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On the Transcendence of Certain Series

CHRISTOPHER F. WOODCOCK AND HABIB SHARIF

Mathematical Institute, The University of Kent, Canterbury, Kent CT2 7NF, England

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1. INTRODUCTION

R. P. Stanley in [5] observed that the series $f(x)=\sum_{n=0}^{\infty}\binom{2n}{n}^t x^n$ is transcendental over $C(x)$ for even $t>1$ and stated that it is unknown for odd $t>1$ whether or not it is transcendental. We show here that (the reduction of) f is algebraic over any field of positive characteristic p and we then deduce (from the explicit equations obtained) that f is transcendental over any field of characteristic zero for any integer $t>1$.

We also give a generalisation of this result in the case of multinomial coefficients.

Note that if $t=1$ then f is algebraic of degree at most 2 over any field (in fact $f=(1-4x)^{-1/2}$) and that for all $t \geqslant 1$, $f=1$ over any field of characteristic 2 so that we may suppose that, in the case of positive characteristic p, $p>2$.

If K is a field, $K((x))$ will denote the field of formal power series with coefficients in K, i.e., $f \in K((x))$ if $f=\sum_{n=-k}^{\infty} a_n x^n$, $a_n \in K$, $k \in \mathbf{N}$. By an *algebraic function* over K we mean an element of $K((x))$ which is algebraic over the field of rational functions $K(x)$. An element which is not algebraic is called *transcendental.*

We intend to prove the following theorem:

THEOREM. *For $t \in \mathbf{N}$, $t>1$, $\sum_{n=0}^{\infty}\binom{2n}{n}^t x^n$ is transcendental over K, where K is any field of characteristic zero.*

Throughout this paper f will denote the series $f(x)=\sum_{n=0}^{\infty}\binom{2n}{n}^t x^n \in \mathbf{Z}[[x]]$, for $t \in \mathbf{N}$ and $t>1$ and $\mathbf{Z}_p=\{0, 1, 2, ..., p-1\}$ will denote the field of integers modulo the prime p.

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2. Preliminaries

PROPOSITION 2.1. *Suppose that K is any field of characteristic zero and $h(x)=\sum_{i=0}^{\infty} h_i x^i \in \mathbf{Z}[[x]]$ is algebraic over K of degree N. Then for any prime p, $\bar{h}(x)=\sum_{i=0}^{\infty} \bar{h}_i x^i \in \mathbf{Z}_p[[x]]$ is algebraic over $\mathbf{Z}_p$ of degree at most N, where $\bar{a}$ is the image of a in $\mathbf{Z}_p$.*

Proof. h is algebraic over $\mathbf{Q}(x)$ of degree N [4, Theorem 6.1]. Hence there exist elements $a_i(x)$, $i=0, 1, 2, ..., N$ in $\mathbf{Q}[x]$ (after clearing the denominators) not all zero such that

$$\sum_{i=0}^{N} a_i(x)\, h^i(x)=0.$$

Clearing all the denominators of the coefficients of the $a_i(x)$ we will have

$$\sum_{i=0}^{N} b_i(x)\, h^i(x)=0$$

for some $b_i(x) \in \mathbf{Z}[x]$, $i=0, 1, 2, ..., N$, and $b_j(x) \neq 0$ for some j.

By cancellation of any common factors we may suppose that for each prime p not all of the coefficients in the b_i, $i=0, 1, 2, ..., N$ have the factor p. We now reduce all the coefficients modulo p and obtain the non-trivial equation

$$\sum_{i=0}^{N} \bar{b}_i(x)\, \bar{h}^i(x)=0.$$

(By assumption $\bar{b}_i \neq 0$ for some i and hence the above equation is non-trivial.) Therefore $\bar{h}$ is algebraic over $\mathbf{Z}_p$ of degree at most N (which is independent of p) as required.

LEMMA 2.2. *Suppose that $A \in \mathbf{N}$, $A>0$. Then there exist infinitely many primes p such that whenever m divides $p-1$ then $m=1, 2$ or $m>A$.*

Proof. Suppose that $p_1, p_2, ..., p_t$ are the distinct odd primes which are not greater than A. By the Chinese Remainder Theorem the system of congruences

$$\begin{aligned} x &\equiv 2 \quad (\operatorname{mod} p_1 p_2 \cdots p_t) \\ x &\equiv 3 \quad (\operatorname{mod} 4) \end{aligned} \tag{2.2.1}$$

has a unique solution modulo $4p_1 p_2 \cdots p_t$. Thus there exists $c \in \mathbf{Z}$ (clearly coprime to $4p_1 p_2 \cdots p_t$) such that the equation $x \equiv c \pmod{4p_1 p_2 \cdots p_t}$ and the system (2.2.1) are equivalent. Therefore by Dirichlet's Theorem on primes in an Arithmetic Progression we can find infinitely many primes p such that

$$p \equiv 2 \pmod{p_1 p_2 \cdots p_t}$$
$$p \equiv 3 \pmod{4}.$$

Now, since $p-1 \equiv 1 \pmod{p_1 p_2 \cdots p_t}$ it follows that p_i does not divide $p-1$, $i=1,2,\ldots,t$ and as $p-1 \equiv 2 \pmod 4$, 4 does not divide $p-1$. Thus if m is a divisor of $p-1$ then $m=1,2$ or $m>A$.

3. Results

We recall that if $a(x)=\sum_{n\geqslant 0} a_n x^n$, $b(x)=\sum_{n\geqslant 0} b_n x^n$, then the Hadamard product $a*b$ of a and b is defined by $a*b(x)=\sum_{n\geqslant 0} a_n b_n x^n$.

Note that $f(x)=\sum_{n=0}^{\infty} \binom{2n}{n}^t x^n = h*h*\cdots*h$ (t times), where $h=\sum_{n=0}^{\infty}\binom{2n}{n} x^n=(1-4x)^{-1/2}$ and hence, since over any field of positive characteristic the Hadamard product of two algebraic formal power series is again an algebraic formal power series (see, for example, [4, The Main Theorem]), it follows that f is algebraic over any field of positive characteristic. However, we will now prove this directly by using Lucas' Theorem: For $m, n \in \mathbf{N}$, p a prime, $\binom{m}{n}^p \equiv \binom{mp}{np} \equiv \binom{m}{n} \bmod p$ and $\binom{mp+i}{np+j} \equiv \binom{m}{n}\binom{i}{j} \bmod p$ for $i, j \in \mathbf{N}$ with $0 \leqslant i, j \leqslant p-1$ (see [3, p. 271]).

Proposition 3.1. *If p is any odd prime then f is algebraic over $\mathbf{Z}_p$.*

Proof. Working modulo p and applying Lucas' Theorem we get

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{2n}{n}^t x^n = \sum_{i=0}^{p-1} \sum_{n=0}^{\infty} \binom{2np+2i}{np+i}^t x^{np+i} \\ &= \sum_{i=0}^{(p-1)/2} \sum_{n=0}^{\infty} \binom{2n}{n}^t \binom{2i}{i}^t x^{np+i} \\ &\quad + \sum_{i=(p+1)/2}^{p-1} \sum_{n=0}^{\infty} \binom{2n+1}{n}^t \binom{2i-p}{i}^t x^{np+i}. \end{aligned}$$

Since $f^p(x)=\sum_{n=0}^{\infty}\binom{2n}{n}^t x^{np}$ and $\binom{m}{n}=0$ for $m<n$ it follows that $f(x)=\left(\sum_{i=0}^{(p-1)/2}\binom{2i}{i}^t x^i\right) f^p(x)$. However, $f(x)\neq 0$ and hence

$$f^{p-1}(x)=\left(\sum_{i=0}^{(p-1)/2}\binom{2i}{i}^t x^i\right)^{-1} \in \mathbf{Z}_p(x).$$

Therefore f is algebraic over $\mathbf{Z}_p$.

Let us denote the degree of f over $\mathbf{Z}_p(x)$ by N_p. We will show that if P is the set of all prime numbers, then $\{N_p\}_{p \in P}$ is unbounded.

PROPOSITION 3.2. *Suppose that $A \in \mathbf{N}$, $A > 0$. Then the degree N_p of f over $\mathbf{Z}_p(x)$ is greater than A for some prime p.*

Proof. Let p be a prime satisfying the condition of Lemma 2.2 with $p > 3^{t-1}$. Suppose that $F(X) = X^{p-1} - \alpha \in \mathbf{Z}_p(x)[X]$, where $\alpha = \sum_{i=0}^{(p-1)/2} \binom{2i}{i}^t x^i \neq 0$. Let K be the splitting field of $F(X)$ over $\mathbf{Z}_p(x)$ and $[K: \mathbf{Z}_p(x)] = r$. Since K is a Kummer Field [2, p. 59], if $F(X) = \prod_{j=1}^k P_j(X)$ is the irreducible factorization of $F(X)$ in $\mathbf{Z}_p(x)[X]$, then each $P_j(X)$ has degree r. In particular f has degree r over $\mathbf{Z}_p(x)$ since f^{-1} is a root of $F(X)$. Clearly $p-1 = \text{degree } F(X) = rk$. By the choice of p, as r divides $p-1$, we get that $r = 1, 2$ or $r > A$.

If we can show that the first two cases are impossible then $N_p = r > A$ as required.

Case 1. If $r = 1$, then $f \in \mathbf{Z}_p(x)$. Suppose $f = b/c$ for $b, c \in \mathbf{Z}_p[x]$, $c \neq 0$. We know that $(1/f(x))^{p-1} = \alpha$, i.e., a polynomial of degree $(p-1)/2$ in x. So $c^{p-1} = \alpha b^{p-1}$. Hence $p-1$ divides the degree of α which contradicts $\deg(\alpha) = (p-1)/2$.

Case 2. If $r = 2$, then the degree of each $P_j(X)$ is 2 for $j = 1, 2, ..., k$. So

$$P_j(X) = (X - \lambda_j \beta)(X - \gamma_j \beta) \in \mathbf{Z}_p(x)[X],$$

where β is a root of $F(X)$ in K and $\lambda_j \neq \gamma_j$ in $\mathbf{Z}_p^*$. It follows that $\lambda_j \gamma_j \beta^2$ and hence $\beta^2 \in \mathbf{Z}_p(x)$. Since $\alpha \in \mathbf{Z}_p[x]$ and $\alpha = (\beta^2)^{(p-1)/2} \in \mathbf{Z}_p(x)^{(p-1)/2}$ and since $\mathbf{Z}_p[x]$ is integrally closed in $\mathbf{Z}_p(x)$ we conclude that $\alpha \in \mathbf{Z}_p[x]^{(p-1)/2}$. Thus $\alpha = \sum_{i=0}^{(p-1)/2} \binom{2i}{i}^t x^i = (a + bx)^{(p-1)/2}$ for some $a, b \in \mathbf{Z}_p^*$.

Working in $\mathbf{Z}_p$ and equating coefficients of $1, x, x^2$ we get $1 = a^{(p-1)/2}$, $2^t = ((p-1)/2)\, a^{((p-1)/2)-1} b$, $6^t = \frac{1}{8}(p-1)(p-3) a^{((p-1)/2)-2} b^2$. These lead to $3^{t-1} = 2^{t-1}$ in $\mathbf{Z}_p$. Therefore since $p > 3^{t-1}$ it follows that $3^{t-1} = 2^{t-1}$ in $\mathbf{Z}$ which is a contradiction as $t > 1$.

THEOREM 3.3. *Let $t \in \mathbf{N}$, $t > 1$. If $f(x) = \sum_{n=0}^{\infty} \binom{2n}{n}^t x^n \in \mathbf{Z}[[x]]$, then f is transcendental over any field of characteristic zero.*

Proof. Suppose otherwise, so that f is algebraic over K of degree N (say). By Proposition 2.1, $\bar{f}$ is algebraic over $\mathbf{Z}_p$ of degree at most N which is independent of p. However, by Proposition 3.2, the degree N_p of $\bar{f}$ over $\mathbf{Z}_p(x)$ is unbounded (for varying p) which is the required contradiction. Hence f is transcendental over K.

Remark 3.4. This method seems to be more generally applicable to the problem of deciding whether or not a given series in $\mathbf{Z}[[x]]$ is transcendental over K, where K is any field of characteristic zero. For example, it is known that $\sum_{m=0}^{\infty} ((3m)!/(m!)^3)\, x^{3m}$ is transcendental over $\mathbf{Q}$ [1, p. 209]. Using the method described above we will prove the following theorem:

THEOREM. *If* $g(x)=\sum_{m=0}^{\infty} \binom{km}{m, m, ..., m}^t x^m$ *where* $t, k \in \mathbf{N}$ *with* $t \geqslant 1$, $k \geqslant 3$, *then g is transcendental over any field of characteristic zero.*

Note that $g=1$ over any field of characteristic p, $p \leqslant k$, so that we may assume that, in the case of positive characteristic p, $p>k$.

Moreover, note that g is an algebraic series over any field of positive characteristic as $g(x_1 x_2 \cdots x_k)=(h_1 * h_2) * (h_1 * h_2) * \cdots * (h_1 * h_2)$ (t times), where $h_1=1/(1-x_1 x_2 \cdots x_k)$, $h_2=1/(1-x_1-x_2- \cdots -x_k)$ and $*$ denotes the Hadamard product operation. However, we will prove this directly by using a generalisation of Lucas' Theorem.

From now on g will denote the series $g(x)=\sum_{m=0}^{\infty} \binom{km}{m, m, ..., m}^t x^m$.

A GENERALISATION OF LUCAS' THEOREM. *Suppose that j, j_i, n, $n_i \in \mathbf{N}$ for $i=1, 2, ..., k$ and p is a prime. If $n=n_1+n_2+ \cdots +n_k$ and $j=j_1+j_2+ \cdots +j_k$ with $0 \leqslant j, j_i \leqslant p-1$, $i=1, 2, ..., k$, then*

(i) $\binom{n}{n_1, n_2, ..., n_k}^p \equiv \binom{np}{n_1 p, n_2 p, ..., n_k p} \equiv \binom{n}{n_1, n_2, ..., n_k} \bmod p$

(ii) $\binom{np+j}{n_1 p+j_1, n_2 p+j_2, ..., n_k p+j_k} \equiv \binom{n}{n_1, n_2, ..., n_k}\binom{j}{j_1, j_2, ..., j_k} \bmod p.$

The results follow easily from the fact that

$$(x_1+x_2+ \cdots +x_k)^{np}=(x_1^p+x_2^p+ \cdots +x_k^p)^n \in \mathbf{Z}_p[x_1, x_2, ..., x_k].$$

Remark 3.5. It is also easily seen that multinomial coefficients not of the form of the left hand side of (ii) are zero in $\mathbf{Z}_p$.

PROPOSITION 3.6. *If $p>k$ is any prime, then g is algebraic over $\mathbf{Z}_p$.*

Proof. Working modulo p and applying the Generalisation of Lucas' Theorem we get

$$\begin{aligned} g &= \sum_{n=0}^{\infty} \binom{kn}{n, n, ..., n}^t x^n \\ &= \sum_{i=0}^{p-1} \sum_{n=0}^{\infty} \binom{k(pn+i)}{pn+i, pn+i, ..., pn+i}^t x^{pn+i} \\ &= \sum_{i=0}^{[(p-1)/k]} \left(\sum_{n=0}^{\infty} \binom{kn}{n, n, ..., n}^t x^n\right)^p \binom{ki}{i, i, ..., i}^t x^i \\ &\quad + \sum_{i=[(p-1)/k]+1}^{p-1} \sum_{n=0}^{\infty} \binom{k(pn+i)}{pn+i, pn+i, ..., pn+i}^t x^{pn+i}. \end{aligned}$$

The second term in the last equality is zero by Remark 3.5. Hence $g = (\sum_{i=0}^{[(p-1)/k]} \binom{ki}{i, i, ..., i}^t x^i) g^p$. However, $g(x) \neq 0$ and hence

$$g^{p-1}(x) = \left(\sum_{i=0}^{[(p-1)/k]} \binom{ki}{i, i, ..., i}^t x^i \right)^{-1} \in \mathbf{Z}_p(x).$$

Therefore $g(x)$ is algebraic over $\mathbf{Z}_p$.

Note that the top coefficient of $\eta = \sum_{i=0}^{[(p-1)/k]} \binom{ki}{i, i, ..., i}^t x^i$ is non-zero in $\mathbf{Z}_p$ as $k[(p-1)/k] < p$, and hence η is a polynomial of degree $[(p-1)/k]$. Thus it is easily shown that Proposition 3.2 similarly holds for g, i.e.,

PROPOSITION 3.7. *Suppose $A \in \mathbf{N}$, $A > 0$. Then the degree N_p (say) of g over $\mathbf{Z}_p(x)$ is greater than A for some prime p.*

Hence we have

THEOREM 3.8. *If $g(x) = \sum_{m=0}^{\infty} \binom{km}{m, m, ..., m}^t x^m$ where t, $k \in \mathbf{N}$ with $t \geqslant 1$, $k \geqslant 3$, then g is transcendental over any field of characteristic zero.*

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