to test a differential equation one has only to compute the exponents, the equation $L^{\circledS^3}(y) = 0$ and rational solutions of some differential equations of order 3. If none of the conditions holds, then $L(y) = 0$ has no liouvillian solution. We thus show in this paper that

$$L(y) = \frac{d^3y}{dx^3} - 4x\frac{dy}{dx} - 2y = 0$$

has no liouvillian solutions. If some of the conditions hold, then the conditions can be used to avoid unnecessary computations in an algorithm.

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