A HUMAN PROOF OF GESSEL'S LATTICE PATH CONJECTURE

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Abstract. Gessel walks are planar walks confined to the positive quarter plane, that move by unit steps in any of the following directions: West, North-East, East and South-West. In 2001, Ira Gessel conjectured a closed-form expression for the number of Gessel walks of a given length starting and ending at the origin. In 2008, Kauers, Koutschan and Zeilberger gave a computer-aided proof of this conjecture. The same year, Bostan and Kauers showed, using again computer algebra tools, that the trivariate generating function of Gessel walks is algebraic. In this article we propose the first "human proofs" of these results. They are derived from a new expression for the generating function of Gessel walks.

1. Introduction

Main results. Gessel walks are planar walks confined to the positive quarter plane $\mathbf{Z}_+^2$, that move by unit steps in any of the following directions: West, North-East, East and South-West, see Figure 1 below. For $(i,j) \in \mathbf{Z}_+^2$ and $n \geqslant 0$, let

$$q(i,j;n) = \#\{\text{Gessel walks of length } n \text{ starting at } (0,0) \text{ and ending at } (i,j)\}.$$

Gessel walks have been puzzling the combinatorics community since 2001, when Ira Gessel conjectured that:

(A) For all $n \geqslant 0$, the following closed-form expression holds for the number of Gessel excursions (Gessel walks starting and ending at the origin)

$$q(0,0;2n) = 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n}, \tag{1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol.

Notice that obviously $q(0,0;2n+1) = 0$ for any $n \geqslant 0$. In 2008, Kauers, Koutschan and Zeilberger [17] gave a computer-aided proof of this conjecture. A second intriguing problem was to decide whether or not:

(B) The (trivariate) generating function (GF) of Gessel walks

$$Q(x,y;z) = \sum_{i,j,n\geqslant 0} q(i,j;n)x^i y^j z^n \tag{2}$$

is holonomic,[1] or even algebraic.

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[1]The function $Q(x,y;z)$ is holonomic if the vector space over $\mathbf{C}(x,y,z)$—the field of rational functions in the three variables x,y,z—spanned by the set of all derivatives of $Q(x,y;z)$ is finite dimensional, see for instance [10, Appendix B.4].
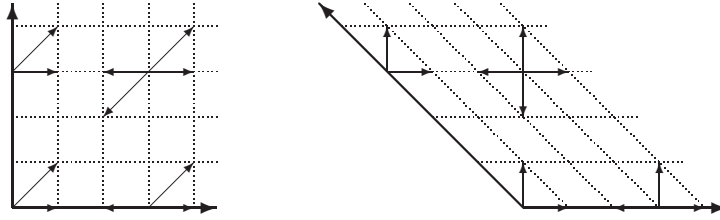
Figure 1. Gessel's model (note that on the boundary, the jumps are the natural ones: those that would take the walk out of $\mathbf{Z}_+^2$ are discarded)

The answer to this question (namely, the algebraicity of $Q(x, y; z)$) was finally obtained by Bostan and Kauers [4], using computer algebra techniques.

> In summary, the only existing proofs for Problems (A) and (B) used heavy computer calculations in a crucial way. In this article we obtain a new explicit expression for $Q(x, y; z)$, from which we derive the first "human proofs" of (A) and (B).

Context of Gessel's conjecture. In 2001, the motivation of considering Gessel's model was twofold. First, by an obvious linear transformation, Gessel's walk can be viewed as the simple walk (i.e., with jumps to the West, North, East and South) constrained to lie in a cone with angle 135°, see on the right in Figure 1. It turns out that before 2001, the simple walk was well studied in different cones. Pólya [28] first studied the simple walk in the whole plane, and remarked that the probability that a simple random walk ever returns to the origin is one. This is a consequence of the fact that there are exactly $\binom{2n}{n}^2$ unrestricted excursions of length $2n$ in the plane $\mathbf{Z}^2$. There also exist formulæ for excursions of length $2n$ evolving in other regions of $\mathbf{Z}^2$: $\binom{2n+1}{n} C_n$ for the half plane, and $C_n C_{n+1}$ for the quarter plane, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number [2]. Gouyou-Beauchamps [14] found a similar formula $C_n C_{n+2} - C_{n+1}^2$ for the cone with angle 45° (the first octant). It was thus natural to consider the cone with angle 135°, and this is what Gessel did.

The second part of the motivation is that Gessel walks are particular instances of walks in the quarter plane. Although it is since 2008 that most of the works concerning walks in the quarter plane appeared, in 2001 there were already famous examples of such models: Kreweras' walk [19, 11, 12] (with jumps to the West, North-East and South) for which the GF (2) is algebraic; Gouyou-Beauchamps' walk [14]; the simple walk [15]. Further, around 2000, the walks in the quarter plane were brought up to date, notably by Bousquet-Mélou and Petkovšek [6, 7]. Indeed, they were used to illustrate the following phenomenon: although the numbers of walks satisfy a (multivariate) linear recurrence with constant coefficients, their GF (2) might be non-holonomic; see [7] for the example of the knight walk (with the same moves of a knight in chess).

Existing results in the literature. After 2001, many approaches to treat walks in the quarter plane appeared. Bousquet-Mélou and Mishna initiated a systematic study of the walks with small steps (this means that the step set, i.e., the set of possible steps for the walk, is a subset of the eight nearest neighbors). Mishna [26, 27] first considered the case of step sets of cardinality three. She presented a complete classification of the GF (2) of these

walks with respect to the classes algebraic, transcendental holonomic and non-holonomic. Bousquet-Mélou and Mishna [5] then considered all 79 small step sets.[2] They considered a functional equation for the GF that counts walks in such a model leading to a group[3] of birational transformations of $\mathbf{C}^2$. In 23 cases out of 79 this group turns out to be finite, and the corresponding functional equations were solved in 22 out of 23 cases (the finiteness of the group being a crucial feature in [5]). The remaining case is the one of Gessel walks. In 2008, a method using computer algebra techniques was proposed by Kauers, Koutschan and Zeilberger [18, 17]. Kauers and Zeilberger [18] first obtained a computer-aided proof of the algebraicity of the GF counting Kreweras' walks. A few months later, this approach was enhanced to cover Gessel's case, and the conjecture (Problem (A)) was proved [17]. At the same time, Bostan and Kauers [4] showed, using heavy computer calculations, that the trivariate GF counting Gessel walks is algebraic (Problem (B)). Using the minimal polynomials obtained by Bostan and Kauers, van Hoeij [4, Appendix] managed to obtain an explicit expression for the GF of Gessel walks.

Purely mathematical analysis of the GF of Gessel walks (but without answering Gessel's conjectures) were proposed in [20, 30, 9, 3, 29, 31]. Kurkova and Raschel [20] obtained an explicit integral representation of $Q(x, y; z)$. This was done by solving a boundary value problem, a method inspired by the book [8]. This approach has been generalized for all models of walks with small steps in the quarter plane, see [30]. In [9], Fayolle and Raschel gave another proof of the algebraicity of the GF (Problem (B)), using probabilistic and algebraic methods of the seventies. In [3], Ayyer proposed a combinatorial approach inspired by representation theory. He interpreted Gessel walks as words on certain alphabets. He then reformulated $q(i, j; n)$ as numbers of words, and computed very particular numbers of Gessel walks. Petkovšek and Wilf [29] stated new conjectures, closed to Gessel's. They found an expression for Gessel's numbers in terms of determinants of matrices, by showing that the numbers of walks are solution to an infinite system of equations. Ping [31] introduced a probabilistic model for Gessel walks, and reduced the computation of $q(i, j; n)$ to the computation of a certain probability. Using then probabilistic methods (such as the reflection principle) he proved two conjectures of Petkovšek and Wilf [29].

Presentation of our method and organization of the article. First of all, we fix $z \in]0, 1/4[$. To solve Problems (A) and (B), we start from the GFs $Q(x, 0; z)$ and $Q(0, y; z)$ and from the functional equation (3) of Bousquet-Mélou and Mishna (valid at any $(x, y; z)$ with $|x| < 1$ and $|y| < 1$)

$$K(x, y; z)Q(x, y; z) =$$
$$K(x, 0; z)Q(x, 0; z) + K(0, y; z)Q(0, y; z) - K(0, 0; z)Q(0, 0; z) - xy. \quad (3)$$

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[2]A priori, there are $2^8 = 256$, but the authors of [5] showed that, after eliminating trivial cases, and also those which can be reduced to walks in a half plane, there remain 79 inherently different models.

[3]Historically, this group was introduced by Malyshev [23, 24, 25] in the seventies. For details on this group we refer to Section 2, in particular to equation (12).
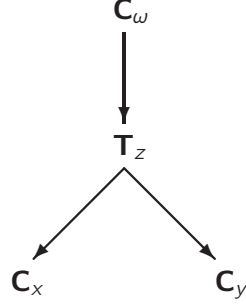
$$\mathbf{C}_\omega$$

$$\downarrow$$

$$\mathbf{T}_z$$

$$\mathbf{C}_x \qquad \mathbf{C}_y$$

Figure 2. Different levels in our proof

Above, $K(x, y; z)$ is the kernel of the walk, given by

$$K(x, y; z) = xyz \left(\sum_{(i,j) \in \mathfrak{G}} x^i y^j - 1/z \right) = xyz(xy + x + 1/x + 1/(xy) - 1/z), \quad (4)$$

where $\mathfrak{G} = \{(1,1), (1,0), (-1,0), (-1,-1)\}$ denotes Gessel's step set. Our main idea is to construct all branches of these functions, in other words, to consider the meromorphic continuations of $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ along any path of the complex plane (and thus not only in their natural domains of definition $\{|x| < 1\}$ and $\{|y| < 1\}$). This idea is motivated by the fact that a function is algebraic if and only if it has a finite number of branches.

To achieve the objective of constructing all branches of the GFs, we need to consider the elliptic curve $\mathbf{T}_z$ defined by the zeros of the kernel $K(x, y; z)$:

$$\mathbf{T}_z = \{(x, y) \in (\mathbf{C} \cup \{\infty\})^2 : K(x, y; z) = 0\}, \quad (5)$$

and to introduce the universal covering of $\mathbf{T}_z$, which is the complex plane $\mathbf{C}_\omega$ with a new variable ω. The functions $x \mapsto Q(x, 0, z)$ and $y \mapsto Q(0, y, z)$ can be lifted on their respective natural domains of definition on $\mathbf{T}_z$ and next on the corresponding domains of the universal covering $\mathbf{C}_\omega$, namely $\{\omega \in \mathbf{C}_\omega : |x(\omega)| < 1\}$ and $\{\omega \in \mathbf{C}_\omega : |y(\omega)| < 1\}$. It turns out that the latter domains are vertical strips. This lifting procedure is illustrated on Figure 2. The first level (at the bottom) represents the complex planes $\mathbf{C}_x$ and $\mathbf{C}_y$, where $Q(x, 0; z)$ and $Q(0, y; z)$ are defined in $\{|x| < 1\}$ and $\{|y| < 1\}$. The second level, where the variables x and y are not independent anymore, is given by $\mathbf{T}_z$. The third level is $\mathbf{C}_\omega$, the universal covering of $\mathbf{T}_z$. All this construction has been first elaborated by Malyshev [23] for stationary probability GFs of random walks in $\mathbf{Z}_+^2$, and has been further developed in [8]. We make it explicit in the context of Gessel walks in Section 2.

The key-point of all our approach is the following: defining the lifted function $r_x(\omega) = K(x(\omega), 0; z)Q(x(\omega), 0; z)$, we have the identity

$$r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega), \qquad \forall \omega \in \mathbf{C}_\omega, \quad (6)$$

where the shift (real positive) vector ω_3 and the function f_x are explicit (and relatively simple). A similar equation holds for the lifted function of $Q(0, y; z)$. Equation (6) has many consequences.

Firstly, due to (6), the function r_x can be continued from its initial domain of definition (a vertical strip) to the whole plane $\mathbf{C}_\omega$. By projecting back on $\mathbf{C}_x$, we shall recover all branches of $Q(x, 0; z)$.

Secondly, we shall notice that there are only a finite number of different branches, which will yield the algebraicity of $Q(x, 0; z)$. Using a similar result on $Q(0, y; z)$ and the functional equation (3), we shall derive in this way the solution to Problem (B), see Section 5.

Thirdly, from (6), we shall deduce the poles of r_x. In general, it is clearly impossible to deduce the expression of a function from the knowledge of its poles. A notable exception is constituted by elliptic functions. In our case, it will happen that the poles of r_x form a two-dimensional lattice, and that the residues are periodic; the function r_x is therefore elliptic (all poles are of order 1). From this fact we shall deduce an explicit expression of r_x in terms of elliptic ζ-functions. By projection on $\mathbf{C}_x$, this will give a new explicit expression of $Q(x, 0; z)$ for Gessel walks as an infinite series. An analogous result will hold for $Q(0, y; z)$, and (3) will then lead to a new explicit expression for $Q(x, y; z)$, see Section 3.

Fourthly, evaluating this expression of $Q(x, 0; z)$ at $x = 0$ and performing several simplifications (using identities concerning special functions [1] and the theory of Darboux covering for tetrahedral hypergeometric equations [32]), we shall obtain the solution of Problem (A), see Section 4.

2. Meromorphic continuation of the GFs

The aim of Section 2 is to prove equation (6), which, as we have seen just above, is the fundamental starting equation for our analysis. In passing, we shall also introduce some useful tools for the sequel. Though crucial, this section does not contain any new result. We thus chose just to state the results, without proof, and we refer to [21, Sections 2–5] fore full details.

Branch points. For brevity, we drop the variable z (which is fixed in $]0, 1/4[$) from the notations when no ambiguity can arise, writing for instance $Q(x, y)$ instead of $Q(x, y; z)$ and $\mathbf{T}$ instead of $\mathbf{T}_z$. The kernel $K(x, y)$ defined in (4) is a second degree polynomial in both x and y. The algebraic function $X(y)$ defined by $K(X(y), y) = 0$ has thus two branches and four branch points, that we call y_i, $i \in \{1, \ldots, 4\}$. They are the roots of the discriminant of the second degree equation $K(x, y) = 0$ in the variable x:

$$\widetilde{d}(y) = (-y)^2 - 4z^2(y^2 + y)(y + 1).$$

We have $y_1 = 0$, $y_4 = \infty = 1/y_1$, and

$$y_2 = \frac{1 - 8z^2 - \sqrt{1 - 16z^2}}{8z^2}, \qquad y_3 = \frac{1 - 8z^2 + \sqrt{1 - 16z^2}}{8z^2} = 1/y_2,$$

in such a way that $y_1 < y_2 < y_3 < y_4$. Since there are four distinct branch points, the Riemann surface of $X(y)$ is a torus $\mathbf{T}^y$ (i.e., a Riemann surface of genus 1). The analogous statement is true for the algebraic function $Y(x)$ defined by $K(x, Y(x)) = 0$. Its four branch points x_i, $i \in \{1, \ldots, 4\}$, are the roots of

$$d(x) = (zx^2 - x + z)^2 - 4z^2x^2. \tag{7}$$

They are real and numbered in such a way that $x_1 < x_2 < x_3 < x_4$:

$$x_1 = \frac{1 + 2z - \sqrt{1 + 4z}}{2z}, \qquad x_2 = \frac{1 - 2z - \sqrt{1 - 4z}}{2z}, \qquad x_3 = 1/x_2, \qquad x_4 = 1/x_1.$$

The Riemann surface of $Y(x)$ is also a torus $\mathbf{T}^x$. Since $\mathbf{T}^x$ and $\mathbf{T}^y$ are equivalent, in what follows we shall consider a single Riemann surface $\mathbf{T}$ with two different coverings $x, y : \mathbf{T} \to \mathbf{S}$; see Figure 2.

Universal covering. The torus $\mathbf{T}$ is isomorphic to a quotient space $\mathbf{C}/(\omega_1 \mathbf{Z} + \omega_2 \mathbf{Z})$, where ω_1, ω_2 are complex numbers linearly independent on $\mathbf{R}$. This set can obviously be thought as the (fundamental) parallelogram $[0, \omega_1] + [0, \omega_2]$, the opposed edges of which are identified (here, all parallelograms will be rectangles). The periods ω_1, ω_2 are unique (up to a unimodular transform) and are found in [8, Lemma 3.3.2]:

$$\omega_1 = i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-d(x)}}, \qquad \omega_2 = \int_{x_2}^{x_3} \frac{dx}{\sqrt{d(x)}}. \tag{8}$$

The universal covering of $\mathbf{T}$ has the form $(\mathbf{C}, \lambda)$, where $\mathbf{C}$ is the complex plane (the union of infinitely many parallelograms

$$\Pi_{m,n} = \omega_1[m, m + 1[+ \omega_2[n, n + 1[, \qquad m, n \in \mathbf{Z},$$

which are glued together) and $\lambda : \mathbf{C} \to \mathbf{T}$ is a non-branching covering map. For any $\omega \in \mathbf{C}$ such that $\lambda\omega = s \in \mathbf{T}$, we have $x(\lambda\omega) = x(s)$ and $y(\lambda\omega) = y(s)$. The uniformization formulæ [8, Lemma 3.3.1] are

$$\begin{cases} x(\omega) & = & x_4 + \dfrac{d'(x_4)}{\wp(\omega) - d''(x_4)/6}, \\ y(\omega) & = & \dfrac{1}{2a(x(\omega))} \left(-b(x(\omega)) + \dfrac{d'(x_4)\wp'(\omega)}{2(\wp(\omega) - d''(x_4)/6)^2} \right), \end{cases} \tag{9}$$

where $a(x) = zx^2$, $b(x) = zx^2 - x + z$, d is defined in (7), and $\wp$ is the Weierstrass elliptic function with periods ω_1, ω_2 (its expansion is given in (44)). Throughout, we shall write $x(\lambda\omega) = x(\omega)$ and $y(\lambda\omega) = y(\omega)$. Due to (9), these quantities are elliptic functions on $\mathbf{C}$ with periods ω_1, ω_2. Clearly

$$K(x(\omega), y(\omega)) = 0, \qquad \forall \omega \in \mathbf{C}. \tag{10}$$

Furthermore, since each parallelogram $\Pi_{m,n}$ represents a torus $\mathbf{T}$ composed of two complex spheres, the function $x(\omega)$ (resp. $y(\omega)$) takes each value of $\mathbf{C} \cup \{\infty\}$ twice within this parallelogram, except for the branch points x_i, $i \in \{1, \ldots, 4\}$ (resp. y_i, $i \in \{1, \ldots, 4\}$). The points $\omega_{x_i} \in \Pi_{0,0}$ such that $x(\omega_{x_i}) = x_i$, $i \in \{1, \ldots, 4\}$ are represented on Figure 3. They are equal to

$$\omega_{x_1} = \omega_2/2, \qquad \omega_{x_2} = (\omega_1 + \omega_2)/2, \qquad \omega_{x_3} = \omega_1/2, \qquad \omega_{x_4} = 0.$$

The points ω_{y_i} such that $y(\omega_{y_i}) = y_i$ are just the shifts of ω_{x_i} by a real vector $\omega_3/2$ (to be defined below): $\omega_{y_i} = \omega_{x_i} + \omega_3/2$ for $i \in \{1, \ldots, 4\}$, see also Figure 3. We refer to [8, Chapter 3] and to [20, 30] for proofs of these facts. The vector ω_3 is defined as

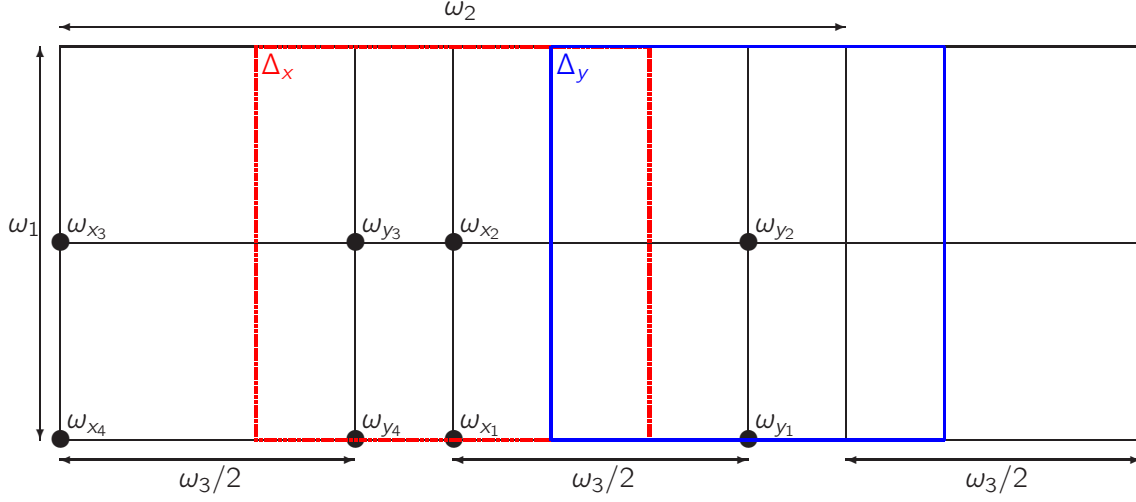$$\omega_3 = \int_{-\infty}^{x_1} \frac{dx}{\sqrt{d(x)}}.$$

Figure 3. The fundamental parallelograms for the functions r_x and r_y, namely, $\Pi_{0,0} = \omega_1[0,1[+\omega_2[0,1[$ and $\Pi_{0,0} + \omega_3/2 = \omega_1[0,1[+\omega_2[0,1[+\omega_3/2,$ and some important points and domains on them

For Gessel's model we have the following important relation [20, Proposition 14], which holds for all $z \in]0, 1/4[$:

$$\omega_3/\omega_2 = 3/4. \tag{11}$$

Galois automorphisms. It is easy to see that the functions ξ and η of $\mathbf{C}^2$

$$\xi(x,y) = \left(x, \frac{1}{x^2 y}\right), \qquad \eta(x,y) = \left(\frac{1}{xy}, y\right) \tag{12}$$

leave invariant the quantity $\sum_{(i,j)\in\mathfrak{S}} x^i y^j$ (and therefore also the set $\mathbf{T}$ in (5) for any fixed $z \in]0, 1/4[$). They span a group $\langle \xi, \eta \rangle$ of birational transformations of $\mathbf{C}^2$, which is a dihedral group. It is of order 8, see [5] (to see this, it suffices to compute all elements). Furthermore,

$$\xi^2 = \eta^2 = \mathrm{id}. \tag{13}$$

This group was first defined in a probabilistic context by Malyshev [23, 24, 25]; it was introduced for the combinatorics of walks with small steps in the quarter plane by Bousquet-Mélou and Mishna [5]. In (12), it is defined as a group on $\mathbf{C}^2 = \mathbf{C}_x \times \mathbf{C}_y$, i.e., at the bottom level of Figure 2. We now lift it to the upper levels of Figure 2.

First, we lift it on the intermediate level $\mathbf{T}$ as the restriction of $\langle \xi, \eta \rangle$ on $\mathbf{T}$. Namely, any point $s \in \mathbf{T}$ admits the two "coordinates" $(x(s), y(s))$, which satisfy $K(x(s), y(s)) = 0$ by construction. For any $s \in \mathbf{T}$, there exists a unique s' (resp. s'') such that $x(s') = x(s)$ (resp. $y(s'') = y(s)$). The values $x(s), x(s')$ (resp. $y(s), y(s'')$) are the two roots of the second degree equation $K(x, y(s)) = 0$ (resp. $K(x(s), y) = 0$) in x (resp. y). The automorphism $\xi : \mathbf{T} \to \mathbf{T}$ (resp. $\eta : \mathbf{T} \to \mathbf{T}$) is defined by the identity $\xi s = s'$ (resp. $\eta s = s''$) and is called

a Galois automorphism, following the terminology of [23, 24, 25, 8]. Clearly by (12) and (13), we have, for any $s \in \mathbf{T}$,

$$x(\xi s) = x(s), \quad y(\xi s) = \frac{1}{x^2(s)y(s)}, \quad x(\eta s) = \frac{1}{y(s)x(s)}, \quad y(\eta s) = y(s), \quad \xi^2(s) = \eta^2(s) = s.$$

Finally $\xi s = s$ (resp. $\eta s = s$) if and only if $x(s) = x_i$, $i \in \{1, \ldots, 4\}$ (resp. $y(s) = y_i$, for some $i \in \{1, \ldots, 4\}$).

There are many ways to lift ξ and η from $\mathbf{T}$ to the universal covering $\mathbf{C}$. We follow the way of [8] and [21], lifting them on $\mathbf{C}$ in such a way that ω_{x_2} and ω_{y_2} stay their fixed points, respectively (see Figure 3):

$$\xi\omega = -\omega + \omega_1 + \omega_2, \qquad \eta\omega = -\omega + \omega_1 + \omega_2 + \omega_3, \qquad \forall \omega \in \mathbf{C}.$$

Let us remind that $\omega_1 + \omega_2 = 2\omega_{x_2}$ and that $\omega_1 + \omega_2 + \omega_3 = 2\omega_{y_2}$. One has

$$x(\xi\omega) = x(\omega), \qquad y(\eta\omega) = y(\omega), \qquad \forall \omega \in \mathbf{C}.$$

Lifting of the GFs on the universal covering. The domains

$$\{\omega \in \mathbf{C} : |x(\omega)| < 1\}, \qquad \{\omega \in \mathbf{C} : |y(\omega)| < 1\}$$

consist of infinitely many curvilinear strips, which differ from translations by a multiple of ω_2. We denote by Δ_x (resp. Δ_y) the strip that is within $\cup_{n \in \mathbb{Z}} \Pi_{0,n}$ (resp. $\cup_{n \in \mathbb{Z}} \Pi_{0,n} + \omega_3/2$). The domain Δ_x (resp. Δ_y) is delimited by vertical lines, see [20, Proposition 26], and is represented on Figure 3. We notice that the function $Q(x(\omega), 0)$ (resp. $Q(0, y(\omega))$) is well defined in Δ_x (resp. Δ_y), by (2). Let us put

$$r_x(\omega) = K(x(\omega), 0)Q(x(\omega), 0), \qquad \forall \omega \in \Delta_x,$$
$$r_y(\omega) = K(0, y(\omega))Q(0, y(\omega)), \qquad \forall \omega \in \Delta_y.$$

The domain $\Delta_x \cap \Delta_y$ is a non-empty open strip, see Figure 3. It follows from (3) and (10) that

$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0, \qquad \forall \omega \in \Delta_x \cap \Delta_y. \qquad (14)$$

Meromorphic continuation of the GFs on the universal covering. Let $\Delta = \Delta_x \cup \Delta_y$. Due to (14), the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued as meromorphic functions on the whole of Δ, by setting

$$r_x(\omega) = -r_y(\omega) + K(0,0)Q(0,0) + x(\omega)y(\omega), \qquad \forall \omega \in \Delta_y,$$
$$r_y(\omega) = -r_x(\omega) + K(0,0)Q(0,0) + x(\omega)y(\omega), \qquad \forall \omega \in \Delta_x.$$

To continue the functions from Δ to the whole of $\mathbf{C}$, we first notice that

$$\bigcup_{n \in \mathbb{Z}} (\Delta + n\omega_3) = \mathbf{C},$$

as proved in [8, 21] and illustrated on Figure 3. Let

$$\begin{cases} f_x(\omega) &= y(\omega)[x(-\omega + 2\omega_{y_2}) - x(\omega)], \\ f_y(\omega) &= x(\omega)[y(-\omega + 2\omega_{x_2}) - y(\omega)]. \end{cases} \qquad (15)$$

The following result holds true, see [21, Theorem 4].

Lemma 1 ([21]). *The functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically to the whole of* **C**. *Further, for any $\omega \in$ **C**, we have*

$$r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega),$$

$$r_y(\omega + \omega_3) = r_y(\omega) + f_y(\omega), \tag{16}$$

$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0, \tag{17}$$

$$\begin{cases} r_x(\xi\omega) = r_x(\omega), \\ r_y(\eta\omega) = r_y(\omega), \end{cases} \tag{18}$$

$$\begin{cases} r_x(\omega + \omega_1) = r_x(\omega), \\ r_y(\omega + \omega_1) = r_y(\omega). \end{cases} \tag{19}$$

3. GFs in terms of Weierstrass Zeta-functions

Statements of the results. The aim of this section is to prove the following results: let ω_1, ω_2 be defined as in (8), and let $\zeta_{1,3}$ be the ζ-Weierstrass function with periods $\omega_1, 3\omega_2$, see Lemma 14 in the appendix for its definition and some of its properties.

Theorem 2. *We have*

$$Q(0,0) = \tag{20}$$

$$\frac{\zeta_{1,3}(\omega_2/4) - 3\zeta_{1,3}(2\omega_2/4) + 2\zeta_{1,3}(3\omega_2/4) + 3\zeta_{1,3}(4\omega_2/4) - 5\zeta_{1,3}(5\omega_2/4) + 2\zeta_{1,3}(6\omega_2/4)}{2z^2}.$$

Theorem 3. *We have, for all $\omega \in$ **C**,*

$$\begin{aligned} r_y(\omega) = c + &\frac{1}{2z}\zeta_{1,3}(\omega - (1/8)\omega_2) - \frac{1}{2z}\zeta_{1,3}(\omega - (3/8)\omega_2) \\ &+ \frac{1}{2z}\zeta_{1,3}(\omega - (1 + 3/8)\omega_2) - \frac{1}{2z}\zeta_{1,3}(\omega - (1 + 5/8)\omega_2) \\ &- \frac{1}{2z}\zeta_{1,3}(\omega - (1 + 7/8)\omega_2) + \frac{1}{z}\zeta_{1,3}(\omega - (2 + 1/8)\omega_2) \\ &- \frac{1}{z}\zeta_{1,3}(\omega - (2 + 5/8)\omega_2) + \frac{1}{2z}\zeta_{1,3}(\omega - (2 + 7/8)\omega_2), \end{aligned} \tag{21}$$

where c is a constant.

Note that the constant c above is immediately made explicit from these two theorems. In fact, the point $\omega_0^y = 7\omega_2/8 \in \Delta_y$ is such that $y(\omega_0^y) = 0$ (see Lemma 5 below). Hence the value of $r_y(7\omega_2/8) = K(0, y(7\omega_2/8))Q(0, y(7\omega_2/8)) = K(0,0)Q(0,0) = zQ(0,0)$ is found in Theorem 2. The value of c then is $c = zQ(0,0) - \widehat{\zeta}_{1,3}(7\omega_2/8)$, where $\widehat{\zeta}_{1,3}(\omega)$ is the sum of the eight ζ-functions in (21).

A similar expression to (21) holds for $r_x(\omega)$ (with another constant c). There are two ways to obtain this expression: the first one consists in doing the same analysis as for r_y; the second one is to express r_x from equation (17) in terms of r_y and to apply Theorems 2 and 3. Presented in terms of ζ-functions, the results of both approaches are rigorously the same.

We shall explain in Section 5 how to deduce from Theorems 2 and 3 explicit expressions for $Q(0, y; z)$ and $Q(x, 0; z)$. Using the functional equation (3), we shall then obtain a new expression for $Q(x, y; z)$.

Preliminary results. The poles of f_y will play a crucial role in our analysis. They are given in the lemma hereafter.

Lemma 4. *In the fundamental parallelogram $\omega_1[0, 1[+\omega_2[0, 1[$, the function f_y has poles at $\omega_2/8$, $3\omega_2/8$, $5\omega_2/8$ and $7\omega_2/8$. These poles are simple, with residues equal to $-1/(2z)$, $1/(2z)$, $1/(2z)$ and $-1/(2z)$, respectively.*

Before proving Lemma 4, we recall from [20, Lemma 28] the following result, dealing with the zeros and poles of $x(\omega)$ and $y(\omega)$:

Lemma 5 ([20]). *In the fundamental parallelogram $\omega_1[0, 1[+\omega_2[0, 1[$, the only poles of x (of order one) are at $\omega_2/8$, $7\omega_2/8$, and its only zeros (of order one) are at $3\omega_2/8$, $5\omega_2/8$. The only pole of y (of order two) is at $3\omega_2/8$, and its only zero (of order two) is at $7\omega_2/8$.*

Proof of Lemma 4. With the definition (15) of the function $f_y(\omega)$ and (9), we derive that

$$f_y(\omega) = \frac{1}{2z}\frac{x'(\omega)}{x(\omega)}.$$

Accordingly, if $x(\omega)$ has a simple zero (resp. a simple pole) at ω_0, then $f_y(\omega)$ has a simple pole at ω_0, with residue $1/(2z)$ (resp. $-1/(2z)$). Lemma 4 then follows from Lemma 5. □

The following lemma will shorten the proof of Theorem 3.

Lemma 6. *The function r_y is elliptic with periods $\omega_1, 3\omega_2$.*

Proof. The function r_y is meromorphic and ω_1-periodic due to (19). Further, by Lemma 1,

$$r_y(\omega + 4\omega_3) - r_y(\omega) = f_y(\omega) + f_y(\omega + \omega_3) + f_y(\omega + 2\omega_3) + f_y(\omega + 3\omega_3), \qquad \forall \omega \in \mathbf{C}. \quad (22)$$

By the analysis of Lemma 4 and the fact (11) that $\omega_3 = 3\omega_2/4$, the elliptic function

$$\mathcal{O}(\omega) = \sum_{k=0}^{3} f_y(\omega + k\omega_3)$$

has no poles on $\mathbf{C}$. Hence, with property (P2) of Lemma 14, $\mathcal{O}(\omega)$ must be a constant c, so that $r_y(\omega + 4\omega_3) = r_y(\omega) + c$ for all $\omega \in \mathbf{C}$. In particular, $r_y(\omega_{y_2} - 4\omega_3) + 2c = r_y(\omega_{y_2} + 4\omega_3)$. But in view of (18), $r_y(\omega_{y_2} - 4\omega_3) = r_y(\omega_{y_2} + 4\omega_3)$, and then $c = 0$. It follows that $r_y(\omega)$ is also $4\omega_3 = 3\omega_2$-periodic, and thus elliptic with periods $\omega_1, 3\omega_2$. □

Note that this lemma also follows from [22, Proposition 10]: by Lemma 4, the assumption of this proposition holds for Gessel's model. Finally, Lemma 6 is proved in [21, Proposition 11] as well, using the representation of $\mathcal{O}(\omega)$ as the so-called orbit-sum:

$$\mathcal{O}(\omega) = \sum_{1 \leqslant k \leqslant 4} (xy)(\omega + k\omega_3) - (xy)(\eta(\omega + k\omega_3))$$

$$= \sum_{1 \leqslant k \leqslant 4} (xy)((\eta\xi)^k\omega) - (xy)(\xi(\eta\xi)^{k-1}\omega)$$

$$= \sum_{\theta \in \langle \xi, \eta \rangle} (-1)^\theta xy(\theta(\omega)),$$

where $(-1)^\theta$ is the signature of θ, i.e., $(-1)^\theta = (-1)^{\ell(\theta)}$, where $\ell(\theta)$ is the smallest ℓ for which we can write $\theta = \theta_1 \circ \cdots \circ \theta_\ell$, with $\theta_1, \ldots, \theta_\ell$ equal to ξ or η.

Proof of Theorems 2 and 3.

Proof of Theorem 3. Since r_y is elliptic with periods $\omega_1, 3\omega_2$ (Lemma 6), and since any elliptic function is characterized by its poles in a fundamental parallelogram, it suffices to study r_y in $\omega_1[-1/2, 1/2[+\omega_2[-5/2, 1/2[$. To do so, we shall use [22, Theorem 6], which gives the poles and the principal parts at these poles of r_y in terms of the function f_y, for any model of walks with small steps in the quarter plane (and rational ω_2/ω_3). Specifically, [22, Theorem 6] says that a pole d of r_y must satisfy $\mathcal{N}_d^- \neq \emptyset$, where (recall that $\Re\omega_{x_1} = \omega_2/2$)

$$\mathcal{N}_d^- = \{n \geq 0 : d + n\omega_3 \text{ is a pole of } f_y \text{ with } -5\omega_2/2 < \Re d + n\omega_3 < \omega_2/2\}.$$

If a point d is such that $\mathcal{N}_d^- \neq \emptyset$, then function r_y has the following principal part $R_{d,y}$ at d:

$$R_{d,y}(\omega) = \sum_{n \in \mathcal{N}_d^-} -(\lfloor n/4 \rfloor + 1) F_{d+n\omega_3, y}(\omega + n\omega_3), \tag{23}$$

where $F_{d+n\omega_3, y}$ is the principal part of f_y at its pole $d + n\omega_3$, and $\lfloor x \rfloor$ is the lower integer part of $x \in \mathbf{R}$.

We thus first need to find the poles d of f_y in $\omega_1[-1/2, 1/2[+\omega_2[-5/2, 1/2[$. By Lemma 4, these are the points of the set

$$P = \{3\omega_2/8 - 2k\omega_2/8 : k \in \{0, \ldots, 11\}\}. \tag{24}$$

We thus have $\mathcal{N}_d^- = \{n \geq 0 : d + n\omega_3 \in P\}$. It is then obvious that the points d of the parallelogram $\omega_2[-5/2, 1/2[+\omega_1[-1/2, 1/2[$ such that $\mathcal{N}_d^- \neq \emptyset$ must be among the points of P. We now study each of them.

For $k \in \{0, 1, 2\}$ in (24), the associated points $d = 3\omega_2/8, \omega_2/8, -\omega_2/8 \in P$ are such that $\mathcal{N}_d^- = \{0\}$. This implies that there is one single term in formula (23), namely $R_{d,y}(\omega) = -F_{d,y}(\omega)$. With Lemma 4 we conclude that r_y has a pole of order 1 at d, with residue $-1/(2z), 1/(2z), 1/(2z)$, respectively.

For $k \in \{3, 4, 5\}$ in (24), we have the points $d = -3\omega_2/8, -5\omega_2/8, -7\omega_2/8 \in P$, which are such that $\mathcal{N}_d^- = \{0, 1\}$. Let us find the principal part at $d = -3\omega_2/8$. With (23) we have

$$R_{-3\omega_2/8, y}(\omega) = -(\lfloor 0/4 \rfloor + 1) F_{-3\omega_2/8, y}(\omega) - (\lfloor 1/4 \rfloor + 1) F_{3\omega_2/8, y}(\omega + 6\omega_2/8)$$

$$= -\frac{1}{2z} \frac{1}{\omega + 3\omega_2/8} - \frac{1}{2z} \frac{1}{\omega + 6\omega_2/8 - 3\omega_2/8}$$

$$= -\frac{1}{z} \frac{1}{\omega + 3\omega_2/8},$$

where the second line above is a consequence of Lemma 4. Hence $d = -3\omega_2/8$ is a simple pole of r_y, with residue $-1/z$. In the same way $R_{-5\omega_2/8, y}(\omega) = 0$, so that $-5\omega_2/8$ is not a pole of r_y, and $R_{-7\omega_2/8, y}(\omega) = (1/z)/(\omega + 7\omega_2/8)$, so that $-7\omega_2/8$ is a simple pole of r_y, with residue $1/z$.

For $k \in \{6, 7, 8\}$ in (24), points $-9\omega_2/8, -11\omega_2/8, -13\omega_2/8$ are with $\mathcal{N}_d^- = \{0, 1, 2\}$. The principal parts (23) of r_y at them are computed as the sums of the principal parts of f_y at three poles. This eventually shows that $-9\omega_2/8, -11\omega_2/8, -13\omega_2/8$ are simple poles of r_y with the residues $-1/(2z), -1/(2z), 1/(2z)$, respectively.

Finally, for $k \in \{9, 10, 11\}$ in (24), points $-15\omega_2/8, -17\omega_2/8, -19\omega_2/8 \in P$ are such that $\mathcal{N}_d^- = \{0, 1, 2, 3\}$. The principal parts of r_y (23) at them are computed as the sums of

the principal parts of f_y at four poles, which all turn out to be zero. These points are thus not poles for r_y.

Applying the property (P6) of Lemma 14, we finally reach the conclusion that

$$r_y(\omega) = c - \frac{1}{2z}\zeta_{1,3}(\omega - 3\omega_2/8) + \frac{1}{2z}\zeta_{1,3}(\omega - \omega_2/8)$$

$$+ \frac{1}{2z}\zeta_{1,3}(\omega + \omega_2/8) - \frac{1}{z}\zeta_{1,3}(\omega + 3\omega_2/8)$$

$$+ \frac{1}{z}\zeta_{1,3}(\omega + 7\omega_2/8) - \frac{1}{2z}\zeta_{1,3}(\omega + 9\omega_2/8)$$

$$- \frac{1}{2z}\zeta_{1,3}(\omega + 11\omega_2/8) + \frac{1}{2z}\zeta_{1,3}(\omega + 13\omega_2/8).$$

Using the fact that $\zeta_{1,3}(\omega + 3\omega_2) = \zeta_{1,3}(\omega) + 2\zeta_{1,3}(3\omega_2/2)$, see property (P8) of the same lemma, we close the proof of Theorem 3, with another constant c. □

Proof of Theorem 2. Equation (17) yields $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + K(0,0)Q(0,0)$. We compute the constant $K(0,0)Q(0,0)$ as $r_y(\omega_0^y) = K(0, y(\omega_0^y))Q(0, y(\omega_0^y))$, where $\omega_0^y \in \Delta_y$ is such that $y(\omega_0^y) = 0$. Lemma 5 gives a unique possibility for ω_0^y, namely, $\omega_0^y = 7\omega_2/8$. Hence $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + r_y(7\omega_2/8)$. Let us substitute $\omega = 5\omega_2/8$ in this equation. The point $5\omega_2/8$ is a zero of $x(\omega)$ that lies in Δ_x, so that

$$r_x(5\omega_2/8) = K(x(5\omega_2/8), 0)Q(x(5\omega_2/8), 0) = K(0,0)Q(0,0) = zQ(0,0).$$

This point is not a pole of $y(\omega)$, in such a way that $x(5\omega_2/8)y(5\omega_2/8) = 0$. We obtain

$$zQ(0,0) = r_y(7\omega_2/8) - r_y(5\omega_2/8). \tag{25}$$

Note in particular that in order to obtain the expression (25) of $Q(0,0)$, there is no need to know the constant c in Theorem 3.

With Theorem 3 and (25), $Q(0,0)$ can be written as a sum of 16 $\zeta_{1,3}$-Weierstrass functions (each of them being evaluated at a rational multiple of ω_2). Using the fact that $\zeta_{1,3}$ is an odd function and using property (P8), we can perform many easy simplifications in (25), and, this way, we obtain (20). □

4. Proof of Gessel's conjecture (Problem (A))

In this section, we shall prove Gessel's formula (1) for the number of Gessel excursions. The starting point is Theorem 2, which expresses the generating function of Gessel excursions as a linear combination of (evaluations at multiples of $\omega_2/4$ of) the ζ-Weierstrass function $\zeta_{1,3}$ with periods $\omega_1, 3\omega_2$. The individual terms of this linear combination are transcendental functions; our strategy is to group them in a way that brings up a linear combination of algebraic hypergeometric functions, from which Gessel's conjecture follows by telescopic summation.

Roadmap of the proof. More precisely, Gessel's formula (1) is equivalent to[4]

$$Q(0,0) = \frac{1}{2z^2}\left(\,_2F_1\left(\begin{bmatrix} -\frac{1}{2}, -\frac{1}{6} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \end{bmatrix}, 16z^2\right) - 1\right).$$

---

[4]This was already remarked by Ira Gessel when he initially formulated the conjecture.

Here, we use the notation $_2F_1([a, b], [c], z)$ for the Gaussian hypergeometric function

$$_2F_1([a, b], [c], z) = \sum_{n=0}^{\infty} \frac{(a)_n \cdot (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{26}$$

where $(x)_n = x(x + 1) \cdots (x + n - 1)$ denotes the Pochhammer symbol.

In view of Theorem 2, Gessel's conjecture is therefore equivalent to

$$F_1 - 3F_2 + 2F_3 + 3F_4 - 5F_5 + 2F_6 = G - 1, \tag{27}$$

where G is the algebraic hypergeometric function $_2F_1([-1/2, -1/6], [2/3], 16z^2)$, and where F_k denotes the transcendental function $\zeta_{1,3}(k\omega_2/4)$ for $1 \leqslant k \leqslant 6$.

Let us denote by $V_{i,j,k}$ the function $F_i + F_j - F_k$. Then, the left-hand side of equality (27) rewrites $4V_{1,4,5} - V_{2,4,6} - V_{1,5,6} - 2V_{1,2,3}$.

To prove (27), our key argument is encapsulated in the following identities:

$$V_{1,4,5} = (2G + H)/3 - K/2, \tag{28}$$

$$V_{2,4,6} = (2G + H)/3 - K, \tag{29}$$

$$V_{1,5,6} = (J + 1)/2, \tag{30}$$

$$V_{1,2,3} = (2G + 2H - J - 2K + 1)/4. \tag{31}$$

Here H, J and K are auxiliary algebraic functions, defined in the following way: H is the hypergeometric function $_2F_1([-1/2, 1/6], [1/3], 16z^2)$, J stands for $(G - K)^2$, and K stands for $zG' = 4z^2 {}_2F_1([1/2, 5/6], [5/3], 16z^2)$.

Gessel's conjecture is a consequence of the equalities (28)–(31). Indeed, by summation, these equalities imply that $4V_{1,4,5} - V_{2,4,6} - V_{1,5,6} - 2V_{1,2,3}$ is equal to $G - 1$, proving (27).

It then remains to prove equalities (28)–(31). To do this, we use the following strategy. Instead of proving the equalities of functions of the variable z, we rather prove that their evaluations at $z = (x(x+1)^3/(4x+1)^3)^{1/2}$ are equal. This choice of algebraic transformation is inspired by the Darboux covering for tetrahedral hypergeometric equations of the Schwarz type $(1/3, 1/3, 2/3)$ [32, §6.1]. Indeed, G, H, K and J belong to this class of hypergeometric functions.

First, we make use of a corollary of the Frobenius-Stickelberger identity (P9) ([34, page 446]), which implies that $V_{i,j,k}$ is equal to the algebraic function $\sqrt{T_i + T_j + T_k}$ as soon as $k = i + j$. Here, T_ℓ denotes the algebraic function $\wp_{1,3}(\ell\omega_2/4)$. Second, using classical properties of the $\wp$- and ζ-Weierstrass functions, we explicitly determine $T_\ell((x(x + 1)^3/(4x + 1)^3)^{1/2})$ for $1 \leqslant \ell \leqslant 6$, then use them to compute $V_{1,4,5}, V_{2,4,6}, V_{1,5,6}$ and $V_{1,2,3}$ evaluated at $z = (x(x+1)^3/(4x+1)^3)^{1/2}$. Finally, equalities (28)–(31) are proved by checking that they hold when evaluated at $z = (x(x + 1)^3/(4x + 1)^3)^{1/2}$.

Preliminary results. We shall deal with elliptic functions with different pairs of periods. We shall denote by $\zeta, \wp$ the elliptic functions with periods ω_1, ω_2, and by $\zeta_{1,3}, \wp_{1,3}$ the elliptic functions with periods $\omega_1, 3\omega_2$, see Lemma 14 for their definition. Further, we recall that elliptic functions are alternatively characterized by their periods (see equation (44)) or by their invariants. The invariants of $\wp$ are denoted by g_2, g_3. They are such that

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3, \qquad \forall \omega \in \mathbf{C}. \tag{32}$$

We recall from [20, Lemma 12] the following result that provides explicit expressions for the invariants g_2, g_3.

Lemma 7 ([20]). *We have*

$$g_2 = (4/3)(1 - 16z^2 + 16z^4), \qquad g_3 = -(8/27)(1 - 8z^2)(1 - 16z^2 - 8z^4). \qquad (33)$$

Likewise, we define the invariants $g_2^{1,3}, g_3^{1,3}$ of $\wp_{1,3}$. To compute them, it is convenient to introduce first an algebraic function called R, which is the unique positive root[5] of

$$X^4 - 2g_2X^2 + 8g_3X - g_2^2/3 = 0. \qquad (34)$$

Using equations (33) and (34), we obtain the local expansion $R(z) = 2 - 16z^2 - 48z^4 + O(z^6)$ in the neighborhood of 0.

The algebraic function R will play an important role in determining the algebraic functions $T_\ell = \wp_{1,3}(\ell\omega_2/4)$. To begin with, the next lemma expresses T_4, as well as the invariants $g_2^{1,3}, g_3^{1,3}$, in terms of R.

Lemma 8. *One has*

$$T_4 = \wp_{1,3}(\omega_2) = R/6,$$
$$g_2^{1,3} = -g_2/9 + 10R^2/27,$$
$$g_3^{1,3} = -35R^3/729 + 7g_2R/243 - g_3/27,$$

where expressions for g_2 and g_3 are given in (33).

Proof. Using the properties (P4) and (P7) from Lemma 14, one can write,

$$\wp(\omega) = -4\wp_{1,3}(\omega_2) - \wp_{1,3}(\omega) + \frac{\wp_{1,3}'(\omega)^2 + \wp_{1,3}'(\omega_2)^2}{2(\wp_{1,3}(\omega) - \wp_{1,3}(\omega_2))^2}, \qquad \forall\omega \in \mathbf{C}. \qquad (35)$$

We then make a local expansion at the origin of the both sides of the equation above, using property (P3) from Lemma 14. We obtain

$$\frac{1}{\omega^2} + \frac{g_2}{20}\omega^2 + \frac{g_3}{28}\omega^4 + O(\omega^6) =$$
$$\frac{1}{\omega^2} + \left(6\wp_{1,3}(\omega_2)^2 - \frac{9g_2^{1,3}}{20}\right)\omega^2 + \left(10\wp_{1,3}(\omega_2)^3 - \frac{3g_2^{1,3}\wp_{1,3}(\omega_2)}{2} - \frac{27g_3^{1,3}}{28}\right)\omega^4 + O(\omega^6).$$

Identifying the expansions above, we obtain two equations for the three unknowns $\wp_{1,3}(\omega_2)$, $g_2^{1,3}$ and $g_3^{1,3}$ (remember that its invariants g_2 and g_3 are known from Lemma 7). We add a third equation by noticing that $\wp_{1,3}(\omega_2)$ is the only real positive solution to (see, e.g., [20, Proof of Lemma 22])

$$X^4 - \frac{g_2^{1,3}}{2}X^2 - g_3^{1,3}X - \frac{(g_2^{1,3})^2}{48} = 0.$$

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[5]To prove this, we need to introduce the discriminant of the fourth-degree polynomial $P(X)$ defined by (34). Since $\deg P(X) = 4$ and since the leading coefficient of $P(X)$ is 1, its discriminant equals the resultant of $P(X)$ and $P'(X)$. Some elementary computations give that it equals $cz^{16}(1 - 16z^2)^2$, where c is a negative constant. The discriminant is thus negative (for any $z \in]0, 1/4[$). On the other hand, the discriminant can be interpreted as $\prod_{1\leqslant i<j\leqslant 4}(R_i - R_j)^2$, where the R_i, $i \in \{1, \ldots, 4\}$, are the roots of $P(X)$. The negative sign of the discriminant implies that $P(X)$ has two complex conjugate roots and two real roots. Further, the product of the roots is clearly negative, see (34), so that one of the two real roots is negative while another one is positive.

We then have a (non-linear) system of three equations with three unknowns. Some easy computations finally lead to the expressions of $\wp_{1,3}(\omega_2)$, $g_2^{1,3}$ and $g_3^{1,3}$ of Lemma 8. $\square$

The next result expresses the algebraic functions T_1, T_2, T_3, T_5 and T_6 in terms of the algebraic function R, and of the invariants g_2 and g_3 (the quantity T_4 has already been found in Lemma 8).

Lemma 9. *One has the following formulæ:*

(i) $T_1 = \wp_{1,3}(\omega_2/4)$ *is the unique solution of*

$$X^3 - \left(\frac{R}{3} + \frac{1 + 4z^2}{3}\right) X^2 + \left(\frac{R(1 + 4z^2)}{9} + \frac{R^2}{108} + \frac{g_2}{18}\right) X$$
$$+ \left(\frac{23R^3}{2916} - \frac{R^2(1 + 4z^2)}{108} + \frac{g_3}{27} - \frac{19Rg_2}{972}\right) = 0 \quad (36)$$

such that in the neighborhood of 0, $T_1 = 1/3 + 4z^2/3 - 4z^6 - 56z^8 + O(z^{10})$.

(ii) $T_2 = \wp_{1,3}(2\omega_2/4)$ *is equal to*

$$T_2 = \frac{R + 1 - 8z^2}{6} - \frac{T_6}{2}. \quad (37)$$

(iii) $T_3 = \wp_{1,3}(3\omega_2/4)$ *is the unique solution of* (36) *such that in the neighborhood of 0,*
$T_3 = 1/3 - 8z^2/3 - 8z^4 - 60z^6 + O(z^8)$.

(iv) $T_5 = \wp_{1,3}(5\omega_2/4)$ *is the unique solution of* (36) *such that in the neighborhood of 0,*
$T_5 = 1/3 - 8z^2/3 - 8z^4 - 64z^6 + O(z^8)$.

(v) $T_6 = \wp_{1,3}(6\omega_2/4)$ *is equal to*

$$T_6 = \frac{R + 1 - 8z^2 - \sqrt{3R^2 - 4R(1 - 8z^2) + 4(1 - 8z^2)^2 - 6g_2}}{9}. \quad (38)$$

Proof. We first prove that for a given value of ω (and thus for a given value of $\wp(\omega)$), the three solutions of

$$X^3 - \left(\frac{R}{3} + \wp(\omega)\right) X^2 + \left(\frac{R\wp(\omega)}{3} + \frac{R^2}{108} + \frac{g_2}{18}\right) X$$
$$+ \left(\frac{23R^3}{2916} - \frac{\wp(\omega)R^2}{36} + \frac{g_3}{27} - \frac{19Rg_2}{972}\right) = 0 \quad (39)$$

are

$$\{\wp_{1,3}(\omega), \wp_{1,3}(\omega + \omega_2), \wp_{1,3}(\omega + 2\omega_2)\}.$$

By property (P7) we find

$$\wp(\omega) = -4\wp_{1,3}(\omega_2) - \wp_{1,3}(\omega) + \frac{\wp'_{1,3}(\omega)^2 + \wp'_{1,3}(\omega_2)^2}{2(\wp_{1,3}(\omega) - \wp_{1,3}(\omega_2))^2}, \qquad \forall \omega \in \mathbf{C},$$

where by Lemma 8, one has $\wp_{1,3}(\omega_2) = R/6$. Then $\wp'_{1,3}(\omega_2)^2 = 4(R/6)^3 - g_2^{1,3}R/6 - g_3^{1,3}$, and following this way, we obtain that $\wp_{1,3}(\omega)$ satisfies (39).

We start the proof of the lemma by showing (i). Using [20, Lemma 19] one has that $\wp(\omega_2/4) = \wp(3\omega_2/4) = (1 + 4z^2)/3$. Then the equation (36) is exactly (39) with $\omega = \omega_2/4$. The three roots of (36) are $\wp_{1,3}(\omega_2/4)$, $\wp_{1,3}(5\omega_2/4)$ and $\wp_{1,3}(9\omega_2/4) = \wp_{1,3}(3\omega_2/4)$. By using standard properties of $\wp$-Weierstrass functions, $\wp_{1,3}(\omega_2/4)$ is the biggest of the three quantities (and this for any $z \in {]0, 1/4[}$). Further, since (39) is a polynomial of degree 3, we

can easily find its roots in terms of the variable z. This way, we find that the three solutions admit the expansions

$$1/3 + 4z^2/3 - 4z^4 - 56z^6 + O(z^8),$$
$$1/3 - 8z^2/3 - 8z^4 - 60z^6 + O(z^8),$$
$$1/3 - 8z^2/3 - 8z^4 - 64z^6 + O(z^8).$$

Accordingly, $T_1 = \wp_{1,3}(\omega_2/4)$ corresponds to the first one, $T_3 = \wp_{1,3}(3\omega_2/4)$ to the second one and $T_5 = \wp_{1,3}(5\omega_2/4)$ to the last one.

We now prove (ii) and (v). Using again [20, Lemma 19], one derives that $\wp(2\omega_2/4) = (1-8z^2)/3$. The three roots of equation (39) with $\omega = 2\omega_2/4$ are $\wp_{1,3}(2\omega_2/4)$, $\wp_{1,3}(6\omega_2/4)$ and $\wp_{1,3}(10\omega_2/4)$. Since $\wp_{1,3}(10\omega_2/4) = \wp_{1,3}(2\omega_2/4)$, (39) with $\omega = 2\omega_2/4$ has a double root (that we call t_1) and a simple root (t_2). It happens to be simpler to deal now with the derivative of the polynomial in the left-hand side of (39). It is an easy exercise to show that the roots of the derivative of a polynomial of degree 3 with a double root at t_1 and a simple root at t_1 are t_1 and $(t_1 + 2t_2)/3$. This way, we obtain expressions for $\wp_{1,3}(2\omega_2/4)$ and $(\wp_{1,3}(2\omega_2/4) + 2\wp_{1,3}(6\omega_2/4))/3$, which are equal to

$$\frac{R + 1 - 8z^2 \pm \sqrt{3R^2 - 4R(1 - 8z^2) + 4(1 - 8z^2)^2 - 6g_2}}{9}. \tag{40}$$

Since $\wp_{1,3}(2\omega_2/4) > \wp_{1,3}(6\omega_2/4)$, $\wp_{1,3}(2\omega_2/4)$ corresponds to the sign $+$ in (40). This way we immediately find expressions for $\wp_{1,3}(2\omega_2/4)$ and $\wp_{1,3}(6\omega_2/4)$, and this finishes the proof of the lemma. $\qquad\square$

The next result derives explicit expressions of T_ℓ evaluated at $z = (x(x+1)^3/(4x+1)^3)^{1/2}$.

Lemma 10. *One has the following formulæ:*

$$T_1\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = \frac{4x^4 + 28x^3 + 30x^2 + 10x + 1}{3(4x+1)^3} + \frac{2x(x+1)(2x+1)}{(4x+1)^{5/2}},$$

$$T_5\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = \frac{4x^4 + 28x^3 + 30x^2 + 10x + 1}{3(4x+1)^3} - \frac{2x(x+1)(2x+1)}{(4x+1)^{5/2}},$$

$$T_2\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = \frac{4x^4 + 16x^3 + 12x^2 + 4x + 1}{3(4x+1)^3},$$

$$T_4\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = \frac{(2x^2 - 2x - 1)^2}{3(4x+1)^3},$$

$$T_3\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = \frac{4x^4 + 4x^3 + 4x + 1}{3(4x+1)^3},$$

$$T_6\left(\sqrt{\frac{x(x+1)^3}{(4x+1)^3}}\right) = -\frac{8x^4 + 8x^3 - 4x - 1}{3(4x+1)^3}.$$

Proof. All equalities are consequences of Lemma 9. We begin with R: we replace z by $\varphi(x) = \sqrt{x(x+1)^3/(4x+1)^3}$ in equation (34), factor the result, and identify the corresponding minimal polynomial of $R(\varphi(x))$ in $\mathbf{Q}(x)[T]$. To do this, we use that the

local expansion of $R(\varphi(x))$ at $x = 0$ is equal to $2 - 16x + O(x^2)$. The minimal polynomial has degree 1, proving the equality

$$R\left(\varphi(x)\right) = \frac{2(2x^2 - 2x - 1)^2}{(4x + 1)^3}.$$

From R, we directly deduce $T_4 = R/6$. Now, replacing z by $\varphi(x)$ in Lemma 9 (v) provides the expression of $T_6(\varphi(x))$. Then T_2 is treated in a similar way using Lemma 9 (ii). Finally, an annihilating polynomial for $T_1(\varphi(x)), T_3(\varphi(x)), T_5(\varphi(x))$ is deduced in a similar manner using Lemma 9 (i). This polynomial in $\mathbf{Q}(x)[T]$ factors as a product of a linear factor and a quadratic factor. Using the local expansions $1/3 + 4/3x - 12x^2 + 80x^3 + O(x^4), 1/3 - 8/3x + 16x^2 - 84x^3 + O(x^4)$ and $1/3 - 8/3x + 16x^2 - 88x^3 + O(x^4)$ allows to conclude. □

Corollary 11. *The algebraic functions* $V_{1,4,5}, V_{2,4,6}, V_{1,5,6}$ *and* $V_{1,2,3}$ *satisfy the following equalities:*

$$V_{1,4,5}\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{2x^2 + 4x + 1}{(4x + 1)^{3/2}},$$

$$V_{2,4,6}\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{2x + 1}{(4x + 1)^{3/2}},$$

$$V_{1,5,6}\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{2x + 1}{4x + 1},$$

$$V_{1,2,3}\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{x}{4x + 1} + \frac{(x + 1)(2x + 1)}{(4x + 1)^{3/2}}.$$

Proof. It is a direct consequence of the definition and of the previous lemma. □

Completing the proof. The last step consists in proving the equalities (28)–(31). The starting point is that the hypergeometric functions $G = {}_2F_1([-1/2, -1/6], [2/3], 16z^2)$ and $H = {}_2F_1([-1/2, 1/6], [1/3], 16z^2)$ are algebraic and satisfy the equations displayed in the following lemma.

Lemma 12. *One has the following formulæ:*

$$G\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{4x^2 + 8x + 1}{(4x + 1)^{3/2}},$$

$$H\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{4x^2 + 2x + 1}{(4x + 1)^{3/2}},$$

$$K\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{4x(x + 1)}{(4x + 1)^{3/2}},$$

$$J\left(\sqrt{\frac{x(x + 1)^3}{(4x + 1)^3}}\right) = \frac{1}{4x + 1}.$$

Proof. The first two equalities are similar to those given by Vidunas in Section 6.1 of his article [32], see also [33]. They correspond to tetrahedral hypergeometric equations of the

Schwarz type $(1/3, 1/3, 2/3)$. The last two equalities are easy consequences of the first two.   □

Now, equalities (28)–(31) evaluated at $z = (x(x+1)^3/(4x+1)^3)^{1/2}$ are easily proven using Lemma 11 and Lemma 12. For instance, equality (28) evaluated at $z = (x(x + 1)^3/(4x + 1)^3)^{1/2}$ reads:

$$\frac{2x^2 + 4x + 1}{(4x + 1)^{3/2}} = \frac{2}{3}\frac{4x^2 + 8x + 1}{(4x + 1)^{3/2}} + \frac{1}{3}\frac{4x^2 + 2x + 1}{(4x + 1)^{3/2}} - \frac{1}{2}\frac{4x(x + 1)}{(4x + 1)^{3/2}}.$$

Similarly, equality (31) evaluated at $z = (x(x + 1)^3/(4x + 1)^3)^{1/2}$ reads:

$$\frac{x}{4x + 1} + \frac{(x + 1)(2x + 1)}{(4x + 1)^{3/2}} = \frac{1}{2}\frac{4x^2 + 8x + 1}{(4x + 1)^{3/2}} + \frac{1}{2}\frac{4x^2 + 2x + 1}{(4x + 1)^{3/2}} - \frac{1}{4}\frac{1}{4x + 1} - \frac{1}{2}\frac{4x(x + 1)}{(4x + 1)^{3/2}} + \frac{1}{4}.$$

The proof of Gessel's formula (1) for the number of Gessel excursions is thus completed. Note that incidentally we proved that the generating series $Q(0, 0)$ for Gessel excursions is *algebraic*. The next section is devoted to the proof that the *complete* generating series of Gessel walks in algebraic.

5. Proof of the algebraicity of the GF (Problem (B))

Branches of the GFs and algebraicity of $Q(x, y)$ in the variables x, y. In this section we prove a weakened version of Problem (B): we show that $Q(x, y)$ is algebraic in x, y (we do not prove here that the latter function is algebraic in x, y, z, which is much stronger). This is not necessary for our analysis, but this illustrates that our approach easily yields to algebraicity results.

We first propose two proofs of the algebraicity of $Q(0, y)$ as a function of y. The first proof is an immediate application of property (P5). The sum of the residues (i.e., the multiplicative factors in front of the ζ-functions) in the formula (21) of Theorem 3 is clearly 0, so that $r_y(\omega)$ is an algebraic function of $\wp_{1,3}(\omega)$, by (P5). Further, by (P7), $\wp_{1,3}(\omega)$ is an algebraic function of $\wp(\omega)$, and finally by (9), $\wp(\omega)$ is algebraic in $y(\omega)$. This eventually implies that $r_y(\omega)$ is algebraic in $y(\omega)$, or equivalently that $Q(0, y)$ is algebraic in y.

The second proof is based on the meromorphic continuation of the GFs on the universal covering, which is done in Section 2. The restrictions of $r_y(\omega)/K(0, y(\omega))$ on the half parallelogram

$$\mathcal{D}_{k,\ell} = \omega_3/2 + \omega_1[\ell, \ell + 1[+\omega_2]k/2, (k + 1)/2]$$

for $k, \ell \in \mathbf{Z}$ provide all branches of $Q(0, y)$ on $\mathbf{C} \setminus ([y_1, y_2] \cup [y_3, y_4])$ as follows:

$Q(0, y) = \{r_y(\omega)/K(0, y(\omega)) : \omega$ is the (unique) element of $\mathcal{D}_{k,\ell}$ such that $y(\omega) = y\}$,

see [21, Section 5.2] for more details. Due to the ω_1-periodicity of $r_y(\omega)$ and $y(\omega)$ (see (19) and (9), respectively), the restrictions of these functions on $\mathcal{D}_{k,\ell}$ do not depend on $\ell \in \mathbf{Z}$, and therefore determine the same branch as on $\mathcal{D}_{k,0}$ for any ℓ. Furthermore, due to equation (18), the restrictions of $r_y(\omega)/K(0, y(\omega))$ on $\mathcal{D}_{-k+1,0}$ and on $\mathcal{D}_{k,0}$ lead to the same branches for any $k \in \mathbf{Z}$. Hence, the restrictions of $r_y(\omega)/K(0, y(\omega))$ on $\mathcal{D}_{k,0}$ with $k \geqslant 1$ provide all different branches of this function. In addition, Lemma 6 says that r_y is $3\omega_2$-periodic. This fact yields that $Q(0, y)$ has (at most) six branches, and is thus algebraic.

An analogous statement holds for (the restrictions of) the function $r_x(\omega)/K(x(\omega), 0)$ and then for $Q(x, 0)$. Using the functional equation (3), we conclude that $Q(x, y)$ is algebraic in the two variables x, y.

In the section below, we refine the previous statement, by proving that $Q(x, y)$ is algebraic in x, y, z (Problem (B)).

Proof of the algebraicity of the trivariate GF. We start by proving the algebraicity of $Q(0, y)$ as a function of y, z. We consider the representation of $r_y(\omega)$ given in Theorem 3 and apply eight times the addition theorem (P4) for ζ-functions, namely (for suitable values of $k \in \mathbf{Z}$ that can be deduced from (21))

$$\zeta_{1,3}(\omega - k\omega_2/8) = \zeta_{1,3}(\omega) - \zeta_{1,3}(k\omega_2/8) + \frac{1}{2} \frac{\wp'_{1,3}(\omega) + \wp'_{1,3}(k\omega_2/8)}{\wp_{1,3}(\omega) - \wp_{1,3}(k\omega_2/8)}.$$

We then make the weighted sum of the eight identities above (corresponding to the good values of k in (21)); this way, we obtain

$$r_y(\omega) = U_1(\omega) + U_2 + U_3(\omega),$$

where $U_1(\omega)$ is the weighted sum of the eight functions $\zeta_{1,3}(\omega)$, U_2 is the sum of c and of the weighted sum of the eight quantities $\zeta_{1,3}(k\omega_2/8)$, and $U_3(\omega)$ is the weighted sum of the eight quantities

$$\frac{\wp'_{1,3}(\omega) + \wp'_{1,3}(k\omega_2/8)}{\wp_{1,3}(\omega) - \wp_{1,3}(k\omega_2/8)}. \tag{41}$$

Since the sum of the residues in the formula (21) equals 0, the coefficients in front of $\zeta_{1,3}(\omega)$ is 0, so that $U_1(\omega)$ is identically zero. To prove that U_2 is algebraic in z, it suffices to use similar arguments as we did to prove that $Q(0, 0)$ is algebraic (see Section 4: we group together different ζ-functions and we use standard identities as the Frobenius-Stickelberger equality (P9) or the addition formula for the ζ-function (P4)); we do not repeat the arguments here. Finally, we show that $U_3(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z. In other words, we show that there exists a non-zero bivariate polynomial P such that

$$P(U_3(\omega), y(\omega)) = 0,$$

where the coefficients of P are algebraic functions in z. This is enough to conclude to the algebraicity of $Q(0, y)$ as a function of y, z.

To prove the latter fact, we shall prove that each term (41) satisfies the property above (with different polynomials P, of course). First, Lemma 13 below implies that $\wp_{1,3}(k\omega_2/8)$ and $\wp'_{1,3}(k\omega_2/8)$ are both algebraic in z. Further, it follows from (P7) that the function $\wp_{1,3}(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z. The same property holds for $\wp'_{1,3}(\omega)$; this comes from the fact above together with the differential equation (32) satisfied by the Weierstrass elliptic functions.

The proof of the algebraicity of $Q(x, 0)$ as a function of x, z is analogous. With equation (3) the algebraicity of $Q(x, y)$ as a function of x, y, z is proved.

Lemma 13. *For any $k \in \mathbf{Z}$ and any $\ell \in \mathbf{Z}_+$, $\wp^{(\ell)}(k\omega_2/8)$ and $\wp^{(\ell)}_{1,3}(k\omega_2/8)$ are (infinite or) algebraic functions of z.*

Proof. First, for any $\ell \in \mathbf{Z}_+$ and $k \in 8\mathbf{Z}$ (resp. $k \in 24\mathbf{Z}$), $\wp^{(\ell)}(k\omega_2/8)$ (resp. $\wp_{1,3}^{(\ell)}(k\omega_2/8)$) is infinite. For other values of k, they are finite. By periodicity and parity, it is enough to prove the algebraicity for $k \in \{1, \ldots, 4\}$ (resp. $k \in \{1, \ldots, 12\}$).

It is important to notice that it suffices to prove the result for $\ell = 0$. Indeed, all the invariants g_2, g_3, $g_2^{1,3}$ and $g_3^{1,3}$ are algebraic functions of z (see Lemmas 7 and 8), so that using inductively the differential equation (32), we obtain the algebraicity for values of $\ell \geqslant 1$ from the algebraicity for $\ell = 0$.

We first consider $\wp$. It is demonstrated in [20, Lemma 19] that $\wp(k\omega_2/8) = \wp^{(0)}(k\omega_2/8)$ is algebraic for $k = 2$ and $k = 4$. For $k = 1$ this follows from the bisection formula (P10) and from the case $k = 2$ (note that $\wp(\omega_1/2)$, $\wp(\omega_2/2)$ and $\wp((\omega_1 + \omega_2)/2)$ are algebraic in z, see [20, Lemma 12]). For $k = 3$, this is a consequence of the addition formula (P4).

We now deal with $\wp_{1,3}$. Let $k \in \{1, \ldots, 12\}$. Using (35), we easily derive that $\wp_{1,3}(\omega_0)$ is algebraic in z as soon as $\wp(\omega_0)$ is algebraic in z: indeed, in (35) the functions $\wp_{1,3}(\omega_2)$ and $\wp_{1,3}'(\omega_2)$ are algebraic in z, as a consequence of Lemma 8. This remark works for all $\omega_0 = k\omega_2/8$, except for $k = 8$, since then $\wp(k\omega_2/8) = \infty$. In fact, for $k = 8$ the situation is also simple, as $\wp_{1,3}(k\omega_2/8) = R/6$ (see Lemma 8) is already known to be algebraic. $\square$

6. Conclusion

Application of our results to other end points. In this article we have presented the first human proofs of Gessel conjecture (Problem (A)) and of the algebraicity of the trivariate GF counting Gessel walks (Problem (B)). We have deduced the closed-form expression (1) of the numbers of walks $q(0,0;n)$ from a new algebraic expression of the GF $\sum_{n\geqslant 0} q(0,0;n)z^n$.

With a very similar analysis, we could obtain an expression for the series $\sum_{n\geqslant 0} q(i,j;n)z^n$, for any fixed couple $(i,j) \in \mathbf{Z}_+^2$. Let us illustrate this fact with the example $(i,j) = (0,j)$, $j \geqslant 0$. Let

$$g_j(z) = \sum_{n\geqslant 0} q(0,j;2n)z^{2n}$$

be the function counting walks ending at the point $(0,j)$ of the vertical axis. We obviously have

$$Q(0,y) = \sum_{j\geqslant 0} y^j g_j(z).$$

In particular, the functions $g_j(z)$ are exactly the successive derivatives of $Q(0,y)$ with respect to (w.r.t.) the variable y evaluated at $y = 0$ (after dividing by $j!$). First, $g_0(z) = Q(0,0)$. Further, one has $r_y(\omega) = z(y(\omega)+1)Q(0,y(\omega))$. Differentiating w.r.t. ω and evaluating at ω_0^y (which is such that $y(\omega_0^y) = 0$), we find

$$g_1(z) = \frac{r_y'(\omega_0^y)}{zy'(\omega_0^y)} - Q(0,0).$$

All quantities above can be computed, and a similar analysis as in Section 4 could lead to a closed-form expression for the numbers of walks $q(0,1;2n)$, $n \geqslant 0$. Similarly, one could compute $g_2(z)$, $g_3(z)$, etc.

New Gessel conjectures. For any $j \geqslant 0$, define

$$f_j(z) = (-1)^j(2j+1)z^j + 2z^{j+1}\sum_{n\geqslant 0} q(0,j;2n)z^n.$$

With this notation, the closed-form expression (1) for the $q(0,0;2n)$ is equivalent to (see [4, 13])

$$f_0\left(z\frac{(1+z)^3}{(1+4z)^3}\right) = \frac{1+8z+4z^2}{(1+4z)^{3/2}}. \tag{42}$$

On March 2013, Ira Gessel [13] proposed the following new conjectures: for any $j \geqslant 1$,

$$f_j\left(z\frac{(1+z)^3}{(1+4z)^3}\right) = (-z)^j\frac{p_j(z)}{(1+4z)^{3/2+3j}}, \tag{43}$$

where $p_j(z)$ is a polynomial of degree $3j+2$ with positive coefficients (due to (42), these new conjectures generalize the original one).

In this article we shall not prove these conjectures. However, we do think that following the method sketched in the first part of Section 6, it could be possible to prove them, at least for small values of $j \geqslant 0$. In particular the fact that the quantity

$$z\frac{(1+z)^3}{(1+4z)^3}$$

is so important and natural for Gessel walks (see indeed (43)) is explained in our Section 4, by the Darboux covering for tetrahedral hypergeometric equations of the Schwarz type $(1/3, 1/3, 2/3)$.

Appendix A. Some properties of elliptic functions

In this appendix, we gather the results we used on $\wp$- and ζ-Weierstrass functions.

Lemma 14. *Let ζ and $\wp$ be the Weierstrass functions with certain periods $\overline{\omega}, \widehat{\omega}$.*

(P1) *We have the expansion*

$$\zeta(\omega) = \frac{1}{\omega} + \sum_{(\overline{n},\widehat{n})\in\mathbb{Z}^2\backslash\{(0,0)\}} \left(\frac{1}{\omega-(\overline{n\omega}+\widehat{n\omega})} + \frac{1}{\overline{n\omega}+\widehat{n\omega}} + \frac{\omega}{(\overline{n\omega}+\widehat{n\omega})^2}\right), \qquad \forall\omega\in\mathbb{C}.$$

As for the expansion of $\wp(\omega)$, it is given in

$$\wp(\omega) = \frac{1}{\omega^2} + \sum_{(\overline{n},\widehat{n})\in\mathbb{Z}^2\backslash\{(0,0)\}} \left(\frac{1}{(\omega-(\overline{n\omega}+\widehat{n\omega}))^2} - \frac{1}{(\overline{n\omega}+\widehat{n\omega})^2}\right), \qquad \forall\omega\in\mathbb{C}. \tag{44}$$

In particular, in the fundamental parallelogram $[0,\overline{\omega}[+[0,\widehat{\omega}[$, ζ (resp. $\wp$) has a unique pole. It is of order 1 (resp. 2), at 0, and has residue 1 (resp. 0, and principal part $1/\omega^2$).

(P2) *An elliptic function with no poles in the fundamental parallelogram $[0,\overline{\omega}[+[0,\widehat{\omega}[$ is constant.*

(P3) *In the neighborhood of 0, the function $\wp(\omega)$ admits the expansion*

$$\wp(\omega) = \frac{1}{\omega^2} + \frac{g_2}{20}\omega^2 + \frac{g_3}{28}\omega^4 + O(\omega^6).$$

(P4) *We have the addition theorems*

$$\zeta(\omega + \widetilde{\omega}) = \zeta(\omega) + \zeta(\widetilde{\omega}) + \frac{1}{2}\frac{\wp'(\omega) - \wp'(\widetilde{\omega})}{\wp(\omega) - \wp(\widetilde{\omega})}, \qquad \forall \omega, \widetilde{\omega} \in \mathbf{C}.$$

and

$$\wp(\omega + \widetilde{\omega}) = -\wp(\omega) - \wp(\widetilde{\omega}) + \frac{1}{4}\left(\frac{\wp'(\omega) - \wp'(\widetilde{\omega})}{\wp(\omega) - \wp(\widetilde{\omega})}\right)^2, \qquad \forall \omega, \widetilde{\omega} \in \mathbf{C}.$$

(P5) *For given* $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_p \in \mathbf{C}$, *define*

$$f(\omega) = c + \sum_{1 \leqslant \ell \leqslant p} r_\ell \zeta(\omega - \widetilde{\omega}_\ell), \qquad \forall \omega \in \mathbf{C}. \tag{45}$$

The function f *above is elliptic if and only if* $\sum_{1 \leqslant \ell \leqslant p} r_\ell = 0$.

(P6) *Let* f *be an elliptic function with periods* $\overline{\omega}, \widehat{\omega}$ *such that in the fundamental parallelogram* $[0, \overline{\omega}[+[0, \widehat{\omega}[$, f *has only poles of order 1, at* $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_p$, *with residues* $r_1, \ldots, r_p$, *respectively. Then there exists a constant* c *such that* (45) *holds.*

(P7) *Let* p *be some positive integer. The Weierstrass elliptic function with periods* $\overline{\omega}, \widehat{\omega}/p$ *can be written in terms of* $\wp$ *as*

$$\wp(\omega) + \sum_{1 \leqslant \ell \leqslant p-1} [\wp(\omega + \ell\widehat{\omega}/p) - \wp(\ell\widehat{\omega}/p)], \qquad \forall \omega \in \mathbf{C}.$$

(P8) *The function* ζ *is quasi-periodic, in the sense that*

$$\zeta(\omega + \overline{\omega}) = \zeta(\omega) + 2\zeta(\overline{\omega}/2), \qquad \zeta(\omega + \widehat{\omega}) = \zeta(\omega) + 2\zeta(\widehat{\omega}/2), \qquad \forall \omega \in \mathbf{C}.$$

(P9) *If* $\alpha + \beta + \gamma = 0$ *then*

$$(\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma))^2 = \wp(\alpha) + \wp(\beta) + \wp(\gamma).$$

(P10) *We have the bisection formula:*

$$\begin{aligned}
\wp(\omega/2) = \wp(\omega) &+ \sqrt{(\wp(\omega) - \wp(\omega_1/2))(\wp(\omega) - \wp(\omega_2/2))} \\
&+ \sqrt{(\wp(\omega) - \wp(\omega_1/2))(\wp(\omega) - \wp((\omega_1 + \omega_2)/2))} \\
&+ \sqrt{(\wp(\omega) - \wp(\omega_2/2))(\wp(\omega) - \wp((\omega_1 + \omega_2)/2))}, \qquad \forall \omega \in \mathbf{C}.
\end{aligned}$$

Proof. All these properties of elliptic functions are classical; they can be found in [1, 16, 34] (more precise references are given in [22]). □

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