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Note

An algebraic identity on q-Apéry numbers

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ABSTRACT

By means of partial fraction decomposition, we establish a q-extension of an algebraic identity on rational function due to Chu [W. Chu, A binomial coefficient identity associated with Beukers' conjecture on Apéry numbers, The Electronic Journal of Combinatorics 11 (2004) #N15]. Its limiting case as $q \to 1$ leads to a harmonic number identity closely related to Beukers' well-known conjecture on Apéry numbers.

The classical harmonic numbers $\{H_n\}$ are defined by

$$H_0 := 0 \quad \text{and} \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n = 1, 2, \ldots.$$

They have extensively been studied (see [6, p. 272] for example) and have important applications in combinatorics, number theory and algorithmic analysis. By means of partial fraction decomposition, Chu [2] has recently established an algebraic identity on rational function

$$\frac{x(1-x)_n^2}{(x)_{n+1}^2} = \frac{1}{x} + \sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \frac{-k}{(x+k)^2} + \frac{1+2kH_{n+k}+2kH_{n-k}-4kH_k}{x+k} \right\}. \tag{1}$$

Its limiting case leads to the following harmonic number identity

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \{1+2kH_{n+k}+2kH_{n-k}-4kH_k\} = 0 \tag{2}$$

which has been shown to imply a congruence on Apéry numbers conjectured by Beukers.

During the past two decades, the research on q-series has been active. For a comprehensive coverage of it and its applications to combinatorics, number theory and special functions, the reader can refer to the monograph by Gasper and Rahman [5]. The q-harmonic congruences have been studied in [1,4]. The purpose of this paper is to find the q-extension of Chu's results by means of partial fraction decomposition.

First, we use the standard notation on q-series. For two indeterminates q and a, the q-shifted factorial is defined by

$$(a;q)_0 := 1 \quad \text{and} \quad (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{for } n = 1, 2, \ldots.$$

The q-binomial coefficient (or the Gauss coefficient) is correspondingly given by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

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The q-counterpart of H_n is given by the q-harmonic numbers

$$\mathcal{H}_n(q) := \sum_{k=1}^{n} \frac{1}{[k]_q}, \quad n = 1, 2, \ldots,$$

where

$$[k] \quad \text{or} \quad [k]_q := \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}.$$

For a natural number n, define the q-Apéry number $\mathcal{A}(n)$ by the following q-binomial sum

$$\mathcal{A}(n) := \sum_{k=0}^{n} q^{k(k-2n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2.$$

With these preparations, we are ready to state our main result of this paper as the following general q-algebraic identity.

Theorem 1. *For a natural number n and an indeterminate x, the following algebraic identity holds*

$$\frac{x^{2n}(q/x;q)_n^2}{(1-x)(qx;q)_n^2} = \frac{1}{1-x} + \sum_{k=1}^{n} q^{k(k-2n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \times \left\{ \frac{q^k-1}{(1-xq^k)^2} + \frac{1 - 4[k]\mathcal{H}_k(q) + 2[k]\mathcal{H}_{n+k}(q) + 2q[k]\mathcal{H}_{n-k}(q^{-1})}{1-xq^k} \right\}. \tag{3}$$

Proof. According to the partial fraction decomposition, $f(x)$ can formally be written as

$$f(x) := \frac{x^{2n}(q/x;q)_n^2}{(1-x)(qx;q)_n^2} = \frac{A}{1-x} + \sum_{k=1}^{n} \left\{ \frac{B_k}{(1-xq^k)^2} + \frac{C_k}{1-xq^k} \right\}$$

where the coefficients A and $\{B_k, C_k\}$ remain to be determined.

First, it is almost trivial to evaluate the coefficient:

$$A = \lim_{x\to 1}(1-x)f(x) = \lim_{x\to 1} \frac{x^{2n}(q/x;q)_n^2}{(qx;q)_n^2} = 1.$$

Then, we can analogously determine the coefficient $\{B_k\}$:

$$\begin{aligned} B_k &= \lim_{x\to q^{-k}} (1-xq^k)^2 f(x) = \lim_{x\to q^{-k}} \frac{x^{2n}(1-x)(q/x;q)_n^2}{(x;q)_k^2 (xq^{k+1};q)_{n-k}^2} \\ &= q^{k(k-2n)} \frac{(q^k-1)(q^{1+k};q)_n^2}{(q;q)_k^2 (q;q)_{n-k}^2} = q^{k(k-2n)}(q^k-1) \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2. \end{aligned}$$

Finally applying the L'Hôspital rule, the coefficients $\{C_k\}$ can be computed as follows:

$$\begin{aligned} C_k &= \lim_{x\to q^{-k}} (1-xq^k) \left\{ f(x) - \frac{B_k}{(1-xq^k)^2} \right\} = \lim_{x\to q^{-k}} \frac{(1-xq^k)^2 f(x) - B_k}{1-xq^k} \\ &= \lim_{x\to q^{-k}} \frac{-1}{q^k} \frac{\mathrm{d}}{\mathrm{dx}} \{(1-xq^k)^2 f(x) - B_k\} = \lim_{x\to q^{-k}} \frac{-1}{q^k} \frac{\mathrm{d}}{\mathrm{dx}} \frac{x^{2n}(1-x)(q/x;q)_n^2}{(x;q)_k^2 (xq^{1+k};q)_{n-k}^2} \\ &= \lim_{x\to q^{-k}} \frac{-1}{q^k} \frac{x^{2n}(1-x)(q/x;q)_n^2}{(x;q)_k^2 (xq^{1+k};q)_{n-k}^2} \left\{ \frac{1}{x-1} + \sum_{i=1}^{n} \frac{2}{x-q^i} + \sum_{\substack{j=0 \\ j\neq k}}^{n} \frac{2q^j}{1-xq^j} \right\} \\ &= q^{k(k-2n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \{1 - 4[k]\mathcal{H}_k(q) + 2[k]\mathcal{H}_{n+k}(q) + 2q[k]\mathcal{H}_{n-k}(q^{-1})\}. \end{aligned}$$

This completes the proof of the theorem. □

As applications, we display two examples of Theorem 1.

Switching $1/(1-x)$ to the left side of Eq. (3) and then letting $x \to 1$, we deduce from Theorem 1 immediately the following q-binomial-harmonic number identity.

Corollary 2.

$$\sum_{k=0}^{n} q^{k(k-2n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \{2\,\mathcal{H}_k(q) - \mathcal{H}_{n+k}(q) - q\,\mathcal{H}_{n-k}(q^{-1})\} = 0. \tag{4}$$

We remark that one of the identities due to Chu [3, Example 2] results in the limiting case $q \to 1$ of this identity.

Multiplying by $1 - x$ across Eq. (3) and then letting $x \to \infty$, we obtain another q-binomial-harmonic number identity.

Corollary 3.

$$\sum_{k=1}^{n} q^{2\binom{n-k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \{1 - 4[k]\mathcal{H}_k(q) + 2[k]\mathcal{H}_{n+k}(q) + 2q[k]\mathcal{H}_{n-k}(q^{-1})\} = 1 - q^{2\binom{n+1}{2}}. \tag{5}$$

The limiting case $q \to 1$ of this last identity reduces to (2) clearly.

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