But since $D^{\kappa+1}\psi(\epsilon, t)$ is bounded (uniformly in ϵ) in (1, 2), it follows that (24) continues to hold for $t \geqslant 1$.

Now suppose that $n \leqslant w \leqslant n+1$, where n is an integer (and where $n \geqslant 1$). It follows from the definition of $\phi(t)$ and from equations (23) and (24) that

$$\int_0^w t^\kappa |D^{\kappa+1}\psi(t)|\,dt \leqslant \int_0^w t^\kappa |D^{\kappa+1} t|\,dt + \sum_{\nu=1}^{n-1} \frac{1}{\nu^\kappa} \int_\nu^w t^\kappa |D^{\kappa+1}\psi(\epsilon_\nu, t-\nu)|\,dt$$

$$+\frac{1}{n^\kappa}\int_n^{n+1} t^\kappa |D^{\kappa+1}\psi(\epsilon_n, t-n)|\,dt$$

$$\leqslant Aw + \sum_{\nu=1}^{n-1} \frac{A}{\nu^\kappa}\left\{(\nu+1)^\kappa + \int_{\nu+1}^w t^\kappa (t-\nu)^{-\kappa-1}\,dt\right\} + \frac{A(n+1)^\kappa}{n^\kappa}$$

$$\leqslant Aw.$$

Since $\phi(w) > w$, it is clear that (1) is satisfied.

On the other hand we have, for $\epsilon_n < u < \delta$,

$$\phi(n+\delta)-\phi(n+u) = \frac{1}{n^\kappa}\left\{\psi(\delta)+n^\kappa\delta - n^\kappa u + \frac{u^\kappa}{\log u}\right\}.$$

It therefore follows from Lemma 5 that, for fixed n,

$$\int_{n+\epsilon_n}^{n+\delta} t^\kappa |D_{n+\delta}^{\kappa+1}\{\phi(n+\delta)-\phi(t)\}^\kappa|\,dt,$$

and, *a fortiori*,

$$\int_0^{n+\delta} t^\kappa |D_{n+\delta}^{\kappa+1}\{\phi(n+\delta)-\phi(t)\}^\kappa|\,dt,$$

can be made arbitrarily large by taking ϵ_n sufficiently small. It is thus clear that we can choose the sequence $\{\epsilon_n\}$ in such a way that (4) is not satisfied, and Theorem 2 now follows at once from Theorem 1.

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SUMS OF EIGHT VALUES OF A CUBIC POLYNOMIAL

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It is a well-known elementary result that every positive integer is a sum of nine pyramidal numbers $\frac{1}{6}(x^3-x)$, $x \geqslant 0$. Hua† has shown that

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† *Math. Annalen*, CXI (1935), 622–628.

every *large* positive integer is a sum of eight such numbers, and, more generally, of eight values of the cubic polynomial $Dx+\frac{1}{6}(x^3-x)$, $x \geqslant 0$. The object of this note is to consider the possibility of omitting the word "large" in this result.

All letters denote rational integers; p and q denote primes greater than 3. $\left(\frac{n}{p}\right)$ is the Legendre symbol. $P(x)$ denotes $\frac{1}{6}(x^3-x)$ and $Q(x)$ denotes $\frac{1}{6}(x^3+5x)$. P_r, Q_r denote, for brevity, sums of r values of $P(x)$, $Q(x)$, respectively, with in either case $x \geqslant 0$.

LEMMA 1. *If* $a \not\equiv 0 \pmod{p}$, *the congruence*

$$x^3+ax \equiv n \pmod{p} \tag{1}$$

is soluble for exactly $[\frac{1}{3}(2p+1)]$ *incongruent values of* n.

Proof. Let N_i, $i=1, 2, 3$, be the number of incongruent n for which (1) has exactly i incongruent solutions. The case $i=2$ can arise only when one solution is a double one, satisfying the derived congruence

$$3x^2+a \equiv 0 \pmod{p}. \tag{2}$$

The three solutions of (1) are then $\pm u$, $\pm u$, $\mp 2u$, where $x \equiv u$ satisfies (2), and we must have $n \equiv \mp 2u^3$. Hence $N_2 = 2$ or 0 according as (2) is soluble or not, that is

$$N_2 = 1+\left(\frac{-3a}{p}\right). \tag{3}$$

Clearly
$$N_1+2N_2+3N_3 = p. \tag{4}$$

Define Q by
$$N_1+4N_2+9N_3 = p+Q. \tag{5}$$

Then $p+Q$ is the number of solutions of

$$x^3+ax \equiv y^3+ay \pmod{p}, \tag{6}$$

and as there are obviously p solutions with $x \equiv y$, Q is the number of solutions of (6) with $x \not\equiv y$; or, putting $x, y \equiv u \pm v$, of

$$3u^2+v^2+a \equiv 0, \tag{7}$$

with $v \not\equiv 0$. But (7) has $p \pm 1$ solutions, including N_2 with $v \equiv 0$; hence

$$Q+N_2 = p \pm 1.$$

This gives, by (5),

$$N_1+5N_2+9N_3 = 2p \pm 1,$$

whence, by (4), $N_2+2N_3=\frac{1}{3}(p\pm 1)$, and hence $N_1+N_2+N_3=\frac{1}{3}(2p\pm 1)$, which is the result required.

LEMMA 2. *If* $(6, m)=1$, *the congruence*

$$x^3-x+y^3-y\equiv n \pmod{m} \tag{8}$$

is soluble for any n. *If further* m *is divisible by the square of some prime* q, *there is always a solution satisfying*

$$0\leqslant x<\tfrac{2}{3}m, \quad 0\leqslant y<\tfrac{2}{3}m. \tag{9}$$

Proof. First suppose m is a prime p. Using the preceding lemma, with $a=-1$, and substituting all values of x, we have at least $\frac{1}{3}(2p-1)$ different congruences for y, at most $\frac{1}{3}(p+1)\leqslant\frac{1}{3}(2p-4)$ of which can be insoluble.

Next suppose m is a prime power p^r, $r\geqslant 2$. Proceeding by induction on r, we assume the existence of a solution $(x_0, y_0) \pmod{p^{r-1}}$ and put

$$x=x_0+up^{r-1}, \quad y=y_0+vp^{r-1}.$$

Since $2(r-1)\geqslant r$, (8) becomes

$$x_0^3-x_0+y_0^3-y_0+(3x_0^2-1)\,up^{r-1}+(3y_0^2-1)\,vp^{r-1}\equiv n \pmod{p^r};$$

and this by the inductive hypothesis must reduce to the form (for some integral n')

$$(3x_0^2-1)\,u+(3y_0^2-1)\,v\equiv n' \pmod{p}.$$

This last congruence must be soluble except, possibly, if $3x_0^2\equiv 1$, $3y_0^2\equiv 1 \pmod{p}$. By the remark at the beginning of the preceding lemma, we can avoid this case, throughout the inductive process, by changing x_0 to $-2x_0$ at the first step. The first part of the lemma follows, for all prime powers p^r, and hence for all m.

If $m=q^2M$, we find in the foregoing manner a solution $(x_1, y_1) \pmod{qM}$, which we may suppose to satisfy

$$0\leqslant x_1<qM, \quad 0\leqslant y_1<qM, \quad 3x_1^2\not\equiv 1, \quad 3y_1^2\not\equiv 1 \pmod{q}. \tag{10}$$

We now seek a solution

$$x_1+uqM, \quad y_1+vqM,$$

to modulus q^2M; it evidently depends on a congruence of the form

$$u+\lambda v\equiv N \pmod{q}, \quad \text{where} \quad \lambda\not\equiv 0 \pmod{q}.$$

This can be solved subject to

$$0\leqslant u\leqslant\tfrac{1}{2}(q-1), \quad 0\leqslant v\leqslant\tfrac{1}{2}(q-1). \tag{11}$$

For among the $\frac{1}{2}(q+1)$ incongruent numbers $N-\lambda v$ with v satisfying $(11)_2$, there must be a number u satisfying $(11)_1$, or we should have $q+1$ incongruent numbers (mod q). By (10) and (11), the solution of (8) thus found satisfies (9), since $q \geqslant 5$.

LEMMA 3. *If* $(6, m) = 1$, *m is divisible by the square of some prime, and*

$$0{\cdot}197m^3 < n < \tfrac{1}{4}m^3, \tag{12}$$

then n is P_8.

Proof. We can choose $x, y \geqslant 0$ so that, defining N by

$$n = P(x)+P(y)+N, \tag{13}$$

we have
$$N \equiv 0 \pmod{m}, \quad \tfrac{1}{8}m^3 < N < \tfrac{1}{4}m^3. \tag{14}$$

For we have only to make x, y satisfy (8), with $6n$ for n, subject to (9), which gives

$$0 \leqslant P(x)+P(y) \leqslant \tfrac{1}{6}(x^3+y^3) < 0{\cdot}072m^3.$$

We need now only show that N satisfying (14) must be P_6. We have by (14), with integral s,

$$48N = 6m^3-24m+ms,$$

where, since m is odd and $m^2 \equiv 1 \pmod{8}$, we must have $s \equiv 18 \pmod{48}$, and we may put $s = 6(8k+3)$, which gives

$$48N = 6m^3-24m+6m(8k+3). \tag{15}$$

By (14), $8k+3$ is positive and less than m^2+4, and consequently $\leqslant m^2+2$. By the classical three square theorem, we have

$$8k+3 = u^2+v^2+w^2,$$

with u, v, w odd, positive and $\leqslant m$. Hence, by (15),

$$48N = f(u)+f(v)+f(w), \tag{16}$$

where
$$\begin{aligned} f(u) &= 2m^3-8m+6mu^2 \\ &= (m+u)^3-4(m+u)+(m-u)^3-4(m-u). \end{aligned}$$

The numbers U, $U' = \frac{1}{2}(m \pm u)$ are non-negative integers, and

$$f(u) = 48P(U)+48P(U'),$$

which, with (16), gives the result.

THEOREM 1. *Every positive integer is a sum of eight pyramidal numbers,* *Proof*. The following is a sequence of values of m satisfying the conditions of Lemma 3:

$$725,\ 775,\ 833 = 49\times 17,\ 875,\ 925,\ 961 = 31^2,\ 1025,\ 1075,\ 1127 = 49\times 23;$$

and $$25(47,\ 49,\ 53,\ 55,\ 59,\ \ldots).$$

The ratio of any of them to its successor is at least

$$49/53 = 0{\cdot}924\ldots > (4\times 0{\cdot}197)^{\frac{1}{3}}.$$

Hence they give overlapping intervals (12), and so prove the result for all integers exceeding $0{\cdot}197\times 725^3$, which is less than $7{\cdot}51\times 10^7$.

If $n < 7{\cdot}51\times 10^7$, define $x \geqslant 0$ by

$$P(x) \leqslant n < P(x+1).$$

We have then $$n = P(x)+n',$$

with $$n' < P(x+1)-P(x) = \tfrac{1}{2}(x^2+x),$$

which cannot exceed $2{\cdot}95\times 10^5$. Repetition of the process gives

$$n = P_2+n'',\quad 0 \leqslant n'' \leqslant 7381.$$

Noting that $P(35) = 7140$, $P(34) = 6545$, we have

$$n = n'''+P_3,\quad 0 \leqslant n''' \leqslant 594.$$

To complete the proof we need only show that all integers up to 594 are P_5. The first fourteen positive values of $P(x)$ are

$$1,\ 4,\ 10,\ 20,\ 35,\ 56,\ 84,\ 120,\ 165,\ 220,\ 286,\ 364,\ 455,\ 560.$$

It may easily be verified that all integers $\equiv 0 \pmod 5$ up to 260 are P_3, except

$$100,\ 110,\ 115,\ 145,\ 180,\ 215,\ 245.$$

The following are some numbers, classified according to their residues $(\bmod 5)$, that are P_2:

70, 140, 240, 340, 440, 490, 560;

1, 56, 121, 221, 321, 406, 506;

4, 84, 124, 204, 304, 364, 459, 539;

2, 57, 112, 287, 342, 572;

8, 88, 168, 368, 448.

Hence all integers up to 594 are P_5, except possibly:

(i) 212, 222, 227, 257;

(ii) 442, 452, 457, 487, 522, 557;

(iii) 268, 278, 283, 313, 348;

(iv) 548, 558, 563, 593.

On subtracting 57, which is P_2, from any of the numbers (i), the remainder is P_3. Similarly by subtracting 287 from the numbers (ii), we dispose of all of them except $557 = 455+56+35+10+1$. By subtracting 88 from the numbers (iii), we deal with all but $268 = 220+2(20+4)$. The numbers (iv) can be dealt with by subtracting 368, except for

$$548 = 455+84+4+4+1.$$

Thus the proof is complete.

By a straightforward modification of the foregoing argument we might obtain Hua's result for any value of D, with an easily calculated bound for the unrepresentable integers, if any. If, however, we exclude, as is natural, negative values of the summand $P(x)+Dx$ as well as of x, then a universal result is possible only if the summand can take the value 1. This is the case only if $D = 0$ or ± 1. Thus we need only consider the polynomials $\frac{1}{6}(x^3+5x)$, for $x \geqslant 0$, and $\frac{1}{6}(x^3-7x)$, for $x \geqslant 3$. I proceed to show that the former polynomial has the same property as has been proved for $P(x)$; a similar result holds in the other case, but the calculations are much heavier.

LEMMA 4. *If* $(6, m) = 1$, *the congruence*

$$x^3+5x+y^3+5y \equiv n \pmod{m} \tag{17}$$

is soluble for any n. *If further* $m \equiv 0 \pmod 5$, *there is a solution satisfying* (9).

Proof. The first part is proved as in Lemma 2; if m has a factor 5, we cannot use Lemma 1, but every integer is a cubic residue (mod 5) and it is easy to solve (17) (mod 5) with neither x nor y divisible by 5, and then proceed by induction, in case 5^r, $r \geqslant 2$, divides m.

The second part is then proved as in Lemma 2 if $m \equiv 0 \pmod{25}$. If not, put $m = 5M$, and if x_0, y_0 is a solution $(\operatorname{mod} M)$ we need only satisfy

$$(x_0+uM)^3+(y_0+vM)^3 \equiv n \pmod 5,$$

with each of u, $v = 0$, 1, or 2. This is easily seen to be possible, since each term on the left has three possible residues (mod 5).

LEMMA 5. *If m is divisible by five and prime to six, and if*

$$0{\cdot}197m^3+1{\cdot}875m < n < \tfrac{1}{4}m^3+\tfrac{15}{4}m, \tag{18}$$

then n is Q_8.

Proof. We choose x, y as in the previous lemma, but with $6n$ for n. We then have

$$0 \leqslant Q(x)+Q(y) \leqslant \tfrac{1}{6}(x+1)^3+\tfrac{1}{6}(y+1)^3 \leqslant 0{\cdot}072m^3.$$

Hence defining N by

$$n = Q(x)+Q(y)+N, \tag{19}$$

we have

$$N \equiv 0 \pmod{m}, \tag{20}$$

$$\tfrac{1}{8}m^3+1{\cdot}875m < N < \tfrac{1}{4}m^3+\tfrac{15}{4}m. \tag{21}$$

We must deduce from (20) and (21) that N is Q_6. As in Lemma 3, we may write

$$48N = 6m^3+120m+6m(8k+3), \tag{22}$$

where

$$-5 < 8k+3 < m^2+10. \tag{23}$$

It follows from (23) that $8k+3$ is positive and $\leqslant m^2+2$; so we have

$$8k+3 = u^2+v^2+w^2,$$

with u, v, w odd, positive and $\leqslant m$. It now suffices to show that

$$2m^3+40m+6mu^2 = 48Q(U)+48Q(U'),$$

with U, $U' = \tfrac{1}{2}(m \pm u)$. This is easily verified.

THEOREM 2. *Every positive integer is a sum of eight values of $\tfrac{1}{6}(x^3+5x)$, with integral $x \geqslant 0$.*

Proof. As in the previous theorem, we obtain overlapping intervals (18) if we take values of m whose ratio is between unity and 49/53. But in this case admissible values of m satisfying this condition are:

$$5(47,\ 49,\ 53,\ 55,\ 59,\ \ldots).$$

Hence if n is not Q_8,

$$n < 0{\cdot}197 \times 235^3+470 < 2{,}567{,}000.$$

Proceeding as in Theorem 1, simple calculations show that it suffices to prove that all numbers up to 210 are Q_5.

The first ten positive values of Q are:

$$1,\quad 3,\quad 7,\quad 14,\quad 25,\quad 41,\quad 63,\quad 92,\quad 129,\quad 175.$$

It may be verified that all positive integers up to 11 are Q_3, and all up to 45, except 37, are Q_4. This disposes of all up to 210, except

$$166 = 92+63+7+3+1.$$

I am greatly indebted to Professor Davenport for reading an earlier draft of this paper, and suggesting a shorter proof of Lemma 1.

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THE COVERING OF n-DIMENSIONAL SPACE BY SPHERES

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1. Suppose that the whole of n-dimensional space is covered by a system of equal spheres, so that every point of space is in (or on the boundary of) one at least of the spheres. Suppose also that the centres of the spheres form a lattice of determinant Δ. Then the number of centres in a large cube of volume V is asymptotically V/Δ, and the total volume of the spheres with these centres is asymptotically VJ/Δ, where J is the volume of one of the spheres. The ratio of this total volume to the volume of the cube has the limit J/Δ as the cube expands to infinity, and it is therefore reasonable to call J/Δ the *density* of the covering. It is obvious that this density, say ϑ, satisfies $\vartheta > 1$. Our object in the present note is to prove that *the density ϑ of any covering of n-dimensional space by equal spheres (the centres forming a lattice) satisfies*†

$$\vartheta > \tfrac{4}{3} - \epsilon_n, \tag{1}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

As regards results in the opposite direction, one of us has recently proved‡ that there exists a lattice-covering for which

$$\vartheta < (1.15)^n,$$

provided that n is sufficiently large. While the method of the present paper might perhaps be refined to give a constant greater than $\frac{4}{3}$ in (1), there seems

* Received 11 July, 1951; read 15 November, 1951.

† The only result in this direction so far given seems to be one due to Hlawka, *Monatshefte für Math.*, 53 (1949), 81–131 [§15].

‡ H. Davenport, *Rend. di Palermo* (in course of publication).