# 第 5 章 多元函数微积分

# Outline

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# 链式法则 1

在一元函数微分学中，有复合函数求导的链式法则.
针对多元复合函数比较常见的情形，我们给出求导（偏导）法则，
其他情形可以用同样的方法得到类似的公式.

## Theorem (链式法则 1)

设函数 $u=\varphi(x), v=\psi(x)$ 都在点 $x$ 处可导，函数 $z=f(u,v)$ 在对应的点 $(u,v)$ 处可微，则复合函数 $z=f(\varphi(x),\psi(x))$ 在点 $x$ 处可导，且有公式

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}.$$

我们将上述定理中出现的导数 $\frac{\mathrm{d}z}{\mathrm{d}x}$ 称“全导数”.

$$\begin{tikzcd}
 & u \arrow[r] & x \\
z \arrow[ur] \arrow[dr] & & \\
 & v \arrow[r] & x
\end{tikzcd}$$

Proof.

因为函数 $z = f(u, v)$ 可微，所以

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + o(\rho),$$

这里 $\rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}$.

设自变量 $x$ 有增量 $\Delta x$，由此引起中间变量 $u, v$ 有增量

$$\Delta u = \varphi(x + \Delta x) - \varphi(x), \qquad \Delta v = \psi(x + \Delta x) - \psi(x),$$

则

$$\begin{aligned}\frac{\Delta z}{\Delta x} &= \frac{\partial z}{\partial u}\frac{\Delta u}{\Delta x} + \frac{\partial z}{\partial v}\frac{\Delta v}{\Delta x} + \frac{o(\rho)}{\Delta x}\\ &= \frac{\partial z}{\partial u}\cdot\frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x} + \frac{\partial z}{\partial v}\cdot\frac{\psi(x + \Delta x) - \psi(x)}{\Delta x} + \frac{o(\rho)}{\Delta x},\end{aligned}$$

当 $\Delta x \to 0$ 时，因为函数 $u = \varphi(x), v = \psi(x)$ 可导，所以

$$\lim_{\Delta x \to 0} \frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x} = \frac{\mathrm{d}u}{\mathrm{d}x},$$

$$\lim_{\Delta x \to 0} \frac{\psi(x + \Delta x) - \psi(x)}{\Delta x} = \frac{\mathrm{d}v}{\mathrm{d}x},$$

而 $\lim_{\Delta x \to 0} \Delta u = 0, \lim_{\Delta x \to 0} \Delta v = 0$ ，故 $\lim_{\Delta x \to 0} \rho = 0,$

从而

$$\lim_{\Delta x \to 0} \frac{o(\rho)}{\Delta x} = \lim_{\Delta x \to 0} \frac{o(\rho)}{\rho} \sqrt{\left(\frac{\Delta u}{\Delta x}\right)^2 + \left(\frac{\Delta v}{\Delta x}\right)^2} = 0,$$

所以

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x}.$$

$\square$

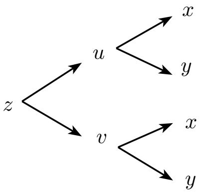
# 链式法则 2

如果中间变量 $u, v$ 分别是自变量 $x, y$ 的二元函数，我们有如下的结论:

## Theorem (链式法则 2)

设函数 $u = \varphi(x, y), v = \psi(x, y)$ 在点 $(x, y)$ 处可偏导，函数 $z = f(u, v)$ 在对应的点 $(u, v)$ 处可微，则复合函数 $z = f(\varphi(x, y), \psi(x, y))$ 在点 $(x, y)$ 处可偏导，且有公式

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$



# 链式法则 3

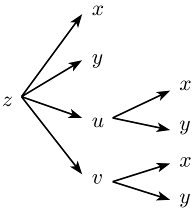
## Theorem (链式法则 3)

设函数 $u=\varphi(x,y), v=\psi(x,y)$ 都在点 $(x,y)$ 处可偏导，函数 $z=f(x,y,u,v)$ 在对应点 $(x,y,u,v)$ 处可微，则复合函数

$$z(x,y)=f(x,y,\varphi(x,y),\psi(x,y))$$

在点 $(x,y)$ 处可偏导，且有公式

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \\
\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.
\end{aligned}$$



## Remark

- 上述公式中等式右端的 $\dfrac{\partial f}{\partial x}$ 与 $\dfrac{\partial f}{\partial y}$ 不能写成 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$ ，
前者表示 $f$ 作为 $x, y, u, v$ 的四元函数对 $x$ 或 $y$ 求偏导数，
后者表示 $z$ 作为 $x, y$ 的二元 (复合) 函数对 $x$ 或 $y$ 求偏导数.
- 链式法则可以推广到具有多个自变量和多个中间变量的情形.
在运用链式法则求多元复合函数偏导数 (或全导数) 时，应注意区分中间变量与自变量，并注意导数符号与偏导数符号的正确使用.

## Example

已知 $y = \mathrm{e}^{uv} + u^2, u = \sin x, v = \cos x$，试求全导数 $\frac{\mathrm{d}y}{\mathrm{d}x}$ .

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Proof.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial y}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial y}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} \\
&= (v\mathrm{e}^{uv} + 2u)\cos x + u\mathrm{e}^{uv}(-\sin x) \\
&= (\mathrm{e}^{\sin x \cos x}\cos x + 2\sin x)\cos x - \mathrm{e}^{\sin x \cos x}\sin^2 x \\
&= \mathrm{e}^{\sin x \cos x}(\cos 2x + \sin 2x).
\end{aligned}$$

□

## Example

设 $z = u^v, u = x^2 + y^2, v = xy$，试求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

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设 $z=u^v, u=x^2+y^2, v=xy$ ，试求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

## Proof.

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2xvu^{v-1} + yu^v \ln u \\
&= 2x^2y(x^2+y^2)^{xy-1} + y(x^2+y^2)^{xy}\ln(x^2+y^2) \\
&= (x^2+y^2)^{xy}\left(\frac{2x^2y}{x^2+y^2} + y\ln(x^2+y^2)\right). \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\
&= 2yvu^{v-1} + xu^v \ln u \\
&= (x^2+y^2)^{xy}\left(\frac{2xy^2}{x^2+y^2} + x\ln(x^2+y^2)\right).
\end{aligned}
$$

□

## Example

设函数 $w = f(x, u, v)$ 可微， $u = x^2 + xy + yz, v = xyz$ ，
求 $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ .

## Example

设函数 $w = f(x,u,v)$ 可微， $u = x^2 + xy + yz, v = xyz$ ，
求 $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ .

## Proof.

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f'_x + (2x+y)f'_u + yzf'_v, \\
\frac{\partial w}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = (x+z)f'_u + xzf'_v, \\
\frac{\partial w}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} = yf'_u + xyf'_v.
\end{aligned}
$$

□

# 一阶全微分的形式不变性

**Theorem (一阶全微分的形式不变性)**

设函数 $u=\varphi(x,y), v=\psi(x,y)$ 在点 $(x,y)$ 处可微，函数 $z=f(u,v)$ 在对应的点 $(u,v)$ 处可微，则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 在点 $(x,y)$ 处的全微分仍可表示为

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

# 一阶全微分的形式不变性

**Theorem (一阶全微分的形式不变性)**

设函数 $u=\varphi(x,y), v=\psi(x,y)$ 在点 $(x,y)$ 处可微，函数 $z=f(u,v)$ 在对应的点 $(u,v)$ 处可微，则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 在点 $(x,y)$ 处的全微分仍可表示为

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

**Proof.**

由链式法则可知

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y},$$

则

$$
\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y \\
&= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\
&= \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v.
\end{aligned}
$$

□

则

$$\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y \\
&= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\
&= \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v.
\end{aligned}$$

□

由此可见，无论 $u, v$ 是自变量还是中间变量，函数 $z = f(u, v)$ 的一阶全微分形式是一样的，这个性质叫做**一阶全微分的形式不变性**.

### Example

已知 $z = \mathrm{e}^u + \sin v,\ u = x + y,\ v = xy$，
利用一阶全微分形式不变性求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

## Example

已知 $z = \mathrm{e}^u + \sin v$, $u = x + y$, $v = xy$，利用一阶全微分形式不变性求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

## Proof.

$$\begin{aligned}
\mathrm{d}z &= \mathrm{d}(\mathrm{e}^u + \sin v) = \mathrm{e}^u \mathrm{d}u + \cos v \mathrm{d}v, \\
\mathrm{d}u &= \mathrm{d}(x+y) = \mathrm{d}x + \mathrm{d}y, \quad \mathrm{d}v = \mathrm{d}(xy) = y\mathrm{d}x + x\mathrm{d}y, \\
\text{所以 } \mathrm{d}z &= \mathrm{e}^{x+y}(\mathrm{d}x + \mathrm{d}y) + \cos xy(y\mathrm{d}x + x\mathrm{d}y) \\
&= \left(\mathrm{e}^{x+y} + y\cos xy\right)\mathrm{d}x + \left(\mathrm{e}^{x+y} + x\cos xy\right)\mathrm{d}y, \\
\text{从而 } \frac{\partial z}{\partial x} &= \mathrm{e}^{x+y} + y\cos xy, \quad \frac{\partial z}{\partial y} = \mathrm{e}^{x+y} + x\cos xy.
\end{aligned}$$

□

Example

设 $z = f\left(x + y, xy, \frac{x}{y}\right)$ , $f$ 的所有二阶偏导数连续，试求 $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ .

# 复合函数的高阶偏导数

Example

设 $z=f\left(x+y,xy,\frac{x}{y}\right)$，$f$ 的所有二阶偏导数连续，试求 $\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial x\partial y}$ .

Proof.

$$\frac{\partial z}{\partial x}=f_1'+yf_2'+\frac{1}{y}f_3'$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\frac{\partial f_1'}{\partial x}+y\frac{\partial f_2'}{\partial x}+\frac{1}{y}\frac{\partial f_3'}{\partial x}.\\&=\left(f_{11}''+yf_{12}''+\frac{1}{y}f_{13}''\right)+y\left(f_{21}''+yf_{22}''+\frac{1}{y}f_{23}''\right)+\frac{1}{y}\left(f_{31}''+yf_{32}''+\frac{1}{y}f_{33}''\right)\\&=f_{11}''+y^2f_{22}''+\frac{1}{y^2}f_{33}''+2yf_{12}''+\frac{2}{y}f_{13}''+2f_{23}''.\end{aligned}$$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= f_1' + y f_2' + \frac{1}{y} f_3' \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial f_1'}{\partial y} + f_2' + y \frac{\partial f_2'}{\partial y} - \frac{1}{y^2} f_3' + \frac{1}{y} \frac{\partial f_3'}{\partial y}. \\
&= \left( f_{11}'' + x f_{12}'' - \frac{x}{y^2} f_{13}'' \right) + f_2' + y \left( f_{21}'' + x f_{22}'' - \frac{x}{y^2} f_{23}'' \right) - \frac{1}{y^2} f_3' \\
&\quad + \frac{1}{y} \left( f_{31}'' + x f_{32}'' - \frac{x}{y^2} f_{33}'' \right) \\
&= f_2' - \frac{1}{y^2} f_3' + f_{11}'' + xy f_{22}'' - \frac{x}{y^3} f_{33}'' + (x+y) f_{12}'' + \left( \frac{1}{y} - \frac{x}{y^2} \right) f_{13}''.
\end{aligned}
$$

$\square$

## Example

设 $z(x,y)=\int_{xy}^{x^2+y^2} f(t)\mathrm{d}t+\varphi(x-y)$，其中 $f(u)$ 一阶连续可微，$\varphi(v)$ 二阶连续可微，求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ .

## Example

设 $z(x,y) = \int_{xy}^{x^2+y^2} f(t)\mathrm{d}t + \varphi(x-y)$，其中 $f(u)$ 一阶连续可微，$\varphi(v)$ 二阶连续可微，求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ .

## Proof.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2xf(x^2+y^2) - yf(xy) + \varphi'(x-y), \\
\frac{\partial^2 z}{\partial x^2} &= 2f(x^2+y^2) + 4x^2 f'(x^2+y^2) - y^2 f'(xy) + \varphi''(x-y), \\
\frac{\partial^2 z}{\partial x \partial y} &= 4xyf'(x^2+y^2) - f(xy) - xyf'(xy) - \varphi''(x-y).
\end{aligned}$$

□

### Example

已知 $z = f(x, xy)$ 求 $\frac{\partial^2 z}{\partial x \partial y}$ .

### Example

设 $z = f(x+y, xy) + \int_{x+y}^{xy} \phi(t)dt$，其中 $f, \phi$ 二阶偏导数连续，

计算 $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$.

## Example

设函数 $u = f(x, y)$ 具有二阶连续偏导数，且满足等式

$$4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0.$$

求 $a, b$ 的值，使得等式在变换 $\xi = x + ay$, $\eta = x + by$ 下化为 $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$.

提示: 把 $u = u(x, y)$ 看成 $u = u(\xi, \eta), \xi = \xi(x, y), \eta = \eta(x, y)$ 复合而成的函数. $(a = -2, b = -\frac{2}{5}$ 或 $a = -\frac{2}{5}, b = -2)$

# 隐函数的偏导数

在本节中，我们将隐函数的概念推广到多元函数的情形，并给出隐函数存在的条件和隐函数求导法则.

## Definition (隐函数)

设给定方程

$$F(x_1, x_2, \cdots, x_n, y) = 0,$$

其中 $(x_1, x_2, \cdots, x_n) \in D \subseteq \mathbb{R}^n$, $y \in I \subseteq \mathbb{R}$.

如果对 $D$ 中每一点 $X(x_1, x_2, \cdots, x_n)$ 都有唯一确定的 $y$ 值与之对应，
使得方程 $F(x_1, x_2, \cdots, x_n, y) = 0$ 恒成立，
即存在一个函数 $y = f(x_1, x_2, \cdots, x_n)$，使得

$$F(x_1, x_2, \cdots, x_n, f(x_1, x_2, \cdots, x_n)) \equiv 0, \quad \forall (x_1, x_2, \cdots, x_n) \in D,$$

则称方程确定了一个定义在 $D$ 上的隐函数 $y = f(x_1, x_2, \cdots, x_n)$.

## Definition (隐函数组)

对于方程组也有类似的定义. 设给定方程组

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, \\ F_2(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, \\ \qquad \cdots \quad \cdots \\ F_m(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, \end{cases}$$

其中 $(x_1, x_2, \cdots, x_n) \in D \subseteq \mathbb{R}^n$, $(y_1, y_2, \cdots, y_m) \in G \subseteq \mathbb{R}^m$ . 如果对 $D$ 中每一点 $X(x_1, x_2, \cdots, x_n)$ 都有唯一确定的 $Y(y_1, y_2, \cdots, y_m)$ :

$$\begin{cases} y_1 = f_1(x_1, x_2, \cdots, x_n) \\ y_2 = f_2(x_1, x_2, \cdots, x_n), \\ \qquad \cdots \quad \cdots \\ y_m = f_m(x_1, x_2, \cdots, x_n), \end{cases}$$

与之对应，使得 $(X,Y)$ 恒满足方程组，即

$$\begin{cases} F_1(x_1,\cdots,x_n,f_1(x_1,x_2,\cdots,x_n),\cdots,f_m(x_1,x_2,\cdots,x_n))\equiv 0, \\ F_2(x_1,\cdots,x_n,f_1(x_1,x_2,\cdots,x_n),\cdots,f_m(x_1,x_2,\cdots,x_n))\equiv 0, \\ \qquad \cdots\ \ \cdots \\ F_m(x_1,\cdots,x_n,f_1(x_1,x_2,\cdots,x_n),\cdots,f_m(x_1,x_2,\cdots,x_n))\equiv 0, \end{cases}$$

则称方程组确定了一个定义在 $D$ 上的隐函数组.

# 由一个方程确定的隐函数

**Theorem (隐函数存在定理 1)**

设 $P_0(x_0, y_0) \in \mathbb{R}^2, G = N_\delta(P_0)$，假设

1) 函数 $F$ 在 $G$ 上连续可微;

2) $F(P_0) = F(x_0, y_0) = 0$;

3) $F'_y(P_0) \neq 0$,

则存在 $x_0$ 的邻域 $I = N_{\delta_1}(x_0)$ 和唯一的函数 $y = f(x)$ 使得:

(1) 对任意 $x \in I$, $F(x, f(x)) = 0$;

(2) $f(x_0) = y_0$;

(3) $f$ 连续可微，且当 $x \in I$ 时有公式 $f'(x) = -\frac{F'_x(x,y)}{F'_y(x,y)}$,
其中 $y = f(x)$.

我们略去此定理的证明，此处进行粗略地解释:

- 函数 $z = F(x, y)$ 为三维空间中的一张曲面，$F(x_0, y_0) = 0$ 表明该曲面与坐标面 $z = 0$ 有一个交点 $(x_0, y_0, 0)$.
- $F'_y(x_0, y_0) \neq 0$ 以及偏导数的连续性保证了在 $(x_0, y_0)$ 附近 $F'_y(x, y)$ 保持固定的符号，即 $F(x, y)$ 作为 $y$ 的函数是严格单调的，曲面 $z = F(x, y)$ 与坐标面 $z = 0$ 之交必定是一条过点 $(x_0, y_0)$ 的曲线. 这条曲线就是我们要求的隐函数 $y = f(x)$ 的图形.

定理可以推广到 $F$ 是三元函数或一般 $n$ 元函数的情况. 对于 $F$ 是三元函数的情形，我们有下面的定理:

## Theorem (隐函数存在定理 2)

设 $P_0(x_0, y_0, z_0) \in \mathbb{R}^3, G = N_\delta(P_0)$，假设

1) 函数 $F$ 在 $G$ 上连续可微;

2) $F(P_0) = F(x_0, y_0, z_0) = 0$;

3) $F'_z(P_0) \neq 0$,

则存在 $U = \{(x, y) \mid |x - x_0| < h, |y - y_0| < k\}$ 和唯一的函数 $z = f(x, y)$ 使得:

(1) 对任意 $(x, y) \in U$, $F(x, y, f(x, y)) = 0$;

(2) $f(x_0, y_0) = z_0$;

(3) $f$ 连续可微，且当 $(x, y) \in U$ 时有公式

$$\frac{\partial f}{\partial x} = -\frac{F'_x(x, y, z)}{F'_z(x, y, z)}, \qquad \frac{\partial f}{\partial y} = -\frac{F'_y(x, y, z)}{F'_z(x, y, z)},$$

其中 $z = f(x, y)$.

### Example

验证方程 $1+y+\sin(x^2+y^2)=\mathrm{e}^{xy}$ 在点 $(0,0)$ 的某邻域内满足隐函数存在定理 1 的条件，从而在 $x=0$ 的某邻域内能唯一确定一个隐函数 $y=y(x)$ ，并求 $\frac{\mathrm{d}y}{\mathrm{d}x}$ .

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### Proof.

令 $F(x,y)=1+y+\sin(x^2+y^2)-\mathrm{e}^{xy}$ ，则

1) 函数 $F(x,y)$ 连续可微;

2) $F(0,0)=0$;

3) $F'_y(0,0)=\left.(1+2y\cos(x^2+y^2)-x\mathrm{e}^{xy})\right|_{(0,0)}=1\neq 0$,

由隐函数存在定理 1 可知方程 $F(x,y)=0$ 在点 $x=0$ 的某邻域内能唯一确定一个隐函数 $y=y(x)$ ，且

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F'_x(x,y)}{F'_y(x,y)}=-\frac{2x\cos(x^2+y^2)-y\mathrm{e}^{xy}}{1+2y\cos(x^2+y^2)-x\mathrm{e}^{xy}}.$$

□

## Example

设 $u + \mathrm{e}^u = xy$ 确定 $u = u(x, y)$ . 试求 $\frac{\partial^2 u}{\partial x \partial y}$ .

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## Proof.

令 $F(x,y,u)=u+\mathrm{e}^u-xy$ ，则

$$\frac{\partial u}{\partial x} = -\frac{F'_x}{F'_u} = -\frac{-y}{1+\mathrm{e}^u} = \frac{y}{1+\mathrm{e}^u},$$

$$\frac{\partial u}{\partial y} = -\frac{F'_y}{F'_u} = -\frac{-x}{1+\mathrm{e}^u} = \frac{x}{1+\mathrm{e}^u},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{1+\mathrm{e}^u}\right) = \frac{1+\mathrm{e}^u - y\frac{\partial(1+\mathrm{e}^u)}{\partial y}}{(1+\mathrm{e}^u)^2} = \frac{1+\mathrm{e}^u - y\mathrm{e}^u\frac{\partial u}{\partial y}}{(1+\mathrm{e}^u)^2}$$

$$= \frac{1+\mathrm{e}^u - y\mathrm{e}^u\frac{x}{1+\mathrm{e}^u}}{(1+\mathrm{e}^u)^2} = \frac{(1+\mathrm{e}^u)^2 - xy\mathrm{e}^u}{(1+\mathrm{e}^u)^3}.$$

□

## Example

设 $z = z(x, y)$ 是由方程 $f(x + y, x + z) = 0$ 确定的隐函数，
其中 $f(u, v)$ 满足隐函数存在定理的条件，
且 $f(u, v)$ 的所有二阶偏导数连续，
试求 $\frac{\partial^2 z}{\partial x^2}$.

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试求 $\frac{\partial^2 z}{\partial x^2}$.

## Proof.

令 $u = x + y$, $v = x + z$，则

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{f'_u + f'_v}{f'_v},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{f'_v \frac{\partial (f'_u + f'_v)}{\partial x} - (f'_u + f'_v) \frac{\partial (f'_v)}{\partial x}}{-(f'_v)^2}$$

$$= \frac{f'_v \left(f''_{uu} + f''_{uv}(1 + \frac{\partial z}{\partial x}) + f''_{vu} + f''_{vv}(1 + \frac{\partial z}{\partial x})\right) - (f'_u + f'_v) \left(f''_{vu} + f''_{vv}(1 + \frac{\partial z}{\partial x})\right)}{-(f'_v)^2}$$

$$= \frac{(f'_v)^2 f''_{uu} - 2 f'_u f'_v f''_{uv} + (f'_u)^2 f''_{vv}}{-(f'_v)^3}.$$

□

**Remark**

设 $F(x,y,z)=0$ 满足隐函数存在定理的条件，且 $F'_x$, $F'_y$, $F'_z$ 皆不等于 *0*，则

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=\Big(-\frac{F'_x}{F'_z}\Big)\Big(-\frac{F'_y}{F'_x}\Big)\Big(-\frac{F'_z}{F'_y}\Big)=-1.$$

这个例子说明偏导数的记法 $\dfrac{\partial z}{\partial x}$ 是一个整体，

不能像导数的记号 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ 一样看成是两部分的商.

# 由方程组确定的隐函数

**Theorem (隐函数存在定理 3)**

设 $P_0 = (x_0, y_0, u_0, v_0) \in \mathbb{R}^4$, $G = N_\delta(P_0)$，假设

1) 函数 $F, H$ 在 $G$ 上连续可微;

2) $F(P_0) = F(x_0, y_0, u_0, v_0) = 0$, $H(P_0) = H(x_0, y_0, u_0, v_0) = 0$;

3) 雅可比 (Jacobi) 行列式

$$\left.\frac{D(F,H)}{D(u,v)}\right|_{P_0} \triangleq \left.\begin{vmatrix} F'_u & F'_v \\ H'_u & H'_v \end{vmatrix}\right|_{P_0} \neq 0,$$

则存在 $U = \{(x,y) \mid |x - x_0| < h, |y - y_0| < k\}$ 和唯一一组函数 $u = u(x,y), v = v(x,y)$，使得:

(1) 对任意 $(x,y)\in U$,
$F(x,y,u(x,y),v(x,y))=0,\ H(x,y,u(x,y),v(x,y))=0$;

(2) $u(x_0,y_0)=u_0,\ v(x_0,y_0)=v_0$ ;

(3) 函数$u(x,y),v(x,y)$连续可微;

(4) 当 $(x,y)\in U$ 时有公式

$$\frac{\partial u}{\partial x}=-\frac{\frac{D(F,H)}{D(x,v)}}{\frac{D(F,H)}{D(u,v)}},\qquad \frac{\partial u}{\partial y}=-\frac{\frac{D(F,H)}{D(y,v)}}{\frac{D(F,H)}{D(u,v)}},$$

$$\frac{\partial v}{\partial x}=-\frac{\frac{D(F,H)}{D(u,x)}}{\frac{D(F,H)}{D(u,v)}},\qquad \frac{\partial v}{\partial y}=-\frac{\frac{D(F,H)}{D(u,y)}}{\frac{D(F,H)}{D(u,v)}}.$$

### Example

验证方程组

$$\begin{cases} x^2 + y^2 + uv = 1, \\ xy + u^2 + v^2 = 1 \end{cases}$$

在点 $(x_0, y_0, u_0, v_0) = (1, 0, 1, 0)$ 的某邻域内满足隐函数存在定理 3 的条件，从而在点 $(x_0, y_0) = (1, 0)$ 的某邻域内存在唯一一组连续可微的函数组 $u = u(x, y), v = v(x, y)$，并求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

**Proof.**

设 $F(x,y,u,v)=x^2+y^2+uv-1,\ H(x,y,u,v)=xy+u^2+v^2-1$,

1) 函数 $F,H$ 连续可微;

2) $F(1,0,1,0)=0,\ H(1,0,1,0)=0$;

3) 雅可比行列式

$$\left.\frac{D(F,H)}{D(u,v)}\right|_{(1,0,1,0)}=\left.\begin{vmatrix} F'_u & F'_v \\ H'_u & H'_v \end{vmatrix}\right|_{(1,0,1,0)}=\left.\begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix}\right|_{(1,0,1,0)}=-2\neq 0,$$

则由隐函数存在定理 3 可知，在点 $(x_0,y_0)=(1,0)$ 的某邻域内存在唯一一组连续可微的函数组 $u=u(x,y),v=v(x,y)$，且

$$\frac{\partial u}{\partial x} = -\frac{\frac{D(F,H)}{D(x,v)}}{\frac{D(F,H)}{D(u,v)}} = -\frac{\begin{vmatrix} 2x & u \\ y & 2v \end{vmatrix}}{\begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix}} = -\frac{4xv - yu}{2v^2 - 2u^2},$$

$$\frac{\partial u}{\partial y} = -\frac{\frac{D(F,H)}{D(y,v)}}{\frac{D(F,H)}{D(u,v)}} = -\frac{\begin{vmatrix} 2y & u \\ x & 2v \end{vmatrix}}{\begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix}} = -\frac{4yv - xu}{2v^2 - 2u^2},$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{D(F,H)}{D(u,x)}}{\frac{D(F,H)}{D(u,v)}} = -\frac{\begin{vmatrix} v & 2x \\ 2u & y \end{vmatrix}}{\begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix}} = -\frac{yv - 4xu}{2v^2 - 2u^2},$$

$$\frac{\partial v}{\partial y} = -\frac{\frac{D(F,H)}{D(u,y)}}{\frac{D(F,H)}{D(u,v)}} = -\frac{\begin{vmatrix} v & 2y \\ 2u & x \end{vmatrix}}{\begin{vmatrix} v & u \\ 2u & 2v \end{vmatrix}} = -\frac{xv - 4yu}{2v^2 - 2u^2}.$$

$\square$

# 几个例子

Example

设函数 $z=f(x,y)$ 由方程 $x^2(y+z)-4\sqrt{x^2+y^2+z^2}=0$ 所确定，
求 $z$ 在点 $P(-2,2,1)$ 处的全微分 $dz$.

Example

设 $z=z(x,y)$ 由方程 $x^2+y^2+z^2=yf(\frac{z}{y})$ 所确定，
求 $(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}$.　(所求结果与 $f(u)$ 无关).

Example

设 $\frac{x}{z}=e^{y+z}$，计算 $\frac{x}{z}\frac{\partial z}{\partial x}-e^{y+z}\frac{\partial z}{\partial y}$.

Example

已知 $z=f(x+y,x-y,xy)$，其中 $f$ 具有二阶连续偏导数，

(1) 求 $dz$; (2) 求 $\frac{\partial^2 z}{\partial x \partial y}$.

Example

已知 $z=f(x-y,\frac{x}{y})$，其中 $f$ 具有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x^2}$.

Example

已知 $u=f(r)+xyz$，其中 $r=\sqrt{x^2+y^2+z^2}$, $f$ 具有二阶偏导数，求 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$