Guessing and Proving*

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There are at least two directions that a teacher might take after reading this article. One might be to follow up the CONTENT of the article by presenting Euler's formula to your class. Two-year college teachers might lead their students into discovering the formula, or perhaps verifying it for several specific cases actually handled in class; there is hardly a better way for students to really learn the meanings of "vertex," "edge," "face" and "polyhedron." Two-year college instructors may want to look into some of the proofs of the formula, to compare them for rigor and completeness and so perhaps gain an appreciation of the developmental aspect of formal axiomatics. A second direction might be to follow up the SPIRIT of the article, by assigning similar classification projects—classification of polynomial functions into four basic shapes, for example, or classification of sets into convex and non-convex, and so on. Only a singularly unimaginative teacher will go uninspired by this article from one of the alltime giants of mathematics education.

In the "commentatio" (Note presented to the Russian Academy) in which his theorem on polyhedra (on the number of faces, edges and vertices) was first published Euler gives no proof.[1] In place of proof, he offers an inductive argument: He verifies the relation in a variety of special cases. There is little doubt that he also discovered the theorem, as many of his other results, inductively. Yet he does not give a direct indication of how he was led to his theorem, of how he "guessed" it, whereas in some other cases he offers suggestive hints about the ways and motives of his inductive considerations.

How was Euler led to his theorem on polyhedra? I think that it is not futile to speculate on this question although, of course, we cannot expect a conclusive answer. The question is relevant pedagogically: The theorem is of high interest in itself, and it is so simple; its understanding requires so little preliminary knowledge that its *rediscovery* can be proposed as a stimulating project to an intelligent teenager. Projects of this kind could give young people a first idea of scientific

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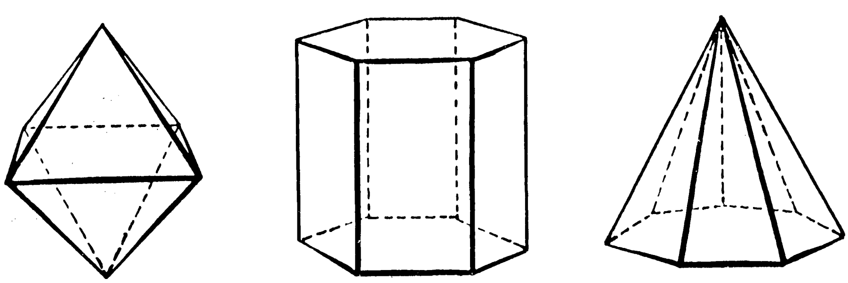
[1] See [1], pp. 72–93. There is a following second Note, pp. 94–108 and a preceding summary, pp. 71–72, which mentions the second Note. See also the remarks of the editor, pp. XIV–XVI. (Numbers in square brackets refer to the bibliography at the end of this paper.)

work, a first insight into the interplay of guessing and proving in the mathematician's mind.

One can imagine various approaches to the discovery (rediscovery) of Euler's theorem. I have presented two different approaches on former occasions.[2] I offer here a third one which, I like to think, could have been Euler's own approach. At any rate, I shall stress in the following some points of contact with Euler's text.[3]

1. Analogy suggests a problem. There is a certain analogy between plane geometry and solid geometry which may appear plausible even to a beginner. A circle in the plane is analogous to a sphere in space; the area enclosed by a curve in the plane is analogous to the volume enclosed by a surface in space; polygons enclosed by straight sides in the plane are analogous to polyhedra enclosed by plane faces in space.

Yet there is a difference. If we look closer the geometry of the plane appears as simpler and easier whereas that of space appears as more intricate and more difficult. Take the polygons and the polyhedra. We have a simple classification of polygons according to the number of their sides. The triangles form the simplest class; they have three sides. Next comes the class of quadrilaterals which have four sides, and so on. The n-sided polygons form a class; two polygons belonging to this class may differ quantitatively, in the lengths of their sides and the openings of their angles, yet they agree in an important respect—should we say "qualitatively" or "morphologically"?[4] We could try to classify the polyhedra according to the number of their faces. Now look at the three polyhedra of Figure 1: the regular octahedron, a prism with a hexagonal base, and a pyramid with a heptagonal base. All three have the same number of faces, namely eight, but they are too different in their whole aspect (morphology) to be classified together.[5]



(1) (2) (3)

Figure 1.

[2] See [2], vol. 1, pp. 35–43 and [3], vol. 2, pp. 149–156, and also the annexed problems and solutions in both books.

[3] The contents of this short note were presented, under various titles and with some variations, in Rome, in Zurich, in Missoula (to a joint session of the MAA and AMS on August 21, 1973), and in several colleges and high schools in the wider neighborhood of Stanford. A detailed abstract appeared in *Periodico di Matematica* (vol. 49 (1973), pp. 77–85) and a shorter one (for limited circulation only) was prepared by Clement E. Falbo and Jean C. Stanek of California State College, Sonoma.

[4] I am intentionally avoiding the standard term which, by the way, did not exist in Euler's time. One of the ugliest outgrowths of the "new math" is the premature introduction of technical terms.

[5] See [1], p. 71.

Here emerges a problem: Let us devise a classification of polyhedra that accomplishes something analogous to the simple classification of polygons according to the number of their sides. Yet in the case of polyhedra taking into account just the number of faces is not enough as the example of Figure 1 shows.[6]

2. A first trial. Here is a remark that could be relevant. Two polygons that have the same number of sides also have the same number of vertices, equal to the number of sides. Yet polyhedra are more complicated. All three polyhedra in Figure 1 have eight faces, yet they have different numbers of vertices, namely six, twelve and eight vertices, respectively. Would it be enough for a good classification to take into account the number of faces F and the number of vertices V?

What should we do to answer this question? Survey as many different forms of polyhedra as we can and count their faces and vertices. Figure 2 offers a short survey. Most polyhedra in Figure 2 are named. The abbreviation "n Pyd" means "Pyramid with an n-sided base", and "n Psm" stands for "Prism with an n-sided base". Thus in polyhedron (5) the 4 Psm is represented by a cube, as quantitative specialization does not matter. Just the last polyhedron is not named; it is a "truncated 3 Psm", and we obtain it from a 3 Psm by cutting off one vertex (in fact, a small tetrahedron topped by that vertex). Polyhedron (6) answers our question, and it answers it negatively: (6) agrees with (5) (the cube) in both numbers considered (of faces and vertices), yet (5) and (6) are essentially (morphologically) different; they should not be put into the same class. The F and V, the numbers of faces and vertices, are *not enough*.[7]

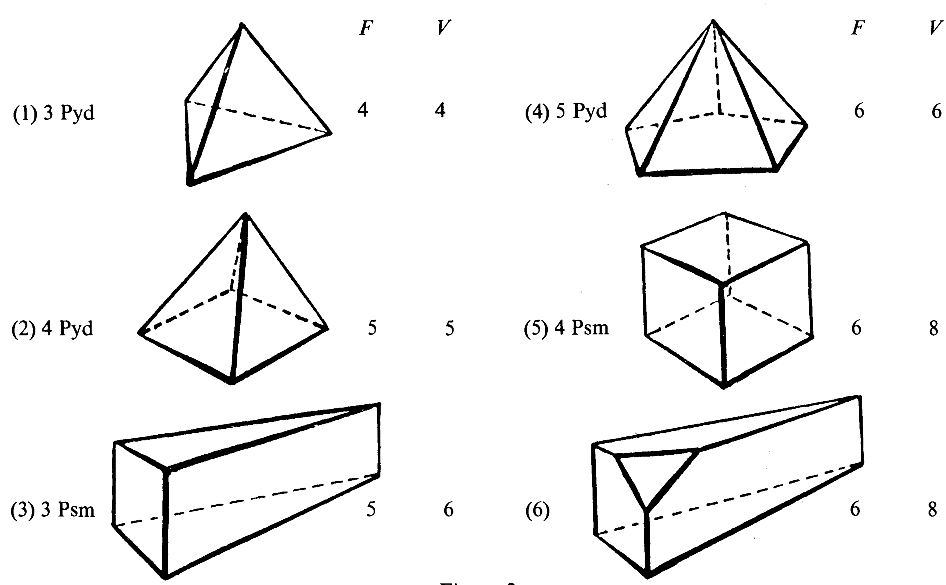


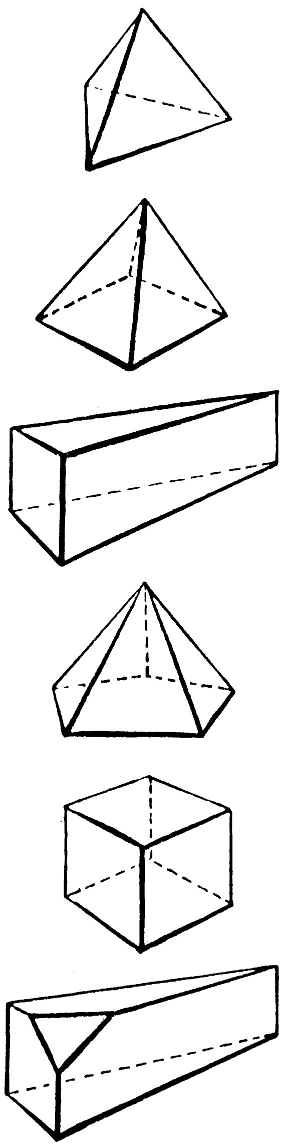
Figure 2.

[6] The classification of polyhedra takes up the major part of the text of Euler's Note. Was this problem of classification his starting point?

[7] A high school student, working on our project, will probably have to survey many more polyhedra before he encounters such a critical pair, providing the negative answer.

3. Another trial. Disappointment and triumph. What else is here, besides the faces and the vertices, to provide a basis for the classification of polyhedra? The edges.[8] Let us look again at the polyhedra we have surveyed, and let us examine their edges.

Figure 3 omits the names shown in Figure 2, but repeats the rest and adds the



	F	V	E
(1)	4	4	6
(2)	5	5	8
(3)	5	6	9
(4)	6	6	10
(5)	6	8	12
(6)	6	8	12

Figure 3.

[8] Euler was the first to introduce the concept of the "edge of a polyhedron" and to give a name to it (*acies*). He emphasizes this fact, mentions it twice; see [1], p. 71 and p. 73. Perhaps Euler introduced edges in the hope of a better classification, and we follow his example here.

number of edges E. Yet it does not help; it provides no distinction between the polyhedra (5) and (6). They agree also in the number of edges, as they have agreed in the number of faces and vertices. And exploring further cases we find invariably: If two polyhedra have the same F and V, they also have the same E. Thus the number of edges contributes nothing to the classification of polyhedra over and above what the faces and vertices have done already. What a disappointment!

Yet there is something else. If the number E of edges is determined by the numbers F and V, of faces and vertices, then E is a function of F and V. Which function? Is it an increasing function? Does E increase whenever F increases? Does E necessarily increase with V? When examples show that neither is the case, the question arises, "Does E increase somehow with F and V jointly?" Such or similar questions (cf. the pages quoted in footnote [2]) may lead to more examples (more than displayed in Figures 2 and 3) and eventually to the guess,

$$E = F + V - 2.$$

An unsuspected, extremely simple relation, unique of its kind. What a triumph!

One is tempted to compare Euler with Columbus. Columbus set out to reach India by a western route across the ocean. He did not reach India, but he discovered a new continent. Euler set out to find a classification of polyhedra. He did not achieve a complete classification, but he discovered, in fact, a new branch of mathematics, topology, to which his theorem on polyhedra properly belongs.

Added in proof: A paper by Jean J. Pedersen on the classification attempted by Euler will appear in *California Mathematics*.

4. Miscellaneous remarks. We have seen a way which Euler may have taken in discovering his theorem on polyhedra, and this was our main topic. There remain several connected points worth considering, but we must consider them very briefly.

(a) Almost a year after his first Note Euler presented to the Russian Academy a second Note on polyhedra in which he gave a proof of his theorem.[9] His proof is invalid, and, looking back at it from our present standpoint, we can easily see why it was fated to be invalid.

It hinges on the concept of "polyhedron". What is a polyhedron? A part of three-dimensional space enclosed by plane faces. Take the case of Figure 4, a cube with a cubical cavity. Between the outer, larger cubical surface and the inner, smaller one there is a part of space, enclosed by plane faces. Is it a polyhedron? It has twice as many faces, vertices and edges as an ordinary cube, namely

$$F = 12,\ V = 16,\ E = 24,$$

and these numbers do not satisfy Euler's relation,

$$24 \neq 12 + 16 - 2.$$

[9] [1], pp. 94–108.

(b) After Euler's death there arose a debate.[10] Some mathematicians were for Euler's theorem, others against it. The opponents produced counterexamples, such as Figure 4. The partisans answered with invective: Figure 4 is not a polyhedron; it is a monster. It is indecent; it is obscene: The big cube with the small cube inside looks like an expectant mother with her unborn baby. An expectant mother is not a counterexample to the proposition that each person has just one head, and, similarly, Figure 4 is inadmissable as a counterexample.

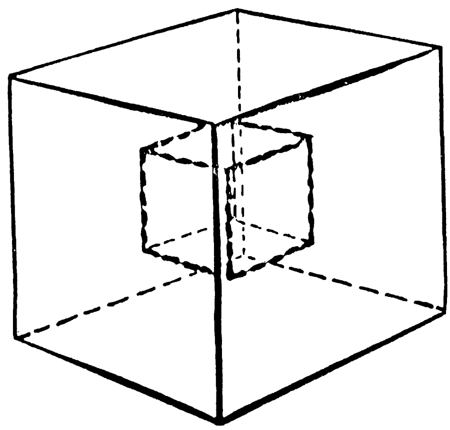


Figure 4.

Which shows that in the heat of debate mathematicians can sound as silly as politicians.

(c) In fact, Euler's theorem cannot be really and truly proved before it is satisfactorily stated. There are two ways to state it more satisfactorily, either by generalizing, by introducing the relevant topological concepts unknown in Euler's time[11], or by specializing, by restricting it to convex polyhedra.

(d) Analogy suggested the approach to Euler's theorem considered in the foregoing sections. Yet analogy is a many-sided thing, and it can suggest other approaches too.

The sum of the angles in an n-sided polygon is $(n-2)180°$. Looking for an analogous fact about polyhedra we can be led to Euler's theorem, and even to a valid proof of it for convex polyhedra, which is intuitive and can be conveniently presented on the high school level.[12]

(e) I said at the beginning that the "rediscovery" of Euler's theorem can be proposed as a project to an intelligent teenager. A more obvious occasion is to

[10] [4] uses the history of Euler's theorem to discuss topics in epistemology and methodology. Rewriting this discussion so that it becomes simpler and more accessible, even if less witty, would be a rewarding task.

[11] See [5], especially pp. 258–259.

[12] See [3], vol. 2, pp. 149–156. This approach, although not the attached proof, was also known to Euler. See [1], especially p. 90, Propositio 9.

develop the theorem in class discussion; the teacher should lead the discussion so that the students have a fair share in the "rediscovery" of the theorem.

(f) In individual projects, or in class discussion, Euler's theorem could and should be used to introduce young people to *scientific method*, to *inductive research*, to the fascination of *analogy*. What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

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