FORMAL RELATIONS BETWEEN ANALYTIC FUNCTIONS

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Abstract. In this paper we give conditions under which the completion of the kernel of a homomorphism of analytic rings $\phi: A \to B$ coincides with the kernel of the corresponding homomorphism of the completions $\hat{\phi}: \hat{A} \to \hat{B}$.

Introduction

Let $y_1(x), \cdots, y_n(x)$ be analytic functions defined in a neighborhood of the origin in $\mathbf{C}^m$. A formal relation between the functions $y_i(x)$ will be a formal power series $F(y_1, \cdots, y_n)$ that vanishes if we put $y_i = y_i(x)$. If F is a convergent series, then it is called an analytic relation between the functions $y_i(x)$.

In [2] M. Artin posed the following question. Assume that there is a (nontrivial) formal relation between the analytic functions $y_i(x)$. Does there exist an analytic relation between these functions? This question can be reformulated as follows. Consider the homomorphisms of rings $\phi: \mathbf{C}\{y\} \to \mathbf{C}\{x\}$ and $\hat{\phi}: \mathbf{C}[[y]] \to \mathbf{C}[[x]]$, defined by putting $y_i = y_i(x)$.(1) Assume that ϕ is injective. Will $\hat{\phi}$ be injective? In a more general situation, suppose that A and B are analytic rings,(2) $\phi: A \to B$ a ring homomorphism, and $\hat{\phi}: \hat{A} \to \hat{B}$ the corresponding homomorphism of the completions. Assume that ϕ is injective. Is $\hat{\phi}$ injective? In this form the question was stated by Grothendieck [6].

In [12] the author constructed an example that gave a negative answer to these questions: he showed four functions of two variables between which no nontrivial analytic relations exist, but a formal relation *does* exist.

However, it turns out that if "sufficiently many" formal relations exist between the analytic functions $y_i(x)$, then all of these are induced by analytic relations. Let $J \subset \mathbf{C}[[y]]$ be the ideal of all formal relations between the functions $y_i(x)$, and $I \subset \mathbf{C}\{y\}$ the ideal of all analytic relations. Put $r_1 = \operatorname{rank}(\partial y_i(x)/\partial x_j)$, $r_2 = \dim \mathbf{C}[[y]]/J$, $r_3 = \dim \mathbf{C}\{y\}/I$, where dim denotes the Krull dimension. It is easy to show that $r_1 \leq r_2 \leq r_3$. *In this paper we prove the following assertion* (Theorem 4.8):

If $r_1 = r_2$, then $r_2 = r_3$ and $J = I \cdot \mathbf{C}[[y]]$. (Note that in the example of [12], $r_1 = 2$, $r_2 = 3$ and $r_3 = 4$.)

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(1) Here $\mathbf{C}\{x\}$ and $[[x]]$ are the rings of convergent and formal power series, respectively.

(2) Recall that an analytic ring is a ring of the form $\mathbf{C}\{x\}/I$, where I is an ideal in $\mathbf{C}\{x\}$.

This theorem is then applied to study the connection between homomorphisms $\phi: A \to B$ of analytic rings, and the corresponding homomorphisms $\hat{\phi}: \hat{A} \to \hat{B}$ of the completions of these rings. If B is an integral domain, we obtain conditions under which

$$\ker \hat{\varphi} \simeq \ker \varphi \otimes_A \hat{A} \tag{1}$$

(Theorem 5.2 and the corollary to it) and

$$\hat{\varphi}(\hat{A}) \cap B \simeq \varphi(A) \tag{2}$$

(Theorem 5.5). The case when B is an arbitrary analytic ring without nilpotent elements easily reduces to the case when B is an integral domain (Proposition 5.6). In particular, the isomorphism (1) occurs if B is a ring without nilpotents and $\dim A \leq 3$ (Theorem 5.7). The example given at the end of the paper shows that each situation can arise if B contains nilpotent elements.

The author thanks V. P. Palamodov for calling his attention to this subject.

§1. Convergence of formal series that depend algebraically on a parameter

Lemma 1.1. *Let K be a field, R a K-algebra (commutative with identity) and x an independent variable. Let*

$$f = \sum_{\nu=0}^{\infty} f_\nu x^\nu \in R[[x]].$$

If f is integral over $K[[x]]$, then all the f_ν are algebraic over K.

The proof is trivial.

Lemma 1.2. *Let A be an integral domain, B an A-algebra,*

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu x^\nu \in B[[x]],$$

$$P(x, z) = z^p + \sum_{j=1}^{p} c_j(x) z^{p-j} \in A[[x]][z], \quad c_j = \sum_{i=0}^{\infty} c_{ij} x^i.$$

Assume that $P(x, f(x)) = 0$ and $P'_z(x, f(x)) \neq 0$ in $B[[x]]$. Put

$$P'_z(x, f(x)) = \sum_{l=0}^{\infty} g_l x^l, \quad g_l \in B$$

(we shall also consider g_l as polynomials of c_{ij} and f_ν with integral coefficients). Let l_0 be the minimal index such that $g_{l_0} \neq 0$ in B. Put $g = g_{l_0}$. For every $l > l_0$ we have

$$f_l = G_l / g^{2(l-l_0)-1}, \tag{1.1}$$

where G_l is a polynomial of c_{ij} $(i \leq l + l_0)$, f_ν $(\nu \leq l_0)$ and g with integral coefficients, *whose degree in g does not exceed* $2(l-l_0-1)$, *and in* c_{ij} *and* f_ν *does not exceed* $2p(l-l_0)-p$. *Moreover, if we assume* c_{ij} *and* f_ν *to be homogeneous of degree i and* ν *respectively, then the generalized degree of* G_l *in* c_{ij} *and* f_ν *does not exceed* $(2l_0+1)(l-l_0)$.

Proof. We write $P(x, f(x))$ in the form $P(x, f(x)) = \Sigma_0^\infty P_k x^k$, where the P_k are polynomials in c_{ij} and f_ν with integral coefficients. Let $\mu, \nu \in \mathbf{N}$, $\mu < \nu$. Then the polynomial $P_{\nu+\mu}$ depends linearly on f_ν. Moreover, $P_{\nu+\mu} = g_\mu f_\nu + Q_{\nu,\mu}$, where $Q_{\nu,\mu}$ does not depend on f_ν. In particular, if $\mu < l_0$, then $g_\mu = 0$ and $P_{\nu+\mu}$ does not depend on f_ν. Now suppose $l > l_0$. Then P_{l+l_0} does not depend on f_ν, if $\nu > l$, since

$$l_0 + l = \mu + \nu, \quad \nu > l \Rightarrow \mu < l_0.$$

Furthermore, $P_{l+l_0} = gf_l + Q_{l,l_0}$. From the equality $P_{l+l_0} = 0$ we get

$$f_l = -Q_{l,l_0}/g. \tag{1.2}$$

We note that Q_{l,l_0} is a polynomial of c_{ij} $(i \le l + l_0)$ and f_ν $(\nu < l)$ of degree at most p and generalized degree $l + l_0$.

Replacing in turn on the right side of (1.2) f_{l-i} $(1 \le i \le l - l_0)$ by $-Q_{l-i,l_0}/g$, we obtain the following expression for f_l:

$$f_l = G_l/g^{m_1(l)}, \tag{1.3}$$

where G_l is a polynomial of c_{ij} $(i \le l + l_0)$, f_ν $(\nu \le l_0)$ and g, whose degree in g is at most $m_1(l) - 1$, in c_{ij} and f_ν is equal to $m_2(l)$, and whose generalized degree in c_{ij} and f_ν is equal to $m_3(l)$. We shall show that

$$\begin{aligned} m_1(l) &\leqslant 2(l-l_0)-1, \\ m_2(l) &\leqslant 2p(l-l_0)-p, \\ m_3(l) &\leqslant (2l_0+1)(l-l_0). \end{aligned} \tag{1.4}$$

We use induction on l. For $l = l_0 + 1$ we have $m_1(l) = 1$, $m_2(l) \le p$ and $m_3(l) = 2l_0 + 1$, and the inequalities (1.4) hold. We assume that they are true for all l, $l_0 < l < l_1$. Obviously $G_{l_1}/g^{m_1(l_1)}$ is obtained from $-Q_{l_1,l_0}/g$ after substituting $f_l = G_l/g^{m_1(l)}$ in Q_{l_1,l_0}. Let Q be some homogeneous polynomial Q_{l_1,l_0}. We consider three cases.

1. Q does not depend on f_l $(l_0 < l < l_1)$. Then the term $-Q/g$ in the expression (1.3) for f_{l_1} has g^1 in the denominator, degree p and generalized degree $l_1 + l_0$, so that the inequalities (1.4) hold for it (granting that $l_1 \ge l_0 + 1$).

2. $Q = f_\nu \cdot Q'$, where Q' does not depend on f_l $(l_0 < l < l_1)$. Substituting $f_\nu = G_\nu/g^{m_1(\nu)}$, in the expression for f_{l_1} we will get a term whose denominator is $g^{m_1(\nu)+1}$, whose degree is at most $p - 1 + m_2(\nu)$, and whose generalized degree is equal to $l_1 + l_0 - \nu + m_3(\nu)$. Since $\nu \le l_1 - 1$, inequalities (1.4) also hold for this term.

3. $Q = Q' \cdot \Pi_{i=1}^{i_0} f_{\nu_i}$ $(l_0 < \nu_i < l_1)$, where Q' does not depend on f_l $(l_0 < l < l_1)$,

and the ν_i are not necessarily different where $i_0 \geq 2$. After the substitution $f_{\nu_i} = G_{\nu_i}/g^{m_1(\nu_i)}$, granting (1.4) we get

$$m_1(l_1) \leqslant 1 + \sum_{i=1}^{i_0} (2(\nu_i - l_0) - 1) \leqslant 1 + 2\sum_{i=1}^{i_0} \nu_i - 4l_0 - 2.$$

Since the generalized degree of Q is equal to $l_1 + l_0$, it follows that $\sum_{i=1}^{i_0} \nu_i \leq l_1 + l_0$, from which we have $m_1(l_1) \leq 2(l_1 - l_0) - 1$. Further,

$$m_2(l_1) \leqslant p - i_0 + \sum_{i=1}^{i_0} (2p(\nu_i - l_0) - p) \leqslant p + 2p\sum_{i=1}^{i_0} \nu_i - 4pl_0 - 2p$$
$$\leqslant p + 2p(l_1 + l_0) - 4pl_0 - 2p = 2p(l_1 - l_0) - p.$$

Finally,

$$m_3(l_1) \leqslant l_1 + l_0 - \sum_{i=1}^{i_0} \nu_i + \sum_{i=1}^{i_0} (2l_0 + 1)(\nu_i - l_0)$$
$$\leqslant l_1 + l_0 + 2l_0 \sum_{i=1}^{i_0} \nu_i - 2(2l_0 + 1) l_0$$
$$\leqslant (2l_0 + 1)(l_1 + l_0) - 2(2l_0 + 1) l_0 = (2l_0 + 1)(l_1 - l_0).$$

The lemma is proved.

Definition 1.3. We denote by $\mathfrak{A}_{x,t}$ the subring of $\mathbf{C}[t][[x]]$ formed by the series

$$c(x, t) = \sum_{i=0}^{\infty} c_i(t) x^i$$

satisfying the following condition:

$$\exists k_1, k_2 : \deg c_i(t) \leqslant k_1 i + k_2 \quad \forall i. \tag{1.5}$$

Lemma 1.4. *The ring $\mathfrak{A}_{x,t}$ is integrally closed.*

Proof. Let f be integral over $\mathfrak{A}_{x,t}$ and $f = g/h$, where g and h belong to $\mathfrak{A}_{x,t}$. Since $\mathfrak{A}_{x,t} \subset \mathbf{C}[t][[x]]$, and $\mathbf{C}[t][[x]]$ is integrally closed, we have $f \in \mathbf{C}[t][[x]]$. We write f in the form

$$f = \sum_{i=0}^{\infty} f_i(t) x^i, \quad f_i(t) \in \mathbf{C}[t].$$

Let $P = z^p + \sum_{j=1}^{p} c_j(x, t) z^{p-j} \in \mathfrak{A}_{x,t}[z]$ be a polynomial annihilating f. Let

$$g = \sum_{i=0}^{\infty} g_i(t) x^i, \quad h = \sum_{i=0}^{\infty} h_i(t) x^i, \quad c_j = \sum_{i=0}^{\infty} c_{ij}(t) x^i.$$

We put $d_i = \max(\deg g_i(t), \deg h_i(t), \deg c_{ij}(t))$. By definition of the ring $\mathfrak{A}_{x,t}$ there exist constants k_1 and k_2 such that $d_i \leq k_1 i + k_2$ for all i. We assume that this is not so. Then $f(x, t)$ contains a monomial of the form $ax^\lambda t^\mu$, where $a \in \mathbf{C}$, $a \neq 0$ and $\mu > k_1\lambda + k_2$. By the change $x = u^{k_1}$ the problem reduces to the case $k_1 = 1$. Finally, replacing f by $u^{k_2}f$, g by $u^{2k_2}g$, h by $v^{k_2}h$ and c_j by $u^{jk_2}c_j$, we are reduced to the case $k_2 = 0$.

Let a be the subring of $\mathfrak{A}_{x,t}$ consisting of all the series satisfying the condition (1.5) with $k_1 = 1$ and $k_2 = 0$. Then f is integral over a and belongs to the field of fractions of a, but does not belong to a. Therefore it suffices to prove that a is integrally closed. But the mapping $(u, t) \to (v, y)$, defined by the formula $v = u$, $y = ut$, establishes an isomorphism $\mathbf{C}[[v, y]] \to$ a, and since $\mathbf{C}[[v, y]]$ is integrally closed, the lemma is proved.

Lemma 1.5. *Let $T(t)$ be a polynomial of degree n and ϵ an arbitrary positive number. Then for all $\tau \in \mathbf{C}^m$*

$$|T(\tau)| \leqslant \left(\frac{|\tau|}{\varepsilon} + 1\right)^n \sup_{|t|\leqslant\varepsilon} |T(t)|.$$

Proof. Let $\alpha_i \in \mathbf{C}$ $(i = 0, \cdots, n)$ be $(n+1)$th roots of unity $(\alpha_0 = 1)$. Then

$$T(\tau) = \sum_{i=0}^{n} T(\varepsilon\alpha_i) \prod_{j:j\neq i} \frac{\frac{\tau}{\varepsilon} - \alpha_j}{\alpha_i - \alpha_j}.$$

But

$$\left|\prod_{j:j\neq i} (\alpha_i - \alpha_j)\right| = |(n+1)\alpha_i^n| = n+1,$$

$$\left|\prod_{j:j\neq i} \left(\frac{\tau}{\varepsilon} - \alpha_j\right)\right| = \left|\alpha_i^n \prod_{j\neq 0} \left(\frac{\tau}{\varepsilon\alpha_i} - \alpha_j\right)\right|$$

$$= \left|\alpha_i^n \sum_{j=0}^{n} \left(\frac{\tau}{\varepsilon\alpha_i}\right)^j\right| \leqslant \left(\frac{|\tau|}{\varepsilon} + 1\right)^n,$$

and therefore

$$|T(\tau)| \leqslant \left(\frac{|\tau|}{\varepsilon} + 1\right)^n \max_i |T(\varepsilon\alpha_i)|,$$

from which the assertion of our lemma follows.

Lemma 1.6. *Let $c(x, t) = \sum_0^\infty c_i(t)x^i \in \mathfrak{A}_{x,t}$. If for each t_0 of some open set $U \subset \mathbf{C}_t^1$ the series $c(x, t_0)$ converges, then the series $c(x, t)$ converges in a neighborhood of any point $(0, \tau) \in \mathbf{C}_{x,t}^2$.*

Proof. For each $t \in U$ there exists a constant M_t such that $|c_i(t)| < M_t^i$ for all $i > 0$. For $j \in \mathbf{N}$ we put $\Lambda_j = \{t \in U: M_t \leq j\}$. Since $U = \bigcup_j \Lambda_j$, there exists a j_0 such that Λ_{j_0} is not nowhere dense. Hence Λ_{j_0} is everywhere dense in some disc $|t - t_0| < \epsilon$. Furthermore, since $c(x, t) \in \mathfrak{A}_{x,t}$, there exist numbers k_1 and k_2 such that $\deg c_i(t) \leq k_1 i + k_2$. Applying Lemma 1.5, we get

$$|c_i(\tau)| \leqslant \left(1 + \frac{|\tau - t_0|}{\varepsilon}\right)^{k_1 i + k_2} \sup_{|t - t_0| < \varepsilon} |c_i(t)|$$

$$\leqslant \left(1 + \frac{|\tau - t_0|}{\varepsilon}\right)^{k_1 i + k_2} \sup_{\Lambda_{j_0}} |c_i(t)| \leqslant j_0^i \left(1 + \frac{|\tau - t_0|}{\varepsilon}\right)^{k_1 i + k_2},$$

from which the assertion of the lemma follows.

Lemma 1.7. *Let $T(t)$ be a polynomial of degree n, and let ϵ and R be positive constants, $\epsilon < \frac{1}{2}$. Then the diameter of each connected component of the set*

$$M = \{t : |t| \leqslant R, \ |T(t)| \leqslant |T(0)| \varepsilon^n\}$$

does not exceed $8R\epsilon$.

Proof. Let $T(t) = a\Pi_1^n(t - t_j)$, and let $|t_j| \leq 2R$ for $j \leq n_1$ and $|t_j| > 2R$ for $n_1 < j \leq n$. Then $\Pi_{j \leq n_1} |t_j| \leq (2R)^{n_1}$, and since

$$|T(0)| = \Big(\prod_{j \leqslant n_1} |t_j|\Big)\Big(|a| \prod_{j > n_1} |t_j|\Big)$$

we have

$$|a| \prod_{j > n_1} |t_j| \geqslant |T(0)|/(2R)^{n_1}.$$

If $t \in M$, then

$$\left|\prod_{j \leqslant n_1} (t - t_j)\right| = \frac{|T(t)|}{|a| \prod\limits_{j > n_1} |t - t_j|} \leqslant \frac{2^{n - n_1} |T(t)|}{|a| \prod\limits_{j > n_1} |t_j|} \leqslant \varepsilon^n 2^n R^{n_1} \leqslant (2R\varepsilon)^{n_1}.$$

Therefore it suffices to prove that the diameter of each connected component of the set

$$M_1 = \left\{t : \left|\prod_{j \leqslant n_1} (t - t_j)\right| \leqslant (2R\varepsilon)^{n_1}\right\}$$

does not exceed $8R\epsilon$.

Assume that the set M_1 has a connected component Λ of diameter $d > 8R\epsilon$. We may assume that the projection of Λ onto the real axis is a segment l of length d. We consider the polynomial $T_1 = \Pi_{j \leq n_1}(t - \operatorname{Re} t_j)$. Since $|\operatorname{Re} t - \operatorname{Re} t_j| \leq |t - t_j|$, the polynomial T_1 on the segment l does not exceed $(2R\epsilon)^{n_1} < (d/4)^{n_1}$ in modulus, which contradicts the theorem on the polynomials deviating the least from zero.

Lemma 1.8 (Malgrange [3], Chapter IV, Lemma 2.3). *Let z and $c_1, \cdots, c_p$ be complex numbers, with*

$$z^p + \sum_{j=1}^{p} c_j z^{p-j} = 0.$$

Then $|z| \leq 2\max|c_j|^{1/j}$.

Lemma 1.9. *Let $P_i = z^p + \Sigma_1^p c_j^{(i)} z^{p-j}$ $(i = 1, 2)$ be polynomials, $z_\nu^{(i)}$ $(\nu = 1, \cdots, p)$*

their roots. Let $|c_j^{(i)}| \leq K^j$ *and* $|c_j^{(1)} - c_j^{(2)}| \leq K^j\delta$. *Then we can renumber the* $z_\nu^{(i)}$ *so that for each* ν

$$|z_\nu^{(1)} - z_\nu^{(2)}| \leqslant 4pK\,\delta^{1/p}.$$

The proof is easily obtained from Lemmas 2.4 and 2.5 of Chapter IV of Malgrange's book [3].

Lemma 1.10. *Let* $P = z^p + \sum_1^p c_j z^{p-j}$ *be a polynomial,* z_ν *its roots. Let* $|c_j| < K^j$ *for* $1 \leq j \leq j_0$, $|c_{j_0}| = K^{j_0}\alpha$, *and* $|c_j| \leq K^j\delta$ *for* $j > j_0$, *where* $\alpha \leq 1$ *and* $\delta^{1/p} < \alpha/16p$. *Then the* z_ν *can be renumbered so that*

$$\max_{\nu > j_0} |z_\nu| < \min_{\nu \leqslant j_0} |z_\nu|.$$

Proof. Let

$$P_1 = z^p + \sum_{j=1}^{j_0} c_j z^{p-j}, \quad Q = z^{j_0} + \sum_{j=1}^{j_0} c_j z^{j_0-j}.$$

Then the roots $z_\nu^{(1)}$ of the polynomial P_1 can be numbered so that $z_\nu^{(1)} = 0$ for $\nu > j_0$ and $Q(z_\nu^{(1)}) = 0$ for $\nu \leq j_0$. Let $\nu \leq j_0$ and $\omega_\nu = 1/z_\nu^{(1)}$. Then

$$\omega_\nu^{j_0} + \sum_{j=1}^{j_0} \frac{c_{j_0-j}}{c_{j_0}}\,\omega_\nu^{j_0-j} = 0$$

(here $c_0 = 1$). Since $|c_{j_0-j}/c_{j_0}| \leq 1/\alpha K^j \leq 1/(K\alpha)^j$, from Lemma 1.8 it follows that $|\omega_\nu| \leq 2/K\alpha$, i.e. $|z_\nu^{(1)}| \geq K\alpha/2$ for $\nu \leq j_0$. We now apply Lemma 1.9 to P_1 and $P_2 = P$. We can renumber the z_ν so that

$$|z_\nu| \leqslant 4pK\delta^{1/p} \qquad \text{for } \nu > j_0,$$
$$|z_\nu| > \frac{K\alpha}{2} - 4pK\,\delta^{1/p} \qquad \text{for } \nu \leqslant j_0,$$

from which the assertion of the lemma follows.

Definition 1.11. Let

$$P = z^p + \sum_{j=1}^{p} c_j(x)\,z^{p-j}, \; S = z^s + \sum_{j=1}^{s} b_j(x)\,z^{s-j}$$

be unitary pseudopolynomials. Let $z_\nu(x)$ be the roots of S and $\sigma_s^{(k)}$ the standard symmetric function of degree k of s variables. We put

$$R_k(P, S) = \sigma_s^k(P(z_1(x)), \ldots, P(z_s(x))).$$

The set $(R_k(P, S))_{1 \leq k \leq s}$ is called the *complete resultant* of the pseudopolynomials P and S.

Remark. It is not hard to prove that $R_k(P, S)$ is a polynomial in c_j and b_j with integral coefficients.

Theorem 1.12. *Let* $\psi(t)$ *be an analytic function in a neighborhood of zero, and suppose that there exists an irreducible polynomial*

$$S(t, z) = z^s + \sum_{\varkappa=0}^{s-1} d_\varkappa(t) z^\varkappa \quad (d_\varkappa(t) \in \mathbf{C}[t]),$$

such that $S(t, \psi(t)) \equiv 0$. *Let*

$$f(x, t) = \sum_{i=0}^{\infty} f_i(t) x^i \in \mathbf{C}\{x, t\} \quad \textit{and} \quad f(x, t) = \sum_{\varkappa=0}^{s-1} f_\varkappa(x, t) \psi(t)^\varkappa,$$

where $f_\kappa = \Sigma_0^\infty f_i^{(\kappa)}(t) x^i \in \mathfrak{A}_{x,t}$. *Then* $f_\kappa(x, t) \in \mathbf{C}\{x, t\}$.

Proof. Since $f \in \mathbf{C}\{x, t\}$, there exist constants τ and M_1 such that

$$\sup_{|t| \leqslant \tau} |f_i(t)| < M_1^i. \tag{1.6}$$

We may assume that $\tau < 1$. We shall show that

$$\sup_{|t| \leqslant \tau} |f_i^{(\varkappa)}| \leqslant M_2^i$$

for some constant M_2. Assume that this is not so. Set

$$N_i^{(\varkappa)} = \sup_{|t| \leqslant \tau} f_i^{(\varkappa)}(t), \quad N_i = \max_\varkappa N_i^{(\varkappa)}.$$

There exists a sequence of indices i_n such that

$$\lim_{n \to \infty} N_{i_n}^{1/i_n} = \infty.$$

Let κ_0 be the largest number such that

$$\varlimsup_{n \to \infty} \left(\frac{N_{i_n}^{(\varkappa_0)}}{N_{i_n}} \right)^{1/i_n} > 0.$$

We may assume that the sequence i_n is chosen such that

1) $N_{i_n}^{1/i_n} \to \infty$;
2) $N_{i_n} < M_3^{i_n} N_{i_n}^{(\kappa_0)}$;
3) $(N_{i_n}^{(\kappa)})^{1/i_n} = o(N_{i_n}^{1/i_n})$ for all $\kappa > \kappa_0$.

Moreover, since the case $\kappa_0 = 0$ is trivial, it can be assumed that $\kappa_0 \geq 1$.

Consider the κ_0-sheeted analytic function $z = \phi_i(t)$, defined by the equation

$$Q_i(t, z) = z^{\varkappa_0} + \sum_{\varkappa=0}^{\varkappa_0 - 1} (f_i^{(\varkappa_0)})^{\varkappa_0 - \varkappa - 1} f_i^{(\varkappa)} z^\varkappa = 0.$$

Let $\phi_{i\mu}(t)$ $(\mu = 1, \cdots, \kappa_0)$ be the values of this function, enumerated in an arbitrary fashion. Then by Lemma 1.8

$$\max_\mu |\varphi_{i\mu}(t)| \leqslant 2 \max_{\varkappa < \varkappa_0} |(f_i^{(\varkappa_0)}(t))^{\varkappa_0 - \varkappa - 1} f_i^{(\varkappa)}(t)|^{\frac{1}{\varkappa_0 - \varkappa}},$$

and therefore

$$\max_{\mu}\sup_{|t|\leqslant\tau}|\varphi_{i\mu}(t)|\leqslant 2N_i. \tag{1.7}$$

Furthermore, we have

$$\prod_{\mu}(f_i^{(\varkappa_0)}\psi(t)-\varphi_{i\mu}(t))=(f_i^{(\varkappa_0)}(t))^{\varkappa_0-1}\sum_{\varkappa=0}^{\varkappa_0}f_i^{(\varkappa)}(t)\,\psi(t)^{\varkappa}$$
$$=(f_i^{(\varkappa_0)}(t))^{\varkappa_0-1}\left(f_i(t)-\sum_{\varkappa=\varkappa_0+1}^{s-1}f_i^{(\varkappa)}\psi(t)^{\varkappa}\right),$$

and from (1.6) and property 3) of the sequence i_n it follows that

$$\sup_{|t|\leqslant\tau}\left|\prod_{\mu}(f_{i_n}^{(\varkappa_0)}(t)\,\psi(t)-\varphi_{i\mu}(t)\right|^{1/i_n}=o(N_{i_n}^{\varkappa_0/i_n}). \tag{1.8}$$

Let t_l be the roots of the discriminant $\Delta(t)$ of the polynomial $S(t, z)$. Put

$$R_1=\min_{t_l\neq t_k}|t_l-t_k|,\quad R_2=\max|t_l|,$$

$$K=\{t\in\mathbf{C}:\ |t|\leqslant 2R_2,\ |t-t_l|>R_1/4\ \ \forall l\}.$$

Then K is a connected compact set, $\pi_1(K)$ generates $\pi_1(C_t\setminus\{\Delta(t)=0\})$ and there exists a constant $c>0$ such that for all $t\in K$ we have

$$\min_{\mu\neq\nu}|z_\mu(t)-z_\nu(t)|\geqslant c. \tag{1.9}$$

Here $z_\nu(t)$ are the roots of the polynomial $S(t, z)$. Moreover, let $C>0$ be chosen so that

$$\sup_{|t|\leqslant 2R_2}\max_{\nu}|z_\nu(t)|\leqslant C. \tag{1.10}$$

Let

$$S_i=z^s+\sum_{\varkappa=0}^{s-1}(f_i^{(\varkappa_0)}(t))^{s-\varkappa-1}d_\varkappa(t)\,z^{\varkappa}.$$

We note that S_i is an irreducible polynomial and its roots are $z_\nu(t)\cdot f_i^{(\kappa_0)}(t)$. Let $(R_{ik}(t))_{k=1,\dots,s}$ be the complete resultant of the polynomials Q_i and S_i. By definition

$$R_{ik}(t)=\sigma_s^{(k)}\left(\prod_{\mu=1}^{\varkappa_0}(f_i^{(\varkappa_0)}(t)\,z_\nu(t)-\varphi_{i\mu}(t))\right)\quad(\nu=1,\ \dots,\ s),$$

and, using (1.7) and (1.10), we obtain

$$\sup_{|t|\leqslant\tau}|R_{ik}(t)|\leqslant M_4N_i^{k\varkappa_0}. \tag{1.11}$$

Further, let $z_\nu(t)$ be numbered so that $z_1(t)=\psi(t)$ for $|t|\leq\tau$. Then

$$R_{is}(t)=\left[\prod_{\mu=1}^{\varkappa_0}(f_i^{(\varkappa_0)}(t)\,\psi(t)-\varphi_{i\mu}(t))\right]\left[\prod_{\nu=2}^{s}\prod_{\mu=1}^{\varkappa_0}(f_i^{(\varkappa_0)}(t)\,z_\nu(t)-\varphi_{i\mu}(t))\right].$$

Using (1.7) and (1.8), we deduce

$$\sup_{|t|\leqslant\tau} |R_{i_n s}(t)|^{1/i_n} = o(N_{i_n}^{\frac{s\varkappa_0}{i_n}}). \tag{1.12}$$

Let k_0 be the largest number such that

$$\overline{\lim_{n\to\infty}} \left(\frac{\sup\limits_{|t|\leqslant\tau} |R_{i_n k_0}(t)|}{N_{i_n}^{k_0\varkappa_0}} \right)^{\frac{1}{i_n}} > 0.$$

(If no such number exists, put $k_0 = 0$.) From (1.12) it follows that $k_0 < s$. We may assume that the sequence i_n is chosen so that

4) $$N_{i_n}^{k_0\varkappa_0} < M_6^{i_n} \sup_{|t|\leqslant\tau} |R_{i_n k_0}(t)|;$$

5) $$\sup_{|t|\leqslant\tau} |R_{i_n k}(t)|^{1/i_n} = o(N_{i_n}^{k\varkappa_0/i_n}) \quad \text{for all } k > k_0.$$

Since $f^{(\kappa)} \in \mathfrak{A}_{x,t}$, and the R_{ik} are polynomials in $f_i^{(\kappa)}$ and d_κ whose degrees do not depend on i, there exist constants k_1 and k_2 such that

$$\max_{\varkappa,k} (\deg f_i^{(\varkappa)}, \deg R_{ik}) \leqslant k_1 i + k_2. \tag{1.13}$$

From (1.11) and Lemma 1.5 it follows then that

$$\sup_{|t|\leqslant 2R_2} |R_{ik}(t)| \leqslant M_7^i N_i^{k\varkappa_0}. \tag{1.14}$$

Further, from property 5) of the sequence i_n and Lemma 1.5 it follows that for $k > k_0$ we have

$$\sup_{|t|\leqslant 2R_2} |R_{i_n k}(t)|^{1/i_n} = o(N_{i_n}^{k\varkappa_0/i_n}). \tag{1.15}$$

Finally from property 4) of the sequence i_n and Lemma 1.7 it follows that we can choose $\epsilon_1 > 0$ (not depending on i) such that the diameter of each connected component of the set

$$K_n = \{|t| \leqslant 2R_2,\ |R_{i_n k_0}(t)| < \varepsilon_1^{i_n} N_{i_n}^{k_0\varkappa_0}\}$$

does not exceed $R_1/4$. In particular, $\pi_1(K\setminus K_n)$ generates $\pi_1(K)$.

Consider the polynomial

$$P_{i_n}(t, z) = z^s + \sum_{k=1}^{s} R_{i_n k}(t) z^{s-k}.$$

Its roots are $Q_{i_n}(t, f_{i_n}^{(\kappa_0)} \cdot z_\nu)$. From (1.14), (1.15) and Lemma 1.10 it follows that if n is sufficiently large, then the roots of the polynomial $P_{i_n}(z, t)$, $t \in K\setminus K_n$, can be ordered so that

$$\max_{\nu > k_0} |Q_{i_n}(t, f_{i_n}^{(\varkappa_0)} z_\nu)| < \min_{\nu\leqslant k_0} |Q_{i_n}(t, f_{i_n}^{(\varkappa_0)} \cdot z_\nu)|. \tag{1.16}$$

If $k_0 > 0$, then (1.16) for each $t \in K\setminus K_n$ defines a nontrivial partition of the set

$S_{i_n}(t, z) = 0$, depending continuously on t, and hence also a partition of the set $S(t, z) = 0$, which contradicts the irreducibility of the set S, since $\pi_1(K \setminus K_n)$ generates $\pi_1(\mathbf{C}_t \setminus \{\Delta(t) = 0\})$.

Suppose $k_0 = 0$. From (1.13), property 2) of the sequence i_n and Lemma 1.7 it follows that we can choose $\epsilon_2 > 0$ (not depending on i) so that the diameter of each connected component of the set

$$L_n = \{|t| \leqslant 2R_2,\ |f_{i_n}^{(\varkappa_0)}(t)| < \varepsilon_2^{i_n} N_{i_n}\}$$

does not exceed $R_1/4$. In particular, $K \setminus L_n$ is nonempty. Further, from (1.15) and Lemma 1.8, applied to $P_{i_n}(t, z)$, it follows that

$$\sup_{|t| \leqslant 2R_2} \max_{\nu} (Q_{i_n}(t, f_{i_n}^{(\kappa_0)} \cdot z_\nu))^{1/i_n} = o(N_{i_n}^{\kappa_0/i_n}).$$

Since

$$Q_i(t, f_i^{(\varkappa_0)} \cdot z_\nu) = \prod_{\mu=1}^{\varkappa_0} (f^{(\varkappa_0)} z_\nu - \varphi_{i\mu}),$$

it follows that

$$\sup_{|t| \leqslant 2R_2} \max_{\nu} \min_{\mu} (f_{i_n}^{(\varkappa_0)}(t) z_\nu(t) - \varphi_{i_n\mu}(t))^{1/i_n} = o(N_{i_n}^{1/i_n}).$$

Hence

$$\sup_{t \in K \setminus L_n} \max_{\nu} \min_{\mu} \left(z_\nu(t) - \frac{\varphi_{i_n\mu}(t)}{f_{i_n}^{(\varkappa_0)}(t)} \right)^{1/i_n} = o(1),$$

and since $\kappa_0 < S$, we are led to a contradiction with (1.9). The theorem is proved.

Theorem 1.13. *Let $P(x, t, z) \in \mathfrak{A}_{x,t}[z]$ be a unitary pseudopolynomial, and let $f(x, t) \in \mathbf{C}\{x, t\}$ and $P(x, t, f(x, t)) = 0$. Then there exist an irreducible polynomial*

$$S(t, z) = z^s + \sum_{j=1}^{s} d_j(t) z^{s-j} \in \mathbf{C}[t, z],$$

polynomials $F(t)$, $\Delta(t) \in \mathbf{C}[t]$, not identically equal to zero, and functions $f_\kappa(x, t) \in \mathfrak{A}_{x,t} \cap \mathbf{C}\{x, t\}$, such that

$$\Delta(t) f(x, t) = \sum_{\varkappa=0}^{s-1} f_\varkappa\left(\frac{x}{F(t)},\ t\right) \psi(t)^\varkappa,$$

where $\psi(t) \in \mathbf{C}\{t\}$ and $S(t, \psi(t)) \equiv 0$.

Proof. Since the ring $\mathfrak{A}_{x,t}$ is integrally closed, on eliminating the multiple factors we may assume that $\Delta_z(P) \not\equiv 0$. In particular, $P_z'(x, t, f(x, t)) \not\equiv 0$. Since $f = \sum f_\nu(t) x^\nu$ is integral over $\mathbf{C}[t][[x]]$, it follows from Lemma 1.1 that all the $f_\nu(t)$ are algebraic over $\mathbf{C}(t)$. We now apply Lemma 1.2, setting $A = \mathbf{C}[t]$ and $B = \mathbf{C}[t][[x]]$. Since g is a polynomial of $f_\nu(t)$ $(\nu \leq l_0)$ and $c_{ij}(t)$ $(i \leq l_0)$ of degree at most $p - 1$ and generalized degree at most l_0 (we use the notation of Lemma 1.2), (1.1) can be rewritten in the form

$$f_l(t) = H_l/g(t)^{2(l-l_0)-1} \quad (l > l_0), \tag{1.17}$$

where H_l is a polynomial with integral coefficients of the c_{ij} $(i \leq l + l_0)$ and the f_ν $(\nu \leq l_0)$ of degree at most $(4p-2)(l-l_0-1)+p$ and generalized degree at most $(4l_0+1)(l-l_0)-2l_0$.

Since the $f_\nu(t)$ are algebraic over $\mathbf{C}(t)$, and $g(t)$ is a polynomial of the f_ν and the c_{ij}, we see that g is algebraic over $\mathbf{C}(t)$. Hence so is g^{-1}. Therefore there exists a polynomial $a(t)$ such that $\phi_\nu(t) = a(t)f_\nu(t)$ for $\nu \leq l_0$ and $\chi(t) = a(t)g^{-1}(t)$ are integral over $\mathbf{C}[t]$. Since the ring $\mathbf{C}\{t\}$ is integrally closed, it follows that $\chi(t) \in \mathbf{C}\{t\}$. Further, there exists a function $\psi(t) \in \mathbf{C}\{t\}$, integral over $\mathbf{C}[t]$, such that

$$\Delta_1(t)\varphi_\nu(t) = Q_\nu(t, \psi(t)), \quad \Delta_1(t)\chi(t) = Q(t, \psi(t)),$$

where $\Delta_1(t) \in \mathbf{C}[t]$ is the discriminant of the minimal polynomial

$$S(t, z) = z^s + \sum_{j=1}^{s} d_j(t)z^{s-j} \in \mathbf{C}[t, z]$$

of the function $\psi(t)$, and Q_ν and Q are polynomials of t and ψ with complex coefficients of degree at most $s-1$ in ψ. Substituting $g(t)^{-1} = \chi(t)/a(t)$ and $f_\nu(t) = \phi_\nu(t)/a(t)$ in the right side of (1.17), and then

$$\varphi_\nu(t) = Q_\nu(t, \psi(t))/\Delta_1(t), \quad \chi(t) = Q(t, \psi(t))/\Delta_1(t)$$

and finally $\psi(t)^s = -\sum_1^s d_j(t)z^{s-j}$, we obtain the expression

$$f_l(t) = \sum_{\varkappa=0}^{s-1} f_l^{(\varkappa)}(t)\psi(t)^\varkappa, \tag{1.18}$$

where

$$f_l^{(\varkappa)}(t) = T_{l\varkappa}/a(t)^{d_1(l)}\Delta_1(t)^{d_2(l)}, \tag{1.19}$$

and $T_{l\kappa}$ are polynomials of t and c_{ij} $(i \leq l + l_0)$ of degree at most $d_3(l)$ and generalized degree in c_{ij} at most $d_4(l)$ $(d_1(l), \cdots, d_4(l)$ are certain linear functions of $l)$. Since $f_\nu(t) = \phi_\nu(t)/a(t)$ for $\nu \leq l_0$, we may assume that (1.18)–(1.19) holds for all l. Since $P \in \mathfrak{A}_{x,t}[z]$ by hypothesis, there exist constants k_1 and k_2 such that $\deg c_{ij} \leq k_1 i + k_2$. Substituting $c_{ij} = c_{ij}(t)$ in the numerator of the right-hand side of (1.19), we obtain

$$f_l^{(\varkappa)}(t) = S_{l\varkappa}(t)/a(t)^{d_1(l)}\Delta_1(t)^{d_2(l)}, \tag{1.20}$$

where $S_{l\kappa}$ is a polynomial of t of degree at most $D(l)$ ($D(l)$ is a linear function of l). Let $d_1(l) = d_1'l + d_1''$ and $d_2(l) = d_2'l + d_2''$. We may assume that $d_1'', d_2'' \geq 0$. We put

$$\Delta(t) = a(t)^{d_1''}\Delta_1(t)^{d_2''}, \quad F(t) = a(t)^{d_1'}\Delta_1(t)^{d_2'}.$$

Then the function $f_*(x, t) = \Delta(t)f(xF(t), t)$ belongs to $\mathbf{C}\{x, t\}$. On the other hand, from (1.18) and (1.20) it follows that

$$f_*(x, t) = \sum_{\varkappa=0}^{s-1} \psi(t)^\varkappa f_\varkappa(x, t), \tag{1.21}$$

where

$$f_\varkappa(x, t) = \sum_{l=0}^{\infty} T_{l\varkappa}(t) x^l.$$

From the estimate of the degree of $T_{l\kappa}(t)$ it follows that $f_\kappa(x, t) \in \mathfrak{A}_{x,t}$. By Theorem 1.12 it follows that $f_\kappa(x, t) \in \mathfrak{A}_{x,t} \cap \mathbf{C}\{x, t\}$. Therefore (1.21) gives the desired extension of f.

Corollary 1.14. *Suppose the conditions of Theorem* 1.13 *hold. Then the function $f(x, t)$ can be analytically continued along any path in*

$$0_x \times (\mathbf{C}_t \setminus \{F(t)\Delta(t)\Delta_1(t) = 0\})$$

(here $\Delta_1(t)$ is the discriminant of $S(t, z)$).

For the proof it suffices to apply Lemma 1.6 to the expansion of the function $f(x, t)$.

Corollary 1.15. *Let $P(x, t, z) \in \mathfrak{A}_{x,t}[z]$ and*

$$Q(x, t, z) = z^q + \sum_{j=1}^{q} b_j(x, t) z^{q-j} \in \mathbf{C}\{x, t\}[z]$$

be unitary pseudopolynomials, and let $P \vdots Q$ in $\mathbf{C}[[x, t]][z]$. Then there exists an irreducible pseudopolynomial

$$S(t, z) = z^s + \sum_{j=1}^{s} d_j(t) z^{s-j} \in \mathbf{C}[t, z],$$

polynomials $F(t)$ and $\Delta(t)$, and functions $b_j^{(\kappa)}(x, t) \in \mathfrak{A}_{x,t} \cap \mathbf{C}\{x, t\}$ such that for $j = 1, \cdots, q$

$$\Delta(t) b_j(x, t) = \sum_{\varkappa=0}^{s-1} b_j^{(\varkappa)}\left(\frac{x}{F(t)}, t\right) \psi(t)^\varkappa,$$

where $\psi(t) \in \mathbf{C}\{t\}$ and $S(t, \psi(t)) \equiv 0$.

For the proof it suffices to note that the $b_j(x, t)$ are integral over $\mathfrak{A}_{x,t}$, and to apply Theorem 1.13.

Remark. The assertion of Theorem 1.13 remains true if $x = (x_1, \cdots, x_n)$ and $t = (t_1, \cdots, t_m)$ have an arbitrary number of variables. However, in order to keep the proofs simple we have restricted ourselves here to the case $x = x_1$, $t = t_1$, which is the only case that we shall need later.

§2. Branching of analytic functions

2.1. **Notation.** Let $x = (x_1, \cdots, x_n)$, and let $\Phi(x)$ be an analytic function in a neighborhood of zero in $\mathbf{C}^n$. The multiplicity of Φ at zero is the largest number k such that $\Phi \in \mathfrak{m}^k$. The multiplicity at zero of the set $\{x: \Phi(x) = 0\}$ will be the multiplicity of the generating ideal of this set. Let $x = (x', x_n)$, where $x' = (x_1, \cdots, x_{n-1})$, and let $Q = x_n^q + \sum_{i=1}^{q} d_i(x') x_n^{q-i}$ be a unitary pseudopolynomial. The discriminant (in x_n) of Q will be denoted by $\Delta(Q)$. The decomposition of Q into irreducible factors will be the representation of Q in the form $Q = \Pi Q_j^{\mu_j}$, where the Q_j are irreducible (in the ring $\mathbf{C}\{x\}$) unitary pseudopolynomials. We put $Q^{(0)} = \Pi Q_j$. We call $\Delta(Q^{(0)})$ the reduced discriminant $\Delta^0(Q)$ of the pseudopolynomial Q. Obviously, $\Delta^0(Q) \not\equiv 0$ for any Q.

Lemma 2.2. *Let $P = z^p + \Sigma_1^p c_i z^{p-i}$ be a polynomial, z_j $(j = 1, \cdots, k)$ its distinct roots $(k > 1)$, $\Delta^0 = \Delta^0(P)$ and $M = 2\max|z_j|$. Then*

$$\min_{i\neq j} |z_i - z_j| \geqslant \sqrt{|\Delta^0|}/M^{\frac{k(k-1)}{2}-1}.$$

The proof is trivial.

Lemma 2.3. *Let $x = (x_1, \cdots, x_n)$; let $P = z^p + \Sigma_1^p c_i(x) z^{p-i}$ be a distinguished pseudopolynomial, $\Delta^0(x)$ its reduced discriminant, and α an arbitrary positive number. Let $z_1(x), \cdots, z_k(x)$ be distinct roots of P. For each $l \in \mathbf{N}$ there exists an $m = m(p, l) \in \mathbf{N}$ such that for sufficiently small x*

$$|\Delta^0(x)| > \alpha |x|^l, \quad |P(x, z)| < |x|^m$$
$$\Rightarrow \exists\, i: \ |z - z_i(x)| < \frac{1}{2} \min_{i \neq j} |z_i(x) - z_j(x)|.$$

Proof. Since P is a distinguished pseudopolynomial, we have $\max|c_i(x)|^{1/i} \leq 4\alpha|x|^{1/p}$, if x is sufficiently small. Therefore, by Lemma 1.8, $2\max|z_j| \leq 4\alpha|x|^{1/p}$. Hence, by Lemma 2.2,

$$\varepsilon(x) = \min_{i \neq j} |z_i(x) - z_j(x)| \geqslant \frac{\sqrt{\Delta^0(x)}}{(4\alpha|x|^{1/p})^{\frac{k(k-1)}{2}-1}} \geqslant c_1 |x|^{\frac{l}{2} - \frac{k(k-1)-2}{2p}}.$$

Now we assume that z does not belong to an $\epsilon(x)/2$-neighborhood of the points $z_i(x)$. Then

$$|P(x, z)| \geqslant (\varepsilon(x)/2)^p \geqslant c_2 |x|^{\frac{lp}{2} - \frac{k(k-1)-2}{2}}.$$

Since $k(k-1) \geq 2$, we can put $m = (lp + 1)/2$.

Lemma 2.4. (**Abhyankar** [4], (39.7)). *Let*

$$P(x, z) = z^p + \sum_{j=1}^{p} c_j(x) z^{p-j}$$

be a unitary pseudopolynomial, $\Delta(x)$ its discriminant, $\Delta \not\equiv 0$. Assume that all the $c_j(x)$ are defined in a polycylinder $D \in \mathbf{C}_x^n$, and let Z be a disc in $\mathbf{C}_z$ such that the set

$$V = \{(x, z) \in D \times \mathbf{C}_z : P(x, z) = 0\}$$

is contained in $D \times Z$. Let $x_0 \in D$ and $\Delta(x_0) \neq 0$. Then

$$\pi_1(\{x_0\} \times Z \setminus V) \to \pi_1(D \times Z \setminus V)$$

is an epimorphism.

Lemma 2.5. *Let $Q = x_n^q + \Sigma_{i=1}^q d_i(x') x_n^{q-i}$ be a distinguished pseudopolynomial, $\Delta^0(x')$ its reduced discriminant, and κ the multiplicity of Δ^0 at zero. Let $\epsilon = (\epsilon', \epsilon_n) \in \mathbf{R}_+^n$, and let D_ϵ be a polycylinder in $\mathbf{C}^n$ with center at zero and polyradius ϵ such that* *all the* $d_i(x')$ *are defined in the polycylinder* $D_{\epsilon'} \subset \mathbf{C}^{n-1}_{x'}$ *and* $Q(x', x_n) \neq 0$ *for* $x' \in D_{\epsilon'}$, $|x_n| = \epsilon_n$. *Then there exists an* $m = m(q, \kappa)$ *such that the mapping*

$$\pi_1(\{x \in D_\varepsilon,\ |Q(x)| > |x|^m\}) \to \pi_1(\{x \in D_\varepsilon,\ Q(x) \neq 0\})$$

is surjective.

Proof. We choose $a > 0$ such that the set $A = \{x' : |\Delta^0(x')| > a|x'|^\kappa\}$ ajoins zero. From Lemma 2.3 applied to Q it follows that for sufficiently small $x'_0 \in A$ and $m = m(q, \kappa)$ the mapping

$$\pi_1(\{x_n : |x_n| < \varepsilon_n,\ |Q(x'_0, x_n)| > |(x'_0, x_n)|^m\})$$
$$\to \pi_1(\{x_n : |x_n| < \varepsilon_n,\ Q(x'_0, x_n) \neq 0\})$$

is surjective. If, moreover, $x'_0 \in D_{\epsilon'}$, then the mapping

$$\pi_1(\{x_n : |x_n| < \varepsilon_n, Q(x'_0, x_n) \neq 0\}) \to \pi_1(\{x \in D_\varepsilon,\ Q(x) \neq 0\})$$

is surjective by Lemma 2.4 applied to $Q^{(0)}$, from which the assertion of the lemma follows.

Definition 2.6. A distinguished pseudopolynomial $Q = x_n^q + \Sigma_1^q d_i(x')x_n^{q-i}$ is said to be *regular* if its multiplicity at zero is equal to q.

Lemma 2.7. *Let* $\Delta(x)$ *be an analytic function in a neighborhood of zero in* $\mathbf{C}^n$, *and let* $Q = x_n^q + \Sigma_1^q d_i(x')x_n^{q-i}$ *be a distinguished pseudopolynomial equivalent to* Δ. *Assume that* Q *is regular. Let* $\Delta_1(x) \in \mathbf{C}\{x\}$ *and* $\Delta_1(x) - \Delta(x) \in \mathfrak{m}^r$, $r > q$, *and let* Q_1 *be a distinguished pseudopolynomial equivalent to* Δ_1. *Then* $Q - Q_1 \in \mathfrak{m}^r$.

The proof is trivial.

Lemma 2.8. *Let* $\Delta(x)$ *and* $\Phi(x)$ *be analytic functions in a neighborhood of zero in* $\mathbf{C}^n$, q *the multiplicity of* $\Delta(x)$ *at zero,* $m > q$, *and* $A = \{x : |\Delta(x)| > |x|^m\}$. *If for some* $C > 0$ *we have* $|\Phi(x)| < C|x|^r$ *for some sufficiently small* $x \in A$, *then* $\Phi(x) \in \mathfrak{m}^r$.

The proof is trivial.

We fix the following notation:

$P = z^p + \Sigma_1^p c_i(x)z^{p-i}$ is a distinguished pseudopolynomial without multiple factors;

$\Delta(x)$ is the discriminant of P;

q is the multiplicity of Δ at zero;

(x', x_n) is a basis of $\mathbf{C}^n$ such that $\Delta(0, x_n) \sim x_n^q$.

$Q = x_n^q + \Sigma_1^q d_i(x')x_n^{q-i}$ is a regular pseudopolynomial equivalent to Δ;

D_ϵ is a polycylinder in $\mathbf{C}^n$ satisfying the conditions of Lemma 2.5;

$\Delta^0(x')$ is the reduced discriminant of Q;

κ is the multiplicity of Δ^0 at zero;

ρ: $\mathbf{C}^{n+1}_{x,z} \to \mathbf{C}^n_x$ is the projection.

Theorem 2.9. *Let* $F = z^p + \Sigma_1^p f_i(x)z^{p-i}$ *and* $S = z^s + \Sigma_1^s b_i(x)z^{s-i}$ *be distinguished* *pseudopolynomials*, $\Delta_1(x)$ *the discriminant of* F, *and* $\{R_j(F, S)\}$ *the complete resultant of* F *and* S (*cf. Definition* 1.11). *There exist numbers* $r_0 = r_0(p, q, \kappa)$ *and* $r_1 = r_1(p, q, \kappa)$ *satisfying the following conditions*:

a) *If* $F - P \in \mathfrak{m}^r$ $(r \geq r_0)$ *and* $R_j(F, S) \in \mathfrak{m}^{jr}$ *for* $j = 1, \cdots, s$, *then there exists a distinguished pseudopolynomial* T *such that* $P \vdots T^{(0)}$ *and* $T - S \in \mathfrak{m}[(r - p + 1)/p]$.

b) *If* $F - P \in \mathfrak{m}^r$ $(r \geq r_1)$ *and the multiplicities at zero of the sets* $\{\Delta = 0\}$ *and* $\{\Delta_1 = 0\}$ *coincide, then there exist decompositions* $P = \Pi P_j$ *and* $F = \Pi F_j$ *into irreducible factors such that* $P_j - F_j \in \mathfrak{m}[(r - p + 1)/p]$ *for all* j.

Proof. Apply Lemma 2.5 to the pseudopolynomial Q. We obtain a number $m_1 = m_1(q, \kappa)$ and a set $U = \{x \in D_\epsilon, |Q(x)| > |x|^{m_1}\}$ such that $\pi_1(U)$ generates $\pi_1(\{x \in D_\epsilon, Q(x) \neq 0\})$.

Now we apply Lemma 2.3 to the pseudopolynomial P, its discriminant $\Delta = \Delta^0(P)$ and the number $l = \max(m_1, q + 1)$ such that the set

$$A = \{x : |\Delta(x)| > a|x|^l\}$$

contains U. We obtain a number $m_2 = m_2(p, q, m_1)$ such that for every point (x, z) of the set $B \cap \rho^{-1}(A)$, where

$$B = \{(x, z) : |P(x, z)| < |x|^{m_2}\},$$

the root $(x, \phi(x, z))$ of the pseudopolynomial P "close to it" is uniquely determined. Obviously the function $\phi(x, z)$ is continuous in $B \cap \rho^{-1}(A)$.

Suppose the condition of a) holds. Put $r_0 = m_2 + p$. Consider the polynomial $t^s + \Sigma_1^s R_j(F, S)t^{s-j}$. By Lemma 1.8 its roots (i.e. the values of F on the roots of S) are bounded in modulus by $2\max_j |R_j(F, S)|^{1/j} < \alpha|x|^r$ (as always it is assumed that x is sufficiently small). But if $|F(x, z)| < |x|^r$, then

$$|P(x,z)| \leqslant |F(x,z)| + |\sum (c_i - f_i)z^{p-i}| < \beta|x|^{r-p+1} \tag{2.1}$$

(since $c_i(x) - f_i(x) \in \mathfrak{m}^{r-p+i}$, $i = 1, \cdots, p$). Therefore $|P(x, z)| < |x|^{m_2}$ on the roots of S (since $m_2 = r_0 - p < r - p + 1$). Hence the set $\{S = 0\}$ is contained in B, and we can define a continuous map

$$\check{\varphi} : (\{S=0\} \cap \rho^{-1}(A)) \to (\{P=0\} \cap \rho^{-1}(A)),$$

where $\check{\phi}(x, z) = (x, \phi(x, z))$. Put $V = A \cap \{\Delta^0(S) \neq 0\}$. Then $\{S = 0\} \cap \rho^{-1}(V)$ and $\{P = 0\} \cap \rho^{-1}(V)$ are coverings of V, and $\check{\phi}$ is a continuous mapping of the coverings. Let L be the image of the mapping $\check{\phi}$. Then L is also a covering of V. Furthermore, $\pi_1(V)$ generates $\pi_1(\{\Delta \neq 0\})$, since $\pi_1(U)$ generates $\pi_1(\{\Delta \neq 0\})$, $U \subset A$, $V = A \setminus \{\Delta^0(S) = 0\}$, and $\operatorname{codim}_{\mathbb{C}}\{\Delta^0(S) = 0\} = 1$. Therefore the set $L \subset \{P = 0\}$ is invariant under the action of $\pi_1(\{\Delta \neq 0\})$ on the roots of P. From this, as is well known, it follows that there exists a distinguished pseudopolynomial without multiple factors $T^{(0)}$ such that $P \vdots T^{(0)}$ and

$$\{T^{(0)}=0\}\cap\rho^{-1}(V)=L.$$

Let $T^{(0)}=\Pi T_j^{(0)}$ be a decomposition of $T^{(0)}$ into multiple factors, and let $L_j=\{T_j^{(0)}=0\}\cap\rho^{-1}(V)$. Let W_{jl} be the connected components of the cover $\check{\phi}^{-1}(L_j)$. Then we can define the multiplicity ν_{jl} of the mapping $\check{\phi}\colon W_{jl}\to L_j$ and the multiplicity μ_{jl} of the pseudopolynomial S at the points $(x, z)\in W_{jl}$. Put $T=\Pi_{j,l}T_j^{\mu_{jl}\nu_{jl}}$. Obviously, $\deg T=s$. For the proof of a) it remains to show that $S-T\in \mathfrak{m}^{[(r-p+1)/p]}$.

Let $x\in V$ and let $z_j(x)$ and $z_j'(x)$ be the roots (counting multiplicities) of the polynomials $S(x, z)$ and $T(x, z)$ respectively. We may assume that $z_j'(x)=\phi(x, z_j(x))$. Furthermore, it follows from (2.1) that $|P(x, z)|\le\beta|x|^{r-p+1}$ on the roots of S. Therefore

$$\max \operatorname{dist}(z_j(x), \{P=0\}\cap\rho^{-1}(x))\leqslant\gamma|x|^{\frac{r-p+1}{p}}.$$

But, since z_j' is the root of P closest to $z_j(x)$,

$$|z_j(x)-z_j'(x)|\leqslant\gamma|x|^{\frac{r-p+1}{p}}. \tag{2.2}$$

Therefore for $x\in V$ and $|z|\le 1$ we have

$$|S(x,z)-T(x,z)|\leqslant c|x|^{\frac{r-p+1}{p}}.$$

By continuity this inequality is also true for all $x\in A$. Since $l>q$, it follows from Lemma 2.8 that $S-T\in\mathfrak{m}^{[(r-p+1)/p]}$.

Suppose the condition of b) holds. Put $r_1=\max(r_0, p+q\kappa, l+p)$. Since $f_i, c_i\in\mathfrak{m}^{r-p+i}$, and the discriminant is a polynomial of the coefficients, it follows that $\Delta_1-\Delta\in\mathfrak{m}^{r-p+i}$. Since $r-p+1>r_0-p\ge q$ and Q is a regular pseudopolynomial, by Lemma 2.7

$$\Delta_1(x)\sim Q_1(x', x_n)=x_n^q+\sum_{i=1}^{q}d_i^{(1)}(x')\,x_n^{q-i},$$

where $Q_1-Q\in\mathfrak{m}^{r-p+1}$. In particular, Q_1 is also a regular pseudopolynomial. Therefore $d_i, d_i^{(1)}\in\mathfrak{m}^i$. Since, moreover, $d_i-d_i^{(1)}\in\mathfrak{m}^{r-p+1-q+i}$, there exists a constant c_1 such that

$$|d_i(x')|<(c_1|x'|)^i,\quad |d_i^{(1)}(x')|<(c_1|x'|)^i,$$
$$|d_i(x')-d_i^{(1)}(x')|<(c_1|x'|)^i|x'|^{r-p+1-q}$$

for sufficiently small x' and $i=1,\cdots,q$. Put $K(x')=c_1|x'|$ and $\delta(x')=|x'|^{r-p+1-q}$. From Lemma 1.9 it follows that for each x' the roots $y_\nu(x')$ and $y_\nu^{(1)}(x')$ of the polynomials $Q(x', x_n)$ and $Q_1(x', x_n)$ can be numbered so that

$$|y_\nu(x')-y_\nu^{(1)}(x')|<c_2|x'|^{\frac{r-p+1}{q}}. \tag{2.3}$$

Furthermore, since Q and Q_1 are regular pseudopolynomials, the line $\{x'=0\}$

does not belong to the tangent cone of the sets $\{Q = 0\}$ and $\{Q_1 = 0\}$. Therefore the pseudopolynomials $Q^{(0)}$ and $Q_1^{(0)}$ are also regular, and, since the multiplicities at zero of the sets $\{Q = 0\}$ and $\{Q_1 = 0\}$ coincide, $\deg Q^{(0)} = \deg Q_1^{(0)}$. Since the sets of roots of the polynomials $Q^{(0)}$ and $Q_1^{(0)}$ coincide with the sets of roots of Q and Q_1 respectively, it follows from (2.3) that

$$|\Delta^0 - \Delta^0(Q_1)| = |\Delta(Q^{(0)}) - \Delta(Q_1^{(0)})| < c_3 |x'|^{\frac{r-p+1}{q}}. \tag{2.4}$$

But since $r \geq r_1 \geq p + q\kappa$, it follows from this that the multiplicity of $\Delta^0(Q_1)$ at zero is equal to κ.

We now turn to the start of the proof of the theorem. Applying Lemma 2.4 to Q_1, we obtain a set $U_1 = \{x \in D_\epsilon, |Q_1(x)| > |x|^{m_1}\}$, whose fundamental group generates $\pi_1(\{x \in D_\epsilon, Q_1(x) \neq 0\})$. Furthermore, since $\Delta_1 - \Delta \in \mathfrak{m}^{r-p+1}$ and $r \geq r_1 \geq l + p$, we may assume that the set A contains U_1. Now we act as in the proof of part a), taking $S = F$. We obtain a mapping

$$\check{\varphi}: (\{F=0\} \cap \rho^{-1}(V)) \to (\{P=0\} \cap \rho^{-1}(V)).$$

Let $(x, z) \in \{F = 0\} \cap \rho^{-1}(U_1)$. Since $P(x, \phi(x, z)) = 0$, we have

$$|F(x, \varphi(x, z))| \leqslant c|x|^{r-p+1} < |x|^{m_2}$$

(this estimate is obtained in the same way as (2.1)). From Lemma 2.3 applied to F it follows that

$$|z - \varphi(x, z)| < \frac{1}{2} \min_{i \neq j} |z_i(x) - z_j(x)|,$$

if $x \in U_1$ is sufficiently small. (Here $z_i(x)$ are the roots of the polynomial $F(x, z)$.) Therefore the mapping $\check{\phi}$ is an isomorphism of coverings over U_1, and hence also over V. Since $\pi_1(V)$ generates $\pi_1(\{\Delta \neq 0\})$ and $\pi_1(\{\Delta_1 \neq 0\})$, the mapping $\check{\phi}$ establishes a one-to-one correspondence between the irreducible factors F_j of the pseudopolynomial F and the irreducible factors P_j of the pseudopolynomial P. From (2.2) it follows that $F_j - P_j \in \mathfrak{m}^{[(r-p+1)/p]}$.

The theorem is proved.

§3. Branching of formal series

We shall use the notation of §2.1, which extends in a natural way to formal power series.

Lemma 3.1. *Let* $\overline{P} = z^p + \sum_1^p \overline{c}_i(x) z^{p-i}$ *be a formal (i.e.* $\overline{c}_i(x) \in \mathbf{C}[[x]]$*) distinguished pseudopolynomial without multiple factors,* $\overline{\Delta}(x)$ *its discriminant. There exists a sequence of analytic pseudopolynomials* $P_j = z^p + \sum_1^p c_{ij} z^{p-i}$ *converging to* $\overline{P}$ *in the Krull topology and satisfying the following condition:*

Let $x = (x', x_n)$ *be a basis of* $\mathbf{C}^n$ *such that*

$$\overline{\Delta}(x) \sim \overline{Q}(x', x_n) = x_n^q + \sum_{i=1}^q \overline{d}_i(x') x_n^{q-i},$$

where $\overline{Q}$ is a regular pseudopolynomial. Then for sufficiently large j the discriminants $\Delta_j(x)$ of the pseudopolynomials P_j are equivalent to the regular pseudopolynomials $Q_j(x', x_n) = x_n^q + \sum_1^q d_{ij}(x')x_n^{q-i}$, and the sequence $Q_j^{(0)}$ converges to $\overline{Q}^{(0)}$ in the Krull topology.

Proof. We write $\overline{\Delta}(x)$ as a polynomial in the coefficients of $\overline{P}$:

$$\overline{\Delta}(x) = \Delta(\overline{c_i}(x)).$$

Furthermore, let $\overline{\Delta}(x) = \Pi(\overline{\Delta}_\nu(x))^{\mu_\nu}$ be a decomposition of $\overline{\Delta}$ into irreducible factors. Consider the equality

$$\Delta(c_i) = \prod_\nu \Delta_\nu^{\mu_\nu} \tag{3.1}$$

as an equation in the unknowns c_i and Δ_ν, whose formal solution is $(\overline{c}_i(x), \overline{\Delta}_\nu(x))$. By Artin's theorem on the approximation of formal solutions by analytic ones [1], there exists a sequence $(c_{ij}(x), \Delta_{\nu j}(x))_{j\in\mathbf{N}}$ of analytic solutions of the equation (3.1), converging to $(\overline{c}_i(x), \overline{\Delta}_\nu(x))$ in the Krull topology. Put $P_j = z^p + \sum_1^p c_{ij}(x)z^{p-i}$. From (3.1) it then follows that $\Delta_j(x) = \Pi_\nu(\Delta_{\nu j}(x))^{\mu_\nu}$ is the discriminant of P_j.

Since $\overline{\Delta}(x) \sim \overline{Q}(x', x_n)$, where $\overline{Q}$ is a regular pseudopolynomial, we have $\overline{\Delta}_\nu(x) \sim \overline{Q}_\nu(x', x_n)$, where $\overline{Q}_\nu$ are regular pseudopolynomials. From Lemma 2.7 (which is of course also true for formal series) it follows that for sufficiently large j

$$\Delta_{\nu j} \sim Q_{\nu j}(x', x_n),$$

where $Q_{\nu j}$ are regular pseudopolynomials, and $Q_{\nu j} \to \overline{Q}_\nu$ in the Krull topology. But since $\overline{Q}_\nu$ is an irreducible pseudopolynomial, $\Delta(\overline{Q}_\nu) \not\equiv 0$. Therefore $\Delta(Q_{\nu j}) \not\equiv 0$ for sufficiently large j. Hence $Q_{\nu j}$ (and hence also $\Delta_{\nu j}$) does not contain multiple factors. Since $\Delta_j \sim Q_j = \Pi_\nu Q_{\nu j}^{\mu_\nu}$, it follows from this that

$$Q_j^{(0)} = \prod_\nu Q_{\nu j} \to \prod_\nu \overline{Q}_\nu = \overline{Q}^{(0)},$$

as required.

Theorem 3.2. *Let $P_j = z^p + \sum_1^p c_{ij}(x)z^{p-i}$ be a sequence of analytic pseudopolynomials convergent to a formal pseudopolynomial $\overline{P}$ without multiple factors and satisfying the condition of Lemma 3.1, and let $\overline{P} = \Pi\overline{P}_l$ be a decomposition of $\overline{P}$ into irreducible factors. Then for sufficiently large j there exist decompositions $P_j = \Pi P_{jl}$ of the P_j into irreducible factors such that $P_{jl} \to \overline{P}_l$ in the Krull topology.*

Proof. Since $Q_j^{(0)}$ converges to $\overline{Q}^{(0)}$ in the Krull topology,

$$\Delta^0(Q_j) = \Delta(Q_j^{(0)}) \to \Delta(\overline{Q}^{(0)}) = \Delta^0(\overline{Q}).$$

Let κ be the multiplicity of $\Delta^0(\overline{Q})$. Then for sufficiently large j the multiplicities of all the $\Delta^0(Q_j)$ are equal to κ.

Let j_0 be an index such that for all $j \geq j_0$ the multiplicity of $\Delta^0(Q_j)$ is equal to κ and $P_j - \overline{P} \in \mathfrak{m}^{r_1(p,q,\kappa)}$ (cf. Theorem 2.9), and let j and l be $\geq j_0$. From Theorem

2.9b) applied to $P = P_j$ and $F = P_l$ it then follows that if $P_j - P_l \in \mathfrak{m}^r$, then there exist decompositions into irreducible factors $P_j = \Pi P_{jk}$ and $P_l = \Pi P_{lk}$ such that $P_{jk} - P_{lk} \in \mathfrak{m}^{[(r-p+1)/p]}$. Therefore the decompositions into factors of the pseudopolynomials P_j converge in the Krull topology to some decomposition $\overline{P} = \Pi \overline{P}_k$ of $\overline{P}$. Moreover, if $P_j - \overline{P} \in \mathfrak{m}^r$, we may assume that $P_{jk} - \overline{P}_k \in \mathfrak{m}^{[(r-p+1)/p]}$. To prove the theorem it remains to show that the $\overline{P}_k$ are irreducible. Assume that for some k there exists a nontrivial decomposition into factors $\overline{P}_k = \overline{S}_1 \overline{S}_2$, where $\overline{S}_1$ and $\overline{S}_2$ are distinguished pseudopolynomials. Let j be an index such that the multiplicity of $\Delta^0(Q_j)$ equals κ and $\overline{P}_{jk} - \overline{P}_k \in \mathfrak{m}^{r_0(p,q,\kappa)}$. Since P_{jk} is a divisor of P_j, $\Delta(P_{jk})$ is a divisor of Δ_j. Therefore the multiplicity of $\Delta(P_{jk})$ does not exceed q. Furthermore, since Q_j is a regular pseudopolynomial, the line $\{x' = 0\}$ does not belong to the tangent cone of the set $\{\Delta_j(x) = 0\}$, and hence it also does not belong to the tangent cone of the set $\{\Delta(P_{jk})(x) = 0\}$. Therefore $\Delta(P_{jk}) \sim Q_{jk}$, where Q_{jk} is a regular pseudopolynomial of degree at most q. Since Q_{jk} is a divisor of Q_j, $\Delta^0(Q_{jk})$ is a divisor of $\Delta^0(Q_j)$. Therefore the multiplicity of $\Delta^0(Q_{jk})$ is at most κ.

Now we apply Theorem 2.9a) to $P = P_{jk}$, $F = \check{S}_1 \check{S}_2$ and $S = \check{S}_1$, where the $\check{S}_i$ $(i = 1, 2)$ are pseudopolynomials with coefficients in $\mathbf{C}[x]$,

$$\check{S}_l \equiv \overline{S}_l \bmod \mathfrak{m}^{r_0(p,q,\kappa)}.$$

(Obviously $P_{jk} - \check{S}_1 \check{S}_2 \in \mathfrak{m}^{r_0}$ and $R_l(\check{S}_1 \check{S}_2, \check{S}_1) \equiv 0$ for $l = 1, \cdots, \deg \check{S}_1$.) We obtain a distinguished pseudopolynomial T such that $P_{jk} \vdots T^{(0)}$ and $\deg T = \deg \check{S}_1$. But $\deg \check{S}_1 = \deg \overline{S}_1$; therefore $0 < \det T < \deg \overline{P}_k = \deg P_{jk}$. Thus $T^{(0)}$ is a nontrivial divisor of P_{jk}, and we are led to a contradiction with the nonirreducibility of P_{jk}.

§4. Formal relations between analytic functions

Lemma 4.1. *Let $P = z^p + \sum_1^p c_i(x_1, x_2) z^{p-i}$ be a unitary analytic pseudopolynomial. Assume that $\Delta(P) \sim x_1^{\mu_1} x_2^{\mu_2}$. Let V be an open set in $\mathbf{C}^2$, containing some deleted neighborhood of zero $\{0 < |x_1| < \epsilon\}$ in the set $\{x_2 = 0\}$, and for $(x_1, x_2) \in V$ let*

$$P(x_1, x_2, z) = T(x_1, x_2, z) \cdot H(x_1, x_2, z),$$

where T and H are unitary pseudopolynomials whose coefficients are analytic in V. Then there exist analytic pseudopolynomials $\check{T}$ and $\check{H}$ such that $P = \check{T} \cdot \check{H}$ and $\check{T}|_V = T$, $\check{H}|_V = H$.

Proof. Let $D = \{|x_1| < \epsilon_1, |x_2| < \epsilon_2\}$ be a bicylinder such that $D \cap \{x_2 = 0\} \subset V$, all the coefficients of P are defined and analytic in D and $\Delta(P) = x_1^{\mu_1} x_2^{\mu_2} G(x_1, x_2)$, where $G(x_1, x_2) \neq 0$ in D. If ϵ_2 is small enough, then the circles $\gamma_1 = \{|x_1| = \epsilon_1/2, x_2 = \epsilon_2/2\}$ and $\gamma_2 = \{x_1 = \epsilon_1/2, |x_2| = \epsilon_2/2\}$ are contained in V. The assertion of the lemma now follows from the fact that γ_1 and γ_2 are generators of the group $\pi_1(D \setminus \{\Delta(P) = 0\})$.

Lemma 4.2. *Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_n)$, and let ϕ: $\mathbf{C}[[y]] \to \mathbf{C}[[x]]$ be a homomorphism of rings, $\phi(y_i) = f_i(x)$. If the rank (in the ring $\mathbf{C}[[x]]$) of the matrix $J_\phi = (\partial f_i(x)/\partial x_j)$ equals n, then ϕ is a closed imbedding in the Krull topology.*

Proof. It suffices to show that ϕ is an imbedding, since the closedness follows from the linear compactness of the ring $\mathbf{C}[[y]]$ (cf. [5], Chapter III, §2). Assume that $\ker \phi \neq 0$. Let k be the greatest number such that $\ker \phi \in \mathfrak{m}^k$. Then there exists a formal series $P(y) \in \ker \phi$, not belonging to $\mathfrak{m}^{k+1}$. We may assume that $\partial P/\partial y_1 \notin \mathfrak{m}^k$, and so $\phi(\partial P/\partial y_1) \neq 0$ in $\mathbf{C}[[x]]$. We define a ring homomorphism ψ: $\mathbf{C}[[z]] \to \mathbf{C}[[y]]$ according to the formula $\psi(z_1) = P(y)$, $\psi(z_i) = y_i$ $(i = 2, \cdots, n)$. Since $\phi\psi(z_1) = 0$, the first row of the matrix $J_{\phi\psi}$ is zero, and in particular the rank of $J_{\phi\psi}$ is less than n. On the other hand, $J_{\phi\psi} = \phi(J_\psi) \cdot J_\phi$ (cf. [5], Chapter III, §4). Since $\det \phi(J_\psi) = \phi(\partial P/\partial y_1) \neq 0$, we have $\operatorname{rank} J_\phi = \operatorname{rank} J_{\phi\psi} < n$, which contradicts the hypothesis.

Let f: $\mathbf{C}^m_x \to \mathbf{C}^n_y$ be an analytic map. Denote by f^* the ring homomorphism $\mathbf{C}\{y\} \to \mathbf{C}\{x\}$ defined by the map f, and by $\hat{f}^*$ the corresponding homomorphism $\mathbf{C}[[y]] \to \mathbf{C}[[x]]$.

Lemma 4.3. *Let f: $\mathbf{C}^2_x \to \mathbf{C}^2_y$ be an analytic map defined by the formula $y_1 = x_1F(x_2)$, $y_2 = x_2$, where $F(0) = 0$. Let $\overline{P}(y, z)$ be a formal pseudopolynomial (in z) without multiple factors, and $P_j(y, z)$ a sequence of analytic pseudopolynomials converging to $\overline{P}$ and satisfying the hypothesis of Lemma 3.1. Then the sequence $P_j(f(x), z)$, converging to $\overline{P}(f(x), z)$, satisfies the condition of Lemma 3.1.*

Proof. By hypothesis there exists a basis of $\mathbf{C}^2_y$ in which the discriminants $\overline{\Delta}(y)$ and $\Delta_j(y)$ of the pseudopolynomials $\overline{P}(y, z)$ and $P_j(y, z)$ are equivalent to regular pseudopolynomials $\overline{Q}(y)$ and $Q_j(y)$, and $Q_j^{(0)} \to \overline{Q}^{(0)}$ in the Krull topology (here and later on it is assumed that j is sufficiently large). Hence $f^*Q_j^{(0)} \to f^*\overline{Q}^{(0)}$ in the (x_2)-adic topology. Let $f^*\overline{Q}^{(0)} = x_2^k\overline{\psi}(x)$, where $\overline{\psi}(x) \notin (x_2)$. Then $f^*Q_j^{(0)} = x_2^k\psi_j(x)$, where $\psi_j(x) \notin (x_2)$, and $\psi_j \to \overline{\psi}$. We shall show that $\overline{\psi}$ does not contain multiple factors.

Since $\overline{Q}^{(0)}$ is a formal series without multiple factors, the ideal $(\partial\overline{Q}^{(0)}/\partial y_1, \partial\overline{Q}^{(0)}/\partial y_2)$ contains some power of the maximal ideal. Hence the ideal $(f^*(\partial\overline{Q}^{(0)}/\partial y_1), f^*(\partial\overline{Q}^{(0)}/\partial y_2))$ contains some power of the ideal (x_2). From the formula for the derivatives of the function $f^*(\overline{Q}^{(0)})$ it then follows that the ideal

$$\left(\frac{\partial}{\partial x_1}(f^*\overline{Q}^{(0)}), \frac{\partial}{\partial x_2}(f^*\overline{Q}^{(0)})\right)$$

contains some power of the ideal (x_2). Therefore the derivatives of the function $f^*\overline{Q}^{(0)}$ cannot have a common factor that is not divisible by x_2, from which it follows that $\overline{\psi}$ does not contain multiple factors.

Now let (x_1', x_2) be a basis in $\mathbf{C}^2_x$ in which $f^*\overline{\Delta} \sim \overline{R}(x_1', x_2)$, where $\overline{R}$ is a regular pseudopolynomial (in x_2). Since $f^*\overline{\Delta} \,\vdots\, \overline{\psi}$, it follows that $\overline{\psi} \sim \overline{T}(x_1', x_2)$, where $\overline{T}$ is a regular pseudopolynomial. Since $\overline{\psi}$ contains no multiple factors and is not divisible by x_2, we have $\overline{R}^{(0)} = x_2\overline{T}$. Furthermore, from Lemma 2.7 it follows that $f^*\Delta_j \sim$

$R_j(x_1', x_2)$ and $\psi_j \sim T_j(x_1', x_2)$, where R_j and T_j are regular pseudopolynomials, where $T_j \to \overline{T}$. Since $\overline{T}$ is a pseudopolynomial without multiple factors, $\Delta(\overline{T}) \not\equiv 0$. Since $\Delta(T_j) \to \Delta(\overline{T})$, we have $\Delta(T_j) \not\equiv 0$, i.e. the T_j are pseudopolynomials without multiple factors. But since $\overline{T} \notin (x_2)$ and $T_j \to \overline{T}$, we see that $T_j \notin (x_2)$. Therefore $R_j^{(0)} = x_2 T_j$, and hence $R_j^{(0)} \to \overline{R}^{(0)}$, as required.

Theorem 4.4. *Let $f\colon \mathbf{C}^2_{x,t} \to \mathbf{C}^3_{y_1,y_2,z}$ be the mapping defined by the formula $y_1 = x$, $y_2 = xt$, $z = f(x, t)$, where $f(x, t) \in \mathbf{C}\{x, t\}$, $f(0, t) \equiv 0$; and let $\overline{P}(y, z) = z^p + \sum_1^p \overline{c}_i(y) z^{p-i}$ be a formal distinguished pseudopolynomial belonging to $\ker \hat{f}^*$. Then there exists an analytic distinguished pseudopolynomial of degree at most p belonging to $\ker f^*$.*

Proof. 1. If $\overline{P} \in \ker \hat{f}^*$, then also $\overline{P}^{(0)} \in \ker \hat{f}^*$. Therefore we may assume that $\overline{P}$ is a pseudopolynomial without multiple factors and its discriminant $\overline{\Delta}(y)$ is different from zero in $\mathbf{C}[[y]]$.

As is well known, the singularities of the ideal can be resolved by a finite number of σ-processes with centers at points. More precisely, we have the following assertion.

There exists a finite sequence of complex algebraic varieties X_ν $(0 \le \nu \le N)$ and regular maps $\phi_\nu\colon X_{\nu+1} \to X_\nu$ $(0 \le \nu \le N-1)$ satisfying the following conditions:

1) $X_0 = \mathbf{C}^2_y$, and $\phi_0^{-1}\colon \mathbf{C}^2_y \to X_1$ is a σ-process with center at zero.

2) Each map $\phi_\nu^{-1}\colon X_\nu \to X_{\nu+1}$ $(1 \le \nu \le N-1)$ is a σ-process with center at the point $y_\nu \in X_\nu$, $\phi_0 \cdots \phi_{\nu-1}(y_\nu) = 0$.

3) Put $X = X_N$ and $\phi = \phi_0 \cdots \phi_{N-1}$. At each point $x \in X$ there exist formal coordinates (x_1, x_2) such that $\hat{\phi}^*(\overline{\Delta}) \sim x_1^{\mu_1} x_2^{\mu_2}$.

Obviously ϕ is a proper map, $\phi^{-1}(0)$ is a connected "graph" of the projective lines W_ν pasted in under the σ-processes ϕ_ν^{-1}, and the mapping $\phi\colon X \setminus \phi^{-1}(0) \to \mathbf{C}^2_y \setminus \{0\}$ is biregular.

We consider the map $\phi_0\colon X_1 \to \mathbf{C}^2_y$. The variety X_1 is a subvariety of $\mathbf{C}P^1 \times \mathbf{C}^2$ and is defined by the equation $t_1 y_1 = t_0 y_2$ (here (t_0, t_1) are homogeneous coordinates in $\mathbf{C}P^1$, and (y_1, y_2) are coordinates in $\mathbf{C}^2$). Put $U = X_1 \setminus \{t_0 = 0\}$. Then U is an affine variety with coordinates $(x' = y_1, t' = t_1/t_0)$ and $\phi_0|_U = (x', t'x')$.

Consider the formal pseudopolynomial

$$\hat{\varphi}_0^* \overline{P}(x', t', z) = z^p + \sum_{i=1}^{p} \hat{\varphi}_0^* (\overline{c}_i) z^{p-i}.$$

Obviously

$$\hat{\varphi}_0^* \overline{P}(x', t', f(x', t')) = \hat{f}^* \overline{P} \equiv 0. \tag{4.1}$$

Furthermore, for every series $\overline{c}(y) = \sum c_{ij} y_1^i y_2^j \in \mathbf{C}[[y]]$ the series $\hat{\phi}_0^* \overline{c} =$

$\Sigma c_{ij} x'^{i+j} t'^{j}$ belongs to $\mathfrak{A}_{x',t'}$. Therefore $\hat{\phi}_0^* \bar{P}(x', t', z) \in \mathfrak{A}_{x',t'}[z]$.

Analogously one can show that if $y \in \phi^{-1}(0)$ is an arbitrary point, $y \in W_\nu$, and $V \cong \mathbf{C}^1$ is an affine neighborhood of the point y in W_ν, then there exist an affine neighborhood $U \cong \mathbf{C}^2$ of y in X and coordinates (x, t) in U such that

$$V = \{(x, t) \in U,\ x = 0\} \text{ and } \hat{\varphi}^* \bar{P}|_U \in \mathfrak{A}_{x,t}[z]. \tag{4.2}$$

2. Put $\check{f} = \phi_{N-1}^* \cdots \phi_1^* f$ (we assume that $f(x', t')$ is defined in an open set in X_1). Then $\check{f}$ is analytic in a neighborhood U_0 of some point $y \in W_0$.

Let $V \subset W_0$ and $U \subset X$ be affine neighborhoods of the point y, and let (x, t) be coordinates in U satisfying the conditions (4.2). From (4.1) it follows that $\hat{\phi}^* \bar{P}(x, t, \check{f}(x, t)) \equiv 0$.

We apply Theorem 1.13. We obtain an analytic function $\psi(t)$ in an open set $V_0 \subset U_0$, an irreducible polynomial in $\mathbf{C}[t, z]$

$$S(t, z) = z^s + \sum_{j=1}^{s} d_j(t) z^{s-j},$$

annihilating $\psi(t)$, polynomials $F(t)$ and $\Delta(t)$ and functions $f_\kappa(x, t) \in \mathfrak{A}_{x,t} \cap \mathbf{C}\{x, t\}$ such that

$$\check{f}(x, t) = \sum_{\varkappa=0}^{s-1} \frac{f_\varkappa(x/F(t), t)}{\Delta(t)} \psi(t)^\varkappa. \tag{4.3}$$

From the condition of the theorem it follows that $f|_{\phi_0^{-1}(0)} \equiv 0$. Therefore also $\check{f}|_{\phi^{-1}(0)} \equiv 0$. Hence $\check{f}(0, t) \equiv 0$. From the uniqueness of the decomposition

$$\Delta(t) \check{f}(0, t) = \sum_{\varkappa=0}^{s-1} f_\varkappa(0, t) \psi(t)^\varkappa$$

it follows that $f_\kappa(0, t) \equiv 0$ for all κ, i.e.

$$f_\varkappa(x, t) = \sum_{\nu>0} f_{\varkappa\nu}(t) x^\nu.$$

We put

$$f'_\varkappa(x, t) = \sum_{\nu>0} f_{\varkappa\nu}(t) \cdot \Delta(t)^{\nu-1} x^\nu.$$

Then

$$f'_\varkappa(x, t) \in \mathfrak{A}_{x,t} \cap \mathbf{C}\{x, t\} \quad \text{and} \quad \check{f}(x, t) = \sum_{\varkappa=0}^{s-1} f'_\varkappa\left(\frac{x}{F(t)\Delta(t)}, t\right).$$

Replacing $F(t)$ by $F(t)\Delta(t)$ and f_κ by f'_κ, we may assume that

$$\check{f}(x, t) = \sum_{\varkappa=0}^{s-1} f_\varkappa\left(\frac{x}{F(t)}, t\right) \psi(t)^\varkappa. \tag{4.3'}$$

Consider the mapping $\eta\colon \mathbf{C}^2_{\xi,\tau} \to \mathbf{C}^2_{x,t}$, defined by the formula $x = \xi F(\tau)$, $t = \tau$.

The formal pseudopolynomial $\overline{Q}(\xi, \tau, z) = \hat{\eta}^* \hat{\phi}^* \overline{P}$ vanishes when we substitute

$$z = Z(\xi, \tau) = \sum_{\varkappa=0}^{s-1} f_\varkappa(\xi, \tau) \psi(\tau)^\varkappa.$$

Let $t_0 \in V_0 \cap V$ be a point such that $\Delta(S)(t_0) \neq 0$, and let $\psi_\nu(t) \in \mathbf{C}\{t\}$ $(\nu = 1, \cdots, s)$ be the roots of S in a neighborhood of the point t_0. We may assume $t_0 = 0$. Put

$$Z_\nu(\xi, \tau) = \sum_{\varkappa=0}^{s-1} f_\varkappa(\xi, \tau) \psi_\nu(\tau)^\varkappa.$$

Let $Q^{(n)}(\xi, \tau, z) \in \mathbf{C}[\xi, \tau, z]$, $Q^{(n)} \equiv \overline{Q} \bmod(\xi^n)$, be unitary pseudopolynomials in z of degree p. Since $Q^{(n)} \to \overline{Q}$ in the (ξ)-adic topology, the sequence of analytic functions $Q^{(n)}(\xi, \tau, Z(\xi, \tau))$ converges to zero in the (ξ)-adic topology. Since S is an irreducible polynomial, for each ν there exists a closed path λ_ν in the set $\{t\colon \Delta(S)(t) \neq 0\}$, after a circuit of which $\psi(t)$ goes into $\psi_\nu(t)$. Analytically continuing the functions $Q^{(n)}(\xi, \tau, Z(\xi, \tau))$ along λ_ν (cf. Corollary 1.14), we obtain that the sequence $Q^{(n)}(\xi, \tau, Z_\nu(\xi, \tau))$ converges to zero in the (ξ)-adic topology. Hence

$$\overline{Q}(\xi, \tau, Z_\nu(\xi, \tau)) \equiv 0 \tag{4.4}$$

for all ν. Let

$$T' = \prod_\nu (z - Z_\nu(\xi, \tau)) \in \mathbf{C}\{\xi, \tau\}[z].$$

Obviously $T' \in \mathfrak{A}_{\xi,\tau}[z]$ (since its coefficients are expressed by d_i and f_κ). Since $\mathfrak{A}_{\xi,\tau}$ is integrally closed (Lemma 1.4), $T = T'^{(0)}$ also belongs to $\mathfrak{A}_{\xi,\tau}[z]$.

From (4.4) it follows that $\overline{Q} \vdots T$ in $\mathbf{C}[[\xi, \tau]][z]$, and since $\overline{Q}$ and T belong to $\mathfrak{A}_{\xi,\tau}[z]$, then $\overline{Q} = T \cdot \overline{H}$, where $\overline{H}$ is a unitary pseudopolynomial belonging to $\mathfrak{A}_{\xi,\tau}[z]$, and the decomposition $\overline{Q} = T \cdot \overline{H}$ occurs at every point $(0, \tau)$.

Let $T = z^k + \sum_1^k v_i(\xi, \tau) z^{k-i}$. Put

$$\eta_* T = z^k + \sum_{i=1}^k v_i(x/F(t), t) z^{k-i}.$$

We wish to prove that all the coefficients of the pseudopolynomial $\eta_* T$ are analytically continued into a neighborhood of the set $\{x = 0\}$ and at every point $(0, t_0)$ we have that $\hat{\phi}^* \overline{P} \vdots \eta_* T$ in the ring $\mathbf{C}[[x, t]][z]$.

Let $P_j(y, z) = z^p + \sum c_{ji}(y) z^{p-i}$ be a sequence of analytic pseudopolynomials converging to $\overline{P}$ and satisfying the conditions of Lemma 3.1. From Lemma 4.3 it follows that the sequence $\hat{\phi}^* P_j$, converging to $\phi^* \overline{P}$, also satisfies the conditions of Lemma 3.1. Now let t_0 be an arbitrary point in V. Replacing t by $t - t_0$, we may assume $t_0 = 0$. If $F(0) \neq 0$, then $\eta_* T$ is obviously analytic at t_0.

Suppose $F(0) = 0$. From Lemma 4.3 it follows that the sequence $Q_j = \hat{\eta}^* \hat{\phi}^* P_j$, converging to $\overline{Q}$, satisfies the conditions of Lemma 3.1. We apply Theorem 3.2. Since

$\overline{Q} = T\overline{H}$, there exist sequences of unitary pseudopolynomials $\{T_j\}$ and $\{H_j\}$, convergent to T and $\overline{H}$ respectively, such that $Q_j = T_j \cdot H_j$. Let

$$T_j = z^k + \sum_{i=1}^{k} v_{ji}(\xi, \tau) z^{k-i}, \quad H_j = z^l + \sum_{i=1}^{l} w_{ji}(\xi, \tau) z^{l-i},$$

where v_{ji} and w_{ji} are analytic on the set $\{|\xi| < \epsilon_j, |\tau| < \epsilon_j\}$. Put

$$\eta_* T_j = z^k + \sum_{i=1}^{k} v_{ji}\left(\frac{x}{F(t)}, t\right) z^{k-i},$$

$$\eta_* H_j = z^l + \sum_{i=1}^{l} v_{ji}\left(\frac{x}{F(t)}, t\right) z^{l-i}.$$

Then the coefficients of $\eta_* T_j$ and $\eta_* H_j$ are analytic in the open set $\{|t| < \epsilon_j, |x| < \epsilon_j |F(t)|\}$, containing the set $\{|x| = 0, 0 < |t| < \epsilon_j\}$, if ϵ_j is sufficiently small. We shall show that we can apply Lemma 4.1 to $\phi^* P_j = \eta_* T_j \cdot \eta_* H_j$.

By property 3) of the mapping ϕ there exist formal coordinates (x_1, x_2) in which $\hat{\phi}^* \overline{\Delta} \sim x_1^{\mu_1} x_2^{\mu_2}$. If $\overline{\Delta}(0) \neq 0$, then $\Delta(\phi^* P_j)(0) \neq 0$ for sufficiently large j and the conditions of Lemma 4.1 trivially hold for $\phi^* P_j$. But if $\overline{\Delta}(0) = 0$, then $\overline{\phi}^* \overline{\Delta} \,\vdots\, t$, and hence we may assume that $x_2 = t$. Since the sequence $\phi^* P_j$ satisfies the conditions of Lemma 3.1 and $\phi^* \Delta(P_j) \,\vdots\, t$, it is not hard to show that for sufficiently large j there exist analytic coordinates $(x_{(j)}, t)$ such that $\phi^* \Delta(P_j) \sim x_{(j)}^{\mu_1} t^{\mu_2}$. Hence also in this case $\phi^* P_j$ satisfy the conditions of Lemma 4.1, i.e. all the coefficients of $\eta_* T_j$ and $\eta_* H_j$ are analytically continued into a neighborhood of zero. Further, since $\eta^*(\eta_* T_j) = T_j$ converges to T, and $\eta^*(\eta_* H_j)$ to $\overline{H}$, it then follows from Lemma 4.2 that $\eta_* T_j$ and $\eta_* H_j$ converge to formal pseudopolynomials R_1 and R_2 such that $\hat{\eta}^* R_1 = T$ and $\hat{\eta}^* R_2 = \overline{H}$.

We shall show that R_1 is an analytic pseudopolynomial. Since $\hat{\eta}^* R_1 = T$ is an analytic pseudopolynomial, it suffices to prove that

$$\hat{\eta}^{*-1}(\mathbf{C}\{\xi, \tau\}) = \mathbf{C}\{x, t\}. \tag{4.5}$$

Let $F(t) = G(t) \cdot t^\nu$, where $G(0) \neq 0$. Replacing t and τ by $G(t)^{1/\nu} t$ and $G(\tau)^{1/\nu} \tau$, we reduce the mapping η to the form $x = \xi \tau^\nu$, $t = \tau$, for which assertion (4.5) is trivial.

Since $\eta^*(R_1) = T$, R_1 is an analytic continuation of $\eta_* T$ in a neighborhood of zero. Since this argument applies at any point $t_0 \in T$, $\eta_* T$ can be analytically continued into a neighborhood of the set $\{x = 0\}$, as required.

3. Now let $W_\nu = V_{\nu 1} \cup V_{\nu 2}$ be an affine cover of the projective line W_ν and $U_{\nu j} \subset X$ $(\nu = 0, \cdots, N-1;\ j = 1, 2)$ be affine sets with the coefficients $(x_{(\nu j)}, t_{(\nu j)})$, satisfying condition (4.2) $(U_{\nu j} \cap W_\nu = V_{\nu j})$. Assume that in a neighborhood of $V_{\nu j}$ a unitary analytic pseudopolynomial T is defined such that $\hat{\phi}^* \overline{P} \,\vdots\, T$ in $\mathbf{C}[[x_{(\nu j)}, t_{(\nu j)}]]$ at every point of $V_{\nu j}$. Let $y \in V_{\nu j} \cap V_{\nu' j'}$. Acting as in part 2 of the proof (with the difference that Corollary 1.15 must be used instead of Theorem 1.13), we obtain a pseudopolynomial T', analytic in a neighborhood of $V_{\nu' j'}$, such that $T' \,\vdots\, T$ in a

neighborhood of y and $\hat{\phi}^*\overline{P} \vdots T'$ at all points of $V_{\nu' j'}$. Since $\phi^{-1}(0)$ is a connected set and $V_{\nu j}$ is a finite cover of $\phi^{-1}(0)$, and the degree of the pseudopolynomial does not exceed the degree of $\overline{P}$, after a finite number of such analytic continuations we obtain a unitary pseudopolynomial T satisfying the following conditions:

1) T is analytic in a neighborhood of $\phi^{-1}(0)$.

2) $\hat{\phi}^* P \vdots T$ at each point of $\phi^{-1}(0)$.

3) $T \vdots (z - \check{f})$.

Since ϕ is a proper map, it follows from 1) that there exists a unitary analytic pseudopolynomial $\phi_* T \in \mathbf{C}\{y_1, y_2\}[z]$ such that $T = \phi^*(\phi_* T)$. It follows from 2) that $\deg T \leq \deg P$. It follows from 3) that

$$\varphi_{N-1}^* \cdots \varphi_1^*(\varphi_* T(x, xt, f(x, t)) \equiv 0$$

and, by Lemma 4.2, $\phi_* T(x, xt, f(x, t)) \equiv 0$, i.e. $\phi_* T \in \ker f^*$. The theorem is proved.

Definition 4.5. Let $\phi\colon A \to B$ be a homomorphism of local rings, and $\hat{\phi}\colon \hat{A} \to \hat{B}$ the corresponding homomorphism of the completions. The homomorphism ϕ is called *analytically regular* if

$$\ker \hat{\varphi} = \hat{A} \otimes_A \ker \varphi.$$

Lemma 4.6. *Assume that A and B are analytic rings, B an integral domain. A homomorphism $\phi\colon A \to B$ is analytically regular if and only if*

$$\dim \hat{A}/\ker \hat{\varphi} = \dim A/\ker \varphi. \tag{4.6}$$

Proof. Since $(A/\ker \phi)\hat{} = \hat{A}/\hat{A} \otimes \ker \phi$, and the dimension does not change under completion, condition (4.6) is necessary for ϕ to be analytically regular.

Conversely, suppose (4.6) holds. Then

$$\operatorname{coht} \hat{A} \otimes \ker \varphi = \operatorname{coht} \ker \varphi = \operatorname{coht} \ker \hat{\varphi}. \tag{4.7}$$

Since $A/\ker \phi$ is an integral domain, from the theorem of Zariski and Nagata [10], Theorem 44.1) it follows that $(A/\ker \phi)\hat{}$ is an integral domain, i.e. $\hat{A} \otimes \ker \phi$ is a prime ideal. Since $\hat{A} \otimes \ker \phi \subset \ker \hat{\phi}$, it follows from (4.7) that these ideals coincide.

Lemma 4.7. *Let $\phi\colon A \to B$ and $\psi\colon B \to C$ be homomorphisms of analytic rings, C an integral domain, B integral over A. For the homomorphism $\psi \circ \phi$ to be analytically regular it is necessary and sufficient that the homomorphism ψ be analytically regular.*

Proof. In fact, $B/\ker \psi$ is an integral extension of the ring $A/\ker(\psi \circ \phi)$, and $\hat{B}/\ker \hat{\psi}$ is an integral extension of $\hat{A}/\ker(\hat{\psi} \circ \hat{\phi})$. From the Cohen-Seidenberg theorem (cf. [8], Chapter III) it follows that

$$\dim A/\ker(\psi \circ \varphi) = \dim B/\ker \psi,$$

$$\dim \hat{A}/\ker(\hat{\psi} \circ \hat{\varphi}) = \dim \hat{B}/\ker \hat{\psi}.$$

Lemma 4.7 now follows from Lemma 4.6.

Theorem 4.8. *Let $g: \mathbf{C}_x^m \to \mathbf{C}_y^n$ be an analytic map, $J(g)$ its Jacobian. Assume that*

$$\dim \mathbf{C}[[y]]/\ker \hat{g}^* = \operatorname{rank} J(g).$$

Then g^* is an analytically regular homomorphism.

Proof. By a sequence of reductions we shall reduce the assertion of Theorem 4.8 to that of Theorem 4.4.

1. *Reduction to the case* corank $J(g) = 1$. Let

$$\operatorname{rank} J(g) = \dim \mathbf{C}[[y]]/\ker \hat{g}^* = r.$$

By Lemma 4.6 it suffices to prove that $\dim \mathbf{C}\{y\}/\ker g^* = r$. Assume that $\dim \mathbf{C}\{y\}/\ker g^* \geq r+1$. Making a linear change of coordinates in $\mathbf{C}_y^n$ if necessary, we may make the following assumptions:

1) $\ker \hat{g}^* \cap \mathbf{C}[[y_1, \cdots, y_{r+1}]] \neq 0$ (we assume that $\mathbf{C}[[y_1, \cdots, y_{r+1}]]$ is embedded in $\mathbf{C}[[y]]$ as the subring of the series independent of $y_{r+2}, \cdots, y_n$).

2) $\ker g^* \cap \mathbf{C}\{y_1, \cdots, y_{r+1}\} = 0$.

3) The rank of the matrix $(\partial g_i/\partial x_j)$ $(i = 1, \cdots, r+1;\ j = 1, \cdots, m)$ is equal to r.

We put $y' = (y_1, \cdots, y_{r+1})$ and $g' = (g_1, \cdots, g_{r+1}): \mathbf{C}_x^m \to \mathbf{C}_{y'}^{r+1}$. We have corank $J(g') = 1$, $\ker g'^* = 0$ and $\ker \hat{g}'^* \neq 0$. Thus the assertion of the theorem reduces to the following:

$$\operatorname{corank} J(g) = 1, \quad \ker \hat{g}^* \neq 0 \Rightarrow \ker g^* \neq 0.$$

2. *Reduction to the case $m = n-1$.* Let the rank of $J(g)$ equal $n-1$, and $\ker \hat{g}^* \neq 0$. There obviously exists a nonsingular $(n-1)$-dimensional surface $L \subset \mathbf{C}_x^m$ such that the rank of $J(g|_L)$ remains equal to $n-1$. Put $g' = g|_L$. Obviously $\ker \hat{g}'^* \supset \ker \hat{g}^*$. Therefore $\ker \hat{g}'^* \neq 0$. We shall show that $\ker g'^* \subset \ker g^*$. In fact, let $\phi \in \ker g'^*$. There exists a point $x_0 \in L$ satisfying the following conditions:

1) The function ϕ is defined and analytic in a neighborhood of the point $g(x_0)$ in $\mathbf{C}_y^n$.

2) The function $g^*\phi$ is defined and analytic in a connected neighborhood of zero $U \subset \mathbf{C}_x^m$ containing x_0.

3) The rank of the matrix $J(g')(x_0)$ equals $n-1$.

4) $g^*\phi|_L = g'^*\phi \equiv 0$ in a neighborhood of x_0 in L.

By the theorem on rank, in a neighborhood of x_0 the space $\mathbf{C}_x^m$ is a fibration with fiber $\{g = \text{const}\}$ and base L. Since $g^*\phi|_L \equiv 0$, it follows that $g^*\phi \equiv 0$ in a neighborhood of x_0 in $\mathbf{C}_x^m$, and, since U is a connected open set containing x_0, $g^*\phi \equiv 0$ in U. Hence $\phi \in \ker g^*$.

Thus $\ker g'^* \subset \ker g^*$. In particular, if $\ker g'^* \neq 0$, then also $\ker g^* \neq 0$. Therefore it suffices to prove the assertion of the theorem for g', i.e. for $m = n-1$.

3. *Reduction to the case $n \leq 3$.*

3.1. Since $\mathbf{C}[[y]]$ is a regular ring of dimension n, since $\ker \hat{g}^*$ is a prime ideal and since $\operatorname{coht} \ker \hat{g}^* = n-1$, it follows that $\ker \hat{g}^*$ is a principal ideal and its generator is an irreducible formal series $\overline{P}$. By making a linear change of coordinates and multiplying through by an invertible formal series, we may assume that

$$\overline{P}=y_n^p+\sum_{i=1}^{p}\bar{c}_i(y_1,\ \dots\ ,y_{n-1})\,y_n^{p-i}$$

is a distinguished formal pseudopolynomial. If $\ker g^*=0$, $\overline{P}$ is a divergent series.

Conversely, suppose $\ker \hat{g}^*$ contains a divergent irreducible distinguished pseudopolynomial. We shall show that then $\ker g^*=0$. In fact, let $g^*\not\equiv 0$. Then $\dim \mathbf{C}\{y\}/\ker g^*<n$; and, since

$$n-1=\dim \mathbf{C}[[y]]/\ker \hat{g}^*\leqslant \dim \mathbf{C}\{y\}/\ker g^*,$$

we have $\dim \mathbf{C}\{y\}/\ker g^*=n-1$, i.e. $\ker g^*$ is a principal ideal. Let Φ be its generator. From Lemma 4.5 it follows that $\overline{P}\in\Phi\cdot\mathbf{C}[[y]]$. Since $\overline{P}$ is irreducible, $\overline{P}\sim\Phi$ in $\mathbf{C}[[y]]$. Since $\overline{P}$ is a distinguished pseudopolynomial, from the uniqueness in the Weierstrass preparation theorem it follows that $\overline{P}\in\mathbf{C}\{y\}$; but this contradicts the assumption of the divergence of P.

3.2. We may assume that all the functions $g_i(x)$ are divisible by x_1 and 0 is a nonsingular point of the set $\{g_1(x)=0\}$. In fact, we replace g by $g\circ f_{v^0}$, where $f_{v^0}\colon \mathbf{C}_v^{n-1}\to\mathbf{C}_x^{n-1}$ is the mapping given by the formula

$$x_1=v_1,\ x_2=v_1(v_2+v_2^0),\ \dots\ ,x_{n-1}=v_1(v_{n-1}+v_{n-1}^0)$$

(here $v^0=(0,v_2^0,\cdots,v_{n-1}^0)$ is some point in $\{v_1=0\}$). Since $\det J(f_{v^0})=v_1^{n-2}\not\equiv 0$ in $\mathbf{C}\{v\}$, from Lemma 4.2 it follows that $\ker \hat{f}^*_{v^0}=0$. Therefore it suffices to prove the assertion for $g\circ f_{v^0}$. On the other hand, all the functions $f^*_{v^0}(g_i)$ are divisible by v_1; and if v^0 is a nonsingular point of $\{f^*_0(g_1)=0\}$, then 0 is a nonsingular point of $\{f^*_{v^0}(g_1)=0\}$.

3.3. Since $g_1(x)\,\vdots\,x_1$ and 0 is a nonsingular point of $\{g_1(x)=0\}$, we have $g_1(x)\sim x_1^k$. Therefore there exists an analytic function $g_1'(x)$ such that $g_1(x)=g_1'(x)^k$. Obviously $g_1'(x)\sim x_1$. Consider the mappings

$$g'=(g_1',g_2,\ \dots\ ,g_n)\colon\mathbf{C}_x^{n-1}\to\mathbf{C}_z^n\ \text{ and }\ h=(z_1^k,z_2,\ \dots\ ,z_n)\colon\mathbf{C}_z^n\to\mathbf{C}_y^n.$$

Since $g^*=g'^*\circ h^*$, from Lemma 4.6 applied to $\mathbf{C}\{y\}\xrightarrow{h^*}\mathbf{C}\{z\}\xrightarrow{g^*}\mathbf{C}\{x\}$ we see that it suffices to prove the assertion of the theorem for g'. Thus we may assume that $g_1(x)\sim x_1$. But then by a nondegenerate change of coordinates $(x_1,\cdots,x_{n-1})\rightsquigarrow(g_1(x),x_2,\cdots,x_{n-1})$ we can reduce to the case $g_1(x)=x_1$.

3.4. Let $g_2=\sum_{j=1}^{\infty}G_j(x_2,\cdots,x_{n-1})x_1^j$, and let j_0 be the smallest index for which $G_{j_0}\not\equiv \text{const}$ (such an index exists since otherwise $g_2=\phi(g_1)$, and $y_2-\phi(y_1)\in\ker g^*$). By making the change of variables

$$y_2\rightsquigarrow y_2-\sum_{j=1}^{j_0-1}G_jy_1^j-ay_1^{j_0},$$

in $\mathbf{C}_y^n$, where a is some constant, we may assume that $g_2 = x_1^{j_0} G(x)$, where $G(x)|_{x_1=0} \not\equiv$ const and $G(0) \not\equiv 0$. As was done in 3.3 for g_1, we can reduce the problem to the case $j_0 = 1$. Furthermore, making the change $y_2 \rightsquigarrow y_2 - G(0)y_1$ in $\mathbf{C}_y^n$, we may assume that $G(0) = 0$. Finally, replacing g by $h \circ g$, where $h = (y_1^2, y_2, \cdots, y_n)$ (such a change is possible as a result of Lemma 4.6), we reduce the problem to the case

$$g_1(x) = x_1^2, \quad g_2(x) = x_1 G(x),$$

where $G(0) = 0$, $G|_{x_1=0} \not\equiv$ const.

3.5. Let $n > 3$ and $\ker g^* = 0$, and let $\overline{P}$ be an irreducible divergent distinguished pseudopolynomial (in y_n) belonging to $\ker \hat{g}^*$. In $\mathbf{C}_y^n$ consider the linear system of hyperplanes $L_c = \{cy_1 - y_2 = 0\}$, $c \in \mathbf{C}$. From the local Bertini's theorem (cf. [9]) it follows that $P|_{L_c}$ is an irreducible formal series for all values of c except for perhaps a finite number.

We shall show that the set of those c for which the series $\overline{P}|_{L_c}$ diverges is everywhere dense in $\mathbf{C}$. For this we consider the mapping $\sigma\colon \mathbf{C}_{\xi,\tau}^n \to \mathbf{C}_y^n$ $(\xi = (\xi_2, \cdots, \xi_n))$ defined by the formula

$$y_1 = \xi_2,\ y_2 = \tau\xi_2,\ y_3 = \xi_3,\ \ldots,\ y_n = \xi_n.$$

Then $\overline{P}|_{L_c} = \hat{\sigma}^*\overline{P}|_{\tau=c}$. Obviously $\hat{\sigma}^*\overline{P} \in \mathfrak{A}_{\xi,\tau}$. From Lemma 1.6 it follows that the set of those points c for which $\hat{\sigma}^*\overline{P}|_{\tau=c}$ diverges is either everywhere dense or empty, where in the second case $\hat{\sigma}^*\overline{P}$ is an analytic function. But then it is obvious that $\overline{P}$ is also an analytic function, which contradicts the assumption $\ker g^* = 0$.

Consider the sets $g^{-1}(L_c)$. They are given by the equations $0 = cg_1 - g_2 = x_1(cx_1 - G(x))$. Therefore $g^{-1}(L_c) = \{x_1 = 0\} \cup M_c$, where $M_c = \{cx_1 - G(x) = 0\}$. The variety M_c is of dimension $n-2$ and nonsingular, if $c \not\equiv (\partial G/\partial x_1)(0)$. Furthermore, for all c, except perhaps a finite number,

$$M_c \not\subset \{x : \operatorname{rank} J(g)(x) < n-1\},$$

and hence $\operatorname{rank} J(g|_{M_c}) = n-2$.

Thus there exists a $c \in \mathbf{C}$ satisfying the following conditions:

1) M_c is a nonsingular $(n-2)$-dimensional variety in a neighborhood of zero in $\mathbf{C}_x^{n-1}$.

2) $\operatorname{rank} J(g|_{M_c}) = n-2$.

3) If $g'\colon M_c \to L_c$ is a map induced by g, then $\overline{P}|_{L_c} \in \ker \hat{g}'^*$ is an irreducible divergent formal series.

4) If the system of coordinates $(y_1, y_3, \cdots, y_n)$ is introduced on L_c, then $\overline{P}|_{L_c}$ is a distinguished pseudopolynomial in the variable y_n.

As is shown in 3.1, it follows from this that $\ker g'^* = 0$. Therefore it suffices to prove the assertion of the theorem for g', i.e. for the mapping $\mathbf{C}^{n-2} \to \mathbf{C}^{n-1}$.

4. *Reduction to Theorem* 4.4. Let $g\colon \mathbf{C}_x^2 \to \mathbf{C}_y^3$ (if $n < 3$, we may use the functions $g_i = x_i$), rank $J(g) = 2$ and ker $\hat{g}^* \neq 0$. Let $\overline{P}$ be a generator of ker $\hat{g}^*$. We may assume that $\overline{P}$ is a distinguished pseudopolynomial in y_3.

As in 3.2, we replace g by $g \circ f_{v^0}$, but in choosing the point v^0 we require additionally that $f_{v^0}^* g_2$ in a neighborhood of v^0 will be equal to $v_1^l \cdot G(v)$, where $(\partial G/\partial v_2)(v^0) \neq 0$. Then the problem reduces to the case $g_1 = x_1^k$, $g_2 = x_1^l \cdot G(x)$, where $(\partial G/\partial x_2)(0) \neq 0$ and $g_3 \vdots x_1$. Furthermore, as in 3.3 and 3.4, the problem reduces to the case $g_1 = x_1$, $g_2 = x_1 G(x)$, where $G(0) = 0$ and $(\partial G/\partial x_2)(0) \neq 0$. By a nondegenerate change of coordinates $(x_1, x_2) \rightsquigarrow (x_1, G(x))$ the problem reduces to the case $g_1 = x_1$, $g_2 = x_1 x_2$, i.e. to Theorem 4.4.

§5. Homomorphisms of analytic rings

Let $\phi\colon A \to B$ be a homomorphism of analytic rings. As is known [7], there exist analytic spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ such that $A = \mathcal{O}_{Y,y_0}$ and $B = \mathcal{O}_{X,x_0}$ (here y_0 and x_0 are points in Y and X respectively) and the homomorphism ϕ is induced by the morphism $(f, \Phi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ taking x_0 to y_0.

Definition 5.1. The *geometric rank* $r(\phi)$ of the homomorphism ϕ is the maximum of the numbers r such that the closure in X of the set $\{x \in X \setminus \operatorname{sing}(X)$, the rank of the map f at the point x is equal to $r\}$ contains x_0.

Theorem 5.2. *Let $\phi\colon A \to B$ be a homomorphism of analytic rings, and B an integral domain. If* $\dim \hat{A}/\ker \hat{\phi} = r(\phi)$, *then ϕ is an analytically regular homomorphism.*

Proof. By Hironaka's theorem on the resolution of singularities [11], there exists an imbedding $\psi\colon B \to C$, where C is a regular analytic ring, such that $r(\psi) = \dim B$. Furthermore, by the definition of an analytic ring there exists an epimorphism $\chi\colon D \to A$, where D is a regular analytic ring. Consider the composite map $\eta = \psi \circ \phi \circ \chi$. Obviously $r(\eta) = r(\phi)$ and $A/\ker \phi \simeq D/\ker \eta$. Furthermore, $\dim \hat{D}/\ker \hat{\eta} \leq \dim \hat{A}/\ker \hat{\phi} = r(\eta)$. On the other hand, from Lemma 4.2 it is not hard to deduce that $\dim \hat{D}/\ker \hat{\eta} \geq r(\eta)$. Therefore $\dim \hat{D}/\ker \hat{\eta} = r(\eta)$, and the assertion of the theorem reduces to the case of regular rings, i.e. to Theorem 4.8.

Corollary 5.3. *Let $\phi\colon A \to B$ be a homomorphism of analytic rings, B an integral domain, and* $\dim A \leq r(\phi) + 1$. *Then ϕ is an analytically regular homomorphism.*

Proof. If $\dim \hat{A}/\ker \hat{\phi} = r(\phi)$, then the assertion follows from Theorem 5.2. But if $\dim \hat{A}/\ker \hat{\phi} > r(\phi)$, then $\dim \hat{A}/\ker \hat{\phi} = \dim A$, and the assertion follows from Lemma 4.6.

Corollary 5.4. *Let $g = (g_1(x), \cdots, g_n(x))$ be a mapping $\mathbf{C}_x^m \to \mathbf{C}_y^n$ satisfying the condition of Theorem 4.8, and $h(x)$ an arbitrary analytic function. Define a map g': $\mathbf{C}_x^m \to \mathbf{C}_{y,z}^{n+1}$ by the formula $y = g(x)$, $z = h(x)$. Then the homomorphism g'^* is analytically regular.*

Proof. By Theorem 4.8, $\dim \mathbf{C}\{y\}/\ker g^* = r(g^*)$. Hence

$$\dim \mathbf{C}\{y, z\}/\ker g^* \cdot \mathbf{C}\{y, z\} = r(g^*) + 1,$$

and so $\dim \mathbf{C}\{y, z\}/\ker g'^* \leq r(g^*) + 1$. Since $r(g^*) \leq r(g'^*)$, it suffices to apply Corollary 5.3 to the homomorphism $\mathbf{C}\{y, z\}/\ker g'^* \to \mathbf{C}\{x\}$.

Theorem 5.5. *Let $\phi: A \to B$ be a homomorphism of analytic rings, B an integral domain, and $\dim \hat{A}/\ker \hat{\phi} = r(\phi)$. Then $\hat{\phi}(\hat{A}) \cap B = \phi(A)$.*

Proof. Let $B = \mathbf{C}\{x\}/I$ and $A = \mathbf{C}\{y\}/J$, and let $\overline{H}(y) \in \hat{A}$ and $\hat{\phi}(\overline{H}) = h(x) \in B$. Put $A' = \mathbf{C}\{y, z\}/J \cdot \mathbf{C}\{y, z\}$, and consider the homomorphism $\phi': A' \to B$ defined by the formula

$$\sum f_i(y) z^i \mapsto \sum \varphi(f_i) h(x)^i.$$

The imbeddings $\mathbf{C}\{y\} \to \mathbf{C}\{y, z\}$ and $\mathbf{C}[[y]] \to \mathbf{C}[[y, z]]$ induce imbeddings $A \to A'$ and $\hat{A} \to \hat{A}'$. Here $A \cap \ker \phi' = \ker \phi$ and $\hat{A} \cap \ker \hat{\phi}' = \ker \hat{\phi}$. Hence we have imbeddings

$$\rho: A/\ker \varphi \to A'/\ker \varphi'$$

and

$$\hat{\rho}: \hat{A}/\ker \hat{\varphi} \to \hat{A}'/\ker \hat{\varphi}'.$$

Since $z - \overline{H}(y) \in \ker \hat{\phi}'$, $\hat{\rho}$ is an isomorphism. In particular, $\dim \hat{A}'/\ker \hat{\phi}' = r(\phi)$. Since $\dim \hat{A}'/\ker \hat{\phi}' \geq r(\phi') \geq r(\phi)$, it follows that $\dim \hat{A}'/\ker \hat{\phi}' = r(\phi')$. From Theorem 5.2 it now follows that

$$\hat{A}'/\ker \hat{\varphi}' = (A'/\ker \varphi')\hat{}.$$

Since, moreover, $\hat{A}/\ker \hat{\phi} = (A/\ker \phi)\hat{}$, we see that

$$\hat{\rho}: (A/\ker \varphi)\hat{} \to (A'/\ker \varphi')\hat{}$$

is an isomorphism. Hence (cf. [7]) ρ is also an isomorphism.

Let $H(y) \in A$ and $H(y) = \rho^{-1}(z)$. Then $\hat{\rho}(H(y) - \overline{H}(y)) = 0$, and hence $H(y) - \overline{H}(y) \in \ker \hat{\phi}$, i.e. $h(x) = \phi(H(y))$, as required.

Proposition 5.6. *Let $\phi: A \to B$ be a homomorphism of analytic rings, B a ring without nilpotents, $\mathfrak{p}_i$ minimal prime ideals of the ring B, and $\pi_i: B \to B/\mathfrak{p}_i$ the natural projections.*

a) *If for each i the homomorphism $\pi_i \circ \phi$ is analytically regular, then ϕ is an analytically regular homomorphism.*

b) *If, moreover, for each i*

$$\hat{\pi}_i \circ \hat{\varphi}(\hat{A}) \cap B/\mathfrak{p}_i = \pi_i \circ \varphi(A),$$

then $\hat{\phi}(\hat{A}) \cap B = \phi(A)$.

Proof. a) The assertion follows from the fact that

$$\ker \varphi = \bigcap \ker (\pi_i \circ \varphi) \text{ and } \ker \hat{\varphi} = \bigcap \ker (\hat{\pi}_i \circ \hat{\varphi}).$$

b) Let $\mathfrak{q}_i = \phi^{-1}(\mathfrak{p}_i)$. Since $\pi_i \circ \phi$ are analytically regular, we have $\hat{\phi}^{-1}(\hat{\mathfrak{p}}_i) = \hat{\mathfrak{q}}_i$. We consider the finite A-module $M = \bigoplus A/\mathfrak{q}_i$ and the natural maps $\rho: A \to M$ and $\hat{\rho}: \hat{A} \to \hat{M} = \bigoplus \hat{A}/\hat{\mathfrak{q}}_i$. Let $\overline{F} \in \hat{A}$ and $\hat{\phi}(\overline{F}) \in B$. By the hypothesis, for each i there exists a function $F_i \in A$ such that $\hat{\phi}(\overline{F} - F_i) \in \hat{\mathfrak{p}}_i$, i.e. $\overline{F} - F_i \in \hat{\mathfrak{q}}_i$. Hence

$$\hat{\rho}(\overline{F}) = (F_i \operatorname{mod} \hat{\mathfrak{q}}_i) \in \hat{\rho}(\hat{A}) \cap M = \rho(A).$$

Therefore there exists a function $F \subset A$ such that $\hat{\rho}(\overline{F} - F) = 0$, i.e. $\overline{F} - F \in \bigcap \hat{\mathfrak{q}}_i = \ker \hat{\phi}$, as required.

Theorem 5.7. *Let $\phi: A \to B$ be a homomorphism of analytic rings, B a ring without nilpotents, and* $\dim A \leq 3$. *Then ϕ is analytically regular.*

Proof. By Proposition 5.6a) it suffices to consider the case when B is an integral domain. If $r(\phi) = 1$, the assertion reduces easily to the case $\dim B = 1$ (as in part 2 of the proof of Theorem 4.8). But then ϕ is a finite homomorphism, and the assertion follows from Lemma 4.7. But if $r(\phi) \geq 2$, it suffices to apply Corollary 5.3.

If B contains nilpotents, the situation is considerably more complicated, as the following example shows.

Example 5.8. Let

$$A = \mathbf{C}\{t_1, \ldots, t_4, y\}/(y, t_3, t_4)^2, \qquad B = \mathbf{C}\{x_1, x_2, v\}/(v^2).$$

We define a homomorphism $\phi: A \to B$ by the formula

$$\varphi(y) = v, \quad \varphi(t_1) = x_1, \quad \varphi(t_2) = x_1 x_2,$$
$$\varphi(t_3) = v x_1 x_2 e^{x_2}, \quad \varphi(t_4) = v \cdot \Phi(x),$$

where

$$\Phi(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(k+j)!} x_1^k x_2^{k+j+1}.$$

We note that

$$A/\mathfrak{N}(A) = \mathbf{C}\{t_1, t_2\}, \qquad B/\mathfrak{N}(B) = \mathbf{C}\{x_1, x_2\}$$

and a homomorphism $\phi: A/\mathfrak{N}(A) \to B/\mathfrak{N}(B)$ is given by the formula

$$\varphi(t_1) = x_1, \quad \varphi(t_2) = x_1 x_2.$$

We shall show that $\ker \phi = 0$. In fact, let

$$H(t, y) = H_1(t_1, t_2) + H_2(t_1, t_2) y + H_3(t_1, t_2) t_3 + H_4(t_1, t_2) t_4$$

belong to $\ker \phi$. Then $H_1(x_1, x_1 x_2) \equiv 0$, and hence $H_1(t_1, t_2) \equiv 0$. Furthermore,

$$H_2(x_1, x_1 x_2) + H_3(x_1, x_1 x_2) \cdot x_1 x_2 e^{x_2} + H_4(x_1, x_1 x_2) \Phi(x) \equiv 0.$$

Hence $z(t) = H_2(t_1, t_2) + H_3(t_1, t_2) t_3 + H_4(t_1, t_2) t_4$ belongs to the kernel of the map constructed in [12] (counterexample (1)). As was shown in [12], it follows from this that $z(t) \equiv 0$. Thus $H(t, y) \equiv 0$, i.e. $\ker \phi = 0$. On the other hand, it is easy to verify that

$$\bar{z}(t) = t_4 - t_3 \sum_{k=1}^{\infty} k!\, t_1^{k-1} + y \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{k!}{(i-1)!} t_1^{k-i} t_2^i$$

belongs to the kernel of $\hat{\phi}$. Hence ϕ is not analytically regular.

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