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On the Parity of $p(n)$, II

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1. INTRODUCTION

Not a great deal is known about the parity of $p(n)$. Since Kolberg [3], we have known that $p(n)$ is infinitely often even, infinitely often odd. Recently, together with M. V. Subbarao [2], I proved that for every r, $p(16n+r)$ is infinitely often even, infinitely often odd. The main results of this note are that

for every r, $p(12n+r)$ is infinitely often even, infinitely often odd

and

for every r, $p(40n+r)$ is infinitely often even, infinitely often odd.

In order to establish these results, we obtain congruences modulo 2 for the generating functions of $p(3n+r)$, $r=0, 1, 2$, and of $p(5n+r)$, $r=0, 1, 2, 3, 4$. These congruences appear in the important recent paper of Frank Garvan and Dennis Stanton [1], but our derivation of them is rather more straightforward, relying only on the triple product identity. We use these congruences to obtain recurrences modulo 2 for $p(12n+r)$ and for $p(40n+r)$ from which we deduce our results via standard "Kolberg-type" arguments.

Garvan and Stanton obtain congruences for the generating functions of $p(7n+r)$, $r=0, 2, 6$. We derive these, and show how they, together with an identity of Ramanujan, yield the result

$p(56n+r)$, $r \equiv 0, 2, 5,$ *or* $6 \bmod 7$, *is infinitely often even, infinitely often odd.*

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2. $p(12n+r)$ Is Infinitely often Even, Infinitely often Odd

We have, modulo 2,

$$\sum p(n)q^n = \frac{1}{(q;q)_\infty} \equiv (q;q^2)_\infty$$

$$= (q;q^6)_\infty\,(q^3;q^6)_\infty\,(q^5;q^6)_\infty$$

$$\equiv \frac{(q;q^6)_\infty\,(q^5;q^6)_\infty}{(q^3;q^3)_\infty}$$

$$\equiv \frac{(q;q^6)_\infty\,(q^5;q^6)_\infty\,(q^6;q^6)_\infty}{(q^3;q^3)^3_\infty}$$

$$\equiv \frac{1}{(q^3;q^3)^3_\infty}\sum q^{3a^2-2a}$$

$$= \frac{1}{(q^3;q^3)^3_\infty}\left\{\sum q^{3(3a)^2-2(3a)} + \sum q^{3(3a+1)^2-2(3a+1)} + \sum q^{3(3a-1)^2-2(3a-1)}\right\}.$$

$$= \frac{1}{(q^3;q^3)^3_\infty}\left\{\sum q^{27a^2-6a} + q\sum q^{27a^2-12a} + q^5\sum q^{27a^2-24a}\right\}.$$

So

$$\sum p(3n)q^n \equiv \frac{1}{(q;q)^3_\infty}\sum q^{9a^2-2a},$$

$$\sum p(3n+1)q^n \equiv \frac{1}{(q;q)^3_\infty}\sum q^{9a^2-4a},$$

$$\sum p(3n+2)q^n \equiv \frac{q}{(q;q)^3_\infty}\sum q^{9a^2-8a}.$$

We now multiply by $(q;q)^4_\infty$.

Since

$$(q;q)_\infty \equiv \sum q^{(3a^2-a)/2}$$

and

$$(q;q)^4_\infty \equiv (q^4;q^4)_\infty,$$

we have

$$\sum q^{2(3a^2-a)} \sum p(3n)q^n \equiv \sum q^{(3a^2-a)/2+(9b^2-2b)} = \sum c_0(n)q^n,$$

$$\sum q^{2(3a^2-a)} \sum p(3n+1)q^n \equiv \sum q^{(3a^2-a)/2+(9b^2-4b)} = \sum c_1(n)q^n,$$

$$\sum q^{2(3a^2-a)} \sum p(3n+2)q^n \equiv \sum q^{(3a^2-a)/2+(9b^2-8b)+1} = \sum c_2(n)q^n.$$

If we now write $p_r(n) = p(12n+r)$, we have, modulo 2,

$$(*) \quad p_r(n)+p_r(n-1)+p_r(n-2)+p_r(n-5)+p_r(n-7)+\cdots$$

$$\equiv \begin{cases} c_0(4n) & \text{if } r=0 \\ c_1(4n) & \text{if } r=1 \\ c_2(4n) & \text{if } r=2 \\ c_0(4n+1) & \text{if } r=3 \\ c_1(4n+1) & \text{if } r=4 \\ c_2(4n+1) & \text{if } r=5 \\ c_0(4n+2) & \text{if } r=6 \\ c_1(4n+2) & \text{if } r=7 \\ c_2(4n+2) & \text{if } r=8 \\ c_0(4n+3) & \text{if } r=9 \\ c_1(4n+3) & \text{if } r=10 \\ c_2(4n+3) & \text{if } r=11. \end{cases}$$

Now, $(3a^2-a)/2+(9b^2-2b) \not\equiv 4$, 17, 30, 43, 56, 69, 95, 108, 121, 134, 147, 160 mod 169, so $c_0(n)=0$ for $n \equiv 4$, 17, 30, 43, 56, 69, 95, 108, 121, 134, 147, 160 mod 169. Similarly

$c_1(n)=0$ for $n \equiv 8$, 21, 34, 47, 60, 73, 86, 99, 112, 125, 151, 164 mod 169,

$c_2(n)=0$ for $n \equiv 12$, 38, 51, 64, 77, 90, 103, 116, 129, 142, 155, 168 mod 169.

It follows that if $n \equiv m_r$ mod 169, where m_r is given by the table

r	0	1	2	3	4	5	6	7	8	9	10	11
m_r	1	2	3	4	5	19	7	8	9	10	11	12

then $(*)$ becomes

$$(**) \quad p_r(n)+p_r(n-1)+p_r(n-2)+p_r(n-5)+p_r(n-7)+\cdots \equiv 0 \bmod 2.$$

Next, let k_r be given by the table

r	0	1	2	3	4	5	6	7	8	9	10	11
k_r	0	0	1	3	0	4	0	3	2	5	6	5

Then $p_r(k_r)$ is odd.

Finally let l_r be given by the table

r	0	1	2	3	4	5	6	7	8	9	10	11
l_r	-1	1	1	-1	-2	3	2	-2	2	-2	-2	2

Then $(3l_r^2+l_r)/2+k_r \equiv m_r \bmod 169$.

Suppose $p_r(n)$ is odd (alternatively even) for $n \geqslant n_0$. We can suppose $n_0 \equiv l_r \bmod 169$, and that $2n_0+1>k_r$.

Let $N=(3n_0^2+n_0)/2+k_r$.

Then $N \equiv (3l_r^2+l_r)/2+k_r \equiv m_r \bmod 169$, and $(**)$ becomes

$$(***) \quad p_r(N)+p_r(N-1)+p_r(N-2)+p_r(N-5)+p_r(N-7)+\cdots$$
$$+\cdots+p_r(n_0+k_r)+p_r(k_r) \equiv 0 \bmod 2.$$

(The condition $2n_0+1>k_r$ ensures that $p_r(k_r)$ is the last term on the left.)

But the left-hand-side of $(***)$ is odd: there is an odd number, $2n_0+1$, of terms of which the last is odd while the others are all odd (alternatively even). So we have a contradiction, and our result is proved.

3. $p(40n+r)$ Is Infinitely often Even, Infinitely often Odd

We have, modulo 2,

$$\begin{aligned}
\sum p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&\equiv (q;q^2)_\infty \\
&= (q;q^{10})_\infty (q^3;q^{10})_\infty (q^5;q^{10})_\infty (q^7;q^{10})_\infty (q^9;q^{10})_\infty \\
&\equiv \frac{(q;q^{10})_\infty (q^3;q^{10})_\infty (q^7;q^{10})_\infty (q^9;q^{10})_\infty}{(q^5;q^5)_\infty} \\
&\equiv \frac{(q;q^{10})_\infty (q^9;q^{10})_\infty (q^{10};q^{10})_\infty \times (q^3;q^{10})_\infty (q^7;q^{10})_\infty (q^{10};q^{10})_\infty}{(q^5;q^5)_\infty^5} \\
&\equiv \frac{1}{(q^5;q^5)_\infty^5} \sum q^{5n^2-4n+5m^2-2m}
\end{aligned}$$

$$= \frac{1}{(q^5;q^5)_\infty^5} \sum q^{(m+2n)^2+(2m-n)^2-2(m+2n)}$$

$$= \frac{1}{(q^5;q^5)_\infty^5} \sum_{b \equiv 2a \bmod 5} q^{a^2-2a+b^2} \qquad \begin{array}{l} a = m+2n \\ b = 2m-n \end{array}$$

$$= \frac{1}{(q^5;q^5)_\infty^5} \Big\{ \sum q^{25a^2-10a+25b^2} + \sum q^{(5a+1)^2-2(5a+1)+(5b+2)^2}$$

$$+ \sum q^{(5a+2)^2-2(5a+2)+(5b-1)^2}$$

$$+ \sum q^{(5a+3)^2-2(5a+3)+(5b+1)^2}$$

$$+ \sum q^{(5a-1)^2-2(5a-1)+(5b-2)^2} \Big\}$$

$$= \frac{1}{(q^5;q^5)_\infty^5} \Big\{ \sum q^{25a^2-10a+25b^2} + q^3 \sum q^{25a^2+25b^2-20b}$$

$$+ q \sum q^{25a^2-10a+25b^2-10b} + q^4 \sum q^{25a^2-20a+25b^2-10b}$$

$$+ q^7 \sum q^{25a^2-20a+25b^2-20b} \Big\}.$$

So

$$\sum p(5n)q^n \equiv \frac{1}{(q;q)_\infty^5} \sum q^{5a^2-2a+5b^2}$$

$$\equiv \frac{1}{(q;q)_\infty^5} \sum q^{5a^2-2a}$$

$$\sum p(5n+1)q^n \equiv \frac{1}{(q;q)_\infty^5} \sum q^{5a^2-2a+5b^2-2b}$$

$$\equiv \frac{1}{(q;q)_\infty^5} \sum q^{10a^2-4a} \qquad \text{(terms } (a, b) \text{ with } a \neq b \text{ cancel in pairs)}$$

$$\sum p(5n+2)q^n \equiv \frac{q}{(q;q)_\infty^5} \sum q^{5a^2-4a+5b^2-4b}$$

$$\equiv \frac{q}{(q;q)_\infty^5} \sum q^{10a^2-8a}$$

$$\sum p(5n+3)q^n \equiv \frac{1}{(q;q)_\infty^5} \sum q^{5a^2+5b^2-4b}$$

$$\equiv \frac{1}{(q;q)_\infty^5} \sum q^{5b^2-4b}$$

$$\begin{aligned}\sum p(5n+4)q^n &\equiv \frac{1}{(q;q)_\infty^5}\sum q^{5a^2-4a+5b^2-2b}\\ &\equiv \frac{1}{(q;q)_\infty^5}(q;q^{10})_\infty(q^9;q^{10})_\infty(q^{10};q^{10})_\infty\\ &\quad\times(q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{10};q^{10})_\infty\\ &= \frac{1}{(q;q)_\infty^5}\cdot(q;q^2)_\infty\cdot\frac{(q^{10};q^{10})_\infty^2}{(q^5;q^{10})_\infty}\\ &\equiv \frac{1}{(q;q)_\infty^6}\cdot(q^5;q^5)_\infty\cdot(q^{10};q^{10})_\infty^2\\ &\equiv \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.\end{aligned}$$

(This also follows from a result of Ramanujan.)

We now multiply by $(q;q)_\infty^8$.

Since

$$(q;q)_\infty^8 \equiv (q^8;q^8)_\infty,$$

$$(q;q)_\infty^3 \equiv \sum q^{2a^2-a}$$

and

$$(q;q)_\infty^2 \equiv (q^2;q^2)_\infty$$

we have

$$\sum q^{4(3a^2-a)}\sum p(5n)q^n \equiv \sum q^{(2a^2-a)+(5b^2-2b)}$$

and so on, and

$$\sum q^{4(3a^2-a)}\sum p(5n+4)q^n \equiv (q^5;q^5)_\infty^5\sum q^{(3a^2-a)}.$$

Now,

$(2a^2-a)+(5b^2-2b)$

$\not\equiv 12, 29, 46, 63, 80, 97, 114, 131, 148, 165, 182, 199, 216, 233, 250, 284$ mod 289

$(2a^2-a)+(10b^2-4b)$

$\not\equiv 5, 22, 39, 56, 73, 90, 107, 124, 141, 158, 175, 192, 226, 243, 260, 277$ mod 289

$$(2a^2-a)+(10b^2-8b)+1$$
$$\not\equiv 15, 32, 49, 66, 83, 100, 117, 134, 168, 185, 202, 219, 236, 253, 270, 287 \mod 289$$
$$(2a^2-a)+(5b^2-4b)$$
$$\not\equiv 8, 25, 42, 59, 76, 110, 127, 144, 161, 178, 195, 212, 229, 246, 263, 280 \mod 289.$$

If we write $p_r(n)=p(40n+r)$, and $r=0$, 1, 2 or 3 mod 5, we can establish the result

$$(**) \quad p_r(n)+p_r(n-1)+p_r(n-2)+p_r(n-5)+p_r(n-7)+\cdots\equiv 0 \mod 2$$

for n in certain residue classes modulo 289.

Indeed, if m_r, k_r, and l_r are given by the table

r	0	1	2	3	5	6	7	8	10	11	12
m_r	10	7	4	1	12	9	23	3	14	11	8
k_r	8	0	2	0	0	4	16	1	2	6	1
l_r	1	2	1	-1	-3	-3	2	1	-3	-2	2

r	13	15	16	17	18	20	21	22	23	25	26
m_r	5	16	13	10	7	1	15	12	9	3	17
k_r	0	14	1	9	0	0	10	5	4	2	5
l_r	-2	1	-3	-1	2	-1	-2	2	-2	-1	-3

r	27	28	30	31	32	33	35	36	37	38
m_r	14	28	5	2	16	13	7	4	1	15
k_r	2	16	3	1	1	1	0	3	0	0
l_r	-3	-3	1	-1	3	-3	2	-1	-1	3

and we work modulo 289, then the proof proceeds as before, establishing our result for $r\equiv 0$, 1, 2 or 3 mod 5.

We have

$$\sum q^{4(3a^2-a)}\sum p(5n+4)q^n\equiv (q^5;q^5)_\infty^5 \sum q^{(3a^2-a)},$$

and $(3a^2-a)\not\equiv 1$, 3 mod 5, so the right-hand-side has no powers congruent to 1 or 3 mod 5. So for $r\equiv 4$ mod 5 we have

$$(**) \quad p_r(n)+p_r(n-1)+p_r(n-2)+p_r(n-5)+p_r(n-7)+\cdots\equiv 0 \mod 2$$

for n in certain residue classes modulo 5.

If m_r, k_r, and l_r are given by the table

r	4	9	14	19	24	29	34	39
m_r	1	4	2	0	3	1	4	2
k_r	0	2	0	3	2	0	2	0
l_r	-1	1	1	1	-1	-1	1	1

and we work modulo 5, then the proof proceeds as before, establishing our result in these remaining cases.

4. $p(56n+r)$, $r \equiv 0, 2, 5$, or 6 mod 7, Is Infinitely often Even, Infinitely often Odd

We have

$$\begin{aligned}
&\sum p(n)q^n \\
&= \frac{1}{(q;q)_\infty} \equiv (q;q^2) \\
&\equiv \frac{1}{(q^7;q^7)_\infty^7} (q;q^{14})_\infty (q^{13};q^{14})_\infty (q^{14};q^{14})_\infty \\
&\quad \times (q^3;q^{14})_\infty (q^{11};q^{14})_\infty (q^{14};q^{14})_\infty \\
&\quad \times (q^5;q^{14})_\infty (q^9;q^{14})_\infty (q^{14};q^{14})_\infty \\
&\equiv \frac{1}{(q^7;q^7)_\infty^7} \sum q^{7k^2+2k+7l^2+4l+7m^2+6m} \\
&= \frac{1}{(q^7;q^7)_\infty^7} \sum q^{(2k+l+m)^2+(-k+l+2m)^2+(k-2l+m)^2+(k+l-m)^2+2(2k+l+m)+2(-k+l+2m)} \\
&= \frac{1}{(q^7;q^7)_\infty^7} \sum q^{a^2+b^2+c^2+d^2+2a+2b},
\end{aligned}$$

where the sum is taken over all quadruples (a, b, c, d) with

$$\begin{aligned}
2a-b+c+d &\equiv 0 \mod 7, \\
a+b-2c+d &\equiv 0 \mod 7, \\
a+2b+c-d &\equiv 0 \mod 7, \\
-a+b+c+2d &\equiv 0 \mod 7.
\end{aligned}$$

These congruences admit 49 solutions, which fall into seven sets of seven according to the residue of $a^2+b^2+c^2+d^2+2a+2b$ modulo 7.

Thus, for example, the seven quadruples for which $a^2+b^2+c^2+d^2+2a+2b\equiv 0 \mod 7$ are $(a, b, c, d)\equiv(-3, 3, 3, -1)$, $(-2, 2, 2, -3)$, $(-1, 1, 1, 2)$, $(0, 0, 0, 0)$, $(1, -1, -1, -2)$, $(2, -2, -2, 3)$, $(3, -3, -3, 1)$.

It follows that

$$\sum p(7n)q^{7n}$$

$$\equiv \frac{1}{(q^7;q^7)_\infty^7}\Big\{\sum q^{(7a-3)^2+(7b-4)^2+(7c+3)^2+(7d-1)^2+2(7a-3)+2(7b-4)}$$
$$+\sum q^{(7a-2)^2+(7b+2)^2+(7c+2)^2+(7d-3)^2+2(7a-2)+2(7b+2)}$$
$$+\sum q^{(7a-1)^2+(7b+1)^2+(7c+1)^2+(7d+2)^2+2(7a-1)+2(7b+1)}$$
$$+\sum q^{(7a)^2+(7b)^2+(7c)^2+(7d)^2+2(7a)+2(7b)}$$
$$+\sum q^{(7a+1)^2+(7b-1)^2+(7c-1)^2+(7d-2)^2+2(7a+1)+2(7b-1)}$$
$$+\sum q^{(7a+2)^2+(7b-2)^2+(7c-2)^2+(7d-3)^2+2(7a+2)+2(7b-2)}$$
$$+\sum q^{(7a-4)^2+(7b-3)^2+(7c-3)^2+(7d+1)^2+2(7a-4)+2(7b-3)}\Big\}.$$

$$= \frac{1}{(q^7;q^7)_\infty^7}\Big\{q^{21}\sum q^{49a^2+49b^2+49c^2+49d^2-28a-42b+42c-14d}$$
$$+q^{21}\sum q^{49a^2+49b^2+49c^2+49d^2-14a+42b+28c-42d}$$
$$+q^{7}\sum q^{49a^2+49b^2+49c^2+49d^2+28b+14c+28d}$$
$$+\sum q^{49a^2+49b^2+49c^2+49d^2+14a+14b}$$
$$+q^{7}\sum q^{49a^2+49b^2+49c^2+49d^2+28a-14c-28d}$$
$$+q^{21}\sum q^{49a^2+49b^2+49c^2+49d^2+42a-14b-28c-42d}$$
$$+q^{21}\sum q^{49a^2+49b^2+49c^2+49d^2-42a-28b-42c+14d}\Big\}$$

$$\equiv \frac{1}{(q^7;q^7)_\infty^7}\sum q^{49a^2+49b^2+49c^2+49d^2+14a+14b}$$

$$\equiv \frac{1}{(q^7;q^7)_\infty^7}\sum q^{49a^2+49b^2+14a+14b}$$

$$\equiv \frac{1}{(q^7;q^7)_\infty^7}\sum q^{98a^2-28a},$$

or,

$$\sum p(7n)q^n \equiv \frac{1}{(q;q)_\infty^7} \sum q^{14a^2-4a}.$$

In the same way we find

$$\sum p(7n+2)q^n \equiv \frac{1}{(q;q)_\infty^7} q^2 \sum q^{14a^2-12a}$$

(the seven quadruples for which $a^2+b^2+c^2+d^2+2a+2b \equiv 2 \mod 7$ are $(a, b, c, d) \equiv (-3, -3, -1, -3)$, $(-2, 3, -2, 2)$, $(-1, 2, -3, 0)$, $(0, 1, 3, -2)$, $(1, 0, 2, 3)$, $(2, -1, 1, 1)$, $(3, -2, 0, -1)$) and

$$\sum p(7n+6)q^n \equiv \frac{1}{(q;q)_\infty^7} \sum q^{14a^2-8a}$$

(the seven quadruples for which $a^2+b^2+c^2+d^2+2a+2b \equiv 6 \mod 7$ are $(a, b, c, d) \equiv (-3, -1, -2, 0)$, $(-2, -2, -3, -2)$, $(-1, -3, 3, 3)$, $(0, 3, 2, 1)$, $(1, 2, 1, -1)$, $(2, 1, 0, -3)$, $(3, 0, -1, 2)$).

In the case of $\sum p(7n+r)q^n$, $r = 1, 3, 4, 5$, we do not find the same sort of simplification.

If we multiply by $(q;q)_\infty^8$, we have

$$\sum q^{4(3a^2-a)} \sum p(7n)q^n \equiv \sum q^{(3a^2-a)/2+(14b^2-4b)}$$

and so on.

Now,

$$(3a^2-a)/2+(14b^2-4b)$$
$$\not\equiv 3, 16, 29, 42, 68, 81, 94, 107, 120, 133, 146, 159 \mod 169$$

$$(3a^2-a)/2+(14b^2-12b)+2$$
$$\not\equiv 12, 25, 38, 51, 64, 77, 90, 116, 129, 142, 155, 168 \mod 169$$

$$(3a^2-a)/2+(14b^2-8b)$$
$$\not\equiv 4, 17, 43, 56, 69, 82, 95, 108, 121, 134, 147, 160 \mod 169.$$

If we define m_r, k_r, l_r by the table

r	0	2	6	7	9	13	14	16	20	21	23	27
m_r	2	8	7	10	3	2	5	11	10	13	19	5
k_r	0	6	0	5	2	0	0	4	3	1	12	3
l_r	1	1	2	-2	-1	1	-2	2	2	-3	2	1
r	28	30	34	35	37	41	42	44	48	49	51	55
m_r	8	14	13	3	9	8	11	4	16	19	64	11
k_r	3	9	1	1	4	3	4	2	1	4	7	4
l_r	-2	-2	-3	1	-2	-2	2	1	3	3	6	2

and we work modulo 169, then the proof proceeds as before, and the result is established for $r \equiv 0$, 2, or 6 mod 7.

Ramanujan gave the identity

$$\sum p(7n+5)q^n = \frac{7(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}.$$

It follows that, modulo 2,

$$\sum q^{4(3a^2-a)} \sum p(7n+5)q^n \equiv (q^7;q^7)_\infty^3 \sum q^{2(3a^2-a)} + q(q^7;q^7)_\infty^7$$

The right-hand-side has no powers congruent to 2, 3, or 5 mod 7.

If m_r, k_r, and l_r are given by the table

r	5	12	19	26	33	40	47	54
m_r	2	1	0	6	5	4	3	2
k_r	0	0	5	1	0	2	2	0
l_r	1	-1	1	-2	-2	1	-1	1

and we work modulo 7, then the proof proceeds as before, establishing our result for $r \equiv 5$ mod 7.

References

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