A COLLECTION OF OPEN PROBLEMS

Edited by
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One of the features of the Second International Conference on Combinatorial Mathematics was an emphasis on open problems. Participants were encouraged to present problems in a variety of ways: in the papers themselves, in the discussion periods following the various presentations, and in two special problem sessions each of which lasted several hours.

The collection below consists, for the most part, of problems presented in these special sessions. No attempt has been made to reproduce in this collection those problems which have been formulated explicitly in the individual papers. Thus, the problem-oriented reader is advised that an abundance of conjectures and open questions are scattered throughout the conference proceedings. Indeed, a number of papers contain fairly explicit lists of open problems. These include the papers by Bari, Biggs, Chung and Graham, Frucht, Haggard, Harary and Kabell, Kennedy and Gordon, Levinson and Silverman, Malkevitch, Nešetřil, and Pless. The paper of Erdös deserves special mention, providing as it does an account of a wide range of challenging combinatorial problems.

Some of the problems posed at the Conference have already been solved. Accounts of these solutions are presented immediately following this collection. No doubt other problems will be solved in the future; anyone desiring information as to the current status of a particular problem in this collection is advised to contact the proposer. For this purpose the name and current address of the proposer have been provided with each problem.

For the convenience of the reader, each problem unit has been assigned a descriptive caption, and an effort has been made to group together problems with similar themes. In many instances, background material and references are included. Again, if additional information is desired (or can be provided!) one should consult the proposer.

1. Genus of Regular Graphs and Cayley Graphs

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Let $N(g, d)$ be the number of finite regular graphs of degree d having (minimal) genus g. It is not hard to show that $N(g, d)$ is finite for any $d > 6$ and that $N(1, d)$ is infinite for $d = 3, 4, 5, 6$. With a bit more effort it can be shown that $N(g, d)$ is also infinite for all $g \geq 1$, $d = 3, 4$.

QUESTION 1. Is $N(g, d)$ infinite for all $g \geq 1$, $d = 5, 6$?

The genus of a group is the least genus of any of its Cayley color graphs. Let $C(g)$ be the number of groups of genus g. Clearly $C(0)$ and $C(1)$ are infinite. We have shown, however, that $C(g)$ is finite for $g > 1$. V. K. Proulx has asked:

QUESTION 2. Is $C(2) = 0$?

It also appears that $C(3) = 1$. Let $CN(g)$ be the number of Cayley graphs of genus g. Unfortunately, $C(g) < \infty$ does not imply $CN(g) < \infty$.

QUESTION 3. Is $CN(g)$ finite for all $g > 1$?

2. Duke's Conjecture for Genus Three Graphs

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Duke's conjecture states that for any connected graph, G, $b(G) \geq 4\gamma(G)$, where $b(G)$ is the betti number $q - p + k$ and $\gamma(G)$ is the genus. Here p, q, and k denote the number of points, lines and components, respectively, in G. The conjecture is known to be true for $\gamma = 0, 1, 2$, and false for $\gamma \geq 4$. The case $\gamma = 3$, i.e., $b(G) \geq 12$ for genus 3 graphs, is unresolved. It can be shown that $K_{5,5}$ is a genus 3 graph which satisfies the conjecture. Furthermore, we have shown the following: Any connected genus 3 graph which is regular of degree at least 7 satisfies Duke's conjecture.

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3. Hamiltonicity in Planar and Toroidal Graphs

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It is known [2] that every 4-connected planar graph is hamiltonian and that every 6-connected graph of genus 1 is hamiltonian [1]. A graph is hamiltonian-connected if there exists a hamiltonian path between every pair of its points. A graph G is 1-hamiltonian if G is hamiltonian, and G-v is hamiltonian, for every $v \in V(G)$. It is known that neither of these properties implies the other.

PROBLEM 1. Is every 4-connected planar graph hamiltonian-connected (1-hamiltonian)?

PROBLEM 2. Is every 6-connected graph of genus 1 hamiltonian-connected (1-hamiltonian)?

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4. Conjectures on a Shortest-Route Algorithm

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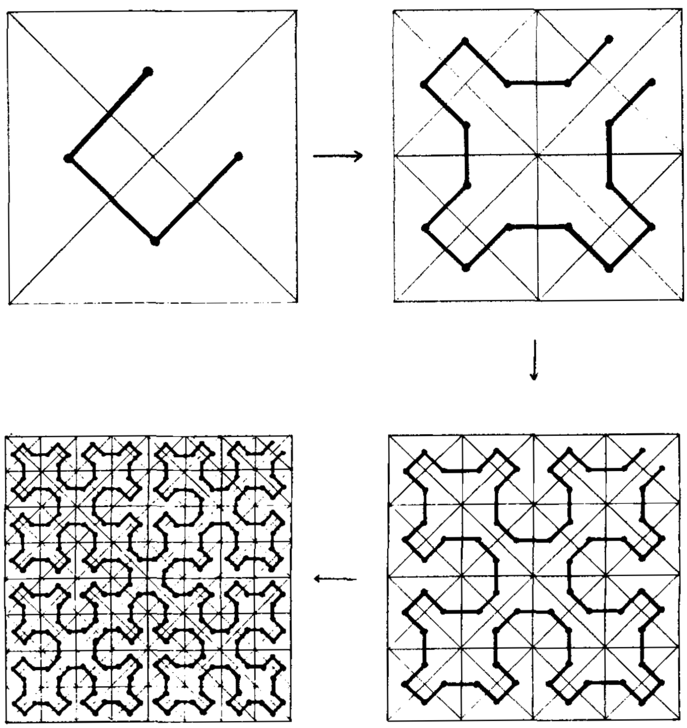
A variety of "pathological" curves can be constructed which cover the plane. One such is the limit of the process illustrated in the figure. Call such a curve C.



Consider an arbitrary pattern on n points distributed in the plane. Let R be a route which visits each point *in the order in which they occur* on curve C. That is, pairs of points are joined by straight lines and R is a circuit that visits each of the n points taken in the prescribed order.

Initiated by Fremlin [1], we conjecture that:

(1) The route R is a close approximation to the shortest route through the n points.

(2) The route R is shorter than any other that can be computed by a small-order polynomial algorithm.

The shortest route (traveling salesman) problem is NP-complete. Approximation algorithms have been described recently [2]. Based on random (Poisson) distribution of up to 8000 points in a plane, Fremlin has used Monte Carlo methods [1] to support conjecture (2).

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5. Hamiltonian Circuits in Polyhedral Graphs

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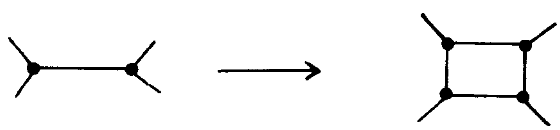
A plane, 3-connected graph G is called a multi-k-gon ($k = 3, 4, 5$) if every face of G has a multiple of k for the number of its sides. There exist 3-valent multi-3-gons which have no hamiltonian circuit. They can be obtained from a 3-valent nonhamiltonian graph by sequentially replacing some vertices by triangles (i.e., vertex truncations).

David Barnette [1] has conjectured that any 3-valent, 3-connected, plane, bipartite graph has a hamiltonian circuit. The answer to Problem 1 below may be negative, even if Barnette's conjecture is false.

Problem 1. Do there exist 3-valent multi-4-gons which have no hamiltonian circuit?

Problem 2. Do there exist 3-valent multi-5-gons which have no hamiltonian circuit?

Problem 3. Can any 3-valent, plane, 3-connected graph be transformed into a multi-4-gon by a sequence of "edge shaves"? (An edge shaving is a transformation of an edge as shown in the figure.)



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6. Extremal Polyhedral Graphs

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If P is a polyhedron in the sense of Steinitz [1, 4], then let $V(P)$, $E(P)$, and $F(P)$ denote the vertex-, edge-, and face-set, respectively, of P. If a face $g \in F(P)$ is bounded by exactly k edges, then we say that g is a k-gon.

The set of all the k-gons in $F(P)$ will be denoted by $F_k(P)$, and we put $f_k(P) = |F_k(P)|$.

Let $\mathfrak{P}$ denote the set of all trivalent polyhedra and let us define the subsets $\mathfrak{P}_i$ of $\mathfrak{P}$ ($i = 0, 1, \ldots, 9$) in the following way:

$$\mathfrak{P}_0 = \mathfrak{P}$$

$$\mathfrak{P}_1 = \{P \in \mathfrak{P} \mid f_k(P) = 0, \text{ when } k \not\equiv 0 \pmod 2\}$$

$$\mathfrak{P}_2 = \{P \in \mathfrak{P} \mid f_k(P) = 0, \text{ when } k \not\equiv 0 \pmod 3\}$$

$$\mathfrak{P}_3 = \{P \in \mathfrak{P} \mid f_3(P) = 0\}$$

$$\mathfrak{P}_4 = \{P \in \mathfrak{P} \mid f_4(P) = 0\}$$

$$\mathfrak{P}_5 = \{P \in \mathfrak{P} \mid f_5(P) = 0\}$$

$$\mathfrak{P}_6 = \{P \in \mathfrak{P} \mid f_6(P) = 0\}$$

$$\mathfrak{P}_7 = \mathfrak{P}_3 \cap \mathfrak{P}_4$$

$$\mathfrak{P}_8 = \mathfrak{P}_3 \cap \mathfrak{P}_5$$

$$\mathfrak{P}_9 = \mathfrak{P}_4 \cap \mathfrak{P}_5$$

Recently solutions of some extremal problems on polyhedra have been found that deal with the values of the functions π, ρ, σ, and τ defined as follows.

Let $(u, v) = e$ be an edge of a polyhedron $P \in \mathfrak{P}$ with the neighborhood as described in the figure. Then we put

$$\pi(e) = m + n$$

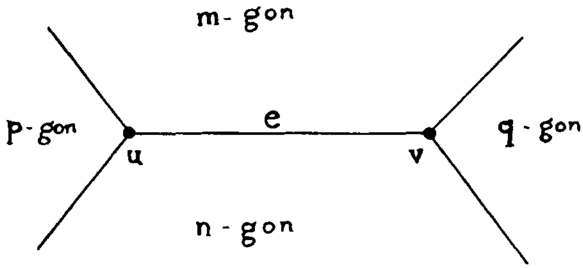
$$\rho(e) = p + q$$

$$\sigma(u) = m + n + p$$

$$\sigma(v) = m + n + q$$

$$\tau(e) = m + n + p + q$$

The functions π, ρ, σ, and τ are defined in the following way: $\sigma(P) = \min\{\sigma(v) \mid v \in V(P)\}$ and, for $\xi \in \{\pi, \rho, \tau\}$, $\xi(P) = \min\{\xi(e) \mid e \in E(P)\}$. Finally, when $\xi \in \{\pi, \rho, \sigma, \tau\}$



and $i \in \{0, 1, \ldots, 9\}$, we define $\xi(\mathfrak{P}_i) = \max\{\xi(P) \mid P \in \mathfrak{P}_i\}$. (We put $\xi(\mathfrak{P}_i) = \infty$ iff for every natural number N there exists P in $\mathfrak{P}_i$ such that $\xi(P) > N$.)

The known results are described in the table below. Several of these results were published earlier: $\pi(\mathfrak{P}_0) = 13$ and $\pi(\mathfrak{P}_3) = 11$ (see [2]), and $17 \leq \sigma(\mathfrak{P}_7) \leq 18$ (see [3]). The remaining results are announced here for the first time.

i	$\pi(\mathfrak{P}_i)$	$\rho(\mathfrak{P}_i)$	$\sigma(\mathfrak{P}_i)$	$\tau(\mathfrak{P}_i)$
0	13	13	∞	∞
1	10	12	∞	∞
2	12	12	27	∞
3	11	13	∞	∞
4	13	$\{12, 13\}$	$\{29, \ldots, 39\}$	∞
5	13	13	∞	∞
6	13	13	∞	∞
7	11	11	17	$\{25, 26\}$
8	11	13	∞	∞
9	13	12	$\{29, 30, 31\}$	∞

PROBLEM 1. Find the exact values of $\xi(\mathfrak{P}_i)$ in the four unknown cases: $\rho(\mathfrak{P}_4)$, $\sigma(\mathfrak{P}_4)$, $\sigma(\mathfrak{P}_9)$, and $\tau(\mathfrak{P}_7)$.

One can very easily prove that for every $\xi \in \{\pi, \rho, \sigma, \tau\}$ and $i \in \{0, 1, \ldots, 9\}$, whenever $\xi(\mathfrak{B}_i) < \infty$ there exists an infinite set of nonisomorphic polyhedra P in $\mathfrak{B}_i$ such that $\xi(P) = \xi(\mathfrak{P}_i)$.

PROBLEM 2. For each $\xi \in \{\pi, \rho, \sigma, \tau\}$, whenever $\xi(\mathfrak{B}_i) < \infty$, find the smallest polyhedron $P \in \mathfrak{P}_i$ with the extremal property $\xi(P) = \xi(\mathfrak{P}_i)$.

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7. Extremal Graphs and Cages of Prescribed Girth

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The extremal problem considered here is the following: Given numbers n and g, find a graph with maximum number of lines among all graphs having n points and girth g. Clearly the extremal graphs of girth 3 are just the complete graphs K_n. Extremal graphs of girth 4 are also known [1]; namely, the complete bipartite graphs $K_{n/2, n/2}$ and $K_{(n-1)/2, (n+1)/2}$ for n even and odd, respectively. For higher girth, hardly anything seems to be known about the structure of extremal graphs. One difficulty is that such extremal graphs are not unique. In an attempt to discover some regularity, extremal graphs of girth 5 were investigated for small n. Several conjectures were disproved; the only persistent feature was a tendency to be regular. This may be formulated as:

CONJECTURE 1. Extremal graphs of prescribed girth admit realizations in which the degrees of any two points differ by at most 1.

It is important to call attention to the "associated" extremal problem: Given a number m, find a simple graph of prescribed girth with minimal number of points having m lines. This problem is interesting because of the existence problem for cages. A (d, g)-cage is a d-regular graph of prescribed girth g having minimum number of points. The idea now is that, in view of CONJECTURE 1, cages are nothing but special solutions to the associated problem. Let $f(n, g)$ denote the number of lines in an extremal graph on n points with girth g. We are interested in the asymptotic behavior of $f(n, g)$, for fixed g. In determining this behavior, the cages play a special role. Note that the (2, 5)-cage, which is C_5, the (3, 5)-cage, which is the Petersen graph, and the (4, 5)-cage are known to be extremal graphs. This supports:

CONJECTURE 2. All (d, g)-cages are extremal graphs of girth g.

Now observe that a $(d, 5)$-cage may be labeled as follows. Give one point label 0, its d adjacent points label 1, the other points adjacent to these points label 2. This yields a tree of diameter 4 with $d^2 + 1$ points. For $d = 2$ and $d = 3$, one needs no more points. For $d = 4$, two points with label 3 are needed to obtain the (4, 5)-cage and four points for $d = 5$. If we denote the number of additional points with label 3 by $a(d)$ we know that $a(7) = 0$. For $d = 6$ it can be shown that $a(6) \geq 2$; we conjecture that, in fact, $a(6) = 4$. This last is equivalent to:

CONJECTURE 3. The (6, 5)-cage has 41 points.

The above procedure was described for cages of girth 5, but carries over unchanged for all odd g. For g even, the labeling starts by giving label 0 to two adjacent points. In each case, we obtain trees of diameter $g - 1$. Let $t(d)$ denote the number of points in such a tree and $a(d)$ the number of additional points needed to obtain a (d, g)-cage. The impression is that $a(d) < t(d + 1) - t(d)$, leading to:

CONJECTURE 4. (d, g)-cages exist for all values of d and g and have the structure described above. The number of points satisfies $t(d) \leq n < t(d + 1)$.

The importance of this conjecture lies in the fact that the number of lines in a (d, g)-cage acts as an upper bound for $f(n, g)$ in case $t(d - 1) \leq n < t(d)$, and as a lower bound for $f(n, g)$ in case $t(d + 1) \leq n < t(d + 2)$. In view of this remark, CONJECTURE 4 would imply that for the asymptotic behavior of $f(n, g)$, we have:

CONJECTURE 5. When g is odd, $f(n, g) \approx (n^r)/2$ with $r = ([g/2] - 1)/[g/2]$; when g is even, $f(n, g) \approx (n/2)^s$, with $s = g/(g - 2)$.

It should be noted that the weakest link in this chain of conjectures, the existence of (d, g)-cages, need not be fatal for CONJECTURE 5 if it fails. The essential point is to have a set of "beacon graphs" for appropriate values of n along the number line that enable tight estimation of $f(n, g)$, for all n.

ACKNOWLEDGMENT

I wish to thank my students H. Beverdam and G. J. van Essen for their enthusiastic cooperation in the investigation that led to these conjectures*.

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8. An Extremal Problem in Communication Networks

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If we represent a communication network by a graph and consider that a message can be transmitted free of garbles over one link with probability $t < 1$, then the probability of the garble-free transmission of a message between any two terminals is $\geq t^d$, where d is the diameter of the graph. To consider the vulnerability of such a system to degradation, it is natural to define the following two parameters:

Persistence: $\pi_0(G)$—the minimum number of points which must be removed in order to increase the diameter of G or yield a trivial graph.

Line persistence: $\pi_1(G)$—the minimum number of lines which must be removed in order to increase the diameter of G.

By convention, we consider the diameter of a disconnected graph to be infinite.

It is clear that π_0 and π_1 are in many ways similar in behavior to the connectivity and line-connectivity of a graph and, in fact, satisfy analogs of Menger's Theorem. The questions which are of a practical nature are extremal. Specifically, what is the minimum number of lines q for fixed number of points p, with $\pi_0 \geq n$? The problem may become more tractable if the diameter d is also fixed, i.e., find the minimum q for fixed p, d and $\pi_0 \geq n$. The same type of questions can also be asked about π_1 and, in each case, we can ask for the maximum p with q fixed.

* A more elaborate version of the reasoning behind these conjectures will be available as an internal report of our institute.

9. Extremal Graphs with Given Automorphism Group

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If A is a group, an A-graph is a graph whose automorphism group is isomorphic to A and an (A, n)-graph is an A-graph on n points. Consider the following extremal problem: Given a finite group A, for each integer n, determine the minimal number $e(A, n)$ of lines possible in an (A, n)-graph. Of course, this only makes sense for $n \geq v(A)$, the smallest number of points possible in an A-graph.

This problem has been solved for several classes of groups; a survey of known results is given in [1]. Also, it is known [2] that, for each group A, there exists a "stability graph" $M(A)$ such that whenever n is sufficiently large $e(A, n)$ is attained by taking the union of $M(A)$ with a standard asymmetric forest. For "small" values of n, the graphs attaining $e(A, n)$ need not be of this form; indeed, their structure may vary considerably with n. As is illustrated in [3], finding such graphs for small n generally involves the solution to a related extremal problem: the determination of $e_c(A, n)$, the minimal number of lines possible in a *connected* (A, n)-graph.

Thus there are several related problems in which we are interested; viz., the determination of $v(A)$, $M(A)$, $e_c(A, n)$, and $e(A, n)$ for various groups A. Below are some problems of each type that are still open.

PROBLEM 1. Determine $e(A, n)$ when A is a wreath product of symmetric groups, i.e., a complete monomial group $S_k[S_m]$ with $k > 1$ and $m > 2$ (see [4] for $m = 2$).

PROBLEM 2. Determine $e_c(A, n)$ when A is cyclic of nonprime order. Do the same when A is dihedral.

PROBLEM 3. Determine the stability graph $M(A)$ when A is an arbitrary finite Abelian group. (Some results on the structure of $M(A)$ are known [1, 2] and the residual combinatorial problem has neither a graph theoretic nor group-theoretic flavor.)

PROBLEM 4. Determine $v(A)$ for any group A which is not Abelian, dihedral, hyperoctahedral or a direct product of symmetric groups.

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10. Automorphism Groups of Graphs and Digraphs

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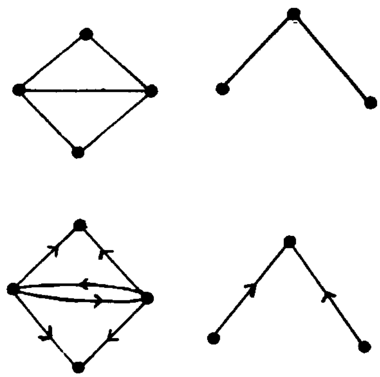
When is the automorphism group of a digraph the same as the automorphism group of the underlying graph? A related, but possibly more tractable, question is to ask: When can a graph be oriented in such a way as to preserve its automorphism group? Clearly it is sufficient in either case that the graph involved be an identity graph, but this is not necessary as shown by the examples in the figure.



(a) (b)

11. Orbits in Random Trees

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A *fixed point* of a graph G is left invariant by each automorphism of G. A graph is *rigid* if its group is the identity, so that all its points are fixed. It is very well known that almost all graphs are rigid—this fact was observed by Pólya and others. (The term *rigid graph* was introduced by category theorists.) More specifically, let g_p be the number of graphs with p points and let r_p be the number of rigid graphs among these. Then lim $r_p/g_p \to 1$ as $p \to \infty$. It follows that the probability that a point of a random graph is fixed approaches 1.

What is the corresponding situation for trees? Using the notation t_p for the number of trees with p points (as introduced by Pólya and employed in our book [1]), the total number of points in all such trees is pt_p. Let f_p be the total number of fixed points among all p-point trees. We recently showed that to the nearest percent

$\lim f_p/pt_p$ as $p \to \infty$ is .70 (see [2]). In words, we say that the probability that a point of a random tree is fixed is about 70%.

UNSOLVED PROBLEM. What is the probability that a point of a random tree belongs to an orbit of cardinality n, for $n = 2, 3, \ldots$?

Let a_n denote this probability. Obviously $\sum_1^\infty a_n = 1$, and we have just mentioned that $a_1 = 0.7$. Another related question is to verify the intuitively obvious inequalities $a_1 > a_2 > a_3 > \cdots$. We may also ask for an elementary justification of the existence of each of the limits a_n.

Finally, what is the minimum n such that $a_n < .01$? Our guess is that this smallest n is 4 or 5.

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12. Enumeration and Structure of Stereoisomers of Alkanes

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In [1] a method was used to study the enumeration, in terms of tertiary and quaternary carbons, of chiral, achiral, and meso alkanes having a given diameter (length of a longest carbon chain) and a carbon skeleton with automorphism group isomorphic to a symmetric group. That is, the alkanes were classified in terms of their symmetry properties, the valency properties of their bonding structure, and their diameter. Furthermore, the structure of carbon skeletons (4-trees) with the given constraints is described in some detail [1, 2].

PROBLEM. Obtain the analogous enumeration, in terms of tertiary and quaternary carbons, of chiral, achiral, and meso alkanes having a given diameter and carbon skeleton with automorphism group other than a symmetric group.

COMMENT. If a method other than that of [1] is used, it is desirable that the investigation should yield at least as much information about the structure of the alkanes being studied as was obtained in [1, 2].

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13. Enumerating Chemical Isomers

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Given a structural skeleton, the stereo and position isomers of polysubstituted amines can be enumerated by a generalized wreath product method [1]. However, the enumeration of structural skeletons containing carbon and nitrogen atoms remains open. The problem can be abstracted and formulated in mathematical terms as follows.

PROBLEM. Enumerate the nonisomorphic connected multigraphs whose vertex set is the union of disjoint sets X, Y such that each element of X, which is distinguishable from elements of Y, has degree at most 4 and each element of Y degree at most 3.

Another problem is to extend the approach of Robinson *et al.* [2] for calculating the number of chiral, achiral, and monosubstituted alkanes to steric trees having elements like carbon, nitrogen, etc.

Finally, the enumeration of cyclic compounds remains open, but see [3].

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14. Epi-universal Uncountable Graphs

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All graphs considered are without isolated vertices. A graph H is a *homomorphic image* of a graph G if there is a map φ from $V(G)$ *onto* $V(H)$ such that for each edge $[y, y'] \in E(H)$, there is an edge $[x, x'] \in E(G)$ with $[\varphi x, \varphi x'] = [y, y']$. If G is connected and H is a homomorphic image of G, then H is connected and $|E(H)| \leq |E(G)|$. Call G *epi-universal* if it is connected and every nonempty connected graph H with $|E(H)| \leq |E(G)|$ is a homomorphic image of G. The only finite epi-universal graphs are K_2 and $K_{1,2}$. If an infinite graph G is epi-universal, then (i) it is bipartite and (ii) diam G is infinite. In the countable case these two conditions are also sufficient, the reason being that rays (= one-way infinite paths) are epi-universal. If G is epi-universal and uncountable ($|G| = \aleph_\alpha$, $\alpha > 0$), then it also satisfies: (iii) For every positive integer n, G contains $\aleph_\alpha$ disjoint geodesic paths of length n. This follows from the fact that any fan of $\aleph_\alpha$ rays having only their origins in common is epi-universal.

PROBLEM. Do conditions (i), (ii), (iii) imply that G is epi-universal?

15. Switching-Complete Graph Properties

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A *switching* in a graph is defined as follows. For any vertices a, b, c, and d for which edges (a, b) and (c, d) are present but (a, c) and (b, d) are not, we can perform a switching by deleting (a, b) and (c, d) and adding (a, c) and (b, d). The operation of switching preserves the degree sequence. Eggleton [2] has proved that given two graphs G and H with the same degree sequence, there is a finite sequence of switchings producing a graph isomorphic to G from H. The application of this to producing a list of all nonisomorphic graphs with a given degree sequence is immediate. However, it should be noted that computationally more efficient algorithms are known [1].

We now consider the problem of listing all nonisomorphic graphs with a given degree sequence satisfying some property P. Define *switching constrained to property* P to be switching with the restriction that the switch can be performed only if the result has property P. We call a property *switching complete*, or *s-complete*, if the following statement holds. For any two graphs G and H with the same degree sequence d, both satisfying property P, there is a finite sequence of switchings constrained to property P which transforms H into a graph isomorphic to G.

PROBLEM. Characterize s-complete graph properties.

Few results are known. The property of "being a tree" is s-complete, but 2-colorability is s-incomplete [1]. Beyond this only a few other trivial s-incompleteness results are known. There are many open problems here; as incentive, consider that, for each property shown to be s-complete, we have a reasonably efficient method for listing the graphs with the property (see [1], Chapter 4).

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16. Critical and Minimal Graphs

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Graph G is k-*critical* (resp. k-*minimal*) with respect to parameter P if whenever any $j \leq k$ vertices (edges) of G are removed, the value of P is reduced by j. We say that G is k-critically (k-minimally) n-P-adic if G is k-critical (k-minimal) with respect to P and $P(G) = n$.

When $k = 1$, we have what are usually called critical and minimal graphs. These have been studied extensively for several parameters. For instance, if P is vertex-connectivity and $k = 1$, we have critically connected and minimally connected graphs.

Graphs that are 1-critical and 1-minimal exist for almost any P and they usually form an interesting class. For $k \geq 2$, k-critical and k-minimal graphs often do not exist at all, or form a dull class. For instance, it is easy to show that 1) the only 2-critical, n-chromatic graph is K_n, and 2) there are no 2-minimally, n-chromatic graphs.

We have looked carefully at k-critical and k-minimal graphs for all the usual sorts of connectivity; for some sorts, these classes are very interesting. For other parameters we have looked at (e.g., chromatic number) these classes have not been interesting. However, we have not systematically studied all parameters.

Open Topic. For $k \geq 2$, determine for which graphical parameters P the class of k-critical graphs or the class of k-minimal graphs is interesting. Study these classes.

For P = vertex connectivity, call G an (n^*, k^*) graph if G is k-critically n-connected, but not $(k + 1)$-critical.

Conjecture (Slater). (n^*, k^*) graphs exist precisely for $k = n$ and $k \leq n/2$.

The unsolved part is to show that there are no (n^*, k^*) graphs for other k. For more information, see our papers.

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17. Avoiding Isolates in Vertex Colorings of Graphs

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Let G be a graph with p vertices. Let the vertices be colored with n colors. A vertex v is *color isolated* if no vertex adjacent to v has the same color as v. A. J. Goldman asked (at the AMS national meeting in Washington, January, 1975): For which graphs G and which numerical partitions $p_1, p_2, \ldots, p_n$ of p can G be colored with n colors so that, for all i, p_i vertices are colored color i and no vertex is color isolated. When such a coloring is possible, we say that G is $(p_1, \ldots, p_n)$-colorable *without isolates*. As shorthand in this note, we simply say that G is *n-colorable*.

Clearly, if some $p_i = 1$, no G is n-colorable. At the other extreme, some graphs are n-colorable for every other $(p_1, \ldots, p_n)$, e.g., complete graphs. Call a partition with no $p_i = 1$ a *1-free* partition.

Open Topic. Give some large classes of graphs such that, for every G in one of these classes and every 1-free partition of the number of vertices, G is n-colorable.

Preferably, membership in these classes and construction of a coloring should be computable by efficient algorithms.

For $n = 2$, I have developed a fairly complete theory for Goldman's problem (see my paper, "Vertex Colorings without Isolates," to appear in JCT Ser. B). In particular, in line with the problem above, I prove:

THEOREM. If G is connected and has minimal degree $\delta \geq 2$, then G is 2-colorable for any 1-free partition.

I give a polynomial algorithm to find such a coloring. A faster algorithm, linear in the number of edges and using depth first search, has been given by F. Thomas Leighton, a Princeton undergraduate (unpublished).

For $n \geq 3$, almost nothing is known except that various versions of the general problem are NP-complete.

CONJECTURE. If G is connected and $\delta \geq n \geq 3$, then for any 1-free partition $(p_1, \ldots, p_n)$, G is $(p_1, \ldots, p_n)$-colorable without isolates.

Everyone I have gotten to work on this conjecture believes it is true, but all attempts to prove it by induction from known results for $n = 2$ have failed. Fresh ideas are needed.

For $\delta < n$, there exist connected graphs G which are not n-colorable. For instance, consider $K_{m,n-1}$, where m is arbitrary. Suppose $p_i > 1$ for all i. At most $n - 1$ colors can be used on the right side of the graph, for there are only $n - 1$ vertices there. Thus each vertex on the left must also be one of these $n - 1$ colors, or else it is isolated.

It should be noted that "almost all" graphs are n-colorable for every 1-free partition. This is because "almost all" graphs are Hamiltonian, and every Hamiltonian graph is easily seen to be n-colorable for every 1-free partition.

18. Nonequivalent 4-Colorings of Large Planar Graphs

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Let S be a sphere of radius 1. The *mesh* of a graph G embedded on S is $\mu(G) =$ the maximum diameter of any region. Let $\lambda(G) =$ the number of nonequivalent 4-colorings of G, that is, colorings which do not differ simply by a permutation of the colors.

CONJECTURE. If $\mu(G) \to 0$, then $\lambda(G) \to \infty$.

Certain families of known uniquely 4-colorable planar graphs are ruled out by the mesh requirement. If the above conjecture is false, the main consequence is: the 4-color conjecture is "barely" true in that some large graphs have unreasonably few 4-colorings.

19. Existence of Uniquely Non-Ramsey Graphs

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Does there exist a (finite) graph G with the property that its edges can be colored red and blue without a monochromatic triangle, but in only one way (except for interchange of colors)? Strangely enough, it is not hard to construct an infinite graph with this property.

For a survey of many other Ramsey type problems and results, see [1].

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20. Self-Centered Graphs

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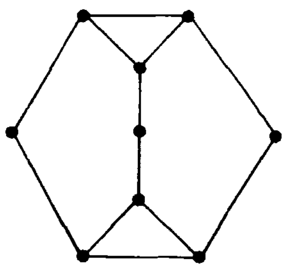
We call a graph self-centered if all of its points are in its center. There are several classes of such graphs: cycles, complete graphs, disconnected graphs, complete n-partite graphs. The self-centered graph in the figure, however, belongs to none of these categories.

It is easy to show that a graph is self-centered iff its diameter and radius are equal. Can you find a more useful characterization?



21. Graphs of Diameter 3 with Unique Local Extremes

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Let G be a connected graph and $v \in V(G)$. The vertex $w \in V(G)$ is a *local extreme* for v if and only if there does not exist a vertex $x \in V(G)$ which is adjacent to w such that $d_G(v, x) > d_G(v, w)$, where $d_G(x, y)$ is the distance between x and y in G. The *diameter* $d(G)$ of G is the maximum distance between two vertices of G. The set of all local extremes for v is denoted by $L(v)$. If $|L(v)| = 1$, for every v, then G is a *GS-graph*. The unique element of $L(v)$, where $|L(v)| = 1$, is denoted by $\bar{v}$ and is called the opposite of v in G.

All GS-graphs G with $d(G) < 4$ are regular and, for every $d > 3$, there exist irregular GS-graphs of diameter d. The only GS-graph with $d = 1$ is K_2. The only GS-graphs with $d = 2$ are $K_{2n} - L$, where L is a linear factor (1-factor) of K_{2n}, $n > 1$.

The following theorem gives a method for constructing an infinite set of GS-graphs with $d = 3$.

THEOREM. Let G be a GS-graph of diameter 3. Let $\{V_1, V_2\}$ be a partition of $V(G)$. Let H be the graph obtained from G by the addition of two vertices w_1 and w_2 and two sets of edges

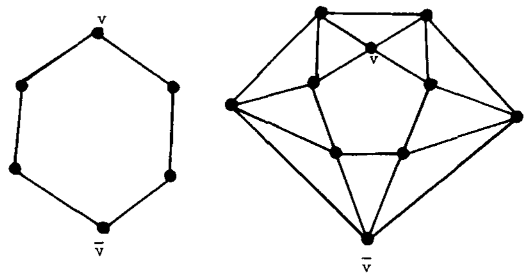
$$E_1 = \{(w_1, x) \mid x \in V_1\} \quad \text{and} \quad E_2 = \{(w_2, x) \mid x \in V_2\}.$$

Then H is a GS-graph of diameter 3 if and only if, for every $v \in V(G)$,

(i) $v \in V_1$ if and only if $\bar{v} \in V_2$, and

(ii) $v \in V_i$ implies that there exists $w \in V_j$, $\{i, j\} = \{1, 2\}$ such that $d_G(v, w) = 1$.

The two GS-graphs, shown in the figure, are not obtainable by the method of the above theorem.



(a) (b)

PROBLEM 1. Are there any other GS-graphs of diameter 3 which are not obtainable by the theorem?

PROBLEM 2. Is the set of GS-graphs of diameter 3 not obtainable by the theorem finite?

PROBLEM 3. It is known that for every $r > 1$ there exists at least one GS-graph of diameter 3 which is regular of degree r. What is the number $N(r)$ of mutually nonisomorphic GS-graphs of diameter 3 which are regular of degree r? Clearly, $r > 1$ implies $N(r) \geq 1$ and for $r = 2, 3, 4$ we have $N(2) = N(3) = 1$ and $N(4) = 3$. The values of $N(r)$ for $r > 4$ are unknown.

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22. Optimal Search Trees

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In [1] typical optimization problems concerning search trees are treated. Let a search tree have N endpoints and let l_i, $i = 1, 2, \ldots, N$, denote the length of the unique path from the root to endpoint i. Also let w_i be a weight associated with endpoint i. One asks for the binary or ternary tree for which $\sum_{i=1}^{N} l_i$ or $\sum_{i=1}^{N} l_i w_i$ attains its minimum. In the latter case, the tree is found by Huffman's algorithm.

If one does not pose the restriction that the trees should be binary or ternary, the problem suddenly becomes a lot more difficult. Paper [2] introduces the concept of *effort*. We define the effort of an internal point with u successors, or sons, as some increasing, positive-valued function of u. Let us denote it by $e(u)$ and let $E(P_i)$ be the sum of the efforts of the internal points on the path P_i to endpoint i. For given effort function $e(u)$ we asked for the trees for which $\sum_{i=1}^{N} E(P_i)$ or $\sum_{i=1}^{N} E(P_i)w_i$ attained its minimum. In the first case, we found that, for $e(u) = u$, ternary trees are optimal. This leaves us with the following

Problem. What is the analog of Huffman's algorithm that gives the trees for which $\sum_{i=1}^{N} E(P_i)w_i$ attains its minimum?

One may, of course, choose $e(u) = u$ or $e(u) = 1$, in which case $E(P_i) = l_i$.

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23. Trees with Integral Eigenvalues

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In a recent manuscript written with Mamoru Watanabe, we show that the following trees have integral eigenvalues:

(i) *Stars.* $K_{1,m}$ is integral if and only if m is a perfect square.

(ii) *Double stars.* The tree $T(m, r)$ of diameter 3 is formed by joining the centers of $K_{1,m}$ and $K_{1,r}$ with a new edge. For $m \le r$, tree $T(m, r)$ is integral if and only if $x^4 - (m + r + 1)x^2 + mr$ factors as $(x^2 - a^2)(x^2 - b^2)$. This occurs if $m = r = k(k + 1)$, $a = k$, and $b = k + 1$. An exhaustive computer search has found 65 solutions with $m < r$, $a \le b$ and $a \le 5000$.

(iii) *Diameter 4.* The tree $S(r, m)$ is formed by joining the centers of r copies of $K_{1,m}$ to a new vertex v. The tree $S(r, m)$ is integral if and only if both m and $m + r$ are perfect squares. Note that case (i) can be subsumed here if we allow $m = 0$.

Question 1. Are there any integral trees with diameter greater than 4?

QUESTION 2. Are there any integral trees of diameter 4 other than those specified in (iii)?

QUESTION 3. We know that all integral trees of diameter less than 4 are included in (i) and (ii). Then (combining QUESTIONS 1 and 2) have we found all integral trees in (i), (ii), and (iii)?

24. Density of Spectrum-Determined Graphs

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Let g_n denote the number of graphs on n vertices and let h_n be the number of n-vertex graphs which are uniquely determined by their spectra.

CONJECTURE. $\lim_{n\to\infty} \frac{h_n}{g_n} = 0.$

The analogous result for trees is proved in [1]. That result was based on a single pair of subtrees whose replacement leaves the spectrum unchanged. Surely the graphical version is more difficult. It would be significant progress to demonstrate that lim sup $h_n/g_n < 1$. Even this modest goal seems to require more than one replaceable pair of subgraphs.

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25. Characteristic Polynomials of Trees

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The characteristic polynomial of a tree can be written in the form

$$\phi(T; x) = \sum_{k=0}^{m} (-1)^k a_{2k} x^{n-2k}$$

From Sachs' Theorem we know that a_{2k} is the number of ways to select k independent edges in T, and m is the size of the largest independent set.

CONJECTURE. The coefficients a_{2k} are unimodal; that is, there exists an r such that

$$a_0 < a_2 < a_4 < \cdots < a_{2r} \geq a_{2r+2} > a_{2r+4} > \cdots > a_{2m}$$

The conjecture has been verified for all trees with at most eleven vertices by exhaustive search.

26. Binary Rank of Adjacency Matrices

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Regard the entries of the vertex adjacency matrix A of graph G as being in the field F_2. Is there any interesting graph theoretical interpretation of the rank of A?

27. Upper-Triangular Matrices

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Let T be an upper-triangular matrix of size n with 1's on the main diagonal. Then T^{-1} exists and is also upper triangular with 1's on the main diagonal. Let αT denote the sum of the entries in the matrix T.

QUESTION. When is $(\alpha T)(\alpha T^{-1}) = n^2$? In particular, does this condition imply that T is the identity matrix?

An affirmative answer would have an application in polymer chemistry.

28. Partitions and a Series-Product Transform

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First some notation. If P is an unordered partition $p_1, p_2, \ldots, p_k$, and $x_1, x_2, \ldots$ is a sequence (in a commutative ring R), let x_P denote the product of x_{p_i}, $1 \leq i \leq k$. We write $\alpha(P)$ for k and $\sigma(P)$ for the sum of the p_i, $1 \leq i \leq k$.

Each formal power series

$$1 + \sum_{m=1}^{\infty} a_m t^m$$

with coefficients in R is uniquely expressible as a formal infinite product

$$\prod_{j=1}^{\infty} (1 + b_j t^j)$$

and vice versa. This is equivalent to the remark that the transformation specified by $a_m = \sum b_P$, where the summation is taken over all partitions P of m into *distinct* summands, is reversible. It is easily seen that the inverse transformation is of the form $b_k = \sum t_{k,Q} a_Q$, where the sum is taken over all partitions Q of k.

PPROBLEM. Learn as much as possible about $t_{k,Q}$.

Clearly $t_{k,Q}$ is an integer. It seems likely that sgn $t_{k,Q} \in \{0, (-1)^{1+\alpha(Q)}\}$.

The nonlinear system considered above may be embedded in a linear one by noting that for each partition P, $a_P = \sum \Gamma_{P,Q}$, where the sum is over all partitions Q for which $\sigma(Q) = \sigma(P)$. This infinite system is reducible to a collection of finite linear systems of order $p(m)$, where $p(m)$ denotes the number of partitions of m. If the partitions are listed in lexicographic order, the matrix Γ is triangular with 1's on the main diagonal and all entries are integers.

Expanded Problem. Learn as much as possible about Γ^{-1}.

Note that the top row of Γ^{-1} corresponds to the original problem.

Problem for Analysts. Find relations between convergence sets for $\sum a_m t^m$ and $\prod (1 + b_k t^k)$.

29. Lines in the Pascal Triangle for q-Binomial Coefficients

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Regard the Pascal triangle as being laid out in the positive quadrant of the xy-plane below the principle diagonal. That is, to each integral lattice point (x, y), with $x \geq y \geq 0$, assign the binomial coefficient $\binom{x}{y}$. Consider a line in the plane which passes through infinitely many such lattice points. The sequence of lattice points encountered on the line (in order of occurrence) can be represented as $(n + at, k + bt)$, for $t = 0, 1, 2, \ldots$, where n, k, a, b are fixed integers such that $n \geq k \geq 0$, $a \geq b \geq 0$, a and b have greatest common divisor 1, and either $n - a < k - b$ or $k - b < 0$. Finally, consider the function $F(x)$ defined by a (formal) Taylor series whose coefficients are the binomial coefficients associated with the lattice points encountered on the line; i.e.,

$$F(x) = \sum_{t=0}^{\infty} \binom{n + at}{k + bt} x^t$$

It turns out that $F(x)$ is an algebraic function. (A sweeping generalization of this was proved by Pólya [1].)

Question. Is the analogous result true when the binomial coefficients are replaced by q-binomial coefficients?

Specifically, if n, k, a, b satisfy the conditions stated earlier, is the function

$$F(x, q) = \sum_{t=0}^{\infty} \begin{bmatrix} n + at \\ k + bt \end{bmatrix}_q x^t$$

algebraic? That is, does there exist a nonzero polynomial $P(x, y, z)$ whose coefficients are constants (say, complex numbers) such that $P(x, q, F(x, q)) = 0$, for all x and q?

Remark. If the line is parallel to an edge of the q-Pascal triangle (i.e., if $a = b$ or $b = 0$), then $F(x, q)$ is actually a rational function of x and q.

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30. The Star of David Identity

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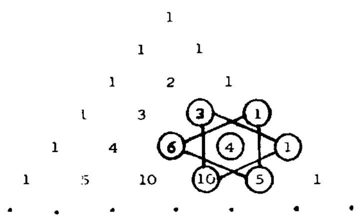
Pascal's triangle for the binomial coefficients is shown below in "isosceles form." If one focuses on some entry not on a side, one finds that the entry is surrounded by six others at the vertices of a hexagon. If a Star of David is formed, as shown in the figure, the product of the coefficients at three of the vertices equals the product of those at the other three. The actual formula is:

$$\binom{n}{r}\binom{n+1}{r+2}\binom{n+2}{r+1} = \binom{n}{r+1}\binom{n+1}{r}\binom{n+2}{r+2}$$

An algebraic proof is trivial. Find a combinatorial proof. I offer $50 for the first one I receive.



31. A Combinatorial Condition for a Complex to Be Finite

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THEOREM. Let K be a simplicial complex which is a connected 3-manifold without boundary. Assume that every 1-simplex of K is a face of at most five 3-simplexes. Then K is a finite complex.

This theorem has been proved by noncombinatorial methods. Prove it combinatorially.

32. Sequences Avoiding Repeated Subsequences

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Let P_n denote the number of sequences of n elements chosen from the set $\{1, 2, 3\}$ having the property that no two contiguous subsequences are identical. Thus, for

$n = 9$, the sequence 1 3 2 3 1 2 1 3 2 would be counted, whereas 1 3 2 3 1 3 2 3 1 would not (because of the repetition of the subsequence 1 3 2 3). Determine the function P_n.

This problem (differently worded) was given in a recent issue of the magazine *Creative Computing* as an exercise in programming. By means of a computer program, I have determined P_n up to $n = 20$, and the values seem to indicate that

$$P_n \sim k^{n+1}$$

where k appears to be a little less than 4/3. Prove or disprove this asymptotic estimate.

33. Lattices of k-ary Words

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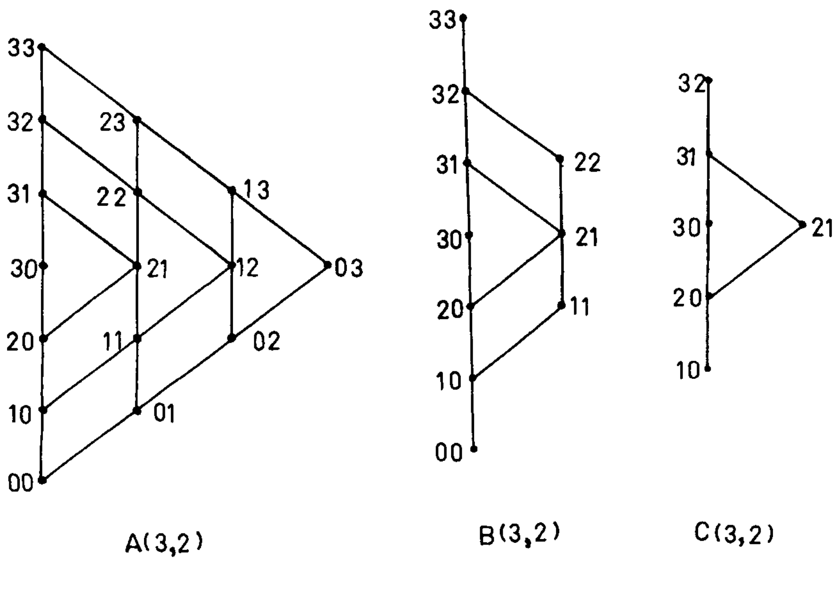
Let n be a nonnegative integer and k a positive integer. Denote $T = \{0, \ldots, n\}$ and define

$$A(n, k) = \{a_1 a_2 \cdots a_k \mid \text{each } a_i \in T\},$$
$$B(n, k) = \{b_1 b_2 \cdots b_k \mid \text{each } b_i \in T \text{ and } b_1 \geq b_2 \geq \cdots \geq b_k\},$$
$$C(n, k) = \{c_1 c_2 \cdots c_k \mid \text{each } c_i \in T \text{ and } c_1 > c_2 > \cdots > c_k\}.$$

We make A, B, and C into finite distributive lattices by specifying that $y_1 \cdots y_k$ covers $x_1 \cdots x_k$ if and only if $\sum_{i=1}^{k} (y_i - x_i) = 1$ and $y_i \geq x_i$, for $i = 1, 2, \ldots, k$. An example is provided in the figure.



A rank function is given by $r(x_1 \cdots x_k) = \sum_{i=1}^{k} x_i$. For each $p \geq 0$ let $a_{n,k}(p)$ = the number of points in $A(n, k)$ of rank p. So $a_{n,k}(p)$ = the number of ordered partitions of p into at most k parts, each part of size at most n. Lastly, for each $q \geq 1$ define $l_q(A)$ = the number of integers p such that $a_{n,k}(p) - q \geq 0$. For example, $l_1(A(3, 2)) = 7$, $l_2(A(3, 2)) = 5$, $l_3(A(3, 2)) = 3$, $l_4(A(3, 2)) = 1$ and $l_i(A(3, 2)) = 0$, if $i > 4$.

PROPOSITION. If $l_1(A), \ldots, l_t(A)$ are the positive $l_i(A)$'s, then $A(n, k)$ can be decomposed into disjoint chains of lengths $l_1(A) - 1$, $l_2(A) - 1, \ldots, l_t(A) - 1$, respectively.

In the above example, in $A(3, 2)$ there are disjoint chains of lengths 6, 4, 2, and 0; namely, the vertical lines in $A(3, 2)$.

The proofs of the proposition and the corresponding statements for B and C are implicit in [1]. However the proofs are not lattice theoretic.

PROBLEM. Do there exist such decompositions for arbitrary finite distributive lattices?

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34. Disjoint Unions of Projective Planes

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Let Ω be a set of size $v = n^2 + n + 1$, with $n = p^a$, p a prime. Let $\binom{\Omega}{n+1}$ denote the collection of all $(n + 1)$-subsets of Ω.

CONJECTURE. If n is sufficiently large, $\binom{\Omega}{n+1}$ can be decomposed as the disjoint union of projective planes of order n.

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35. Generalizations of Kirkman's Schoolgirl Problem

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Kirkman [1] posed the problem: Can the blocks of a BIBD with block-size 3 be resolved into parallel classes of blocks? The problem was solved by Ray-Chaudhuri and Wilson [2], and later a solution was given for a similar problem with block-size 4 [3].

Very little is known about further generalizations for block-size 5. Clearly, the number of elements in the BIBD must be $v \equiv 5 \pmod{20}$. For $v = 5^n$, $n = 1, 2, \ldots$, the solution is simple. Perhaps somebody could give a construction of such resolvable designs for $v = 45, 65, 85$, or 105.

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36. Deleting Elements from Block Designs

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Easy Question. Let D be a (b, v, r, k, λ) design. If it is possible to remove exactly one element from each block so that each element now occurs in $r - 1$ blocks and each pair of elements occurs in $\lambda - 1$ blocks, what can you say about D?

Answer: Design D has parameters $(4t - 1, 2t, t)$ and forms a skew-equivalent Hadamard matrix.

Question. If you can remove e elements from each block so that each element now occurs $r - i$ times and each pair of elements occurs $\lambda - j$ times, what can you say about D?

37. Maximal Set Packings

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A set packing $r(v, k, \lambda)$ consists of a set of blocks $B(i)$ such that $|B(i)| = k$ with elements selected from the set $S = \{1, 2, \ldots, v\}$. In addition, the blocks must pairwise intersect in at most λ elements. $R(v, k, \lambda)$ is defined to be the cardinality of the largest

$r(v, k, \lambda)$ packing. A set packing may be viewed as a balanced binary block code, a Johnson association scheme, or a vertex independent set in the graph of the binomial coefficient $\binom{v}{k}$. Other related combinatorial configurations include BIBD, PBD, and (r, λ)—systems. At present, many values of $R(v, k, \lambda)$ remain unknown. The following cases, which are still open, are interesting because the range of possible values is relatively small:

$$42 \leq R(14, 7, 4) \leq 51$$

$$16 \leq R(16, 7, 3) \leq 22$$

$$21 \leq R(17, 7, 3) \leq 31$$

The divergence between upper and lower bounds is dramatic even for small increases in one of the parameters. For example, $90 \leq R(16, 7, 4) \leq 156$. The list above is of some small open cases which seem reasonable to investigate first. Many others are tabulated in [1] and [2]. Similar open problems exist for the restricted class of set packings $s(v, k, \lambda)$, where blocks must pairwise intersect in exactly λ elements. A discussion of this is contained in [2].

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38. Ranking Systems

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Good ranking systems usually involve paired comparisons, at best triple comparisons. But when treating a large number of items, e.g., 30–150, after first sorting into smaller sets of 10–50, the ranking of the items in a set by eight or nine judges is not computationally practical by these methods. Alternatively, there are weighted systems of the Borda type, but these are particularly susceptible to manipulation by judges. As far as I know, there is no acceptable model for a complete ranking of k items. What then is the best procedure currently available?

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