Combinatorial Identities

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To my wife Josephine
for her never-failing
support and enthusiasm

"How do I love thee?
Let me count the ways."

Combinatorial Identities

A STANDARDIZED SET OF TABLES

LISTING 500 BINOMIAL COEFFICIENT SUMMATIONS

"Scientia non habet inimicum nisi ignorantiam"

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Revised Edition

Morgantown, W. Va. 1972

INTRODUCTION

Anyone who has taken the time to read far into the vast literature of mathematics will be aware of the fact that summation formulas for binomial coefficients are very widely scattered in books and journals. The situation is parallel with what we should have if no table of integrals existed for our daily use and convenience. It is our object here to attempt to fill this gap and we present what we believe is the beginning of a standard set of tables for binomial coefficient summations.

It would, of course, be a hopeless task to list all known or all possible formulas, and so the best which an aspiring table-maker may hope to achieve is to list at a given date those formulas which seem of frequent occurrence, or which are interesting for some special reason, or which are very general in nature. It might be possible to produce some type of minimal list of formulas together with a set of operating rules whereby other formulas could be obtained, yet we shall not attempt this as such an attempt would destroy the utility of the kind of tables we propose.

Certain lists of identities which exist in the literature deserve special mention here. The lists of Hagen, Netto, Schwatt, Dougall, and Bateman seem very important. The Bateman Manuscript Project, which resulted in the series on special functions, is a monumental achievement in bringing about order in a chaotic field. Bateman himself had projected a book on binomial coefficients apart from their connection with the hypergeometric functions. The author has worked his "shoe box" notes into publishable form. When a truly complete index of formulas is contemplated, such notes and material will be taken into account. See bibliography below.

Binomial coefficient summations have always proved a popular subject in problem departments of mathematical journals. Much interest was stirred up also after A. C. Dixon first found a closed expression for the sum of an alternating series of cubes of binomiel coefficients. This led to the great contributions of [illegible], Franel, and MacMahon on the summation of powers of the binomial coefficients. Many newcomers to this work are, however, unaware of Franel's paper in <u>l'Intermédiaire des Mathématiciens</u>.

The author's interest in this work began twenty-seven years ago with the study of problems in the <u>American Mathematical Monthly</u>. It would be impossible to give complete references for the tables of formulas he has collected and in some cases extended, for he would have to list virtually all the journals which have been at his disposal, and this would include journals and books at the University of Virginia, Duke University, University of North Carolina, the Library of Congress, and West Virginia University. The author was warned long ago that he might well flounder in formulas, if only in one small field, such as Stirling numbers, or Bernoulli numbers, not to mention this vast subject of binomial coefficient summations, and that is one reason he will feel some sense of accomplishment if even a beginning is made in the present tables.

It must be pointed out that we omit proofs in the present work. In most instances the author has worked out proofs which differ from the known proofs; however, to include proofs in the present text would magnify the length of this volume by about ten. The author's personal interest is not the mere tabulation of formulas, but this emphasis is desired in the tables we give here.

PROOFS

As far as proofs are concerned, it is important to indicate the most important methods. These seem to be any one or a combination of the following: comparison of coefficients in series expansions, use of differentiation or integration, finite difference methods, solution of difference equations or recurrence formulas, mathematical induction, enumeration of lattice points, theory of combinations and permutations, and general series transformations. The use of the finite Taylor's series for polynomials is rather important also. In the last part of our index will be found some useful transformations which are especially important in handling the algebra of series. Many of these will be found in intricate detail in Schwatt's Introduction to the Operations with Series [72] .

NOTATIONS AND DEFINITIONS

In the body of our tables, we have used the letters x,y,z for real numbers except in a few instances where some note to the contrary appears. The letters j,m,n,r,s generally have been reserved for non-negative integers. In most cases k has been used for the dummy variable of summation and appears in no other usage. The binomial coefficient $\binom{x}{n}$ is defined by the following:

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} = (-1)^n \prod_{k=1}^{n}\left(1 - \frac{x+1}{k}\right),$$

a polynomial of degree n in x, and we define $x! = \Gamma(x+1)$ in a few cases, but generally we feel it best to avoid use of the standard gamma function. The gamma function of Euler is only one of an infinity of ways of extending the factorial concept.

By extensive use of relations such as

$$\binom{x}{n} = \binom{x}{x-n} = \frac{x}{n}\binom{x-1}{n-1},$$

$$\binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1},$$

$$\binom{-x}{n} = (-1)^n \binom{x+n-1}{n},$$

$$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{2n}{n} 2^{-2n},$$

many variations of the listed formulas we give may be found. A few of the more useful binomial coefficient relations of this type are collected together in our last table.

Although we do not include formulas which sum series involving Bernoulli numbers, Stirling numbers, Euler numbers, etc., we do need to define the Stirling numbers inasmuch as many sums involving binomial coefficients can only be given in terms of these. We shall define the <u>Stirling numbers of the first kind</u> by the coefficient C_k^n in the expansion

$$\binom{x}{n} = \sum_{k=0}^{n} C_k^n \; x^k .$$

The <u>Stirling numbers of the second kind</u> we define by the expansion

$$x^n = \sum_{k=0}^{n} B_k^n \binom{x}{k} .$$

See the author's papers (second bibliography) [11], [18], [37], [41], [44], [62] for discussions of notation and other properties. A better notation $S_1(n,k)$, $S_2(n,k)$ was introduced in [11].

In particular it follows that

$$B_k^n = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n ,$$

and

$$(*) \qquad n!\, C_{n-k}^n = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j} \binom{k-n}{k+j} \frac{1}{j!} B_j^{j+k} ,$$

the latter being perhaps the simplest expression known for the Stirling numbers of the first kind, given by L. Schläfli and O. Schlömilch. The reader should consult [27], [33], [37], [45], [61] for details regarding the Stirling numbers. In [37] the author proved a formula inverse to (*) above.

Iterated summations of terms have been omitted, since their inclusion would necessitate a tabulation of much greater length.

The reception accorded the first printing (1959-60) of these tables has been so overwhelmingly favorable and continued, that the demand for copies has long since exhausted the supply. This corrected reprint has been undertaken to meet the need for such tables. Rather than retype the entire manuscript and run the risk of introducing new errors, it has been decided to correct those misprints which are easy to change. At the end will be found a set of errata tables, bringing together those errata which have been called to the attention of the author in the intervening years. These lists will be useful to anyone who has access to one of the original printings of these tables. The bibliography has not been altered; however a list of the author's papers is included so that the reader can consult those items for a general idea of the trend of research since 1960. Most of this work has been supported by research grants from the National Science Foundation since 1960.

A few new formulas have been introduced, just those which seem to be absolutely essential. All binomial formulas are kept on file in a card catalogue, and proofs are kept in running volumes of notes now covering several thousand pages. It is abundantly clear that this reprint will serve a useful purpose until the author completes a manuscript in preparation for a completely new treatise on the subject of combinatorial identities.

Such a new treatise will offer many new identities, a complete revision of the present tables, model proofs, hundreds of references to the literature and a whole chapter on the generalized Vandermonde and Abel convolutions which have occupied much of the author's attention.

The author is obliged to Drs. T. A. Botts and Reed Dawson who originally suggested the compilation of the tables. He must express his deepest thanks to the following who have made many suggestions and in many cases supplied corrections to the tables or suggested new formulas: L. Carlitz, John Riordan, Donald Knuth, Eldon R. Hansen, R. E. Greenwood, Michael P. Drazin, Verner E. Hoggatt, Jr., Julius Lieblein, Emory P. Starke, and many others too numerous to mention.

The author would be pleased to hear from users of the tables, and welcomes any summation problem involving binomial coefficients whether the user knows the sum or not. All suggestions for improvement will be carefully studied.

The attention of the reader is called to a very important recent book COMBINATORIAL IDENTITIES, by John Riordan, published by John Wiley and Sons, New York, 1968. This is the first book-length treatise dealing entirely with the theory of combinatorial identities. The subject matter covers: recurrences, inverse relations (two chapters), generating functions, partition polynomials, and operators. Anyone dealing with our subject matter should consult Riordan's book. Some of this writer's work on inverse binomial series relations is treated in detail in Riordan's book. There has been a great flood of books recently dealing with combinatorial theory, but the subject of combinatorial identities remains scattered in the literature of several hundred journals, in portions of books, in unpublished manuscripts and files, etc. It is expected that this situation will shortly change. There is a need for pedagogical treatises on the subject. In THE ART OF COMPUTER PROGRAMMING, by Donald Knuth, Addison-Wesley, 1968, Vol.1, pp.53-73, the reader will find a good, leisurely account of techniques for attacking a problem of summing a binomial series.

A NOTE ABOUT THE SYSTEM OF INDEXING

It is important in a compilation such as this to make use of a system of indexing which will be easy to use and flexible enough to allow the addition of a few formulas from time to time without completely destroying the old ordering.

We therefore draw up a sequence of tables of formulas, numbered 0, 1, 2, 3, etc. Each table lists formulas of the type S:p/q for certain values of p and q, where p denotes the number of binomial coefficients appearing in the general numerator term of the summation, and q has the same meaning for the general denominator term.

The tables are to be found arranged in the ordering

0/0, 1/0, 0/1, 2/0, 1/1, 0/2, 3/0, 2/1, 1/2, 0/3, 4/0, 3/1, ...etc.

It is easy to prove that the table number, n, assigned to each type formula, S:p/q, is given uniquely according to the formula

$$n = q + \sum_{k=0}^{p+q} k = q + \tfrac{1}{2}(p+q)(p+q+1) \quad ..$$

Thus, in this system, all formulas for sums of form S:4/3 should be listed in table number 31. The proof of this is related to the usual proof of the countability of the rational numbers.

Conversely, if we desire to find out what type of sum is to be found in a table of number n we may determine this from the formulas

$$p = \tfrac{1}{2}(N-1)(N+2) - n \quad , \qquad\qquad q = n - \tfrac{1}{2}N(N-1) \quad ,$$

where $N = \left[\dfrac{1+\left[\sqrt{8n+1}\right]}{2}\right]$, and where $[x]$ means the integral part of x. The proof depends on the fact that every odd square is of the form $8n+1$. Thus for table 31, $N = 8$, so $p = \tfrac{1}{2}(7)(10) - 31 = 35 - 31 = 4$, and $q = 31 - \tfrac{1}{2}(8)(7) = 31 - 28 = 3$.

In this second edition of our formulary, the following tables are listed and the indicated approximate number of summations of specified type are to be found within each table:

TABLE NR.	Form S:p/q listed	Approx. nr. sums listed
1	1/0	135
2	0/1	26
3	2/0	180
4	1/1	29
5	0/2	2
6	3/0	52
7	2/1	48
8	1/2	1
10	4/0	9
11	3/1	3
12	2/2	8
16	4/1	3
17	3/2	5
21	6/0	1
22	5/1	2
23	4/2	1
24	3/3	1
31	4/3	2
71	6/5	1
97	7/6	1
X	p/0	15
Z	General theorems	30
		TOTAL: 555

As might be expected, the sums of form S:2/0, S:1/0, S:2/1, and S:3/0 are most well known. In general sums of the form S:0/q , with large values of q, are relatively rare in the literature.

An important factor to keep in mind is that there is no unique form S:p/q in which a formula of some basic type is always written, because by use of elementary binomial coefficient identities we may rewrite a given formula using more or fewer binomial coefficients. Thus there will always exist a large degree of what is really cross-classification. However we do not propose to draw up a minimal list of formulas. An axiomatic study is planned later.

TABLE 1

SUMMATIONS OF THE FORM S:1/0

(1.1) $$\sum_{k=0}^{\infty} \binom{x}{k} z^k = (1+z)^x ,$$ Binomial Theorem ,

Valid for arbitrary complex x, and complex $|z|<1$, where the principal value of $(1+z)^x$ is taken.

Convergence is irrelevant when we think of this as a generating function.

(1.2) $$\sum_{k=0}^{\infty} (-1)^k \binom{x}{k} = 0 , \qquad R(x) > 0 ,$$

(1.3) $$\sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \frac{1}{(1-x)^{n+1}} , \qquad |x|<1 ,$$

(1.4) $$\sum_{k=a}^{n} (-1)^k \binom{x}{k} = (-1)^a \binom{x-1}{a-1} + (-1)^n \binom{x-1}{n} ,$$

(1.5) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k} = (-1)^n \binom{x-1}{n} = \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right) ,$$

(1.6) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k} k^r = \sum_{k=0}^{r} (-1)^k \binom{x}{k} \binom{n-x}{n-k} B_k^r ,$$

(1.7) $$\lim_{n\to\infty} n^{x-r} \sum_{k=0}^{n-1} (-1)^k \binom{x}{k} k^r = \frac{1}{(r-x)\,\Gamma(-x)} ,$$

(Prob. 4551, Amer.Math.Monthly,1953,p.482)

(1.8) $$\sum_{k=0}^{n} \binom{x+1}{k} z^k = \sum_{k=0}^{n} \binom{x-n+k}{k} z^k (1+z)^{n-k} ,$$

$$(1.9)\quad \sum_{k=0}^{n} \binom{x}{k} y^k = \sum_{k=0}^{n} \binom{n-x}{k} (1+y)^{n-k} (-y)^k ,$$

$$(1.10)\quad \sum_{k=0}^{n-1} \binom{z}{k} x^{n-k-1} = \sum_{k=1}^{n} \binom{z-k}{n-k} (x+1)^{k-1} ,$$

$$(1.11)\quad \sum_{k=0}^{n-1} \binom{z}{k} \frac{x^{n-k}}{n-k} = \sum_{k=1}^{n} \binom{z-k}{n-k} \frac{(x+1)^k - 1}{k} ,$$

$$(1.12)\quad \sum_{k=0}^{n} \binom{n}{k} \frac{x^{r+k}}{r+k} = \sum_{k=1}^{r} (-1)^{r-k} \binom{r-1}{r-k} \frac{(x+1)^{n+k} - 1}{n+k} ,$$

$$(r \geq 1)$$

$$(1.13)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j \begin{aligned} &= 0 , && 0 \leq j < n \\ &= (-1)^n n! , && j = n , \end{aligned}$$

$$(1.14)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n+1} = \frac{2x-n}{2} (n+1)! ,$$

$$(1.15)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n+2} = \frac{3n^2 + n + 12x^2 - 12nx}{24} (n+2)! ,$$

$$(1.16)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n+j} = \sum_{k=0}^{j} \binom{x-n}{k} B_{n+k}^{n+j} ,$$

Let $S_j = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+j}$.

Then

$$S_j = (-1)^n \frac{(n+j)!}{j!} B_j^{(-n)} ,$$

(1.17)

$$= (-1)^n (n+j)! \sum_{k=0}^{j} \binom{j-n}{j-k}\binom{n}{k} \frac{1}{(k+j)!} B_k^{k+j} ,$$

Special cases:

$$S_0 = (-1)^n n! , \qquad S_1 = (-1)^n (n+1)! \cdot \frac{n}{2} ,$$

(1.18) $$S_2 = (-1)^n \frac{(n+2)!}{2} \cdot \frac{n(3n+1)}{12} ,$$

(1.19) $$S_3 = (-1)^n \frac{(n+3)!}{6} \cdot \frac{n^2(n+1)}{8} ,$$

(1.20) $$S_4 = (-1)^n \frac{(n+4)!}{24} \cdot \frac{n(15n^3+30n^2+5n-2)}{240} ,$$

(1.21) $$S_5 = (-1)^n \frac{(n+5)!}{120} \cdot \frac{n^2(3n^3+10n^2+5n-2)}{96} ,$$

(1.22) $$S_6 = (-1)^n \frac{(n+6)!}{6!} \cdot \frac{n(63n^5+315n^4+315n^3-91n^2-42n+16)}{4032}$$

(Values of S_5 and S_7 printed in [72] were incorrect, see page 102 that text.)

(1.23) $$\sum_{k=0}^{\infty} \binom{n+k}{k} 2^{-k} = 2^{n+1} ,$$

(1.24) $$\sum_{k=0}^{n} \binom{n}{k} = 2^n ,$$

$$(1.25)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 , & n \neq 0 , \\ 1 , & n = 0 . \end{cases}$$

$$(1.26)\quad \sum_{k=0}^{n} \binom{n}{k} \cos kx = 2^n \cdot \cos \frac{nx}{2} \cdot \left(\cos \frac{x}{2}\right)^n ,$$

$$(1.27)\quad \sum_{k=0}^{n} \binom{n}{k} \sin kx = 2^n \cdot \sin \frac{nx}{2} \cdot \left(\cos \frac{x}{2}\right)^n ,$$

$$(1.28)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos kx = (-1)^n \cdot 2^n \cdot \left(\sin \frac{x}{2}\right)^n \cdot \cos \frac{n(x+\pi)}{2} ,$$

$$(1.29)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sin kx = (-1)^n \cdot 2^n \cdot \left(\sin \frac{x}{2}\right)^n \cdot \sin \frac{n(x+\pi)}{2} ,$$

$$(1.30)\quad \sum_{k=0}^{[n/2]} \binom{n}{k} \cos(n-2k)x = 2^{n-1} \cdot \cos^n x + \frac{1}{2} \binom{n}{n/2} \frac{1+(-1)^n}{2} ,$$

$$(1.31)\quad \sum_{k=0}^{[n/2]} \binom{n}{2k} \cos kx = 2^{n-1} \left\{ \cos^n \left(\frac{x}{4}\right) \cdot \cos \frac{nx}{4} + (-1)^n \cdot \sin^n \left(\frac{x}{4}\right) \cdot \cos \frac{n(2\pi+}{4} \right.$$

$$(1.32)\quad \sum_{k=0}^{[n/2]} \binom{n}{2k} \sin kx = 2^{n-1} \left\{ \cos^n \left(\frac{x}{4}\right) \cdot \sin \frac{nx}{4} + (-1)^n \cdot \sin^n \left(\frac{x}{4}\right) \cdot \sin \frac{n(2\pi+}{4} \right.$$

$$(1.33)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \cos(2k+1)x$$

$$= 2^{n-1}\left\{\cos^n(x/2)\cos(nx/2) - (-1)^n \sin^n(x/2)\cos\frac{n(\pi+x)}{2}\right\}$$

$$(1.34)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \sin(2k+1)x$$

$$= 2^{n-1}\left\{\cos^n(x/2)\sin(nx/2) - (-1)^n \sin^n(x/2)\sin\frac{n(\pi+x)}{2}\right\}$$

$$(1.35)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} = 2^{n-1} + \frac{1}{2}\binom{n}{n/2}\frac{1+(-1)^n}{2},$$

$$(1.36)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} = 2^{n-1} - \frac{1 \mp (-1)^n}{4}\binom{n}{n/2},$$

$$(1.37)\quad \sum_{k=0}^{n} \binom{n}{k}\frac{x^k}{k+1} = \frac{(x+1)^{n+1}-1}{(n+1)x},$$

$$(1.38)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\frac{x^{2k}}{2k+1} = \frac{(x+1)^{n+1}-(1-x)^{n+1}}{2(n+1)x},$$

$$(1.39)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}\frac{x^{2k}}{k+1} = \frac{(x+1)^{n+1}+(1-x)^{n+1}-2}{(n+1)x^2},$$

$$(1.40)\quad \sum_{k=0}^{\infty} (-1)^k\binom{x}{k}\frac{1}{z+k} = \frac{\Gamma(z)\,\Gamma(x+1)}{\Gamma(x+z+1)} = \frac{1}{z\binom{x+z}{x}}$$

$$R(x) > -1$$

$$(1.41)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x}{x+k} = \frac{1}{\binom{x+n}{n}} ,$$

Closely related to this is the following sum of form S:1/1 :

$$(1.42)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{\binom{x+k}{k}} = \frac{x}{x+n} ,$$

The proof is related to theorem (Z.23).

$$(1.43)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{x-k} = \frac{(-1)^n}{(x-n)\binom{x}{n}} ,$$

$$(1.44)\quad \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{\binom{k+1/a}{k}} = \sum_{k=0}^{n} \frac{1}{ak+1} ,$$

$$(1.45)\quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} ,$$

The following sum is an S:1/0 or a special instance of an S:1/1 :

$$(1.46)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+3)(k+4)(k+7)}$$

$$= \frac{1}{n(n+1)} \left\{ \frac{1}{2(n+3)} + \frac{1}{2\binom{n+7}{5}} - \frac{1}{\binom{n+4}{2}} \right\} ,$$

$$(1.47)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^j}{x+k} = (-1)^j \frac{x^{j-1}}{\binom{x+n}{n}} , \qquad (j \leq n)$$

$$(1.48)\quad \sum_{k=0}^{n} \binom{k+x}{r} = \binom{n+x+1}{r+1} - \binom{x}{r+1} ,$$

$$(1.49)\quad \sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n} ,$$

$$(1.50)\quad \sum_{k=1}^{n} \binom{x+k}{k} \frac{1}{x+k} = \frac{1}{x}\left\{\binom{x+n}{n} - 1\right\} ,$$

$$(1.51)\quad \sum_{k=a}^{n} \binom{k}{x} = \binom{n+1}{x+1} - \binom{a}{x+1} ,$$

$$(1.52)\quad \sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1} ,$$

$$(1.53)\quad \sum_{k=0}^{\left[\frac{n-a}{r}\right]} \binom{n}{a+kr} x^{a+kr} = \frac{1}{r}\sum_{j=1}^{r} (\omega_r{}^j)^{-a} (1+x\omega_r{}^j)^n ,$$

where $\omega_r = e^{2\pi i/r}$. $(r-1 \geqslant a)$ $(a = \text{pos. or neg. integer})$

$$(1.54)\quad \sum_{k=0}^{\left[\frac{n-a}{r}\right]} \binom{n}{a+kr} = \frac{1}{r}\sum_{j=1}^{r} \left(2\cos\frac{\pi j}{r}\right)^n \cos\frac{(n-2a)j\pi}{r} ,$$

$(n \geqslant a \geqslant 0,\ r-1 \geqslant a)$

$$(1.55)\quad \sum_{k=0}^{\left[\frac{n}{r}\right]} \binom{n}{rk} = \frac{1}{r}\sum_{j=1}^{r} (1+\omega_r{}^j)^n = \frac{2^n}{r}\sum_{j=1}^{r}\left(\cos\frac{\pi j}{r}\right)^n \cos\frac{n\pi j}{r}$$

$$(1.56)\quad \sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{n}{3k} = \frac{1}{3}\left\{2^n + 2\cos\frac{n\pi}{3}\right\} ,$$

$$(1.57)\quad \sum_{k=0}^{\left[\frac{n-1}{3}\right]} \binom{n}{3k+1} = \frac{1}{3}\left\{2^n + 2\cos\frac{(n-2)\pi}{3}\right\},$$

$$(1.58)\quad \sum_{k=0}^{\left[\frac{n}{4}\right]} \binom{n}{4k} = \frac{1}{4}\left\{2^n + 2(\sqrt{2})^n \cos\frac{n\pi}{4}\right\},$$

$$(1.59)\quad \sum_{k=0}^{n} \binom{4n}{4k} = \frac{1}{4}\left\{2^{4n} + (-1)^n\, 2^{2n+1}\right\},$$

$$(1.60)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x-y},$$

$$(1.61)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} 2^{n-2k} z^k = \frac{x^{n+1} - y^{n+1}}{x-y}$$

where: $x = 1+\sqrt{z+1}$, $y = 1-\sqrt{z+1}$

$$(1.62)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (2\cos x)^{n-2k} = \frac{\sin(n+1)x}{\sin x},$$

$$(1.63)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{(2\cos x)^{n-2k}}{n-k} = \frac{2}{n}\cos nx,$$

$$(1.64)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \frac{1}{n-k} \left(\frac{z}{4}\right)^k = \frac{1}{n\,2^{n-1}} \cdot \frac{x^n + y^n}{x+y},$$

where: $x = 1+\sqrt{z+1}$, $y = 1-\sqrt{z+1}$

$$(1.65)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{2^{2n-2k}}{n-k} = \frac{2^{n+1}}{n}, \quad (n \geqslant 1)$$

$$(1.66)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{(2\cos x)^{n-2k}}{k+1} = \frac{(2\cos x)^{n+2} - 2\cos(n+2)x}{n+2},$$

$$(1.67)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{2^{2n-2k}}{k+1} = \frac{4^{n+1} - 2^{n+1}}{n+2},$$

$$(1.68)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{1}{n-k} = \begin{cases} (-1)^n \frac{2}{n}, & \text{if } 3 \mid n, \\ (-1)^{n-1} \frac{1}{n}, & \text{if } 3 \nmid n, \end{cases}$$

(Cambridge Math. Tripos, 1932; Hardy, Pure Math., page 445.)

$$(1.69)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \frac{6^k}{n-k} = \frac{3^n + (-1)^n 2^n}{n}, \quad (n \geqslant 1)$$

$$(1.70)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} z^k = \frac{(2z+1 \pm \sqrt{1+4z})^{n+1} + (-1)^n (2z)^{n+1}}{2^{n/2}(2z+1 \pm \sqrt{1+4z})^{n/2}(4z+1 \pm \sqrt{1+4z})},$$

$$(1.71)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} z^k = \frac{x^{n+1} - 1}{(x-1)(1+x)^n}, \quad \text{where: } z = \frac{-x^2}{(1+x)^2},$$

$$(1.72)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} 2^{n-2k} = n+1,$$

$$(1.73)\sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} \frac{1}{2^{2k}} = \frac{2n+1}{2^{2n}} ,$$

$$(1.74)\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

$= a_n$ = n-th Fibonacci number, where $a_0 = 1 = a_1$,
and $a_{n+1} = a_n + a_{n-1}$.

$$(1.75)\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} = \frac{(-1)^{\left[\frac{n}{3}\right]} + (-1)^{\left[\frac{n+1}{3}\right]}}{2} ,$$

$$(1.76)\sum_{k=0}^{n} \binom{n+k}{2k} = \sum_{k=0}^{n} \binom{2n-k}{k} = a_{2n} \text{ using (1.74)} ,$$

$$(1.77)\sum_{k=0}^{n} \binom{n+k}{2k} 2^{n-k} = \frac{2^{2n+1}+1}{3} ,$$

$$(1.78)\sum_{k=0}^{n} \binom{n+k}{k} \left\{(1-x)^{n+1} x^k + x^{n+1}(1-x)^k\right\} = 1 ,$$

$$(1.79)\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^n ,$$

$$(1.80)\sum_{k=1}^{\infty} \binom{2n+k}{n} 2^{-k} = 2^{2n} ,$$

$$(1.81)\quad \sum_{k=0}^{n} \binom{2n-k}{n} 2^{k} = 2^{2n} \quad ,$$

$$(1.82)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{2n}{n+2k} = 2^{2n-2} + \binom{2n-1}{n} \quad , (n \geqslant 1)$$

$$(1.83)\quad \sum_{k=0}^{n} \binom{2n+1}{k} = 2^{2n} \quad ,$$

$$(1.84)\quad \sum_{k=0}^{n} \binom{2n-1}{k} = 2^{2n-2} + \binom{2n-1}{n}, \quad (n \geqslant 1)$$

$$(1.85)\quad \sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \binom{2n-1}{n} \quad , \quad (n \geqslant 1)$$

$$(1.86)\quad \sum_{k=0}^{n} (-1)^{k} \binom{2n}{k} = (-1)^{n} \binom{2n-1}{n} \quad ,$$

$$(1.87)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} x^{k} = \frac{(1+\sqrt{x})^{n} + (1-\sqrt{x})^{n}}{2} \quad ,$$

$$(1.88)\quad \sum_{k=0}^{n} (-1)^{k} \binom{2n}{2k} = (-1)^{n/2}\, 2^{n}\, \frac{1+(-1)^{n}}{2} \quad ,$$

$$(1.89)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} = 2^{n-1} \quad , \quad (n \geqslant 1)$$

$$(1.90)\quad \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} = \frac{(1+i)^n + (1-i)^n}{2} = (\sqrt{2})^n \cos \frac{n\pi}{4} ,$$

$$(1.91)\quad \sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=0}^{n} \binom{2n}{2k+1} = 2^{2n-1} , \quad (n \geq 1)$$

$$(1.92)\quad \sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{2k} = 2^{2n-2} + \frac{1+(-1)^n}{2} \binom{2n-1}{n} , \quad (n \geq 1)$$

$$(1.93)\quad \sum_{k=0}^{n} \binom{2n+1}{2k} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n} ,$$

$$(1.94)\quad \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} = (-1)^{[\frac{n+1}{2}]} 2^n ,$$

$$(1.95)\quad \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} x^k = \frac{(1+\sqrt{x})^n - (1-\sqrt{x})^n}{2\sqrt{x}} , \quad (n \geq 1)$$

$$(1.96)\quad \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} = \frac{(1+i)^n - (1-i)^n}{2i} = (\sqrt{2})^n \sin \frac{n\pi}{4} ,$$

$$(1.97)\quad \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} = 2^{n-1} ,$$

$$(1.98)\quad \sum_{k=0}^{[\frac{n-1}{2}]} \binom{2n}{2k+1} = 2^{2n-2} + \frac{1-(-1)^n}{2} \binom{2n-1}{n} ,$$

(1.99) $$\sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} = (-1)^{\left[\frac{n}{2}\right]} 2^n \frac{1-(-1)^n}{2} ,$$

(1.100) $$\sum_{k=0}^{n} \binom{2n+1}{2k+1} k = (2n-1)\, 2^{2n-2} , \quad (n \geq 1)$$

(1.101) $$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} = (-1)^{\left[\frac{n}{2}\right]} 2^n ,$$

(1.102) $$\sum_{k=0}^{n} (-1)^k \binom{n+x}{n-k} \frac{1}{k+1} = \binom{B(x)+n}{n} , \text{ symbolically,}$$

where $B^n(x)$ means $B_n(x)$ = Bernoulli polynomial.

(1.103) $$\sum_{k=0}^{n} \binom{x}{k} \frac{k \cdot k!}{x^{k+1}} = 1 - \binom{x}{n+1} \frac{(n+1)!}{x^{n+1}} ,$$

(1.104) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{(1-2k)\, 2^{2k}} = \sqrt{1-x} , \quad (|x| \leq 1)$$

(1.105) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^{2k}} = \frac{1}{\sqrt{1-x}} , \quad (|x| < 1, \text{ also valid for } x = -1$$

(1.106) $$\sum_{k=1}^{\infty} (-1)^{k-1} \binom{2k}{k} \frac{x^k}{k\, 2^{2k+1}} = \log \frac{1+\sqrt{1+x}}{2} , \quad (|x| < 1)$$

(1.107) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{(2k+1)^2\, 2^{2k}} = \int_0^1 \frac{\arcsin x}{x}\, dx = \frac{\pi}{2} \log 2 ,$$

$$(1.108)\qquad \sum_{k=0}^{n} \binom{2k}{k} \frac{k^r}{2^{2k}} = S_r = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{k=0}^{r} \binom{n}{k} \frac{B_k^r}{2k+1} ,$$

$$(1.109)\qquad S_0 = \frac{2n+1}{2^{2n}} \binom{2n}{n} = \binom{n+\frac{1}{2}}{n} ,$$

$$(1.110)\qquad S_1 = \frac{n}{3} S_0 ,$$

$$(1.111)\qquad S_2 = \frac{n(3n+2)}{3(5)} S_0 ,$$

$$(1.112)\qquad S_3 = \frac{n(15n^2+18n+2)}{3(5)(7)} S_0 ,$$

$$(1.113)\qquad \sum_{k=0}^{n} \binom{2k}{k} \frac{k \cdot k!}{2^k} = \binom{2n+2}{n+1} \frac{(n+1)!}{2^{n+2}} - \frac{1}{2} ,$$

BERNSTEIN POLYNOMIALS:

$$(1.114)\qquad B_n^f(x) = \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k f\left(\frac{k}{n}\right) ,$$

GENERALIZED LAGUERRE POLYNOMIALS:

$$(1.115)\qquad L_n^{(a)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+a}{n-k} \frac{x^k}{k!} ,$$

$$= \frac{1}{n!} x^{-a} e^x D_x^n \left\{ x^{a+n} e^{-x} \right\} ,$$

Ordinary Laguerre Polynomials: $L_n(x) = L_n^{(0)}(x)$.

$$(1.116)\qquad \sum_{k=0}^{n} \binom{p-k}{n-k} x^k = x^{p+1} (x-1)^{n-p-1} , \text{ provided } 0 \le p \le n-1 ,$$

(1.117) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{n-k} (k+1)^{k-1} = (x-1)^n \quad ,$$

(This and the next two are special cases of a general formula due to Abel.)

(1.118) $$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{k} (yk)^{n-k} (x-yk)^k = \frac{x^n}{n} \quad , \qquad \text{(Abel)}, \; n \geq 2$$

(1.119) $$\sum_{k=1}^{n} \binom{n}{k} (yk)^{n-k} (x-yk)^{k-1} = x^{n-1} \quad , \qquad \text{(Abel)}$$

(1.120) $$\sum_{k=0}^{\infty} \binom{a+bk}{k} z^k = \frac{x^{a+1}}{(1-b)x+b} \quad , \quad z = \frac{x-1}{x^b} \quad , \quad |z| < \left| \frac{(b-1)^{b-1}}{b^b} \right|$$

(Polya and Szegö)

(1.121) $$\sum_{k=0}^{\infty} \frac{a}{a+bk} \binom{a+bk}{k} z^k = x^a$$,(same conditions as in above.)

(1.122) $$\sum_{k=-\infty}^{+\infty} \binom{n+ak}{b+ck}$$, a discussion has been given by D. Dickinson [17].

(1.123) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{(2k+1)^n \; 2^{2k}}$$ was studied by S. Ramanujan, in whose collected works one may find evaluations of this.

(1.124) $$\sum_{k=0}^{n} \binom{n}{k} (x+kz)^k (y-kz)^{n-k} = n! \sum_{k=0}^{n} \frac{(x+y)^k}{k!} z^{n-k}$$

(1.125) $$\sum_{k=0}^{n} B_k(x,z)\, B_{n-k}(y,z)\, (p+qk) = \frac{p(x+y)+qnx}{x+y} B_n(x+y,z),$$

where $B_k(x,z) = \frac{x}{k!} (x+kz)^{k-1}$.

(This is a Generalized Abel Convolution. Compare with (3.142)-(3.148).)

$$(1.126) \qquad \sum_{k=0}^{n} \binom{n}{k} k^r \; x^k$$

$$= (1+x)^n \sum_{j=0}^{r} (-1)^j \binom{n}{j} \frac{x^j}{(1+x)^j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^r \quad ,$$

(1.127) (Definition of Eulerian numbers)

$$x^n = \sum_{k=0}^{n} \binom{x+k-1}{n} A_{nk} \quad ,$$

and explicitly

$$A_{nk} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \quad ,$$

(Cf. Carlitz, Math. Mag., Vol.32(1959), p.247)

These numbers of Euler (also Worpitzky) are a special case of numbers due to Nielsen and which we might define as follows:

$$(1.128) \qquad B^n_{r,q} = \sum_{k=0}^{r} (-1)^k \binom{q}{k} (r-k)^n \quad .$$

From this it follows that $x^n = (-1)^{m+n} \sum_{k=0}^{m+1} B^n_{k,m+1} \binom{x+k-1}{m}$,

for all integers $m \geq n$.

$$(1.129)\quad \sum_{k=0}^{n}\binom{2k+1}{j} = \frac{(-1)^{j-1}}{2^{j+2}}\left\{\sum_{k=0}^{j+1}(-1)^k\binom{2n+3}{k}2^k + 1\right\},$$

(Schwatt [72], p.48)

$$(1.130)\quad \sum_{k=0}^{n}(-1)^k\binom{j+k}{j} = \frac{(-1)^j}{2^{j+1}}\left\{(-1)^n\sum_{k=0}^{j}(-1)^k\binom{n+j+1}{k}2^k + (-1)^j\right\}$$

(Schwatt [72], p. 51)

Schwatt gives various other more complicated examples of this, such as formulas for

$$\sum_{k=0}^{n}(-1)^k\binom{2k+1}{j} \quad \text{and} \quad \sum_{k=0}^{n}\binom{3k-2}{j}, \text{ etc.}$$

$$(1.131)\quad \sum_{j=k}^{n}\binom{n-a+1}{j-a+1}(1-x)^{n-j}x^j = \sum_{j=k}^{n}\binom{j-a}{k-a}(1-x)^{j-k}x^k, \qquad \text{(Gould)}$$

$$(1.132)\quad \sum_{k=m}^{n-1}\binom{k-1}{m-1}\frac{1}{n-k} = \binom{n-1}{m-1}\sum_{k=m}^{n-1}\frac{1}{k}, \qquad \text{(Chung)}$$

$$(1.133)\quad \sum_{k=0}^{n}\binom{x-k}{n-k} = \binom{x+1}{n},$$

$$(1.134)\quad \sum_{k=a}^{n}(-1)^{k-a}\binom{n}{k}\frac{1}{k} = \sum_{k=a}^{n}\binom{k-1}{a-1}\frac{1}{k}, \qquad \text{(Saalschütz)}$$

$$(1.135)\quad \left\{\sum_{k=0}^{n}\binom{-1/2}{k}\right\}^2 = \binom{-1/2}{n}\sum_{k=0}^{n}\binom{-1/2}{n-k}\frac{2n+1}{2k+1},$$

(Paul Bruckman)

TABLE 2

SUMMATIONS OF THE FORM S:0/1

1) $$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{x}{k}} = \frac{x+1}{x+2}\left\{1 + \frac{(-1)^n}{\binom{x+1}{n+1}}\right\} ,$$

2) $$\sum_{k=0}^{n} \frac{1}{\binom{x+k}{k}} = \frac{x}{x-1}\left\{1 - \frac{1}{\binom{x+n}{n+1}}\right\} ,$$

3) $$\sum_{k=j}^{n} \frac{1}{\binom{k}{j}} = \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{1}{k} , & j = 1 , \\ \displaystyle\frac{j}{j-1}\left\{1 - \frac{1}{\binom{n}{j-1}}\right\}, & j \neq 1 \end{cases}$$

4) $$\sum_{k=0}^{n} \frac{x^k}{\binom{n}{k}} = (n+1)\left(\frac{x}{1+x}\right)^{n+1} \sum_{k=1}^{n+1} \frac{1}{k}\,\frac{1+x^k}{1+x}\left(\frac{1+x}{x}\right)^k ,$$

$$= S_n(x) ,$$

5) $$\left(1 + \frac{1}{x}\right) S_{n+1}(x) = \frac{n+2}{n+1} S_n(x) + x^{n+1} + \frac{1}{x} ,$$

6) $$\sum_{k=1}^{r} \frac{n-2k}{\binom{n}{k}} = 1 - \frac{n+1}{\binom{n+1}{r+1}} ,$$

$$(2.7)\quad \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\binom{2n}{k}} = \frac{1}{2(n+1)} + \frac{(-1)^{n-1}}{2\binom{2n}{n}} ,$$

$$(2.8)\quad \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{k}{\binom{2n}{k}} = \frac{n}{n+1} ,$$

$$(2.9)\quad \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} = \frac{2^{2n}}{\binom{2n}{n}} - 1 ,$$

$$(2.10)\quad \sum_{k=n}^{\infty} \frac{1}{\binom{k}{r}} = \frac{n}{r-1} \; \frac{1}{\binom{n}{r}} , \quad (r>1),$$

$$(2.11)\quad \sum_{k=1}^{\infty} \frac{1}{k\binom{k+n}{k}} = \frac{1}{n} , \quad (n \geqslant 1) ,$$

$$(2.12)\quad \sum_{k=1}^{\infty} \frac{1}{k^2\binom{k+n}{k}} = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} , \quad (n \geqslant 1) ,$$

$$(2.13)\quad \sum_{k=0}^{\infty} \frac{x^{k+1}}{\binom{z}{k}} = \sum_{k=0}^{\infty} \frac{z+1}{z+1-k} \left(\frac{x}{x+1}\right)^{k+1} ,$$

$$R(x) < \tfrac{1}{2} , \quad z \neq -1, 0, 1, 2, \ldots$$

$$(2.14)\quad \sum_{k=0}^{\infty} \frac{x^{k+1}}{\binom{z+k}{k}} = \sum_{k=0}^{\infty} (-1)^k \frac{z}{z+k} \left(\frac{x}{1-x}\right)^{k+1} ,$$

$$R(x) < \tfrac{1}{2} , \quad z \neq 0, -1, -2, \ldots .$$

$$\sum_{k=0}^{\infty} \frac{x^k}{\binom{x+k}{k} k!} = x \sum_{k=0}^{\infty} \frac{k^k}{k!} \frac{e^{-k}}{x+k} \quad ,$$

16) $$\sum_{k=0}^{\infty} \frac{1}{\binom{x+k}{k} k!} = ex \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{1}{x+k} \quad ,$$

17) $$\sum_{k=1}^{n} \frac{1}{k^2} \left\{ 1 - \frac{1}{\binom{k+r}{k}} \right\} = \sum_{k=1}^{r} \frac{1}{k^2} \left\{ 1 - \frac{1}{\binom{k+n}{k}} \right\} ,$$

18) $$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{\binom{2n}{k}} \sum_{j=1}^{k} \frac{1}{j} = \frac{n}{2(n+1)^2} + \frac{1}{2n+2} \sum_{k=1}^{2n} \frac{1}{k} ,$$

19) $$\sum_{k=0}^{2n-1} \frac{(-1)^k}{(k+1)\binom{2n}{k}} = (2n+1) \cdot \sum_{k=1}^{n} \frac{1}{(k+n)^2} \quad , \text{(Ljunggren)}$$

20) $$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} = \frac{\pi^2}{18} \quad ,$$

21) $$\sum_{k=1}^{\infty} \frac{2^{2k-1}}{k(2k+1)\binom{2k}{k}} x^{2k+1} = x - \sqrt{1-x^2} \cdot \operatorname{Arcsin} x \quad ,$$

$$\left(|x| < 1, \text{ also valid for } x = \pm 1 \right)$$

22) $$\sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}}{k(2k+1)\binom{2k}{k}} x^{2k+1} = x - \sqrt{1+x^2} \log\left(x + \sqrt{1+x^2}\right) ,$$

$$\left(|x| < 1, \text{ also valid for } x = \pm 1 \right)$$

23) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 \binom{2k}{k}} = 4 \sum_{k=0}^{\infty} \frac{(\sqrt{5}-2)^{2k+1}}{(2k+1)^2}$$

$$= \frac{\pi^2}{6} - 3 \log^2\left(\frac{\sqrt{5}-1}{2}\right),$$

24) $$\sum_{k=1}^{\infty} \frac{1}{\binom{kn}{\cancel{n}\,k}} = n \int_0^1 \frac{(1-x)^{n-1}}{1-x^n}\, dx$$

$$= n! \sum_{k=0}^{\infty} \frac{1}{(kn+1)(kn+2)\cdots\cdots(kn+n)}$$

$$= \sum_{k=1}^{n-1} (-\omega_k)(1-\omega_k)^{n-1} \log\left(\frac{1-\omega_k}{-\omega_k}\right), \quad \omega_k = e^{\frac{ik\pi}{n}},$$

(E. H. Clarke)

(2.25) $$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$ (Tor B. Staver)

(Also special case of (2.4))

(2.26) $$\sum_{k=0}^{n} \frac{1}{\binom{x+k}{r}} = \frac{r}{r-1} \left\{ \frac{1}{\binom{x-1}{r-1}} - \frac{1}{\binom{n+x}{r-1}} \right\},$$

(natural extension of (2.2); Gould)

TABLE 3

SUMMATIONS OF THE FORM S:2/0

1) $$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}\,,$$ (Vandermonde Convolution)

2) $$\sum_{k=0}^{n} \binom{x+k}{k}\binom{y+n-k}{n-k} = \binom{x+y+n+1}{n}\,,$$

3) $$\sum_{k=r}^{n-s} \binom{k}{r}\binom{n-k}{s} = \binom{n+1}{r+s+1}\,,$$

4) $$\sum_{k=0}^{n} \binom{n}{k}\binom{x+y-n}{x-k} = \binom{x+y}{x}\,,$$

5) $$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{k} = \sum_{k=0}^{n} (-1)^k \binom{n-x}{k}\binom{y+n-k}{n}\,,$$

6) $$\sum_{k=0}^{n} \binom{x}{k}\binom{x}{2n-k} = \sum_{k=0}^{n} \binom{x}{n-k}\binom{x}{n+k} = \frac{1}{2}\left\{\binom{2x}{2n} + \binom{x}{n}^2\right\},$$

7) $$\sum_{k=0}^{n-1} \binom{x}{k}\binom{x}{2n-1-k} = \sum_{k=0}^{n-1} \binom{x}{n-1-k}\binom{x}{n+k} = \frac{1}{2}\binom{2x}{2n-1}\,,$$

8) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{x}{2k}\binom{x}{n-2k} = \frac{1}{2}\binom{2x}{n} + \frac{(-1)^{\frac{n}{2}}}{2}\binom{x}{n/2}\,\frac{1+(-1)^n}{2}\,,$$

9) $$\sum_{k=0}^{n} \binom{x}{2k}\binom{x}{2n-2k} = \frac{1}{2}\binom{2x}{2n} + \frac{(-1)^n}{2}\binom{x}{n} ,$$

.10) $$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{x}{2k+1}\binom{x}{n-2k-1} = \frac{1}{2}\binom{2x}{n} - \frac{(-1)^{n/2}}{2}\binom{x}{n/2}\frac{1+(-1)^n}{2} ,$$

.11) $$\sum_{k=0}^{n-1} \binom{x}{2k+1}\binom{x}{2n-2k-1} = \frac{1}{2}\binom{2x}{2n} - \frac{(-1)^n}{2}\binom{x}{n} ,$$

3.12) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{x}{2k}\binom{2n-x}{n-2k} = \frac{1}{2}\left\{\binom{2n}{n} + (-1)^n\, 2^{2n}\binom{\frac{x-1}{2}}{n}\right\} ,$$

3.13) $$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{x}{2k+1}\binom{2n-x}{n-2k-1} = \frac{1}{2}\left\{\binom{2n}{n} - (-1)^n\, 2^{2n}\binom{\frac{x-1}{2}}{n}\right\} ,$$

3.14) $$\sum_{k=0}^{r} \binom{x}{k}\binom{-x}{n-k} = -\frac{x-r}{n}\binom{x}{r}\binom{-x-1}{n-r-1} = \frac{n-r}{n}\binom{x-1}{r}\binom{-x}{n-r} ,$$

$$(n \geqslant 1, \quad 0 \leq r \leq n)$$

(Erik Sparre Andersen)

$$= -\sum_{k=r+1}^{n} \binom{x}{k}\binom{-x}{n-k} ,$$

3.15) $$\sum_{k=0}^{r} \binom{x}{k}\binom{1-x}{n-k} = \frac{(n-1)(1-x)-r}{n(n-1)}\binom{x-1}{r}\binom{-x}{n-r-1} ,$$

$$(n \geqslant 2 , \quad 0 < r \leq n-1)$$

(Erik Sparre Andersen)

$$(3.16)\quad \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}\binom{\frac{1}{2}}{k} = \binom{-\frac{1}{2}}{n}\binom{-\frac{3}{2}}{n} = (2n+1)\binom{-\frac{1}{2}}{n}^{2} ,$$

$$(3.17)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} z^{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{x+n-k}{n}(z-1)^{k} ,$$

$$(3.18)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{z+k}{k}(x-y)^{n-k}y^{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{z}{k}x^{n-k}y^{k} ,$$

(Ljunggren)

$$(3.19)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} 2^{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{n} ,$$

$$(3.20)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k+r} = \binom{n+x}{n+r} ,$$

$$(3.21)\quad \sum_{k=0}^{n} \binom{x}{k}\binom{y+k}{n-k} 4^{k} = \sum_{k=0}^{n}\binom{2x}{k}\binom{y}{n-k}2^{k}$$

$$= \sum_{k=0}^{n}\binom{2x}{k}\binom{y+2x-k}{n-k} ,$$

(Karl Goldberg)

$$(3.22)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{x}{k}\binom{x-k}{n-2k} 2^{2n-2k} = \sum_{k=\left[\frac{n}{2}\right]}^{n}\binom{x}{k}\binom{k}{n-k}2^{2k} = 2^{n}\binom{2x}{n} ,$$

$$(3.23)\quad \sum_{k=0}^{n} (-1)^{k}\binom{n+x}{n-k}\binom{k+x+1}{k} = \begin{cases} 1, & n=0, \\ -1, & n=1, \\ 0, & n \geqslant 2 , \end{cases}$$

$$(3.24)\quad \sum_{k=0}^{n} \binom{x}{2k}\binom{x+n-k-1}{n-k} = \binom{x+2n-1}{2n},$$

$$(3.25)\quad \sum_{k=0}^{n} \binom{x}{2k+1}\binom{x+n-k-1}{n-k} = \binom{x+2n}{2n+1},$$

$$(3.26)\quad \sum_{k=0}^{n} \binom{2x}{2k}\binom{x-k}{n-k} = \frac{x}{x+n}\binom{x+n}{2n}2^{2n} = \frac{2^{2n}}{(2n)!}\prod_{k=0}^{n-1}(x^2-k^2),$$

$$(3.27)\quad \sum_{k=0}^{n} \binom{2x+1}{2k+1}\binom{x-k}{n-k} = \frac{2x+1}{2n+1}\binom{x+n}{2n}2^{2n} = \frac{2x+1}{(2n+1)!}\prod_{k=0}^{n-1}\left\{(2x+1)^2-(2k+1)^2\right\}$$

$$(3.28)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{n+2x}{k+x} = \binom{2x+2n}{x+n},$$

$$(3.29)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k-r} = \binom{n+x}{n-r},$$

$$(3.30)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}k = n\binom{x+n-1}{n},$$

$$(3.31)\quad \sum_{k=0}^{n} (-1)^k\binom{x}{k}\binom{y}{n-k} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k\binom{x}{k}\binom{y-x}{n-2k},$$

$$(3.32)\quad \sum_{k=0}^{n} (-1)^k\binom{x}{k}\binom{x}{n-k} = (-1)^{\frac{n}{2}}\binom{x}{n/2}\frac{1+(-1)^n}{2},$$

$$= \binom{x}{n}\frac{2^n\,(x-n)!\,\sqrt{\pi}}{(x-n/2)!\,(-n/2-\frac{1}{2})!},$$

$$(3.33)\quad \sum_{k=r}^{n-r} (-1)^k \binom{k}{r}\binom{n-k}{r} = (-1)^r \binom{n/2}{r} \frac{1+(-1)^n}{2} ,$$

$$(3.34)\quad \sum_{k=0}^{2n} (-1)^k \binom{x}{k}\binom{x}{2n-k} = (-1)^n \binom{x}{n} ,$$

$$(3.35)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{x}{2n-k} = \frac{(-1)^n}{2}\left\{\binom{x}{n}+\binom{x}{n}^2\right\} ,$$

$$(3.36)\quad \sum_{k=0}^{n} (-1)^k \binom{x+k}{k}\binom{x+n-k}{n-k} = \binom{x+n/2}{n/2} \frac{1+(-1)^n}{2} ,$$

$$(3.37)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{n-k}\binom{x}{n+k} = \frac{1}{2}\left\{\binom{x}{n}+\binom{x}{n}^2\right\} ,$$

$$(3.38)\quad \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k}\binom{2r}{r-k} = \frac{\binom{2n}{n}\binom{2r}{r}}{\binom{n+r}{n}} = \frac{(2n)!\ (2r)!}{(n+r)!\ n!\ r!} ,$$

(K. v. Szily)

$$(3.39)\quad \sum_{k=0}^{\infty} (-1)^k \binom{x}{k}\binom{-x}{k} = \frac{\sin \pi x}{\pi}\int_0^1 u^{x-1}\left(\frac{1+u}{1-u}\right)^x du ,$$

$$(3.40)\quad \sum_{k=0}^{n} (-1)^k \binom{-x}{k}\binom{x}{n-k} = \sum_{k=0}^{n-1} \binom{x}{k+1}\binom{n-1}{k} 2^{k+1} ,$$

$$(3.41)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{x}{k}\binom{-x}{n-2k} = (-1)^n \binom{x}{n} ,$$

$$(3.42)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2n-x}{n-k} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k}\binom{2n-2x}{n-2k}$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n-k}{n-k}\binom{2n-x}{k} 2^k .$$

$$= \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-x) = (-1)^n\ 2^{2n} \binom{\frac{x-1}{2}}{n} ,$$

$$(3.43)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y-2k}{n-k} 2^k = \sum_{k=0}^{[\frac{n}{2}]} \binom{x}{k}\binom{y-2x}{n-2k} ,$$

$$(3.44)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2x-2k}{n-k} 2^{2k} = (-1)^n \binom{2x}{n} ,$$

$$(3.45)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{4x-2k}{n-k} 2^{2k} = (-1)^{\frac{n}{2}} \binom{2x}{n/2} \frac{1+(-1)^n}{2} ,$$

$$(3.46)\quad \sum_{k=0}^{n} (-1)^k \binom{2x+1}{k}\binom{2n-2x-1}{n-k} = (-1)^n\ 2^{2n}\binom{x}{n} ,$$

$$(3.47)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{r} = (-1)^n\binom{x}{r-n} ,$$

$$(3.48)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{r+k} = (-1)^n\binom{x}{n+r} ,$$

9) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x-k}{r} = \binom{x-n}{r-n} ,$$

0) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2n-k}{n} = \binom{2n-x}{n} ,$$

1) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y-2k}{n-k} 3^k = \sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{x}{k}\binom{y-3x}{n-3k} ,$$

52) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{3x-2k}{n-k} 3^k = (-1)^n \binom{x}{n/3} \frac{(-1)^{\left[\frac{n}{3}\right]} - (-1)^{\left[\frac{n-1}{3}\right]}}{2} ,$$

$$= \begin{cases} 0 , & 3 \nmid n , \\ \binom{x}{n/3} , & 3 \mid n , \end{cases}$$

53) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2n-k}{n} 2^k \frac{k}{2n-k} = 2^{2n} \binom{-\frac{x}{2}+n-1}{n} ,$$

54) $$\sum_{k=0}^{n} (-1)^k \binom{x}{n-k}\binom{n+k}{k} \frac{n-k}{n+k} \cdot \frac{1}{2^{n+k}} = \binom{\frac{x}{2}}{n} ,$$

55) $$\sum_{k=0}^{n} (-1)^k \binom{n+1}{k}\binom{2n-2k+x}{n} = \binom{n-x+1}{n} ,$$ (Generalizes a formula proposed by B.C.Wong)

56) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n+1}{k}\binom{2n-2k}{n} = n+1 ,$$ (B. C. Wong)

$$(3.57)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+2x}{k+x} = (-1)^n \binom{2n}{n} \frac{\binom{2n+2x}{x}}{\binom{n+x}{n}} ,$$

$$(3.58)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2x-n}{x-k} = (-1)^{n/2}\, \frac{1+(-1)^n}{2} \cdot \binom{x}{n/2} \frac{\binom{2x}{x}}{\binom{2x}{n}} ,$$

$$(3.59)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2x-2n}{x-k} = (-1)^n \frac{\binom{x}{n}\binom{2x}{x}}{\binom{2x}{2n}} ,$$

$$(3.60)\quad \sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{2x-2n}{x-k} = \frac{(-1)^n}{2} \left\{\binom{x}{n}+\binom{x}{n}^2\right\} \frac{\binom{2x}{x}}{\binom{2x}{2n}} ,$$

$$(3.61)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2x-k}{n-k} 2^k = (-1)^{n/2} \binom{x}{n/2} \frac{1+(-1)^n}{2} ,$$

$$(3.62)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2x-2k}{n-k} 2^k = \frac{1+(-1)^n}{2} \binom{x}{n/2} ,$$

$$(3.63)\quad \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k}\binom{2x-2k}{n-2k} = \binom{x}{n} 2^n ,$$

$$(3.64)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{j} = (-1)^n \binom{n}{j-n} 2^{2n-j} ,$$

$$(3.65)\quad \sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n}(x-1)^k ,$$

$$(3.66)\quad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} ,$$

$$(3.67)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k}^2 = \frac{1}{2}\binom{2n}{n} - \frac{1+(-1)^n}{4}\binom{n}{n/2}^2 ,$$

$$(3.68)\quad \sum_{k=0}^{n} \binom{2n}{k}^2 = \frac{1}{2}\binom{4n}{2n} + \frac{1}{2}\binom{2n}{n}^2 ,$$

$$(3.69)\quad \sum_{k=0}^{n} \binom{2n+1}{k}^2 = \frac{1}{2}\binom{4n+2}{2n+1} ,$$

$$(3.70)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}^2 = \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{n/2}}{2}\binom{n}{n/2}\frac{1+(-1)^n}{2} ,$$

$$(3.71)\quad \sum_{k=0}^{n} \binom{2n}{2k}^2 = \frac{1}{2}\binom{4n}{2n} + \frac{(-1)^n}{2}\binom{2n}{n} ,$$

$$(3.72)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{2n}{2k}^2 = \frac{1}{4}\binom{4n}{2n} + \frac{(-1)^n}{4}\binom{2n}{n} + \frac{1+(-1)^n}{4}\binom{2n}{n}^2 ,$$

$$(3.73)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}^2 = \frac{1}{2}\binom{2n}{n} - \frac{(-1)^{n/2}}{2}\binom{n}{n/2}\frac{1+(-1)^n}{2} ,$$

$$(3.74)\quad \sum_{k=0}^{n-1} \binom{2n}{2k+1}^2 = \frac{1}{2}\binom{4n}{2n} - \frac{(-1)^n}{2}\binom{2n}{n},$$

$$(3.75)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{2n}{2k+1}^2 = \frac{1}{4}\binom{4n}{2n} - \frac{(-1)^n}{4}\binom{2n}{n} + \frac{1-(-1)^n}{4}\binom{2n}{n}^2,$$

$$(3.76)\quad \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k}^2 (n-2k)^2 = n\binom{2n-2}{n-1},$$

$$(3.77)\quad \sum_{k=0}^{n} \binom{n}{k}^2 k^r = \sum_{k=0}^{r} \binom{n}{k}\binom{2n-k}{n} B_k^r = S_r,$$

$$(3.78)\quad S_0 = \binom{2n}{n}, \quad S_1 = \frac{n}{2}\binom{2n}{n} = (2n-1)\binom{2n-2}{n-1},$$

$$(3.79)\quad S_2 = n^2\binom{2n-2}{n-1}, \quad S_3 = \frac{n^2(n+1)}{2}\binom{2n-2}{n-1},$$

$$(3.80)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^{n/2}\binom{n}{n/2}\frac{1+(-1)^n}{2}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-k}{n} 2^k,$$

$$(3.81)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n\binom{2n}{n},$$

$$(3.82)\quad \sum_{k=0}^{n} (-1)^k \binom{2n}{k}^2 = \frac{(-1)^n}{2}\left\{\binom{2n}{n} + \binom{2n}{n}^2\right\},$$

$$(3.83)\quad \sum_{k=0}^{n} \binom{n}{k}^2 2^k = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{3n-2k}{2n} ,$$

$$(3.84)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\left(\frac{z}{4}\right)^k = \sum_{k=0}^{n} \binom{n}{k}\binom{-\frac{1}{2}}{k} z^k$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k}\binom{2k}{k}(z-1)^k ,$$

$$(3.85)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\frac{1}{2^{2k}} = \binom{2n}{n}\frac{1}{2^{2n}} ,$$

$$(3.86)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k} = (-1)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k} ,$$

$$(3.87)\quad 2^{2n} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k}\binom{2k}{k} 3^k ,$$

$$(3.88)\quad 2^{2n} \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 5^k ,$$

$$(3.89)\quad \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k}\binom{2k}{k}\frac{2k+1}{2^{2k}} = \binom{2n}{n}\frac{1}{(2n-1)\,2^{2n}} ,$$

$$(3.90)\quad \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} = 2^{2n} ,$$

(3.91) $$\sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k}\binom{2k}{k} = \frac{1+(-1)^n}{2}\binom{n}{n/2}2^n ,$$

(3.92) $$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k-1} = \begin{cases} 0, & n \geqslant 1 , \\ -1, & n = 0 , \end{cases}$$

(3.93) $$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k-1)} = (-1)^n 2^{2n}\binom{1}{n} = \begin{cases} 1, & n = 0, \\ -4, & n = 1 , \\ 0, & n \geqslant 2, \end{cases}$$

(3.94) $$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k+1)} = \frac{2^{4n}}{2n(2n+1)\binom{2n}{n}} ,(n \geqslant 1),$$

(3.95) $$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{x}{x+k} = 2^{2n}\frac{\binom{n+x-\frac{1}{2}}{n}}{\binom{x+n}{n}} , \ (n \geqslant 0) ,$$

$$= 2^n \frac{(2x+1)(2x+3)\cdots(2x+2n-1)}{(x+1)\cdots\cdots(x+n)} , \ (n \geqslant 1) ,$$

(3.96) $$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k+1} = \frac{2^{4n}}{(2n+1)\binom{2n}{n}} = \frac{2^{2n}}{\binom{n+\frac{1}{2}}{n}} ,$$

(3.97) $$\sum_{k=0}^{n} \binom{4n-4k}{2n-2k}\binom{4k}{2k} = 2^{2n-1}\binom{2n}{n} + 2^{4n-1} ,$$

(3.98) $$\sum_{k=0}^{n-1} \binom{4n-4k-2}{2n-2k-1}\binom{4k+2}{2k+1} = 2^{4n-1} - 2^{2n-1}\binom{2n}{n} , \ (n \geqslant 1) ,$$

) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k} 2^{n-2k} = \binom{2n}{n},$$

0) $$\sum_{k=0}^{n} (-1)^k \binom{n+k}{2k}\binom{2k}{k} \frac{x}{x+k} = (-1)^n \frac{\binom{x-1}{n}}{\binom{x+n}{n}},$$

01) $$\sum_{k=r}^{n} (-1)^k \binom{n}{k}\binom{2k}{k-r} 2^{n-k} = \frac{(-1)^r + (-1)^n}{2} \binom{n}{\frac{n-r}{2}},$$

2) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{2n-2k}{n+r} = 2^{n-r}\binom{n}{r},$$

03) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+2r+k}{n+r} 2^{n-k} = (-1)^{n/2} \frac{1+(-1)^n}{2} \frac{\binom{n+r}{n/2}\binom{n+2r}{r}}{\binom{n+r}{n}},$$

(Grosswald)

104) $$\sum_{k=0}^{n-r} (-1)^k \binom{n}{k+r}\binom{n+k+r}{k} 2^{n-r-k} = (-1)^{\frac{n-r}{2}} \binom{n}{\frac{n-r}{2}} \frac{1+(-1)^{n-r}}{2},$$

(Grosswald)

105) $$\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k}\binom{n+k+r}{n} 2^{n-r-k} = (-1)^{\frac{n-r}{2}} \frac{\binom{n}{\frac{n-r}{2}}\binom{n+r}{n}}{\binom{n}{r}} \frac{1+(-1)^{n-r}}{2},$$

(Grosswald)

106) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+2k}{n+k} 3^{2n-k} = \binom{2n}{n},$$

107) $$\sum_{k=r}^{\left[\frac{n}{2}\right]} \binom{n-r}{k-r}\binom{n-k}{k} 2^{n-2k} = \binom{2n-2r}{n},$$

3.108) $$\sum_{k=0}^{n}\binom{k+j}{k}\binom{k+j+m+1}{m}=\sum_{k=0}^{m}\binom{k+j}{k}\binom{k+j+n+1}{n} ,$$

3.109) $$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}2^{2n-2k}=\binom{4n+1}{2n} ,$$

3.110) $$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}\frac{k}{(2n-k)\,2^{k}}=(-1)^{n}\,2^{2n}\binom{-\frac{1}{4}}{n} ,$$

3.111) $$\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^{k}\binom{n}{k}\binom{2n-2k-1}{n-1}=1 , \quad (n\geqslant 1) ,$$

3.112) $$\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^{k}\binom{n+1}{k}\binom{2n-2k-1}{n}=\tfrac{1}{2}\,n(n+1), \quad (n\geqslant 1) ,$$

3.113) $$\sum_{k=0}^{\left[\frac{r-1}{r}n\right]}(-1)^{k}\binom{n+1}{k}\binom{rn-rk}{n}=\binom{n+r-1}{n} ,$$

3.114) $$\sum_{k=0}^{n}(-1)^{k}\binom{2n}{n-k}\binom{2n+2k+1}{2k}=(-1)^{n}(n+1)\,2^{2n} ,$$

3.115) $$\sum_{k=0}^{n}\binom{4n+1}{2n-2k}\binom{k+n}{n}=2^{2n}\binom{3n}{n} ,$$

3.116) $$\sum_{k=0}^{n}\binom{4n}{2n-2k}\binom{k+n}{n}=\frac{2}{3}\,2^{2n}\binom{3n}{n} ,$$

$$(1.117)\quad \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{2n-2k}{n} = 2^n ,$$

$$(1.118)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{k}{j} x^k = \binom{n}{j} x^j (1+x)^{n-j} ,$$

$$(1.119)\quad \sum_{k=j}^{n} (-1)^k \binom{n}{k}\binom{k}{j} = \begin{cases} 0 , & j \neq n , \\ (-1)^n, & j = n , \end{cases}$$

$$(1.120)\quad \sum_{k=j}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{k}{j} = 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} ,$$

$$(1.121)\quad \sum_{k=j}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j} \binom{n-j}{j} ,$$

$$(1.122)\quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}\binom{n+k}{k} \frac{1}{k} = 2 \sum_{k=1}^{n} \frac{1}{k} ,$$

$$(1.123)\quad \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \sum_{j=1}^{k} \frac{1}{j} = \frac{(-1)^n}{n} ,$$

$$(1.124)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \sum_{j=1}^{n+k-1} \frac{1}{j} = \frac{(-1)^n}{n} , \quad \text{(R.R,Goldberg)}$$

$$(1.125)\quad \sum_{k=1}^{n} \binom{n}{k}^2 \sum_{j=1}^{k} \frac{1}{j} = \binom{2n}{n} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=n+1}^{2n} \frac{1}{j} \right\}$$

126) $$\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}^2 \frac{1}{2n+2k} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}^2 \frac{1}{2n-2k+1}, \quad (n \geq 1),$$

127) $$\sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{z}{4}\right)^{2k} = \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - z^2 \sin^2 x}},$$

128) $$\sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{z}{4}\right)^{2k} \frac{1}{1-2k} = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{1 - z^2 \sin^2 x}\, dx,$$

BESSEL POLYNOMIALS.

129) $$y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} k! \left(\frac{x}{2}\right)^k,$$

$$= 2^{-n} e^{2/x} D_x^n \left\{ x^{2n} e^{-2/x} \right\},$$

HERMITE POLYNOMIALS:

130) $$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \binom{n-k}{k} k! \, (2x)^{n-2k},$$

$$= (-1)^n e^{x^2} D_x^n e^{-x^2},$$

JACOBI POLYNOMIALS:

131) $$P_n^{(a,b)}(x) = \sum_{k=0}^{n} \binom{n+a}{k} \binom{n+b}{n-k} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k,$$

$$= \frac{(-1)^n}{2^n n!} (1-x)^{-a} (1+x)^{-b} D_x^n \left\{ (1-x)^{a+n} (1+x)^{b+n} \right\},$$

Definition by formula of Rodrigues:

$$(3.132)\qquad P_n(x) = \frac{1}{2^n\, n!}\, D_x^n\, (x^2 - 1)^n \qquad ,$$

In terms of generating functions,

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-\frac{1}{2}} \qquad ,$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) = e^{tx} J_0\left(t\sqrt{1 - x^2}\right) \qquad ,$$

In terms of the Hypergeometric function,

$$P_n(x) = F(n + 1, -n, 1; \frac{1 - x}{2}) \qquad ,$$

Many summations of the form S:2/0 in the literature involve $P_n(x)$, and the following table of equivalent forms of $P_n(x)$ may be found of use.

$$(3.133)\qquad P_n(x) = 2^{-n} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{2n - 2k}{n} x^{n - 2k} \qquad ,$$

$$(3.134)\qquad P_n(x) = \left(\frac{x - 1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x + 1}{x - 1}\right)^k \qquad ,$$

$$(3.135)\qquad P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n + k}{k} \left(\frac{x - 1}{2}\right)^k \qquad ,$$

$$(3.136)\quad P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k} 2^{-k} (x^2-1)^{k/2} \left(x - (x^2-1)^{\frac{1}{2}}\right)^{n-k} ,$$

$$(3.137)\quad P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}\binom{2k}{k} 2^{-2k} x^{n-2k} (x^2-1)^{k} \, . \, ,$$

$$(3.138)\quad \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} x^{2k} = 2^{2n} x^n P_n\left(\tfrac{1}{2}(x+x^{-1})\right) , \qquad \text{(R.P.Kelisky)}$$

$$= 2^{2n} \frac{2}{\pi} \int_0^{\pi/2} (x^2\sin^2 t + \cos^2 t)^n \, dt ,$$

$$(3.139)\quad \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k} x^{2k} = (-1)^n x^n P_n\left(\tfrac{1}{2}(x+x^{-1})\right) ,$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{k-\frac{1}{2}}{n} x^{2k} ,$$

Integral of Laplace:

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2-1}\cdot\cos t)^n \, dt ,$$

Finite series of I. J. Good:

$$(3.140)\quad P_n(x) = \frac{1}{t} \sum_{k=0}^{t-1} \left(x + \sqrt{x^2-1}\cdot\cos\frac{2\pi k}{t}\right)^n ,$$

(valid for integral $t > n$)

Integral form of L. Schläfli:

$$P_n(x) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n (t - x)^{n+1}} \, dt \quad ,$$

where C is a circle counterclockwise about the point $t = x$.

In general one may define $P_n(x)$ for all real values of n, and relations such as the following are found:

$$(3.141) \quad (1 - x)^{n+1} \sum_{k=0}^{\infty} \binom{-n-1}{k}^2 x^k = (1 - x)^{-n} \sum_{k=0}^{\infty} \binom{n}{k}^2 x^k \quad ,$$

or if one defines $f(z) = P_z\left(\frac{1+x}{1-x}\right) = (1 - x)^{-z} \sum_{k=0}^{\infty} \binom{z}{k}^2 x^k$,

then the above reads $f(n) = f(-n-1)$.

$$(3.142)\quad \sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k} \frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}$$

(Hagen / Rothe)

(contains (1.125) as special case)

$$(3.143)\quad \sum_{k=0}^{n} \binom{x+kz}{k}\binom{y+(n-k)z}{n-k} = \sum_{k=0}^{n}\binom{x+t+kz}{k}\binom{y-t+(n-k)z}{n-k},$$

$$(3.144)\quad \sum_{k=0}^{n} \binom{x+kz}{k}\binom{y-kz}{n-k} = \sum_{k=0}^{n}\binom{x+y-k}{n-k} z^k,$$ (Jensen)

$$(3.145)\quad \sum_{k=0}^{n} \binom{x+kz}{k}\binom{p-x-kz}{n-k} = \begin{cases} z^{p+1}(z-1)^{n-p-1}, & 0 \leq p \leq n-1, \\ \dfrac{z^{n+1}-1}{z-1}, & p = n, \end{cases}$$

(Jensen)

$$(3.146)\quad \sum_{k=0}^{n} \binom{x+kz}{k}\binom{y-kz}{n-k} \frac{p+qk}{(x+kz)(y-kz)}$$

$$= \frac{p(x+y-nz)+nxq}{x(x+y)(y-nz)}\binom{x+y}{n},$$

(Hagen/Rothe)

$$(3.147)\quad \sum_{k=1}^{n-1} \binom{kz}{k}\binom{nz-kz}{n-k} \frac{1}{kz(nz-kz)} = \frac{2}{nz}\binom{nz}{n}\sum_{k=1}^{n-1}\frac{1}{nz-n+k},$$

(Van der Corput)

$$(3.148)\quad \binom{nd}{n} = d(d-1)\sum_{k=1}^{n} \frac{(dk-2)!}{(k-1)!\,(dk-k)!}\binom{nd-kd}{n-k},$$

(Chung)
(cf. (7.18))

ADDITIONS TO TABLE 3

(3.149) $$\sum_{k=0}^{n} \binom{2x+1}{2k}\binom{x-k}{n-k} = 2^{2n}\binom{x+n}{2n}, \quad \text{cf. } (3.27)$$

(3.150) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+kz}{j} = \begin{cases} 0, & 0 \le j < n, \\ (-1)^n z^n, & j = n, \end{cases}$$

(3.151) $$\sum_{k=0}^{n} \binom{x+k}{m}\binom{x-k}{m} k = \frac{m+1}{2}\left\{\binom{x+1}{m+1}\binom{x}{m+1} - \binom{x+n+1}{m+1}\binom{x-n}{m+1}\right\}$$

(Knuth)

(3.152) $$\sum_{k=0}^{m} \binom{x}{k}\binom{y}{n-k}\left\{nx - (x+y)k\right\} = (m+1)(n-m)\binom{x}{m+1}\binom{y}{n-m}$$

(Knuth) (extends (3.14))

(3.153) $$\sum_{k=0}^{m} \binom{-x}{k}\binom{x}{n+k} = \frac{x}{x-n}\binom{-x-1}{m}\binom{x-1}{n+m}, \quad \text{(Knuth)}$$

(3.154) $$\sum_{k=0}^{m} \binom{-x}{a-k}\binom{x}{b-k} = \binom{x+a}{b}\binom{b-x}{a}, \quad m = \min(a,b)$$

(special case of Stanley's (6.52)) (Gould)

(3.155) $$\sum_{k=0}^{s-1} \binom{k}{n}\binom{k+m}{m} = \binom{s}{n}\binom{s+m}{m}\frac{s-n}{m+n+1}, \quad \text{(Knuth)}$$

(3.156) $$\sum_{k=a}^{n} \binom{x}{k-a}\binom{y}{n-k} = \binom{x+y}{-a+n}, \quad \text{(variation of (3.1))}$$

(3.157) $$\sum_{k=0}^{n-1} \binom{2x}{2k+1}\binom{x-k-1}{n-k-1} = \frac{n}{x+n} 2^{2n}\binom{x+n}{2n},$$

(3.158) $$\sum_{k=0}^{n} \binom{2x}{2k+1}\binom{x-1-k}{n-k} = \frac{x+n}{2n+1} 2^{2n+1}\binom{x-1+n}{2n},$$

(3.159) $$\sum_{k=0}^{n} \binom{2x}{2k+1}\binom{x-k}{n-k} = \frac{2x^2+n}{(2n+1)(x+n)} \binom{x+n}{2n} 2^{2n},$$

(3.160) $$\sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} \frac{n}{n+k} 2^{2k} = (-1)^n \binom{n+a}{2a} \frac{n}{n+a} 2^{2a},$$

(3.161) $$\sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} \frac{2n+1}{2k+1} = (-1)^n \binom{n+a}{2a} 2^{2a},$$

(3.162) $$\sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} = (-1)^n \binom{n+a}{2a} 2^{2a} \frac{2n+1}{2a+1},$$

(3.163) $$\sum_{k=0}^{n} \binom{n}{k}\binom{k/2}{m} = \frac{n}{m}\binom{n-m-1}{m-1} 2^{n-2m}, \quad (m \geqslant 1)$$ (Carlitz)

(3.164) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{k/2}{m} = (-1)^m 2^{n-2m}\left\{\binom{2m-n-1}{m-1} - \binom{2m-n-1}{m}\right\},$$

(Rosenstock, Gray, Riordan)

(3.165) $$\sum_{k=0}^{n} \binom{p-k}{p-n}\binom{q+k+1}{m} = \sum_{k=0}^{m} \binom{q-k}{q-m}\binom{p+k+1}{n},$$

(E. Catalan, 1842)

(3.166) $$\sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{2n-2k}{n-k} (x^2+y^2)^{2k+1}(xy)^{2n-2k} = \sum_{k=0}^{2n+1} \binom{2n+1}{k}^2 (x^2)^k (y^2)^{2n+1-k},$$

(Joel L. Brenner)

(3.167) $$\sum_{k=0}^{n} \binom{2n}{2k}\binom{2n-2k}{n-k} (x^2+y^2)^{2k} (xy)^{2n-2k} = \sum_{k=0}^{2n} \binom{2n}{k}^2 (x^2)^k (y^2)^{2n-k},$$ (Brenner)

$$(3.168)\quad \sum_{k=0}^{n} \frac{2k+1}{n+k+1}\binom{x-1-k}{n-k}\binom{x+k}{n+k} = \binom{x}{n}^2 ,$$

(equiv.to result of Graham and Riordan)

$$(3.169)\quad \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1}\binom{x+k}{n} = \binom{2x}{n} ,$$

$$(3.170)\quad \sum_{k=0}^{[(n+1)/2]} \binom{n+1}{2k}\binom{x+k}{n} = \binom{2x+1}{n} ,$$

$$(3.171)\quad \sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{x+k}{2n} = \binom{2x}{2n} ,$$

(cf. Graham and Riordan)

$$(3.172)\quad \sum_{k=0}^{n} \binom{2n+2}{2k+1}\binom{x+k}{2n+1} = \binom{2x}{2n+1} ,$$

$$(3.173)\quad \sum_{k=0}^{n} \binom{2n+1}{2k}\binom{x+k}{2n} = \binom{2x+1}{2n} ,$$

(Graham and Riordan)

$$(3.174)\quad \sum_{k=0}^{n+1} \binom{2n+2}{2k}\binom{x+k}{2n+1} = \binom{2x+1}{2n+1} ,$$

$$(3.175)\quad \sum_{k=0}^{n} \binom{x}{2k}\binom{x-2k}{n-k} 2^{2k} = \binom{2x}{2n} ,$$

(Machover and Gould)

$$(3.176)\quad \sum_{k=0}^{n} \binom{x+1}{2k+1}\binom{x-2k}{n-k} 2^{2k+1} = \binom{2x+2}{2n+1} ,$$

(Machover and Gould)

$$(3.177)\quad \sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p} ,$$

"Moriarty", H.T.Davis,et al,

$$(3.178)\quad \sum_{k=0}^{n-p} \binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p} 2^{2n-2p} ,$$

(3.177)-(3.178) are equiv.to (3.120)-(3.121).

79) $$\sum_{k=r}^{[n/2]} (-1)^k \binom{n-k}{k}\binom{k}{r} 2^{n-2k} = (-1)^r \binom{n+1}{2r+1},$$

(Marcia Ascher)

)) $$\sum_{k=r}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k}\binom{k}{r} 2^{n-2k-1} = (-1)^r \binom{n}{2r},$$

(companion-piece to (3.179)

- these are inverse Moriarty formulas -

1) $$\sum_{k=0}^{n} (-1)^k \binom{x+k}{k+1}\binom{x}{n-k} = \binom{x}{n+1}, \quad \text{(Brill)}$$

32) $$\sum_{k=0}^{[t]} (-1)^k \binom{2a+2b+1}{k}\binom{3a+2b+1-2k}{2a}$$

$$= (-1)^b \frac{(2a+2b+1)!(2b)!}{(a+2b+1)!b!(a+b)!} = (-1)^b \binom{2a+2b+1}{a} \frac{\binom{2b}{b}}{\binom{a+b}{b}},$$

(A. Brill, Math.Annalen, 36(1890),361-370)

ere $2t = 3a + 2b + 1$

Discussions, proofs, extensions by Gould, Mathematica Monongaliae No. 3, August 1961)

183) $$\sum_{k=0}^{m} (-1)^k \binom{m}{k}\binom{m-1}{a+m-k} = (-1)^{m+n+a} \binom{2n-a}{n},$$

where $m = 2n + 1 - a$

(equivalent to Brill's sum)

TABLE 4

SUMMATIONS OF THE FORM S:1/1

.1) $$\sum_{k=j}^{n} \frac{\binom{z}{k}}{\binom{x}{k}} = \frac{x+1}{x-z+1}\left\{\frac{\binom{z}{j}}{\binom{x+1}{j}} - \frac{\binom{z}{n+1}}{\binom{x+1}{n+1}}\right\},$$

.2) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{\binom{b+k}{c}} = \frac{c}{n+c} \quad \frac{1}{\binom{n+b}{b-c}}, \quad (b \geqslant c > 0),$$

(R. Frisch)

.3) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+y)^k}{\binom{y+k}{k}} = \sum_{k=0}^{n} \binom{n}{k} (x-k)^k (1-x+k)^{n-k} \frac{y}{y+k},$$

.4) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^r}{\binom{x+k}{k}} = \frac{1}{\binom{-x-1}{n}} \sum_{k=0}^{r} \binom{-x}{n-k} B_k^r = S_r(x),$$

$$= \frac{1}{\binom{x+n}{n}} \sum_{k=0}^{r} (-1)^k \binom{x+n-k-1}{n-k} B_k^r,$$

.5) $S_1(n) = \dfrac{-n}{2(2n-1)}$, $S_0(x) = \dfrac{x}{x+n}$ which is (1.42) again.

.6) $$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} = \frac{2x+n+1}{(2x+1)\binom{2x}{x}},$$

.7) $$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{1}{\binom{2n+1+2x}{k+x}} = 0,$$

4.8) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{1}{\binom{2n+2x}{k+x}} = \frac{\binom{x+n}{n}}{\binom{2x}{x}\binom{2x+2n}{2n}} = \frac{\binom{2n}{n}}{\binom{x+n}{n}\binom{2x+2n}{x+n}},$$

4.9) $$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{2n-1}{k}} = 2,$$

4.10) $$\sum_{k=0}^{n} k \frac{\binom{n}{k}}{\binom{2n-1}{k}} = 2 \frac{n}{n+1},$$

4.11) $$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{2^{2k}}{k\binom{2k}{k}} = 2 \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k},$$

4.12) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{1-2n},$$

4.13) $$\sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{\binom{k+r}{k}} = 1 + \frac{(x+1)^{n+r} - \sum_{k=0}^{r} \binom{n+r}{k} x^k}{x^r \binom{n+r}{n}}$$

4.14) $$\sum_{k=0}^{n} \binom{n}{k} \frac{(2k+1)\left\{1+(-1)^{n+k}\right\} P_k(x)}{(n-k+1)\, 2^{k+1} \binom{\frac{n+k+1}{2}}{k}} = x^n,$$

) $$\sum_{k=0}^{2n+1} (-1)^k \frac{\binom{3n+2}{n+k+1}}{\binom{2n+2k+1}{n+k}} 2^{2k} = \frac{1}{3} \frac{\binom{3n+2}{n+1}}{\binom{2n+1}{n}} ,$$

) $$\sum_{k=0}^{n} \frac{\binom{2n}{n+k}}{\binom{2n+2k}{n+k}} \frac{4k+1}{2n+2k+1} 2^{2k} = 1 ,$$

) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(2k+1)\binom{k+n+1}{k}} = \frac{n+1}{\binom{n+\frac{1}{2}}{n}^2} ,$$

This may be rewritten as an infinite series actually:

) $$1 - \frac{1-n}{1+n} \frac{1}{3} + \frac{(1-n)(2-n)}{(1+n)(2+n)} \frac{1}{5} - \cdots\cdots = \frac{1}{4n} \left\{ \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \right\}^2 ,$$

(H. F. Sandham)

) $$1 + 2 \sum_{k=1}^{\infty} \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{1}{\binom{x-\frac{1}{2}}{x}} = \frac{2^{2x}}{\binom{2x}{x}} ,$$

$R(x) \geq 0$

) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^{2p}}{\binom{k+n}{k}} = 0 ,$$ provided $1 < 2p < 2n-1$, p being integral.

21) $$\sum_{k=0}^{n} \frac{\binom{x}{k}}{\binom{x+n-k}{n}} \cdot \frac{x+n+1-2k}{x+n+1-k} = 1,$$ (René Lagrange)

(sp.case of general form when $y = n$) See (7.48)

) $$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x+n+k}{n+1}} = \frac{2^n \cdot (n+1)!}{x(x+2)(x+4)\cdots(x+2n)}$$ (cf. 徐-王《分析方法及例题选讲》P.

ADDITIONS TO TABLE 4

$$(4.22)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n}{2k}^{-1} = \frac{1+(-1)^n}{2} \cdot \frac{2n+1}{n+1} = S_n,$$

$$(4.23)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n+1}{2k+1}^{-1} = \frac{1-(-1)^n}{2} \cdot \frac{1}{n+2} + (-1)^n = T_n ,$$

$$(4.24)\quad \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \binom{2n}{2k+1}^{-1} = U_n = ?$$

$$(4.25)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n+1}{2k}^{-1} = (-1)^n T_n = \begin{cases} 1, & 2|n , \\ \frac{n+1}{n+2}, & 2\nmid n , \end{cases} = V_n,$$

$$(4.26)\quad \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} \binom{n}{k}^{-1} = \frac{1+(-1)^n}{2} \cdot \frac{1}{1-n} = A_n ,$$

$$(4.27)\quad \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \binom{n}{k}^{-1} = 1 + \frac{1}{n} \cdot \frac{1-(-1)^n}{2} = B_n,$$

$$(4.28)\quad \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \binom{n}{k}^{-1} = C_n = B_n - A_n = \begin{cases} \frac{n}{n-1} , & 2|n, \\ \frac{n+1}{n} , & 2\nmid n, \end{cases}$$

$$(4.29)\quad \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} \binom{n}{k}^{-1} = D_n = (-1)^n B_n ,$$

The first of the above eight sums arises naturally in a statistical problem; it amounts to the evaluation of the moments of a certain distribution.

(Gould)

$$(4.30)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} = \frac{x}{x+n} ,$$

this is also listed as (1.42) (because of intimate relation with (1.41)). This is also $r = 0$ in (4.4).

TABLE 5

SUMMATIONS OF THE FORM S:0/2

.1)
$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{k+n}{k}^2} = \binom{2n}{n} \left\{ \frac{\pi^2}{6} - 3 \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} \right\},$$

(5.2)
$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2} = \frac{3(n+1)^2}{2n+3} \frac{1}{\binom{2n+2}{n+1}} \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k},$$

(Tor B. Staver)

TABLE 6

SUMMATIONS OF THE FORM S:3/0

1) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2x}{x-n+k}\binom{2z}{z-n+k}$$

$$= (-1)^n \frac{(n+x+z)!\,(2n)!\,(2x)!\,(2z)!}{(n+x)!\,(n+z)!\,(x+z)!\,n!\,x!\,z!}$$, (Fjeldstad)

which may be restated as

2) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2x+2n}{x+k}\binom{2z+2n}{z+k}$$

$$= (-1)^n \binom{2n}{n}\binom{2x+2n}{x+n}\binom{2z+2n}{z+n} \frac{\binom{x+z+3n}{n}}{\binom{x+2n}{n}\binom{z+2n}{n}} ,$$

or in the symmetrical form

3) $$\sum_{k=-m}^{m} (-1)^k \binom{m+n}{m+k}\binom{n+p}{n+k}\binom{p+m}{p+k} = \frac{(m+n+p)!}{m!\,n!\,p!}$$, (Th. Bang)

4) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{x}{k}\binom{x}{2n-k} = \binom{x}{n}\binom{-x-1}{n} = (-1)^n \binom{2n}{n}\binom{x+n}{2n} ,$$

5) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x}{k}\binom{x}{n-k} = \binom{x}{n/2}\binom{-x-1}{n/2}\frac{1+(-1)^n}{2} ,$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k}\binom{x+n-k}{n-k} ,$$

6) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n}\binom{3n}{n} = (-1)^n \frac{(3n)!}{n!^3}$$, (A.C.Dixon)

7) $$\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} x^k y^k (x+y)^{n-2k} ,$$ (MacMahon)

8) $$\sum_{0}^{\infty} \binom{-x}{k}^3 \frac{x+2k}{x} = \frac{\sin \pi x}{\pi x} , \quad (x \leq 1/3) ,$$ (Dougall)

9) $$\sum_{k=0}^{\infty} (-1)^k \binom{-x}{k}^3 = \cos \frac{\pi x}{2} \cdot \frac{\left(-\frac{3x}{2}\right)!}{\left(-\frac{x}{2}\right)!^3} , \quad R(x) < 2/3 ,$$ (Dougall)

10) $$\sum_{k=0}^{\infty} (-1)^k \binom{-x}{k}^3 \frac{x+2k}{x} = \frac{\sin \pi x}{\pi x} \cdot \frac{\left(\frac{x-1}{2}\right)! \left(\frac{-3x-1}{2}\right)!}{\left(\frac{-x-1}{2}\right)!^2} ,$$ (Dougall)

11) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k} = \binom{2n}{n}^2 ,$$

12) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{1+(-1)^n}{2}\binom{n}{n/2}^2 ,$$

13) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{2n-2k}{n}\binom{n-2k}{r} = 2^{n-r}\binom{n}{r}\binom{n+r}{r} ,$$

14) $$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}\binom{k}{j} = \binom{x}{j}\binom{y+x-j}{n-j} ,$$

15) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{-x+j-1}{k+r}\binom{x}{j-r-k} = (-1)^r \binom{j+n}{j-r}\binom{x}{j} ,$$

$$(6.16)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{-x+r-1}{r+k}\binom{x-r-n}{j-r-k} = (-1)^r \binom{j+n}{j-r}\binom{x}{j},$$

$$(6.17)\quad \sum_{k=0}^{n} \binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n} = \binom{x}{m}\binom{y}{n}, \qquad \text{(Nanjundiah)}$$

$$(6.18)\quad \sum_{k=0}^{n} \binom{x+1}{k}\binom{x+a-1}{k+a-1}\binom{x-k}{n-k} = \binom{x+a-1}{n+a-1}\binom{x+n+a}{n},$$

(which generalizes a formula of P. Tardy, Gior.di Mat.,V.3(1865),pp.1-3.)

$$(6.19)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{r}{k}\binom{x+n+r-k}{n+r} = \binom{x+r}{r}\binom{x+n}{n}, \qquad \text{(Surányi)}$$

$$(6.20)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-k}{n}\binom{n+k}{n},$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k}\binom{2k}{k}\binom{2n-k}{n},$$

$$(6.21)\quad \sum_{k=j}^{n} (-1)^k \binom{n}{k}\binom{2n-k}{n}\binom{k}{j} = (-1)^j \binom{n}{j}^2,$$

$$(6.22)\quad \sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{2n-k}{n}^2 \frac{x}{x+k} = \frac{\binom{2n}{n}\binom{2n+x}{n}}{\binom{x+n}{n}},$$

.23) $$\sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{2n-k}{n}^2 \frac{2n+1}{2n+1+k} = \frac{\binom{2n}{n}\binom{4n+1}{n}}{\binom{3n+1}{n}},$$

.24) $$\sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{2n-k}{n}^2 \frac{2n+1}{2n+1-k} = 1,$$

[illegible] $$\sum_{k=0}^{n} (-1)^k \binom{2n}{k}^2\binom{2n}{n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{4n-k}{2n}\binom{2n+k}{n},$$

$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{4n-k}{3n}\binom{2n+k}{2n},$$

.26) $$\sum_{k=0}^{n} (-1)^k \binom{2r}{k}\binom{2n-2r}{n-k}^2 = (-1)^r \frac{\binom{2r}{r}\binom{2n-r}{n}^2}{\binom{2n-r}{r}},$$ (Carlitz)

which may be restated as

.27) $$\sum_{k=-n}^{n} (-1)^k \binom{2r}{r-k}^2 \binom{2n}{n-k} = \binom{2r+n}{r}\frac{(2n)!\,(2r)!}{(n+r)!\,n!\,r!},$$ (Carlitz)

.28) $$\sum_{k=0}^{n} \binom{n}{k}^2\binom{k}{n-j} = \binom{n}{j}\binom{n+j}{j},$$

.29) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{2x-n}{x-k} = (-1)^{n/2}\,\frac{1+(-1)^n}{2}\binom{n}{n/2}\binom{x+\frac{n}{2}}{n}\frac{\binom{2x}{x}}{\binom{2x}{n}},$$

$$6.30)\quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+k}{2n} = \binom{x}{n}^2 ,$$

(This and the next two formulas are essentially the one given in Math. Rev., V.17(1956),pp.653-4)

$$6.31)\quad \sum_{k=0}^{n} \binom{x-n}{k}^2 \binom{2x-n-k}{n-k} = \binom{x}{n}^2 ,$$

$$6.32)\quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+2n-k}{2n} = \binom{x+n}{n}^2$$

, (Math.Rev.,V.18(1957),p.4)
(Known to Le Jen-Shoo, 1867)

$$6.33)\quad \sum_{k=j}^{[\frac{n}{2}]} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} 2^{n-2k} = \binom{n}{j}\binom{2n-2j}{n} ,$$

$$6.34)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\binom{2k}{k+j} 2^{2n-2k}$$

$$= \binom{n+j}{\frac{n+j}{2}}\binom{n-j}{\frac{n-j}{2}} \frac{(-1)^n + (-1)^j}{2} ,$$

(Generalization of (6.35))

$$6.35)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\binom{2k}{k} 2^{2n-2k} = \frac{1+(-1)^n}{2}\binom{n}{n/2}^2 ,$$

$$= 2^{2n}\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{-n-1}{k}\binom{-\frac{1}{2}}{k} ,$$

(E.T.Bell)

$$6.36)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{n+k}{k}^2 = \binom{2n}{n} ,$$

(6.37) $$\sum_{k=0}^{n}\binom{n}{k}^2\binom{3n+k}{2n}=\binom{3n}{n}^2 ,$$

(6.38) $$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{k}{k+2nx}\binom{k+2nx}{2n}\;\frac{2n-k}{2n-k+2nx}\binom{2n-k+2nx}{2n}$$

$$=(-1)^n x\binom{2n}{n}\frac{n}{n+2nx}\binom{n+2nx}{2n} ,$$

$$=\frac{(-1)^n}{2\cdot n!^2}\prod_{k=0}^{n-1}\left\{(2nx)^2-k^2\right\} ,\quad (n\geqslant 1) ,$$

(6.39) $$\sum_{k=0}^{n}\binom{n}{k}^3=\text{coef. of } x^n \text{ in } (1-x^2)^n P_n\left(\frac{1+x}{1-x}\right)$$ (Carlitz)

(6.40) $$D_x^r P_n(x)=\frac{r!}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\binom{n-2k}{r}x^{n-2k-r}$$

(6.41) $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{j}\binom{x-a-bk}{n-j}=(-1)^j\binom{n}{j}b^n$$

(Gould; special case $a = 0$, $b = 2$, $x = 2n$ suggested by Samuel Karlin)

(6.42) $$\sum_k\binom{b}{k}\binom{c}{k-d}\binom{a+k}{b+c}=\binom{a}{b-d}\binom{a+d}{c+d} ,$$ (M.T.L.Bizley)

(6.43) $$\sum_k\binom{b}{k}\binom{c}{d-k}\binom{a+k}{b+c}=\binom{a}{b+c-d}\binom{a-c+d}{d} ,$$ (Bizley)

ADDITIONS TO TABLE 6:

$$(6.44)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{m+n-k}{m-k}\binom{x}{m+n-k} = \binom{x}{m}\binom{x}{n}, \quad \text{(Riordan)}$$

$$(6.45)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{m}{n-k}\binom{x+n-k}{n+m} = \binom{x}{m}\binom{x}{n}, \quad \text{(Riordan)}$$

$$(6.46)\quad \sum_{k=0}^{n} \binom{x}{k}\binom{y}{k}\binom{x+y+n-k}{n-k} = \binom{x+n}{n}\binom{y+n}{n},$$

(equiv.. to formula of Surányi)

$$(6.47)\quad \sum_{k=0}^{n} \binom{y}{k}\binom{r+y+z-k}{n-k}\binom{r+z-n}{r-k} = \binom{r+z}{r}\binom{y+z}{n},$$

(equiv. to formula posed by H. L. Krall)

$$(6.48)\quad \sum_{k=0}^{n} (-1)^{n-k}\binom{x+y+1}{n-k}\binom{x+k}{k}\binom{y+k}{k} = \binom{x}{n}\binom{y}{n},$$

(extension of formula of David Zeitlin; equiv. to formula of Surányi.)

$$\cancel{(6.49)}\quad \cancel{\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} x^k y^k (x+y)^{n-2k}},$$

((6.20) ~~is sp. case~~) (MacMahon)

$$(6.50)\quad \sum_{k=a}^{n} \binom{k+a-1}{k}\binom{2n}{n-k}\binom{k}{a} = \binom{2n-1}{n}\binom{n}{a}$$

(C. Van Ebbenhorst Tengbergen, 1913)

$$(6.51)\quad \sum_{k=0}^{n} \binom{n}{k}\binom{r}{k}\binom{x+n+r+k}{n+r} = \binom{x+n+r}{n}\binom{x+n+r}{r}$$

(Contrast with (6.19)) (Gould)

$$(6.52)\quad \sum_{k=0}^{m} \binom{x+y+k}{k}\binom{y}{a-k}\binom{x}{b-k} = \binom{x+a}{b}\binom{y+b}{a},$$

(case q = 1 of identity found by R. P. Stanley)
$m = \min(a,b)$. (Gould, J.Combinatorial Theory)

TABLE 7

SUMMATIONS OF THE FORM S:2/1

(7.1) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{z}{k}}{\binom{y}{k}} = \frac{\binom{y-z}{n}}{\binom{y}{n}} ,$$

(7.2) $$\sum_{k=0}^{n} \binom{n}{k}\binom{z}{k} \frac{1}{\binom{x+k}{k}} = \frac{\binom{x+z+n}{n}}{\binom{x+n}{n}} ,$$

(7.3) $$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{x}{k}}{\binom{-x+n-1}{k}} = \frac{2^n (x-n)! \sqrt{\pi}}{(x-n/2)!\,(-n/2-\tfrac{1}{2})!} ,$$

(7.4) $$\sum_{k=0}^{n} (-1)^k \frac{\binom{x}{k}\binom{x+1}{k}}{\binom{2x}{k}} = (-1)^n \frac{\binom{x-1}{n}\binom{x}{n}}{\binom{2x}{n}} ,$$

(P. Tardy, Giornale di Matematiche, V.3(1865), pp.1-3.)

(7.5) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{2kx}{2j}}{\binom{kx}{j}} = \begin{cases} 0 , & 0 \leq j < n , \\ (-1)^n\, 2^{2n} \dfrac{x^n}{\binom{2n}{n}} , & j = n , \end{cases}$$

(7.6) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k} \frac{1}{\binom{j+k}{k} 2^{2k}} = \frac{\binom{2n+2j}{n+j}}{\binom{2j}{j} 2^{2n}} ,$$

Relation (7.4) easily generalizes to

(7.4.1) $$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{k}\binom{x+y-1}{k}^{-1} = (-1)^n \binom{x-1}{n}\binom{y-1}{n}\binom{x+y-1}{n}^{-1}$$

(Knuth)

7.7) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2k}{k} \frac{1}{\binom{n+k}{k} 2^{2k}} = \frac{\binom{6n}{3n}}{\binom{2n}{n}} 2^{-4n} ,$$

7.8) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2k}{k} \frac{2k+1}{\binom{n+k}{k}(n+k+1)} = 1 ,$$

7.9) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k} \frac{2^{2k}}{\binom{2k}{k}(2k+1)} = \frac{2^{2n}}{\binom{2n}{n}(2n+1)} \binom{n-x-\frac{1}{2}}{n},$$

7.10) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \frac{2^{2k}}{\binom{2k}{k}(2k+1)} = \frac{(-1)^n}{2n+1} ,$$

7.11) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k} \frac{2^{2k}}{\binom{2k}{k}(x+k)} = (-1)^n \frac{\binom{2x}{2n}}{x\binom{x}{n}} ,$$

7.12) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \frac{2^{2k}}{\binom{2k}{k}(n+k)} = \frac{(-1)^n}{n} , \quad (n \geqslant 1) ,$$

7.13) $$\sum_{k=0}^{n} \binom{n}{k}\binom{-n-1}{n-k} \frac{1}{\binom{k+r}{k}} = (-1)^n \, r \sum_{k=0}^{n} \binom{n}{k}^2 \frac{1}{k+r} ,$$

$$14)\quad \sum_{k=0}^{n} \frac{\binom{2n}{n+k}\binom{2k}{k}}{(2n+2k+1)\binom{2n+2k}{n+k}} = (-1)^n \frac{2^{4n}\binom{-3/4}{n}}{(4n+1)\binom{4n}{2n}},$$

$$15)\quad \sum_{k=1}^{n} \binom{n}{k}\binom{z}{k} \frac{1}{\binom{x+k}{k}} \sum_{j=1}^{k} \frac{1}{j+x}$$

$$= \frac{\binom{x+z+n}{n}}{\binom{x+n}{n}} \left\{ \sum_{j=1}^{n} \frac{1}{j+x} - \sum_{j=1}^{n} \frac{1}{j+x+z} \right\},$$

$$16)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \frac{1}{\binom{rn+n}{k}} = \frac{(rn)!^2}{(rn-n)!\,(rn+n)!},$$

$$17)\quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}^2 \frac{1}{\binom{m}{k}k} = 2\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k}, \quad (m \geqslant n),$$

$$18)\quad \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{(j-1)n}{(j-1)k}}{\binom{jn}{jk}} \cdot \frac{1}{jk-1} = \frac{j-2}{1-j}, \quad n \geqslant 1 \text{ (Equiv. to a formula of K. L. Chung) (3.148)}$$

$$= -1 \text{ for } n = 0.$$

$$19)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+2x}{k+x} \frac{1}{\binom{2n+2y}{k+y}} = \frac{\binom{2n}{n}\binom{n+x-y-1}{n}\binom{2n+2x}{x}}{\binom{x+n}{n}\binom{y+n}{n}\binom{2n+2y}{n+y}},$$

Many closed expressions for sums of the form S:2/1 are known from the theory of the Hypergeometric function.

Definition:

(7.20) $$F(a, b, c; z) = \sum_{k=0}^{\infty} (-1)^k \frac{\binom{-a}{k}\binom{-b}{k}}{\binom{-c}{k}} z^k ,$$

where a,b,c,z are complex numbers.

(7.21) $$F(a, b, c; 1) = \frac{(c-1)!\,(c-a-b-1)!}{(c-a-1)!\,(c-b-1)!} , \quad R(c) > R(a+b),$$

(7.22) $$\sum_{k=0}^{\infty} \binom{-n/2}{k}\binom{n/2}{k} \frac{(2\sin x)^{2k}}{\binom{2k}{k}} = \cos nx , \quad (n \text{ real}, \ |x| \leq \pi/2) ,$$

(7.23) $$n \sum_{k=0}^{\infty} \binom{-n/2-1}{k}\binom{n/2-1}{k} \frac{2^{2k}(\sin x)^{2k+1}}{(2k+1)\binom{2k}{k}} = \frac{\sin nx}{\cos x} ,$$

$$(n \text{ real}, \ |x| < 1)$$

(7.24) $$n \sum_{k=0}^{\infty} \binom{-n/2-\frac{1}{2}}{k}\binom{n/2-\frac{1}{2}}{k} \frac{2^{2k}(\sin x)^{2k+1}}{(2k+1)\binom{2k}{k}} = \sin nx ,$$

$$(n \text{ real}, \ |x| \leq \pi/2)$$

(7.25) $$\sum_{k=0}^{\infty} \binom{-n/2-\frac{1}{2}}{k}\binom{n/2-\frac{1}{2}}{k} \frac{(2\sin x)^{2k}}{\binom{2k}{k}} = \frac{\cos nx}{\cos x} ,$$

$$(n \text{ real}, \ |x| < 1)$$

$$(7.26)\quad \sum_{k=0}^{\infty} (-1)^k \binom{-2a}{k}\binom{-2b}{k} \frac{2^{-k}}{\binom{-a-b-\frac{1}{2}}{k}} = \frac{(a+b-\frac{1}{2})!\,(-\frac{1}{2})!}{(a-\frac{1}{2})!\,(b-\frac{1}{2})!},$$

$$(a+b+\tfrac{1}{2} \neq 0,-1,-2,\ldots)$$

$$(7.27)\quad \sum_{k=0}^{\infty} (-1)^k \binom{-2a}{k}\binom{-2b}{k} \frac{2^{-k}}{\binom{-a-b-1}{k}}$$

$$= \frac{\sqrt{\pi}}{a-b}\,(a+b)!\left\{\frac{1}{(b-\frac{1}{2})!\,(a-1)!} - \frac{1}{(b-1)!\,(a-\frac{1}{2})!}\right\},$$

$$(a+b+1 \neq 0,-1,-2,\ldots)$$

$$(7.28)\quad \sum_{k=0}^{\infty} (-1)^k \binom{-a}{k}\binom{a-1}{k} \frac{2^{-k}}{\binom{-b}{k}} = \frac{(b-1)!\,(-\frac{1}{2})!\,2^{1-b}}{(a/2+b/2-1)!\,(b/2-a/2-\frac{1}{2})!},$$

$$(b \neq 0,\ -1,\ -2,\ldots)$$

- -

$$(7.49)\quad \sum_{k=1}^{n} \frac{1}{k}\, \frac{\binom{x}{k-1}\binom{y-k}{n-k}}{\binom{n}{k}} = \frac{1}{x+1}\binom{y}{n} \sum_{k=1}^{n} \frac{\binom{x+1}{k}}{\binom{y}{k}}$$

$$= \frac{\binom{x}{n}-\binom{y}{n}}{x-y},$$

called Capelli's relation by Harry Bateman,
Notes on Binomial Coefficients.
Replacing x and y by x/z and y/z and multiplying through by z^n, then letting $z \rightarrow 0$, this gives the limiting form $y^n \sum_{k=1}^{n} x^{k-1}y^{-k} = (x^n-y^n)/(x-y)$. Cf.(11.4),(11.5).

Definition:

.29) $$S_j^n = F(-2n, \tfrac{1}{2}, n+j+1; 4) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2k}{k} \frac{1}{\binom{n+j+k}{k}},$$

.30) $$R_j^n = F(-2n-1, \tfrac{1}{2}, n+j+1; 4) = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}\binom{2k}{k} \frac{1}{\binom{n+j+k}{k}},$$

Then these sums satisfy the recurrence relations

1.31) $$2\,\frac{2n+2j+1}{n+j+1}\,S_{j+1}^n = 3\,S_j^n + R_j^n \quad ,$$

1.32) $$2\,\frac{2n+2j+1}{n+j+1}\,R_{j+1}^n = 3\,R_j^n + S_{j-1}^{n+1} \quad ,$$

Some special values of these summations are as follows:

1.33) $$S_{-1}^n = 3\,, \quad (n \geq 1) \quad ,$$

1.34) $$S_0^n = 1\,, \quad (n \geq 0) \quad ,$$

1.35) $$S_1^n = \frac{n+1}{2n+1} \quad ,$$

1.36) $$S_2^n = \frac{3(n+1)(n+2)}{2(2n+1)(2n+3)} \quad ,$$

1.37) $$S_3^n = \frac{(n+2)(n+3)(11n+10)}{4(2n+1)(2n+3)(2n+5)} \quad ,$$

1.38) $$S_4^n = \frac{(n+2)(n+3)(n+4)(43n+35)}{8(2n+1)(2n+3)(2n+5)(2n+7)} \quad ,$$

1.39) $$R_{-1}^n = -\frac{5n+2}{n}\,, \quad (n \geq 1) \quad ,$$

1.40) $$R_0^n = -1\,, \quad (n \geq 0) \quad ,$$

$$.41) \qquad R_1^n = 0 ,$$

$$.42) \qquad R_2^n = \frac{n+2}{2(2n+3)} ,$$

$$.43) \qquad R_3^n = \frac{5(n+2)(n+3)}{4(2n+3)(2n+5)} ,$$

$$.44) \qquad R_4^n = \frac{21(n+2)(n+3)(n+4)}{8(2n+3)(2n+5)(2n+7)} ,$$

In general one has transformations of the following type.

$$.45) \qquad F(-n, \tfrac{1}{2}, j+1; 4z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k} \frac{z^k}{\binom{j+k}{k}}$$

$$= \frac{2^{2j}}{\binom{2j}{j}} \int_0^1 (\cos 2\pi x)^{2j} (1 - 4z \sin^2 2\pi x)^n \, dx ,$$

$$.46) \qquad = \frac{2^{2j}}{\binom{2j}{j}} \frac{1}{t} \sum_{k=0}^{t-1} \left(\cos\frac{2\pi k}{t}\right)^{2j} \left\{1 - 4z\left(\sin\frac{2\pi k}{t}\right)^2\right\}^n , \quad (\text{valid for } t > 2n+2j)$$

$$.47) \qquad = \frac{1}{\binom{2j}{j}} \sum_{k=0}^{n} \binom{n}{k}\binom{2j+2k}{j+k} 2^{-2k} (4z)^k (1-4z)^{n-k} ,$$

(These relations, (7.29) - (7.47) appear in a paper by the author pending publication.)

$$7.48) \qquad \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{y-k}{n-k}}{\binom{x+y-k}{n}} \cdot \frac{x+y+1-2k}{x+y+1-k} = 1, \quad (\text{R. Lagrange})$$

(7.49) - See page 62.

TABLE 8

SUMMATIONS OF THE FORM S:1/2

(8.1) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{1}{\binom{2n+2x}{k+x}\binom{2n+2y}{k+y}}$$

$$= \binom{2n}{n}\binom{2n+x+y+1}{n} \frac{1}{\binom{n+x}{n}\binom{n+y}{n}\binom{2n+2x}{n+x}\binom{2n+2y}{n+y}} ,$$

TABLE 10

SUMMATIONS OF THE FORM S:4/0

1) $$\sum_{k=0}^{\infty} \binom{-x}{k}^4 \frac{x+2k}{x} = \frac{\sin \pi x}{\pi x} \cdot \frac{(-2x)!}{(-x)!^2}$$, $(x < \frac{1}{2})$, (Dougall)

2) $$\sum_{k=0}^{n} \binom{n}{k}^4 k = \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k}^4 ,$$ (Dougall / Staver)

3) $$\sum_{k=0}^{n} \binom{n}{k}^4 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-k}{n} \sum_{j=0}^{k} (-1)^j \binom{n}{j}^2 \binom{k}{j} ,$$

4) $$\sum_{k=0}^{2n} (-1)^k \binom{3n}{k}\binom{3n-k}{n}^3 = \binom{3n}{2n}\binom{2n}{n} ,$$

5) $$\sum_{k=0}^{2n} (-1)^k \binom{3n+1}{k}\binom{3n-k}{n}^3 = 1 ,$$

6) $$P_n(x)\, P_r(x) = \left(\frac{x-1}{2}\right)^{n+r} \sum_{k=0}^{n+r} \left(\frac{x+1}{x-1}\right)^k \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2 ,$$

7) $$\sum_{k=0}^{n+r} \frac{(-1)^k}{\binom{n+r}{k}} \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2 = \binom{0}{n-r} = \begin{cases} 0 , & n \neq r , \\ 1 , & n = r , \end{cases}$$

8) $$\sum_{k=0}^{n+r} (-1)^k \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2 = (-1)^{\frac{n+r}{2}} \binom{n}{n/2}\binom{r}{r/2} \cdot \frac{1+(-1)^n}{2} \cdot \frac{1+(-1)^r}{2} ,$$

9) $$\sum_{k=0}^{n} \binom{n}{k}^4 = \text{coef. of } x^n \text{ in } (1-x)^{2n} \left\{ P_n\left(\frac{1+x}{1-x}\right) \right\}^2$$, (Carlitz)

TABLE 11

SUMMATIONS OF THE FORM S:3/1

$$(11.1)\quad \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{x}{k}\binom{y}{k+r}}{\binom{x+y+n}{k}} = \frac{\binom{x+r+n}{n}\binom{y+n}{n+r}}{\binom{x+y+n}{n}},$$

$$(11.2)\quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{4n+2k+1}{2k} \frac{1}{(2k+1)\binom{2n+k}{k}} = \frac{\binom{4n+1}{2n}}{2n+1},$$

$$(11.3)\quad \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{y}{k}\binom{z}{n-k}}{\binom{x+y+z}{k}} = \frac{\binom{x+z}{n}\binom{y+z}{n}}{\binom{x+y+z}{n}},$$

(equiv. to a formula of H.L.Krall)

$$(11.4)\quad \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x+y-2k-1}{n-2k}\binom{x}{k}\binom{y}{k} \frac{1}{\binom{n+1}{k}} \cdot \frac{1}{n+1-k}$$

$$= \frac{\binom{x}{n+1}-\binom{y}{n+1}}{x-y},$$

--Harry Bateman (Notes on Binomial Coefficients)

The limiting case of this, when we replace x by x/z and y by y/z and multiply through with z^{n+1}, letting $z \longrightarrow 0$ is precisely (1.60). Bateman does a similar thing with a series of form S:2/1. The result suggests analogies between power relations and binomial sums.

$$(11.5)\quad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{k}\binom{x+y-2k-1}{n-2k-1} \frac{1}{\binom{n-1}{k}} = n\,\frac{\binom{x}{n}-\binom{y}{n}}{x-y},$$

-- Bateman, Notes on Binomial Coefficients.

TABLE 12

SUMMATIONS OF THE FORM S:2/2

$$(12.1)\qquad 1+2\sum_{k=1}^{\infty}\frac{\binom{x}{k}\binom{y}{k}}{\binom{x+k}{k}\binom{y+k}{k}}=\frac{\binom{2x+2y}{2x}}{\binom{x+y}{x}^{2}},$$

valid for $R(x+y)>-\frac{1}{2}$,

$$(12.2)\qquad 1+2\sum_{k=1}^{\infty}(-1)^{k}\frac{\binom{x}{k}\binom{y}{k}}{\binom{x+k}{k}\binom{y+k}{k}}=\frac{1}{\binom{x+y}{x}},$$

valid for $R(x+y)\geq-\frac{1}{2}$,

$$(12.3)\qquad \sum_{k=0}^{n}\frac{\binom{n}{k}\binom{x}{k}}{\binom{n+k}{k}\binom{x+k}{k}}=\frac{1}{2}\left\{1+\frac{\binom{2x+2n}{2n}}{\binom{x+n}{n}^{2}}\right\},$$

$$(12.4)\qquad \sum_{k=0}^{n}(-1)^{k}\frac{\binom{n}{k}\binom{x}{k}}{\binom{n+k}{k}\binom{x+k}{k}}=\frac{1}{2}\left\{1+\frac{1}{\binom{x+n}{n}}\right\},$$

$$(12.5)\qquad \sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\binom{x-1}{k}}{\binom{-n-1}{k}\binom{-x-1}{k}}=\frac{(-1)^{n}}{2}\,\frac{\binom{-x-\frac{1}{2}}{n-1}}{\binom{-x-1}{n-1}\binom{-\frac{1}{2}}{n}},$$

(Equiv. to a problem of H. F. Sandham: No. 4519, in Amer. Math. Monthly.)

.6) $$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{2k+1}{\binom{\frac{n+k+1}{2}}{k}^2 2^{2k}(n-k+1)^2} \cdot \frac{1+(-1)^{n+k}}{2}$$

$$= \frac{1}{2n+1} \quad ,$$

.7) $$\sum_{k=0}^{n} \binom{2n}{2k}^2 \frac{4k+1}{\binom{n+k+\frac{1}{2}}{2k}^2 2^{4k}(2n-2k+1)^2} = \frac{1}{4n+1} \quad ,$$

This may be rewritten in the form

.8) $$\sum_{k=0}^{n} \frac{\binom{2n}{n+k}^2 (4k+1) \cdot 2^{4k}}{\binom{2n+2k}{n+k}^2 (2n+2k+1)^2} = \frac{1}{4n+1} \quad ,$$

2.9) $$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{y+k}{k}}{\binom{x+k}{k}\binom{x+y+k+n+1}{n}} \cdot \frac{x+y+2k+1}{x+y+k+1}$$

$$= \frac{1}{\binom{x+n}{n}} \quad ,$$

Harry Bateman (notes on binomial coefficients); see Gould, Duke Math.J., 32(1965), p.706.

TABLE 16

SUMMATIONS OF THE FORM S:4/1

$$(16.1)\quad \sum_{k=0}^{n} (-1)^k \frac{1}{\binom{n}{k}} \binom{-x}{k}\binom{x-1}{k}\binom{-y}{n-k}\binom{y-1}{n-k}$$

$$= (-1)^n\, 2^{2n} \binom{\frac{n-x-y}{2}}{n}\binom{\frac{n+x-y-1}{2}}{n},$$

(Equiv. to a result of Bailey.)

$$(16.2)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2a+2n}{a+k}\binom{2b+2n}{b+k}\binom{2c+2n}{c+k} \frac{1}{\binom{2d+2n}{d+k}}$$

$$= (-1)^n \frac{(2a+2n)!\,(2b+2n)!\,(2c+2n)!\,(n+d)!\,(2n)!\,d!}{n!\,(a+n)!\,(b+n)!\,(c+n)!\,(2d+2n)!\,(a+2n)!} \cdot$$

$$\cdot \frac{(a+b+3n)!\,(a+c+3n)!\,(b+c+3n)!}{(b+2n)!\,(c+2n)!\,(a+b+2n)!\,(a+c+2n)!\,(b+c+2n)!}$$

with a, b, c real and where $d = a+b+c+3n$.

(Equivalent to a formula of W. A. Al-Salam; see (22.1).)

$$(16.3)\quad \sum_{k=0}^{n} \binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k}\binom{n}{k}^{-1} = \binom{x+z}{n}\binom{y+z}{n},$$

(H.L.Krall)

TABLE 17

SUMMATIONS OF THE FORM S:3/2

(17.1) $$\sum_{k=0}^{\infty} (-1)^k \binom{-z}{k} \frac{\binom{x}{k}\binom{y}{k}}{\binom{x+z+k}{k}\binom{y+z+k}{k}}$$

$$= \frac{(x+z)!\ (y+z)!\ (z/2)!\ (x+y+z/2)!}{(x+z/2)!\ (y+z/2)!\ (x+y+z)!\ z!},$$

for $R(x+y+z/2) > -1$.

(Dougall / Dixon)

(17.2) $$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{x}{k}\binom{y}{k}}{\binom{x+y+z+n}{k}\binom{z+k}{k}} = \frac{\binom{x+z+n}{n}\binom{y+z+n}{n}}{\binom{z+n}{n}\binom{x+y+z+n}{n}}$$

(Saalschütz)

(17.3) $$\sum_{k=0}^{\infty} (-1)^k \frac{\binom{-x}{k}\binom{-y}{k}\binom{-z}{k}}{\binom{-\frac{x+y+z}{2}}{k}\binom{-2z}{k}}$$

$$= \frac{(-\tfrac{1}{2})!\ (z-\tfrac{1}{2})!\ \left(\frac{x+y-1}{2}\right)!\ \left(z-\frac{x+y+1}{2}\right)!}{\left(\frac{x-1}{2}\right)!\left(\frac{y-1}{2}\right)!\ \left(z-\frac{x+1}{2}\right)!\left(z-\frac{y+1}{2}\right)!},$$ (Watson)

Let $$f(x,y) = (1-z)^{2x-1} \sum_{k=0}^{\infty} (-1)^k \frac{\binom{1-2x}{k}\binom{-x-\frac{1}{2}}{k}\binom{y-x+\frac{1}{2}}{k}}{\binom{\frac{1}{2}-x}{k}\binom{-x-y-\frac{1}{2}}{k}} z^k$$

(17.4)

and then it follows that $f(x,y) = f(y,x)$. (Bailey.)

(17.5) $$\sum_{k=0}^{\infty}(-1)^k \frac{\binom{-2x}{k}\binom{-2y}{k}\binom{-x-y}{k}}{\binom{-2x-2y}{k}\binom{-x-y-\frac{1}{2}}{k}} z^k = \left\{\sum_{k=0}^{\infty} (-1)^k \frac{\binom{-x}{k}\binom{-y}{k}}{\binom{-x-y-\frac{1}{2}}{k}} z^k\right\}^2,$$

(Th. Clausen)

TABLE 21

SUMMATIONS OF THE FORM S:6/0

$$(21.1)\quad \sum_{k=0}^{2n} (-1)^k \binom{3n}{k}^3 \binom{3n-k}{n}^3 = (-1)^n \binom{2n}{n}\binom{3n}{2n}^4 \quad ,$$

TABLE 22

SUMMATIONS OF THE FORM S:5/1

$$(22.1)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2a}{a-n+k}\binom{2b}{b-n+k}\binom{2c}{c-n+k}\binom{d+k}{k}\frac{1}{\binom{d+2n}{k}}$$

$$= (-1)^n \frac{(2a)!\,(2b)!\,(2c)!\,(n+d)!\,(2n)!\,(n+a+b)!}{n!\,(n+a)!\,(n+b)!\,(n+c)!\,(d+2n)!\,a!\,b!\,c!}\cdot$$

$$\cdot\frac{(n+a+c)!\,(n+b+c)!}{(a+b)!\,(a+c)!\,(b+c)!},$$

where $d = a+b+c$

By use of the identity

$$\frac{\binom{x+k}{k}}{\binom{x+2n}{k}} = \frac{\binom{2x+2n}{x}}{\binom{2x+2n}{x+k}}$$

it is possible to transform the above S:5/1 into a series of form S:4/1 and the result is tabulated under that heading.

Special case:

$$(22.2)\quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^4 \binom{3n+k}{k}\binom{5n}{k}^{-1} = (-1)^n \binom{3n}{n}^3 \frac{(4n)!\,(2n)!}{(5n)!\,n!},$$

TABLE 23

SUMMATIONS OF THE FORM S:4/2

(23.1) $$\sum_{k=0}^{n} \frac{\binom{x}{k}\binom{y}{k}\binom{x}{n-k}\binom{y}{n-k}}{\binom{x+y-\frac{1}{2}}{k}\binom{x+y-\frac{1}{2}}{n-k}} = \frac{\binom{2x}{n}\binom{2y}{n}\binom{x+y}{n}}{\binom{2x+2y}{n}\binom{x+y-\frac{1}{2}}{n}},$$

(Follows from (17.5).)

TABLE 24

SUMMATIONS OF THE FORM S:3/3

$$24.1)\quad 1+2\sum_{k=1}^{\infty}(-1)^k \frac{\binom{x}{k}\binom{y}{k}\binom{z}{k}}{\binom{x+k}{k}\binom{y+k}{k}\binom{z+k}{k}}$$

$$=\frac{\binom{x+y+z}{y}}{\binom{x+y}{y}\binom{y+z}{y}} = \frac{x!\ y!\ z!\ (x+y+z)!}{(x+y)!\ (y+z)!\ (z+x)!},$$

valid for $R(x+y+z) > -1$, (Dougall)

(The author shows in a paper awaiting publication, that this relation implies Fjeldstad's relation - (6.1).)

TABLE 31

SUMMATIONS OF THE FORM S:4/3

(31.1) $$\sum_{k=0}^{\infty} (-1)^k \frac{c+2k}{c}\binom{c+k-1}{k} \frac{\binom{x}{k}\binom{y}{k}\binom{z}{k}}{\binom{x+c+k}{k}\binom{y+c+k}{k}\binom{z+c+k}{k}}$$

$$= \frac{(x+c)!\,(y+c)!\,(z+c)!\,(x+y+z+c)!}{(y+z+c)!\,(z+x+c)!\,(x+y+c)!\,c!}\,, \qquad \text{(Dougall)}$$

valid for $R(x+y+z+c) > -1$,

(31.2) $$\sum_{k=0}^{\infty} \frac{\binom{-x}{k}\binom{-1-x/2}{k}\binom{-y}{k}\binom{-z}{k}}{\binom{-x/2}{k}\binom{-1-x+y}{k}\binom{-1-x+z}{k}} = \frac{(x-y)!\,(x-z)!}{x!\,(x-y-z)!}\,,$$

(Bailey)

TABLE 71

SUMMATIONS OF THE FORM $S:6/5$

One of the most general identities known is that of Dougall, which may be expressed as an $S:6/5$ or as a ${}_7F_6$ with unit argument.

$$(71.1)\quad \sum_{k=0}^{n}(-1)^k \frac{c+2k}{c} \cdot \frac{\binom{c+k-1}{k}\binom{n}{k}\binom{x}{k}\binom{y}{k}\binom{z}{k}\binom{x+y+z+n+2c+k}{k}}{\binom{n+c+k}{k}\binom{x+c+k}{k}\binom{y+c+k}{k}\binom{z+c+k}{k}\binom{x+y+z+c+n}{k}}$$

$$= \frac{\binom{x+y+c+n}{n}\binom{y+z+c+n}{n}\binom{z+x+c+n}{n}\binom{c+n}{n}}{\binom{x+c+n}{n}\binom{y+c+n}{n}\binom{z+c+n}{n}\binom{x+y+z+c+n}{n}},$$

TABLE 97

Expressed as a ${}_7F_6$ one has the equivalent form of Dougall's formula

$$(97.1)\quad {}_7F_6\left[\begin{matrix} a,\ 1+a/2,\ b,\ c,\ d,\ e,\ -n\ ; \\ a/2,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n\ (1+a-b-c)_n\ (1+a-b-d)_n\ (1+a-c-d)_n}{(1+a-b)_n\ (1+a-c)_n\ (1+a-d)_n\ (1+a-b-c-d)_n},$$

where $1+2a = b+c+d+e-n$, $n = 0,1,2,3,\ldots.$

and $\quad (x)_n = \dfrac{(x+n-1)!}{(x-1)!} = n!\dbinom{x+n-1}{n}.$

TABLE X

SUMMATIONS OF THE FORM S:p/0

An asymptotic formula:

1) $$\sum_{k=0}^{n} \binom{n}{k}^{p} \sim \frac{2^{pn}}{\sqrt{p}} \left(\frac{2}{\pi n}\right)^{\frac{p-1}{2}} \quad \text{as } n \to \infty,$$

for p = integer $\geqslant 1$.
(Polya and Szegö.)

2) $$\sum_{k=0}^{n} \binom{x}{k}^{p} \frac{k!^{p}}{x^{(k+1)p}} \left\{(x-k)^{p} - x^{p}\right\} = \binom{x}{n+1}^{p} \frac{(n+1)!^{p}}{x^{(n+1)p}} - 1 ,$$

(Gould)

3) $$\sum_{k=0}^{rp-r+1} \binom{x+k-1}{rp} \sum_{j=0}^{k-1} (-1)^{j} \binom{rp+1}{j} \binom{k+r-j-1}{r}^{p} = \binom{x}{r}^{p} ,$$

valid for integral $r \geqslant 1$, $p \geqslant 1$.

4) $$\sum_{k=0}^{rp} (-1)^{k} \binom{rp+1}{k} \binom{rp-k+y+r-1}{r}^{p} = \binom{1-y}{r}^{p} ,$$

5) $$\sum_{k=0}^{rp-r+1} (-1)^{k} \binom{rp+1}{k} \binom{rp-k}{r}^{p} = 1 ,$$

(The three relations immediately above are connected with very general expansions of Worpitzky, as shown by Carlitz, and were 'rediscovered' by various modern writers, in particular by Shanks [8] .)

(X.6) Definition: $S_n(q) = \sum_{k=0}^{n} \binom{n}{k}^q$, (Staver)

(X.7) Definition: $S_{n,p}(q) = \sum_{k=0}^{n} \binom{n}{k}^q k^p$, (Staver)

Then the following formulas are known:

(X.8) $S_{n,1}(3) = \frac{n}{2} S_n(3)$,

(X.9) $S_{n,2}(3) = \frac{n^2}{6} S_n(3) + \frac{2n^2}{3} S_{n-1}(3)$,

(X.10) $S_{n,3}(3) = n^3 S_{n-1}(3)$,

(X.11) $S_{n,4}(3) = \frac{n^3(n+1)}{2} S_{n-1}(3)$,

(X.12) $2 S_{n,5}(3) = \frac{n^5}{6} S_n(3) - \frac{5n^4}{6}(n-3) S_{n-1}(3)$,

---- (Staver)

(X.13) $n^2 S_n(3) = (7n^2 - 7n + 2) S_{n-1}(3) + 8(n-1)^2 S_{n-2}(3)$,

(Franel)

This relation just given was first proved by J. Franel, in answer to a question posed by C. A. Laisant in the first volume (1894) of the journal <u>l'Intermédiaire des Mathématiciens</u>. Franel also found the following formula as well as other interesting results:

(X.14) $n^3 S_n(4) = 2(2n-1)(3n^2 - 3n + 1) S_{n-1}(4)$

$+ (4n-3)(4n-4)(4n-5) S_{n-2}(4)$, (Franel / Staver)

(X.15) $\sum_{k=0}^{n} \binom{n}{k}^q x^{n-k} y^k = \sum_{k=0}^{[n/2]} C_{n,k}^{(q)} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k$, where: $$C^{(q+1)}_{n,k} = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} C^{(q)}_{n,j} \quad , \qquad \text{(Nanjundiah)}$$

'his general result includes several special cases in the preceding ıbles.) Thus:

$$C^{(-1)}_{n,k} = (-1)^k \frac{\binom{n+1}{k}}{\binom{n}{k}} \quad , \qquad C^{(0)} = (-1)^k \quad ,$$

$$C^{(1)} = \binom{0}{k} \quad , \qquad C^{(2)} = \binom{n}{k} \quad , \qquad C^{(3)} = \binom{n}{k}\binom{n+k}{k} \; .$$

MacMahon [53] has given very general generating functions for $S_n(q)$, owever this is too extensive to list in the present tables.

(X.16) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{a_1+b_1k}{m_1}\binom{a_2+b_2k}{m_2} \cdots \binom{a_r+b_rk}{m_r} = 0 \quad ,$$

when $0 \leq m_1 + m_2 + \cdots + m_r < n$.
Follows trivially from (Z.8).

(X.17) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\binom{a_1k}{b_1}}{c_1}\binom{\binom{a_2k}{b_2}}{c_2} \cdots \binom{\binom{a_rk}{b_r}}{c_r} = 0 \; ,$$

when $b_1c_1 + b_2c_2 + \cdots + b_rc_r < n$.
(again a trivial consequence of (Z.8))

(X.18) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a_1k^{b_1}}{c_1}\binom{a_2k^{b_2}}{c_2} \cdots \binom{a_rk^{b_r}}{c_r} = 0,$$

when $b_1c_1 + b_2c_2 + \cdots + b_rc_r < n$.
(another trivial consequence of (Z.8))

TABLE Z

In this table we list some general expansions, operations on series, and a few useful binomial coefficient identities. In formulas (Z.1) through (Z.12) $f(x)$ is understood to be any polynomial of degree less than or equal to n in x, that is,

$$f(x) = \sum_{i=0}^{n} a_i x^i .$$

(Z.1) (Lagrange series)

Let
$$g(x) = \prod_{i=0}^{n} (x - x_i) , \qquad g'(x) = D_x g(x),$$

and let distinct real numbers $x_0, x_1, \ldots, x_n$ be given.

Then
$$f(x+y) = \sum_{k=0}^{n} f(x_k + y) \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$$= g(x) \sum_{k=0}^{n} \frac{f(x_k + y)}{(x - x_k)\, g'(x_k)} .$$

(Z.2) (Taylor series)

$$f(x+y) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(y) .$$

(Z.3) (Newton series)

$$f(x+y) = \sum_{k=0}^{n} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(k - j + y) .$$

(Z.4) (Worpitzky - Nielsen series)

$$f(x+y) = (-1)^m \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} f(j-k+y) \quad ,$$

provided $m \geq n$.

(Z.5) (Melzak, prob. nr. 4458, Amer. Math. Monthly, V. 58(1951), p.636; a special case of (Z.1).)

$$f(x+y) = y \binom{y+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} \quad ,$$

(Z.6) $$f(x+y) \sum_{k=0}^{n} \frac{1}{y+k} - D_y f(x+y)$$

$$= y \binom{y+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{(y+k)^2} \quad ,$$

(Z.7) $$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{f(x-k)}{k} = f(x) \sum_{k=1}^{n} \frac{1}{k} - f'(x) \quad ,$$

(Z.8) $$\sum_{k=0}^{r} (-1)^k \binom{r}{k} f(k) \doteq \begin{cases} 0, \text{ if } n < r, \\ (-1)^n n!\, a_n \text{ , if } n = r. \end{cases}$$

(Z.9) $$\sum_{k=0}^{\infty} \frac{x^k}{k!} f(k) = e^x \sum_{j=0}^{n} \frac{x^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} f(j-k) \quad ,$$

Relation (Z.8) is very useful; we have numerous interesting cases by choosing f; e.g. (1.13),(6.41), (X.16), (X.17), (X.18), etc.

$$(Z.10)\qquad \sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\frac{f(y+k^2)}{x^2-k^2}$$

$$=(-1)^n\frac{f(x^2+y)}{2x(x-n)\binom{x+n}{2n}}+\frac{1}{2}\binom{2n}{n}\frac{f(y)}{x^2},$$

$$(Z.11)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(y+k^2)}{(x^2-k^2)\binom{n+k}{k}}$$

$$=(-1)^n\frac{f(x^2+y)}{2x(x-n)\binom{2n}{n}\binom{x+n}{2n}}+\frac{f(y)}{2x^2},$$

which is a restatement of (Z.10).

$$(Z.12)\qquad \frac{f(x)}{\prod_{i=1}^{r}(x-u_i)}=\sum_{k=1}^{r}\frac{f(u_k)}{\prod_{\substack{i=1\\ i\neq k}}^{r}(u_k-u_i)}\cdot\frac{1}{x-u_k},$$

provided $r > n$.

$$(Z.13)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^m}{\prod_{i=1}^{r}(k-u_i)}$$

$$=-\sum_{k=1}^{r}\frac{u_k^{m-1}}{\binom{n-u_k}{n}\prod_{\substack{i=1\\ i\neq k}}^{r}(u_k-u_i)},\ \text{for } r > m.$$

Z.14) Let $f(x)$ be a polynomial of degree $n+1$ in x. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k)}{(k+x)(k+z)}$$

$$= \frac{1}{(n+1)(x-z)} \left\{ \frac{f(y+z)}{\binom{z+n}{n+1}} - \frac{f(y+x)}{\binom{x+n}{n+1}} \right\},$$

Z.15) Let $f(x)$ be a polynomial of degree $n+2$ in x. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k)}{(k+u)(k+v)(k+w)}$$

$$= \frac{u-1}{(n+1)(n+2)(u-v)(u-w)} \left\{ \frac{f(y+u)}{\binom{u+n}{n+2}} - \frac{f(y+w)}{\binom{w+n}{n+2}} \right\}$$

$$- \frac{v-1}{(n+1)(n+2)(u-v)(v-w)} \left\{ \frac{f(y+v)}{\binom{v+n}{n+2}} - \frac{f(w+y)}{\binom{w+n}{n+2}} \right\}$$

Using partial fractions one may evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k)}{\prod_{i=1}^{r} (k+u_i)}$ in general.

(Z.16) Let $f(x) = \sum_{i=0}^{n+r-1} a_i x^i$.

Then
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k)}{\binom{k+r}{k}} = - \sum_{k=1}^{r} (-1)^k \binom{r}{k} \frac{f(y+k)}{\binom{k+n}{k}} ,$$

(Z.17) In the preceding relation, let $r = n$. Then if $f(x)$ is of degree $2n-1$ in x,
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k) + f(y+k)}{\binom{k+n}{k}} = f(y) .$$

(Z.18) If $f(x) = f(-x)$, $f(0) = 0$, and $f(x)$ is of degree $2n-2$ in x, then
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k)}{\binom{k+n}{k}} = 0.$$

(Z.19) A useful operational formula:
$$\left(x^n D_x^n\right)^r y = (n!)^r \sum_{k=1}^{n(r-1)+1} \frac{x^{k+n-1}}{(k+n-1)!} \cdot A_k \cdot D_x^{k+n-1} y,$$
where
$$A_k = \sum_{j=0}^{k+n-1} (-1)^j \binom{k+n-1}{j} \binom{k+n-1-j}{n}^r ,$$

(Cf. Carlitz, Amer. Math. Monthly, Vol.37(1930), pp. 472-9)

A very useful general technique in summing series is to make use of the identity $\left(x\, D_x\right)^r x^k = k^r x^k$. This is what is used in all the series in these tables where a series contains the term $k^r x^k$ and x appears in no other connection in the series. Schwatt gives a very broad discussion of its uses.

In terms of the general numbers of Nielsen type (see (1.1 28)above) we have

$$(Z.20) \qquad \left(x\, D_x\right)^r y = (-1)^{m+n} \sum_{k=0}^{m+1} B^n_{k,m+1} \sum_{j=0}^{n} \binom{k-1}{m-j} \frac{x^j}{j!}\, D_x^j\, y \; .$$

and $m \geqslant n$.

In terms of the Stirling numbers of second kind (differences of zero), two other useful general formulas are

$$(Z.21) \qquad \left(x \underset{x,z}{\Delta}\right)^r f(x) = \sum_{k=0}^{n} \binom{\frac{x}{z} + k - 1}{k} \cdot z^k \cdot B^n_k \cdot \underset{x,z}{\Delta}^{k} f(x) \; ,$$

$$(Z.22) \qquad \left(x\, D_x\right)^n f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \cdot B^n_k \cdot D^k_x\, f(x).$$

(Z.23) Let $F(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k)$,

and suppose that f(k) does not depend on n.

Then the series may be inverted to give

$$f(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} F(k),$$

and also

$$\sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} F(k) .$$

(Z.24) Let $F(k) = \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} f(j)$;

then

$$\sum_{k=0}^{n} (-1)^{k} \binom{x}{k} F(k) = (-1)^{n} \binom{x-1}{n} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{x\, f(k)}{x-k} ,$$

(Gould, [34])

(Z.25) $$\sum_{k=0}^{n} \binom{x}{k} f(k) = \sum_{k=0}^{n} (-1)^{k} \binom{n-x}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} f(j+k) .$$

(Z.26) Suppose that $D_x U_n(x) = n\ U_{n-1}(x)$.

Then the summation $S_n(x) = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} U_k(x)\, U_{n-k}(x)$

has the property that it is independent of x, i.e., $D_x S_n(x) = 0$. Hence the series may be evaluated by letting x take any convenient value. (Math. Tripos, 1896, or see Hardy, Course of Pure Mathematics, 9th Ed., 1944, page 274.)

(Z.27) $$\sum_{k=0}^{n} f(kr) = \frac{1}{r} \sum_{j=1}^{r} \sum_{k=0}^{rn} \omega_r^{jk} f(k),$$

where $\omega_r = e^{\frac{2\pi i}{r}}$.

(Z.28) $$\sum_{k=0}^{\left[\frac{n}{r}\right]} \binom{n}{rk} f(rk) = \frac{1}{r} \sum_{j=1}^{r} \sum_{k=0}^{n} \binom{n}{k} f(k) \cos \frac{2\pi jk}{r},$$

$(r \geq 1,\ n \geq 0)$

(Z.29) Let $$F(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{f(k)}{n-k},$$

and suppose that $f(k)$ is independent of n.

Then
$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} \frac{f(k+1)}{k+1} = \frac{f(0)}{n+2} - F(n+2).$$

(Z.30) Let $F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k)$, and suppose $f(k)$ is

independent of n. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k+1)}{k+1} = \frac{f(0) - F(n+1)}{n+1}.$$

(Z.31) $$\sum_{k=0}^{n} \binom{n}{k} \binom{n+2x}{k+x} f(k) = \frac{\binom{2n+2x}{n+x}}{\binom{2n+2x}{n}} \sum_{k=0}^{n} \binom{n+x}{k} \binom{n+x}{n-k} f(k).$$

(Z.32) $$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} f(k) = \frac{(-1)^n}{\binom{2x}{x}\binom{2x+n}{n}} \sum_{k=0}^{n} \binom{-x-1}{k}\binom{-x-1}{n-k} f(k).$$

(Z.33) $$\sum_{k=a}^{n} f(k) = \sum_{k=\left[\frac{a+1}{2}\right]}^{\left[\frac{n}{2}\right]} f(2k) + \sum_{k=\left[\frac{a}{2}\right]}^{\left[\frac{n-1}{2}\right]} f(2k+1) \quad .$$

(Z.34) $$\sum_{k=a}^{n} (-1)^k f(k) = \sum_{k=\left[\frac{a+1}{2}\right]}^{\left[\frac{n}{2}\right]} f(2k) - \sum_{k=\left[\frac{a}{2}\right]}^{\left[\frac{n-1}{2}\right]} f(2k+1) \quad .$$

(Z.35) $$\sum_{j=0}^{r-1} \sum_{k=\left[\frac{a+r-1-j}{r}\right]}^{\left[\frac{n-j}{r}\right]} f(rk+j) = \sum_{k=a}^{n} f(k) \quad .$$

(Thisis the general case for (Z.33).)

(Z.36) $$\sum_{k=0}^{rn-1} f(k) = \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} f(rk+j) , \quad (r \geq 1, n \geq 1)$$

(Z.37) $$\sum_{k=1}^{n} \left\{ f(k) - f(k+r) \right\} = \sum_{k=1}^{r} \left\{ f(k) - f(k+n) \right\} \quad ,$$

(Z.38) $$\sum_{k=1}^{n} k\, f(k) = (n+1) \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} \sum_{j=1}^{k} f(j) \quad ,$$

(Z.39) $$\sqrt{\pi}\,(2n)! = 2^{2n}\, n!\, (n - \tfrac{1}{2})! \quad ,$$

(Z.40) $$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{2n}{n} 2^{-2n} \quad ,$$

(Z.41) $$(-n - \tfrac{1}{2})!\, (n - \tfrac{1}{2})! = (-1)^n \pi \quad ,$$

(Z.42) $$\frac{\binom{z}{n}}{\binom{x}{n}} = \frac{x+1}{x-z+1} \left\{ \frac{\binom{z}{n}}{\binom{x+1}{n}} - \frac{\binom{z}{n+1}}{\binom{x+1}{n+1}} \right\} ,$$

(Z.43) $$\binom{\frac{2n+1}{2}}{2n+1} = (-1)^n \binom{2n}{n} 2^{-4n-1} \quad ,$$

(Z.44) $$\binom{2n+\frac{1}{2}}{n} = \binom{4n+1}{2n} 2^{-2n} \quad ,$$

In the following identities it is assumed that $0 \leq k \leq n$ (integers).

(Z.45) $$\binom{n-\frac{1}{2}}{k} = \binom{2n}{n}\binom{n}{k} \frac{1}{\binom{2n-2k}{n-k} 2^{2k}} \quad ,$$

(Z.46) $$\binom{n+\frac{1}{2}}{n-k} = \frac{2n+1}{2k+1} \binom{2n}{n}\binom{n}{k} \frac{2^{2k-2n}}{\binom{2k}{k}}$$

(Z.47) $$\binom{-n-\frac{1}{2}}{n-k} = (-1)^{n-k}\, 2^{2k-2n} \frac{\binom{4n-2k}{2n-k}\binom{n}{k}}{\binom{2n}{k}}$$

(Z.48) $$\binom{n}{\frac{1}{2}} = \frac{2^{2n+1}}{\pi \binom{2n}{n}} \quad ,$$

(Z.49) $$\binom{n}{\frac{n}{2}} = \frac{2^{2n}}{\pi\binom{n}{\frac{n-1}{2}}} ,$$

(Z.50) $$\binom{-2n-3/2}{k} = (-1)^k \binom{2k}{k} \frac{\binom{4n+2k+1}{2k}}{\binom{2n+k}{k} 2^{2k}} ,$$

(Z.51) $$\binom{2n+\frac{1}{2}}{k} = \binom{4n+1}{2k}\binom{2k}{k} \frac{1}{\binom{2n}{k} 2^{2k}} ,$$

(Z.52) $$\binom{k}{\frac{1}{2}}\binom{n-k}{\frac{1}{2}}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{2^{2n+2}}{\pi^2} ,$$

(Z.53) $$\binom{k-n}{k+j}\binom{k+n}{k-j} = (-1)^k \binom{n-1}{k}\binom{k+j}{k}^{-1} \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{-n-i}{j-i} ,$$

(Z.54) $$\binom{2n-2k}{n-k}\binom{2n+2k}{n+k} = (-1)^{n+k}\binom{2n}{n+k}\binom{n-k-\frac{1}{2}}{2n} 2^{4n} ,$$

(Z.55) $$\binom{2k}{k}\binom{2n-2k}{n-k} = (-1)^{n-k}\binom{n}{k}\binom{k-\frac{1}{2}}{n} 2^{2n} = \binom{n}{k}^2 \binom{2n}{n}\binom{2n}{2k}^{-1} ,$$

(Z.56) $$\binom{x}{k}\binom{x}{n-k} = \binom{n}{k}\binom{2x}{n}\binom{2x-n}{x-k}\binom{2x}{x}^{-1} ,$$

(Z.57) $$\binom{x}{2n-1}\binom{2n}{n} = \frac{2n}{x+1}\binom{\frac{x+1}{2}}{n}\binom{\frac{x}{2}}{n}\,2^{2n} ,$$

(Z.58) $$\binom{2n+2x}{2n}\binom{2n}{n} = 2^{2n}\binom{n+x}{n}\binom{n+x-\frac{1}{2}}{n} ,$$

(Z.59) $$\binom{x+n}{n}\binom{2x+2n}{x+n} = \binom{x+2n}{n}\binom{2x+2n}{x} ,$$

(Z.60) $$D_x\binom{x+n}{n} = \binom{x+n}{n}\sum_{k=1}^{n}\frac{1}{k+x} ,$$

(It is by this differentiation that the finite harmonic series can be introduced into a series of binomial coefficients; e.g., see relations of this type such as that obtained from (1.44) by use of (Z.23), or Goldberg's relation (3.124), etc.)

(Z.61) $$\binom{x}{n}\binom{-x-1}{n} = (-1)^n\binom{2n}{n}\binom{x+n}{2n} ,$$

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ERRATA

p. 3 of main tables, 3rd line up from bottom, read [72] instead of 72 .

p. 5, in (1.36), on right read $1 + (-1)^n$ instead of $1 - (-1)^n$.

p. 8, in (1.58), read $\frac{n\pi}{4}$ instead of $\frac{n}{4}$.

p. 9, in (1.71), read $z = \dfrac{-x}{(1+x)^2}$ instead of $z = \dfrac{-x^2}{(1+x)^2}$.

p. 12, in (1.93), on left read $\binom{2n+1}{2k}$ instead of $\binom{2n+1}{k}$.

p. 14.1, in (1.118), add condition that $n \geqslant 2$.

p. 17, in (2.16), on right read $\frac{1}{k!}$ instead of $\frac{1}{k}$.

p. 18, in (2.24), read $\binom{kn}{n}$ instead of $\binom{kn}{k}$.

p. 21, in (3.23), on right hand side read $n \geqslant 2$ instead of $n \quad 2$.

p. 27, in (3.72), read $\frac{(-1)^n}{4}$ instead of $\frac{(-1)^n}{2}$.

p. 30, in (3.95), read $(x+1)\cdots(x+n)$ instead of $x(x+1)\cdots(x+n)$

p. 40, (4.10), the sum should be $$\sum_{k=0}^{n} k \frac{\binom{n}{k}}{\binom{2n-1}{k}} .$$

p. 40, in (4.13), read $n+r$ instead of $n \quad r$.

p. 50, in (7.11), read $\binom{2x}{2n}$ instead of $\binom{2x}{x}$.

p. 71, line 3, read (Z.12) instead of (Z.13).

p. 79, change statement after (Z.35) to read "This is the general case for (Z.33)."

p. 80, in (Z.39), on left read $\sqrt{\pi}\,(2n)!$ instead of $(2n)!$.

p. 80, in (Z.48), on right read $\pi\binom{2n}{n}$ instead of $\binom{2n}{n}$.

p. 81, in (Z.52), read k instead of kk .

page 6 formula (1.46)

$$\text{for} \quad \frac{1}{24(n+3)} + \frac{1}{24\binom{n+7}{5}} - \frac{1}{12\binom{n+4}{2}}$$

$$\text{read} \quad \frac{1}{n(n+1)} \left\{ \frac{1}{2(n+3)} + \frac{1}{2\binom{n+7}{5}} - \frac{1}{\binom{n+4}{2}} \right\}$$

page 18 formula (2.24), line 2

$$\text{for} \quad n!\sum_{k=1}^{\infty} \quad \text{read} \quad n!\sum_{k=0}^{\infty} .$$

page 21 formula (3.22) in left-most sum

$$\text{for} \quad 2^{-2k} \quad \text{read} \quad 2^{2n-2k} .$$

page 35, in third displayed formula from top,

$$\text{for} \quad \frac{t^n}{n} \quad \text{read} \quad \frac{t^n}{n!} .$$

Errata for: Gould, "Combinatorial Identities, ..."

page 4, eq.(1.31). Replace $\sin^n(\frac{x}{2})$ by $\sin^n(\frac{x}{4})$

page 13, eq.(1.106). Replace $(-1)^k$ by $(-1)^{k-1}$

page 26, eq.(3.57). Multiply the right member by $\frac{(2x)!(2n)!}{x!(x+2n)!}$

page 26, eq.(3.58). Multiply the right member by 2^n

page 31, eq.(3.103). Replace 2^{n-r} by 2^{n-k}

page 43, eq.(6.5). Replace $\cos(n\pi/2)$ by $\frac{1}{2}[1+(-1)^n]$

page 44, eq.(6.8). Delete $(-1)^k$

page 44, eq.(6.10). Insert $(-1)^k$ into the summand.

page 47, eq.(6.35). Multiply the second sum by 2^{2n}

page 51, eq.(7.19). Replace $\binom{n-x-y-1}{n}$ by $\binom{n+x-y-1}{n}$

page 66, eq.(31.1). Replace $(x+y)!$ by $(x+c)!$

---compiled by Eldon R. Hansen, August, 1969.

FURTHER ERRATA

p.7, in (1.53), add: where $\omega_r = e^{2\pi i/r}$.

p.14.1, in (1.117), for $(x + 1)^{k-1}$ read $(k + 1)^{k-1}$.

p.26, in (3.57), for $\binom{2n+2x}{2n}$ read $\binom{2n+2x}{x}$. Cf. Hansen's form of this correction.

p.38, in (3.147), for $\sum_{k=0}^{n-1}$ read $\sum_{k=1}^{n-1}$.

p.51, in (7.18), the sum = -1 for n = 0. Indicated sum correct for $n \geqslant 1$.

p. 51, in (7.17), in left-hand sum, insert the factor $\frac{1}{k}$.

Also, for $n \geqslant m$ read $m \geqslant n$.

p. 65, for (2.41) read (24.1).

p.86, item #51, for 'Todskrift' read 'Tidskrift'.

p. 87, item #60, add ref. to Second edition, 1927.

p.3, introduction, line 10, for 'indes' read 'index'.

p. 69, in (X.13), for $(n-1)^3$ read $(n-1)^2$.

p. 14.2, in def. stated with (1.127), delete the factor $(-1)^k$.

H. W. Gould
1 May, 1972

Research and Publications, selected list: H. W. Gould

1. Note on a paper of Grosswald, Amer. Math. Monthly, 61(1954), 251-253.

2. Some generalizations of Vandermonde's convolution, Amer. Math. Monthly, 63(1956), 84-91.

3. The Stirling numbers, and generalized difference expansions, University of Virginia Master's Thesis, June 1956. iii+66pp.

4. Final analysis of Vandermonde's convolution, Amer. Math. Monthly, 64(1957), 409-415.

5. A theorem concerning the Bernstein polynomials, Math. Magazine, 31(1958), 259-264.

6. Note on a paper of Sparre Andersen, Math. Scand., 6(1958), 226-230.

7. Dixon's series expressed as a convolution, Nordisk Mat. Tidsk., 7(1959), 73-76.

8. Note on a paper of Steinberg, Math. Magazine, 33(1959), 46-48.

9. Combinatorial Identities, a standardized set of tables listing 500 binomial coefficient summations, Morgantown, W.Va. 1959-60. 100pp.

10. Generalization of a theorem of Jensen concerning convolutions, Duke Math. J., 27(1960), 71-76.

11. Stirling number representation problems, Proc. Amer. Math. Soc., 11(1960), 447-451.

12. The Lagrange interpolation formula and Stirling numbers, Proc. Amer. Math. Soc., 11(1960), 421-425.

13. A q-binomial coefficient series transform, Bull. Amer. Math. Soc., 66(1960), 383-387.

14. Fjeldstad's series and the q-analog of a series of Dougall, Math. Scand., 8(1960), 198-200.

15. The q-series generalization of a formula of Sparre Andersen, Math. Scand., 9(1961), 90-94.

16. Note on a paper of Klamkin concerning Stirling numbers, Amer.Math. Monthly, 68(1961), 477-479.

17. A series identity for finding convolution identities, Duke Math. J., 28(1961), 193-202.

18. The q-Stirling numbers of first and second kinds, Duke Math. J., 28(1961), 281-289.

19. Some relations involving the finite harmonic series, Math. Mag., 34(1961), 317-321.

20. A generalization of a problem of L. Lorch and L. Moser, Canadian Math. Bull., 4(1961), 303-305.

21. A binomial identity of Greenwood and Gleason, Math. Student, 29(1961), 53-57.

22. (with A.T.Hopper) Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J., 29(1962), 51-63.

23. Congruences involving sums of binomial coefficients and a formula of Jensen, Amer. Math. Monthly, 69(1962), 400-402.

24. A new convolution formula and some new orthogonal relations for inversion of series, Duke Math. J., 29(1962), 393-404.

25. Generalization of an integral formula of Bateman, Duke Math. J., 29(1962), 475-479.

26. Theory of binomial sums, Proc. W. Va. Acad. Sci., 34(1962), 158-162 (1963).

27. Note on two binomial coefficient sums found by Riordan, Ann. Math. Stat., 34(1963), 333-335.

28. The q-analogue of a formula of Szily, Scripta Mathematica, 26(1961), 155-157 (1963).

29. Generating functions for products of powers of Fibonacci and Lucas numbers, 1(1963), No.2, 1-16.

30. Solution to Problem 470, Math. Mag., 36(1963), 206.

31. Solution to Problem 505, Math. Mag., 36(1963), 267-268.

32. Sums of logarithms of binomial coefficients, Amer. Math. Monthly, 71(1964), 55-58.

33. The functional operator $Tf(x) = f(x+a)f(x+b) - f(x)f(x+a+b)$, Math. Mag., 37(1964), 38-46.

34. Solution to Problem 521, Math. Mag., 37(1964), 61.

35. (with L. Carlitz) Bracket function congruences for binomial coefficients, Math. Mag., 37(1964), 91-93.

36. A new series transform with applications to Bessel, Legendre, and Tchebycheff polynomials, Duke Math. J., 31(1964), 325-334.

37. The operator $(a^x\Delta)^n$ and Stirling numbers of the first kind, Amer. Math. Monthly, 71(1964), 850-858.

38. Problem 5231, Amer. Math. Monthly, 71(1964), 923. See solutions, ibid., 72(1965), 921-923.

39. Binomial coefficients, the bracket function, and compositions with relatively prime summands, Fibonacci Quarterly, 2(1964), 241-260.

40. The construction of orthogonal and quasi-orthogonal number sets, 72(1965), 591-602.

41. An identity involving Stirling numbers, Ann. Inst. Stat. Math., 17(1965), 265-269.

42. A variant of Pascal's triangle, Fibonacci Quarterly, 3(1965), 257-271. Errata, 4(1966), 62.

43. Inverse series relations and other expansions involving Humbert polynomials, Duke Math. J., 32(1965), 697-711.

44. Note on recurrence relations for Stirling numbers, Publ. Inst. Math. (Beograd), N.S., 6(20)(1966), 115-119.

45.(with J. Kaucký) Evaluation of a class of binomial coefficient summations, J. Combinatorial Theory, 1(1966), 233-247. Errata, ibid., 12(1972), 309-310.

46. Note on a q-series transform, J. Combinatorial Theory, 1(1966), 397-399.

47. (with C. A. Church, Jr.) Lattice point solution of the generalized problem of Terquem and an extension of Fibonacci numbers, Fibonacci Quarterly, 5(1967), 59-68.

48. Comment on Problem 602, Math. Mag., 40(1967), 107-108.

49. (with Maurice Machover) Problem E 1975, Amer. Math. Monthly, 74(1967), 437-438. See solutions, ibid., 75(1968), 682-683.

50. Note on a combinatorial identity in the theory of bi-colored graphs, Fibonacci Quart., 5(1967), 247-250.

51. The bracket function, q-binomial coefficients, and some new Stirling number formulas, Fibonacci Quart., 5(1967), No.5, 401-423. Errata, ibid., 6(1968), 59.

52. (with Franklin C. Smith) Problem 5612, Amer. Math. Monthly, 75(1968), 791. See solution, ibid., 76(1969), 705-707.

53. Problem H-142, Fibonacci Quart., 6(1968), No.4, 252. See solution, ibid., 8(1970), 275-276.

54. Note on a paper of Paul Byrd, and a solution of problem P-3, Fibonacci Quart., 6(1968), 318-321.

55. Note on two binomial coefficient identities of Rosenbaum, J. Math. Physics, 10(1969), 49-50.

56. The bracket function and Fontené-Ward generalized binomial coefficients with application to Fibonomial coefficients, Fibonacci Quart., 7(1969), No.1, 23-40, 55.

57. Power sum identities for arbitrary symmetric arrays, SIAM J. Applied Math., 17(1969), 307-316.

58. Invariance and other simple techniques as a genesis of binomial identities, Nieuw Archief voor Wiskunde, (3)17(1969), 214-217.

59. Remarks on a paper of Srivastava, J. Math. Physics, 12(1971), 1378-1379.

60. Comment on a paper of Chiang, J. Math. Physics, 12(1971), 1743.

61. Reviews #2033 - 2034, Math. Reviews, 38(1969).

62. Noch einmal die Stirlingschen Zahlen, Bemerkungen zu Noten von Johannes Thomas und F. Klein-Barmen, Jber. Deutsch. Math.-Verein., 73(1971), 149-152.

63. Solution of Problem 783, Math. Mag., 44(1971), 289-291.

64. Equal products of generalized binomial coefficients, Fibonacci Quart., 9(1971), No.4, 337-346.

65. Explicit formulas for Bernoulli numbers, Amer. Math. Monthly, 79(1972), 44-51.

66. Solution of Problem 70-21, SIAM Review, 14(1972), 166-169.

67. A new symmetrical combinatorial identity, J. Comb. Theory,

68. The case of the strange binomial identities of Professor Moriarty, Fibonacci Quart.,

69. Some combinatorial identities of Bruckman - a systematic treatment with relation to the older literature, Fibonacci Quart.,

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71. A new primality criterion of Mann and Shanks and its relation to a theorem of Hermite, Fibonacci Quart.,

72. A remarkable combinatorial formula of Heselden, Proc. W.Va. Acad. of Sci.,

73. Evaluation of the finite hypergeometric series F(-n,1/2;j+1;4),

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77. Polynomials associated with an integral of Euler,

78. Types of orthogonality for arrays of numbers,

79. Special functions associated with the higher derivatives of $y = x^x$,

80. Brill's series and some related expansions obtainable in closed form, Mathematica Monongaliae, #3, August, 1961. 14pp.

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82. Notes on a calculus of Turán operators, Mathematica Monongaliae, #6, May, 1962. 25pp.

83. The Bateman notes on binomial coefficients, edited by H.W.Gould, August, 1961. 150pp. to appear. Based on 560 page manuscript comprising three versions of the original work.