Necessary Conditions for Liouvillian Solutions of (Third Order) Linear Differential Equations¶

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October 16, 1993

Abstract

In this paper we show how group theoretic information can be used to derive a set of necessary conditions on the coefficients of $L(y)$ for $L(y) = 0$ to have a liouvillian solution. The method is used to derive (and improve in one case) the necessary conditions of the Kovacic algorithm and to derive an explicit set of necessary conditions for third order differential equations.

1 Introduction

In our previous work [20], [21], we have shown how group theoretic techniques can be used to develop effective algorithms to calculate Galois groups of second and third order homogeneous linear differential equations and to decide questions about the algebraic nature of the solutions of such equations (e.g., solvability in terms of liouvillian functions or in terms of linear differential equations of lower order).

¶A weaker version of these results were announcened in *Liouvillian Solutions of Third Order Linear Differential Equations: New Bounds and Necessary Conditions*, Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation, ACM Press

*Partially supported by NSF Grant 90-24624

**Partially supported by Deutsche Forschungsgemeinschaft , while on leave from Universität Karlsruhe. This paper was written during the two year visit of the second author at North Carolina State University. The second author would like to thank North Carolina State University for its hospitality and support during the preparation of this paper.

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In [12], Kovacic gave an algorithm to decide if a second order homogeneous linear differential equation has liouvillian solutions. In the process of doing this, Kovacic derived a very strong set of necessary conditions for the existence of such solutions. In this paper, we show how one can derive these conditions from general group theoretic properties. We also derive similar conditions for the third order case.

We shall assume that the reader is familiar with the elementary concepts of differential Galois theory and elementary properties of linear operators. The necessary facts are reviewed in [20]. We shall need results from [21], but these will be stated in full for the convenience of the reader.

The rest of the paper is organized as follows. In section 2, we give some elementary results concerning exponents and monodromy groups. Section 3 contains the necessary conditions. The last section deals with linear differential equations having 3 singular points and examples.

We wish to thank the referees for pointing out a mistake in the original proof of Theorem 3.5 and for suggestions that allowed us to sharpen our original version of the necessary conditions for case 2.

2 Exponents and Monodromy Groups

In this section we review some facts about formal and analytic solutions of linear differential equations. There is a well developed general theory of formal solutions at any point (of both systems and single equations) as well as an asymptotic theory (c.f. [1, 5, 14]). We have restricted ourselves to the most basic facts in order to give an elementary and direct exposition of the results that we need.

In what follows we will frequently wish to consider solutions of a linear differential equation that can be expressed as a series in (not necessarily integral) powers of x. This motivates the following discussion.

Let
$$L(y) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y$$
be a linear differential equation with coefficients in the field $C((x))$ of formal Laurent series and let $a_i(x) = \sum_{j\geq i_0} a_{i,j}x^j$ where $a_{i,i_0} \neq 0$. If $L(y) = 0$ has a solution of the form $y = x^\rho \sum_{j\geq 0} c_j x^j$, $c_0 \neq 0$, then formally substituting this expression into $L(y) = 0$ and examining the coefficient of the smallest power of x, one sees that ρ will satisfy the equation
$$P(\rho) = \sum a_{k,k_0}(\rho)_k = 0$$
where $(\rho)_j = \rho(\rho-1)\ldots(\rho-(j-1))$ and the sum is over all k with $k_0 - k = \min_{0\leq i\leq n}\{i_0 - i\}$. $P(\rho)$ is called the *indicial equation of* $L(y)$ *(at* 0*)*. We shall refer to the roots of the indicial polynomial as *exponents of* $L(y)$ *(at* 0*)*.

If $L(y)$ has coefficients in $C(x)$, then for any $c \in C$ we can expand these coefficients in powers of $(x-c)$. Analogously, we can define the indicial equation at c and exponents at c. Via the transformation $x = 1/t$, $\frac{d}{dx} = -t^2 \frac{d}{dt}$ we can also define the indicial equation and exponents at infinity by considering the point $t = 0$ of the transformed equation. We say that $x = c$ is an *ordinary point* if, for $0 \le i \le n, a_i(x)/a_n(x)$ is analytic at $x = c$ (for $c = \infty$ this property is related to the point $t = 0$ of the transformed equation). The following lemma contains facts that are easily verified by computation (cf. [5], Ch. 4,5; [14], Ch. IV,V).

Lemma 2.1 *Let $L(y) = 0$ have coefficients in $C(x)$ and let $c \in C$.*

1. *If $L(y) = 0$ has a formal solution of the form*

$$y = (x-c)^{\rho} \sum_{i \ge 0} c_i (x-c)^{r_i},\ r_i \in R,\ 0 = r_0 < r_1 < \dots,\ c_0 \neq 0,$$

   then ρ is an exponent of $L(y)$ at c.

2. *The degree of $P(\rho)$ is at most n.*

3. *If c is an ordinary point then the exponents at c are $\{0, \dots, n-1\}$.*

The converse of part 1 of the above lemma can be false in the case when log terms occur in formal solutions. For example, $L(y) = x^2y'' - (3-2x)xy' + (3-6x)y$ has exponents $\{1, 3\}$ at 0 and at this point it has a fundamental set of solutions of the form $\{x^3, -4x^3 \ln x + x\phi(x)\}$, where $\phi(x)$ is a power series in x with $\phi(0) \neq 0$. No linear combination of these two solutions can be of the form $x\psi(x)$ where $\psi(x)$ is a power series with $\psi(0) \neq 0$.

If a point c is not an ordinary point, it is called a *singular point*. If, in addition, the indicial polynomial at this point has degree n, we say that the point is a *regular singular point*. An obviously equivalent way of saying this is that the order of the pole of $a_{n-i}(x)/a_n(x)$ at c is at most i. A linear differential equation with only regular singular points (including ∞) is called a *fuchsian linear differential equation*. We gather some facts about regular singular points in the following lemma, but first give the following definition:

Definition 2.1 *Let $\{y_1, \dots, y_n\}$ be a fundamental set of solutions of the linear differential equation $L(y) = 0$. The m^{th} symmetric power $L^{Ⓢm}(y)$ of $L(y)$ is the differential equation of smallest order, such that the solution space of $L^{Ⓢm}(y) = 0$ is spanned by $\{y_1^m, y_1^{m-1}y_2, \dots, y_n^m\}$.*

Note that this definition is independent of the choice of fundamental solutions and that $L^{Ⓢm}(y)$ is the linear differential equation of smallest order satisified by all homogeneous forms $F(y_1, \dots, y_n)$ of degree m with constant coefficients in solutions $y_1, \dots, y_n$ of $L(y) = 0$. See [17] for a further discussion of this concept. An algorithm to construct the equation $L^{\circledS m}(y)$ is given in [17] and [20], Section 3.2.2.

Lemma 2.2 *Let $L(y) = 0$ be of order n and have coefficients in $C(x)$ and let $c \in C$.*

1. *If c is a regular singular point and ρ is a root of multiplicity r of the indicial equation, then $L(y) = 0$ has r independent solutions of the form*
$$y_i = (x-c)^\rho \left(\sum_{j=0}^{i} \phi_{i,j}(x) \left(\log(x-c)\right)^j \right) \tag{1}$$
where $i = 0, \dots, r-1$, the $\phi_{i,j}(x)$ are analytic at c and for each i there is some j such that $\phi_{i,j}(c) \neq 0$. Furthermore, if $\rho + N$ is not an exponent of $L(y) = 0$ for any positive $N \in Z$, then $L(y) = 0$ has a solution of the form $(x-c)^\rho \varphi(x)$, where $\varphi(x)$ is analytic at c and $\varphi(c) \neq 0$.

2. *c is a regular singular point if and only if in any open angular sector Ω at c, all solutions of $L(y) = 0$ analytic in Ω, satisfy $\lim_{x \to c}(x-c)^N y = 0$ for some $N \geq 0$.*

3. *If all solutions of $L(y) = 0$ are algebraic over $C(x)$, then $L(y) = 0$ is a fuchsian equation and at any singular point c there are n* distinct *rational exponents. Furthermore, at any point there is a fundamental set of solutions of the form $y_i = (x-c)^{a_i} \phi_i(x)$ where $\phi_i(x)$ is analytic at c, $\phi_i(c) \neq 0$, and $a_1, \dots, a_n$ are the exponents.*

4. *Let $L(y) = 0$ be a fuchsian equation and $a_1, \dots, a_n$ exponents at some point c. Then the m^{th} symmetric power $L^{\circledS m}(y)$ is fuchsian and the exponents at c are in the set $\left\{ \left(\sum_{j=1}^{m} a_{i_j} \right) + t \mid i_j \in \{1, \dots, n\};\ t \in Z; t \geq 0 \right\}$.*

Proof: 1. and 2. are proved in [5], Ch. 5, §1-5 and [14] Ch. V, §16,17. Lemma 2.2.2 is usually refered to as Fuchs' Criterion (cf. [5], p. 124 and [14], pp. 65-68). Lemma 2.2.2 implies that if all solutions are algebraic, the equation must be fuchsian. To prove 3., let $c = 0$ and let $y_1, \dots, y_n$ be solutions of $L(y) = 0$ as described in 1. Since these are algebraic over $C(x)$, they are also algebraic over $C((x))$. Let k be the differential field $C((x))(x^{\rho_1}, \dots, x^{\rho_n})$, where $\rho_1, \dots, \rho_n$ are the exponents at 0. We claim that for each solution y_i of the above form, we must have $\phi_{i,j} = 0$ for $j = 1, \dots, i$. If not, then $\log x$ would be algebraic over k. The Kolchin-Ostrowski Theorem ([11]) implies that $\log x$ would be in $C((x))$. Since there is no element y in $C((x))$ such that $y'/y = 1/x$, we have a contradiction. Therefore each y_i is of the form $x^{\rho_i} \phi_i(x)$, for some $\phi_i(x)$, $\phi_i(0) \neq 0$ in $C((x))$. Since each y_i is algebraic over $C((x))$, we have that each ρ_i is rational. We now claim that all the exponents of $L(y)$ are distinct. Consider the set of bases $B = (y_1, \ldots, y_n)$ where each $y_i = x^{\rho_i}\phi_i(x)$, $\phi_i(0) \neq 0$ where $\rho_1 \leq \ldots \leq \rho_n$ and define an order on this set by letting $B < \tilde{B}$ if $(\rho_1, \ldots, \rho_n) < (\tilde{\rho}_1, \ldots, \tilde{\rho}_n)$ in the lexicographical order. Let $\hat{B}$ be a maximal element of this set, then for this set the $\hat{\rho}_i$ are distinct. This will show that there are n distinct exponents. Assume not and let j be the smallest index so that all $\hat{\rho}_i$ with $i > j$ are distinct. We then have that $\hat{\rho}_j = \hat{\rho}_{j+1}$. Therefore, there is a constant d such that $z = y_j - dy_{j+1} = x^{\rho}\psi(x)$ for some $\rho > \hat{\rho}_j$. Replacing y_{j+1} by z gives a new basis which (after we perhaps rearrange the elements) is larger in the ordering. This contradicts the maximality of $\hat{B}$.

To prove 4., consider a basis of the solution space of $L(y) = 0$ of the form $y_i = (x-c)^{a_i}\left(\sum_{j=0}^{k_i} \phi_{i,j}(x)\,(\log(x-c))^j\right)$ where the $\phi_{i,j}(x)$ are analytic at c, some $\phi_{i,j}(c) \neq 0$, and $a_1, \ldots, a_n$ are the exponents of $L(y) = 0$ at c. Note that $L^{\circledS m}(y)$ has a basis of solutions of the form $y_1^{i_1} y_2^{i_2} \cdots y_n^{i_n}$, where $i_1 + \ldots + i_n = m$. The second assertion of the Lemma implies that c is at worst a regular singular point of $L^{\circledS m}(y)$. Let ρ be an exponent of $L^{\circledS m}(y)$ at c. From 1. we get that $L^{\circledS m}(y) = 0$ has a solution of the form $y = (x-c)^{\rho}\left(\sum_{r=0}^{h} \psi_r(x)\,(\log(x-c))^r\right)$, where $\psi_r(x)$ is analytic at c, and some $\psi_r(c) \neq 0$. Therefore there exist constants c_I such that

$$\begin{aligned} y &= \sum_{I=(i_1,\ldots,i_n)} c_I y_1^{i_1} \cdot \ldots \cdot y_n^{i_n} \\ &= \sum_{I=(i_1,\ldots,i_n)} c_I (x-c)^{a_1 i_1 + \ldots + a_n i_n} \left(\sum_{j=0}^{k} \varphi_{I,j}(x)\,(log(x-c))^j\right) \end{aligned}$$

where the sum is over all $I = (i_1, \ldots, i_n)$ with $i_1 + \ldots + i_n = m$, the $\varphi_{I,j}(x)$ are analytic at c and for some (I, j), $\varphi_{I,j}(c) \neq 0$. Since the $\varphi_{I,j}(x)$ contain only nonnegative powers of x, comparing the lowest power of $\log(x)$ whose coefficient is not zero and taking into account possible cancelation, yields the result. ∎

For non fuchsian equations very little can be said regarding the relationship between the exponents of a linear differential equation and those of its symmetric power. The following example shows that one can have a linear differential equation with no exponents at 0 while the symmetric square has an exponent at 0. Let

$$L(y) = y'' - \frac{2}{x(x-2)}y' + \frac{2x^2 - 3x + 2}{x^4(x-2)}y$$

This equation has a basis of solutions $\{x\exp(1/x), \exp(-1/x)\}$ and no exponents at 0. The second symmetric power is

$$y''' - \frac{6}{x(x-2)}y'' + \frac{4(3x^3 - 6x^2 + 8x - 4)}{x^4(x-2)^2}y' - \frac{4(3x^3 - 6x^2 + 8x - 4)}{x^5(x-2)^2}y$$

and this has 1 as an exponent since x is a solution.

We do note that the theory of formal solutions at a point does allow one to give apriori bounds on the exponents of the symmetric power even in the non-fuchsian case (c.f. [5], Ch. 5, Sec. 2 and [1, 19]). This involves calculating formal exponents, a task that is more difficult than calculating exponents defined above. Since this will not be needed, we have stated Lemma 2.2.4 just for the fuchsian case.

Consider the class of linear differential equation with coefficients in $C(x)$. For these equations we can use analytic considerations to define a group called the *monodromy group* that is a subgroup of the differential Galois group $\mathcal{G}(L)$ of $L(y) = 0$ over $C(x)$ (see e.g. [10, 20]). Let $c_1, \ldots, c_m$ be the singular points of $L(y) = 0$ (including infinity if it is a singular point) and let c_0 be an ordinary point of the equation. We consider these points as lying on the Riemann Sphere S^2. Let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of $L(y) = 0$ analytic at c_0 and let γ be a closed path in $S^2 - \{c_1, \ldots, c_m\}$ that begins and ends at c_0. One can analytically continue $\{y_1, \ldots, y_n\}$ along γ and get new fundamental solutions $\{\tilde{y}_1, \ldots, \tilde{y}_n\}$ analytic at c_0. These two sets must be related via $(\tilde{y}_1, \ldots, \tilde{y}_n)^T = M_\gamma(y_1, \ldots, y_n)^T$ where $M_\gamma \in GL(n, C)$. One can show that M_γ depends only on the homotopy class of γ and that the map $\gamma \mapsto M_\gamma$ defines a group homomorphism from $\pi_1(S^2 - \{c_1, \ldots, c_m\})$ to $GL(n, C)$. The image of this map depends on the choice of c_0 and $\{y_1, \ldots, y_n\}$ but is unique up to conjugacy and is called the *monodromy group of $L(y)$*. In general the image of this group will be a proper subgroup of $\mathcal{G}(L)$, but when $L(y)$ is fuchsian, the Zariski closure of this group will be the full differential Galois group $\mathcal{G}(L)$ (c.f. [23]). In particular if $\mathcal{G}(L)$ is finite (i.e., all solutions of $L(y) = 0$ are algebraic) then the map is surjective and the monodromy and Galois groups coincide. The fact that the monodromy group is a subgroup of the differential Galois group allows us to prove the following:

Lemma 2.3 *Let $L(y) = 0$ be a linear differential equation with coefficients in $C(x)$, assume all solutions of $L(y) = 0$ are algebraic over $C(x)$. If a is an exponent at any point c, then $Na \in Z$ for some integer N that is the order of an element of $\mathcal{G}(L)$.*

Proof: Lemma 2.2 implies that there is a fundamental set of solutions of the form $y_i = (x - c)^{a_i}\phi_i(x)$ where $\phi_i(x)$ is analytic at c, $\phi_i(c) \neq 0$, and $a_1, \ldots, a_n$ are the exponents. If we continue analytically along a small loop around c, we get the solutions $\tilde{y}_i = e^{(2\pi\sqrt{-1})a_i}(x - c)^{a_i}\phi_i(x)$. Therefore the diagonal matrix $g = \mathrm{diag}(e^{(2\pi\sqrt{-1})a_1}, \ldots, e^{(2\pi\sqrt{-1})a_n})$ is conjugate to an element of the differential Galois group. Since $g^N = 1$, we have $Na_i \in Z$ for $i = 1, \ldots, n$. ∎

The relationship between the exponents of $L(y)$ at c and the eigenvalues of the monodromy matrix M_γ corresponding to a simple loop γ around c, containing no other singular point, is given by the following:

Lemma 2.4 *Let $L(y) = 0$ be a linear differential equation with coefficients in $C(x)$, c a regular singular point of $L(y) = 0$ and $a_1, \ldots, a_n$ the exponents at c. The* *eigenvalues of the monodromy matrix M_γ, corresponding to a loop γ around c are* $e^{2\pi\sqrt{-1}a_1}, \ldots, e^{2\pi\sqrt{-1}a_n}$.

Proof: For convenience let $c = 0$. Separate the exponents into different sets $S_1, \ldots, S_r$, such that the elements of each set differ by integers, and elements of different sets differ by non-integers. Let $m_j = |S_j|$ and $n = m_1 + \ldots + m_r$. For each set S_j, we denote by a_j an element with the property that for any integer $m > 1$, $a_j + m \notin S_j$. From Lemma 2.2 we get that there exists a solution of the form $x^{a_j}\phi_j(x)$, where $\phi_j(x)$ is analytic at 0 and $\phi_j(0) \neq 0$. Therefore $\lambda_j = e^{2\pi\sqrt{-1}a_j}$ is an eigenvalue of the monodromy matrix. We claim that each λ_j is an eigenvalue of multiplicity at least m_j. Since $n = m_1 + \ldots + m_r$, this will imply that λ_j is of multiplicity m_j. Since for any $a \in S_j$, $\lambda_j = e^{2\pi\sqrt{-1}a}$, we will have established the conclusion of the Lemma. To prove the claim, let $a_1, \ldots, a_t$ be the *distinct* elements of S_j and assume that $a_i \in S_j$ is a root of multiplicity s_i of the indicial polynomial. For each a_i there are s_i independent solutions of the form (1). If $y = x^{a_i}\left(\sum_{j=0}^{h}\phi_j(x)(\log(x))^j\right)$ with $0 \leq h < s_i$ is one of these solutions, then $(M_\gamma - e^{2\pi\sqrt{-1}a_i}Id)(y)$

$$
\begin{aligned}
&= e^{2\pi\sqrt{-1}a_i}x^{a_i}\left(\sum_{j=0}^{h}\phi_j(x)(\log(x) + 2\pi\sqrt{-1})^j\right) - e^{2\pi\sqrt{-1}a_i}x^{a_i}\left(\sum_{j=0}^{h}\phi_j(x)(\log(x))^j\right) \\
&= x^{a_i}\left(\sum_{j=0}^{h-1}\psi_j(x)(\log(x))^j\right),
\end{aligned}
$$

where the $\psi_j(x)$ are analytic at 0. This shows that $(M_\gamma - e^{2\pi\sqrt{-1}a_i}Id)^{s_i-1}(y) = x^{a_i}\psi(x)$, where $\psi(x)$ is analytic at 0, and thus we get $(M_\gamma - e^{2\pi\sqrt{-1}a_i}Id)^{s_i}(y) = 0$. Note that for distinct a_i we get independent solutions of the form (1). Therefore the generalized eigenspace corresponding to $\lambda_j = e^{2\pi\sqrt{-1}a_j}$ has dimension at least $s_1 + \ldots + s_t = m_j$. ∎

3 Necessary Conditions

In this section we show how the results of [21] can be used to derive strong necessary conditions for a third order linear differential equation with coefficients in $C(x)$ to have a solution liouvillian over $C(x)$ (see e.g. [10, 11, 12, 18, 20, 21] for definitions). We will also derive the necessary conditions of the Kovacic algorithm using our approach, and improve them.

In practice, one is given a linear differential equation $L(y) = 0$ whose coefficients lie in $F(x)$, where F is a finitely generated extension of Q. In order to do our calculations, we need to assume that one has algorithms to perform the field operations and to factor polynomials over F. When this is the case, it is known that the same is true for the algebraic closure $\overline{F}$ of F ([7]). Although we give necessary conditions for certain behavior over $C(x)$, it is clear from our procedures, that all calculations can be done in $\overline{F}(x)$. We have not addressed the issue of finding the most efficient way to calculate in this field. In particular, we assume that all polynomials over the field F can be factored into linear factors whenever needed (for example, to consider individually each singular point of the differential equation). Reducing the need to factor (and work in algebraic extensions) would increase the efficiency of our procedures. Techniques for attacking this problem are discussed in [2, 3].

We shall continue using the philosophy of [20] and [21], that is, to distinguish 3 different cases (for second order differential equations these correspond to the first 3 cases of the Kovacic algorithm). If a third order equation $L(y) = 0$ has a liouvillian solution, then one of the following holds (cf. [20]):

1. The differential Galois group is a reducible subgroup of $SL(3, C)$ (cf. case 1 in the Kovacic algorithm [12]). In this case the computation of the liouvillian solutions of $L(y) = 0$ can be reduced by factorisation to the computation of the liouvillian solutions of some second order linear differential equation over $C(x)$ (cf. [21], Section 2).

2. The differential Galois group is an imprimitive subgroup of $SL(3, C)$ (cf. case 2 in the Kovacic algorithm). In this case all solutions of the third order equation $L(y) = 0$ are Liouvillian, and $L(y) = 0$ has a fundamental system of solutions $\{y_1, y_2, y_3\}$ whose logarithmic derivatives y_i'/y_i are algebraic of degree 3 over $C(x)$ (cf. [21], Section 3).

3. The differential Galois group is a finite primitive subgroup of $SL(3, C)$ (cf. case 3 in the Kovacic algorithm). In this case all solutions of $L(y) = 0$ are algebraic over $C(x)$ (cf. [21], Section 4).

If none of the above cases holds, then the differential Galois group is an infinite primitive subgroup of $SL(3, C)$ and $L(y) = 0$ has no liouvillian solutions. Case 1 must be tested first, since this case is assumed to not hold in the other cases. (Note that the above classification can be used for differential equations of arbitrary order.)

In the following, we will assume that the differential Galois group of $L(y) = 0$ is unimodular (i.e. $\mathcal{G}(L) \subseteq SL(3, C)$), or equivalently (cf. [10], Ch. VI, Sec. 24) that $L(y) = 0$ is of the form

$$L(y) = y''' + Ay'' + By' + Cy = 0, \qquad A, B, C \in C(x),$$

where A is of the form $\sum_i \frac{n_i}{x - \alpha_i}$, with $n_i \in Z$ and $\alpha_i \in C$. If $\mathcal{G}(L)$ is not unimodular, we can always transform the equation by a change of variable to an equation where $A = 0$ (cf. [10], Ch. VI, Sec. 24). The new differential equation has liouvillian solutions if and only if the original differential equation does.

3.1 Reducible Case

In this section we give conditions on the coefficients of $L(y) = 0$ for this case to occur.

3.1.1 First step

We start by showing that for a given differential equation $L(y) = 0$ of arbitrary order n and coefficients in $C(x)$, all solutions of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i, can be computed by only finding polynomial solutions of some n-th order linear differential equations with coefficients in $C(x)$.

We first note the following necessary condition:

Lemma 3.1 (cf. [16], §178) *If a linear differential equation $L(y) = 0$ of degree n has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are the singular points of $L(y) = 0$, a_i are exponents at c_i and $P(x)$ is a polynomial, then there is an exponent e_∞ at ∞ such that the sum $(\sum_i a_i) + e_\infty$ is a non-positive integer. In particular, if $L(y) = 0$ has no singularities other than ∞ and has such a solution, then it must have a non-positive integer exponent at ∞.*

Proof: Each a_i is an exponent at the singular point c_i. Expanding $P(x)\prod_i (x-c_i)^{a_i}$ in increasing powers of x^{-1}, we see that $-deg(P) - \sum_i a_i$ is the exponent of the leading term. Therefore this must be an exponent e_∞ at infinity. ∎

Lemma 3.2 *Let $c_i, a_i \in C$ and y an m-times differentiable function. then*

$$\left(y \prod_i (x-c_i)^{a_i}\right)^{(m)} = \left(\sum_{j=0}^{m} \left(q_j(x)\, y^{(j)}\right)\right) \prod_i (x-c_i)^{a_i},$$

where $q_j(x) \in C(x)$.

Proof: The result is true for $m = 0$. For $1 \leq m$ we get:

$$\begin{aligned}
&\left(y \prod_i (x-c_i)^{a_i}\right)^{(m)} \\
&= \left(y' \prod_i (x-c_i)^{a_i}\right)^{(m-1)} + \left(y \sum_i \left(a_i (x-c_i)^{a_i-1} \prod_{j\neq i} (x-c_i)^{a_i}\right)\right)^{(m-1)} \\
&= \left(y' \prod_i (x-c_i)^{a_i}\right)^{(m-1)} + \left(\left(y \sum_i \frac{a_i}{x-c_i}\right) \prod_i (x-c_i)^{a_i}\right)^{(m-1)}
\end{aligned}$$

Now the result follows by induction by using y' and $y \sum_i \frac{a_i}{x-c_i}$ instead of y. ∎

For a given equation $L(y) = 0$ of degree n, there are only a finite number of singularities and at each singularity c_i, there are at most n possible exponents a_i. If $L(y) = 0$ has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i, then there are at most a finite number of possibilities for $\prod_i (x-c_i)^{a_i}$. For each possible term $\prod_i (x-c_i)^{a_i}$, we consider the differential equation:

$$\tilde{L}(y) \quad := \quad \frac{L\left(y \prod_i (x-c_i)^{a_i}\right)}{\prod_i (x-c_i)^{a_i}} = 0.$$

From Lemma 3.2 we get that the coefficients of $\tilde{L}(y)$ belong to $C(x)$. If $L(y) = 0$ has a solution of the form $P(x)\prod_i (x-c_i)^{a_i}$, then $\tilde{L}(y) = 0$ has a solution $P(x) \in C[x]$. We thus have to compute a basis $\{p_1(x), \cdots, p_k(x)\}$ $(k \leq n)$ of the polynomial solutions of $\tilde{L}(y) = 0$ to get a basis of the solutions of the form $P(x)\prod_i (x-c_i)^{a_i}$.

3.1.2 Necessary conditions for case 1

If the differential Galois group of a differential equation $L(y) = 0$ of order n is a reducible subgroup of $GL(n, C)$, then $L(y)$ factors as a differential operator and algorithms performing such a factorisation are known (cf. [20], Section 3.2.1). For a third order equation, this leads either to a right factor of order one or to a left factor of order one. Since a left factor of order one leads to a right factor of order one of the adjoint differential operator $L^*(y)$ (cf. [14], p. 38), testing reducibility leads to the computation of a right factor of order one of $L(y)$ or $L^*(y)$. If an equation has a right factor of order one, then it has a solution y whose logarithmic derivative $u = y'/y$ is rational. If $L(y) = 0$ is of fuchsian type, then any solution whose logarithmic derivative is rational must be of the form $P(x)\prod_i (x-c_i)^{a_i}$, where $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i (cf. [16], §178). Using Lemma 3.1 and the fact that the adjoint of a differential equation of fuchsian type is also of fuchsian type (This follows from the facts that the solutions of $L^*(y) = 0$ are contained in a Picard-Vessiot extension K associated with $L(y) = 0$ (cf. [5], p. 101, Ex. 19; [14], p. 43, Ex. 12 and p. 38) and any element of K must have the growth properties described in Lemma 2.2.2) we get:

Corollary 3.3 *Let $L(y) = 0$ be a third order differential equation which is of fuchsian type. If $L(y) = 0$ is reducible, then for either $L(y) = 0$ or $L^*(y) = 0$ at each finite singular point c_i there are exponents a_i such that for some exponent e_∞ at ∞, the sum $(\sum_i a_i) + e_\infty$ is a non-positive integer.*

We now consider the non fuchsian case. The riccati equation associated with $L(y) = y''' + Ay'' + By' + Cy = 0$ is

$$R(u) = u'' + 3uu' + Au' + u^3 + Au^2 + Bu + C = 0.$$

and the adjoint $L^*(y) = y''' - Ay'' + (B - 2A')y' + (-C + B' - A'')y$ of $L(y)$ has riccati equation:

$$R^*(u) = u'' + 3uu' - Au' + u^3 - Au^2 + (B - 2A')u + (-C + B' - A'') = 0.$$

In our computations we will usually assume that the finite singularity is at 0. For the Laurent series at 0 or ∞ we introduce the following notation:

$$\begin{aligned}
A &= \alpha x^a + \cdots \quad \text{(higher order terms)} \\
B &= \beta x^b + \cdots \quad \text{(higher order terms)} \\
C &= \gamma x^c + \cdots \quad \text{(higher order terms)} \\
A &= \alpha_\infty x^{a_\infty} + \cdots \quad \text{(lower order terms)} \\
B &= \beta_\infty x^{b_\infty} + \cdots \quad \text{(lower order terms)} \\
C &= \gamma_\infty x^{c_\infty} + \cdots \quad \text{(lower order terms)}
\end{aligned}$$

For $A = A_1/A_2$ where $A_i \in C[x]$, a_∞ denotes $deg_x(A_1) - deg_x(A_2)$.

If a Puiseux or Laurent series of a solution y of $L(y) = 0$ or a solution u of $R(u) = 0$ exists at the point 0 or ∞, we denote them by:

$$\begin{aligned}
y &= \rho x^r + \cdots \quad \text{(higher order terms)} \\
u &= \eta x^h + \cdots \quad \text{(higher order terms)} \\
y &= \rho_\infty x^{r_\infty} + \cdots \quad \text{(lower order terms)} \\
u &= \eta_\infty x^{h_\infty} + \cdots \quad \text{(lower order terms)}
\end{aligned}$$

Necessary conditions for case 1 *Let $L(y) = y''' + Ay'' + By' + Cy$ be a third order linear differential equation with coefficients in $C(x)$ such that $L(y) = 0$ and the adjoint $L^*(y) = 0$ have no solutions of the form $P(x)\prod_i (x - c_i)^{a_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and a_i are exponents of $L(y) = 0$ at c_i. If $L(y)$ is reducible over $C(x)$, then one of the following holds:*

1. *$C = 0$ and*

$$L(y) = \left(\frac{d^2}{dx^2} + A\frac{d}{dx} + B\right)\left(\left(\frac{d}{dx}\right)(y)\right)$$

2. *$C \neq 0$ and at some finite singular point of $L(y) = 0$ the coefficients A, B and C have exponents a, b and c such that one of the following holds:*
   - *(a) $A \neq 0$, $a \leq -2$, $3a \leq c$ and (if $B \neq 0$ then $2a \leq b$).*
   - *(b) $B \neq 0$, $b \leq -4$, $b \in 2Z$, $b \leq \frac{2}{3}c$ and (if $A \neq 0$ then $b \leq 2a$).*
   - *(c) $c \leq -6$, $c \in 3Z$, (if $A \neq 0$ then $c \leq 3a$) and (if $B \neq 0$ then $c \leq \frac{3}{2}b$).*

(d) $AB \neq 0$, $a < -2$, $b < -4$, $b - a < -1$, $2a < b$ and $2b - a \leq c$.

(e) $A \neq 0$, $a < -2$, $c < -6$, $c - a < -4$, $(c - a) \in 2Z$, $3a < c$ and (if $B \neq 0$ then $a + c \leq 2b$).

(f) $B \neq 0$, $c < -6$, $b < -4$, $c \leq b - 2$, $3b < 2c$ and (if $A \neq 0$ then $2b < a + c$).

3. $C \neq 0$ and for A, B and C, one of the following holds:

(a) $A \neq 0$, $a_\infty \geq 0$, $3a_\infty \geq c_\infty$ and (if $B \neq 0$ then $2a_\infty \geq b_\infty$).

(b) $B \neq 0$, $0 \leq b_\infty$, $b_\infty \in 2Z$, $b_\infty \geq \frac{2}{3}c_\infty$ and (if $A \neq 0$ then $b_\infty \geq 2a_\infty$).

(c) $0 \leq c_\infty \in 3Z$, (if $A \neq 0$ then $c_\infty \geq 3a_\infty$) and (if $B \neq 0$ then $c_\infty \geq \frac{3}{2}b_\infty$).

(d) $AB \neq 0$, $0 < a_\infty$, $a_\infty \leq b_\infty$ and $2b_\infty - a_\infty \geq c_\infty$.

(e) $A \neq 0$, $0 \leq c_\infty - a_\infty$, $c_\infty - a_\infty \in 2Z$, $0 < c_\infty < 3a_\infty$ and (if $B \neq 0$ then $a_\infty + c_\infty \geq 2b_\infty$).

(f) $B \neq 0$, $0 < b_\infty \leq c_\infty$, $3b_\infty > 2c_\infty$ and (if $A \neq 0$ then $2b_\infty > a_\infty + c_\infty$).

Example: We use the above necessary conditions to show that for the following equation (which is the second symmetric power of the Airy equation)

$$L(y) \quad = \quad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y,$$

case 1 (of a reducible group $\mathcal{G}(L) \subseteq SL(3, C)$) cannot occur. The point ∞ is the only singular point of $L(y)$, and the only exponent at ∞ is $1/2$. From Lemma 3.1 we get that $L(y) = 0$ has no solution of the form $P(x)\prod_i (x - c_i)^{a_i}$. The first condition above cannot hold, since the coefficient of y is not 0. Since $L(y) = 0$ has no finite singular point, the above second condition does not hold. Finally, for $A = 0$, $b_\infty = 1$ and $c_\infty = 0$ the third condition above does not hold. Thus, since $L(y)$ is selfadjoint, $L(y) = 0$ is an irreducible equation. ∎

Proof of the Necessary conditions for case 1: We first note that if $C = 0$, then $L(y) = 0$ has a right factor of order one of the form stated. We assume from now on that $C \neq 0$. If $L(y) = 0$ is a reducible third order differential equation, then either $L(y)$ or $L^*(y)$ has a right factor of order 1 ([20]).

1. First assume $L(y) = L_1(L_2(y))$, where $L_2(y)$ is of order 1, then $R(u)$ has a solution $u \in C(x)$.

   (a) If u has a pole of order bigger than 1 then u must have a pole of order > 1 at some singularity of $L(y) = 0$ which we assume to be 0. We have $u = \eta x^h + \ldots$ (higher order terms) and $h < -1$. Plugging u into $R(u) = 0$ we get:

$$\left(\eta h(h-1)x^{h-2} + \ldots\right) + \left(3\eta^2 h x^{2h-1} + \ldots\right) + \left(\eta\alpha h x^{a+h-1} + \ldots\right)$$
$$+ \left(\eta^3 x^{3h} + \ldots\right) + \left(\eta^2 \alpha x^{2h+a} + \ldots\right) + \left(\beta\eta x^{b+h} + \ldots\right) + (\gamma x^c + \ldots) = 0$$

Since $h < -1$ we get $h - 2 > 3h$, $2h - 1 > 3h$ and $a + h - 1 > a + 2h$. For the lowest term to cancel one of the following must hold:

i. If $3h$ is the lowest exponent, then:

A. if $3h = 2h + a$, in which case $A \neq 0$, then $a \leq -2$, $3a \leq c$ and (if $B \neq 0$ then $2a \leq b$).

B. if $3h = h + b$, in which case $B \neq 0$, then $b \leq -4$, $b \in 2Z$, $b \leq \frac{2}{3}c$ and (if $A \neq 0$ then $b \leq 2a$).

C. if $3h = c$, then $c \leq -6$, $c \in 3Z$, (if $A \neq 0$ then $c \leq 3a$) and (if $B \neq 0$ then $c \leq \frac{3}{2}b$).

ii. If $a + 2h$ is the lowest exponent but not $3h$, then:

A. if $a + 2h = h + b$, in which case $AB \neq 0$, then $a < -2$, $b < -4$, $b - a < -1$, $2a < b$ and $2b - a \leq c$.

B. if $a + 2h = c$, in which case $A \neq 0$, then $a < -2$, $c < -6$, $c - a < -4$, $(c - a) \in 2Z$, $3a < c$ and (if $B \neq 0$ then $a + c \leq 2b$).

iii. If $a + 2h$ and $3h$ are not the lowest exponents, then $B \neq 0$, $c < -6$, $b < -4$, $c \leq b - 2$, $3b < 2c$ and (if $A \neq 0$ then $2b < a + c$).

(b) If u has no pole of order bigger than 1, then u is of the form

$$u = \sum \frac{\gamma_i}{x - c_i} + Q(x),$$

where $Q(x) \in C[x]$. If $Q(x) = 0$, then $L(y) = 0$ has a solution of the form $P(x)\prod_i (x - c_i)^{\gamma_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and γ_i are exponents of $L(y) = 0$ at c_i. Since we assume that this is not the case, we get $h_\infty \geq 0$. Plugging u into $R(u) = 0$ we get an equation similar to the one in 1(a) above (the only difference is that we have added the subscript ∞).

Since $h_\infty \geq 0$ we get $h_\infty - 2 < 3h_\infty$, $2h_\infty - 1 < 3h_\infty$, $a_\infty + h_\infty - 1 < a_\infty + 2h_\infty$. For the highest term to cancel one of the following must hold:

i. If $3h_\infty$ is the highest exponent, then:

A. if $3h_\infty = a_\infty + 2h_\infty$, in which case $A \neq 0$, then $a_\infty \geq 0$, $3a_\infty \geq c_\infty$ and (if $B \neq 0$ then $2a_\infty \geq b_\infty$).

B. if $3h_\infty = b_\infty + h_\infty$, in which case $B \neq 0$, then $0 \leq b_\infty$, $b_\infty \in 2Z$, $b_\infty \geq \frac{2}{3}c_\infty$ and (if $A \neq 0$ then $b_\infty \geq 2a_\infty$).

C. if $3h_\infty = c_\infty$, then $0 \leq c_\infty \in 3Z$, (if $A \neq 0$ then $c_\infty \geq 3a_\infty$) and (if $B \neq 0$ then $c_\infty \geq \frac{3}{2}b_\infty$).

ii. If $a_\infty + 2h_\infty$ is the highest exponent but not $3h_\infty$, then:

A. if $a_\infty + 2h_\infty = b_\infty + h_\infty$, in which case $AB \neq 0$, then $0 < a_\infty$, $a_\infty \leq b_\infty$ and $2b_\infty - a_\infty \geq c_\infty$.

B. if $a_\infty + 2h_\infty = c_\infty$, in which case $A \neq 0$, then $0 \leq c_\infty - a_\infty$, $c_\infty - a_\infty \in 2Z$, $0 < c_\infty < 3a_\infty$ and (if $B \neq 0$ then $a_\infty + c_\infty \geq 2b_\infty$).

iii. If $b_\infty + h_\infty$ is the highest exponents, but not $3h_\infty$ or $a_\infty + 2h_\infty$, then $B \neq 0, 0 < b_\infty \leq c_\infty, 3b_\infty > 2c_\infty$ and (if $A \neq 0$ then $2b_\infty > a_\infty + c_\infty$).

2. Now assume $L^*(y)$ has a right factor of order one. Note that $L^*(y)$ has the same singular points as $L(y) = 0$. When one expands the coefficients appearing in $R^*(u)$ and applies the above arguments, one gets the same conditions on a, b and c (resp. a_∞, b_∞ and c_∞). ∎

3.2 Case of an imprimitive unimodular differential Galois group

If a third order linear differential equation $L(y) = 0$ with coefficients in $C(x)$ has a differential Galois group which is an imprimitive subgroup of $SL(3, C)$, then all the solutions of $L(y) = 0$ are liouvillian. In fact in this case $L(y) = 0$ has a solution z such that $u = z'/z$ is algebraic over $C(x)$ of degree at most 3 (Theorem 3.3, [21]). The minimal polynomial of u can be computed by the method described in Section 3.2 of [21]. We now derive a necessary condition for this case to hold.

Since an imprimitive subgroup of prime degree (e.g. 3) is a monomial group (cf. [24], Definition 3.2), we will now derive necessary conditions for the differential Galois group to be monomial. For third (or just prime) order equations, these are necessary conditions for the imprimitive case.

Necessary conditions for case 2 *Let $L(y) = 0$ be an irreducible linear differential equation of order n over $C(x)$ with monomial differential Galois group $\mathcal{G}(L) \subseteq SL(n, C)$. The n-th symmetric power $L^{ⓢn}(y) = 0$ of $L(y) = 0$ must have a non trivial solution of the form $P(x)\prod_i (x - c_i)^{\alpha_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are singular points of $L^{ⓢn}(y) = 0$ with exponents $\alpha_i \in \frac{1}{2}Z$. In particular, any singularity of $L^{ⓢn}(y) = 0$ must have an exponent of the form $b/2$, where $b \in Z$.*

If, in addition, $L(y) = 0$ is of fuchsian type, then at each of the m singular points a_i of $L(y) = 0$ on the Riemann Sphere there must be exponents $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}$ of $L(y) = 0$ at a_i such that

- $(\sum_{j=1}^{n} \alpha_{i,j}) \in \frac{1}{2}Z$, *and*
- $\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{i,j}\right) \in Z$ *and is non positive.*

Note that the elements $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}$ are not necessarily distinct. Furthermore, note that the above conditions imply that $L(y) = 0$ has a solution whose square is a rational function. For third order equations, the condition that $L^{ⓢ3}(y) = 0$ has a solution whose square is rational is in fact necessary and sufficient (cf. [20], Theorem 4.6).

Example: We now use the above necessary conditions to show that, for the second symmetric power of the Airy equation

$$L(y) \;=\; \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y,$$

the case 2 (of an imprimitive group $\mathcal{G}(L) \subseteq SL(3, C)$) cannot hold. Since $L(y) = 0$ is not of fuchsian type, we have to compute $L^{ⓢ3}(y)$ which gives:

$$\begin{aligned} L^{ⓢ3}(y) \;=\; & \frac{\mathrm{d}^7 y}{\mathrm{d}x^7} - 56x\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} - 140\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 784x^2\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2352x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ & -4(576x^3 - 295)\frac{\mathrm{d}y}{\mathrm{d}x} - 3456x^2 y. \end{aligned}$$

The only possible exponent of $L^{ⓢ3}(y)$ at the unique singular point ∞ is $3/2$, which does not rule out the possibility of an imprimitive group $\mathcal{G}(L) \subseteq SL(3, C)$. We thus have to test the stronger condition that if $\mathcal{G}(L) \subseteq SL(3, C)$ is an imprimitive group, then $L^{ⓢ3}(y) = 0$ must have a solution of the form $P(x)\prod_i (x - c_i)^{a_i}$, where $P(x) \in C[x]$, $c_i \neq \infty$ are singular points of $L(y) = 0$ and $a_i \in \frac{1}{2}Z$. Since ∞ is the only singular point and $3/2$ is the only exponent, Lemma 3.1 implies that $L^{ⓢ3}(y) = 0$ has no non-zero solution of this form. Thus case 2 (in which case we always have a liouvillian solution) does not hold. ∎

Proof of the Necessary conditions for case 2: The "non fuchsian" part of our necessary conditions follows from:

Theorem 3.4 (cf. Proposition 3.6, [20]) *If an irreducible linear differential equation $L(y) = 0$ of order n with coefficients in $C(x)$ has a monomial differential Galois group $\mathcal{G}(L) \subseteq SL(n, C)$, then the n-th symmetric power $L^{ⓢn}(y) = 0$ of $L(y) = 0$ has a solution which is the square root of an element of $C(x)$. At any singularity, $L^{ⓢn}(y) = 0$ must have an exponent of the form $a/2$, where $a \in Z$.*

When $L(y)$ is fuchsian we can do better. If for some solutions y_i of $L(y) = 0$ we have $(y_1 y_2 \cdots y_n)^2$ is a rational function, then $(y_1 y_2 \cdots y_n)$ must be of the form

$$P(x)\prod (x - a_i)^{\frac{b_i}{2}},$$

where $P(x) \in C[x]$, a_i are singular points of $L(y) = 0$ and $\frac{b_i}{2} = e_i$ with $b_i \in Z$ is an exponent of $L^{ⓢn}(y) = 0$ at a_i (the apparent singularities may also contribute to $P(x)$). Thus, for some exponent e_∞ of $L^{ⓢn}(y) = 0$ at ∞ we have that $-deg(P) - \sum_i e_i = e_\infty$. Thus $e_\infty + \sum_i e_i$ is a non-positive integer. Using Lemma 2.2 we can express the exponents of $L^{ⓢn}(y) = 0$ at a_i in terms of the exponents α_k of $L(y) = 0$ at a_i. We get that there exists non negative integers $m_1, \ldots, m_n$ with $\sum_{j=1}^{n} m_j = n$ such that

$$\left(\sum_{j=1}^{n} m_j \alpha_{\infty,j}\right) + t_\infty + \sum_i \left(\left(\sum_{j=1}^{n} m_j \alpha_{i,j}\right) + t_i\right)$$

(where t_j are positive integer) is non positive and in Z. In particular

$$\left(\sum_{j=1}^{n} m_j \alpha_{\infty,j}\right) + \sum_i \left(\sum_{j=1}^{n} m_j \alpha_{i,j}\right),$$

is non positive and in Z. Since $e_i = \sum_{j=1}^{n} m_j \alpha_{i,j} + t_i$ and $e_i = \frac{b_i}{2}$ with $b_i \in Z$, we get that $\sum_{j=1}^{n} m_j \alpha_{i,j} \in \frac{1}{2} Z$. This proves the fuchsian part of the Theorem. ∎

For a second order differential equation $L(y) = y'' - ry$ $(r \in C(x))$, one gets the condition of the Kovacic algorithm by looking at the riccati equation of the third order linear differential equation $L^{ⓢ 2}(y) = 0$ which is (cf. [12], p. 10 and [6]):

$$\theta'' + 3\theta'\theta + \theta^3 \quad = \quad 4r\theta + 2r'$$

For a third order differential equation $L(y) = 0$ the order of $L^{ⓢ 3}(y)$, which is less than or equal to 10, is not known in advance (from [20], Lemma 3.5 we get that the order can be 7, 9 or 10). This makes a general discussion as in [12] difficult.

3.3 Case of a finite primitive unimodular differential Galois group

There are, up to isomorphism, only 8 primitive finite subgroups of $SL(3, C)$. Following [20] section 2.2, we denote them $A_6^{SL_3}$, A_5, $A_5 \times C_3$, G_{168}, $G_{168} \times C_3$, $H_{216}^{SL_3}$, $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$. We note that the last 3 groups are solvable, G_{168} is the simple group of 168 elements and $A_6^{SL_3}$ is a central extension of A_6. By Jordan's Theorem, such a finite list exists for any degree.

The order of a one dimensional character ζ is the smallest integer i, such that $(\zeta)^i$ is the trivial character. If G is a group and V a G-module, then we denote the m^{th} symmetric power of V, which is also a G-module, by $\mathcal{S}^m(V)$ (c.f. [13], p. 586).

Our necessary conditions in this case are based on the following Theorem:

Theorem 3.5 *Let $L(y) = 0$ be a differential equation of degree n whose differential Galois group is a finite subgroup G of $SL(n, C)$. We denote by V the solution space of $L(y) = 0$. If $\mathcal{S}^m(V)$ has a $G-$summand of dimension 1 whose character has order i, then, at each point c including ∞, there there exists positive integers $m_1, \ldots, m_n$, with $\sum_{j=1}^{n} m_j = m$, such that $i(\sum_{j=0}^{n} m_j e_j) \in Z$, where $e_1, \ldots, e_n$ are the exponents of $L(y) = 0$ at c.*

Proof: There is a natural map Φ_m of $\mathcal{S}^m(V)$ into the Picard-Vessiot extension for $L(y) = 0$ given by sending $z_1 ⓢ \ldots ⓢ z_m$ to $z_1 \cdot \ldots \cdot z_m$. Since a finite group is completely reducible, the solution space of $L^{ⓢ m}(y) = 0$ is G-isomorphic to a direct summand of $\mathcal{S}^m(V)$ (cf. [20] Lemma 3.5). Let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of $L(y) = 0$. Since G is a finite group and $\mathcal{S}^m(V)$ has a one dimensional summand, there is a homogeneous polynomial $P(Y_1, \ldots, Y_n)$ of degree m whose (possibly trivial) image $P(y_1,\dots,y_n)$ under Φ_m is a semi-invariant of G and such that $P(y_1,\dots,y_n)^i$ is rational (cf. [21], Proposition 1.6).

If $P(y_1,\dots,y_n)\neq 0$, then the exponents of $P(y_1,\dots,y_n)$ at any point c are of the form $(\sum_{j=0}^{n} m_j e_j)+h$, where h and m_j are positive integers and $\sum_{j=1}^{n} m_j = m$ (cf. Lemma 2.2.4). Since $(P(y_1,\dots,y_n))^i$ is rational, we have $i\left((\sum_{j=0}^{n} m_j e_j)+h\right)\in Z$, thus $i(\sum_{j=0}^{n} m_j e_j)\in Z$.

Now assume $P(y_1,\dots,y_n)=0$ and let ζ be the character of the one dimensional summand of $\mathcal{S}^m(V)$. In this case we will show that there is a linear differential equation $\tilde{L}(y)=0$ with the following properties:

i. $L(y)$ and $\tilde{L}(y)$ have the same differential Galois group G,

ii. The solution space $\tilde{V}$ of $\tilde{L}(y)=0$ is G-isomorphic to V,

iii. For some fundamental set of solutions $\{\tilde{y}_1,\dots,\tilde{y}_n\}$ of $\tilde{L}(y)=0$, $P(\tilde{y}_1,\dots,\tilde{y}_n)\neq 0$ and $P(\tilde{y}_1,\dots,\tilde{y}_n)$ generates a one dimensional invariant subspace of the solution space of $\tilde{L}^{\circledS m}(y)=0$ corresponding to the character ζ,

iv. For any point p on the Riemann Sphere, the sets of exponents $\mathcal{E}_p$ of $L(y)$ and $\tilde{\mathcal{E}}_p$ of $\tilde{L}(y)$ are the same mod Z

Assuming we have constructed $\tilde{L}(y)$, we can finish the proof. We apply the argument of the preceeding paragraph to the equation $\tilde{L}(y)$ and conclude that, at each point c there exist positive integers $m_1,\dots,m_n$, with $\sum_{j=1}^{n} m_j = m$, such that $i(\sum_{j=0}^{n} m_j\tilde{e}_j)\in Z$, where $\tilde{e}_1,\dots,\tilde{e}_n$ are the exponents of $\tilde{L}(y)=0$ at c. Since the exponents of $\tilde{L}(y)$ and $L(y)$ coincide mod Z, we achieve the conclusion of the theorem.

To construct $\tilde{L}(y)$ we proceed as follows. Let $R_1,\dots,R_n$ be new variables and consider the substitution

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_n' & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$$

Under this substitution, the polynomial $P(Y_1,\dots,Y_n)$ becomes $\hat{P}(R_1,\dots,R_n)$, a polynomial with coefficients in K, the Picard-Vessiot extension associated with $L(y)$. Since the matrix $\left(y_i^{(j)}\right)$ is nonsingular, we see that $\hat{P}(R_1,\dots,R_n)$ is a nonzero polynomial. Let $\hat{Q}(R_1,\dots,R_n)$ be the image of the differential polynomial $det(Wr(Y_1,\dots,Y_n))$ (where Wr is the wronskian matrix) under this substitution. One can select the $r_i\in C(x)$ so that $\hat{P}(r_1,\dots,r_n)\cdot\hat{Q}(r_1,\dots,r_n)\neq 0$ (see [15], p.

35). Let

$$\begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_n' & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

We have that $\tilde{y}_1, \ldots, \tilde{y}_n$ belong to K and, by construction, these are linearly independent over C. Applying elements of the Galois group G to $\tilde{y}_1, \ldots, \tilde{y}_n$, one sees that the C-span $\tilde{V}$ of these elements is left invariant by G. Furthermore, these elements form a fundamental set of solutions of the linear differential equation $\tilde{L}(y) = det(Wr(y, \tilde{y}_1, \ldots, \tilde{y}_n))/det(Wr(\tilde{y}_1, \ldots, \tilde{y}_n))$ whose coefficients are left fixed by G and so lie in $C(x)$.

Note that the matrix $S = (Wr(y_1, \ldots, y_n)) \cdot (Wr(\tilde{y}_1, \ldots, \tilde{y}_n))^{-1}$ is left invariant by the differential Galois group and so has entries in $C(x)$. Comparing first rows of $(Wr(y_1, \ldots, y_n))$ and $S \cdot Wr(\tilde{y}_1, \ldots, \tilde{y}_n)$ we have that $y_i = \sum_{j=0}^{n-1} s_j \tilde{y}_i^{(j)}$ with $s_j \in C(x)$. The map $y \mapsto \sum_{j=0}^{n-1} r_j y_i^{(j)}$ (resp. $\tilde{y} \mapsto \sum_{j=0}^{n-1} s_j \tilde{y}_i^{(j)}$) will take solutions of $L(y) = 0$ to solutions of $\tilde{L}(y) = 0$ (resp. solutions of $\tilde{L}(y) = 0$ to solutions of $L(y) = 0$). Therefore, the extension $C(x) < \tilde{y}_1, \ldots, \tilde{y}_n >$ coincides with K. In particular, $L(y)$ and $\tilde{L}(y)$ have the same differential Galois group, so property i. holds. To verify that property ii. holds, one observes that the matrix of any $\sigma \in G$ is the same with respect to the bases $\{y_1, \ldots, y_n\}$ and $\{\tilde{y}_1, \ldots, \tilde{y}_n\}$. Property iii. follows by construction. To verify property iv., let $p = 0$ and let $y = x^\rho \sum_n a_n x^n$ be a solution of $L(y) = 0$. We then have that $\tilde{y} = \sum_{j=0}^{n-1} r_j y^{(j)}$ will be a solution of $\tilde{L}(y)$ and furthermore it will be of the form $\tilde{y} = x^\rho \sum_n b_n x^n$ (the r_i are at worst meromorphic at 0). Therefore, if ρ is an exponent of $L(y) = 0$ at 0, then $\rho + N$ will be an exponent of $\tilde{L}(y)$ at 0. Therefore elements of $\mathcal{E}_0$ are congruent to elements of $\tilde{\mathcal{E}}_0$ mod Z. Reversing the roles of $L(y)$ and $\tilde{L}(y)$ and arguing simillarly, one establishes the last property. ∎

We now state the necessary conditions for third order equations in this case:

Necessary conditions for case 3 *Let $L(y)$ be an irreducible third order linear differential equation with Galois group a finite primitive group $\mathcal{G}(L) \subset SL(3, C)$. Then $L(y) = 0$ must be a differential equation of fuchsian type having 3 distinct rational exponents at any singularity. Furthermore, if c is any singularity of $L(y) = 0$ and e_1, e_2, e_3 are the exponents at c, then one of the following holds:*

1. *If $\mathcal{G}(L) \cong A_6^{SL_3}$, then*
   - *(a) each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in Z$ and $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$ and*
   - *(b) there exists non negative integers n_1, n_2, n_3, such that $\sum_{i=1}^{3} n_i = 6$ and $\sum_{i=1}^{3} n_i e_i \in Z$.*

2. *If* $\mathcal{G}(L) \cong A_5$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 5\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 2$ *and* $\sum_{i=1}^{3} n_i e_i \in Z$.

3. *If* $\mathcal{G}(L) \cong A_5 \times C_3$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 5, 6, 15\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 2$ *and* $3(\sum_{i=1}^{3} n_i e_i) \in Z$.

4. *If* $\mathcal{G}(L) \cong G_{168}$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 7\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 4$ *and* $\sum_{i=1}^{3} n_i e_i \in Z$.

5. $\mathcal{G}(L) \cong G_{168} \times C_3$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 7, 12, 21\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 4$ *and* $3(\sum_{i=1}^{3} n_i e_i) \in Z$.

6. $\mathcal{G}(L) \cong H_{216}^{SL_3}$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 9, 12, 18\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 6$ *and* $3(\sum_{i=1}^{3} n_i e_i) \in Z$.

7. $\mathcal{G}(L) \cong H_{72}^{SL_3}$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 12\}$, *and*

   (b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 6$ *and* $\sum_{i=1}^{3} n_i e_i \in Z$.

8. $\mathcal{G}(L) \cong F_{36}^{SL_3}$, *then*

   (a) *each* e_i *is of the form* a_i/m_i, *where* $(a_i, m_i) = 1$, $a_i, m_i \in Z$ *and* $lcm(m_1, m_2, m_3) \in \{1, 2, 3, 4, 6, 12\}$, *and*

(b) *there exists non negative integers* n_1, n_2, n_3, *such that* $\sum_{i=1}^{3} n_i = 3$ *and* $4(\sum_{i=1}^{3} n_i e_i) \in Z$.

In order to exclude this case it is enough to show that none of the conditions on the exponents is satisfied.

Example: The second symmetric power of the Airy equation

$$L(y) \;=\; \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y$$

has an unimodular differential Galois group and is not of fuchsian type. Thus none of the necessary conditions 1, 2 and 3 hold and case 1, 2 and 3 can not occur for $L(y) = 0$. This proves that $L(y) = 0$ has no liouvillian solutions. ∎

Proof of Necessary conditions for case 3: The conditions that $L(y) = 0$ is of fuchsian type and that all exponents are rational are a consequence of the result that any solution of $L(y) = 0$ has to be algebraic in this case (cf. Lemma 2.2.3).

From Lemma 2.3 we get that all exponents are of the form a/m, where m is the order of an element of $\mathcal{G}(L)$. The possible set given in the necessary conditions is just the set of orders of elements of $\mathcal{G}(L)$.

For each irreducible three dimensional character χ of a finite primitive group G, using the recurence relation given in [20] Section 2.3 we can compute the character χ_m of $\mathcal{S}^m(V)$. The following are the decompositions of the χ_m into irreducible characters of G in which a one dimensional summand appears for the first time (For example, in 7., χ_6 is the sum of the trivial character 1, 3 non trivial different one dimensional characters $\zeta_{1,1}$, $\zeta_{1,2}$ and $\zeta_{1,3}$, and 3 times the same irreducible eight dimensional character ζ_8):

1. For $\mathcal{G}(L) \cong A_6^{SL_3}$ we get $\chi_6 = 1 + \zeta_{5,1} + \zeta_{5,2} + \zeta_8 + \zeta_9$.
2. For $\mathcal{G}(L) \cong A_5$ we get $\chi_2 = 1 + \zeta_5$
3. For $\mathcal{G}(L) \cong A_5 \times C_3$ we get $\chi_2 = \zeta_1 + \zeta_5$, and ζ_1 is of order 3.
4. For $\mathcal{G}(L) \cong G_{168}$ we get $\chi_4 = 1 + \zeta_6 + \zeta_8$
5. For $\mathcal{G}(L) \cong G_{168} \times C_3$ we get $\chi_4 = \zeta_1 + \zeta_6 + \zeta_8$ and ζ_1 is of order 3.
6. For $\mathcal{G}(L) \cong H_{216}^{SL_3}$ we get $\chi_6 = \zeta_1 + \zeta_3 + \zeta_{8,1} + 2\zeta_{8,2}$, and ζ_1 is of order 3.
7. For $\mathcal{G}(L) \cong H_{72}^{SL_3}$ we get $\chi_6 = 1 + \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3} + 3\zeta_8$, and $\zeta_{1,i}$ is of order 2.
8. For $\mathcal{G}(L) \cong F_{36}^{SL_3}$ we get $\chi_3 = \zeta_{1,1} + \zeta_{1,2} + \zeta_{4,1} + \zeta_{4,2}$, and $\zeta_{1,i}$ is of order 4.

Theorem 3.5 now gives the result. ∎

We now show how our approach can be used to get and to improve the necessary conditions of the Kovacic algorithm in this case (case 3 in [12]). The finite primitive subgroups of $SL(2,C)$ are the tetrahedral, octahedral and icosahedral group, which are denoted resp. $A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$ (see e.g. [20] section 2.2)

Theorem 3.6 *Let $L(y)$ be an irreducible second order linear differential equation whose differential Galois group is a finite primitive group $\mathcal{G}(L) \subseteq SL(2,C)$. Then $L(y)=0$ must be a differential equation of fuchsian type having 2 distinct rational exponents at any singularity. Furthermore, if c is any singularity of $L(y)=0$ and e_1, e_2 are the exponents at c, then one of the following holds:*

1. *$\mathcal{G}(L) \cong A_5^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in Z$ and $lcm(m_1, m_2) \in \{1,2,3,4,5,6,10\}$, and*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 12$ and $n_1 e_1 + n_2 e_2 \in Z$.*
2. *$\mathcal{G}(L) \cong S_4^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in Z$ and $lcm(m_1, m_2) \in \{1,2,3,4,6,8\}$.*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 6$ and $2(n_1 e_1 + n_2 e_2) \in Z$.*
3. *$\mathcal{G}(L) \cong A_4^{SL_2}$, then*
   (a) *each e_i is of the form a_i/m_i, where $(a_i, m_i) = 1$, $a_i, m_i \in Z$ and $lcm(m_1, m_2) \in \{1,2,3,4,6\}$.*
   (b) *there exists non negative integers n_1, n_2, such that $n_1 + n_2 = 4$ and $3(n_1 e_1 + n_2 e_2) \in Z$.*

Proof: The proof is similar to the proof of the Necessary Conditions 3. For any irreducible character χ of degree 2 of $A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$, we have to find the decomposition of the character χ_m of $\mathcal{S}^m(V)$ in which a one dimensional summand appears for the first time. Using the same notation as in proof of the Necessary Conditions 3 we get:

1. For $A_4^{SL_2}$ we get $\chi_4 = \zeta_{1,1} + \zeta_{1,2} + \zeta_3$, where $\zeta_{1,i}$ is of degree 1 and order 3.
2. For $S_4^{SL_2}$ we get $\chi_6 = \zeta_1 + 2\zeta_3$, where ζ_1 is of degree 1 and order 2.
3. For $A_5^{SL_2}$ we get $\chi_{12} = 1 + \zeta_3 + \zeta_4 + \zeta_5$.

We conclude as in Necessary Conditions 3. ∎

The above improvement of the necessary conditions given in [12] (see also [6]), which impose divisibility conditions on the denominator of the exponents, can also be found in a paper of L. Fuchs (cf. [8]). We note that for the above result one does not need to compute the semi invariants, but only the order of a one dimensional character.

3.4 Solvability in terms of lower order equations

If a second order equation with unimodular differential Galois group has no non-zero liouvillian solutions, then its differential Galois group is $SL(2, C)$ (cf. [12, 20]). If a third order equation $L(y) = 0$ with unimodular differential Galois group has no non-zero liouvillian solutions, then its differential Galois group $\mathcal{G}(L)$ is either $SL(3, C)$ or is congugate to $\rho_2(SL(2, C))$ or $\rho_2(SL(2, C)) \times C_3$, where

$$\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

and C_3 is the center of $SL(3, C)$ which is cyclic of order 3 (cf. [20]; Note that $\rho_2(SL(2, C)) \cong PGL(2, C)$). If $\mathcal{G}(L)$ is not $SL(3, C)$, then $L(y) = 0$ is solvable in terms of second order equations (cf. [18]).

If λ_1 and λ_2 are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the eigenvalues of $\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $\{{\lambda_1}^2, 1, {\lambda_2}^2\}$ (since $\lambda_1 \lambda_2 = 1$). This observation yields:

Lemma 3.7 *Let $L(y) = 0$ be a fuchsian third order linear differential equation whose differential Galois group is conjugate to a subgroup of $\rho_2(SL(2, C)) \times C_3$. Then at each singular point, some exponent is of the form $\frac{n}{3}$ for some $n \in Z$.*

Proof: The monodromy matrix at any point is an element of the differential Galois group. Since the elements of C_3 commute with the elements of $\rho_2(SL(2, C))$ one sees that the eigenvalues of a monodromy matrix are of the form $\{\omega{\lambda_1}^2, \omega, \omega{\lambda_2}^2\}$, where $\lambda_1 \lambda_2 = 1$ and $\omega^3 = 1$. Since $L(y) = 0$ is fuchsian, Lemma 2.4 implies that these must be of the form $\{e^{2\pi\sqrt{-1}a_1}, e^{2\pi\sqrt{-1}a_2}, e^{2\pi\sqrt{-1}a_3}\}$. So $a_2 = \frac{n}{3}$ with $n \in Z$. ∎

In the fuchsian case, the above yields necessary conditions useful in testing solvability in terms of second order equation and in computing differential Galois groups. This will be used in the next section.

4 Differential equations with three singular points

In the previous section we derived necessary conditions on the exponents of a differential equation $L(y) = 0$ with coefficients in $C(x)$ to have a liouvillian solution. By assuming that we have only three singular points, we will be able to sharpen the necessary conditions for the finite primitive groups.

Theorem 4.1 *Let $L(y) = 0$ be a third order linear differential equation over $C(x)$ having three singular points s_1, s_2 and s_3 and whose differential Galois group $\mathcal{G}(L)$ is a finite primitive subgroup of $SL(3, C)$. Then for each possible group, all lists $[lcm_1, lcm_2, lcm_3]$ of possible least common multiples lcm_i of the denominators $d_{i,1}$, $d_{i,2}$, $d_{i,3}$ of the rational exponents $\frac{n_{i,1}}{d_{i,1}}, \frac{n_{i,2}}{d_{i,2}}, \frac{n_{i,3}}{d_{i,3}}$, where $(n_{i,j}, d_{i,j}) = 1$, at s_i can be derived.*

1. *If $\mathcal{G}(L) \cong A_6^{SL_3}$, we get:* $[2,4,15]$, $[2,5,12]$, $[2,5,15]$, $[2,12,15]$, $[2,15,15]$, $[3,3,4]$, $[3,3,5]$, $[3,3,12]$, $[3,3,15]$, $[3,4,5]$, $[3,4,15]$, $[3,5,5]$, $[3,5,12]$, $[3,5,15]$, $[3,12,15]$, $[3,15,15]$, $[4,4,4]$, $[4,4,5]$, $[4,4,15]$, $[4,5,5]$, $[4,5,6]$, $[4,5,12]$, $[4,5,15]$, $[4,6,15]$, $[4,12,12]$, $[4,12,15]$, $[4,15,15]$, $[5,5,5]$, $[5,5,6]$, $[5,5,12]$, $[5,5,15]$, $[5,6,12]$, $[5,6,15]$, $[5,12,12]$, $[5,12,15]$, $[5,15,15]$, $[6,12,15]$, $[6,15,15]$, $[12,12,12]$, $[12,12,15]$, $[12,15,15]$ *or* $[15,15,15]$.
2. *If $\mathcal{G}(L) \cong A_5$, we get:* $[2,3,5]$, $[2,5,3]$, $[2,5,5]$, $[3,3,5]$, $[3,5,5]$ *or* $[5,5,5]$.
3. *If $\mathcal{G}(L) \cong A_5 \times C_3$, we get:* $[2,3,15]$, $[2,15,15]$, $[3,3,5]$, $[3,3,15]$, $[3,5,6]$, $[3,5,15]$, $[3,6,15]$, $[3,15,15]$, $[5,6,15]$, $[5,15,15]$, $[6,15,15]$ *or* $[15,15,15]$.
4. *If $\mathcal{G}(L) \cong G_{168}$, we get:* $[2,3,7]$, $[2,4,7]$, $[2,7,7]$, $[3,3,4]$, $[3,3,7]$, $[3,4,4]$, $[3,4,7]$, $[3,7,7]$, $[4,4,4]$, $[4,4,7]$, $[4,7,7]$ *or* $[7,7,7]$.
5. *If $\mathcal{G}(L) \cong G_{168} \times C_3$, we get:* $[2,3,21]$, $[2,12,21]$, $[2,21,21]$, $[3,3,4]$, $[3,3,7]$, $[3,3,12]$, $[3,3,21]$, $[3,4,12]$, $[3,4,21]$, $[3,6,7]$, $[3,6,21]$, $[3,7,12]$, $[3,7,21]$, $[3,12,12]$, $[3,12,21]$, $[3,21,21]$, $[4,6,21]$, $[4,12,12]$, $[4,12,21]$, $[4,21,21]$, $[6,7,12]$, $[6,7,21]$, $[6,12,21]$, $[6,21,21]$, $[7,12,12]$, $[7,12,21]$, $[7,21,21]$, $[12,12,12]$, $[12,12,21]$, $[12,21,21]$ *or* $[21,21,21]$.
6. *If $\mathcal{G}(L) \cong H_{216}^{SL_3}$, we get:* $[3,3,4]$, $[3,3,12]$, $[3,3,18]$, $[3,4,9]$, $[3,4,18]$, $[3,9,12]$, $[3,9,18]$, $[3,12,18]$, $[4,9,18]$, $[4,18,18]$, $[9,12,18]$, $[12,18,18]$ *or* $[18,18,18]$.
7. *If $\mathcal{G}(L) \cong H_{72}^{SL_3}$, we get:* $[4,4,4]$, $[4,4,12]$, $[4,12,12]$ *or* $[12,12,12]$.
8. *If $\mathcal{G}(L) \cong F_{36}^{SL_3}$, we get:* $[2,4,12]$, $[2,12,12]$, $[3,4,4]$, $[3,4,12]$, $[3,12,12]$, $[4,4,6]$, $[4,6,12]$ *or* $[6,12,12]$.

Proof: We use the monodromy group of $L(y) = 0$, which is introduced in the first section. To the singular points at s_1, s_2 and s_3 correspond the matrices M_{s_1}, M_{s_2}, and M_{s_3} of the monodromy group. The product $M_{s_1}M_{s_2}M_{s_3}$ corresponds to the zero path on the punctured Riemann Sphere and thus must be the identity. Since M_{s_1}, M_{s_2} and M_{s_3} generate the monodromy group and $M_{s_1}M_{s_2} = M_{s_3}^{-1}$, we get that the group $\mathcal{G}(L)$ is generated by two elements. Using the group theory system CAYLEY (see [4]) we can compute all possible sets of two generators for each finite primitive subgroup of $SL(3,C)$ and compute their order and the order of their product. From the proof of Lemma 2.3 we get that the order of M_{s_i} is the least common multiple of the denominator of the exponents at s_i. This leads to the possibilities listed above. ∎

We note that the above possibilities often allow one to distinguish the groups A_5 from $A_5 \times C_3$ and G_{168} from $G_{168} \times C_3$.

Example: The following equation due to Hurwitz (see [9]):

$$\begin{aligned} H(y) &= x^2(x-1)^2y''' + (7x-4)x(x-1)y'' + \\ & (\frac{72}{7}(x^2-x) - \frac{20}{9}(x-1) + \frac{3}{4}x)y' + (\frac{792}{7^3}(x-1) + \frac{5}{8} + \frac{2}{63})y \end{aligned}$$

is of fuchsian type and has a unimodular differential Galois group. The exponents of $H(y) = 0$ at the regular singular points 0, 1 and ∞ are respectively $\{0, -\frac{1}{3}, -\frac{2}{3}\}$, $\{\frac{1}{2}, 0, -\frac{1}{2}\}$ and $\{\frac{11}{7}, \frac{9}{7}, \frac{8}{7}\}$.

We now use the necessary conditions to test the possible structure of the differential Galois group $\mathcal{G}(H) \subseteq SL(3,C)$ of $H(y) = 0$:

1. *Reducibility:* Since $H(y)$ is fuchsian, we can use Corollary 3.3. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $H(y) = 0$ has no right factor of order one. The adjoint $H^*(y) = 0$ of $H(y) = 0$ is:

$$\begin{aligned} & y''' + \left(\frac{-7x+4}{x(x-1)}\right)y'' + \left(\frac{\frac{170}{7}x^2 - \frac{6995}{252}x + \frac{92}{9}}{x^2(x-1)^2}\right)y' \\ & + \left(\frac{\frac{12650}{343}x^3 - \frac{1561655}{24696}x^2 + \frac{1143403}{24696}x + \frac{112}{9}}{x^3(x-1)^3}\right)y \end{aligned}$$

   The exponents of $H^*(y) = 0$ at 0, 1 and ∞ are respectively $\{\frac{8}{3}, 2, \frac{7}{3}\}$, $\{\frac{5}{2}, 2, \frac{3}{2}\}$ and $\{-\frac{22}{7}, -\frac{23}{7}, -\frac{25}{7}\}$. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $H^*(y) = 0$ has no right factor of order one. This shows, that the differential equation $H(y) = 0$ is irreducible.

2. *Imprimitivity:* We note that at ∞, the only sum of three (possibly repeated) exponents which is in $\frac{1}{2}Z$ is the sum of the three exponents $\frac{11}{7}$, $\frac{9}{7}$ and $\frac{8}{7}$ whose sum is 4. At 0 the possible sums of three exponents which are in $\frac{1}{2}Z$ are 0, -1 and -2. Since the sum of three exponents at 1 is $\geq -\frac{3}{2}$, we get that no sum

of the form prescribed in the Necessary Conditions 2 will be a non positive element of Z. Thus the differential Galois group $\mathcal{G}(H)$ of this equation can not be an imprimitive group.

3. *Finite primitive groups:* Since the list of least common multiples of the denominators of the exponents at the singularities is $[2,3,7]$, we get from the above theorem, that if $\mathcal{G}(H)$ is a finite primitive group, then $\mathcal{G}(H)$ is isomorphic to G_{168} (note that the Necessary Conditions 3 would lead to the two possibilities $\mathcal{G}(H) \cong G_{168}$ and $\mathcal{G}(H) \cong G_{168} \times C_3$).

4. *Infinite primitive groups:* From Lemma 3.7 it follows that the differential Galois group of $H(y) = 0$ cannot be a subgroup of $\rho_2(SL(2,C)) \times C_3$, since at ∞ no exponent is one third of an integer.

Therefore our necessary conditions show that the group $\mathcal{G}(H)$ is isomorphic to either G_{168} or $SL(3,C)$. In [22] we use the results of [20] and the fact that $H^{\circledS 4}(y) = 0$ is only of order 14 to deduce that $\mathcal{G}(H)$ is not isomorphic to $SL(3,C)$ and thus that $\mathcal{G}(H) \cong G_{168}$. ∎

Example: The general third order linear differential equation $L_p(y) = 0$ of fuchsian type having respectively exponents $\{0, \frac{1}{6}, \frac{5}{6}\}$, $\{\frac{1}{12}, \frac{1}{3}, \frac{7}{12}\}$ and $\{\frac{1}{12}, \frac{1}{6}, \frac{3}{4}\}$ at 0, 1 and ∞ is of the form:

$$\begin{aligned} L_p(y) &= y''' + \left(\frac{2}{x-1} + \frac{2}{x}\right) y'' + \left(\frac{\frac{13}{48}}{(x-1)^2} + \frac{\frac{43}{24}}{x-1} + \frac{\frac{5}{36}}{x^2} + \frac{\frac{-43}{24}}{x}\right) y' \\ &\quad + \left(\frac{\frac{-7}{432}}{(x-1)^3} + \frac{p + \frac{23}{864}}{(x-1)^2} + \frac{-2p - \frac{23}{864}}{x-1} + \frac{p}{x^2} + \frac{2p + \frac{23}{864}}{x}\right) y \end{aligned}$$

where p is an arbitrary parameter.

We now use the necessary conditions to test the possible structure of the differential Galois group $\mathcal{G}(L_p) \subseteq SL(3,C)$ of $L_p(y) = 0$:

1. *Reducibility:* Since $L_p(y)$ is fuchsian, we can use Corollary 3.3. Since all exponents of $L_p(y) = 0$ are non-negative and at 1 and ∞ in fact positive, no sum of the prescribed form can be a negative integer. Thus $L_p(y) = 0$ has no right factor of order one. The adjoint $L_p^*(y) = 0$ of $L_p(y) = 0$ is:

$$\begin{aligned} & y''' + \left(\frac{-2}{x-1} + \frac{-2}{x}\right) y'' + \left(\frac{\frac{205}{48}}{(x-1)^2} + \frac{\frac{43}{24}}{x-1} + \frac{\frac{149}{36}}{x^2} + \frac{\frac{-43}{24}}{x}\right) y' \\ & + \left(\frac{\frac{-1955}{432}}{(x-1)^3} + \frac{-p - \frac{1571}{864}}{(x-1)^2} + \frac{2p + \frac{23}{864}}{x-1} + \frac{\frac{-67}{18}}{x^3} + \frac{-p + \frac{43}{24}}{x^2} + \frac{-2p - \frac{23}{864}}{x}\right) y \end{aligned}$$

The exponents of $L_p^*(y) = 0$ at 0, 1 and ∞ are respectively $\{\frac{11}{6}, 2, \frac{7}{6}\}$, $\{\frac{5}{3}, \frac{17}{12}, \frac{23}{12}\}$ and $\{\frac{-25}{12}, \frac{-11}{4}, \frac{-13}{6}\}$. The sum of three exponents corresponding to three different singular points is never a non positive integer. Thus $L_p^*(y) = 0$ has no right factor of order one. This shows, that for any value of p the differential equation $L_p(y) = 0$ is irreducible.

2. *Imprimitivity:* According to the the Necessary Conditions 2 the differential Galois group $\mathcal{G}(L_p)$ of this equation can not be an imprimitive group, since, because all exponents are non-negative and at 1 positive, the sum of the (possibly) repeated sum of three exponents at each singular point will never be non positive.

3. *Finite primitive groups:* Since the list of least common multiples of the denominators of the exponents at the singularities is $[6, 12, 12]$, we get from the above theorem, that if $\mathcal{G}(L_p)$ is a finite primitive group, then $\mathcal{G}(L_p)$ is isomorphic to $F_{36}^{SL_3}$ (note that the Necessary Conditions 3 would only exclude the cases $\mathcal{G}(L_p) \cong A_5$, $\mathcal{G}(L_p) \cong A_5 \times C_3$ and $\mathcal{G}(L_p) \cong G_{168}$).

4. *Infinite primitive groups:* From Lemma 3.7 it follows that the the differential Galois group of $L_p(y) = 0$ cannot be a subgroup of $\rho_2(SL(2, C)) \times C_3$, since at ∞ no exponent is one third of an integer.

Therefore our necessary conditions show that for any p, the group $\mathcal{G}(L_p)$ is isomorphic to either $F_{36}^{SL_3}$ or $SL(3, C)$. ∎

For second order differential equations with three singular points, the method of the above Theorem always leads to at most one possible finite primite group, as the conditions found are exclusive:

Theorem 4.2 *Let $L(y) = 0$ be a second order linear differential equation over $C(x)$ having three singular points s_1, s_2 and s_3 and whose differential Galois group $\mathcal{G}(L)$ is a finite primitive subgroup of $SL(3, C)$. Then for each possible group, all lists $[lcm_1, lcm_2, lcm_3]$ of possible least common multiples lcm_i of the denominators $d_{i,1}, d_{i,2}$ of the rational exponents $\frac{n_{i,1}}{d_{i,1}}, \frac{n_{i,2}}{d_{i,2}}$, where $(n_{i,j}, d_{i,j}) = 1$, at s_i can be derived.*

1. *If $\mathcal{G}(L) \cong A_5^{SL_2}$, we get:* $[3, 3, 10]$, $[3, 4, 5]$, $[3, 4, 10]$, $[3, 5, 5]$, $[3, 5, 6]$, $[3, 5, 10]$, $[3, 10, 10]$, $[4, 5, 5]$, $[4, 5, 6]$, $[4, 5, 10]$, $[4, 6, 10]$, $[4, 10, 10]$, $[5, 5, 6]$, $[5, 5, 10]$, $[5, 6, 10]$, $[6, 6, 10]$, $[6, 10, 10]$ *or* $[10, 10, 10]$.

2. *If $\mathcal{G}(L) \cong S_4^{SL_2}$, we get:* $[3, 4, 8]$, $[3, 8, 8]$, $[4, 6, 8]$ *or* $[6, 8, 8]$.

3. *If $\mathcal{G}(L) \cong A_4^{SL_2}$, we get:* $[3, 3, 4]$,$[3, 3, 6]$, $[3, 4, 6]$, $[4, 6, 6]$ *or* $[6, 6, 6]$.

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