A CLASSIFICATION OF DIFFERENTIAL EQUATIONS OF FUCHSIAN CLASS

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1. Introduction. In the present paper we investigate some implications of modern results about linear groups, that is, subgroups of the general linear group, $GL(n, \mathbf{C})$, for the theory of the n^{th} order homogeneous linear differential equations of Fuchsian class. For $n > 2$, the link between these subjects was forged by Poincaré ([10], [11]). In particular, he introduced the notion of a *Zetafuchsian system of solutions* associated with a pair, (r, m), of representations of the fundamental group of the domain, Z, formed by ordinary points of the equation. Here, r is supposed to be a faithful representation whose image lies in the group of fractional linear transformations $PSL(2, \mathbf{R})$ and acts discontinuously on the upper half plane. The representation m, which is *not* required to be faithful, lies in $GL(n, \mathbf{C})$ and is called the *monodromy group* of the equation. (In modern parlance, a Zetafuchsian system is called a *relatively automorphic function for a flat factor of automorphy*; see, [4], page 15). Hopefully, the algebraic structure of linear groups has interesting ramifications for equations of Fuchsian class. Here, we concentrate our efforts on a classification of these equations in light of a celebrated theorem of this nature due to Tits, [17].

It transpires that the equations of Fuchsian class can be partitioned into two classes. First, we have those that are *solvable by quadratures* (In the present paper we use this classical, but imprecise, term to mean that *all* the solutions of the equation are *Liouvillian functions* as defined in [14], page 665, or [5], page 39). A description of the remaining Fuchsian equations is more complicated. Roughly speaking (see Theorem 2 for the precise statement), such an equation may be characterized by the fact that, upon the introduction of a suitable transcendental change of the independent variable, it is transformed into an equation whose solutions all belong to one of the three families of functions defined by the functional equations:

(i) $\mathbf{w}(v + 2\pi i) = T\mathbf{w}(v)$

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or

$$\text{(ii)}\quad \begin{aligned} \mathbf{w}(\nu + 1) &= S\mathbf{w}(\nu) \\ \mathbf{w}(\nu + \tau) &= S^m T S^{-m}\mathbf{w}(\nu) \end{aligned}$$

or

$$\text{(iii)}\quad \begin{aligned} \mathbf{w}(\nu + 2) &= S\mathbf{w}(\nu) \\ \mathbf{w}(1/(2\nu + 1)) &= T\mathbf{w}(\nu). \end{aligned}$$

In each case, S and T generate a free subgroup of $\mathrm{GL}(n, \mathbf{C})$ and $\mathbf{w}(\nu)$ denotes a column vector whose entries form a fundamental system of solutions of the equation. The domains of the variable ν corresponding to cases (i) and (ii) are illustrated in Figures 3 and 4 respectively. In case (iii), the domain of ν is the upper half plane.

It would be interesting to be able to decide, say from its coefficients, to which class a given equation, $L(w) = 0$, belongs. In this connection, we first recall that there is an algorithm, [14], which determines if all the solutions of $L(w) = 0$ are Liouvillian when the coefficients belong to a finite algebraic extension of $\mathbf{Q}(z)$, where $\mathbf{Q}$ denotes the field of rational numbers.

Next, we note that it turns out that (iii) occurs if and only if the original equation, $L(w) = 0$, has three singularities and m is a faithful representation. Thus, if $L(w) = 0$ is a second order equation, then it is a hypergeometric equation whose monodromy group is free of rank two. For example, the equation

$$(*) \qquad z(1 - z)w'' + [c - (a + b + 1)z)w' - abw = 0$$

satisfies these conditions when $a = b = 1/2$ and $c = 1$. Now, suppose that the order of $L(w) = 0$ exceeds two. In view of the ubiquity of the hypergeometric equation, we may ask whether the solutions of $L(w) = 0$ can be expressed as combinations of hypergeometric functions. For example, if w_1 and w_2 are a basis for the solutions to (*), then w_1^2, $w_1 w_2$ and w_2^2 span the solution space of a third order equation of Fuchsian class. More complicated examples may involve combinations of hypergeometric functions defined by distinct choices for the parameters a, b, c in (*). Thus, we are led to ask:

When can an n^{th} order differential equation of Fuchsian class be solved by combinations of solutions to lower order equations of Fuchsian class?

This question, which can be formulated precisely in the context of differential Galois theory, is the subject of the authors' papers [15] and [16]. Here, we only mention two results of those investigations. First, for any integer $n > 2$, there is an n^{th} order equation of Fuchsian class with three singularities whose solutions cannot be expressed as combinations of solutions to second order equations of Fuchsian class. Second, in [15] an algorithm is presented which enables one to decide if a given third order equation of Fuchsian class can be solved by solutions to second order equations of Fuchsian class.

Finally, we note that, for the sake of simplicity, we have followed the custom (see, Poole, [12]) of discussing equations of Fuchsian class defined on the Riemann sphere. However, it is in fact the case that *our results are valid for equations of Fuchsian class which are defined on any compact Riemann surface.*

2. A Classification of Differential Equations of Fuchsian Type.

Let

$$L(w) = a_0(z)w^{(n)} + \cdots + a_n(z)w = 0$$

be a differential equation of Fuchsian type on the Riemann sphere $\hat{\mathbf{C}}$. Thus, the coefficients $a_j(z)$ are rational functions and the singularities $z = b_1, \ldots, b_{t+1}$ of the equation are all regular singular points. Setting $Z = \hat{\mathbf{C}} - \{b_1, \ldots, b_{t+1}\}$ and selecting a base point $z = a$ in Z, we obtain a fundamental group $\pi_1(Z, a)$ which is free of rank t. We shall suppose that $t \geq 2$ and that $b_{t+1} = \infty$. Thus, we may apply the uniformization theorem [13] to identify the universal covering of Z with the upper half, $\mathfrak{h}$, of the ζ-plane. Denoting the covering projection by $z = z(\zeta)$ and selecting a base point $\zeta = \alpha$ such that $a = \zeta(\alpha)$, we obtain a faithful representation of $\pi_1(Z, a)$ as a Fuchsian group of the first kind, Γ, acting on $\mathfrak{h}$ so that $\mathfrak{h}/\Gamma$ is conformally equivalent to Z.

Now, let $\mathbf{w}(z - a)$ be a column vector whose entries, $w_j(z - a), j = 1, \ldots, n$, form a basis for the space of solutions of $L(w) = 0$ in a neighborhood of $z = a$. Analytic continuation of $\mathbf{w}(z - a)$ to the end point $z = b$ of a path, σ, in Z yields a vector, $\mathbf{w}(z - b; \sigma)$, whose components form a basis for the solutions to $L(w) = 0$ in a neighborhood of $z = b$. It follows from the monodromy theorem that $\mathbf{w}(z - b; \sigma)$ only depends on the homotopy class, $[\sigma]$, of σ. Thus, analytic continuation along a loop in Z representing an element, γ, of Γ yields a column vector $\mathbf{w}(z - a; \gamma)$ of the form $m(\gamma)\mathbf{w}(z - a)$, where $m(\gamma)$ belongs to $\mathrm{GL}(n, \mathbf{C})$, the group of invertible complex matrices of degree n. Of course, $m(1)$ is the identity, so we shall continue to write $\mathbf{w}(z - a)$ instead of $\mathbf{w}(z - a; 1)$. Now, each path in Z beginning at $z = a$ has a unique lift to a path in $\mathfrak{h}$ beginning at $\zeta = \alpha$. Thus, the holomorphic vector defined in a neighborhood of $\zeta = \alpha$ by $\mathbf{w}(\zeta - \alpha) = \mathbf{w}(z(\zeta) - a)$ may be continued analytically to yield a holomorphic vector, $\mathbf{w}(\zeta)$, on $\mathfrak{h}$ such that

$$(*) \qquad \mathbf{w}(\zeta \cdot z) = m(\gamma)\mathbf{w}(\zeta), \qquad \gamma \in \Gamma,\ \zeta \in \mathfrak{h}.$$

Moreover, since Γ is of the first kind, the real axis is the natural boundary of $\mathbf{w}(\zeta)$.

The correspondence $\gamma \to m\ (\gamma)$ affords a representation of Γ in $\mathrm{GL}(n, \mathbf{C})$ whose image and kernel will be denoted by M and K respectively. We call M the *monodromy group* of $L(w) = 0$; but it is only determined up to conjugacy in $\mathrm{GL}(n, \mathbf{C})$ because choices were made in the selection of the base points $z = a$, $\zeta = \alpha$ and the initial basis of the solution space at $z = a$. Poincaré, [11], referred to vectors satisfying (*) as *Zetafuchsian systems* of functions with respect to the pair of groups (Γ, M).

Since Γ is finitely generated, M is also a finitely generated group and we may apply the following theorem of Tits, [17], to it.

THEOREM (Tits). *Suppose that M is a finitely generated subgroup of $\mathrm{GL}(n, k)$, k an algebraically closed field of characteristic zero. Then, either*

(I) *M contains a solvable subgroup, S, of finite index*

or

(II) *M contains a free subgroup, L, of rank two.*

Now, suppose that Case I applies to our monodromy group M. Then, we have the following:

PROPOSITION 1. *Suppose the monodromy group, M, of $L(w) = 0$ contains a solvable subgroup S of finite index. Then $L(w) = 0$ is solvable by quadratures.*

Proof. According to one of the main results in differential Galois theory (see, [5], Theorem 5.11), our conclusion will be established if we prove that the differential Galois group of the Picard-Vessiot extension $\mathbf{C}\langle z, w_1, \ldots, w_n\rangle$ of $\mathbf{C}(z)$ defined by $L(w) = 0$ has a solvable connected component of the identity. Proposition 3 of [18] asserts that the differential Galois group in question is the closure, $\bar{M}$, of M in $\mathrm{GL}(n, \mathbf{C})$ with respect to the Zariski topology. Now, the index $|\bar{M}:\bar{S}|$ is at most $|M:S|$, so it is finite. Therefore, $\bar{S}$ contains $\bar{M}^\circ$, the connected component of the identity in $\bar{M}$. Moreover, $\bar{S}$ is solvable because it is the closure of the solvable group S; so $\bar{M}^\circ$ is also solvable. This completes our proof.

Next, suppose that Case II applies to our monodromy group M and that $L \leq M$ is a free subgroup of rank two. Applying the isomorphism theorem, we may write $L \cong \Lambda/K$, where Λ is a subgroup of Γ. Because Λ is an extension of K by a free group, it is a *splitting extension* (see [6], *2*, p. 149). In other words, Λ contains a subgroup Δ such that $\Lambda = \Delta K$, $\Delta \cap K = 1$ and Δ is free of rank two. According to Schreier's formula ([6], *2*, p. 36),

$$\operatorname{rank}(\Delta) = |\Gamma:\Delta|(\operatorname{rank}(\Gamma) - 1) + 1,$$

provided all the terms are finite. By hypothesis, the rank of Γ is at least two, so $|\Gamma:\Delta|$ is one or infinity. In the former case $\Gamma = \Delta$ and M is free of rank two.

Now, suppose that Γ coincides with Δ, so $\mathfrak{h}/\Gamma$ is the Riemann sphere with three points removed. Applying a preliminary fractional linear transformation, we may suppose that $L(w) = 0$ has singularities at $z = 0$, 1 and ∞; then $\Gamma = \Gamma(2)$, the principal congruence subgroup of level two. Recall that $\Gamma(2)$ is the subgroup of index six in the modular group, $\mathrm{PSL}(2, \mathbf{Z})$, which is freely generated by the transformations $\zeta \to \zeta + 2$ and $\zeta \to 1/(2\zeta + 1)$ (see, for example, [7], p. 112). Denoting the corresponding elements of the monodromy group M by S and T, we obtain:

Proposition 2. *Suppose that the monodromy group, M, of $L(w) = 0$ is free of rank two. Then the solutions of $L(w) = 0$ are Zetafuchsian systems which satisfy the functional equations*

$$\mathbf{w}(\zeta + 2) = S\mathbf{w}(\zeta), \qquad \zeta \in \mathfrak{h}$$

$$\mathbf{w}\left(\frac{1}{2\zeta + 1}\right) = T\mathbf{w}(\zeta), \qquad \zeta \in \mathfrak{h}$$

where S and T are complex matrices of degree n which freely generate the group M.

Finally, suppose that Δ, the free group of rank two, has infinite index in Γ. Then we may apply the following Lemma:

LEMMA. *Let Γ be a finitely generated Fuchsian group of the first kind and let $\Delta \leq \Gamma$ be a finitely generated subgroup of infinite index. Moreover, suppose that the universal covering of $\mathfrak{h}/\Gamma$ is conformally equivalent to $\mathfrak{h}$. Then, Δ is a Fuchsian group of the second kind.*

Proof. Since Γ acts discontinuously on $\mathfrak{h}$, so does Δ. Let $Z = \mathfrak{h}/\Gamma$ and let $U = \mathfrak{h}/\Delta$ and denote the covering projections by $z = z(\zeta)$ and $u = u(\zeta)$ respectively. The inclusion $\Delta \subset \Gamma$ induces a covering projection, $z = z(u)$, of U onto Z such that $z(\zeta) = z(u(\zeta))$. Since Γ is finitely generated, the set of points on Z above which $\mathfrak{h}$ is branched is finite (see, for example, [2], page 209). Of course, this is also true of $\mathfrak{h}$ qua branched covering of U. It follows that the set of points on Z above which U is branched is also finite.

Since Γ is finitely generated and of the first kind, we may compactify Z by "filling in the punctures" determined by the parabolic elements occurring in a finite set of generators of Γ. The complement of Z in the resulting Riemann surface, W, is a finite set of points.

Now, if we suppose that Δ is of the first kind, the reasoning utilized above also shows that U is the complement of a finite point set in a compact Riemann surface, $\tilde{W}$. Moreover, under these circumstances we shall shortly see that the covering projection of U onto Z admits an extension to a holomorphic mapping $f: \tilde{W} \to W$. Now, if $z = a$ is a point of Z above which f is not branched, then $f^{-1}(a)$ consists of $|\Gamma:\Delta|$ points, so it is an infinite set. On the other hand, $f^{-1}(a)$ must be finite because it is a discrete subset of the compact Riemann surface $\tilde{W}$. This contradiction shows that Δ is of the second kind.

In order to establish the existence of the holomorphic mapping $f: \tilde{W} \to W$, we select conformal disks with pairwise disjoint closures about each of the finitely many points of $\tilde{W}$ not in U. Each of these disks meets U in a domain conformally equivalent to the punctured unit disk $D^* = \{t \in \mathbf{C} \mid 0 < |t| < 1\}$ and the projection mapping $z = z(u)$ defines a holomorphic map of each of them into Z. Now, $z = z(u)$ can be extended across the center of each of these disks by employing the following generalization of Picard's Big Theorem due to Ohtsuka ([8], [9]):

THEOREM (Ohtsuka). *Let f be a holomorphic mapping of the punctured disk $D^* = \{t \in \mathbf{C} \mid 0 < |t| < 1\}$ to a Riemann surface Z with* *universal covering* $\mathfrak{h}$. *Then there is a mapping*, F, *of* $D = \{t \in \mathbf{C} \mid |t| < 1\}$ *such that* $F(t) = f(t)$ *for* t *in* D^* *and* $F(0)$ *is either*: (i) *an interior point of* Z, *or* (ii) *an isolated point on the ideal boundary of* Z *which has a neighborhood in* Z *conformally equivalent to* D^*.

This completes the proof of our Lemma.

We wish to remark that the Lemma can also be established without the aid of Ohtsuka's theorem if we use more of the theory of Fuchsian groups.

Returning to the groups Γ and Δ arising from our differential equation, we see that Δ is a Fuchsian group of the second kind which is free of rank two. Thus, a normal fundamental polygon for Δ must have at least one edge on the boundary of $\mathfrak{h}$, and U is obtained from a compact Riemann surface by deleting the closure of at least one conformal disk. More precisely, it follows from the classification theorem for Riemann surfaces [1] that U is conformally equivalent to either:

(i) the complement of the closure, $\bar{D}$, of a conformal disk, D, in a complex torus T

or

(ii) the unit disk with a pair of disjoint "components" removed. Each of these components may be a point or the closure of a conformal disk.

In case (ii), a famous theorem of Koebe (see, for example, [3], page 114) asserts that U is conformally equivalent to the domain $D(r, s, c) = \{u \in \mathbf{C} \mid r < |u| < 1, s < |u - c|\}$ where the constants r, s and c satisfy the inequalities $0 \leq r < c < 1$ and $0 \leq s < \mathrm{Min}(|1 - c|, |r - c|)$. It is convenient to let $D = \{u \mid |u - c| < s\}$.

We now return to the Zetafuchsian system $\mathbf{w}(\zeta)$ defined on $\mathfrak{h}$ with respect to (Γ, M) by the multi-valued holomorphic vector on Z with branches $\{\mathbf{w}(z - a; \gamma), \gamma \in \Gamma\}$ at $z = a$. Of course, $\mathbf{w}(\zeta)$ is also a Zetafuchsian system with respect to the pair (Δ, L). This system, in turn, corresponds to a multi-valued holomorphic vector on U whose branches at $u_0 = u(\alpha)$ are $\{\mathbf{w}(u - u_0, \delta) = \mathbf{w}(z(u) - a, \delta), \delta \in \Delta\}$. It follows that the substitution $z = z(u)$ allows us to view solutions of $L(w) = 0$ as multi-valued vectors on a domain, U, whose fundamental group is free of rank two and is faithfully represented by the monodromy group of these vectors. Moreover, the boundary of U is a natural boundary for the corresponding multi-valued vectors. Indeed, if such a vector admits analytic continuation to a domain properly containing U, then $\mathbf{w}(\zeta)$ admits analytic continuation across some boundary point of $\mathfrak{h}$. Of course, such behaviour is conceivable for a Zetafuchsian system involving a discontinuous group of the second kind, but it is impossible in the present situation because $\mathbf{w}(\zeta)$ is also Zetafuchsian with respect to the pair (Γ, M) and Γ is of the first kind.

Combining Propositions 1 and 2 with the discussion above we obtain:

THEOREM 1. *Suppose $L(w) = 0$ is an n^{th} order homogenous linear differential equation of Fuchsian type on the Riemann z-sphere with singularities at $z = b_1, \ldots, b_t, \infty$, $t \geq 2$. Then, either*

(i) *$L(w) = 0$ is solvable by quadratures, that is, all of its solutions are Liouvillian functions*

or

(ii) *By a suitable change of variables, $z = z(u)$, the solutions of $L(w) = 0$ may be viewed as solutions of a homogeneous linear differential equation $L^*(w) = 0$, $w = w(z(u))$, on a domain U whose fundamental group is free of rank two and faithfully represented by the monodromy group of $L^*(w) = 0$. In fact, U is either the Riemann sphere with the points $u = 0, 1, \infty$ deleted or a complex torus T with a conformal disk removed or a domain conformally equivalent to $D(r, s, c) = \{u \in \mathbf{C} \mid r < |u| < 1, s < |u - c|\}$, where $0 \leq r < c < 1$* and *$0 \leq s <$* Min$(|1 - c|, |r - c|)$*. Moreover, the boundary of U is the natural boundary of solutions to $L^*(w) = 0$.*

Remarks.

1) The domains U occurring in Case (ii) belong to two families, each depending on three real parameters, if $U \neq \hat{\mathbf{C}} - \{0, 1, \infty\}$.
2) Recalling that a singularity is called *apparent* if all of the solutions are single-valued and holomorphic in a neighborhood of it, we see that the transformation $z = z(u)$ is transcendental if at least four singularities of $L(w) = 0$ are not apparent.

Now, suppose that $L(w) = 0$ is not solvable by quadratures, so Case (ii) of Theorem 1 applies to it. Moreover, suppose that U is not $\hat{\mathbf{C}} - \{0, 1, \infty\}$. By introducing yet another transcendental change of variables, $u = u(v)$, we shall see that the solutions of the resulting differential equation admit certain interesting transformation properties. The required change of variables is the projection mapping of a covering, V, of U which we shall now describe.

We have just seen that U is the complement of the closure of a conformal disk D in a Riemann surface, U^*, which may be a complex torus, T, or an annulus $A(r) = \{u \in \mathbf{C} \mid r < |u| < 1\}$. Let V be the complement in the universal covering, $\tilde{U}^*$, of the inverse image of the closed disk $\bar{D} \subset U^*$. Thus, V is obtained from $\tilde{U}^*$ by deleting countably many disjoint copies of $\bar{D}$. Moreover, V is obviously a connected subspace of $\tilde{U}^*$ which is invariant under the action of $\pi_1(U^*, u_0)$ and projects onto U. Thus, V is the covering of U defined by the kernel of the homomorphism $i_*: \pi_1(U, u_0) \to \pi_1(U^*, u_0)$ induced by the inclusion mapping $i: U \to U^*$. This kernel can

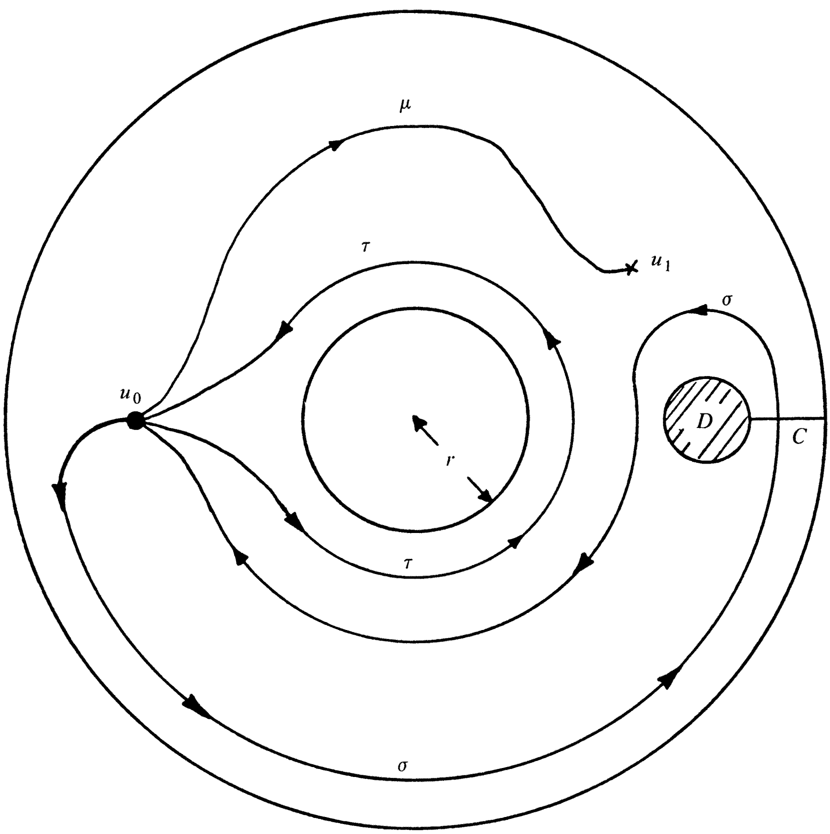

Figure 1

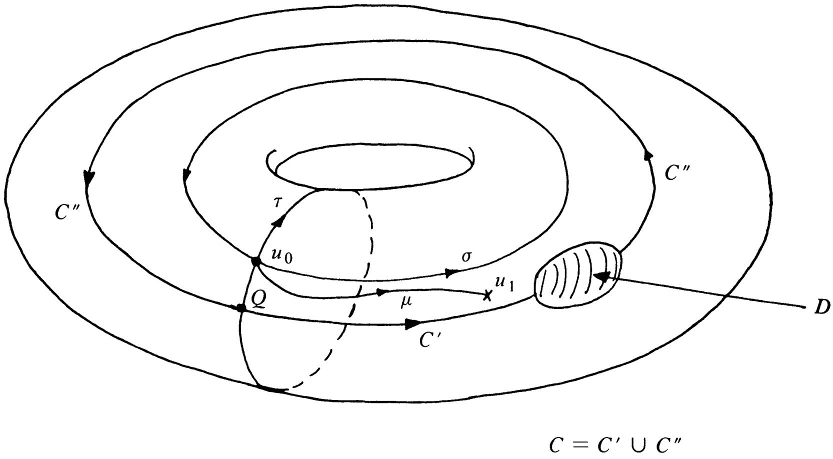

Figure 2

be described explicitly. If U^* is a torus, then it fundamental group is free abelian of rank two, and the kernel of i_* is $\pi_1(U, u_0)'$, the commutator subgroup of $\pi_1(U, u_0)$. If U^* is an annulus, then $\pi_1(U, u_0)$ is a splitting extension of $\ker(i_*)$ by the infinite cyclic group generated by the homotopy class of the loop labeled τ in Figure 1; and $\ker(i_*)$ is the normal closure in $\pi_1(U, u_0)$ of the homotopy class of the loop labeled σ in Figure 1. Identifying $\pi_1(U, u_0)$ with Δ, we see that V is the quotient of $\mathfrak{h}$ by a subgroup of Δ. We shall denote the resulting covering projection by $\nu = \nu(\zeta)$ and use $\nu_0 = \nu(\alpha)$ as the base point for V. It will also be convenient to abuse notation slightly and let σ and τ stand for the elements of Δ determined by the loops σ and τ in the relevant Figure 1 or 2. Thus, V is defined by the normal closure, $\bar{\sigma}$, of σ in Δ when U^* is an annulus and by Δ', the commutator subgroup of Δ, when U^* is a complex torus. Finally, we recall that the restriction of the monodromy representation $m\colon \Gamma \to \mathrm{GL}(n, \mathbf{C})$ to the subgroup Δ is faithful, and we set $S = m(\sigma)$ and $T = m(\tau)$.

Of course, the solutions of $L(w) = 0$ may be viewed multi-valued vectors on V and an analogue of Theorem 1 may be formulated. However, we now prefer to follow a traditional procedure and render V simply connected by introducing certain "branch cuts". The solutions of $L(w) = 0$ will then be single-valued vectors on the resulting domain and their transformation properties will be presented. Of necessity, our discussion is in two parts, corresponding to the alternative possibilities that U^* is an annulus or a complex torus.

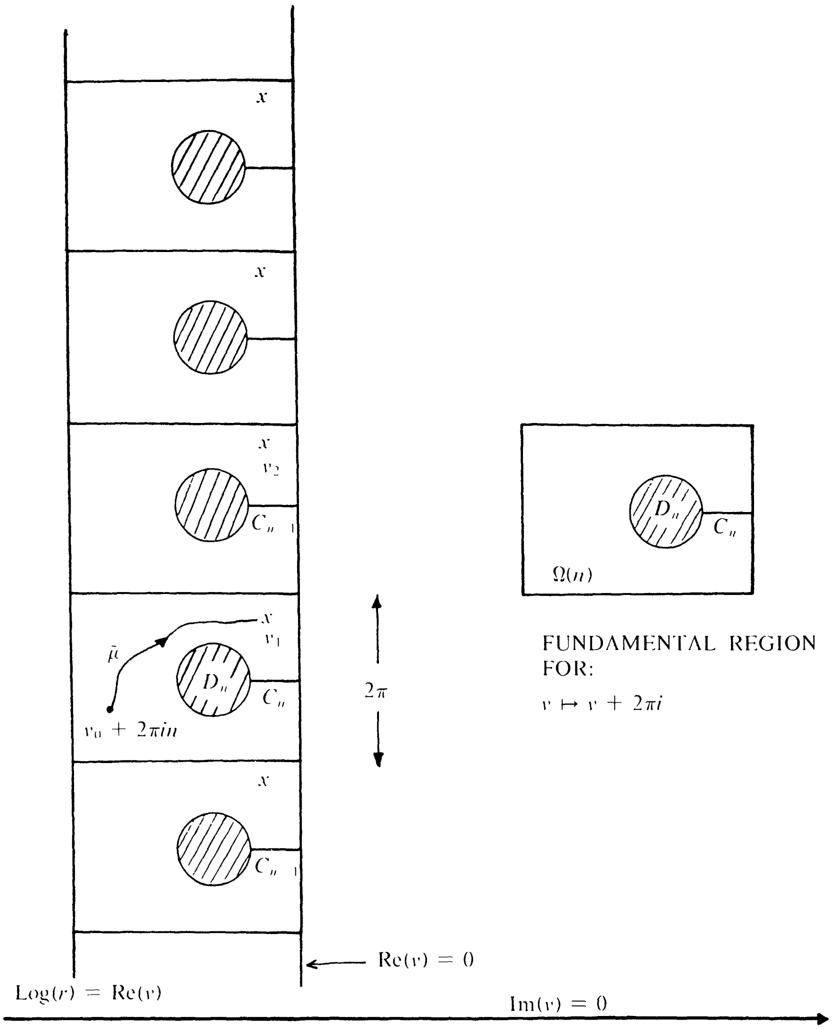

Figure 3

First, suppose that U^* is the annulus $A(r)$, so its universal covering, $\tilde{U}^*$, is the strip in the v-plane defined by $\log(r) < \mathrm{Re}(v) < 0$. The covering projection is given by $u = e^v$ and the action of $\pi_1(U^*, u_0)$ on $\tilde{U}^*$ is generated by the transformation $v \to v + 2\pi i$. Since U is the complement in $A(r)$ of the closed disk $\bar{D}$ with center at $u = c$ and radius s, we see that V is the complement in $\tilde{U}^*$ of the union of the closed sets $\bar{D}_n = \{v = \log(u) +$

$2\pi in \mid u \in \bar{D}, -\pi < \arg(\log(u)) \leq \pi$, n an integer$\}$. Next, we denote the closed segment $\{u \in \mathbf{R} \mid c + s \leq u \leq 1\}$ by C and interpret the arcs $C_n = \{v = \log(u) + 2\pi in \mid u \in C, -\pi < \arg(\log(u)) \leq \pi, n \text{ an integer}\}$ as "branch cuts". A simply connected domain, Ω, results if we delete the totality of the C_n from V. Note that the simply connected subspace $\Omega(n)$ consisting of all points v in Ω such that $(n - 1)\pi < \operatorname{Im} v \leq (n + 1)\pi$ is a fundamental region for the action of the translation $v \to v + 2\pi i$ on V. Finally, let X denote the image of Ω in U. Using this notation, we may now prove:

PROPOSITION 3. *Let* $\mathbf{w}(v, \delta)$ *denote the single-valued holomorphic vector on* Ω *defined by the branch* $\mathbf{w}(v - v_0, \delta) = \mathbf{w}(z(u(v)) - a, \delta)$, $\delta \in \bar{\sigma}$, *of a system of solutions to* $L(w) = 0$. *Then,*

$$\mathbf{w}(v + 2\pi i, \delta) = T\mathbf{w}(v, \delta) \quad \textit{for all } v \textit{ in } \Omega.$$

Proof. Suppose that $\tilde{\lambda}$ is a path joining v_0 to $v = v_1$ on Ω. The power series expansion of $\mathbf{w}(v, \delta)$ at $v = v_1$ is obtained from $\mathbf{w}(v - v_0, \delta)$ by analytic continuation along $\tilde{\lambda}$. This series may also be obtained by continuing $\mathbf{w}(u - u_0, \delta) = \mathbf{w}(z(u) - a, \delta)$ from u_0 to $u_1 = u(v_1)$ along the projection, λ, of $\tilde{\lambda}$ to X and applying the substitution $u = u(v)$ to expand $u - u_1$ in powers of $v - v_1$.

Now, let $v_2 = v_1 + 2\pi i$, let $\tilde{\rho}$ be a path on Ω joining v_0 to v_2, and let ρ denote the projection of $\tilde{\rho}$ to X. Since $u_1 = u(v_1) = u(v_2) = u_2$, the expansions of $u - u_1$ in powers of $v - v_1$ and $v - v_2$ have the same coefficients. Denoting the series obtained from $\mathbf{w}(u - u_0, \delta)$ by continuation along λ and ρ by $\mathbf{w}(u - u_1, \lambda, \delta)$ and $\mathbf{w}(u - u_1, \rho, \delta)$ respectively, we see that Proposition 3 will be proved if we show that $\mathbf{w}(u - u_1, \rho, \delta) = T\mathbf{w}(u - u_1, \lambda, \delta)$.

Since Ω is simply connected, the paths on it joining v_0 to v_1 form a single homotopy class whose projection to X has a particularly useful representative. Namely, suppose that v_1 is in $\Omega(n)$ and that $\tilde{\mu}$ is a path in $\Omega(n)$ joining $v_0 + 2\pi ni$ to v_1. Then, the homotopy class of λ has a unique representative of the form $\mu\tau^n$, where μ is the projection of $\tilde{\mu}$ to X.

It follows that $\mathbf{w}(u - u_1, \lambda, \delta)$ can be obtained by first continuing $\mathbf{w}(u - u_0, \delta)$ along τ^n and then continuing along μ. Recalling that the monodromy transformation $m(\tau) = T$, we see that $\mathbf{w}(u - u_1, \lambda, \delta)$ is obtained by continuing $T^n\mathbf{w}(u - u_0, \delta)$ along μ. However, T is a matrix of constants, so the same result is obtained if we continue $\mathbf{w}(u - u_0, \delta)$ along μ and multiply the resulting vector of power series by T^n.

Finally, we observe that ρ is homotopic to $\mu\tau^{n+1}$, so $\mathbf{w}(u - u_1, \rho, \delta)$ is obtained from the continuation of $\mathbf{w}(u - u_0, \delta)$ along μ by multiplication by T^{n+1}. Thus, $\mathbf{w}(u - u_1, \rho, \delta) = T\mathbf{w}(u - u_1, \lambda, \delta)$, and our proof of Proposition 3 is complete.

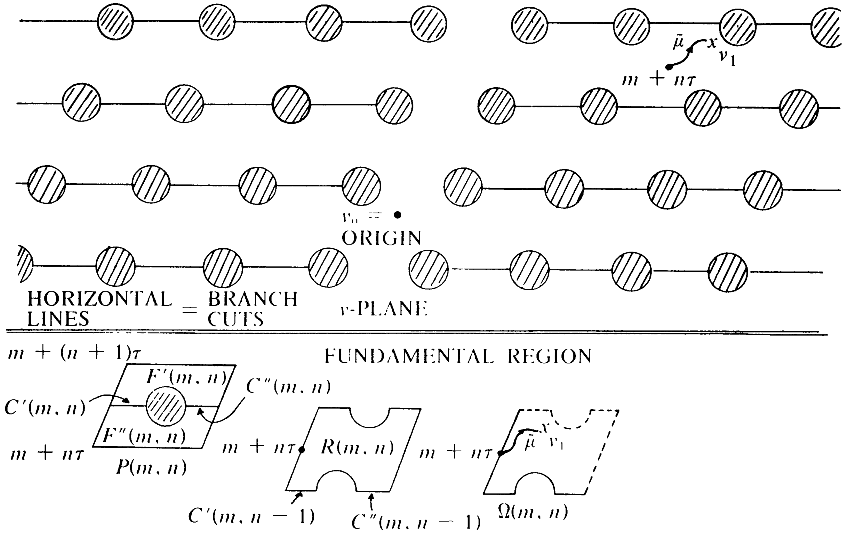

Figure 4

Next, suppose that U^* is a complex torus and that its universal covering, $\tilde{U}^*$, is identified with the complex v-plane. Then $\pi_1(U^*, u_0)$ admits a faithful representation as a lattice, $\mathfrak{L}$, generated by two complex numbers, $v = \omega_1$ and $v = \omega_2$. As is well known (for example, see [13], pages 40–47), we may suppose that $\omega_1 = 1$, $-1/2 \leq \operatorname{Re} \omega_2 \leq 1/2$, $|\omega_2| \geq 1$ and $\operatorname{Re} \omega_2 \leq 0$ when $|\omega_2| = 1$. To simplify notation, we shall also suppose, as indeed we may, that the base point, v_0, of V is at the origin, $v = 0$, and that the parallelogram $P(m, n)$ with vertices $m + n\omega_2$, $(m + 1) + n\omega_2$, $(m + 1) + (n + 1)\omega_2$, $m + (n + 1)\omega_2$ contains in its interior the closure, $\bar{D}(m, n)$, of an open set, $D(m, n)$, which is mapped conformally onto $D \subset U^*$ by the covering projection. Removing $\bar{D}(m, n)$ from $P(m, n)$, we obtain a fundamental region, $F(m, n)$, for the action of $\mathfrak{L}$ on the subspace V of $\tilde{U}^*$. Finally, we shall abuse notation again and write τ in place of ω_2. Thus, the lattice $\mathfrak{L}$ has a basis given by the complex numbers $\nu = 1$ and $\nu = \tau$ corresponding, respectively, to the loops σ and τ on the torus U^*.

We shall now introduce a system of "branch cuts" on V whose complement, Ω, is a simply connected domain. For this purpose, let C be an oriented simple path on U^* such as the one illustrated in Figure 2. Notice that C has the following properties: (i) distinct end points on the boundary of $\bar{D}$, (ii) C does not meet σ, (iii) C crosses τ at a single point, Q. Of course, τ cuts C into two connected paths, C' and C'', as illustrated. Let $C'(m, n)$ and $C''(m, n)$ denote the unique paths on $P(m, n)$ which project onto C' and C'' respectively. These paths divide the fundamental region $F(m, n)$ into an upper half, $F'(m, n)$, and a lower half, $F''(m, n)$, as illustrated. The union of $F''(m, n)$ and $F'(m, n-1)$ forms a simply connected fundamental region, $R(m, n)$, for the action of $\mathfrak{L}$ on V. It will be convenient to denote the simply connected subsapce of $R(m, n)$ consisting of points which do not also belong to $R(m+1, n)$ or $R(m, n+1)$ by $\Omega(m, n)$. Finally we shall denote the complement in V of the $C'(m, n)$, $m \neq 0$, and the $C''(m, n)$, $m \neq -1$, by Ω. Using this notation, we may now prove:

Proposition 4. *Let* $\mathbf{w}(\nu, \delta)$ *denote the single valued holomorphic vector defined on* Ω *by the branch* $\mathbf{w}(\nu - \nu_0, \delta) = \mathbf{w}(z(u(\nu)) - a, \delta)$, $\delta \in \Delta'$, *of a system of solutions to* $L(w) = 0$. Then we have

$$\mathbf{w}(\nu + 1), \delta) = S\mathbf{w}(\nu, \delta), \qquad \nu \in \Omega, \textit{ provided } \nu + 1 \in \Omega$$

$$\mathbf{w}(\nu + \tau, \delta) = S^m T S^{-m} \mathbf{w}(\nu, \delta), \qquad \nu \in \Omega\ (m, n).$$

Proof. We proceed as in the proof of Proposition 3. Thus, the power series expansion of $\mathbf{w}(\nu, \delta)$ at $\nu = \nu_1$ is obtained from $\mathbf{w}(\nu - \nu_0, \delta)$ by analytic continuation along a path, $\tilde{\lambda}$, joining ν_0 to ν_1. Again, this series may also be obtained by continuing $\mathbf{w}(u - u_0, \delta)$ along the projection of $\tilde{\lambda}$ to U and applying the substitution $u = u(\nu)$ to expand $u - u_1$ in terms of $\nu - \nu_1$.

Now, suppose that $\nu_1 \in \Omega(m, n)$ and that $\tilde{\mu}$ is a path in $\Omega(m, n)$ joining $m + n\tau$ to ν_1. Projecting $\tilde{\lambda}$ and $\tilde{\mu}$ to U, we obtain paths λ and μ; and it is easy to see that λ is homotopic to a unique path of the form $\mu\sigma^m\tau^n$. Once again, the power series expansion of $\mathbf{w}(\nu, \delta)$ at ν_1 is obtained by continuing $\mathbf{w}(u - u_0, \delta)$ along μ, multiplying the resulting vector of power series by $S^m T^n$, and expanding $u - u_1$ in powers of $\nu - \nu_1$ by means of the substitution $u = u(\nu)$. We shall denote the series obtained from $\mathbf{w}(u - u_0, \delta)$ by continuation along λ and μ by $\mathbf{w}(u - u_1, \lambda, \delta)$ and $\mathbf{w}(u - u_1, \mu, \delta)$ respectively. Then, $\mathbf{w}(u - u_1, \lambda, \delta) = S^m T^n \mathbf{w}(u - u_0, \mu, \delta)$.

If v_2 belongs to Ω and $u(v_2) = u_2 = u_1 = u(v_1)$, the expansions of $u - u_1$ in powers of $v - v_1$ and $v - v_2$ have the same coefficients. Thus, Proposition 4 may be proved by relating $\mathbf{w}(u - u_1, \rho, \delta)$ to $\mathbf{w}(u - u_1, \lambda, \delta)$, where $\mathbf{w}(u - u_1, \rho, \delta)$ is obtained from $\mathbf{w}(u - u_1, \delta)$ by analytic continuation along the projection, ρ, of a path, $\tilde{\rho}$, joining v_0 to v_2 in Ω.

If $v_2 = v_1 + 1$ lies in Ω (that is, if v_1 is not on $C''(-1, n)$ or $C'(0, n)$), then ρ is obviously homotopic to $\mu\sigma^{m+1}\tau^n$. Thus, $\mathbf{w}(u - u_1, \rho, \delta) = S^{m+1}T^n\mathbf{w}(u - u_1, \mu, \delta) = S\mathbf{w}(u - u_1, \lambda, \delta)$. Similarly, if $v_2 = v_1 + \tau$ then ρ is homotopic to $\mu\sigma^m\tau^{n+1}$ and $\mathbf{w}(u - u_1, \rho, \delta) = S^m T^{n+1}\mathbf{w}(u - u_1, \mu, \delta) = S^m T S^{-m}\mathbf{w}(u - u_1, \lambda, \delta)$. This completes the proof of Proposition 4.

Now, by repeating the argument employed in the paragraph preceding Theorem 1, we may conclude that the boundary of V is a natural boundary for the multi-valued holomorphic vectors defined by the branches, $\mathbf{w}(v - v_0, \delta)$, of solutions of $L(w) = 0$. On the other hand, the single-valued holomorphic vectors $\mathbf{w}(v, \delta)$ admit analytic continuation across the branch cuts defining Ω. We summarize the results of such continuations in the following:

PROPOSITION 5. *Analytic continuation of* $\mathbf{w}(v, \delta)$ *across* C_n *in the positive vertical direction yields* $\mathbf{w}(v, \tau^{-n}\sigma\tau^n\delta)$. *Analytic continuation of* $\mathbf{w}(v, \delta)$ *across* $C'(m, n)$ *or* $C''(m - 1, n)$ *yields* $\mathbf{w}(v, \tau^{-n-1}\sigma^{-m}\tau\sigma^m\tau^n\delta)$.

Proof. Analytic continuation of $\mathbf{w}(v, \delta)$ across any of the indicated branch cuts yields a holomorphic vector on Ω whose expansion at v_0 is obtained from $\mathbf{w}(v - v_0, \delta)$ by continuation along a loop which belongs to Ω except at the point where it crosses the branch cut. It is clear from the figures that such loops define the homotopy classes $\tau^{-n}\sigma\tau^n$ and $\tau^{-n-1}\sigma^{-m}\tau\sigma^m\tau^n$. This completes the proof.

Finally, we summarize our discussion in the following:

THEOREM 2. *Suppose that* $L(w) = 0$ *is an* n^{th} *order homogeneous linear differential equation of Fuchsian type on the Riemann z-sphere. Moreover, suppose that the equation has at least three actual, as opposed to apparent, singularities. Then, either*:

(i) $L(w) = 0$ *is solvable by quadratures, that is, all of its solutions are Liouvillian functions* *or*

(ii) *By a suitable transcendental change of variables*, $z = z(\zeta)$, *bases for the solutions of* $L(w) = 0$ *may be viewed as Zetafuchsian systems of functions which satisfy the functional equations*

$$\mathbf{w}(\zeta + 2) = S\mathbf{w}(\zeta)$$

and

$$\mathbf{w}\left(\frac{1}{2\zeta + 1}\right) = T\mathbf{w}(\zeta).$$

Here, ζ *belongs to the upper half plane*, $\mathfrak{h}$, *and* S *and* T *are complex matrices of degree* n *which generate a free group of rank two.*

or

(iii) *By a suitable transcendental change of variables*, $z = z(v)$, *the solutions to* $L(w) = 0$ *may be viewed as solutions to a differential equation* $L^*(w) = 0$, $w = w(z(v))$, *on one of the domains* V, *of the types illustrated in Figures* 3 *and* 4. *Restricting a basis for these solutions to the simply connected domain* Ω *obtained from* V *by deleting the illustrated "branch cuts", we obtain a family of holomorphic vectors*, $\mathbf{w}(v, \delta)$, *which satisfy the functional equations*:

(a) $\mathbf{w}(v + 2\pi i, \delta) = T\mathbf{w}(v, \delta)$, $v \in \Omega$, *if* V *is of the type illustrated in Figure* 3.

or

(b) $\mathbf{w}(v + 1, \delta) = S\mathbf{w}(v, \delta)$, v *and* $v + 1$ *in* Ω, *and* $\mathbf{w}(v + \tau, \delta) = S^m T S^{-m}\mathbf{w}(v, \delta)$, $v \in \Omega(m, n)$, *if* V *is of the type illustrated in Figure* 4.

In either case, S *and* T *are complex matrices of degree* n *which belong to the monodromy group of the original equation*, $L(w) = 0$, *and generate a free group of rank two. The parameter* δ *ranges over the Fuchsian group which defines* V *as a quotient of* $\mathfrak{h}$ *and, upon analytic continuation across a branch cut, changes as indicated in Proposition* 5. *Finally, the boundary of* V *is a natural boundary for the vectors* $\mathbf{w}(v, \delta)$.

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