BABES BOLYAI UNIVERSITY
Faculty of Mathematics and Computer Science
Department of Mathematics

DOCTORAL THESIS

Bi-criteria optimization with applications in economy (Optimizare bi-criteriala cu aplicatii in economie)

Author:
Traian Ionut LUCA

Supervisor:
prof. univ. dr. Dorel I. DUCA

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Declaration of Authorship

I, Traian Ionut LUCA, declare that this thesis titled, “Bi-criteria optimization with applications in economy (Optimizare bi-criteriala cu aplicatii in economie)” and the work presented in it are my own. I confirm that:

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- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Date:

"A nation which depends upon others for its basic scientific knowledge will be slow in its industrial progress and weak in its competition in world trade, regardless of its mechanical skills"

Vannevar Bush - *"The Endless Frontier"* Report

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Abstract

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Bi-criteria optimization with applications in economy
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by Traian Ionut LUCA

Global context is generating challenges to optimize production of energy. Production of energy is a complex process with long, medium and short term objectives, regulated and deregulated markets. Analyzing a daily production and consumption chart, a fluctuation is visible. We compare this fluctuation with a spread, the peak load being the most extreme point.

Our objective, for the research study which has generated this thesis, is: *To create, solve and validate a mathematical model which will shave the peak load by minimizing fluctuation of energy and maximizing the economic performance.*

Shaving the peak load will generate a supra-production of energy which, according to the strategy of power plant, might be addressed for example by energy storage, demand side management, electric vehicles strategy or by human interference to adjust the production plan. Shifting the production will generate changes in the production diagram, which might reduce the economic performance of the power plant. To mitigate this risk, a bi-criteria optimization problem might be used, the first component of the objective function being focused on peak-load, while the second component being focused on economic performance.

First part of our thesis (Chapters 2 and 3) is dedicated to analysis and development of mathematical tools necessary during our research. Measures for fluctuation and methods to solve bi-criteria optimization problems are evaluated and developed. To solve bi-criteria optimization problems we have used equivalent parametric problems with Kuhn-Tucker conditions and approximate problems. Chapter 3 is dedicated to approximate problems and their connection to the initial problem. Invexity, incavity and avexity are used to prove conditions such that an efficient solution of an initial bi-criteria optimization problem remain efficient also for the approximate problem and reciprocally.

Key to shave the peak load, by minimizing fluctuation, is to find a proper measure of spread able to target directly the most extreme point. Few measures of spread have been evaluated, conclusion being that maximum absolute deviation satisfies our request.

Turnover is employed as measure for economic performance and simple technical constraints, which limit the amount of energy to be produced, are used.

Minimax measure for fluctuation of energy was defined starting from maximum absolute deviation. Energy price is included in the measure. Minimax measure for fluctuation is generating minimax model for energy. Using some equivalent bi-criteria and parametric problems, efficient solution is computed and verified using real data. Efficient production plan proved to have good and excellent behavior compared to real production data.

Input data required for minimax measure are increasing the complexity of the model. To reduce complexity, we have created a new measure for fluctuation of energy and called it index measure for fluctuation. A new model for peak-load shaving, index model, is generated. Using again some equivalent bi-criteria and parametric problems we manage to compute the efficient solution. Efficient production plan proved to have good behavior compared to real production data.

Minimax and index models use only simple technical constraints. Inclusion of more complex technical constraints, profit as measure of economic performance, a better estimation for input data and a new approach for dealing with the transition from night period to day period will open new research directions and might improve the accuracy of solution.

Our thesis is structured in six chapters.

First chapter is dedicated to the general context of energy which generates challenges for optimization.

Second chapter is dedicated to evaluate some mathematical tools used in our research. Bi-criteria optimization problems and Kuhn-Tucker conditions are presented. Some measures of spread are presented and evaluated for finding a starting point in defining our measures for fluctuation of energy.

Chapter 3 is dedicated to development of conditions such that efficient solution of initial bi-criteria optimization problem will remain efficient also for the approximate problem and reciprocally. Approximation is creating a proper environment for solving in a more efficient way complex energy problems which might arise due to technical constraints.

Chapter 4 is dedicated to the minimax energy model. It presents the minimax measure for fluctuation of energy, development, solving and validation of minimax model.

Chapter 5 is dedicated to index energy model. Its development is stimulated by the complexity of input data required by minimax model. A new measure for fluctuation of energy is introduced. Development, solving and validation of index model are presented. Index is less complex than minimax and easier to be applied, but performances are less accurate.

Chapter 6 presents some conclusions of our research, an analysis and comparison for the two models and some possibilities to extend and improve the two models developed.

Our contributions to this thesis might be summarized as: a new approach, based on bi-criteria optimization problems, for shaving the peak load of energy; Theorems 3.3.1, 3.3.2, 3.3.5, 3.3.6, 3.3.9, 3.3.10, 3.3.11, 3.3.12, 3.4.3, 3.4.4, 3.4.5, 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.10, 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10, 3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6, 3.6.7, 3.6.8; Examples 3.3.3, 3.3.7, 3.4.11, 3.4.12, 3.5.11, 3.5.12, 3.5.13; definition of minimax measure for fluctuation (4.1) and index measure for fluctuation (5.1); bi-criteria problems (4.2) and (5.2) used for shaving the peak load; Lemma 4.4.1 and Lemma 5.4.1 used to transform energy problems (4.2) and (5.2) in equivalent problems easier to be solved; Theorem 4.4.11 which proves that order of scenarios does not change the solution; Theorem 4.4.5 which computes the optimal solution for parametric model (4.4); Theorem 4.4.12 which computes the efficient solution for minimax model (4.2); Theorem 5.4.5 which computes the optimal solution for parametric model (5.4); Theorem 5.4.6 which computes the efficient solution for index energy model (5.2); tests for minimax and index models performed using real data and some economic analysis for Kuhn-Tucker multipliers; classification, using Definitions 4.4.6, 4.4.7 and 4.4.8, of Kuhn-Tucker multipliers based on to their capacity to generate feasible and optimal solutions for parametric optimization problem.

Results presented in this thesis were disseminated at two international conferences:

> *Bi-criteria problems for energy optimization*, presented at Conference: International Conference on Approximation Theory and its Applications, organized in Sibiu, Romania during 26-29 May 2016.

> *Bi-criteria models for energy markets*, presented at Conference: Management International Conference, organized in Monastier di Treviso, Italy, during 24-27 May 2017.

and in nine articles:

> *Minimax rule for energy optimization* [98], published in Computers and Fluids, an ISI journal with an Impact Factor of 2.221 and a 5-years Impact Factor of 2.610.

> *Index model for peak-load shaving in energy production* [92], submitted to Engineering Optimization, an ISI Journal with an Impact Factor of 1.728.

> *Bi-criteria models for peak-load shaving* [90], accepted for publication by Journal of Academy of Business and Economics, a journal indexed in EBSCO, EconLit, Ulrich's, Index Copernicus, Research Bible.

> *Approximations of objective function in bi-criteria optimization problems* [96], accepted for publication by European International Journal of Science and Technology, a journal indexed in Google Scholar, NewJour, Hochschulbibliothek Reutlingen, CrossRef.

Approximations of objective function and constraints in bi-criteria optimization problems [95], submitted to Journal of Numerical Analysis and Approximation Theory, a journal indexed in Mathematical Reviews, Zentralblatt MATH.

Approximations of bi-criteria optimization problem [94], submitted to Studia Universitatis Babes-Bolyai Mathematica, a journal indexed in Mathematical Reviews, Zentralblatt MATH, EBSCO, ProQuest Ulrichsweb.

Relations between η - approximation problems of a bi-criteria optimization problem [97], submitted to Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, a journal indexed in Mathematical Reviews, Zentralblatt MATH, American Mathematical Society.

Bi-criteria problems for energy optimization [39], published in General Mathematics, a journal indexed in Zentralblatt MATH, EBSCO, Mathematical Reviews, Index Copernicus.

Portfolio optimization algorithms [93], published in Studia Universitatis Babes-Bolyai Negotia, a journal indexed in EBSCO, Index Copernicus, ERIH PLUS.

and are supported by our previous work reflected in books, articles and conferences:

Matematici economice. Elemente de programare liniara si teoria probabilitatilor [19], a book published by Presa Universitara Clujeana.

Strict fixed points results for multivalued contractions on gauge spaces [122], published in Fixed Point Theory, an ISI journal with an Impact Factor of 1.030 in 2010.

Uniqueness algebraic conditions in the study of second order elliptic systems [18], published in International Journal of Pure and Applied Mathematics, a journal indexed in Scopus.

Maximum principles for a class of second order parabolic systems in divergence form [20], published in Journal of Nonlinear Functional Analysis and Differential Equations, a journal indexed in Scopus, Web of Science, Zentralblatt MATH.

Automotive industry and performances of US Economy [125], published in International Journal of Finance and Economics, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Consumer's inflation expectations in Romania [147], published in International Journal of Business Research, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Comparative analysis of low-cost airlines websites [77], published in Proceedings of IABE - 2009 Las Vegas - Annual Conference, a volume indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Economic applications of dynamic optimization, presented at Conference: 10th International Symposium on Generalized Convexity and Monotonicity, organized in Cluj Napoca, Romania, during 22-27 August 2011.

A relation between transportation problems and profit, presented at Conference: Current Issues of Regional Development, organized in Sec, Czech Republic, during 26-27 June 2007.

Key words: *approximation theorems, generalized convexity, peak load shaving, bi-criteria problems, energy fluctuation minimization, economic performance maximization, parametric optimization, Kuhn-Tucker conditions*.

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Chapter 1

General context of energy optimization

Humanity is dependent on energy, a very complex and challenging resource, characterized by:

strategical power provided by energy;

impact of energy production on climate change (an increase of average global temperature with 0.8^o *Celsius [115])* due to significant emissions of green house gases (42% of total CO_2 emissions are due to electricity and heat production [66]) resulted from burning fuels in thermal power plants (67.4% of fuels are coal, oil and gas [66]);

projected increase of energy consumption with an average of 24% until 2040 [151] (15% in case of Low Economic Growth and 37% in case of High economic Growth);

intermittent production feature of renewable sources;

massive investments associated to power plants and power grids;

increased frequency and magnitude of peak loads.

In the context of an projected increase for population by 2.2 billion persons until 2050 compared with current figure [150], all characteristics presented above will generate more pressure on the system and thus a rational approach, to optimize the energy, becomes necessary.

Doukopoulos [37] is explaining the system used to optimize the production of energy in France, synthesizing the objectives and activities on short, medium and long term. They are presented in Figure 1.1.

Each time horizon has its own characteristics:

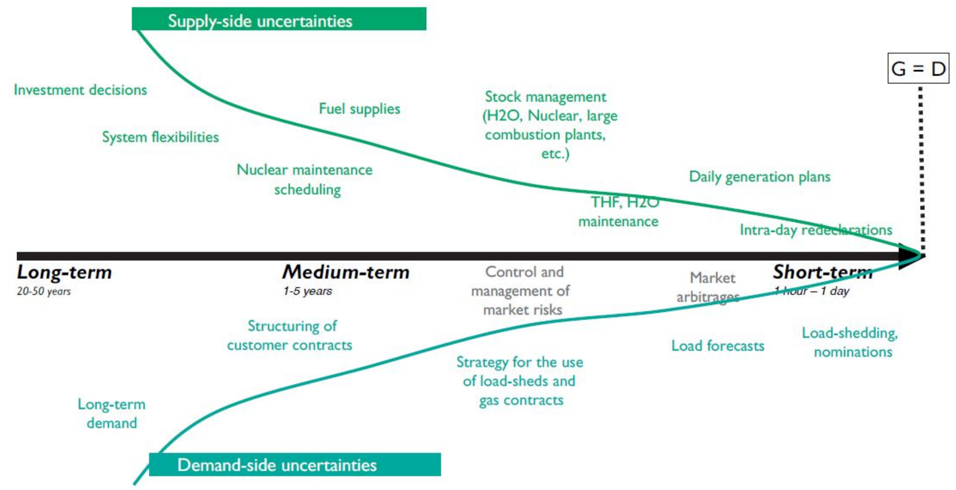


FIGURE 1.1: Short, medium and long term objectives of an energy system

- **Long term (5 to 20 years)**
  - high uncertainty;
  - technical constrains are not important;
  - objectives are: planning investments, defining the mix of power plants to be operated, forecasting polluting emissions.
- **Medium term (1 to 5 years)**
  - operational planning for nuclear power plants (maintenance, refueling);
  - strategy for "row material" (water reservoirs, fuel stocks and price);
  - evaluation of polluting emissions.
- **Short term (hours and days)**
  - demand constraint;
  - technical constrains are very important;
  - uncertainty is evaluated using probability and statistic;
  - objectives are: computation of optimal production plan and marginal cost of the system (will provide price for energy).

On short term, computation of optimal amount of energy to be produced at each time moment, involves highly *complex problems* (up to 10^6 variables and 10^6 constraints in the model used in France, according to Doukopoulos [37] or 400.000 binary variables, 650.000 continuous variables and 1.3000.000 constraints in case of model proposed by Nurnberg [118]), *difficult to solve* and *with a few challenges*:

- covering demand to avoid black-outs;
- extremely short time, available to compute the optimal solution (15 minutes in France according to [37]);
- optimizing different objectives of actors involved;
- climate change and related implications.

Complexity is increased also by the existence of two types of markets: regulated and deregulated.

On regulated markets, a vertically-integrated entity is controlling all activities of generation, transmission and distribution of energy. Energy rates, established based on cost, are approved and regulated. Objective on these markets is to minimize cost of production, an optimal production plan being computed, using mathematical models, for each power plant.

On deregulated markets, the entities in charge with distribution of energy to consumers are independent by the entities responsible to generation and transmission of electricity. Several distributors might act on these markets, each one establishing its own price. Energy supplied is split in two: (a) energy supplied based on fixed contracts and (b) energy supplied based on flexible contracts (day-ahead markets). Producers and distributors are estimating daily production and consumption curves, while an Independent System Operator is determining the energy rates and the hourly amount of energy to be delivered by each producer. On deregulated markets, price is established based on cost and demand. Objectives on these markets might be: minimizing production cost, minimum level of production to guarantee continuity of activity for producers, maximizing profit, maximizing profit and minimizing financial risk. Day-ahead markets are important (65% of energy consumed in 2007 in Scandinavia was dispatched via day-ahead markets [80]) and challenging through uncertainties related to demand and behavior of competitors.

Demand constraint, also known as static constraint, is imposed by the market. It assumes that total energy delivered in the power grid equals demand.

Technical constraints, also known as dynamic constraints, may refer to *(a) restrictions imposed by the type of power plant (hydro, thermal, nuclear or renewable):* start-up/shut-down curves, duration of each procedure, maximum number of start-ups/shut-downs in a time interval, level of variations/day, boundaries for energy produced, turbine or pumping activity, continuity of production; or *(b) restrictions imposed by authorities:* simultaneous turbine and pumping is forbidden, certain time interval between two opposite activities is required (pumping and turbine), environmental regulations (level of water in reservoirs, polluting emissions); or *(c) restrictions imposed by energy storage:* storage cost, efficiency and speed of energy recovery.

Environmental constraints are determined either by the necessity to deliver energy in remote areas where renewable energy is the most feasible or by the necessity to reduce impact of energy production on environment.

Mathematical models, used for optimizing the production plan of a power plant, have been created and solved, among others, by: Frangioni [50], Martinez [102], Philpott [123] – for thermal-plants; Bor-ghetti [13] – for hydro-plants; Belloni [8], Redondo [127], Lemarechal [84], Gollmer [55], Nowak [117], Nürnberg [118] – for systems of thermo and hydro plants; Wen [154], Zhang [161], Gross [58], Conejo [27], [28], Ladurantaye [81], Gonzalez [56], Eichhorn [44] and Dicorato [36] – for deregulated markets; Cormio [30], Islam [67], Akella [1], Martins [103], Watson [153], Nakata [114], Dudhani [41], Babu [7], Duic [42], Morais [108], Marcato [100] and Mahalov et all [87, 130, 131, 132] – for models with environmental restrictions.

Electricity demand and load varies during a day, generating a peak. Information referring to Romanian electricity market, provided by Transelectrica [148] and presented in Figure 1.2, emphasize this fluctuation of energy during 24 hours, with a peak load in the afternoon (chart is realized for winter time). These fluctuations and peak loads are even more visible in arid and semiarid areas, like: California, Arizona, Nevada, New Mexico, Texas, Middle East, being related to extreme heat events.

Increased consumption of electricity, combined with increased magnitude, frequency and duration of extreme heat events, due to climate change, will result in higher peak loads, creating pressure on:

– producers which have to: (i) anticipate peak loads and adapt the production to avoid power failure – see situation from Azerbaijan in 3rd of July 2018 [146] and (ii) deal with projected increase of operation and maintenance costs for power plants [9, 23, 138];
– power grids which are close to reaching their capacity, requiring huge investments for extension and adaptation;
– price of electricity on deregulated markets – for example, during a heat wave recorded in Texas in 2011, the 15-minutes real time average price has increased from 45$/MWh, during normal period, to 1937$/MWh, during peak hours [132].

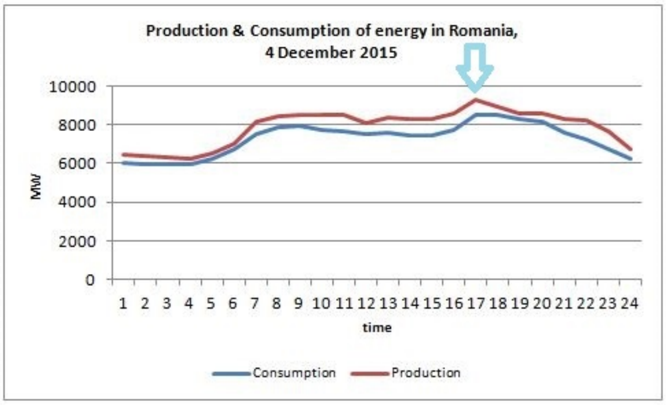


FIGURE 1.2: Evolution of electricity in Romania in 4th of December 2015

Conventional approach for dealing with peak load is based on non-economically feasible solutions, like for example peak-plants. They are used only for short time intervals and have high fuel consumption and CO_2 emissions [107]. A more preferable approach, which has become an important research area is peak load shaving [149].

Peak load shaving might be defined as the process of flattening the load curve by reducing the peak amount of load and shifting it to times of lower load [116]. Three main strategies might be applied to realize a peak load shaving: (i) energy storage systems – ESS, (ii) electric vehicles – EV and (iii) demand side management – DSM.

Among benefits generated by peak load shaving we might address: replacement of expensive peak-plants with more efficient solutions [68, 157], postpone of expensive investments required to update and extend power grids [33, 47], increased power quality [51], efficient energy utilization–Load Factor [57] and system efficiency [70], less CO_2 emissions [134] and reduced costs for end user [15].

ESS is considered the most potential strategy [149] for peak load shaving, several studies being realized for: optimum operation of ESS [23, 24, 82, 128, 162], optimum size of ESS [24, 49, 70, 72, 87, 88, 111, 130] or feasibility of ESS [26, 47, 139, 156, 157]. The consulting company Yole Development [158] is estimating that "stationary storage market will increase from 1 billion US$ in 2015 to 13.5 billion US$ in 2023", highlighting the potential of ESS and peak load shaving.

EVs are not widely used. A collection of vehicles is required to obtain good results, with synchronization and infrastructure challenges [149].

DSM refers to programs that might influence consumers behavior to balance energy consumption with generation capability. Results in this field have been obtained, among others, by [113, 120, 144]. The main challenges for DSM, identified by [149] are: rigidity of people to accept changes, limited availability of advanced tools in the energy system and projected increase of system's complexity.

In our research, we will consider fluctuation of energy as a spread of values, the most distant one being the peak load. By controlling the spread it might be possible to reduce the magnitude of peak load. As side-effects for this control of spread we have identified supra-production of energy at other time moments and modifications in the production diagram [141, 145] with potential impact on the economic performance of the power plant. Extra volumes generated by supra-production might be stored (pumping–turbine principle, kinetic or thermal energy, batteries, compressed or liquified air) and reused when necessary. Storage cost, efficiency and power recovery time depends on the storage method.
To ensure that economic performances of the power plant will not decrease due to production diagram modification, a bi-criteria optimization problem might be employed, the first component of the

objective being related to spread – fluctuation of energy (to be minimized) and the second component being related to economic performance of the power plant (to be maximized).

The novelty of our research consists in using a bi-criteria optimization problem for peak load shaving through power diagram modification.

Considering all these, we define the objective for our research as: **to create, solve and validate a mathematical model able to shave the peak load, by minimizing fluctuation of energy and maximizing economic performance of power plant**.

The challenges to reach our objective are: (a) to define proper measures for fluctuation of energy and economic performance; (b) to create the mathematical model and to solve it.

Chapter 2

Mathematical tools

2.1 Introduction

In our research we will work with a bi-criteria optimization problem. The first component of the objective function is fluctuation of energy. We have to define a proper measure for fluctuation of energy (able to target the peak load). Let's recall that fluctuation of energy might be compared with a spread, for which mathematics has several measures. We will investigate some measures of spread an evaluate if they are able to generate that proper measure which we need.
The second component of the objective function is economic performance. Literature and practice offer several measures for economic performance, like turnover, profit, cost, rate of return.

After clarifying measures issue, we will be able to develop a mathematical model for our problem. Solving it will become, in that step, our challenge. We will investigate methods used to solve optimization problems.

2.2 Multi-criteria optimization problem and efficient points

As mentioned before, we will employ, during our research, bi-criteria problems to create mathematical models which aim to shave the peak load of energy.

Definition 2.2.1 *Let $X \subseteq \mathbb{R}^n$ and $f = (f_1, f_2, ...f_m)^T : X \to \mathbb{R}^m$. The optimization problem*

$$\begin{cases} \min f(x) \\ x \in X \end{cases} \tag{2.1}$$

is called a multi-criteria optimization problem.

Remark 2.2.2 *If $m = 2$ the multi-criteria optimization problem defined above is called bi-criteria optimization problem.*

In our study, the two components of the objective function will refer to fluctuation of energy and economic performance of the power plant. Naturally, fluctuation of energy has to be minimized, while economic performance has to be maximized. Maximizing the economic performance is the same as minimizing the economic performance multiplied with (-1).

By minimizing fluctuation of energy we aim to shave the peak load. Thus, the first component of the objective function from bi-criteria optimization problem is the key factor for achieving or missing our objective.
The second component of the objective function will influence the accuracy of solution.

The objective function of a multi-criteria optimization problem is a vector function, which implies that the optimal solution has to realize a compromise for all its components. This particular type of optimal solution was introduced by the Italian engineer and economist Vilfredo Pareto and is often called efficient point or efficient solution.

Definition 2.2.3 *A feasible solution $x^* \in X$ of problem (2.1) is an efficient solution if $\nexists x \in X$ such that*

$$\begin{aligned} f(x) &\leq f(x^*) \\ f(x) &\neq f(x^*). \end{aligned}$$

Several problems, generated by real life, require dealing with multiple objectives. In this context, multi-criteria optimization problems, in general, and bi-criteria optimization problems, in particular, have several applications in areas like engineering, economics and life science. As examples, we might mention, among others: financial investments (minimization of risk and maximization profit [16, 76], design optimization of turbine blades (minimization of pressure lost and variation of pressure distribution on blade surface), determination for optimal number of clusters in data clustering, evolution of biological systems that are sufficient robust to mutations but in the same time have potential to innovate [69], data analysis [21], logistics [119].

2.3 Solving a bi-criteria problem

Optimization problems are solved either by analytical methods (compute the exact solution based on mathematical proofs) or by numerical methods (approximate the solution using appropriate iterations).

"Scalarization" methods [17] (weighting problem, k^{th} objective Lagrangian problem, k^{th} objective ε - constrained problem) are often used for solving bi-criteria optimization problems. They consists of transforming the bi-criteria optimization problem into two inter- correlated optimization problems with restrictions or transforming the bi-criteria optimization problem into an equivalent parametric optimization problem. The equivalent problems are solved, providing an optimal solution (if exists), which generates also the efficient solution for the bi-criteria optimization problem. In trade-off methods the importance degree of each component of the objective function might be established *a priori* or during the process. [142, 106, 62] present other contributions to the development of trade-off methods.

In case of multi-objective optimization problems, the numerical methods might be divided [32] into algorithms inspired from biology [64, 74, 34, 71], from physics [60, 85], from geography [54, 52] or from social culture [155, 31].

Another direction for solving a bi-criteria optimization problem consists of computing an approximate problem which can be solved easier. In some conditions, the initial and the approximate problem have the same solution.

In the following subsections we will provide some mathematical techniques helpful for the first approach (transforming the initial problem in an equivalent one), while the next chapter is dedicated to development of conditions such that efficient solution for the initial bi-criteria optimization problem remains efficient for the approximate problem and reciprocally.

2.3.1 Equivalence of Yu

Yu [160] has proved the following result

Theorem 2.3.1 (Yu [160]) .

Let $f_1, f_2, g_1, g_2, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ *linear functions and* $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = \overline{1,m}\}$.

A point $x^* \in X$ is an efficient solution for bi-criteria optimization problem

$$\begin{cases} \min\left(f_1\left(x\right), f_2\left(x\right)\right) \\ \qquad x \in X \end{cases}$$

if and only if $\exists\lambda \in (0,1)$ such that x^* is an optimal solution for parametric optimization problem

$$\begin{cases} \min\left(\lambda f_1\left(x\right) + \left(1-\lambda\right) f_2\left(x\right)\right) \\ \qquad x \in X \end{cases}$$

Using this Theorem, the bi-criteria optimization problem is transformed into an equivalent parametric optimization problem.

One of the methods used to solve optimization problems is based on Lagrange conditions (when constraints are equalities) or Karush-Kuhn-Tucker conditions (when constraints are inequalities). Associated multipliers (Lagrange or Kuhn-Tucker) are representing shadow price, showing the marginal behavior of the objective function with respect to the constant value of the corresponding constraint. This sensitivity analysis is an important information, offering support to the decision maker to determine which constraint has the highest influence in a potential re-optimization process. This feature is not provided by particle swarm optimization, a very common optimization method used in energy optimization.

Considering that constraints imposed for our energy models will be inequalities, we will present the Karush-Kuhn-Tucker conditions.

2.3.2 Kuhn-Tucker Theorems

Let us consider the following nonlinear optimization problem

$$\begin{cases} \min f\left(x\right) \\ g_i\left(x\right) \leq 0, \quad i=\overline{1,m} \\ x \in X \end{cases} \tag{2.2}$$

where $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R},\ i=\overline{1,m}$.

Definition 2.3.2 *The constraint $g_i\left(x\right) \leq 0$, with $i=\overline{1,m}$, is called active at x^0 if $g_i\left(x^0\right) = 0$ and inactive at x^0 if $g_i\left(x^0\right) < 0$.*

The following *Theorem* is presenting **necessary Kuhn-Tucker conditions** for existence of solution for nonlinear optimization problem (2.2).

Theorem 2.3.3 (Kuhn-Tucker [73], [79]: necessary conditions) .
Let $X \subseteq \mathbb{R}^n$ be an open and nonempty set, $x^* \in X$ a feasible solution for problem (2.2) *and functions $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R}$, $i = \overline{1,m}$ differentiable at x^*. Suppose that gradient vectors $\bigtriangledown g_i(x^*)$ corresponding to active constraints are linear independent. If $x^* \in X$ is an optimal solution for problem* (2.2)*, then there exists multipliers $\lambda_i \in \mathbb{R}$, $i = \overline{1,m}$ such that*

$$
\begin{aligned}
&\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0 \\
&\lambda_i g_i(x^*) = 0, \quad i = \overline{1,m} \\
&\lambda_i \geq 0, \quad i = \overline{1,m}.
\end{aligned}
$$

Proof. For a detailed proof, reader may refer to Theorem 1 [79] on page 484, from Kuhn-Tucker's original article "Nonlinear programming" . The original problem analyzed by Kuhn and Tucker is a maximum one, so also the proof in [79] is for the maximum case. Due to the fact that in our work we are using minimum problems, the theorem was presented for this case. ■

Which is the meaning of these three types of conditions? The first one emphasis that x^* is a critical point for the Lagrangian of problem (2.2). Conditions two and three emphasize that in case of an inactive constraint in x^*, the corresponding multiplier is 0 (zero), while in case of an active restriction in x^*, the corresponding multiplier is positive.

It is well known that, in general, the necessary Kuhn-Tucker conditions for optimum are not sufficient too. Adding some additional requirements, the necessary conditions become sufficient too.

The following *Theorem* is presenting **sufficient Kuhn-Tucker conditions** for existence of solution for nonlinear optimization problem (2.2).

Theorem 2.3.4 (Kuhn-Tucker [61], [79]: sufficient conditions) .
Let $X \subseteq \mathbb{R}^n$ be an open and nonempty set, $x^* \in X$ a feasible solution for problem (2.2) *and functions $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R}$, $i = \overline{1,m}$ differentiable in x^* and convex. If there exists multipliers $\lambda_i \in \mathbb{R}$, $i = \overline{1,m}$*

such that

$$\begin{aligned}
&\nabla f\left(x^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \nabla g_{i}\left(x^{*}\right)=0 \\
&\lambda_{i} g_{i}\left(x^{*}\right)=0, \quad i=\overline{1, m} \\
&\lambda_{i} \geq 0, \quad i=\overline{1, m}.
\end{aligned}$$

then x^* is an optimal solution for optimization problem (2.2).

Proof. For a detailed proof, reader may refer to Theorem 3 [79] on page 486 from Kuhn-Tucker's original article "Nonlinear programming". In the original article the maximum problem is analyzed and thus we have replaced concavity with convexity. ∎

During time, mathematicians have been preoccupied to replace convexity in sufficient Kuhn-Tucker conditions, with a weaker one. In 1965, Mangasarian [99] has introduced, **pseudo-convexity** and **cvasi-convexity**. He proved that sufficient Kuhn-Tucker conditions remain valid if objective function is pseudo-convex and constraints are cvasi-convex. In 1981, Hanson [61] has defined **invexity** and has proved that invexity of objective function and constraints are enough for validity of sufficient Kuhn-Tucker conditions.

Formulation of necessary Kuhn-Tucker conditions in 1951 opened the field of Nonlinear Programming. Kuhn-Tucker conditions are a generalization of Lagrangian for optimization problems with inequalities constraints.

As important for mathematics these conditions are, is their origin interesting. We will present further on a short history of these famous theorems.

After the Second World War the American society understood the necessity to continue the research activity started during the war period. Under the coordination of US Navy and US Air Force, several entities financed by US Congress were founded, in order to coordinate and finance research activities. In this context *Office of Naval Research* is founded and in 1948 Harold Kuhn and Albert Tucker started to work for this institution. As a result of their research activity, the first version of Kuhn-Tucker theorem is formulated and presented in 1950 at a seminar of RAND Corporation. Due to lack of regularity conditions, Tompkins developed a counter-example. After adding the regularity conditions (gradient vectors $\bigtriangledown g_i\left(x^*\right)$ corresponding to active constraints are linear independent), Kuhn and Tucker have obtained the well known Kuhn-Tucker Theorem 2.3.3, published in

Nonlinear programming at *Second Berkeley Symposium on Mathematical Statistics and Probability*, held in 1951 [73].

2.3.3 Karush Theorem

In 1939, William Karush is presenting under supervision of professor Graves, at University of Chicago, his dissertation *"Minima of functions of Several Variables with Inequalities as Side Conditions"*. Main idea of dissertation was to determine necessary and sufficient conditions for existence of minimum, for a function $f = f(x_1, x_2, ...x_n)$, knowing that constraints $g_\alpha(x_1, x_2, ...x_n) \geq 0,\ \alpha = \overline{1,m}$ are fulfilled and functions f and $g_\alpha,\ \alpha = \overline{1,m}$ are subject to continuity and differentiability. [73]

The following *Theorem* presents a preliminary result of Karush.

Theorem 2.3.5 (Karush theorem [73]: preliminary result) .
If $f(x_0)$ is a minimum then there exists multipliers l_0, l_α not all zero such that the derivatives F_{x_i} of the function

$$F(x) = l_0 f(x) + \sum_{\alpha=1}^{m} l_\alpha g_\alpha(x)$$

all vanish at x^0. [Karush, 1939, pp. 12-13]

Computing the Lagrangian in the above *Theorem* the multiplier l_0 is used for the objective function and no sign restrictions for multipliers are involved. More than that, l_0 might be zero. To avoid this situation, additional restrictions called *regularity conditions* are necessary. To introduce regularity conditions, Karush defines *admissible direction* and *admissible arc*.

Definition 2.3.6 *A vector $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$ is called admissible direction if*

$$\sum_{i=1}^{n} \frac{\partial g_\alpha}{\partial x_i}\left(x^0\right)\lambda_i \geq 0, \quad \alpha = \overline{1,m}.$$

In natural language, this means that by moving from x_0 in the direction indicated by vector λ we remain in the set of admissible solutions.

Definition 2.3.7 *An arc $x : [0, t_0] \to \mathbb{R}^n$ is called admissible, if $g_\alpha(x(t)) \geq 0$, for any $\alpha = \overline{1,m}$ and $t \in [0, t_0]$.*

In natural language, this means that by moving along an admissible arc we remain in the set of admissible solutions.

Definition 2.3.8 *An arc $x : [0, t_0] \to \mathbb{R}^n$ starts from x^0 to direction λ, if*

$$\begin{aligned} x_i(0) &= x_i^0, \quad i = \overline{1, n} \\ x_i^{'}(0) &= \lambda_i, \quad i = \overline{1, n}. \end{aligned}$$

Using these regularity conditions, Karush is creating the necessary environment to use *Farkas Lemma* for proving the existence of multipliers and is formulating the main theorem of his dissertation, which is presented in the following:

Theorem 2.3.9 (Karush [73]) .
Suppose that for each admissible direction λ there is an admissible arc issuing from x^0 in the direction λ. Then a first necessary condition for x^0 to be a minimum is that there exist multipliers $l_\alpha \leq 0, \quad \alpha = \overline{1, m}$ such that the derivatives F_{x_i} of the function

$$F = f + \sum_{\alpha=1}^{m} l_\alpha g_\alpha$$

all vanish at x^0. [Karush, 1939, pp.13].

Considering the conditions to elaborate his dissertation (a study for a finite dimensional case in variational calculus), Karush will not publish the above theorem. Kjeldsen is presenting in [73] the way Kuhn found out about work done early by Karush. The main idea is presented in the next lines. In 1974, Takayama is presenting Karush's theorem in his book "Mathematical Economics" and so Kuhn finds this way about the result obtained in 1939 by Karush. In a letter addressed to Karush in 1975, Kuhn is writing:

> *"First let me say that you have clear priority on the result known as Kuhn-Tucker conditions (including the constraint qualification). I intend to set the record as straight as I can in my talk."* [73]

In 1976, during the American Mathematical Society Symposium, Kuhn is partially publishing Karush's dissertation, emphasizing his contribution to the development of nonlinear programming [78].
This is the moment when Kuhn-Tucker conditions will be renamed as Karush-Kuhn-Tucker conditions.

2.3.4 John Theorem

One of the research areas for Fritz John was convexity theory, being preoccupied among others by extending Lagrange multipliers to cases when constraints were inequalities. The following theorem belongs to him.

Theorem 2.3.10 (John [73]) .
Let R be a set of points in $\mathbb{R}^n$, S a set of points in R and $R^{'}$ the set of all points $x \in R$, which satisfy

$$G(x, y) \geq 0, \textit{for all } y \in S$$

where $G : R \times S \to \mathbb{R}$.
Let $F : R \to \mathbb{R}$ and x^0 be an interior point of R and a point of $R^{'}$with

$$F\left(x^0\right) = \min_{x \in R'} F(x) .$$

Then there exists a finite set of points $y^1 ... y^s$ in S and numbers $\lambda_0, \lambda_1, ... \lambda_s$ which do not all vanish, such that

$$\begin{aligned}
&G\left(x^0, y^r\right) = 0, \quad r = \overline{1, s} \\
&\lambda_0 \geq 0 \\
&\lambda_1 > 0, ... \lambda_s > 0, \quad 0 \leq s \leq n
\end{aligned}$$

and the function

$$\phi(x) = \lambda_0 F(x) - \sum_{r=1}^{s} \lambda_r G(x, y^r)$$

has a critical point at x^0.

Due to geometrical context in which John has developed his theorem, he was using parameters $y \in S$ which are not present in the theorems formulated by Karush or Kuhn-Tucker. More than that, considering the same geometrical context, John does not need regularity conditions. John tries to publish his result in *Duke Mathematical Journal*, but the paper was not accepted for publication and finally he published the theorem in a book for the 60 anniversary of his PhD supervisor Richard Courant [73].

In the mathematical community theorem of Karush and theorem of Kuhn-Tucker are considered identical.

2.4 Measures of spread

2.4.1 Variance

The best known measure of spread is variance. It measures the spread of some values around their average or expected value, being defined as:

$$\sigma = E\left[(X-\mu)^2\right], \tag{2.3}$$

where X is a random variable and μ is its average or expected value.

Quadratic form of variance makes it very complex and difficult to be applied. Several times it is replaced by standard deviation, calculated as square root of variance.

However, variance is successfully used, as measure of risk, in the well known Mean Variance model of Markowitz [101], but in the same time is generating several complex situations and critics. In order to improve its behavior as measure of risk, scientists have tried to extend, modify or simplify variance, by: developing utility functions around variance, considering transaction costs to reduce impact of spread, simplifying it by using an index concept. Some examples, found by us in the literature, are: Merton [104, 105], Smith [140], Mossin [110], Samuelson [133], Fama [48], Hakkanson [59], Elton and Gruber [45, 46], Li and Ng [86], Constantinides [29], Perold [121], Amihud and Mendelson [3], Dumas and Luciano [43], Best and Grauer [10, 11], Chopra, Hensel and Turner[22], Sharpe [135, 136, 137], Stone [143], von Hohenbalken [63], Lee, Finnerty and Wort [83], Huang and Qiao [65].
Despite all this effort, results provided by models developed around variance are considered, by practitioners, not enough accurate.

Variance is measuring how far the values are spread around the expected value, offering an average distance of spread. Thus variance is not targeting directly the most extreme value and by minimizing variance there is no guarantee that magnitude of the most extreme point (peak load) will be reduced.

Attempts (mentioned above) to extend, modify or simplify variance were not welcomed by practitioners, so we will not put effort to adapt variance to our purpose.

As a conclusion we might say that despite its popularity, variance is not suitable for our objective.

2.4.2 Mean absolute deviation

Quadratic form of variance, which is creating several challenges and complex situations, is required to avoid compensation between positive and negative values which appear when deviations of values from expected value are computed.

Module function has a similar behavior with quadratic function, as it is visible in Figure 2.1 and its effect on the before mentioned deviations is the same, but challenges associated to quadratic form are eliminated.

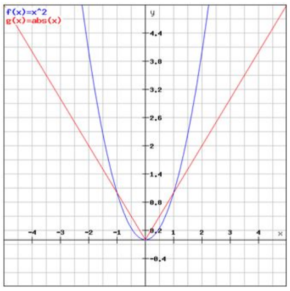


FIGURE 2.1: Comparison between quadratic and module functions

Thus, by replacing quadratic form of variance with module, the mean absolute deviation defined as:

$$\sigma = E\left[|X - \mu|\right] \tag{2.4}$$

is obtained, where X is a random variable and μ is its average or expected value.

It was successfully used, among others, by Konno [75] or by Konno and Yamazaki [76], but still inherits that property to calculate an average distance for spread. Thus, by minimizing mean absolute deviation there is no guarantee that the magnitude of the most extreme point (peak load) will decrease.

We might say that mean absolute deviation is performing better than variance but still is not suitable for our objective.

2.4.3 Maximum absolute deviation

Maximum absolute deviation is defined as

$$\sigma = \max\left(|X - \mu|\right), \tag{2.5}$$

where X is a random variable and μ is its average or expected value.

It was successfully applied in some models for financial investments [16], [159].

To calculate it, absolute deviation of each point from the expected value is calculated and the maximum of these deviations is considered. From its definition, maximum absolute deviation is addressing the most extreme point. By minimizing maximum absolute deviation the magnitude of the most extreme point (peak load) will decrease.

Considering this potential of maximum absolute deviation and its other successfully applications, we might consider it as a starting point for developing a proper measure for fluctuation of energy.

Chapter 3

Approximation theorems for bi-criteria optimization problems

3.1 Introduction

Practical problems arising from different fields of activity might generate highly complex multi-criteria optimization problems. Approximation problems are representing one of the method used to solve these complex practical problems. Antczak [4, 5, 6], Duca [38, 12, 25, 124, 40], Popovici [126, 2] have contributed, among others, to this method of solving optimization problems. In this chapter we are studying conditions such that efficient solutions of the approximate problem will remain efficient also for the initial problem and reciprocally.

3.2 Basic concepts

Let X be a set in $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$. If f is differentiable at x_0 then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0)$$

and call it first $\eta-$approximation of f
and if f is twice differentiable at x_0 then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0)\,.$$

and call it second $\eta-$approximation of f.

Definition 3.2.1 *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:*
invex *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$

avex *at x_0 with respect to η if it is both invex and incave at x_0 w.r.t. η.*

If function f is invex, respectively incave or avex we denote invex^1, respectively incave^1 or avex^1.

Definition 3.2.2 *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function twice differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:*
second order invex *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

second order incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

second order avex *at x_0 with respect to η if it is both second order invex and second order incave at x_0 w.r.t. η.*

If function f is second order invex, respectively second order incave or second order avex we denote invex^2, respectively incave^2 or avex^2.

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

We consider the bi-criteria optimization problem $\left(P_0^{0,0}\right)$, defined as:

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

Assuming that functions f_1, f_2, are differentiable of order $i,\ j \in \{1,\ 2\}$ and functions $g_t,\ (t \in T),\ h_s,\ (s \in S)$ are differentiable of order $k \in \{0, 1, 2\}$, we will approximate original problem $\left(P_0^{0,0}\right)$ by problems $\left(P_k^{i,j}\right)$:

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ G_t^k\left(x\right) \leq 0,\ t \in T \\ H_s^k\left(x\right) = 0,\ s \in S \end{cases}$$

where $(i, j) \in \{(1,0),\ (1,1),\ (2,0),\ (2,1),\ (2,2)\}$, $k \in \{0, 1, 2\}$ and $F_1^0 = f_1,\ F_2^0 = f_2, G_t^0 = g_t\ \ (t \in T), H_s^0 = h_s\ \ (s \in S)$.

We denote by

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k\left(x\right) \leq 0,\ t \in T,\ H_s^k\left(x\right) = 0,\ s \in S\right\},\ k \in \{0, 1, 2\}$$

the set of feasible solutions for bi-criteria optimization problem $\left(P_k^{i,j}\right)$, where $(i, j) \in \{(1,0),\ (1,1),\ (2,0),\ (2,1),\ (2,2)\}$ and $k \in \{0, 1, 2\}$.

3.3 First and second η - approximations for components of objective function on the same feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_0^{1,0}\right)$, $\left(P_0^{1,1}\right)$, $\left(P_0^{2,0}\right)$, $\left(P_0^{2,1}\right)$ and $\left(P_0^{2,2}\right)$

will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Theorem 3.3.1 (Luca and Duca [96]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is differentiable at x_0 and invex[1] at x_0 with respect to η,*

b) $\eta(x_0,\ x_0) = 0.$

If x_0 is an efficient solution for $\left(P_0^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{1,0}\right)$ implies that $\nexists x \in \mathcal{F}^0$ *s.t.*

$$\begin{array}{l}\left(F_1^1(x), f_2(x)\right) \leq \left(F_1^1(x_0), f_2(x_0)\right)\\ \left(F_1^1(x), f_2(x)\right) \neq \left(F_1^1(x_0), f_2(x_0)\right).\end{array}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{0,0}\right)$. It means $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{array}{l}\left(f_1(y), f_2(y)\right) \leq \left(f_1(x_0), f_2(x_0)\right)\\ \left(f_1(y), f_2(y)\right) \neq \left(f_1(x_0), f_2(x_0)\right)\end{array}$$

which implies that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\left\{\begin{array}{l} f_1(y) < f_1(x_0)\\ f_2(y) \leq f_2(x_0)\end{array}\right. \tag{3.1}$$

or

$$\left\{\begin{array}{l} f_1(y) \leq f_1(x_0)\\ f_2(y) < f_2(x_0).\end{array}\right. \tag{3.2}$$

Because f_1 is invex[1] at x_0 with respect to η we get $F_1^1(y) \leq f_1(y)$, $\forall y \in \mathcal{F}^0$. Because $\eta(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^1(x_0)$. Thus from (3.1) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\left\{\begin{array}{l} F_1^1(y) < F_1^1(x_0)\\ f_2(y) \leq f_2(x_0)\end{array}\right.$$

which contradicts the efficiency of x_0 for $\left(P_0^{1,0}\right)$ and from (3.2) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\left\{\begin{array}{l} F_1^1(y) \leq F_1^1(x_0)\\ f_2(y) < f_2(x_0)\end{array}\right.$$

which contradicts the efficiency of x_0 for $\left(P_0^{1,0}\right)$.
In conclusion x_0 is an efficient solution for $\left(P_0^{0,0}\right)$. ■

Theorem 3.3.2 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is differentiable at x_0 and incave[1] at x_0 with respect to η,*

b) $\eta\,(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{1,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{0,0}\right)$ implies that $\nexists x \in \mathcal{F}^0$ *s.t.*

$$\begin{array}{l}(f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)) \\ (f_1(x), f_2(x)) \neq (f_1(x_0), f_2(x_0)).\end{array}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{1,0}\right)$. It means $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{array}{l}(F_1^1(y), f_2(y)) \leq (F_1^1(x_0), f_2(x_0)) \\ (F_1^1(y), f_2(y)) \neq (F_1^1(x_0), f_2(x_0))\end{array}$$

which implies that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^1(y) < F_1^1(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases} \tag{3.3}$$

or

$$\begin{cases} F_1^1(y) \leq F_1^1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3.4}$$

Because f_1 is incave[1] at x_0 with respect to η we get $f_1(y) \leq F_1^1(y)$, $\forall y \in \mathcal{F}^0$. Because $\eta\,(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^1(x_0)$. Thus from (3.3) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases}$$

which contradicts the efficiency of x_0 for $\left(P_0^{0,0}\right)$ and from (3.4) we get that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} f_1(y) \leq f_1(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts the efficiency of x_0 for $\left(P_0^{0,0}\right)$.
In conclusion x_0 is an efficient solution for $\left(P_0^{1,0}\right)$. ■

Example 3.3.3 (Luca and Duca [96]) .
If condition a) from Theorem 3.3.2 is not satisfied, it might be possible either that initial and approximate problems have the same solution or different. The following example is presenting the case when the two problems have different solutions.
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min f(x) = \left(x_1^2 + x_2^2; x_1 - 2x_2\right) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.5}$$

$x^0 = (1,1) \in \mathcal{F}^0$ is an efficient solution for problem (3.5) and the value of function f is $f(1,1) = (2,-1)$.
To compute the approximate problem $\left(P_0^{1,0}\right)$ we have to calculate the first η-approximation of f_1. Thus

$$F_1^1(x) = f_1\left(x^0\right) + \nabla f\left(x^0\right)\eta\left(x, x^0\right).$$

Considering $\eta\left(x, x^0\right) = x - x^0$, we obtain

$$F_1^1(x) = 2x_1 + 2x_2 - 2.$$

and the approximate problem $\left(P_0^{1,0}\right)$ will be

$$\begin{cases} \min F(x) = \left(2x_1 + 2x_2 - 2; x_1 - 2x_2\right) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.6}$$

Thus

$$\begin{array}{l} F\left(x^0\right) = F(1,1) = (2,-1) \geq (2,-4) = F(0,2) \\ F\left(x^0\right) \neq F(0,2) \end{array}$$

which proves that $x^0 = (1,1) \in \mathcal{F}^0$ is not an efficient solution for problem $\left(P_0^{1,0}\right)$. In conclusion, efficient solution of problem (3.5) doesn't remain *efficient also for approximate problem (3.6), proving that incavity of f_1 from condition a) of Theorem 3.3.2 is essential.*

Remark 3.3.4 *Conditions such that efficient solution of problem $\left(P_0^{1,1}\right)$ will remain efficient for problem $\left(P_0^{0,0}\right)$ and reciprocally have been studied in [40].*

Theorem 3.3.5 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_0^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.3.1*. ■

Theorem 3.3.6 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,0}\right)$.

Proof. Proof is similar with *Theorem 3.3.2*. ■

Example 3.3.7 (Luca and Duca [96]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(x_1^2 + x_1x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2\right) \\ -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\ 4x_1 + x_2 - 20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.7}$$

An efficient solution for problem (3.7) is $x^0 = (3, 8)$. Starting from this efficient solution of problem (3.7), the following approximation in x^0 for the first component of objective function has to be computed

$$F_1^2(x) = f_1(x^0) + \nabla f_1(x^0)\, \eta(x, x^0) + \frac{1}{2}\eta(x, x^0)^T \nabla^2 f_1(x^0)\, \eta(x, x^0).$$

Considering $\eta(x, x^0) = x - x^0$ we obtain

$$F_1^2(x) = x_1^2 + x_1x_2 + x_2^2 - 19.25x_1 - 19.875x_2$$

and the corresponding approximate problem $(P_0^{2,0})$ is:

$$\begin{cases} \min\left(x_1^2 + x_1x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2\right) \\ -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\ 4x_1 + x_2 - 20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.8}$$

which is identical with initial problem (3.7) and thus they have the same efficient solution.

Remark 3.3.8 *Example 3.3.7 shows that if second order incavity of f_1 from condition a) of Theorem 3.3.6 is not satisfied it might be possible to obtain the same efficient solution.*

Theorem 3.3.9 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

c) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $(P_0^{2,1})$, then x_0 is an efficient solution for $(P_0^{0,0})$.

Proof. Proof is similar with *Theorem 3.3.1*. ■

Theorem 3.3.10 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,1}\right)$.

Proof. Proof is similar with *Theorem 3.3.2*. ■

Theorem 3.3.11 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.3.1*. ■

Theorem 3.3.12 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) *f_2 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,2}\right)$.

Proof. Proof is similar with *Theorem* 3.3.2. ■

3.4 First and second η-approximations for components of objective function on the first η -approximation of feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_1^{1,0}\right)$, $\left(P_1^{2,0}\right)$, $\left(P_1^{2,1}\right)$ and $\left(P_1^{2,2}\right)$ will remain efficient also for initial problem $\left(P_0^{0,0}\right)$ and reciprocally. Conditions for the relation $\left(P_0^{0,0}\right)$ vs. $\left(P_1^{1,1}\right)$ have been studied in [40] so we will not analyze them anymore.

By approximating also the feasible set it is important to determine conditions such that $\mathcal{F}^0 \subseteq \mathcal{F}^1$ and $\mathcal{F}^1 \subseteq \mathcal{F}^0$. These inclusions were studied in [40]. We will use them in our work, so we will briefly present the Theorems stating these inclusions.

Theorem 3.4.1 (Duca and Ratiu [40]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that:

a) *for each $t \in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^1.$$

Theorem 3.4.2 (Duca and Ratiu [40]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that

a) *for each $t \in T$, the function g_t is differentiable at x_0 and incave*[1] *at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*[1] *at x_0 with respect to η,*

then

$$\mathcal{F}^1 \subseteq \mathcal{F}^0.$$

Theorem 3.4.3 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex*[1] *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*[1] *at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex*[2] *at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_1^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_1^{2,0}\right)$, implies that $\nexists x \in \mathcal{F}^1$ $s.t.$

$$\begin{aligned} &\left(F_1^2\left(x\right), f_2\left(x\right)\right) \leq \left(F_1^2\left(x_0\right), f_2\left(x_0\right)\right) \\ &\left(F_1^2\left(x\right), f_2\left(x\right)\right) \neq \left(F_1^2\left(x_0\right), f_2\left(x_0\right)\right). \end{aligned}$$

Conditions $b)$ and $c)$ imply that

$$\mathcal{F}^0 \subseteq \mathcal{F}^1$$

and thus $\nexists x \in \mathcal{F}^0$ $s.t.$

$$\begin{aligned} &\left(F_1^2\left(x\right), f_2\left(x\right)\right) \leq \left(F_1^2\left(x_0\right), f_2\left(x_0\right)\right) \\ &\left(F_1^2\left(x\right), f_2\left(x\right)\right) \neq \left(F_1^2\left(x_0\right), f_2\left(x_0\right)\right). \end{aligned} \tag{3.9}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{0,0}\right)$. Then $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{gathered}\left(f_1(y), f_2(y)\right) \leq \left(f_1(x_0), f_2(x_0)\right)\\ \left(f_1(y), f_2(y)\right) \neq \left(f_1(x_0), f_2(x_0)\right)\end{gathered}$$

which implies that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases} \tag{3.10}$$

or

$$\begin{cases} f_1(y) \leq f_1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3.11}$$

Because f_1 is invex[2] at x_0 with respect to η we get $F_1^2(y) \leq f_1(y)$, $\forall y \in \mathcal{F}^0$. Because $\eta(x_0, x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$. Thus from (3.10) we get that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases}$$

which contradicts (3.9) and from (3.11) we get that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} F_1^2(y) \leq F_1^2(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts (3.9).
In conclusion x_0 is an efficient solution for $\left(P_0^{0,0}\right)$. ■

Theorem 3.4.4 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave[2] at x_0 with respect to η,*

e) $\eta(x_0,\ x_0) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{2,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{0,0}\right)$, implies that $\nexists x \in \mathcal{F}^0$ $s.t.$

$$\begin{aligned}(f_1(x), f_2(x)) &\leq (f_1(x_0), f_2(x_0))\\ (f_1(x), f_2(x)) &\neq (f_1(x_0), f_2(x_0)).\end{aligned}$$

Conditions $b)$ and $c)$ imply that

$$\mathcal{F}^1 \subseteq \mathcal{F}^0$$

and thus $\nexists x \in \mathcal{F}^1$ $s.t.$

$$\begin{aligned}(f_1(x), f_2(x)) &\leq (f_1(x_0), f_2(x_0))\\ (f_1(x), f_2(x)) &\neq (f_1(x_0), f_2(x_0)).\end{aligned} \tag{3.12}$$

Let's assume that x_0 is not an efficient solution for $\left(P_1^{2,0}\right)$. Then $\exists y \in \mathcal{F}^1$ $s.t.$

$$\begin{aligned}(F_1^2(y), f_2(y)) &\leq (F_1^2(x_0), f_2(x_0))\\ (F_1^2(y), f_2(y)) &\neq (F_1^2(x_0), f_2(x_0))\end{aligned}$$

which implies that $\exists y \in \mathcal{F}^1$ $s.t.$

$$\begin{cases} F_1^2(y) < F_1^2(x_0)\\ f_2(y) \leq f_2(x_0)\end{cases} \tag{3.13}$$

or

$$\begin{cases} F_1^2(y) \leq F_1^2(x_0)\\ f_2(y) < f_2(x_0).\end{cases} \tag{3.14}$$

Because f_1 is incave[2] at x_0 with respect to η we get $f_1(y) \leq F_1^2(y),\ \forall y \in \mathcal{F}^1$. Because $\eta(x_0,\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$. Thus from (3.13) we get that $\exists y \in \mathcal{F}^1$ $s.t.$

$$\begin{cases} f_1(y) < f_1(x_0)\\ f_2(y) \leq f_2(x_0)\end{cases}$$

which contradicts (3.12) and from (3.14) we get that $\exists y \in \mathcal{F}^1$ $s.t.$

$$\begin{cases} f_1(y) \leq f_1(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts (3.12).
In conclusion x_0 is an efficient solution for $\left(P_1^{2,0}\right)$. ■

Theorem 3.4.5 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and invex[1] at x_0 with respect to η,*

e) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_1^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.4.3*. ■

Theorem 3.4.6 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and incave[1] at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{1,0}\right)$.

Proof. Proof is similar with *Theorem 3.4.4*. ■

Theorem 3.4.7 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex[2] at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and invex[1] at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_1^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.4.3*. ■

Theorem 3.4.8 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{2,1}\right)$.

Proof. Proof is similar with *Theorem 3.4.4*. ∎

Theorem 3.4.9 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex^1 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_1^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.4.3*. ∎

Theorem 3.4.10 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and incave*[1] *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*[1] *at* x_0 *with respect to* η,

d) f_1 *is twice differentiable at* x_0 *and incave*[2] *at* x_0 *with respect to* η,

e) f_2 *is twice differentiable at* x_0 *and incave*[2] *at* x_0 *with respect to* η,

f) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $\left(P_0^{0,0}\right)$, *then* x_0 *is an efficient solution for* $\left(P_1^{2,2}\right)$.

Proof. Proof is similar with *Theorem 3.4.4*. ■

In the above theorems, conditions referring to invexity, incavity or avexity of functions are essential to ensure that efficient solution of the initial problem remains efficient for the approximate problem and reciprocally. If those conditions are not fulfill it is possible either that efficient solution of initial problem remains efficient for the approximate problem (and reciprocally) or it does not remain efficient.

Example 3.4.11 (Luca and Duca [95]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ *be:*

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1x_2 + 1 \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ *is* $x_0 = (1,\ 1) \in \mathcal{F}^0$ *and the value of the objective function in* x_0 *is* $f(1,\ 1) = (-1,\ 2)$. *First and second approximate functions for the components of the objective function in* $x_0 = (1,\ 1)$ *are:*

$$F_p^1(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0), p \in \{1, 2\}$$

and

$$F_p^2(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 f_p(x_0)\,\eta(x, x_0), p \in \{1, 2\},$$

while first approximate functions for the constraint is:

$$G_t^1(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0), t \in \{1, 2, 3\}.$$

Considering $\eta(x, x_0) = x - x_0$ *we get:*

$$F_1^i(x) = F_1^i(x) = x_1 - 2x_2, \; i \in \{0, 1, 2\}$$

$$F_2^j(x) = F_2^j(x) = x_1 + x_2, \; j \in \{0, 1, 2\}$$

and

$$G_1^1(x) = -x_1 - x_2 + 2, \; G_2^1(x) = -x_1, \; G_3^1(x) = -x_2$$

Consequently, the approximate problems $\left(P_1^{i,j}\right)$*, with* $(i, j) \in \{(1, 0), (1, 1), (2, 0), (2, 1), (2, 2)\}$ *are:*

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1 - x_2 + 2 \le 0 \\ x_1; x_2 \ge 0. \end{cases}$$

Calculating the value of objective function for problem $\left(P_1^{i,j}\right)$ *in* $x = (0,\ 2) \in \mathcal{F}^1$ *we obtain:*

$$\begin{aligned} &\left(F_1^i, F_2^j\right)(0,\ 2) = (-4,\ 2) \le (-1,\ 2) = \left(F_1^i, F_2^j\right)(1,\ 1) \\ &\left(F_1^i, F_2^j\right)(0,\ 2) \ne \left(F_1^i, F_2^j\right)(1,\ 1) \end{aligned}$$

where $(i, j) \in \{(1, 0), (1, 1), (2, 0), (2, 1), (2, 2)\}$*, which proves that* $x_0 = (1,\ 1) \in \mathcal{F}^1$ *is not an efficient solution for approximate problem* $\left(P_1^{i,j}\right)$*.*

Example 3.4.12 (Luca and Duca [95]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ *be:*

$$\begin{cases} \min\left(x_1^2 + (x_2 - \pi - 1)^2;\ (x_1 + \frac{1}{10})^2 - \frac{1}{2}(x_2 + 1)^2\right) \\ -x_1 - \sin x_1 + x_2 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ *is* $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \in \mathcal{F}^0$ *and the value of the objective function in* x_0 *is* $f(\frac{\pi}{2},\ 1 + \frac{\pi}{2}) = (\frac{\pi^2}{2};\ \frac{\pi^2}{8} - \frac{9\pi}{10} - \frac{199}{100})$*.*

To compute the approximate problem $\left(P_1^{1,1}\right)$ *in* x_0 *we have to calculate:*

$$F_p^1(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0),\ p \in \{1, 2\}$$

and

$$G_t^1(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0),\ t \in \{1, 2, 3, 4\}.$$

Considering $\eta(x, x_0) = x - x_0$ *we get:*

$$F_1^1(x) = \pi x_1 - \pi x_2 + \pi + \frac{\pi^2}{2},$$

$$F_2^1(x) = (\pi + \frac{1}{5})x_1 - (\frac{\pi}{2} + 2)x_2 - \frac{\pi^2}{8} + \frac{\pi}{2} + \frac{1}{100},$$

$$G_1^1(x) = -x_1 + x_2 - 1,\ G_2^1(x) = x_1 - \frac{5\pi}{2},\ G_3^1(x) = -x_1,\ G_4^1(x) = -x_2.$$

Thus, the approximate problem $\left(P_1^{1,1}\right)$ *is:*

$$\begin{cases} \min\left(\pi x_1 - \pi x_2 + \pi + \frac{\pi^2}{2};\ (\pi + \frac{1}{5})x_1 - (\frac{\pi}{2} + 2)x_2 - \frac{\pi^2}{8} + \frac{\pi}{2} + \frac{1}{100}\right) \\ -x_1 + x_2 - 1 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0. \end{cases}$$

Calculating the value for the objective function of problem $\left(P_1^{1,1}\right)$ *in* $x = (\frac{5\pi}{2},\ 1 + \frac{5\pi}{2}) \in \mathcal{F}^1$ *we get:*

$$\begin{aligned} F^1(\frac{5\pi}{2},\ 1 + \frac{5\pi}{2}) &= (\frac{\pi^2}{2},\ \frac{9\pi^2}{8} - \frac{9\pi}{2} - \frac{199}{100}) \\ &\le (\frac{\pi^2}{2};\ \frac{\pi^2}{8} - \frac{9\pi}{10} - \frac{199}{100}) \\ &= F^1(\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \\ F^1(\frac{5\pi}{2},\ 1 + \frac{5\pi}{2}) &\ne F^1(\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \end{aligned}$$

and thus we have proved that $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2})$ *is not an efficient solution for problem* $\left(P_1^{1,1}\right)$.

3.5 First and second η - approximations for components of objective function on the second η - approximation of feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$ and $\left(P_2^{2,2}\right)$ will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Case $\left(P_2^{1,1}\right)$ was studied in [12], where also conditions such that $\mathcal{F}^0 \subseteq \mathcal{F}^2$ and $\mathcal{F}^2 \subseteq \mathcal{F}^0$ were analyzed. We will use them in our work, so we will briefly present the Theorems stating these inclusions.

Theorem 3.5.1 (Duca and Boncea [12]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that:

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^2.$$

Theorem 3.5.2 (Duca and Boncea [12]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^0.$$

Theorem 3.5.3 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

f) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_2^{2,1}\right)$, implies that $\nexists x \in \mathcal{F}^2$ $s.t.$

$$\begin{aligned}(F_1^2(x), F_2^1(x)) &\leq (F_1^2(x_0), F_2^1(x_0))\\ (F_1^2(x), F_2^1(x)) &\neq (F_1^2(x_0), F_2^1(x_0)).\end{aligned}$$

Conditions $b)$ and $c)$ imply that

$$\mathcal{F}^0 \subseteq \mathcal{F}^2$$

and thus $\nexists x \in \mathcal{F}^0$ $s.t.$

$$\begin{aligned}(F_1^2(x), F_2^1(x)) &\leq (F_1^2(x_0), F_2^1(x_0))\\ (F_1^2(x), F_2^1(x)) &\neq (F_1^2(x_0), F_2^1(x_0)).\end{aligned} \tag{3.15}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{0,0}\right)$. Then $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{aligned}(f_1(y), f_2(y)) &\leq (f_1(x_0), f_2(x_0))\\ (f_1(y), f_2(y)) &\neq (f_1(x_0), f_2(x_0))\end{aligned}$$

which implies that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} f_1(y) < f_1(x_0)\\ f_2(y) \leq f_2(x_0)\end{cases} \tag{3.16}$$

or

$$\begin{cases} f_1(y) \leq f_1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3.17}$$

Because f_1 is invex^2 at x_0 with respect to η we get $F_1^2(y) \leq f_1(y)$, $\forall y \in \mathcal{F}^0$. Because f_2 is invex^1 at x_0 with respect to η we get $F_2^1(y) \leq f_2(y)$, $\forall y \in \mathcal{F}^0$. Because $\eta(x_0\, x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$ and $f_2(x_0) = F_2^1(x_0)$. Thus from (3.16) we get that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ F_2^1(y) \leq F_2^1(x_0) \end{cases}$$

which contradicts (3.15) and from (3.17) we get that $\exists y \in \mathcal{F}^0$ $s.t.$

$$\begin{cases} F_1^2(y) \leq F_1^2(x_0) \\ F_2^1(y) < F_2^1(x_0) \end{cases}$$

which contradicts (3.15).

In conclusion x_0 is an efficient solution for $\left(P_0^{0,0}\right)$. ■

Theorem 3.5.4 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*

f) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,1}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{0,0}\right)$, implies that $\nexists x \in \mathcal{F}^0$ $s.t.$

$$\begin{aligned} &\left(f_1\left(x\right), f_2\left(x\right)\right) \leq \left(f_1\left(x_0\right), f_2\left(x_0\right)\right) \\ &\left(f_1\left(x\right), f_2\left(x\right)\right) \neq \left(f_1\left(x_0\right), f_2\left(x_0\right)\right). \end{aligned}$$

Conditions $b)$ and $c)$ imply that

$$\mathcal{F}^2 \subseteq \mathcal{F}^0$$

and thus $\nexists x \in \mathcal{F}^2$ $s.t.$

$$\begin{aligned} &\left(f_1\left(x\right), f_2\left(x\right)\right) \leq \left(f_1\left(x_0\right), f_2\left(x_0\right)\right) \\ &\left(f_1\left(x\right), f_2\left(x\right)\right) \neq \left(f_1\left(x_0\right), f_2\left(x_0\right)\right). \end{aligned} \tag{3.18}$$

Let's assume that x_0 is not an efficient solution for $\left(P_2^{2,1}\right)$. Then $\exists y \in \mathcal{F}^2$ $s.t.$

$$\begin{aligned} &\left(F_1^2\left(y\right), F_2^1\left(y\right)\right) \leq \left(F_1^2\left(x_0\right), F_2^1\left(x_0\right)\right) \\ &\left(F_1^2\left(y\right), F_2^1\left(y\right)\right) \neq \left(F_1^2\left(x_0\right), F_2^1\left(x_0\right)\right) \end{aligned}$$

which implies that $\exists y \in \mathcal{F}^2$ $s.t.$

$$\begin{cases} F_1^2\left(y\right) < F_1^2\left(x_0\right) \\ F_2^1\left(y\right) \leq F_2^1\left(x_0\right) \end{cases} \tag{3.19}$$

or

$$\begin{cases} F_1^2\left(y\right) \leq F_1^2\left(x_0\right) \\ F_2^1\left(y\right) < F_2^1\left(x_0\right). \end{cases} \tag{3.20}$$

Because f_1 is incave^2 at x_0 with respect to η we get $f_1\left(y\right) \leq F_1^2\left(y\right), \forall y \in \mathcal{F}^2$. Because f_2 is incave^1 at x_0 with respect to η we get $f_2\left(y\right) \leq F_2^1\left(y\right), \forall y \in \mathcal{F}^2$. Because $\eta\left(x_0\, x_0\right) = 0$ we get $f_1\left(x_0\right) = F_1^2\left(x_0\right)$ and $f_2\left(x_0\right) = F_2^1\left(x_0\right)$. Thus from (3.19) we get that $\exists y \in \mathcal{F}^2$ $s.t.$

$$\begin{cases} f_1\left(y\right) < f_1\left(x_0\right) \\ f_2\left(y\right) \leq f_2\left(x_0\right) \end{cases}$$

which contradicts (3.18) and from (3.20) we get that $\exists y \in \mathcal{F}^2$ $s.t.$

$$\begin{cases} f_1\left(y\right) \leq f_1\left(x_0\right) \\ f_2\left(y\right) < f_2\left(x_0\right) \end{cases}$$

which contradicts (3.18).

In conclusion x_0 is an efficient solution for $\left(P_2^{2,1}\right)$. ■

Theorem 3.5.5 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_2^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.5.3. ■

Theorem 3.5.6 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and incave^1 at $x_0$with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{1,0}\right)$.

Proof. Proof is similar with *Theorem* 3.5.4. ■

Theorem 3.5.7 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.5.3*. ■

Theorem 3.5.8 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,0}\right)$.

Proof. Proof is similar with *Theorem 3.5.4*. ■

Theorem 3.5.9 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

f) $\eta\,(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem 3.5.3*. ■

Theorem 3.5.10 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

f) $\eta\,(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,2}\right)$.

Proof. Proof is similar with *Theorem 3.5.4*. ■

Example 3.5.11 (Luca and Duca [94]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1-\sin x_1+x_2\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ is $x_0=(\frac{\pi}{2},\ 1+\frac{\pi}{2})\in\mathcal{F}^0$.
Second order approximate functions for the constraints are:

$$G_t^2\left(x\right)=g_t\left(x_0\right)+\nabla g_t\left(x_0\right)\eta\left(x,x_0\right)+\frac{1}{2}\eta\left(x,x_0\right)^T\nabla^2 g_t\eta\left(x,x_0\right),t\in\{1,2,3,4\}$$

Considering $\eta\left(x,x_0\right)=x-x_0$ we get:

$$G_1^2\left(x\right)=-x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1,$$

$$G_2^2\left(x\right)=x_1-\frac{5\pi}{2},\ G_3^2\left(x\right)=-x_1,\ G_4^2\left(x\right)=-x_2.$$

Consequently, the approximate problem $\left(P_2^{0,0}\right)$ is:

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

Calculating the values of objective function for problem $\left(P_2^{0,0}\right)$ in $x_0=(\frac{\pi}{2},\ 1+\frac{\pi}{2})\in\mathcal{F}^2$ and $x=(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32})\in\mathcal{F}^2$ we obtain:

$$f(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32})=(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32});\ 1-\frac{\pi^2}{32})$$

and

$$f(\frac{\pi}{2},\ 1+\frac{\pi}{2})=(-\frac{\pi^2}{50},\ 1).$$

Because $(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32};\ 1-\frac{\pi^2}{32})<(-\frac{\pi^2}{50},\ 1)$ it follows that $x_0=(\frac{\pi}{2},\ 1+\frac{\pi}{2})$ is not an efficient solution for approximate problem $\left(P_2^{0,0}\right)$.

Example 3.5.12 *Let's consider the same initial problem as in Example 3.5.11. First order approximations for the components of the objective function are*

$$F_p^1(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x,x_0),\ p \in \{1,2\}.$$

Considering $\eta(x,x_0) = x - x_0$ *we get:*

$$F_1^1(x) = -\frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5}$$

and

$$F_2^1(x) = -x_1 + x_2.$$

Approximate functions for the constrains are the same computed at Example 3.5.11. Consequently the approximate problem $\left(P_2^{1,1}\right)$ *is:*

$$\begin{cases} \min\left(-\frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5};\ -x_1 + x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0. \end{cases}$$

Calculating the values for the objective function of problem $\left(P_2^{1,1}\right)$ *in* $x_0 = \left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right) \in \mathcal{F}^2$ *and in* $x = \left(\frac{3\pi}{4};\ \frac{3\pi}{4} + 1 - \frac{\pi^2}{32}\right) \in \mathcal{F}^2$ *we get that*

$$F^1\left(\frac{3\pi}{4};\ \frac{3\pi}{4} + 1 - \frac{\pi^2}{32}\right) < F^1\left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right)$$

which proves that $x_0 = \left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right)$ *is not an efficient solution for problem* $\left(P_2^{1,1}\right)$.

Example 3.5.13 (Luca and Duca [94]) .
Let's consider the same initial problem as in Example 3.5.11. Second order approximations for the components of the objective function are

$$F_p^2(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x,x_0) + \frac{1}{2}\eta(x,x_0)^T\,\nabla^2 f_p(x_0)\,\eta(x,x_0),\ p \in \{1,2\}.$$

Considering $\eta(x,x_0) = x - x_0$ *we get:*

$$F_1^2(x) = -\frac{\pi}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2 - 1 - \frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5}$$

and

$$F_2^2(x) = -x_1 + x_2.$$

Approximate functions for the constrains are the same computed at Example 3.5.11. Consequently the approximate problem $(P_2^{2,2})$ is:

$$\begin{cases} \min\left(-\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2-\frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

Calculating the values for the objective function of problem $(P_2^{2,2})$ in $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2}) \in \mathcal{F}^2$ and in $x = (\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}) \in \mathcal{F}^2$ we get that

$$F^2(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}) < F^2(\frac{\pi}{2},\ 1+\frac{\pi}{2})$$

which proves that $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2})$ is not an efficient solution for problem $(P_2^{2,2})$.

3.6 Relations between η - approximation problems for $\left(P_0^{0,0}\right)$

After establishing conditions such that efficient solutions of $(P_k^{i,j})$ remain efficient solutions for $(P_0^{0,0})$ and reciprocally, it is interesting to study relations between different order η - approximation problems.

We will split this part in three subsection which will study relations between $(P_0^{i,j})$ vs $(P_1^{i,j})$, $(P_0^{i,j})$ vs $(P_2^{i,j})$ and $(P_1^{i,j})$ vs $(P_2^{i,j})$.

3.6.1 Relations between $\left(P_0^{i,j}\right)$ and $\left(P_1^{i,j}\right)$

Theorem 3.6.1 (Luca and Duca [97]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{1,0}\right)$, then x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$.*

2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,0}\right)$, then x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$.*

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$, then x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$, then x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$.*

Proof. x_0 being an efficient solution for $\left(P_0^{1,0}\right)$ implies that $\nexists x \in \mathcal{F}^0$ $s.t.$

$$\left(F_1^1(x), f_2(x)\right) \leq \left(F_1^1(x_0), f_2(x_0)\right)$$
$$\left(F_1^1(x), f_2(x)\right) \neq \left(F_1^1(x_0), f_2(x_0)\right).$$

Conditions $b)$ and $c)$ imply that

$$\mathcal{F}^1 \subseteq \mathcal{F}^0$$

which means that $\nexists x \in \mathcal{F}^1$ $s.t.$

$$\left(F_1^1(x), f_2(x)\right) \leq \left(F_1^1(x_0), f_2(x_0)\right)$$
$$\left(F_1^1(x), f_2(x)\right) \neq \left(F_1^1(x_0), f_2(x_0)\right)$$

and thus x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$.
Similar logic is used to prove the statements 2, 3 and 4. ■

Theorem 3.6.2 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and g_t, $h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{1,0}\right)$.*
2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,0}\right)$.*
3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$.*
4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$.*

Proof. Proof is similar with *Theorem 3.6.1*. ■

3.6.2 Relations between $\left(P_0^{i,j}\right)$ and $\left(P_2^{i,j}\right)$

Theorem 3.6.3 (Luca and Duca [97]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{1,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$.*
2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$.*
3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$.*

Proof. Proof is similar with *Theorem 3.6.1*. ■

Theorem 3.6.4 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{1,0}\right)$.*

2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,0}\right)$.*

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$.*

Proof. Proof is similar with *Theorem 3.6.1*. ■

3.6.3 Relations between $\left(P_1^{i,j}\right)$ and $\left(P_2^{i,j}\right)$

Theorem 3.6.5 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that:

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t(x_0)$ is negative semi-definite,*

b) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s(x_0)$ is null definite,*

then

$$\mathcal{F}^1 \subseteq \mathcal{F}^2.$$

Proof. Let $x \in \mathcal{F}^1$.
Then

$$\begin{cases} G_t^1(x) \leq 0, \ \forall t \in T \\ H_s^1(x) = 0, \ \forall s \in S. \end{cases}$$

We know that

$$\begin{aligned} G_t^2(x) &= g_t(x_0) + \nabla g_t(x_0)\,\eta(x,x_0) + \frac{1}{2}\eta(x,x_0)^T \nabla^2 g_t(x_0)\,\eta(x,x_0), \ \forall t \in T \\ H_s^2(x) &= h_s(x_0) + \nabla h_s(x_0)\,\eta(x,x_0) + \frac{1}{2}\eta(x,x_0)^T \nabla^2 h_s(x_0)\,\eta(x,x_0), \ \forall s \in S. \end{aligned}$$

Because $\nabla^2 g_t(x_0)$ is negative semi-definite, it follows that

$$G_t^2(x) \leq G_t^1(x), \ \forall t \in T.$$

Because $\nabla^2 h_s(x_0)$ is null definite, it follows that

$$H_s^2(x) = H_s^1(x), \ \forall s \in S.$$

We get

$$\begin{cases} G_t^2(x) \leq 0, \ \forall t \in T \\ H_s^2(x) = 0, \ \forall s \in S, \end{cases}$$

which implies that $x \in \mathcal{F}^2$ and thus $\mathcal{F}^1 \subseteq \mathcal{F}^2$. ∎

Theorem 3.6.6 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.
If

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t(x_0)$ is positive semi-definite,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and $\nabla^2 h_s(x_0)$ is null definite,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^1.$$

Proof. Let $x \in \mathcal{F}^2$.
Then

$$\begin{cases} G_t^2(x) \leq 0, \ \forall t \in T \\ H_s^2(x) = 0, \ \forall s \in S. \end{cases}$$

We know that

$$\begin{aligned} G_t^2(x) &= g_t(x_0) + \nabla g_t(x_0)\, \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 g_t(x_0)\, \eta(x, x_0), \ \forall t \in T \\ H_s^2(x) &= h_s(x_0) + \nabla h_s(x_0)\, \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 h_s(x_0)\, \eta(x, x_0), \ \forall s \in S. \end{aligned}$$

Because $\nabla^2 g_t(x_0)$ is positive semi-definite, it follows that

$$G_t^1(x) \leq G_t^2(x), \ \forall t \in T.$$

Because $\nabla^2 h_s(x_0)$ is null definite, it follows that

$$H_s^1(x) = H_s^2(x), \ \forall s \in S.$$

We get

$$\begin{cases} G_t^1(x) \leq 0, \ \forall t \in T \\ H_s^1(x) = 0, \ \forall s \in S, \end{cases}$$

which implies that $x \in \mathcal{F}^1$ and thus $\mathcal{F}^2 \subseteq \mathcal{F}^1$. ■

Theorem 3.6.7 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t, \ h_s : X \to \mathbb{R}$, $(t \in T, \ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t(x_0)$ is positive semi-definite,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s(x_0)$ is null definite.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$.*
2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$.*
3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$.*
4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$.*

Proof. Proof is similar with *Theorem 3.6.1*. ■

Theorem 3.6.8 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t(x_0)$ is negative semi-definite,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s(x_0)$ is null definite.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$.*
2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$.*
3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$.*

Proof. Proof is similar with *Theorem 3.6.1*. ■

3.7 Conclusions

This chapter was dedicated to study of some conditions such that efficient solution of a bi-criteria optimization problem remains efficient also for the approximate problem and reciprocally. Practical application of this research is visible in situation when a highly complex bi-criteria optimization problem has to be solved. It might be replaced with an approximate problem, easier to be solved and under certain conditions (referring to invexity, incavity or avexity of functions involved) the efficient solution of one problem remains efficient also for the other.
To be noticed that in case of the above mentioned conditions are not satisfied, the efficient solution of one problem might remain efficient for the other one or not.
Our research was divided and presented in three sections, dedicated to different type of approximate problems associated to the initial one and another section dedicated to the study of relations which might be established between different type of approximate problems.
We have studied conditions such that efficient solution of the initial problem $\left(P_0^{0,0}\right)$ remains efficient for the following type of approximate problems $\left(P_0^{i,j}\right)$, $\left(P_1^{i,j}\right)$, $\left(P_2^{i,j}\right)$, where $(i,j) \in \{(1,0),(1,1),(2,0),(2,1),(2,2)\}$ and reciprocally.
Some counter-examples, showing the importance of "invexity-type conditions" are presented. Our contributions to this Chapter, disseminated in [96], [95], [94] and [97] consisting of 32 Theorems and 7 counter-examples, might be synthesized as:

- 8 Theorems for conditions such that efficient solution of problem $\left(P_0^{0,0}\right)$ remains efficient for problems $\left(P_0^{1,0}\right)$, $\left(P_0^{2,0}\right)$, $\left(P_0^{2,1}\right)$, $\left(P_0^{2,2}\right)$ and reciprocally: *Theorems 3.3.1, 3.3.2, 3.3.5, 3.3.6, 3.3.9, 3.3.10, 3.3.11, 3.3.12*;

- 8 Theorems for conditions such that efficient solution of problem $\left(P_0^{0,0}\right)$ remains efficient for problems $\left(P_1^{1,0}\right)$, $\left(P_1^{2,0}\right)$, $\left(P_1^{2,1}\right)$, $\left(P_1^{2,2}\right)$ and reciprocally: *Theorems 3.4.3, 3.4.4, 3.4.5, 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.10*;

- 8 Theorems for conditions such that efficient solution of problem $\left(P_0^{0,0}\right)$ remain efficient for problems $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$, $\left(P_2^{2,2}\right)$ and reciprocally: *Theorems 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10*;

  8 Theorems to study relations between different order approximate problems: *Theorems 3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6, 3.6.7, 3.6.8*;

- 7 counter-examples to emphasize the importance of "invexity-type conditions": *Examples 3.3.3, 3.3.7, 3.4.11, 3.4.12, 3.5.11, 3.5.12, 3.5.13*.

Chapter 4

Minimax model for energy optimization

4.1 Introduction

This chapter is presenting minimax, our first energy optimization model. It is a bi-criteria problem, which aims to shave the peak load by minimizing fluctuation of energy and maximizing economic performance of power plant. Its efficient solution represents the amount of energy to be produced at each time moment, such that fluctuation is minimized and turnover maximized. Additionally, this solution owns the capacity to shave the peak load and thus the objective of our research will be fulfilled.

Targeting the peak load and shaving it require a proper measure for fluctuation of energy. Previous chapter shows that maximum absolute deviation might be a starting point to develop that measure.

Economic literature and practice are considering profit, turnover, cost or rate of return as possible measures for economic performance of a company. When choosing the appropriate measure, we have to consider which data will be required to perform tests for validating minimax. Of course profit would provide the most accurate solution, but access to specific costs of production and storage is impossible for us at this moment. Therefor we will use turnover, being aware that accuracy of solution will suffer.

A simple technical constraint, which limits the amount of energy to be produced at each time moment, will be employed. More complex technical constraints require specific knowledge and competences in energy field.

To solve the minimax model, we will transform it in an equivalent parametric problem, which will be solved using Kuhn-Tucker conditions. Convexity of parametric problem is ensuring that Kuhn-Tucker conditions are both necessary and sufficient.

Tests on real data will be performed to evaluate performances of minimax and to validate it.

4.2 Minimax measure for fluctuation of energy

In our attempt to shave the peak load, we will try to push it down. Thus we will need a measure for fluctuation able to target the peak load and when fluctuation will be minimized the magnitude of peak load will decrease, generating the desired shaving.

From the three measures evaluated in Chapter 2, only maximum absolute deviation proved to be suitable for addressing the peak load.

Energy is a special commodity, with a price very sensitive to quantity, especially around peak load range, as it was exemplified during the heat wave from 2011 in Texas [132]. Is this hypothesis regarding price confirmed also by other studies? According to [152] own-price elasticity for energy has short-run values of -0.12 (year 1), -0.21 (year 2) and -0.24 (year 3) and long- run value of -0.4 (year 25). The meaning of them is that a price increase of 10% will generate for example a 1.2% decrease in consumption over the next year. Deryugina et all [35] have presented, during a conference hosted by Energy Institute at Haas Berkeley, that some evidence were found to sustain a symmetric demand response for price increase and price decrease. This analysis is providing enough arguments, for short and long time horizon, to include price of energy in the measure for fluctuation of energy. It will balance between low consumption hours and high consumption hours and will support the optimization process. Measuring unit for fluctuation of energy will be money. It has a connection with practice, because when a customer is modifying an order, a manager is calculating the impact in money and not in quantity (pieces).

Let's consider a predefined level of energy around which we aim to minimize fluctuation of energy.

Definition 4.2.1 *Minimax fluctuation of energy is the maximum, over all time periods, for difference between energy produced at certain time moments and a predefined level of energy, multiplied with price of energy at the corresponding time moment.*

Remark 4.2.2 *Predefined level of energy might be for example a random value chosen by energy plant or the average amount of energy produced during a certain period from the past. Of course, when the average is employed, it might be necessary to adjust it with a factor to cover the projected increase of demand.*

Denoting by $1, 2, ..., i, ..., n$ the time horizon considered and

x_i - energy produced at time moment i, $i = \overline{1,n}$,

p_i - price of energy at time moment i, $i = \overline{1,n}$,

r - predefined level of energy,

ε - minimum level of energy assumed by the power plant to be delivered in the power grid,

ρ - maximum level of energy assumed by the power plant to be delivered in the power grid,

and considering that $\epsilon \leq r \leq \rho$ we have the following mathematical expression for *minimax fluctuation of energy*

$$\max_{i=\overline{1,n}} |p_i x_i - p_i r| . \tag{4.1}$$

4.3 Problem formulation

An energy plant focuses the problem of determining the amount of energy to be produced at certain time moments, such that peak load will be shaved by minimizing fluctuation of energy and maximizing economic performance.

It is a bi-criteria problem, where fluctuation has to be minimized and economic performance maximized.

As measure for fluctuation of energy we will use minimax measure (4.1), while as measure for economic performance we will use turnover. Reason for choosing turnover was explained in Section 3.1 (Introduction).

We will use technical constraints which refer to boundaries of energy produced at certain time moments. More advanced technical constraints require specific knowledge and competences in energy field.

Thus we have:

turnover

$$\sum_{i=1}^{n} p_i x_i$$

constraints

$$\varepsilon \leq x_i \leq \rho, \; i = \overline{1,n}$$

and using minimax measure for fluctuation of energy (4.1) we obtain the following mathematical model for our problem

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} \left| p_i x_i - p_i r \right|, \; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.2}$$

and define it *minimax energy model*.

4.4 Computing the solution

We recall that for a problem

$$\begin{cases} \min f(x) \\ x \in X \end{cases}$$

where $X \subseteq \mathbb{R}^n$ and $f = (f_1, f_2, ...f_m)^T : X \to \mathbb{R}^m$, a feasible solution $x^* \in X$ is said to be efficient solution if $\nexists x \in X$ such that

$$\begin{aligned} f(x) &\leq f(x^*) \\ f(x) &\neq f(x^*). \end{aligned}$$

In order to determine the efficient solution for problem (4.2) we will introduce the following bi-criteria equivalent problem

$$\begin{cases} \min \left(y, \ -\sum_{i=1}^{n} p_i x_i \right)^T \\ |p_i x_i - p_i r| \leq y, \quad i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.3}$$

Equivalence between problems (4.2) and (4.3) is shown in the following Lemma.

Lemma 4.4.1 *[98] Let's consider the bi-criteria optimization problems (4.2) and (4.3).*

a) If $x \in \mathbb{R}^n$ is an efficient solution for problem (4.2), then $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} |p_i x_i - p_i r|$ is an efficient solution for problem (4.3).

b) If $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} |p_i x_i - p_i r|$ is an efficient solution for problem (4.3), then $x \in \mathbb{R}^n$ is an efficient solution for problem (4.2).

Proof.

a) Let $x \in \mathbb{R}^n$ be an efficient solution for problem (4.2).

Then $\nexists x^0 \in \mathbb{R}^n$, with $\varepsilon \leq x_i^0 \leq \rho, \ i = \overline{1,n}$ such that

$$\begin{aligned} \max_{i=\overline{1,n}} \left| p_i x_i^0 - p_i r \right| &\leq \max_{i=\overline{1,n}} |p_i x_i - p_i r| \\ \sum_{i=1}^{n} p_i x_i^0 &\geq \sum_{i=1}^{n} p_i x_i, \ i = \overline{1,n} \end{aligned}$$

and at least one inequality holds strictly.

Suppose $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} |p_i x_i - p_i r|$ is not an efficient solution for problem (4.3).

Then there exists $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}$ a feasible solution for problem (4.3), with

$$\begin{aligned} y^0 &\leq y \\ \sum_{i=1}^{n} p_i x_i^0 &\geq \sum_{i=1}^{n} p_i x_i, \ i = \overline{1,n} \end{aligned}$$

and at least one inequality holds strictly.

Because $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}$ is a feasible solution for problem (4.3), it follows that

$$\left|p_i x_i^0 - p_i r\right| \leq y^0,\ i = \overline{1,n}$$

which means that

$$\max_{i=\overline{1,n}} \left|p_i x_i^0 - p_i r\right| \leq y^0.$$

Thus we obtain that

$$\begin{aligned} \max_{i=\overline{1,n}} \left|p_i x_i^0 - p_i r\right| &\leq y^0 \leq y = \max_{i=\overline{1,n}} \left|p_i x_i - p_i r\right| \\ \sum_{i=1}^{n} p_i x_i^0 &\geq \sum_{i=1}^{n} p_i x_i,\ i = \overline{1,n} \end{aligned}$$

and at least one inequality holds strictly.

This way we obtain a contradiction for the efficiency of $x \in \mathbb{R}^n$ as solution for (4.2).

In conclusion, $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max_{i=\overline{1,n}} \left|p_i x_i - p_i r\right|$ is an efficient solution for problem (4.3).

b) Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max_{i=\overline{1,n}} \left|p_i x_i - p_i r\right|$ be an efficient solution for problem (4.3).

Suppose $x \in \mathbb{R}^n$ is not an efficient solution for problem (4.2) and let x^0 be a feasible solution, with

$$\begin{aligned} \max_{i=\overline{1,n}} \left|p_i x_i^0 - p_i r\right| &\leq \max_{i=\overline{1,n}} \left|p_i x_i - p_i r\right|,\ i = \overline{1,n} \\ \sum_{i=1}^{n} p_i x_i^0 &\geq \sum_{i=1}^{n} p_i x_i,\ i = \overline{1,n} \end{aligned}$$

and at least one inequality holds strictly.

Denoting $y^0 = \max_{i=\overline{1,n}} \left|p_i x_i^0 - p_i r\right|$, it follows

$$\begin{aligned} y^0 &\leq y \\ \sum_{i=1}^{n} p_i x_i^0 &\geq \sum_{i=1}^{n} p_i x_i,\ i = \overline{1,n} \end{aligned}$$

and at least one inequality holds strictly, which contradicts the efficiency of $(x, y) \in \mathbb{R}^n \times \mathbb{R}$.

In conclusion $x \in \mathbb{R}^n$ is an efficient solution for problem (4.2) and this ends our proof.

■

Using Theorem 2.3.1 of Yu [160] and similar results of Bot et all [14] and Geoffrion [53] the bi-criteria problem (4.3) is equivalent to the following parametric optimization problem

$$\begin{cases} \min \left\{ \lambda y - (1-\lambda) \sum\limits_{i=1}^{n} p_i x_i \right\} \\ |p_i x_i - p_i r| \leq y, \quad i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.4}$$

with $\lambda \in (0, 1)$ and the following Lemma holds

Lemma 4.4.2 *[98] $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is an efficient solution for bi-criteria problem (4.3) if and only if $\exists \lambda \in (0, 1)$ such that $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution for parametric optimization problem (4.4)*

The meaning of λ in this context is the sensitivity of energy plant for reducing the fluctuation. The bigger λ is, the energy plant is more interested in reducing the fluctuation. The smaller λ is, the energy plant is less interested in reducing the fluctuation and more interested in increasing the turnover.

Remark 4.4.3 *Considering the equivalence between problems (4.2) and (4.3), respectively problems (4.3) and (4.4), it follows from transitivity that problems (4.2) and (4.4) are equivalent. This means that in order to compute the efficient solution for (4.2) we have to determine the optimal solution for (4.4).*

Remark 4.4.4 *In the process of computing the optimal solution, we will split the set $\{1, 2, ..., n\}$ in subsets like $\{1, 2, ..., l\}$ and $\{l + 1, l + 2, ..., n\}$, or $\{1, 2, ..., l\}$, $\{l + 1, l + 2, ..., m\}$ and $\{m + 1, m + 2, ..., n\}$. If price is constant on such a set or subset, it will be denoted by $\overline{p}$.*

Theorem 4.4.5 (Luca and Mahalov [98]; *parametric minimax*) .
The optimal solutions for parametric optimization problem (4.4) are:

1. *If* $\lambda < \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

or

- *if* $\overline{p_1} \le p_j,\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

2. *If* $\lambda = \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i\,(\rho - r)\}\,. \end{cases}$$

3. *If* $\lambda > \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0. \end{cases}$$

4. *If* $\lambda < \frac{l}{l+1}$, *then*

- *if* $p_j < \overline{p},\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

5. *If* $\lambda = \frac{l}{l+1}$, *then*

- *if* $p_j < p_i, i = \overline{1,l}, \;\; j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \{p_i(\rho - r)\}, \; if \; p_j < p_i. \end{cases}$$

- *else problem has no solution*

6. *If* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, *then*

- *if* $p_j \leq \overline{p} \leq p_i, \; i = \overline{1,l}, \; j = \overline{l+1,m}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p}(\rho - r) \end{cases}$$

where $\overline{p_3} = p_k, \; k = \overline{m+1,n}$.

- *else problem has no solution.*

The proof of this theorem is a 4 steps process.

At first step, for a fixed i from $\{1, 2, ..., n\}$ all combinations for Kuhn-Tucker multipliers will be identified and their behavior related to complementarity slackness and dual feasibility conditions ((KT3) to (KT6)) will be analyzed. The end result for this step will be the *possible combinations.*

Definition 4.4.6 Possible combinations *are the combinations of Kuhn-Tucker multipliers, determined for a fixed* $i \in \{1, 2, ..., n\}$ *for which complementarity slackness and dual feasibility conditions are fulfilled.*

At step two, for $i = \overline{1,n}$, the behavior of possible combinations related to the gradient of Lagrangian ((KT1) and (KT2)) is analyzed. The end result are the *feasible combinations.*

Definition 4.4.7 Feasible combinations *are those possible combinations for which the gradient of Lagrangian is zero.*

At step three, for $i = \overline{1,n}$, the combining capacity of feasible combinations is analyzed. The end result are the *critical combinations.*

Definition 4.4.8 Critical combinations *are those feasible combinations for which a solution does not exist if they are combined.*

At step four, the optimal solutions are computed based on feasible and critical combinations.

Proof. Let $\lambda \in (0,1)$ fixed. Problem (4.4) is equivalent with the following

$$\begin{cases} \min\left\{\lambda y - (1-\lambda)\sum\limits_{i=1}^{n} p_i x_i\right\} \\ -y \leq p_i x_i - p_i r, \quad i = \overline{1,n} \\ p_i x_i - p_i r \leq y, \quad i = \overline{1,n} \\ \varepsilon \leq x_i, \quad i = \overline{1,n} \\ x_i \leq \rho, \quad i = \overline{1,n} \end{cases}$$

which is a convex optimization problem with inequality constraints.

The associated Kuhn-Tucker conditions are

$$\frac{\partial L}{\partial x_i} = -(1-\lambda)p_i - a_i p_i + b_i p_i - c_i + d_i = 0, \quad i = \overline{1,n} \tag{KT1}$$

$$\frac{\partial L}{\partial y} = \lambda - \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0 \tag{KT2}$$

$$(-y^* - p_i x_i^* + p_i r)\, a_i = 0, \quad a_i \geq 0, \quad i = \overline{1,n} \tag{KT3}$$

$$(p_i x_i^* - p_i r - y^*)\, b_i = 0, \quad b_i \geq 0, \quad i = \overline{1,n} \tag{KT4}$$

$$(\varepsilon - x_i^*)\, c_i = 0, \quad c_i \geq 0, \quad i = \overline{1,n} \tag{KT5}$$

$$(x_i^* - \rho)\, d_i = 0, \quad d_i \geq 0, \quad i = \overline{1,n} \tag{KT6}$$

where

$$\begin{aligned} L &: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \\ L(x,y,a,b,c,d) &= \lambda y - (1-\lambda)\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} a_i(-y - p_i x_i + p_i r) + \\ &\quad + \sum_{i=1}^{n} b_i (p_i x_i - p_i r - y) + \\ &\quad + \sum_{i=1}^{n} c_i(\varepsilon - x_i) + \sum_{i=1}^{n} d_i(x_i - \rho) \end{aligned}$$

is the associated Lagrangian and $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ is the optimal solution.

Remark 4.4.9 *Due to the fact that the optimization problem is a convex one, it follows that Kuhn-Tucker conditions are both necessary and sufficient.*

We have to compute $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ in order to determine the optimal solution for our parametric problem (4.4).

Step 1

Let $i \in \{1, 2, ...n\}$ fixed. Then, the possible scenarios for Kuhn-Tucker multipliers are presented in Tabel 4.1

	Kuhn-Tucker multipliers			
Scenarios	a_i	b_i	c_i	d_i
1	=0	=0	=0	=0
2	> 0	=0	=0	=0
3	=0	> 0	=0	=0
4	=0	=0	> 0	=0
5	=0	=0	=0	> 0
6	> 0	> 0	=0	=0
7	> 0	=0	> 0	=0
8	> 0	=0	=0	> 0
9	=0	> 0	> 0	=0
10	=0	> 0	=0	> 0
11	=0	=0	> 0	> 0
12	> 0	> 0	> 0	=0
13	> 0	> 0	=0	> 0
14	=0	> 0	> 0	> 0
15	> 0	=0	> 0	> 0
16	> 0	> 0	> 0	> 0

TABLE 4.1: Combination of KT multipliers for minimax model

We will analyze the behavior of each scenario related to complementarity slackness and dual feasibility conditions (Kuhn-Tucker conditions (KT3) to (KT6)) and we will determine the solution for each scenario if it exists.

Scenario 1

$a_i = 0$	$b_i = 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \leq p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* \geq \varepsilon \\ x_i^* \leq \rho. \end{cases} \tag{4.5}$$

From the first two inequalities of the system we have

$$\begin{aligned} x_i^* &\geq r - \frac{y^*}{p_i} \\ x_i^* &\leq r + \frac{y^*}{p_i} \end{aligned}$$

and considering the last two inequalities of the system it follows that

$$\begin{aligned} y^* &\leq p_i (r - \varepsilon) \\ y^* &\leq p_i (\rho - r). \end{aligned}$$

Thus, the solution for system (4.5) is

$$\begin{cases} x_i^* \in \left[r - \frac{y^*}{p_i},\ r + \frac{y^*}{p_i} \right] \\ y^* \leq p_i (r - \varepsilon) \\ y^* \leq p_i (\rho - r). \end{cases}$$

Scenario 2

$a_i > 0$	$b_i = 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* \geq \varepsilon \\ x_i^* \leq \rho. \end{cases}$$

From the first equation of the system we have

$$x_i^* = r - \frac{y^*}{p_i}$$

and replacing it in the other 3 inequations it follows

$$\begin{cases} p_i\left(r-\frac{y^*}{p_i}\right)-p_ir\le y^* \\ r-\frac{y^*}{p_i}\ge\varepsilon \\ r-\frac{y^*}{p_i}\le\rho \end{cases}$$

which is equivalent to

$$\begin{cases} -y^*\le y^* \\ y^*\le p_i\left(r-\varepsilon\right) \\ y^*\ge p_i\left(r-\rho\right). \end{cases}$$

Because first and third relations are obvious, the following solution is obtained

$$\begin{cases} x_i^*=r-\frac{y^*}{p_i} \\ y^*\le p_i\left(r-\varepsilon\right). \end{cases}$$

Scenario 3

$a_i=0$	$b_i>0$	$c_i=0$	$d_i=0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^*\le p_ix_i^*-p_ir \\ p_ix_i^*-p_ir=y^* \\ x_i^*\ge\varepsilon \\ x_i^*\le\rho. \end{cases}$$

From the second equation of the system we have

$$x_i^*=r+\frac{y^*}{p_i}$$

and replacing it in the other 3 inequations it follows

$$\begin{cases} -y^*\le p_i\left(r+\frac{y^*}{p_i}\right)-p_ir \\ r+\frac{y^*}{p_i}\ge\varepsilon \\ r+\frac{y^*}{p_i}\le\rho \end{cases}$$

which is equivalent to

$$\begin{cases} -y^*\le y^* \\ y^*\ge p_i\left(\varepsilon-r\right) \\ y^*\le p_i\left(\rho-r\right). \end{cases}$$

Because first two relations are obvious, the following solution is obtained

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i} \\ y^* \le p_i \left(\rho - r\right). \end{cases}$$

Scenario 4

$a_i = 0$	$b_i = 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \le p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \le y^* \\ x_i^* = \varepsilon \\ x_i^* \le \rho \end{cases}$$

Considering

$$x_i^* = \varepsilon$$

it's obvious that 4th inequality of the system is satisfied. Replacing in the first two it follows

$$\begin{cases} -y^* \le p_i \left(\varepsilon - r\right) \\ p_i \left(\varepsilon - r\right) \le y^* \end{cases}$$

which is equivalent to

$$\begin{cases} y^* \ge p_i \left(r - \varepsilon\right) \\ y^* \ge p_i \left(\varepsilon - r\right). \end{cases}$$

Because the last relation is obvious, the solution for the system is

$$\begin{cases} x_i^* = \varepsilon \\ y^* \ge p_i \left(r - \varepsilon\right). \end{cases}$$

Scenario 5

$a_i = 0$	$b_i = 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \le p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \le y^* \\ x_i^* \ge \varepsilon \\ x_i^* = \rho \end{cases}$$

Considering

$$x_i^* = \rho$$

it's obvious that 3rd inequality of the system is satisfied. Replacing in the first two it follows

$$\begin{cases} -y^* \leq p_i\left(\rho - r\right) \\ p_i\left(\rho - r\right) \leq y^* \end{cases}$$

which is equivalent to

$$\begin{cases} y^* \geq p_i\left(r - \rho\right) \\ y^* \geq p_i\left(\rho - r\right). \end{cases}$$

Because the first relation is obvious, the solution for the system is

$$\begin{cases} x_i^* = \rho \\ y^* \geq p_i\left(\rho - r\right). \end{cases}$$

Scenario 6

$a_i > 0$	$b_i > 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* \geq \varepsilon \\ x_i^* \leq \rho \end{cases}$$

Adding the first two equations of the system we obtain that

$$x_i^* = r$$

and

$$y^* = 0.$$

It's obvious that $x_i^* = r$ verifies the last two inequalities of the system and thus the solution is

$$\begin{cases} x_i^* = r \\ y^* = 0. \end{cases}$$

Scenario 7

$a_i > 0$	$b_i = 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* = \varepsilon \\ x_i^* \leq \rho \end{cases}$$

Considering

$$x_i^* = \varepsilon$$

it's obvious that 4th inequality of the system is satisfied. Replacing in the first equality it follows

$$-y^* = p_i (\varepsilon - r)$$

which is equivalent to

$$y^* = p_i (r - \varepsilon)$$

Replacing in second inequality of the system it follows

$$p_i(\varepsilon - r) \leq y^* = p_i(r - \varepsilon)$$

which is obvious and therefore the solution is

$$\begin{cases} x_i^* = \varepsilon \\ y^* = p_i(r - \varepsilon) . \end{cases}$$

Scenario 8

$a_i > 0$	$b_i = 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* \geq \varepsilon \\ x_i^* = \rho \end{cases}$$

Considering

$$x_i^* = \rho$$

it's obvious that 3rd inequality of the system is satisfied. Replacing in the first equality of the system it follows

$$-y^* = p_i(\rho - r)$$

which is equivalent to

$$y^* = p_i(r - \rho)$$

From second inequality of the system we obtain that

$$p_i(\rho - r) \leq y^* = p_i(r - \rho)$$

which is a contradiction and thus system has no solution.

Scenario 9

$a_i = 0$	$b_i > 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \leq p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* = \varepsilon \\ x_i^* \leq \rho \end{cases}$$

Considering

$$x_i^* = \varepsilon$$

it's obvious that 4th inequality of the system is satisfied. Replacing in the second equality of the system it follows

$$y^* = p_i (\varepsilon - r)$$

which is equivalent to

$$-y^* = p_i (r - \varepsilon)$$

From first inequality of the system we obtain that

$$p_i (r - \varepsilon) = -y^* \leq p_i (\varepsilon - r)$$

which is a contradiction and thus system has no solution.

Scenario 10

$a_i = 0$	$b_i > 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \leq p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* \geq \varepsilon \\ x_i^* = \rho \end{cases}$$

Considering

$$x_i^* = \rho$$

it's obvious that 3rd inequality of the system is satisfied. Replacing in the second equality of the system it follows

$$y^* = p_i (\rho - r)$$

which is equivalent to

$$-y^* = p_i (r - \rho)$$

From first inequality of the system we obtain that

$$p_i (r - \rho) = -y^* \leq p_i (\rho - r)$$

which is obvious and thus the solution for the system is

$$\begin{cases} x_i^* = \rho \\ y^* = p_i \left(\rho - r \right). \end{cases}$$

Scenario 11

$a_i = 0$	$b_i = 0$	$c_i > 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \leq p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* = \varepsilon \\ x_i^* = \rho \end{cases}$$

From 3rd and 4th equations it follows that

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

Scenario 12

$a_i > 0$	$b_i > 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* = \varepsilon \\ x_i^* \leq \rho \end{cases}$$

Considering

$$x_i^* = \varepsilon$$

it's obvious that 4th inequality of the system is satisfied. Replacing in the first equality of the system it follows

$$-y^* = p_i \left(\varepsilon - r \right)$$

which is equivalent to

$$y^* = p_i \left(r - \varepsilon \right)$$

From second equality of the system we obtain that

$$y^* = p_i \left(\varepsilon - r \right)$$

which is a contradiction and thus system has no solution.

Scenario 13

$a_i > 0$	$b_i > 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* \geq \varepsilon \\ x_i^* = \rho \end{cases}$$

Considering

$$x_i^* = \rho$$

it's obvious that 3rd inequality of the system is satisfied. Replacing in the first equality of the system it follows

$$-y^* = p_i (\rho - r)$$

which is equivalent to

$$y^* = p_i (r - \rho)$$

From second equality of the system we obtain that

$$y^* = p_i (\rho - r)$$

which is a contradiction and thus system has no solution.

Scenario 14

$a_i = 0$	$b_i > 0$	$c_i > 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* \leq p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* = \varepsilon \\ x_i^* = \rho \end{cases}$$

From 3rd and 4th equations it follows that

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

Scenario 15

$a_i > 0$	$b_i = 0$	$c_i > 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r \leq y^* \\ x_i^* = \varepsilon \\ x_i^* = \rho \end{cases}$$

From 3rd and 4th equations it follows that

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

Scenario 16

$a_i > 0$	$b_i > 0$	$c_i > 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

$$\begin{cases} -y^* = p_i x_i^* - p_i r \\ p_i x_i^* - p_i r = y^* \\ x_i^* = \varepsilon \\ x_i^* = \rho \end{cases}$$

From 3rd and 4th equations it follows that

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

From the 16 scenarios, we have proved that only 8 are *possible combinations*. These are 1, 2, 3, 4, 5, 6, 7 and 10.

Step 2

For $i = \overline{1,n}$ we will analyze the behavior of possible combinations related to the gradient of Lagrangian (Kuhn-Tucker conditions (KT1) and (KT2)).

Scenario 1

$a_i = 0$	$b_i = 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\, p_i = 0, \quad i = \overline{1,n} \\ \lambda = 0 \end{cases}$$

and thus $\lambda = 0$ and $p_i = 0,\ i = \overline{1,n}$, which is impossible.

Scenario 2

$a_i > 0$	$b_i = 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i - a_i p_i = 0, & i = \overline{1,n} \\ \lambda - \sum\limits_{i=1}^{n} a_i = 0 \end{cases}$$

and thus $a_i = -(1-\lambda) < 0$, which is impossible.

Scenario 3

$a_i = 0$	$b_i > 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i + b_i p_i = 0, & i = \overline{1,n} \\ \lambda - \sum\limits_{i=1}^{n} b_i = 0. \end{cases}$$

Then $b_i = (1-\lambda)\,,\ i = \overline{1,n}$, and thus $\sum\limits_{i=1}^{n} b_i = n\,(1-\lambda)\,.$ Replacing in the last equation of the system it follows that

$$\lambda = \frac{n}{n+1}.$$

Scenario 4

$a_i = 0$	$b_i = 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i - c_i = 0, & i = \overline{1,n} \\ \lambda = 0. \end{cases}$$

and thus $c_i = -(1-\lambda)\,p_i < 0,\ i = \overline{1,n}$, which is impossible.

Scenario 5

$a_i = 0$	$b_i = 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i + d_i = 0, & i = \overline{1,n} \\ \lambda = 0. \end{cases}$$

and thus $d_i = (1-\lambda)\,p_i,\ i = \overline{1,n}$. But $\lambda = 0$ and the system has no solution.

Scenario 6

$a_i > 0$	$b_i > 0$	$c_i = 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i - a_i p_i + b_i p_i = 0, \quad i = \overline{1,n} \\ \lambda - \sum\limits_{i=1}^{n} a_i - \sum\limits_{i=1}^{n} b_i = 0 \end{cases}$$

and thus

$$a_i = b_i - (1-\lambda)\,,\ i = \overline{1,n}.$$

It follows that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i - n\,(1-\lambda)\,.$$

Replacing in the last equation we obtain

$$\lambda - \sum_{i=1}^{n} b_i + n\,(1-\lambda) - \sum_{i=1}^{n} b_i = 0$$

which is equivalent to

$$\lambda + n\,(1-\lambda) - 2\sum_{i=1}^{n} b_i = 0.$$

But $a_i > 0,\ i = \overline{1,n}$ and then $b_i > (1-\lambda)\,,\ i = \overline{1,n}$, which means that

$$\sum_{i=1}^{n} b_i > n\,(1-\lambda)\,.$$

Thus we have

$$\lambda + n\,(1-\lambda) - 2n\,(1-\lambda) > 0$$

equivalent to

$$\lambda - n\,(1-\lambda) > 0.$$

In conclusion

$$\lambda > \frac{n}{n+1}.$$

Scenario 7

$a_i > 0$	$b_i = 0$	$c_i > 0$	$d_i = 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i - a_i p_i - c_i = 0, & i=\overline{1,n} \\ \lambda - \sum\limits_{i=1}^{n} a_i = 0 \end{cases}$$

and thus $c_i = -p_i\,[a_i + (1-\lambda)] < 0,\ i=\overline{1,n}$, which is impossible.

Scenario 10

$a_i = 0$	$b_i > 0$	$c_i = 0$	$d_i > 0$

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

$$\begin{cases} -(1-\lambda)\,p_i + b_i p_i + d_i = 0, & i=\overline{1,n} \\ \lambda - \sum\limits_{i=1}^{n} b_i = 0. \end{cases}$$

Then

$$d_i = p_i\,[(1-\lambda) - b_i]\,,\ i=\overline{1,n}$$

and because $d_i > 0, i=\overline{1,n}$ it follows

$$b_i < (1-\lambda)$$

and thus

$$\sum_{i=1}^{n} b_i < n\,(1-\lambda)\,.$$

From the last equation of the system we obtain

$$\lambda < \frac{n}{n+1}.$$

Synthesizing the behavior of the 8 possible combinations, related to Kuhn-Tucker conditions (KT1) and (KT2), the situation presented in Tabel 4.2 is obtained

Scenario	Solution	(KT1)	(KT2)
1	$\nexists$	$\times$	$\times$
2	$\nexists$	$\times$	$\times$
3	$\exists,\ if\ \lambda = \frac{n}{n+1}$	$\checkmark$	$\checkmark$
4	$\nexists$	$\times$	$\times$
5	$\nexists$	$\checkmark$	$\times$
6	$\exists,\ if\ \lambda > \frac{n}{n+1}$	$\checkmark$	$\checkmark$
7	$\nexists$	$\times$	$\times$
10	$\exists,\ if\ \lambda < \frac{n}{n+1}$	$\checkmark$	$\checkmark$

TABLE 4.2: Behavior of possible combinations related to gradient of Lagrangian for minimax model

Remark 4.4.10 *For scenario 5 we notice that (KT2) is not satisfied, which means that scenario 5 will not generate a solution by its own, but combined with other scenarios, it might generate a solution.*

Thus, the feasible scenarios which will generate the optimal solution for (4.4) are 3, 5, 6 and 10.

Step 3

We state that the critical combinations are 6 with 5 and 6 with 10. We will prove only one, the other being similar.

For $i = \overline{1,l}$ and $j = \overline{l+1,n}$ combination 6 with 5 is composed from

scenario 6 which has the solution

$$\begin{cases} x_i^* = r, & i = \overline{1,l} \\ y^* = 0 \end{cases}$$

and scenario 5 which has the solution

$$\begin{cases} x_j^* = \rho, & j = \overline{l+1,n} \\ y^* \geq p_j\left(\rho - r\right), & j = \overline{l+1,n}. \end{cases}$$

It follows that the conditions to be fulfilled by y^* are

$$\begin{cases} y^* = 0 \\ y^* \geq p_j\left(\rho - r\right), & j = \overline{l+1,n} \end{cases}$$

which means that

$$p_j\left(\rho - r\right) \leq 0,\ j = \overline{l+1,n}$$

but this is impossible.

Step 4

Before computing optimal solutions based on feasible combinations and critical combinations we will prove that the order of scenarios in a combination does not change the solution. We will work with combination 10 with 3, the others being similar.

For combination 10 with 3, as we will notice in the following part of the demonstration for our Theorem, the solution is

$$\begin{cases} x_i^* = \rho, & i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, & j = \overline{l+1,n} \\ y^* = \overline{p_1}\left(\rho - r\right),\ if\ \overline{p_1} \leq p_j \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

What we notice here is that the small prices are constant and on that interval the energy is produced at maximum level to be delivered in the power grid. On the other price interval, energy is computed based on $x_j^* = r + \frac{y^*}{p_j},\ j = \overline{l+1,n}$.

For combination 3 with 10 we will analyze in detail the computation of solution. For 3, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, & i = \overline{1,l} \\ y^* \le p_i\left(\rho - r\right), & i = \overline{1,l} \end{cases}$$

and for 10 with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, & j = \overline{l+1,n} \\ y^* = p_j\left(\rho - r\right), & j = \overline{l+1,n}. \end{cases}$$

Because $y^* = p_j\left(\rho - r\right),\ j = \overline{l+1,n}$ we state that all prices are constant on the set $\{l+1, l+2, ...n\}$ and denote $\overline{p_2} = p_j,\ j = \overline{l+1,n}$. In order to have all conditions for y^* fulfilled it is necessary that

$$\overline{p_2} \le p_i,\ i = \overline{1,l}.$$

Chosing

$$y^* = \overline{p_2}\left(\rho - r\right)$$

a solution is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, & i = \overline{1,l} \\ x_j^* = \rho, & j = \overline{l+1,n} \\ y^* = \overline{p_2}\left(\rho - r\right),\ if\ \overline{p_2} \le p_i. \end{cases}$$

We notice again that the small prices are constant and on that interval the energy is produced at maximum level to be delivered in the power grid. On the other price interval, energy is computed based on $x_i^* = r + \frac{y^*}{p_i},\ i = \overline{1,l}$.

In conclusion we have proved that order of combination does not influence the solution, result which will be formulated as a Theorem and consequently used and addressed.

The combinations based on which we will compute the optimal solution for (4.4), are: 3, 6, 10, 6+3, 10+3, 10+5, 3+5 and 3+5+10.

For 3, denoting by

$$y^* = \min_{i=\overline{1,n}} \{p_i (\rho - r)\}$$

an optimal solution for (4.4) is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i (\rho - r)\} \end{cases}$$

if $\lambda = \frac{n}{n+1}$.

For 6, an optimal solution for (4.4) is

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0 \end{cases}$$

if $\lambda > \frac{n}{n+1}$.

For 10, from $y^* = p_i (\rho - r)$, $i = \overline{1,n}$ we state that all prices are constant and denoting by $\overline{p} = p_i,\ i = \overline{1,n}$, the optimal solution for (4.4) is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p} (\rho - r) \end{cases}$$

if $\lambda < \frac{n}{n+1}$.

For combination 6 with 3, if $i = \overline{1,l}$ and $j = \overline{l+1,n}$, then Kuhn-Tucker conditions (KT1) and (KT2) will be

$$\begin{cases} -(1-\lambda) p_i - a_i p_i + b_i p_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda) p_j + b_j p_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \sum\limits_{i=1}^{l} a_i - \sum\limits_{i=1}^{l} b_i - \sum\limits_{j=l+1}^{n} b_j = 0. \end{cases}$$

Thus

$$a_i = b_i - (1-\lambda), \ i = \overline{1,l}$$

and

$$\sum_{i=1}^{l} a_i = \sum_{i=1}^{l} b_i - l (1-\lambda).$$

Because $a_i > 0,\ i = \overline{1,l}$ it follows that

$$\sum_{i=1}^{l} b_i > l (1-\lambda).$$

Also

$$b_j = (1-\lambda)\,,\; j=\overline{l+1,n}$$

and then

$$\sum_{j=l+1}^{n} b_j = (n-l)\,(1-\lambda)\,.$$

Using now the last equation of the system it follows

$$\lambda > \frac{n}{n+1}.$$

For 6, with $i=\overline{1,l}$ the solution is

$$\begin{cases} x_i^* = r, & i=\overline{1,l} \\ y^* = 0 \end{cases}$$

and for 3, with $j=\overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = r+\frac{y^*}{p_j}, & j=\overline{l+1,n} \\ y^* \le p_j\,(\rho-r)\,, & j=\overline{l+1,n}. \end{cases}$$

Chosing $y^*=0$ it follows that

$$x_j^* = r,\; j=\overline{l+1,n}$$

and

$$0 \le p_j\,(\rho-r)\,,\; j=\overline{l+1,n}$$

which is obvious.

Then, an optimal solution for (4.4) is

$$\begin{cases} x_i^* = r, & i=\overline{1,n} \\ y^* = 0. \end{cases}$$

if $\lambda > \frac{n}{n+1}$.

For combination 10 with 3, if $i=\overline{1,l}$ and $j=\overline{l+1,n}$, then Kuhn-Tucker conditions (KT1) and (KT2) will be

$$\begin{cases} -(1-\lambda)\,p_i + b_i p_i + d_i = 0, & i=\overline{1,l} \\ -(1-\lambda)\,p_j + b_j p_j = 0, & j=\overline{l+1,n} \\ \lambda - \sum\limits_{i=1}^{l} b_i - \sum\limits_{j=l+1}^{n} b_j = 0. \end{cases}$$

Thus

$$d_i = p_i \left[(1-\lambda) - b_i \right], \; i = \overline{1,l}$$

and because $d_i > 0$, $i = \overline{1,l}$ it follows that

$$\sum_{i=1}^{l} b_i < l\,(1-\lambda)\,.$$

Also

$$b_j = (1-\lambda)\,, \; j = \overline{l+1,n}$$

and then

$$\sum_{j=l+1}^{n} b_j = (n-l)\,(1-\lambda)\,.$$

Using now the last equation of the system it follows

$$\lambda < \frac{n}{n+1}.$$

For 10, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ y^* = p_i\,(\rho - r)\,, \quad i = \overline{1,l} \end{cases}$$

and for 3, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* \leq p_j\,(\rho - r)\,, \quad j = \overline{l+1,n}. \end{cases}$$

Because $y^* = p_i\,(\rho - r)\,, \; i = \overline{1,l}$ we state that all prices are constant on the set $\{1, 2, ...l\}$ and denote $\overline{p_1} = p_i, \; i = \overline{1,l}$. In order to have all conditions for y^* fulfilled it is necessary that

$$\overline{p_1} \leq p_j, \; j = \overline{l+1,n}.$$

Chosing

$$y^* = \overline{p_1}\,(\rho - r)$$

an optimal solution for (4.4) is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1}\,(\rho - r)\,, \; if\; \overline{p_1} \leq p_j, \end{cases}$$

if $\lambda < \frac{n}{n+1}$.

For combination 10 with 5, if $i = \overline{1,l}$ and $j = \overline{l+1,n}$, then Kuhn-Tucker conditions (KT1) and (KT2) will be

$$\begin{cases} -(1-\lambda)\,p_i + b_i p_i + d_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\,p_j + d_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \sum\limits_{i=1}^{l} b_i = 0. \end{cases}$$

Thus

$$d_i = p_i\left[(1-\lambda) - b_i\right],\ i = \overline{1,l}$$

and because $d_i > 0$, $i = \overline{1,l}$ it follows that

$$\sum_{i=1}^{l} b_i < l\,(1-\lambda)\,.$$

Using now the last equation of the system it follows that

$$\lambda < \frac{l}{l+1}.$$

For 10, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ y^* = p_i\,(\rho - r)\,, \quad i = \overline{1,l} \end{cases}$$

and for 5, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* \geq p_j\,(\rho - r)\,, \quad j = \overline{l+1,n}. \end{cases}$$

Because $y^* = p_i\,(\rho - r)\,,\ i = \overline{1,l}$ we state that all prices are constant on the set $\{1, 2, ...l\}$ and denote $\overline{p_1} = p_i,\ i = \overline{1,l}$. In order to have all conditions for y^* fulfilled it is necessary that

$$p_j \leq \overline{p_1},\ j = \overline{l+1,n}.$$

Chosing

$$y^* = \overline{p_1}\,(\rho - r)$$

an optimal solution for (4.4) is

$$\begin{cases} x_i^* = \rho, & i = \overline{1,l} \\ x_j^* = \rho, & j = \overline{l+1,n} \\ y^* = \overline{p_1}\left(\rho - r\right), & if\ p_j < \overline{p_1} \end{cases}$$

if $\lambda < \frac{l}{l+1}$.

For combination 3 with 5, if $i = \overline{1,l}$ and $j = \overline{l+1,n}$, then Kuhn-Tucker conditions (KT1) and (KT2) will be

$$\begin{cases} -\left(1-\lambda\right)p_i + b_i p_i = 0, & i = \overline{1,l} \\ -\left(1-\lambda\right)p_j + d_j = 0, & j = \overline{l+1,n} \\ \lambda - \sum\limits_{i=1}^{l} b_i = 0. \end{cases}$$

Thus

$$b_i = \left(1-\lambda\right),\ i = \overline{1,l}$$

and it follows that

$$\sum_{i=1}^{l} b_i = l\left(1-\lambda\right).$$

Using now the last equation of the system it follows that

$$\lambda = \frac{l}{l+1}.$$

For 3, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, & i = \overline{1,l} \\ y^* \leq p_i\left(\rho - r\right), & i = \overline{1,l} \end{cases}$$

and for 5, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, & j = \overline{l+1,n} \\ y^* \geq p_j\left(\rho - r\right), & j = \overline{l+1,n}. \end{cases}$$

The conditions to be fulfilled by y^* are

$$y^* \leq p_i\left(\rho - r\right), \quad i = \overline{1,l}$$
$$y^* \geq p_j\left(\rho - r\right), \quad j = \overline{l+1,n}.$$

If $p_i \leq p_j,\ i = \overline{1,l},\ j = \overline{l+1,n}$, then $\nexists y^*$. If $p_i > p_j,\ i = \overline{1,l},\ j = \overline{l+1,n}$, then by chosing

$$y^* = \min_{i=\overline{1,l}} \left\{ p_i \left(\rho - r \right) \right\}$$

an optimal solution for (4.4) is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \left\{ p_i \left(\rho - r \right) \right\}, \ if\ p_j < p_i, \end{cases}$$

if $\lambda = \frac{l}{l+1}$.

For combination 3 with 5 and with 10, if $i = \overline{1,l},\ \ j = \overline{l+1,m}$ and $k = \overline{m+1,n}$, then Kuhn-Tucker conditions (KT1) and (KT2) will be

$$\begin{cases} -\left(1-\lambda\right) p_i + b_i p_i = 0, \quad i = \overline{1,l} \\ -\left(1-\lambda\right) p_j + d_j = 0, \quad j = \overline{l+1,m} \\ -\left(1-\lambda\right) p_k + b_k p_k + d_k = 0, \quad k = \overline{m+1,n} \\ \lambda - \sum\limits_{i=1}^{l} b_i - \sum\limits_{k=m+1}^{n} b_k = 0. \end{cases}$$

Thus

$$b_i = \left(1-\lambda\right),\ i = \overline{1,l}$$

and it follows that

$$\sum_{i=1}^{l} b_i = l\left(1-\lambda\right).$$

Also

$$d_k = p_k \left[\left(1-\lambda\right) - b_k \right],\ k = \overline{m+1,n}$$

and because $d_k > 0$, $k = \overline{m+1,n}$ it follows that

$$\sum_{k=m+1}^{n} b_k < \left(n-m\right)\left(1-\lambda\right).$$

Using now the last equation of the system it follows that

$$\lambda < \frac{l + \left(n-m\right)}{l + \left(n-m\right) + 1}.$$

For 3, with $i = \overline{1, l}$ the solution is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1, l} \\ y^* \leq p_i \left(\rho - r\right), \quad i = \overline{1, l} \end{cases}$$

for 5, with $j = \overline{l+1, m}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1, m} \\ y^* \geq p_j \left(\rho - r\right), \quad j = \overline{l+1, m} \end{cases}$$

and for 10, with $k = \overline{m+1, n}$ the solution is

$$\begin{cases} x_k^* = \rho, \quad k = \overline{m+1, n} \\ y^* = p_k \left(\rho - r\right), \quad k = \overline{m+1, n}. \end{cases}$$

Because $y^* = p_k \left(\rho - r\right), \ k = \overline{m+1, n}$ we state that all prices are constant on the set $\{m+1, m+2, ...n\}$ and denote $\overline{p_3} = p_k, \ k = \overline{m+1, n}$. In order to have all conditions for y^* fulfilled it is necessary that

$$\begin{aligned} \overline{p_3} &\leq p_i, \ i = \overline{1, l} \\ p_j &\leq \overline{p_3}, \ j = \overline{l+1, m}. \end{aligned}$$

Chosing

$$y^* = \overline{p_3} \left(\rho - r\right)$$

an optimal solution for (4.4) is

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, m} \\ x_k^* = \rho, \quad j = \overline{m+1, n} \\ y^* = \overline{p_3} \left(\rho - r\right), \ if \ p_j \leq \overline{p_3} \leq p_i, \end{cases}$$

if $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$.

Thus, the proof of our theorem is complete. ■

Theorem 4.4.11 (Luca and Mahalov [98]; *order of scenarios*) .
The order of scenarios, used to compute an optimal solution for parametric optimization problem (4.4)*, does not influence the solution.*

Proof. The proof of this theorem was made during the proof of Theorem 4.4.5, at the beginning of Step 4. ■

In the following, TR (total revenue) is the notation for turnover. Based on *Remark* 4.4.3 and *Theorem* 4.4.5 the following result is true.

Theorem 4.4.12 (Luca and Mahalov [98]; *energy minimax*) .
The efficient solution for bi-criteria energy optimization problem (4.2) is

1. *If* $\lambda < \frac{n}{n+1}$ *and there is a single price for energy, then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}\left(\rho - r\right) \\ TR = n\overline{p}\rho \end{cases}$$

or

- *if energy is sold against two different prices during 24 hours and* $\overline{p_1} \leq p_j,\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1}\left(\rho - r\right) \\ TR = l\overline{p_1}\rho + r\sum\limits_{j=l+1}^{n} p_j + \left(n - l\right)y^*. \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

2. *If* $\lambda = \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}}\left\{p_i\left(\rho - r\right)\right\} \\ TR = r\sum\limits_{i=1}^{n} p_i + ny^*. \end{cases}$$

3. *If* $\lambda > \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0 \\ TR = r\sum\limits_{i=1}^{n} p_i. \end{cases}$$

4. *If* $\lambda < \frac{l}{l+1}$, *then*

- *if* $p_j < \overline{p_1},\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\,(\rho - r) \\ TR = l\overline{p}\rho + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

5. *If* $\lambda = \frac{l}{l+1}$, *then*

- *if* $p_j < p_i, i = \overline{1,l},\ \ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \{p_i\,(\rho - r)\} \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

- *else problem has no solution.*

6. *If* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, *then*

- *if* $p_j \leq \overline{p_3} \leq p_i,\ i = \overline{1,l}, j = \overline{l+1,m}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p}\,(\rho - r) \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j + (n-m)\,\overline{p}\rho. \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

- *else problem has no solution.*

Proof. The values for $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ are computed based on Theorem 4.4.5. Values for turnover TR are computed by replacing $x^* \in \mathbb{R}^n$ into $\sum\limits_{i=1}^{n} p_i x_i^*$. ■

4.5 Model validation and conclusions

4.5.1 Testing of solution

Due to confidential character of production data specific for power plants, we have to use data corresponding to the entire Romanian energy market. Thus, we will consider the Romanian electricity system as being a single producer. Transelectrica [148] is providing real time data for production and consumption of electricity on Romanian market. In our analysis we will use data from the period 10th of November 2007, (time 22:18:00) to 4th of December 2015, (time 23:52:55). Considering the records performed every 10 to 15 minutes, a sample of 415789 values is generated.

The first step in our analysis is to check if data referring to production and consumption contain anomalies (outliers and extreme outliers). A Box & Whisker test is performed to identify them. It is based on computing the average and quartiles and identifying values situated outside the interval

$$(Q_1 - 1.5 * IQR; Q_3 + 1.5 * IQR),$$

where $IQR = Q_3 - Q_1$.
Test performed on the sample, using Statgraphics Centurion software, is showing that both outliers and extreme outliers are present in our sample. Figures 4.1 and 4.2 are presenting the Box & Whisker diagrams for production respectively consumption, emphasizing the existence of the above mentioned anomalies.

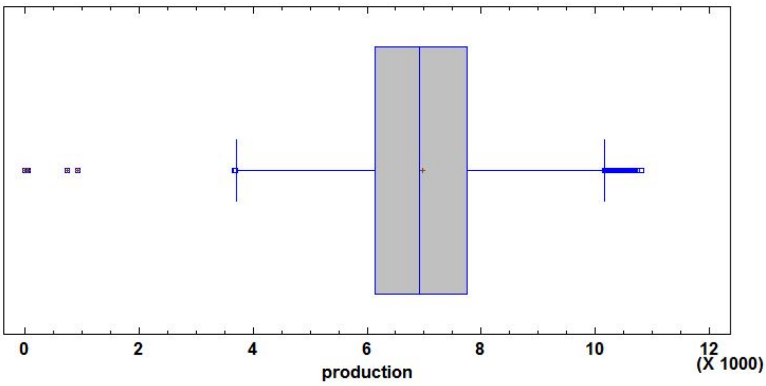


FIGURE 4.1: Box & Whisker diagram for production of energy

The second step in our analysis is to evaluate if anomalies might be excluded from the sample. Performing a Grubbs test, six outliers

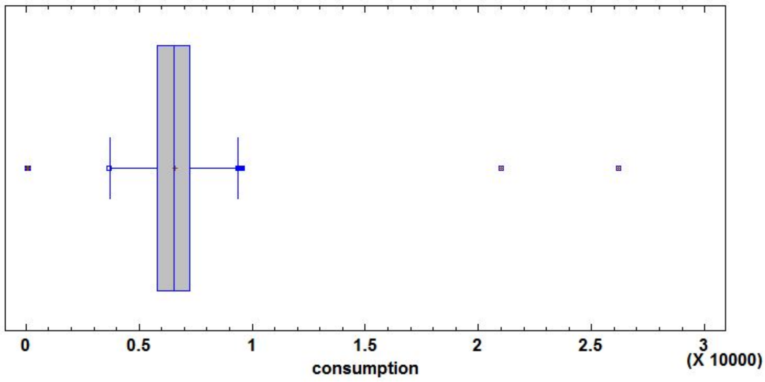


FIGURE 4.2: Box & Whisker diagram for consumption of energy

and extreme outliers for production, respectively consumption are identified, with a 95% confidence level and eliminated. Due to the fact that they correspond to the same time moment, we might assume that there were some faults in data registration. Thus, for the validated data, the extreme values are 3666 MW and 9551 MW - minimum respectively maximum consumption and 3671 MW and 10808 MW - minimum respectively maximum production.

We can proceed now to the third steep, which represents testing of solutions capacity to shave the peak load. We will need some input data for the algorithm – values for ε, ρ, r and price levels.

The following values for input parameters of *Minimax model* will be used:

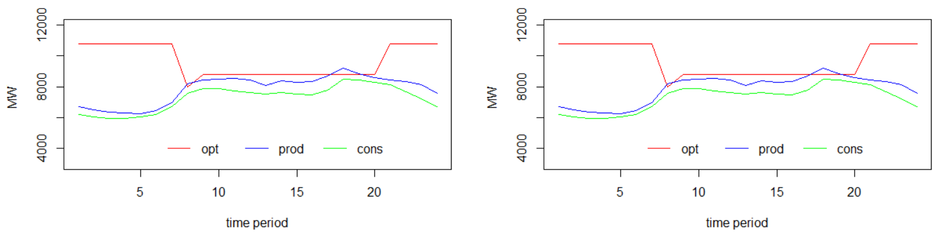
- $\varepsilon = 3666$ MW (minimum consumption, based on validated values, over the entire time period)
- $\rho = 10808$ MW (maximum production, based on validated values, over the entire time period)
- $r = 6970$ MW (average production, based on validated values, over the entire time period)

For price of electricity the values established by Order 40 issued in 21st of June 2013 by Romanian Energy Regulatory Authority [129] will be used. They are presented in Appendix A.

To evaluate the performances, optimal production plan provided by minimax will be compared with real data from 4th of December 2015. For a professional and efficient evaluation, results will be divided in three categories: weak, good and excellent.

Weak result means that efficient solution is far from real data.
Good result means that efficient solution is shaving the peak and flattering the production curve.
Excellent result means that efficient solution is shaving the peak and following the production curve as flat as possible.

Theorem 4.4.5 is providing an optimal production plan. Considering the six solutions provided by the theorem and the possibility to choose different price plans according to conditions required by the theorem, twelve different analysis were performed. A chart was generated for each result for a better visualization. Red color is used to represent the optimal production plan, blue represents the real production curve and green the real consumption curve. They are presented in the following figures and some explanations are presented.

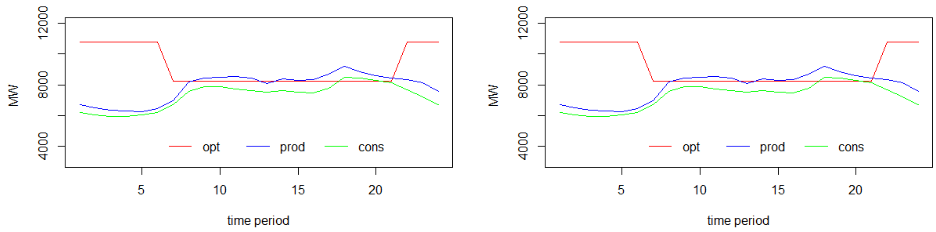


(A) Minimax 1 with Plan III (B) Minimax 2 with Plan III

FIGURE 4.3: Minimax excellent results

Using Price Plan III solutions 1 and 2 from Theorem 4.4.5 are providing the excellent results. Analyzing both charts from Figure 4.3, corresponding to excellent results, it is obvious that our objective to shave the peak load is achieved. Of course fluctuation of energy is reduced and the optimized production plans are maximizing turnover. Additionally, the optimized production curve is following the real production curve as flat as possible. During gap periods, the low price of energy is forcing a boost of production in order to satisfy the maximization of turnover. Shaving the peak load is determining a shift of production and a supra-production of energy at some time moments. According to the strategy of producer, he might choose: to store the energy and to reuse it for covering demand when necessary (ESS), to manually interfere for adjusting the production plan or to initiate a campaign to influence and change the behavior of consumers (DSM).

Let's notice that both excellent results are obtained for Plan III. We will come back to this topic later in our work.



(A) Minimax 1 with Plan II (B) Minimax 2 with Plan II

FIGURE 4.4: Minimax good results

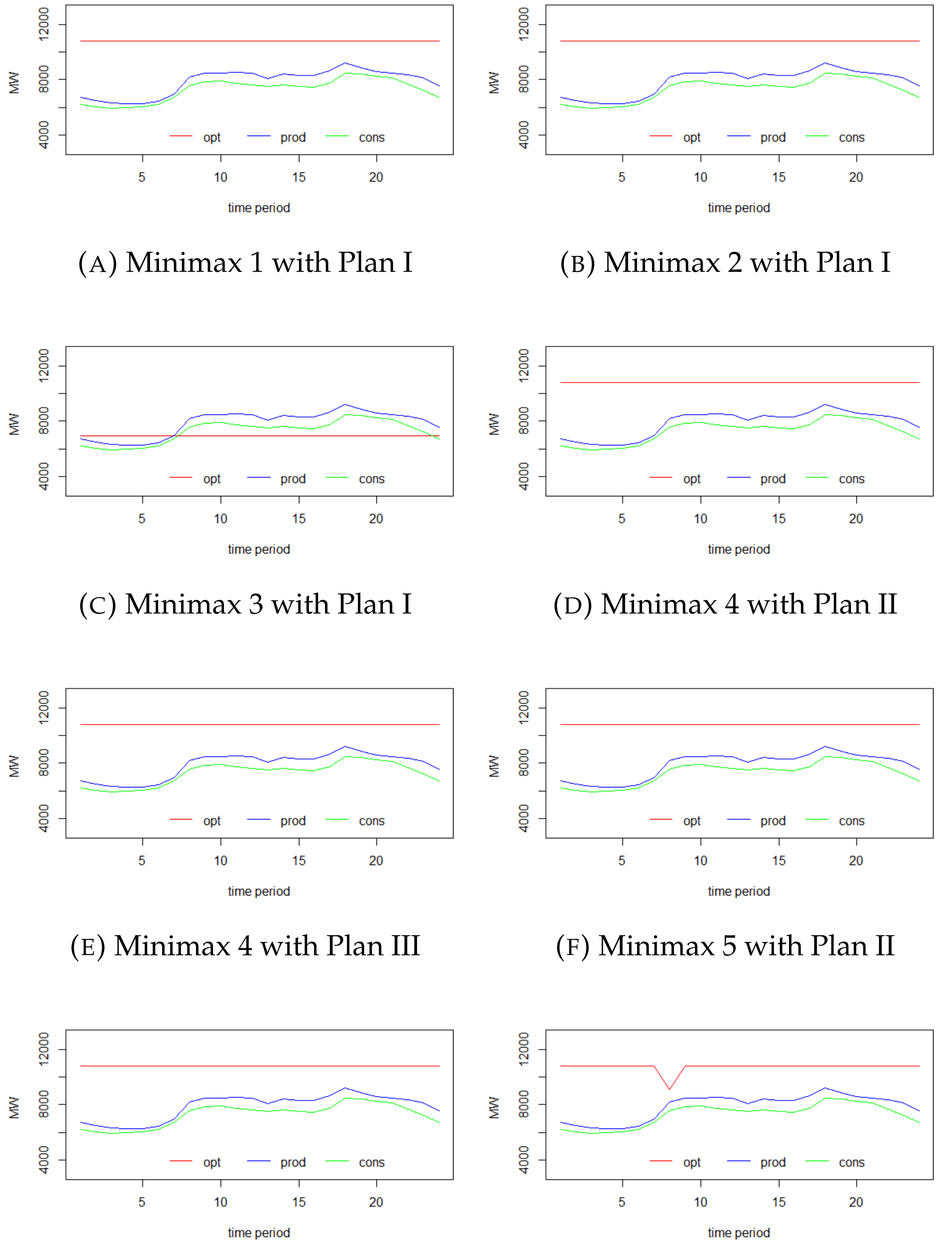
Using Price Plan II, solutions 1 and 2 are providing again results evaluated, this time, with good. Figure 4.4 is presenting the charts for them. Again it is visible that peak load of production is shaved. Thus we might conclude that our objective is realized.

A deeper analysis reveals interesting aspects. Optimized production curve is not following the real production curve as consistent as previous results. It is due to the reduced variety of tariffs in the pricing plan. This is proving that our decision to include price in the measure of fluctuation is helpful in the optimization process and it was a correct one. The area of inconsistent follow is generating additional over production which has to be addressed according to the strategy of producer.

Corresponding to the night hours, when price is low, over production is present again. It is a similar situation with gap period from Plan III and is generated by the correlation between price, quantity and turnover. Possibilities to address this aspect, presented for the previous results, might be used also in this case.

Optimized production plan is situated in some points below the consumption curve which is very risky. A fine tune of input data, and especially predefined level r, in this case, will improve the solution and mitigate the risk. In this point we can have a first glance to the sensitivity aspects of the model.

Charts from Figure 4.5 are presenting solutions evaluated with weak results. Optimal production plan is computed either to minimum or maximum levels assumed by the power plant, or to the predefined level. It is obvious that they can not be accepted from practical point of view. Interesting to be mentioned is that solution 1



(A) Minimax 1 with Plan I

(B) Minimax 2 with Plan I

(C) Minimax 3 with Plan I

(D) Minimax 4 with Plan II

(E) Minimax 4 with Plan III

(F) Minimax 5 with Plan II

(G) Minimax 5 with Plan III

(H) Minimax 6 with Plan III

FIGURE 4.5: Minimax weak results

and 2, combined with Plan I offers weak results. So again price is a critical key.

Tests performed for minimax model and part of the analysis presented in this subsection were disseminated in [90].

Minimax model has generated results evaluated in all three categories, excellent, good and weak. Minimax is realizing the objectiv of our research: to shave the peak load of energy by minimizing fluctuation of energy and maximizing economic performance.

Thus it can be validate as a reliable model for shaving the peak load. Besides planning and economic advantages associated to peak load shaving, as visible in Figures 4.3 and 4.4, we consider that minimax model generates several additional advantages like: (i) incentives for power plants to develop storage systems, (ii) incentives to invest in green energy to compensate the production shift, to reduce CO_2 emissions and to reduce production cost, (iii) reduced fatigue, maintenance and development costs for equipments of power plant (iv) increased efficiency for power plant or (v) a more stable and efficient power grid.

Model is sensitive to input data and price has a key role in optimization. The more price range are available, the better the behavior of solutions is.

4.5.2 Conclusions

Minimax is a bi-criteria optimization problem which aims to shave the peak load by minimizing fluctuation of energy around a predefined level and maximizing economic performance of power plant.

A proper measure to address the peak load was developed, starting from maximum absolute deviation. Price of energy was introduced in the measure, due to its sensitivity to quantity and its elasticity. Defined by 4.1 it was called minimax measure for fluctuation of energy.

Economic performance was evaluated using turnover.

As constraints, we have used only a limitation for energy production, between a minimum and a maximum amount assumed by the power plant to be delivered in the power grid.

Being a bi-criteria optimization problem, a Pareto efficient solution was computed.

Using several equivalences, the minimax energy model (4.2) was transformed in the parametric problem (4.4), for which Theorem 4.4.5 is providing the optimal solution. It computes the amount of energy to be produced at each time moment and the corresponding fluctuation. It is the central theorem of Chapter 4 and its proof is based on

the Kuhn-Tucker conditions, being a 4 steps process. Because parametric problem (4.4) is a convex one, the Kuhn-Tucker conditions are necessary and sufficient.
At first step we have evaluated, at a fixed moment of time (a fixed value for $i = \overline{1, n}$), the existence of solutions for all Kuhn-Tucker multipliers. The combinations which have generated a solution were called *possible combinations*.
The second step is evaluating, over the entire time period, the behavior of possible Kuhn-Tucker combinations related to the Lagrangian. Those for which the Lagrangian is zero were called *feasible combinations*.
The third step is analyzing the combining capacity of feasible combinations and *critical combinations* are identified.
Forth step is computing the optimal solution for the parametric problem (4.4), if the same *feasible combination* is multiplied over the entire time period or if different *feasible combinations* are combined during the time horizon.
During the forth step, we had to prove that the order of combining the scenarios does not influence the solution, result presented as Theorem 4.4.11.
Returning to the initial minimax energy problem (4.2), Theorem 4.4.12 is computing the efficient frontier.

Theoretical results obtained in Theorems 4.4.5 and 4.4.12 were tested using real data. The confidential character of data related to energy production has determined us to use data corresponding to the entire Romanian market. They were provided by Transelectrica [148] and correspond to the period 10th of November 2007, (time 22:18:00) to 4th of December 2015, (time 23:52:55) and by ANRE [129].
Using Box & Whisker and Grubbs tests, production and consumption data were analyzed and validated for generating inputs in the mimax model. Optimal production plan provided by minimax was compared to real production and consumption curves from 4th of December 2015. According to the their behavior, results were evaluated as excellent, good or weak.
Section 3.4.1 is providing a detailed test and analysis. Minimax model is realizing the objectives of our research, by shaving the peak load, flattering the production and maximizing economic performances. Excellent performances are obtained for price plans with three tariffs

and good performances are obtained for price plans with two tariffs. Shaving the peak load is generating a shift in production with associated supra-production, which can be addressed by methods presented in Chapter 1.

A simple visualization of solutions provided by Theorem 4.4.5 shows that solutions 3 and 4 will never generate good or excellent results, due to the fact that optimal production plan is placed either to minimum or maximum amount assumed or to the predefined level.

Minimax model has three types of input data, making it more complex to be used in practice, but results provided are confirmed, by tests on real data, to be good and excellent. Model is sensitive to input data, but in the same time, a fine tune of input data, has the capacity to improve even more the behavior of solution.

Adding additional complex technical constraints or using profit as measure for economic performance the complexity of the model will increase, but solutions will be more accurate.

Our contributions to this chapter, presented at International Conference on Approximation Theory and its Applications, organized in Sibiu, Romania during 26-29 May 2016 and disseminated in [90] and [98], might be synthesized in:

- using of bi-criteria optimization problems in research for shaving the energy peak load,
- introduction of minimax measure of fluctuation (4.1),
- development of minimax energy model (4.2),
- Lemma 4.4.1, which together with other known results of Yu [160], Bot [14], Geoffrion [53] prove the equivalency between minimax energy model (4.2) and parametric model (4.4),
- Theorem 4.4.5 which computes the optimal solution for parametric model (4.4),
- Theorem 4.4.11 which proves that order of scenarios does not change the solution,
- Theorem 4.4.12 which computes the efficient solution for minimax energy model (4.2),
- validation of minimax model after performing tests on real data.

Chapter 5

Index model for energy optimization

5.1 Introduction

This Chapter is presenting index model. It is a bi-criteria optimization problem, which aims to shave the peak load by minimizing fluctuation of energy and maximizing economic performance.

Presence of predefined level in minimax measure for fluctuation (4.1) increases the complexity of input data for Minimax model (4.2), making it difficult to be implemented. Skipping the predefined level will reduce complexity of the model, but in the same time will eliminate the reference point for peak load. We will replace this reference point (the predefined level) with another indicator, more friendly for practitioners.

Skipping the predefined level might increase the range of solution, because maximum absolute deviation, as measure for fluctuation of energy, combined with turnover as measure for economic performance, will never allow a point situated under the predefined level to be Pareto efficient. It is due to absolute value considered in the measure of fluctuation, which makes that a point situated under the predefined level, will always generate a smaller turnover compared with its symmetric, while the fluctuation is the same.

For the same reasons presented in Chapter 4, turnover will measure the economic performance of the power plant and only simple technical constraints, limiting the amount of energy to be produced at certain time moments, will be employed.

Index, our second model for shaving the peak load, is developed in the effort of simplifying minimax and making it more friendly for practitioners.

The idea for the new reference point for peak load and thus for the new measure of energy fluctuation was inspired by production environment, where is common to report loading of an equipment or department to a fixed capacity.

Index will be tested using the same real data as for minimax.

A comparison between minimax and index models, emphasizing the conceptual differences, has been presented by the author during the *12th Edition of ICATA 2016 Conference* and published in [39].

5.2 Index measure for fluctuation of energy

Our aim, is to skip that predefined level around which the fluctuation of energy is minimized and to introduce a new indicator as reference for the peak load. This way complexity of input data will be reduced, range of solution will be extended and a more friendly and easy to be implemented model will be developed.

The optimization process is related to production capacities, for which is quite common to report nominal capacity to a fixed capacity (in general the maximum capacity), as mentioned by Morrison [109] and Muchiri [112].

Using the notations introduced in Chapter 4, and the previous mentioned idea, we propose, as a measure for fluctuation the following

$$\max_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}. \tag{5.1}$$

and call it *index measure for fluctuation of energy*.

Based on sensitivity of price to quantity and elasticity of energy price, explained in the Chapter 4, price is maintained also in the new measure. The measuring unit for index measure of energy fluctuation is money.

5.3 Problem formulation

To avoid burdens generated by the peak load, a power plant aims to shave it by minimizing fluctuation of energy and maximizing economic performance.

As measure for fluctuation of energy we will use index measure (5.1), while turnover is expressed as

$$\sum_{i=1}^{n} p_i x_i.$$

The only constraints imposed for the amount of energy to be produced are

$$\varepsilon \leq x_i \leq \rho,\ i = \overline{1,n}.$$

Thus, the mathematical model for index energy problem is

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho,\ i = \overline{1,n}. \end{cases} \tag{5.2}$$

5.4 Computing the solution

In order to determine the efficient solution for the problem (5.2) we introduce the following bi-criteria optimization problem

$$\begin{cases} \min \left(y; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \frac{x_i}{\rho} p_i \leq y,\ i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho,\ i = \overline{1,n} \end{cases} \tag{5.3}$$

The equivalence between (5.2) and (5.3) is established by the following Lemma.

Lemma 5.4.1 *[92] Let's consider the bi-criteria optimization problems (5.2) and (5.3).*

a) If $x \in \mathbb{R}^n$ is an efficient solution for problem (5.2), then $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ is an efficient solution for problem (5.3).

b) If $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ is an efficient solution for problem (5.3), then $x \in \mathbb{R}^n$ is an efficient solution for problem (5.2).

Proof. Proof is similar with the one of Lemma 4.4.1 from Chapter 4, so we will not present it in details here. ■

Using again the idea to transform the bi-criteria problem into a parametric one and based on Theorem 2.3.1 of Yu [160] and similar results of Bot et all [14] and Geoffrion [53] the bi-criteria optimization problem (5.3) is equivalent to the following parametric optimization problem

$$\begin{cases} \min \left\{ \lambda y - (1-\lambda) \sum_{i=1}^{n} p_i x_i \right\} \\ \frac{x_i}{\rho} p_i \leq y, \; i = \overline{1,n} \\ \varepsilon \leq x_i, \; i = \overline{1,n} \\ x_i \leq \rho, \; i = \overline{1,n} \end{cases} \tag{5.4}$$

with $\lambda \in (0,1)$ and the following is true

Lemma 5.4.2 *[92] $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ is an efficient solution for bi-criteria problem (5.3) if and only if $\exists \lambda \in (0,1)$ such that $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution for parametric optimization problem (5.4).*

The meaning of λ is again the desire of power plant to reduce fluctuation. A big λ means that power plant is focusing on reducing the fluctuation, which might affect turnover, while a small λ means that power plant is focusing more on turnover, accepting a higher fluctuation of energy produced.

Remark 5.4.3 *Based on the equivalence between problems (5.2) and (5.3) proved by Lemma 5.4.1 and the equivalence between (5.3) and (5.4) proved by Lemma 5.4.2, we state that problems (5.2) and (5.4) are equivalent. Therefore, in order to determine the efficient solution for problem (5.2) it is sufficient to determine the optimal solution for problem (5.4).*

Remark 5.4.4 *In the process of computing the optimal solution for parametric optimization problem (5.4) we will split the set $\{1, 2, ..., n\}$ in several subsets like $\{1, 2, ...l\}$, $\{l+1, l+2, ...m\}$, $\{m+1, m+2, ...t\}$ and $\{t+1, t+2, ...n\}$. If on such a set or subset the price is constant we will denote it by $\overline{p_*}$.*

The following *Theorem* presents an optimal solution for the parametric optimization problem (5.4).

Theorem 5.4.5 (Luca and Duca [92]; parametric index) .

An optimal solution $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ for parametric optimization problem (5.4) is:

1. *If* $\lambda = \frac{\rho n}{1+\rho n}$ *and* $\frac{\min\limits_{i=\overline{1,n}} p_i}{\max\limits_{i=\overline{1,n}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,n} \\ y^* \in \left[\max\limits_{i=\overline{1,n}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,n}} p_i\right] \end{cases}$$

else solution does not exist.

2. *If* $\lambda > \frac{\rho n}{1+\rho n}$, *then*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,n} \\ y^* = \frac{\varepsilon}{\rho} \overline{p}, \end{cases}$$

where $\overline{p} = p_i,\ i = \overline{1,n}$,
else solution does not exist.

3. *If* $\lambda < \frac{\rho n}{1+\rho n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}, \end{cases}$$

where $\overline{p} = p_i,\ i = \overline{1,n}$,
else solution does not exist.

4. *If* $\lambda = \frac{\rho l}{1+\rho l}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \begin{cases} \overline{p_2}, & \text{if } \min\limits_{i=\overline{1,l}} p_i = p_j, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{j=\overline{l+1,n}} p_j; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq p_j < \min\limits_{i=\overline{1,l}} p_i, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } p_j < \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i, \quad j = \overline{l+1,n} \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,n}$,
else solution does not exist.

5. *If* $\lambda < \frac{\rho l}{1+\rho l}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1}, \text{ if } \frac{\overline{p_1}}{p_j} \geq 1, \quad j = \overline{l+1,n} \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1, l}$,
else solution does not exist.

6. *If* $\lambda > \frac{\rho l}{1+\rho l}$, *then*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1}, \ \text{if } \frac{\overline{p_1}}{p_j} \geq \frac{\rho}{\varepsilon}, \quad j = \overline{l+1, n} \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1, l}$,
else solution does not exist.

7. *If* $\lambda < \frac{\rho n}{1+\rho n}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, n} \\ y^* = \overline{p_2}, \ \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_2} \\ \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1, n}$,
else solution does not exist.

8. *If* $\lambda > \frac{\rho l}{1+\rho l}$ *and* $\frac{\overline{p_2}}{\overline{p_1}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1} = \overline{p_2} \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1, l}$ *and* $\overline{p_2} = p_j,\ j = \overline{l+1, n}$,
else solution does not exist,

9. *If* $\lambda > \frac{\rho n}{1+\rho n}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1, l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1, n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2}, \ \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,n}$*,*
else solution does not exist.

10. *If* $\lambda > \frac{\rho m}{1+\rho m}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *and* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, \ \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ *and* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$*,*
else solution does not exist.

11. *If* $\lambda > \frac{\rho l + \rho(n-m)}{1+\rho l+\rho(n-m)}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \varepsilon, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3}, \ \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,n}$*,*
else solution does not exist.

12. *If* $\lambda < \frac{\rho l + \rho(n-m)}{1+\rho l+\rho(n-m)}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p_3}, \ \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,n}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_3} \\ \overline{p_3} \leq \min\limits_{i=\overline{1,n}} p_i \\ p_j \leq \overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,n}$*,*
else solution does not exist.

13. *If* $\lambda > \frac{\rho(m-l)}{1+\rho(m-l)}$ *and* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \rho, & i = \overline{1,l} \\ x_j^* = \varepsilon, & j = \overline{l+1,m} \\ x_k^* = \rho, & k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, & \text{if } p_i \leq \frac{\varepsilon}{\rho}\overline{p_2},\ i = \overline{1,l} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ *and* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$, *else solution does not exist.*

14. *If* $\lambda > \frac{\rho l+\rho(t-m)}{1+\rho l+\rho(t-m)}$, $\frac{\min_{i=\overline{1,l}} p_i}{\max_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *and* $\frac{\overline{p_4}}{\overline{p_3}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i}y^*, & i = \overline{1,l} \\ x_j^* = \rho, & j = \overline{l+1,m} \\ x_k^* = \varepsilon, & k = \overline{m+1,t} \\ x_s^* = \rho, & s = \overline{t+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3} = \overline{p_4}, & \text{if } \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,t}$ *and* $\overline{p_4} = p_s,\ s = \overline{t+1,n}$, *else solution does not exist.*

Proof. For a fixed $\lambda \in (0,1)$, the Lagrangian associated to problem (5.4) is

$$\begin{aligned} L &: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \\ L(x,y,a,b,c) &= \lambda y - (1-\lambda)\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} a_i (x_i p_i - \rho y) + \\ &+ \sum_{i=1}^{n} b_i (\varepsilon - x_i) + \sum_{i=1}^{n} c_i (x_i - \rho) \end{aligned}$$

and the corresponding Kuhn-Tucker conditions are

$$\frac{\partial L}{\partial x_i} = -(1-\lambda)p_i + a_i p_i - b_i + c_i = 0,\ i = \overline{1,n} \tag{5.5}$$

$$\frac{\partial L}{\partial y} = \lambda - \rho\sum_{i=1}^{n} a_i = 0 \tag{5.6}$$

$$(p_i x_i^* - \rho y^*)\, a_i = 0, \quad a_i \geq 0, \quad i = \overline{1,n} \tag{5.7}$$

$$(\varepsilon - x_i^*)\, b_i = 0, \quad b_i \geq 0, \quad i = \overline{1,n} \tag{5.8}$$

$$(x_i^* - \rho)\, c_i = 0, \quad c_i \geq 0, \quad i = \overline{1,n} \tag{5.9}$$

where $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution.

Step 1

For a fixed $i \in \{1, 2, ...n\}$ the combinations for Kuhn-Tucker multipliers are presented in Table 5.1

	KT Multipliers		
Scenarios	a_i	b_i	c_i
1	=0	=0	=0
2	>0	=0	=0
3	=0	>0	=0
4	=0	=0	>0
5	>0	>0	=0
6	=0	>0	>0
7	>0	=0	>0
8	>0	>0	>0

TABLE 5.1: Combination of KT multipliers for index model

Based on this, we will analyze the existence of solution for each scenario.

Scenario 1

$a_i = 0$	$b_i = 0$	$c_i = 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i \leq \rho y^* \\ \varepsilon \leq x_i^* \\ x_i^* \leq \rho. \end{cases}$$

From last two inequalities we obtain

$$x_i^* \in [\varepsilon; \rho]$$

and from first it follows

$$y^* \geq \frac{x_i^*}{\rho} p_i.$$

Considering that the smallest value for x_i^* is ε, we get

$$y^* \geq \frac{\varepsilon}{\rho} p_i.$$

Thus, the solution for the system is

$$\begin{cases} x_i^* \in [\varepsilon;\ \rho] \\ y^* \geq \frac{\varepsilon}{\rho} p_i. \end{cases}$$

Scenario 2

$a_i > 0$	$b_i = 0$	$c_i = 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i = \rho y^* \\ \varepsilon \leq x_i^* \\ x_i^* \leq \rho. \end{cases}$$

From the first equality it follows

$$x_i^* = \frac{\rho}{p_i} y^*.$$

Replacing in the next two inequalities we obtain

$$\begin{cases} \varepsilon \leq \frac{\rho}{p_i} y^* \\ \frac{\rho}{p_i} y^* \leq \rho \end{cases}$$

which is equivalent to

$$y^* \in \left[\frac{\varepsilon}{\rho} p_i;\ p_i\right].$$

Thus, system's solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^* \\ y^* \in \left[\frac{\varepsilon}{\rho} p_i;\ p_i\right]. \end{cases}$$

Scenario 3

$a_i = 0$	$b_i > 0$	$c_i = 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i \leq \rho y^* \\ \varepsilon = x_i^* \\ x_i^* \leq \rho. \end{cases}$$

From second equation it follows

$$x_i^* = \varepsilon.$$

Third inequality is obvious and replacing in the first one we obtain

$$\varepsilon p_i \leq \rho y^*$$

equivalent to

$$y^* \geq \frac{\varepsilon}{\rho} p_i$$

and the system has the following solution

$$\begin{cases} x_i^* = \varepsilon \\ y^* \geq \frac{\varepsilon}{\rho} p_i. \end{cases}$$

Scenario 4

$a_i = 0$	$b_i = 0$	$c_i > 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i \leq \rho y^* \\ \varepsilon \leq x_i^* \\ x_i^* = \rho. \end{cases}$$

From third equality we obtain

$$x_i^* = \rho.$$

Second inequality is obvious and replacing in the first one it follows

$$\rho p_i \leq \rho y^*$$

equivalent to

$$y^* \geq p_i$$

which generates the following solution for the system

$$\begin{cases} x_i^* = \rho \\ y^* \geq p_i. \end{cases}$$

Scenario 5

$a_i > 0$	$b_i > 0$	$c_i = 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i = \rho y^* \\ \varepsilon = x_i^* \\ x_i^* \leq \rho. \end{cases}$$

From second equality it follows

$$x_i^* = \varepsilon.$$

Third inequality is obvious and from the first equality it follows that

$$y^* = \frac{\varepsilon}{\rho} p_i.$$

In conclusion, system's solution is

$$\begin{cases} x_i^* = \varepsilon \\ y^* = \frac{\varepsilon}{\rho} p_i. \end{cases}$$

Scenario 6

$a_i = 0$	$b_i > 0$	$c_i > 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i \leq \rho y^* \\ \varepsilon = x_i^* \\ x_i^* = \rho. \end{cases}$$

From last two equalities it follows

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

Scenario 7

$a_i > 0$	$b_i = 0$	$c_i > 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i = \rho y^* \\ \varepsilon \leq x_i^* \\ x_i^* = \rho. \end{cases}$$

From last equality it follows

$$x_i^* = \rho.$$

Second inequality is obvious and from the first equality we get

$$y^* = p_i.$$

Thus the corresponding solution is

$$\begin{cases} x_i^* = \rho \\ y^* = p_i. \end{cases}$$

Scenario 8

$a_i > 0$	$b_i > 0$	$c_i > 0$

The system generated by Kuhn-Tucker conditions (5.7) to (5.9) is

$$\begin{cases} x_i^* p_i = \rho y^* \\ \varepsilon = x_i^* \\ x_i^* = \rho. \end{cases}$$

From last two equalities of the system we obtain

$$x_i^* = \varepsilon = \rho$$

which is impossible and thus system has no solution.

Table 5.2 is presenting the solutions for all possible scenarios.

Completing the analyze for the 8 scenarios, we may conclude that only 6 of them are generating solutions, those being the possible combinations: 1, 2, 3, 4, 5, 7.

Step 2

At these step we will determine the feasible combinations by analyzing, for $i = \overline{1,n}$, the behavior of possible combinations related to the gradient of Lagrangian.

The system generated by *scenario 1* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\, p_i = 0, \quad i = \overline{1,n} \\ \lambda = 0 \end{cases}$$

and thus $\lambda = 0$ or $p_i = 0,\ i = \overline{1,n}$, which is impossible.

Scenarios	Solution
1	$\begin{cases} x_i^* \in [\varepsilon;\ \rho] \\ y^* \geq \frac{\varepsilon}{\rho} p_i \end{cases}$
2	$\begin{cases} x_i^* = \frac{\rho}{p_i} y^* \\ y^* \in \left[\frac{\varepsilon}{\rho} p_i;\ p_i\right] \end{cases}$
3	$\begin{cases} x_i^* = \varepsilon \\ y^* \geq \frac{\varepsilon}{\rho} p_i \end{cases}$
4	$\begin{cases} x_i^* = \rho \\ y^* \geq p_i \end{cases}$
5	$\begin{cases} x_i^* = \varepsilon \\ y^* = \frac{\varepsilon}{\rho} p_i \end{cases}$
6	$\nexists$ solution
7	$\begin{cases} x_i^* = \rho \\ y^* = p_i \end{cases}$
8	$\nexists$ solution

TABLE 5.2: Possible combinations for index model

The system generated by *scenario 2* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\, p_i + p_i a_i = 0, \quad i = \overline{1,n} \\ \lambda - \rho \sum_{i=1}^{n} a_i = 0 \end{cases}$$

Then $a_i = 1 - \lambda,\ i = \overline{1,n}$ and thus $\sum_{i=1}^{n} a_i = n\,(1-\lambda)$. Replacing in the last equation of the system in follows that

$$\lambda = \frac{\rho n}{1 + \rho n}.$$

The system generated by *scenario 3* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\, p_i - b_i = 0, \quad i = \overline{1,n} \\ \lambda = 0 \end{cases}$$

It follows that $b_i = -(1-\lambda)\, p_i < 0,\ i = \overline{1,n}$, which is impossible.

The system generated by *scenario 4* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\, p_i + c_i = 0, \quad i = \overline{1,n} \\ \lambda = 0 \end{cases}$$

and then $c_i = (1-\lambda)\, p_i > 0,\ i = \overline{1,n}$, which is obvious.

The system generated by *scenario 5* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\,p_i + p_i a_i - b_i = 0, \quad i = \overline{1,n} \\ \lambda - \rho \sum\limits_{i=1}^{n} a_i = 0 \end{cases}$$

Then $b_i = [a_i - (1-\lambda)]\,p_i,\ i = \overline{1,n}$, which has to be positive.
It follows that $a_i - (1-\lambda) > 0,\ i = \overline{1,n}$ which means that $a_i > (1-\lambda)\,,\ i = \overline{1,n}$ and then $\sum\limits_{i=1}^{n} a_i > n\,(1-\lambda)$.
From the last equation of the system it follows that

$$\lambda > \frac{\rho n}{1+\rho n}.$$

The system generated by *scenario 7* in relation with (5.5) and (5.6) is

$$\begin{cases} -(1-\lambda)\,p_i + p_i a_i + c_i = 0, \quad i = \overline{1,n} \\ \lambda - \rho \sum\limits_{i=1}^{n} a_i = 0. \end{cases}$$

Then $c_i = [(1-\lambda) - a_i]\,p_i,\ i = \overline{1,n}$, which has to be positive.
It follows that $(1-\lambda) - a_i > 0,\ i = \overline{1,n}$ which means that $(1-\lambda) > a_i,\ i = \overline{1,n}$ and then $\sum\limits_{i=1}^{n} a_i < n\,(1-\lambda)$.
From the last equation of the system it follows that

$$\lambda < \frac{\rho n}{1+\rho n}.$$

The end results are synthesized in the Table 5.3

Scenario	Solution	(5.5)	(5.6)
1	$\nexists$	$\times$	$\times$
2	$\exists,\ if\ \lambda = \frac{\rho n}{1+\rho n}$	$\checkmark$	$\checkmark$
3	$\nexists$	$\times$	$\times$
4	$\nexists$	$\checkmark$	$\times$
5	$\exists,\ if\ \lambda > \frac{\rho n}{1+\rho n}$	$\checkmark$	$\checkmark$
7	$\exists,\ if\ \lambda < \frac{\rho n}{1+\rho n}$	$\checkmark$	$\checkmark$

TABLE 5.3: Behavior of possible combinations related to gradient of Lagrangian for index model

For *scenario 4* we notice that (5.6) is not satisfied, which means that *scenario 4* will not generate a solution by its own, but may be combined with others to generate solution.

Step 2 of the proof ends with identifying the feasible combinations for the problem. These are 2, 4, 5 and 7.

Step 3

At these step, we identify critical combinations. Analyzing the combining ability for 5 and 7, we notice that prices have to be constant on sets $\{1, 2, ...l\}$, respectively $\{l+1, l+2, ...n\}$. Denoting by $\overline{p_1} = p_i,\ i = \overline{1, l}$ and $\overline{p_2} = p_j,\ j = \overline{l+1, n}$, it follows that combination 5 and 7 is not critical if and only if

$$\frac{\overline{p_2}}{\overline{p_1}} = \frac{\varepsilon}{\rho}.$$

Also, for 2 we notice that

$$\frac{\varepsilon}{\rho} p_i \leq y^* \leq p_i,\ i = \overline{1, n}$$

which means that

$$y^* \in \left[\max_{i=\overline{1,n}} \frac{\varepsilon}{\rho} p_i;\ \min_{i=\overline{1,n}} p_i\right]$$

and in order that this interval exists it's necessary that

$$\frac{\min_{i=\overline{1,n}} p_i}{\max_{i=\overline{1,n}} p_i} \geq \frac{\varepsilon}{\rho}.$$

Step 4

At this step we will compute an optimal solution for problem (5.4), based on the following combinations:

$$\begin{aligned}
&2, 7, 5\\
&2+4,\ 7+4,\ 5+4,\ 2+7,\ \mathbf{5+7},\ 2+5\\
&\mathbf{2+5+7},\ 2+4+5,\ 2+4+7,\ \mathbf{4+5+7}\\
&\mathbf{2+4+5+7}
\end{aligned}$$

To compute the optimal solutions we will use the information, stated by Theorem 4.4.11, that order of scenarios does not influence the solution.

These combinations are valid only if 5 + 7 is not critical and the interval for 2 exists. If 5 + 7 is critical, then all combinations which contain 5 + 7 have to be eliminated. If the interval for 2 doesn't exist, then all combinations which contain 2 have to be eliminated.

For 2, if $\lambda = \frac{\rho n}{1+\rho n}$ and $\frac{\min\limits_{i=\overline{1,n}} p_i}{\max\limits_{i=\overline{1,n}} p_i} \geq \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,n} \\ y^* \in \left[\max\limits_{i=\overline{1,n}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,n}} p_i \right] \end{cases}$$

else solution does not exist.

For 7, from $y^* = p_i,\ i = \overline{1,n}$, we state that all prices are constant on the set $\{1, 2, ...n\}$. Denoting by $\overline{p} = p_i,\ i = \overline{1,n}$, we obtain that if $\lambda < \frac{\rho n}{1+\rho n}$ then an optimal solution for (5.4) is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p} \end{cases}$$

else solution does not exist, where $\overline{p} = p_i,\ i = \overline{1,n}$.

For 5, using a similar principle with 7, we conclude that if $\lambda > \frac{\rho n}{1+\rho n}$ then an optimal solution for (5.4) is

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,n} \\ y^* = \frac{\varepsilon}{\rho} \overline{p} \end{cases}$$

else solution does not exist, where $\overline{p} = p_i,\ i = \overline{1,n}$.

For combination 2 + 4, with $i = \overline{1,l}$ and $j = \overline{l+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\, p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\, p_j + c_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i = 0. \end{cases}$$

From the first set of l equations, it follows that $\sum\limits_{i=1}^{l} a_i = l\,(1-\lambda)$ and replacing in the last equation we obtain

$$\lambda = \frac{\rho l}{1 + \rho l}.$$

For 2, with $i=\overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i=\overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i\right] \end{cases}$$

and for 4, with $j=\overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j=\overline{l+1,n} \\ y^* \geq p_j, \quad j=\overline{l+1,n}. \end{cases}$$

In order to have all conditions for y^* fulfilled it is necessary that

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq y^*, \quad j=\overline{l+1,n}. \end{cases}$$

If $\min\limits_{i=\overline{1,l}} p_i < p_j,\ j=\overline{l+1,n}$, then $\nexists\, y^*$.

If $\min\limits_{i=\overline{1,l}} p_i = p_j,\ j=\overline{l+1,n}$, then we state that prices are constant on the set $\{l+1, l+2, ...n\}$, and denoting $\overline{p_2} = p_j,\ j=\overline{l+1,n}$ it follows that

$$y^* = \overline{p_2} = \min\limits_{i=\overline{1,l}} p_i.$$

If $\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i < p_j < \min\limits_{i=\overline{1,l}} p_i$, $j=\overline{l+1,n}$, then

$$y^* \in \left[\max\limits_{j=\overline{l+1,n}} p_j; \min\limits_{i=\overline{1,l}} p_i\right].$$

If $\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i = p_j$, $j=\overline{l+1,n}$, then we state that prices are constant on the set $\{l+1, l+2, ...n\}$, and denoting $\overline{p_2} = p_j,\ j=\overline{l+1,n}$ it follows that

$$y^* \in \left[\max\limits_{j=\overline{l+1,n}} p_j; \min\limits_{i=\overline{1,l}} p_i\right].$$

If $p_j < \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i$, $j=\overline{l+1,n}$, then

$$y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i; \min\limits_{i=\overline{1,l}} p_i\right].$$

As a conclusion, if $\lambda = \frac{\rho l}{1+\rho l}$ and $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \begin{cases} \overline{p_2} = \min\limits_{i=\overline{1,l}} p_i, & \text{if } \min\limits_{i=\overline{1,l}} p_i = p_j, & j = \overline{l+1,n} \\ \in \left[\max\limits_{j=\overline{l+1,n}} p_j ; \min\limits_{i=\overline{1,l}} p_i \right], & \text{if } \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq p_j < \min\limits_{i=\overline{1,l}} p_i, & j = \overline{l+1,n} \\ \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i ; \min\limits_{i=\overline{1,l}} p_i \right], & \text{if } p_j < \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i, & j = \overline{l+1,n} \\ \nexists, & \text{if } \min\limits_{i=\overline{1,l}} p_i < p_j, & j = \overline{l+1,n} \end{cases} \end{cases}$$

else solution does not exist, where $\overline{p_2} = p_j, \ j = \overline{l+1,n}$.

For combination 7+ 4, with $i = \overline{1,l}$ and $j = \overline{l+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\, p_i + a_i p_i + c_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\, p_j + c_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i = 0. \end{cases}$$

From the first set of l equations, it follows that $\sum\limits_{i=1}^{l} a_i < l\,(1-\lambda)$ and replacing in the last equation we obtain

$$\lambda < \frac{\rho l}{1+\rho l}.$$

For 7, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ y^* = p_i, \quad i = \overline{1,l} \end{cases}$$

and for 4, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* \geq p_j, \quad j = \overline{l+1,n}. \end{cases}$$

From $y^* = p_i, \ i = \overline{1,l}$, we conclude that all prices are constant on the set $\{1, 2, ..., l\}$. Denoting by $\overline{p_1} = p_i, \ i = \overline{1,l}$, the conditions which have to be fulfilled by y^* are

$$\overline{p_1} \geq p_j, \quad j = \overline{l+1, n}.$$

In conclusion, if $\lambda < \frac{\rho l}{1+\rho l}$ then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \rho, & i = \overline{1, l} \\ x_j^* = \rho, & j = \overline{l+1, n} \\ y^* = \overline{p_1}, & \text{if } \frac{\overline{p_1}}{p_j} \geq 1, \quad j = \overline{l+1, n} \end{cases}$$

else solution does not exist.

For combination 5 + 4, with $i = \overline{1, l}$ and $j = \overline{l+1, n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\, p_i + a_i p_i - b_i = 0, & i = \overline{1, l} \\ -(1-\lambda)\, p_j + c_j = 0, & j = \overline{l+1, n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i = 0. \end{cases}$$

From the first set of l equations, it follows that $\sum\limits_{i=1}^{l} a_i > l\,(1-\lambda)$ and replacing in the last equation we obtain

$$\lambda > \frac{\rho l}{1+\rho l}.$$

For 5, with $i = \overline{1, l}$ the solution is

$$\begin{cases} x_i^* = \varepsilon, & i = \overline{1, l} \\ y^* = \frac{\varepsilon}{\rho} p_i, & i = \overline{1, l} \end{cases}$$

and for 4, with $j = \overline{l+1, n}$ the solution is

$$\begin{cases} x_j^* = \rho, & j = \overline{l+1, n} \\ y^* \geq p_j, & j = \overline{l+1, n}. \end{cases}$$

From $y^* = \frac{\varepsilon}{\rho} p_i$, $i = \overline{1, l}$, we conclude that all prices are constant on the set $\{1, 2, ...l\}$. Denoting by $\overline{p_1} = p_i$, $i = \overline{1, l}$, the conditions which have to be fulfilled by y^* are

$$\frac{\varepsilon}{\rho} \overline{p_1} \geq p_j, \quad j = \overline{l+1, n}.$$

In conclusion, if $\lambda > \frac{\rho l}{1+\rho l}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1}, \ \text{ if } \frac{\overline{p_1}}{p_j} \geq \frac{\rho}{\varepsilon}, \quad j = \overline{l+1,n} \end{cases}$$

else solution does not exist.

For combination 2 + 7, with $i = \overline{1,l}$ and $j = \overline{l+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\,p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\,p_j + a_j p_j + c_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \rho\sum\limits_{i=1}^{l} a_i - \rho\sum\limits_{j=l+1}^{n} a_j = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i = l\,(1-\lambda)$.

From the next set of n-l equations it follows that $\sum\limits_{j=l+1}^{n} a_j < (n-l)\,(1-\lambda)$.

Replacing in the last equation we obtain

$$\lambda < \frac{\rho n}{1+\rho n}.$$

For 2, with $i = \overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ then the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i\right] \end{cases}$$

and for 7, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = p_j, \quad j = \overline{l+1,n}. \end{cases}$$

From $y^* = p_j,\ j = \overline{l+1,n}$, we conclude that all prices are constant on the set $\{l+1, l+2, ...n\}$. Denoting by $\overline{p_2} = p_j,\ j = \overline{l+1,n}$, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ y^* = \overline{p_2} \end{cases}$$

which implies that

$$\begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon} \overline{p_2} \\ \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases}$$

As a conclusion, if $\lambda < \frac{\rho n}{1+\rho n}$ and $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_2}, \quad \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon} \overline{p_2} \\ \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

else solution does not exist.

For combination 5 + 7, with $i = \overline{1,l}$, $j = \overline{l+1,n}$, if non-critical condition is fulfilled ($\overline{p_1} = p_i$, $i = \overline{1,l}$, $\overline{p_2} = p_j$, $j = \overline{l+1,n}$ and $\frac{\overline{p_2}}{\overline{p_1}} = \frac{\varepsilon}{\rho}$), then Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\overline{p_1} + a_i \overline{p_1} - b_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\frac{\varepsilon}{\rho}\overline{p_1} + a_j \frac{\varepsilon}{\rho}\overline{p_1} + c_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i - \rho \sum\limits_{j=l+1}^{n} a_j = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i > l(1-\lambda)$.
From the next set of n-l equations it follows that

$$\sum_{j=l+1}^{n} a_j < (n-l)(1-\lambda). \tag{5.10}$$

Replacing in the last equation we obtain

$$\lambda - \rho l (1-\lambda) - \rho \sum_{j=l+1}^{n} a_j > 0$$

$$\rho \sum_{j=l+1}^{n} a_j < \lambda - \rho l \left(1 - \lambda\right). \tag{5.11}$$

Because $\exists\, j \in \{l+1, l+2, ..., n\}$ such that $a_j > 0$ it follows that

$$\sum_{j=l+1}^{n} a_j > 0. \tag{5.12}$$

From (5.10) and (5.12) it follows that $(n-l)(1-\lambda) > 0$, which is obvious.
From (5.11) and (5.12) it follows that $\lambda - \rho l (1-\lambda) > 0$, which leads to

$$\lambda > \frac{\rho l}{1+\rho l}.$$

For 5, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1} \end{cases}$$

and for 7, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_2}. \end{cases}$$

It follows that if $\lambda > \frac{\rho l}{1+\rho l}$ and non-critical condition is fulfilled, then an optimal solution for the problem (5.4) is

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1} = \overline{p_2} \end{cases}$$

else solution does not exist.

For combination 2 + 5, with $i = \overline{1,l}$ and $j = \overline{l+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -\left(1-\lambda\right) p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -\left(1-\lambda\right) p_j + a_j p_j - b_j = 0, \quad j = \overline{l+1,n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i - \rho \sum\limits_{j=l+1}^{n} a_j = 0. \end{cases}$$

From the first set of l equations it follows that $\sum_{i=1}^{l} a_i = l\,(1-\lambda)$.

From the next set of n-l equations it follows that $\sum_{j=l+1}^{n} a_j > (n-l)\,(1-\lambda)$.

Replacing in the last equation we obtain

$$\lambda > \frac{\rho n}{1+\rho n}.$$

For 2, with $i = \overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i\right] \end{cases}$$

and for 5, with $j = \overline{l+1,n}$ the solution is

$$\begin{cases} x_j^* = \varepsilon, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho} p_j, \quad j = \overline{l+1,n}. \end{cases}$$

From $y^* = \frac{\varepsilon}{\rho} p_j,\ j = \overline{l+1,n}$, we conclude that all prices are constant on the set $\{l+1, l+2, \ldots n\}$. Denoting by $\overline{p_2} = p_j,\ j = \overline{l+1,n}$, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ y^* = \frac{\varepsilon}{\rho} \overline{p_2} \end{cases}$$

which implies that

$$\begin{aligned} &\max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ &\frac{\varepsilon}{\rho} \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i. \end{aligned}$$

If $\lambda > \frac{\rho n}{1+\rho n}$ and $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho} \overline{p_2}, \quad \text{if} \ \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho} \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

else solution does not exist.

For combination 2 + 5 + 7, with $i = \overline{1,l}$, $j = \overline{l+1,m}$ and $k = \overline{m+1,n}$, if non-critical condition is fulfilled ($\overline{p_2} = p_j,\ j = \overline{l+1,m}$; $\overline{p_3} = p_k,\ k = \overline{m+1,n}$ and $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$), then Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -\left(1-\lambda\right)p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -\left(1-\lambda\right)\overline{p_2} + a_j\overline{p_2} - b_j = 0, \quad j = \overline{l+1,m} \\ -\left(1-\lambda\right)\overline{p_3} + a_k\overline{p_3} + c_k = 0, \quad k = \overline{m+1,n} \\ \lambda - \rho\sum\limits_{i=1}^{l} a_i - \rho\sum\limits_{j=l+1}^{m} a_j - \rho\sum\limits_{k=m+1}^{n} a_k = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i = l\left(1-\lambda\right)$.
From the next set of m-l equations it follows that

$$\sum_{j=l+1}^{m} a_j > \left(m-l\right)\left(1-\lambda\right).$$

From the next set of n-m equations it follows that

$$\sum_{k=m+1}^{n} a_k < \left(n-m\right)\left(1-\lambda\right). \tag{5.13}$$

Replacing in the last equation we obtain

$$\rho\sum_{k=m+1}^{n} a_k < \lambda - \rho l\left(1-\lambda\right) - \rho\left(m-l\right)\left(1-\lambda\right). \tag{5.14}$$

Because $\exists\ k \in \{m+1, m+2, ..., n\}$ such that $a_k > 0$ it follows that

$$\sum_{k=m+1}^{n} a_k > 0. \tag{5.15}$$

From (5.13) and (5.15) it follows that $\left(n-m\right)\left(1-\lambda\right) > 0$, which is obvious.
From (5.14) and (5.15) it follows that $\lambda - \rho l\left(1-\lambda\right) - \rho\left(m-l\right)\left(1-\lambda\right) > 0$, which leads to

$$\lambda > \frac{\rho m}{1+\rho m}.$$

For 2, with $i = \overline{1, l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1, l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i ; \ \min\limits_{i=\overline{1,l}} p_i \right] \end{cases}$$

for 5, with $j = \overline{l+1, m}$ the solution is

$$\begin{cases} x_j^* = \varepsilon, \quad j = \overline{l+1, m} \\ y^* = \frac{\varepsilon}{\rho} \overline{p_2} \end{cases}$$

and for 7, with $k = \overline{m+1, n}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad k = \overline{m+1, n} \\ y^* = \overline{p_3}. \end{cases}$$

Under this assumptions, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ y^* = \frac{\varepsilon}{\rho} \overline{p_2} \\ y^* = \overline{p_3} \end{cases}$$

which implies that

$$\begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho} \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \\ \overline{p_3} = \frac{\varepsilon}{\rho} \overline{p_2}. \end{cases}$$

If $\lambda > \frac{\rho m}{1+\rho m}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ and non-critical condition is fulfilled, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1, l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1, m} \\ x_k^* = \rho, \quad k = \overline{m+1, n} \\ y^* = \frac{\varepsilon}{\rho} \overline{p_2} = \overline{p_3}, \ \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho} \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

else solution does not exist, where $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ and $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

For combination 2 + 4 + 5, with $i = \overline{1,l},\ j = \overline{l+1,m}$ and $k = \overline{m+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\,p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\,p_j + c_j = 0, \quad j = \overline{l+1,m} \\ -(1-\lambda)\,p_k + a_k p_k - b_k = 0, \quad k = \overline{m+1,n} \\ \lambda - \rho \sum\limits_{i=1}^{l} a_i - \rho \sum\limits_{k=m+1}^{n} a_k = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i = l\,(1-\lambda)$.
From the next set of m-l equations it follows that $c_j = (1-\lambda)\,p_j > 0,\ j = \overline{l+1,m}$, which is obvious.
From the next set of n–m equations it follows that $\sum\limits_{k=m+1}^{n} a_k > (n-m)\,(1-\lambda)$.
Replacing in the last equation we obtain

$$\lambda > \frac{\rho l + \rho\,(n-m)}{1 + \rho l + \rho\,(n-m)}.$$

For 2, with $i = \overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i\right]. \end{cases}$$

For 4, with $j = \overline{l+1,m}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,m} \\ y^* \geq p_j, \quad j = \overline{l+1,m}. \end{cases}$$

For 5, with $k = \overline{m+1,n}$ the solution is

$$\begin{cases} x_k^* = \varepsilon, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho} p_k, \quad k = \overline{m+1,n}. \end{cases}$$

From $y^* = \frac{\varepsilon}{\rho} p_k,\ k = \overline{m+1,n}$, we state that all prices are constant on $\{m+1, m+2, ...n\}$. Denoting by $\overline{p_3} = p_k,\ k = \overline{m+1,n}$, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \le y^* \\ y^* \le \min\limits_{i=\overline{1,l}} p_i \\ y^* \ge p_j, \quad j=\overline{l+1,m} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3} \end{cases}$$

which implies

$$\begin{cases} \max\limits_{i=\overline{1,l}} p_i \le \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \le \min\limits_{i=\overline{1,l}} p_i \\ p_j \le \frac{\varepsilon}{\rho}\overline{p_3}, \quad j=\overline{l+1,m}. \end{cases}$$

In conclusion, if $\lambda > \frac{\rho l+\rho(n-m)}{1+\rho l+\rho(n-m)}$ and $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \ge \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i=\overline{1,l} \\ x_j^* = \rho, \quad j=\overline{l+1,m} \\ x_k^* = \varepsilon, \quad k=\overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3}, \ \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \le \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \le \min\limits_{i=\overline{1,l}} p_i \\ p_j \le \frac{\varepsilon}{\rho}\overline{p_3}, \quad j=\overline{l+1,m} \end{cases} \end{cases}$$

else solution does not exist.

For combination 2 + 4 + 7, with $i=\overline{1,l}$, $j=\overline{l+1,m}$, $k=\overline{m+1,n}$, the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\,p_i + a_i p_i = 0, \quad i=\overline{1,l} \\ -(1-\lambda)\,p_j + c_j = 0, \quad j=\overline{l+1,m} \\ -(1-\lambda)\,p_k + a_k p_k + c_k = 0, \quad k=\overline{m+1,n} \\ \lambda - \rho\sum\limits_{i=1}^{l} a_i - \rho\sum\limits_{k=m+1}^{n} a_k = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i = l\,(1-\lambda)$.
From the next set of m-l equations it follows that $c_j = (1-\lambda)\,p_j > 0,\ j=\overline{l+1,m}$, which is obvious.
From the next set of n–m equations it follows that $\sum\limits_{k=m+1}^{n} a_k < (n-m)\,(1-\lambda)$.

Replacing in the last equation we obtain

$$\lambda < \frac{\rho l + \rho (n-m)}{1 + \rho l + \rho (n-m)}.$$

For 2, with $i = \overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i \right]. \end{cases}$$

For 4, with $j = \overline{l+1,m}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,m} \\ y^* \geq p_j, \quad j = \overline{l+1,m}. \end{cases}$$

For 7, with $k = \overline{m+1,n}$ the solution is

$$\begin{cases} x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = p_k, \quad k = \overline{m+1,n}. \end{cases}$$

From $y^* = p_k,\ k = \overline{m+1,n}$, we state that all prices are constant on $\{m+1, m+2, ...n\}$. Denoting by $\overline{p_3} = p_k,\ k = \overline{m+1,n}$, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ y^* \geq p_j, \quad j = \overline{l+1,m} \\ y^* = \overline{p_3} \end{cases}$$

which implies

$$\begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon} \overline{p_3} \\ \overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \overline{p_3}, \quad j = \overline{l+1,m}. \end{cases}$$

If $\lambda < \frac{\rho l+\rho(n-m)}{1+\rho l+\rho(n-m)}$ and $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p_3}, \quad \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_3} \\ \overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

else solution does not exist.

For combination *4 + 5 + 7*, with $i = \overline{1,l}$, $j = \overline{l+1,m}$ and $k = \overline{m+1,n}$, if non-critical condition is fulfilled ($\overline{p_2} = p_j,\ j = \overline{l+1,m}$, $\overline{p_3} = p_k,\ k = \overline{m+1,n}$ and $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$), the Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\, p_i + c_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\, \overline{p_2} + a_j \overline{p_2} - b_j = 0, \quad j = \overline{l+1,m} \\ -(1-\lambda)\, \overline{p_3} + a_k \overline{p_3} + c_k = 0, \quad k = \overline{m+1,n} \\ \lambda - \rho \sum\limits_{j=l+1}^{m} a_j - \rho \sum\limits_{k=m+1}^{n} a_k = 0. \end{cases}$$

From the first set of l equations it follows that $c_i = (1-\lambda)\, p_i > 0,\ i = \overline{1,l}$.

From the next set of *m-l* equations it follows that

$$\sum_{j=l+1}^{m} a_j > (m-l)\,(1-\lambda)\,.$$

From the next set of *n–m* equations it follows that

$$\sum_{k=m+1}^{n} a_k < (n-m)\,(1-\lambda)\,. \tag{5.16}$$

Replacing in the last equation we obtain

$$\rho \sum_{k=m+1}^{n} a_k < \lambda - \rho\,(m-l)\,(1-\lambda)\,. \tag{5.17}$$

Because $\exists\, k \in \{m+1, m+2, ..., n\}$ such that $a_k > 0$ it follows that

$$\sum_{k=m+1}^{n} a_k > 0. \tag{5.18}$$

From (5.16) and (5.18) it follows that $(n-m)(1-\lambda) > 0$, which is obvious.
From (5.17) and (5.18) it follows that $\lambda - \rho(m-l)(1-\lambda) > 0$, which leads to

$$\lambda > \frac{\rho(m-l)}{1+\rho(m-l)}.$$

For 4, with $i = \overline{1,l}$ the solution is

$$\begin{cases} x_i^* = \rho, & i = \overline{1,l} \\ y^* \geq p_i, & i = \overline{1,l}, \end{cases}$$

for 5, with $j = \overline{l+1,m}$ the solution is

$$\begin{cases} x_j^* = \varepsilon, & j = \overline{l+1,m} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} \end{cases}$$

and for 7, with $k = \overline{m+1,n}$ the solution is

$$\begin{cases} x_k^* = \rho, & k = \overline{m+1,n} \\ y^* = \overline{p_3}. \end{cases}$$

Under this assumptions, the conditions which have to be fulfilled by y^* are

$$\begin{cases} p_i \leq y^*, & i = \overline{1,l} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} \\ y^* = \overline{p_3} \end{cases}$$

which implies that

$$\begin{cases} p_i \leq \frac{\varepsilon}{\rho}\overline{p_2}, & i = \overline{1,l} \\ \overline{p_3} = \frac{\varepsilon}{\rho}\overline{p_2}. \end{cases}$$

If $\lambda > \frac{\rho(m-l)}{1+\rho(m-l)}$ and the non-critical condition is fulfilled, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, \ \text{ if } \ p_i \le \frac{\varepsilon}{\rho}\overline{p_2}, \quad i = \overline{1,l} \end{cases}$$

else solution does not exist.

For combination 2+ 4 + 5 + 7, with $i = \overline{1,l}$, $j = \overline{l+1,m}$, $k = \overline{m+1,t}$ and $s = \overline{t+1,n}$, if non-critical condition is fulfilled ($\overline{p_3} = p_k,\ k = \overline{m+1,t}$, $\overline{p_4} = p_s,\ s = \overline{t+1,n}$ and $\frac{\overline{p_4}}{\overline{p_3}} = \frac{\varepsilon}{\rho}$), then Kuhn-Tucker conditions (5.5) and (5.6) will be

$$\begin{cases} -(1-\lambda)\,p_i + a_i p_i = 0, \quad i = \overline{1,l} \\ -(1-\lambda)\,p_j + c_j = 0, \quad j = \overline{l+1,m} \\ -(1-\lambda)\,\overline{p_3} + a_k\overline{p_3} - b_k = 0, \quad k = \overline{m+1,t} \\ -(1-\lambda)\,\overline{p_4} + a_s\overline{p_4} + c_s = 0, \quad s = \overline{t+1,n} \\ \lambda - \rho\sum\limits_{i=1}^{l} a_i - \rho\sum\limits_{k=m+1}^{t} a_k - \rho\sum\limits_{s=t+1}^{n} a_s = 0. \end{cases}$$

From the first set of l equations it follows that $\sum\limits_{i=1}^{l} a_i = l\,(1-\lambda)$.
From the next set of m-l equations it follows that $c_j = (1-\lambda)\,p_j > 0,\ j = \overline{l+1,m}$, which is obvious.
From the next set of t-m equations it follows that

$$\sum_{k=m+1}^{t} a_k > (t-m)\,(1-\lambda)\,.$$

From the next set of n–t equations it follows that

$$\sum_{s=t+1}^{n} a_s < (n-t)\,(1-\lambda)\,. \tag{5.19}$$

Replacing in the last equation we obtain

$$\rho\sum_{s=t+1}^{n} a_s < \lambda - \rho l\,(1-\lambda) - \rho\,(t-m)\,(1-\lambda)\,. \tag{5.20}$$

Because $\exists\ s \in \{t+1, t+2, ..., n\}$ such that $a_s > 0$ it follows that

$$\sum_{s=t+1}^{n} a_s > 0. \tag{5.21}$$

From (5.19) and (5.21) it follows that $(n-t)(1-\lambda) > 0$, which is obvious.
From (5.20) and (5.21) it follows that $\lambda - \rho l(1-\lambda) - \rho(t-m)(1-\lambda) > 0$, which leads to

$$\lambda > \frac{\rho(l+t-m)}{1+\rho(l+t-m)}.$$

For 2, with $i = \overline{1,l}$, if $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, then the solution is

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ y^* \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i;\ \min\limits_{i=\overline{1,l}} p_i\right] \end{cases}.$$

For 4, with $j = \overline{l+1,m}$ the solution is

$$\begin{cases} x_j^* = \rho, \quad j = \overline{l+1,m} \\ y^* \geq p_j, \quad j = \overline{l+1,m}. \end{cases}$$

For 5, with $k = \overline{m+1,t}$ the solution is

$$\begin{cases} x_k^* = \varepsilon, \quad k = \overline{m+1,t} \\ y^* = \frac{\varepsilon}{\rho} \overline{p_3} \end{cases}$$

and for 7, with $s = \overline{t+1,n}$ the solution is

$$\begin{cases} x_s^* = \rho, \quad s = \overline{t+1,n} \\ y^* = \overline{p_4}. \end{cases}.$$

Under this assumptions, the conditions which have to be fulfilled by y^* are

$$\begin{cases} \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq y^* \\ y^* \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq y^*, \quad j = \overline{l+1,m} \\ y^* = \frac{\varepsilon}{\rho} \overline{p_3} \\ y^* = \overline{p_4} \end{cases}$$

which implies

$$\begin{cases} \max\limits_{i=\overline{1,l}} p_i \le \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \le \min\limits_{i=\overline{1,l}} p_i \\ p_j \le \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \\ \overline{p_4} = \frac{\varepsilon}{\rho}\overline{p_3}. \end{cases}$$

If $\lambda > \frac{\rho l+\rho(t-m)}{1+\rho l+\rho(t-m)}, \frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \ge \frac{\varepsilon}{\rho}$ and non-critical condition is fulfilled, then an optimal solution for problem (5.4) is

$$\begin{cases} x_i^* = \frac{\rho}{p_i}y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \varepsilon, \quad k = \overline{m+1,t} \\ x_s^* = \rho, \quad s = \overline{t+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3} = \overline{p_4}, \quad \text{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \le \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \le \min\limits_{i=\overline{1,l}} p_i \\ p_j \le \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

else solution does not exist.

This ends the proof of our theorem. ■

Based on Lemma 5.4.1, Theorem 5.4.5 is providing an efficient solution $x^* \in \mathbb{R}^n$ for problem (5.2).

The values for the objective function of problem (5.2) are provided by: (a) Theorem 5.4.5 in case of fluctuation, knowing that $\max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\} = y^*$ and (b) the following Theorem in case of turnover, where TR denotes the turnover of the power plant.

Theorem 5.4.6 (Luca and Duca [92]; energy index) .

The values for second member of the objective function from problem (5.2) are:

1. $TR = n\rho y^*$, *where y^* is defined by Theorem 5.4.5 item 1;*
2. $TR = \varepsilon \sum\limits_{i=1}^{n} p_i$;
3. $TR = \rho \sum\limits_{i=1}^{n} p_i$;
4. $TR = \rho l y^* + \rho \sum\limits_{j=l+1}^{n} p_j$, *where y^* is defined by Theorem 5.4.5 item 4;*

5. $TR = l\rho\overline{p_1} + \rho \sum_{j=1+1}^{n} p_j$, *where* $\overline{p_1} = p_i,\ i = \overline{1, l}$;

6. $TR = l\varepsilon\overline{p_1} + \rho \sum_{j=l+1}^{n} p_j$, *where* $\overline{p_1} = p_i,\ i = \overline{1, l}$;

7. $TR = n\rho\,\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1, n}$;

8. $TR = n\varepsilon\overline{p_1}$, *where* $\overline{p_1} = p_i,\ i = \overline{1, l}$;

9. $TR = n\varepsilon\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1, n}$;

10. $TR = n\varepsilon\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1, m}$;

11. $TR = (n - m + l)\varepsilon\overline{p_3} + \rho \sum_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1, n}$;

12. $TR = (n - m + l)\rho\overline{p_3} + \rho \sum_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1, n}$;

13. $TR = (n - l)\epsilon\overline{p_2} + \rho\sum_{i=1}^{l} p_i$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1, m}$;

14. $TR = (n - m + l)\varepsilon\,\overline{p_3} + \rho \sum_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1, t}$.

Proof. Values for turnover are computed by replacing x^* obtained in Theorem 5.4.5 into $\sum_{i=1}^{n} x_i^* p_i$. ■

5.5 Model validation and conclusions

5.5.1 Testing of solution

For testing solutions provided by index model (5.2) we will use the same validated data as for minimax model (4.2). Also energy prices will be the same, being detailed in Appendix A.

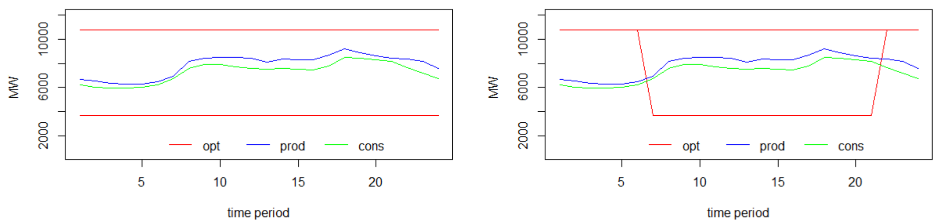
To keep homogeneity in our analysis, we will refer to the same three categories of performances for results, as for minimax, namely weak, good and excellent.

Two different tests have been performed for index model. The difference between them is given by the values used as input data, ε and ρ. To validate an hypothesis, another test, with different values for ε and ρ, was performed, but only for a particular solution.

The input data used for the first test are the same as for minimax. Important to be mentioned is that the predefined level r, present in the minimax and with a key role there, is missing from the index model. Thus, implementation of index model is requiring less effort. The following values will be used as input data:

- $\varepsilon = 3666$ MW (minimum consumption, based on validated values, over the entire time period)
- $\rho = 10808$ MW (maximum production, based on validated values, over the entire time period)

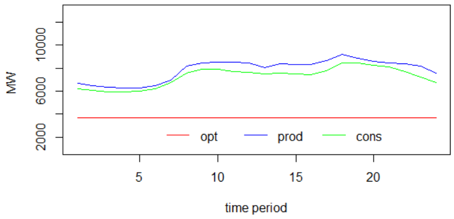
Theorem 5.4.5 is providing an optimal production plan. The fourteen different solutions combined with the three possible price plans have generated eight possible results. Testing of solution 14 was not possible, due to the fact that it requires a price plan with four different tariffs, which was not stipulated by ANRE. We notice the limited number of results compared to minimax, where for the same input data we had twelve. It is due to the fact that non-critical and existence conditions of solutions are not always satisfied. It is the case of solutions 7, 8, 9, 10 and 13. No excellent results have been obtained during this test. Charts for good and weak results are presented below.



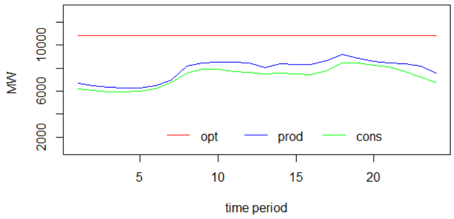
(A) Index.1 1 with Plan I (B) Index.1 4 with Plan II

FIGURE 5.1: Index.1 good results

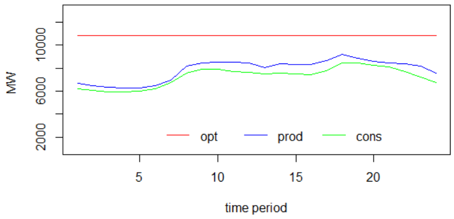
Results provided by both solutions evaluated with good performance and presented in Figure 5.1 are very general. They offer an interval for optimal production plan, which includes the real production and consumption curves. From mathematical point of view results might be good, but from practical point of view they require a lot of additional work to generate an applicable production plan. To be noticed is that in both situations, when deciding for the production plan, the decision factor has to consider that consumption must be covered.



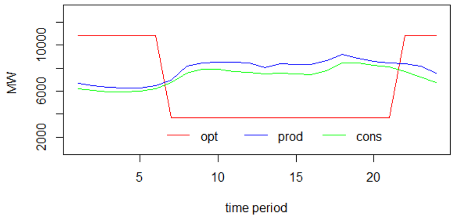
(A) Index.1 2 with Plan I



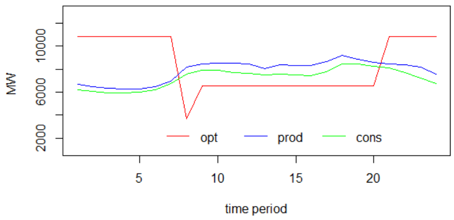
(B) Index.1 3 with Plan I



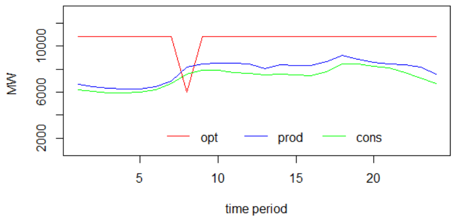
(C) Index.1 5 with Plan II



(D) Index.1 6 with Plan II



(E) Index.1 11 with Plan III



(F) Index.1 12 with Plan III

FIGURE 5.2: Index.1 weak results

Figure 5.2 is presenting charts of solutions evaluated with weak results. These solutions are placing the production plan to the minimum or maximum values assumed (ε and ρ), or as it is the case of solution 11 with Plan III, presented in 5.2e, to low to cover demand. Let's keep in mind that this solution (11) has the proper configuration for an excellent behavior.

Big number of solutions for which non-critical and existence conditions are not satisfied and especially the behavior of solution 11 with Plan III, refereed above, were incentives for the second test. Input data were chosen in such a way that non-critical conditions are satisfied. Thus we will use the following values: $\varepsilon = 8500$ MW and $\rho = 26154$ MW.

Before starting the simulations we have to make the remark that by choosing the value of 26154 MW for ρ, the maximum installed capacity of Romania (24513 MW) is exceeded. In case of applying the algorithm to a power plant it is similar with assuming to deliver more energy than the own production capacity. Is this situation acceptable? We have enough arguments to say yes. As it will be visible in the charts, the optimal production plan will be placed to the maximum assumed value only during night period, when anyhow consumption is much lower and manual interference to adjust the solution is required. We are dealing here with a typical over planing situation which is supporting the optimization process to improve the solution. It is a common situation in real life, with examples like: over booking in airplane industry to assure the maximization of airplanes loading or over planing in production to assure increase of efficiency.

Let's proceed now to testing and analysis of obtained results. Of course the same three categories for evaluation of results performances will be maintained. Noticeable to be mentioned is that the number of solutions for which the non-critical conditions are not satisfied has decreased from five to three.

Unfortunately, also during this test, no excellent results were obtained.

Figure 5.3 is presenting charts of solutions evaluated with good behavior. For a better visualization we have realized, where it was necessary, also a zoom in on the critical part of the charts.

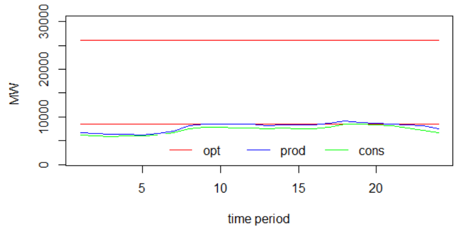
Solution 1 with Plan I is generating an interval. As it is visible in the zoom, the lover limit of the interval is a flat line shaving the peak load. Over production generated for night period is significantly reduced compared with all other previous results. Important to be mentioned is that consumption is always covered.

Solution 1 with plan II has a similar behavior with the lower part of the interval from solution 1 with Plan I. Difference is that the over production during night hours is higher. Manual adjustment is required and consumption is always covered.

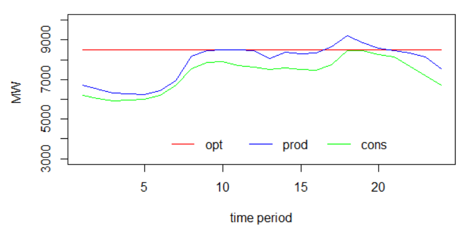
Solution 2 with Plan I did not require a zoom in. Behavior is identical with the lower part of the interval from solution 1 with Plan I.

Solutions 4, 7, 8 and 9 with Plan II have an identical behavior with Solution 1 with Plan II.

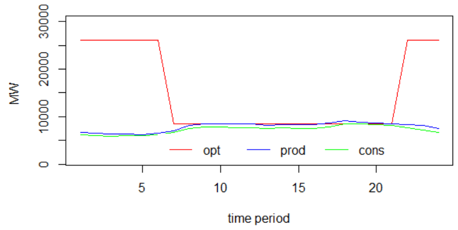
All these eight solutions which are providing results evaluated with a good behavior prove that our objective was realized. They all



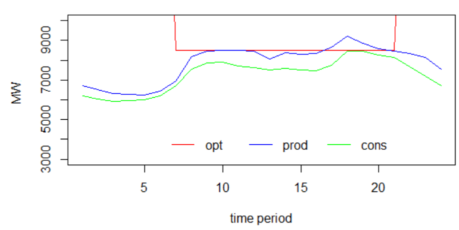
(A) Index.2 1 with Plan I



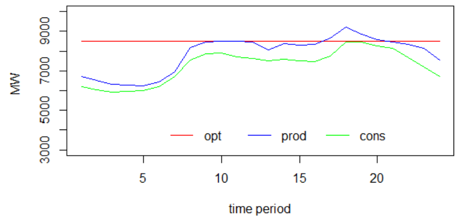
(B) Zoom on Index.2 1 with Plan I



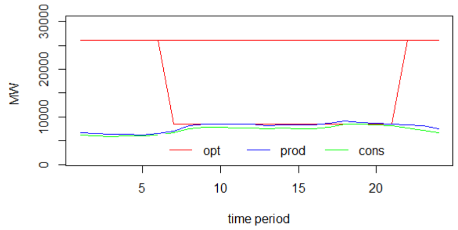
(C) Index.2 1 with Plan II



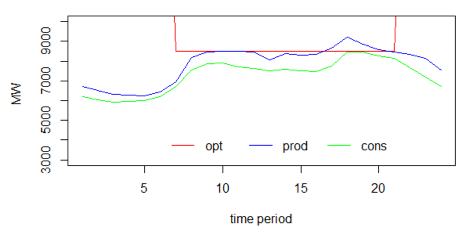
(D) Zoom on Index.2 1 with Plan II



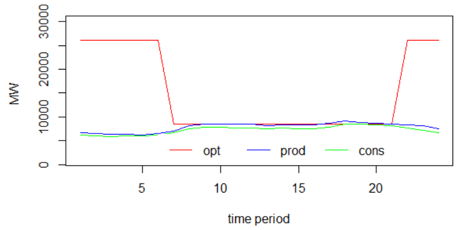
(E) Index.2 2 with Plan I



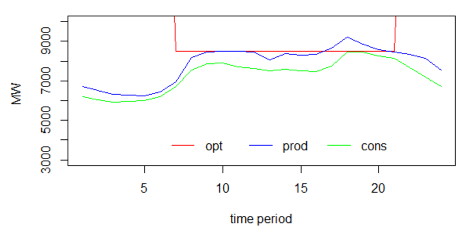
(F) Index.2 4 with Plan II



(G) Zoom on Index.2 4 with Plan II

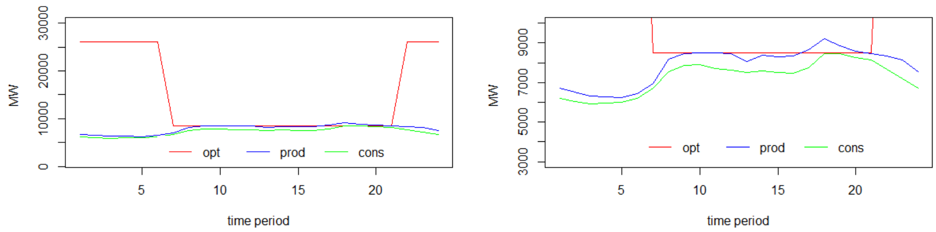


(H) Index.2 7 with Plan II

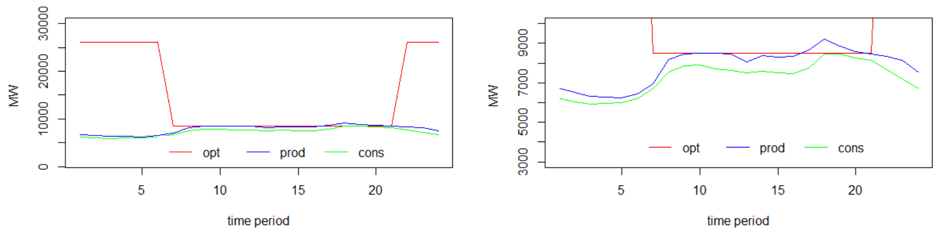


(I) Zoom on Index.2 7 with Plan II

FIGURE 5.3: Index.2 good results



(J) Index.2 8 with Plan II (K) Zoom on Index.2 8 with Plan II



(L) Index.2 9 with Plan II (M) Zoom on Index.2 9 with Plan II

FIGURE 5.3: Index.2 good results

shave the peak load, flatter the production curve and maximize economic performances. In case of intervals, by choosing the minimum level, turnover is sacrificed for a reduced fluctuation. Point A of the Figure 5.4 is corresponding to this combination of sacrificed turnover for a reduced fluctuation.

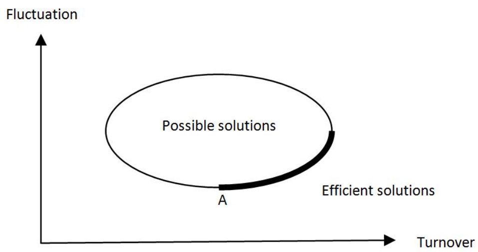
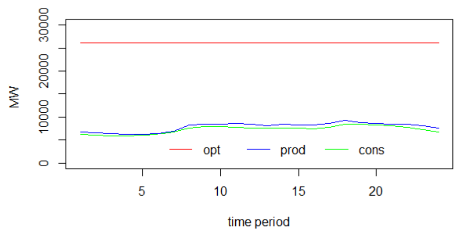
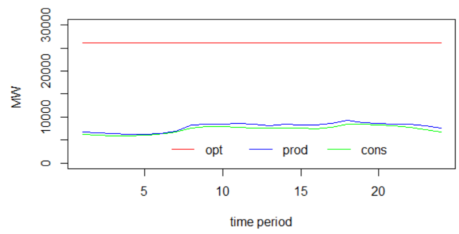


FIGURE 5.4: Efficient frontier

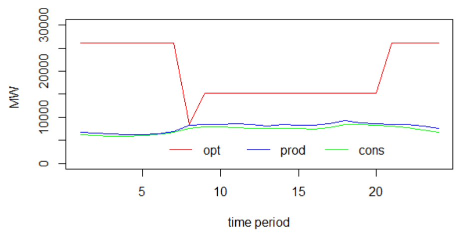
Figure 5.5 is presenting charts corresponding to solutions which are providing results evaluated with weak behavior. Solutions 3 with Plan I, 5 with Plan II and 12 with Plan III are placing the optimal production plan to the maximum assumed level, which of course can't be accepted from practical point of view. Solution 11 with Plan III displays a change of levels which might make us think to an excellent behavior, but due to input data it is situated to high compared with real production and consumption curves.



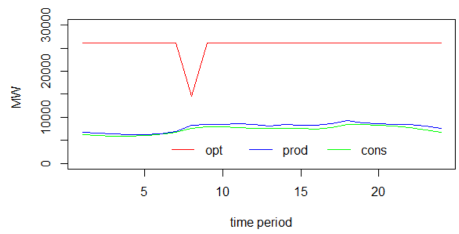
(A) Index.2 3 with Plan I



(B) Index.2 5 with Plan II



(C) Index.2 11 with Plan III



(D) Index.2 12 with Plan III

FIGURE 5.5: Index.2 weak results

As a conclusion for the second test of index model, we might say that objective was again realized and performances are better that those obtained during the first test.

A third test was performed to check if Solution 11 with Plan III might provide excellent behavior. At first test the optimized production curve was to low, while at second test it was to high. We have estimated that a value of 5000 MW for ε might be acceptable for our purpose and in order to satisfy the non-critical and existence conditions we have computed that ρ must be 15385 MW. Thus after performing all computations, we have obtained a solution for which the chart is displayed in Figure 5.6. It shows that peak load is shaved and production curve is flattered, but during gap period the production of energy drops dramatically, even under the level of consumption. This drop is generated by the high price of electricity (805.9 lei/MWh) during gap period. Therefore upgrade of Solution 11 with Plan III from good to excellent can't be performed.

Solution 14 of the index model could not be tested. The other solutions are generating results evaluated with good and weak. Excellent results were not achieved. Solution 11 with Plan III displays a potential for excellent behavior, but during gap period production is dropping too much.

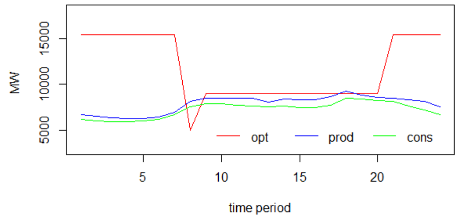


FIGURE 5.6: Index.3 11 with Plan III

Tests performed for index model prove its capacity to shave the peak load. All additional advantages associated to minimax model and presented in Section 4.5.1 remain valid also for index model. More over, index model requires less input data that minimax.

5.5.2 Conclusions

Presence of predefined level r in the minimax model has two disadvantages. Model has a higher complexity and an efficient solution will never fall below it. To reduce complexity of minimax model we have developed a new model. Index model is also a bi-criteria optimization problem which aims to shave the peak load of energy production by minimizing fluctuation of energy and maximizing the economic performance.

Index measure for fluctuation of energy was defined. Idea was inspired by the procedure of reporting loading of an equipment to a fixed value, as mentioned in [109] and [112]

Turnover, as measure for economic performance and simple technical constraints, limiting the amount of energy to be produced are employed.

Index model was solved following the same logic as for minimax. Index energy model (5.2) is transformed in an equivalent parametric problem (5.4). Theorem 5.4.5 is computing the optimal solution for parametric problem (5.4). Proof is based on Kuhn-Tucker conditions and follows the same four steps. Because parametric problem (5.4) is a convex one, the Kuhn-Tucker conditions are necessary and sufficient.

Possible combinations of Kuhn-Tucker multipliers are determined, at first step, based on their capacity to satisfy, at a fixed moment of time, the complementarity slackness and dual feasibility conditions.

At second step, *feasible combinations* are selected from the possible combinations, the criterion being that Lagrangian is zero.
At third step the critical combinations are identified, providing the non-critical conditions.
At last step, the optimal solution for parametric problem (5.4) is computed, if the same combination is multiplied over the entire time period or if different *non-critical combinations* are combined. Theorem 4.4.11 is used when dealing with order of scenarios in a solution.
Returning to index energy model (5.2), Theorem 5.4.6 is used to compute the efficient solution.

Test were conducted to verify the results provided by index model and to validate it as a reliable method for shaving the peak load. Same validated data provided by Transelectrica [148] and ANRE [129] were used. Production plan computed using index model was compared with real production and consumption curves from 4th of December 2015. Based on their behavior, results were evaluated again with excellent, good and weak.
Two tests were performed. Solution 14 could not be simulated in no one of the tests, due to the lack of price plans with four tariffs.

For the first test we have used the same inputs as for minimax model. For several solutions, non-critical conditions were not satisfied, so they could not be tested. Two solutions have generated good results, the rest generating weak results. No excellent result was obtained. For solution 11 with Plan III we notice the potential for excellent result.

For the second test, input data were chosen considering also the non-critical conditions. Performances have improved. The number of solutions which could not be tested has decreased, while the number of solutions which have generated good results has increased. Again no excellent result was obtained, but for the same solution, 11 with Plan III, the potential for excellent results was noticed.

A third test, with fine tuned input data, was performed to evaluate solution 11 with Plan III. The result was again unsatisfactory, so solution 11 with Plan III could not be upgraded to excellent.

After performing these test we can confirm that index model has achieved the objective of shaving the peak load by minimizing fluctuation of energy and maximizing economic performance. Index model might be validated, being reliable for shaving the peak load.

The price of simplifying the measure for fluctuation is reflected in the accuracy. Model is easier and quicker to be implemented but no excellent results were obtained. Additional complex technical constraints or profit as measure of economic performance might increase complexity of index model, but solution would be more accurate.

Our contributions to this chapter, disseminated in [92] and [90], might be synthesized in:

- introduction of index measure of fluctuation.
- development of index energy model (5.2),
- Lemma 5.4.1, used in a similar way as Lemma 4.4.1 to prove the equivalency between index energy model (5.2) and parametric model (5.4),
- Theorem 5.4.5 which computes the optimal solution for parametric model (5.4),
- Theorem 5.4.6 which computes the efficient solution for index energy model (5.2),
- Tests for index model performed using real data.

Chapter 6

Conclusions

Global context of energy production characterized by: *human dependency on energy*, *strategical power provided by electricity*, *impact on climate change* due to CO_2 emissions resulted from burning fuels in thermal power plants, *projected increase of energy consumption and population*, *massive investments associated to power plants and power grids*, *intermittent production feature of renewable sources* and *increased frequency and magnitude of peak loads*, is generating challenges to optimize production of energy.

Energy field is very complex due to: different objectives and actions required on long, medium and short term, specific procedures for regulated and deregulated markets, complexity and challenges generated by static, dynamic (technical) and environmental restriction.

Mathematical models, used for optimizing the production plan of a power plant, have been created and solved, among others, by: Frangioni [50], Martinez [102], Philpott [123] – for thermal-plants; Bor-ghetti [13] – for hydro-plants; Belloni [8], Redondo [127], Lemarechal [84], Gollmer [55], Nowak [117], Nürnberg [118] – for systems of thermo and hydro plants; Wen [154], Zhang [161], Gross [58], Conejo [27], [28], Ladurantaye [81], Gonzalez [56], Eichhorn [44] and Dicorato [36] – for deregulated markets; Cormio [30], Islam [67], Akella [1], Martins [103], Watson [153], Nakata [114], Dudhani [41], Babu [7], Duic [42], Morais [108], Marcato [100] and Mahalov et all [130], [132], [87], [131]. – for models with environmental restrictions.

Analyzing a daily production and consumption curve, like for example the one presented in Figure 1.2, shows a fluctuation of energy during 24 hours with a peak load.

Current trends are projecting an increase for fluctuation of energy and magnitude of peak loads, creating pressure on: *producers* – which have to anticipate peak loads and adapt the production plans; *power grids* – which are close to reaching their capacity and *energy price on deregulated markets*.

We aim to address that peak load and to shave it. Scientific research for shaving the peak load have been conducted, among others by: Nourai [116], Jayasekara [68], Yang [157], Dasgupta [33], Eyer [47], Gajduk [51], Gotham [57], Uddin [149], Chua [23, 24], Leadbetter [82], Reihani [128], Zheng [162], Fossati [49], Kalkhambkar [70], Kerdphol [72], Lu [88], Motalleb [111], Comodi [26], Sigrist [139], Yan [156], Muratori [113], Paterakis [120] and Tascikaraoglu [144].

Fluctuation of energy might be regarded as a spread of values, the most distant one being the peak load. By controlling the spread it might be possible to reduce the magnitude of peak load. In the same time controlling the spread will influence the amount of energy produced and thus the economic performance of the power plant might decrease, situation which again has to be controlled. This is creating a proper context for a bi-criteria optimization problem.

Considering all these, we define the objective for our research as: **to create, solve and validate a mathematical model able to shave the peak load, by minimizing fluctuation of energy and maximizing economic performance of the power plant**.

As consequence of shaving the peak load, a supra-production of energy will appear, which might be stored as: pumping–turbine principle, kinetic energy, thermal energy, batteries, compressed or liquified air. Storage cost, efficiency and power recovery time depends on the storage method.

A measure for fluctuation of energy is proper for our objective if will allow us to target the peak load and to reduce its magnitude when fluctuation is minimized.

To create a mathematical model for shaving the peak load we have to define fluctuation of energy, economic performance and some constraints. Fluctuation of energy has the key role in achieving or missing our objective, while economic performance and constraints will influence accuracy of solution.

Let's recall that fluctuation of energy might be regarded as a spread, for which mathematics offers several measures. We have evaluated variance, mean absolute deviation and maximum absolute deviation

and concluded that maximum absolute deviation is satisfactory for our purpose to target directly the most extreme point. It will be the starting point for developing minimax measure for fluctuation of energy.
Choosing turnover as a measure for economic performance was determined also by the possibility to obtain real data for testing the model in order to validate it. Of course profit would increase the accuracy of solution.
Simple technical constraints, which limit the amount of energy produced, are used.

Minimax measure for fluctuation of energy (4.1) is defined starting from maximum absolute deviation. It requires a predefined level of energy around which fluctuation will be calculated. Behavior of energy price related to quantity (on short term) and its elasticity (on long term), have determined us to introduce price in the minimax measure.

Minimax model (4.2), developed based on minimax measure for fluctuation (4.1) and turnover as indicator for economic performance, aims to shave the peak load by minimizing fluctuation of energy around a predefined level r and maximizing turnover. Efficient solution represents the amount of energy to be produced at each time moment, such that fluctuation is minimized and turnover maximized. Additionally, this solution owns the capacity to shave the peak load.

To solve the minimax energy model (4.2) it was transformed, using Lemma 4.4.1, in the equivalent bi-criteria optimization problem (4.3). Based on Theorem 2.3.1 of Yu [160] and similar results of Bot et all [14] and Geoffrion [53], Lemma 4.4.2 was established and used to transform the equivalent bi-criteria optimization problem (4.3) in the parametric optimization problem (4.4).

Theorem 4.4.5, is computing the optimal solution for parametric optimization problem (4.4). To prove this theorem we have used Kuhn-Tucker Theorem 2.3.4 and a result, formulated as Theorem 4.4.11, which states that the order of scenarios in a solution is negligible.
Prove of Theorem 4.4.5 is a four steps process.

Let's call the objective of problem (4.4), addressed by Theorem 4.4.5 an hypothetical result. Minimizing of the hypothetical result means that fluctuation of energy is minimized and turnover maximized. Because problem (4.4) is a convex one, the Kuhn- Tucker conditions are necessary and sufficient.

For parametric problem (4.4) we have four sets of Kuhn-Tucker multipliers, meaning that for each time moment i, $i = \overline{1,n}$ we have four multipliers: a_i, b_i, c_i and d_i with the following interpretation:

if $a_i > 0$ then we get the maximum negative fluctuation (lowest amount of energy, at time i under the predefined level),

if $b_i > 0$ then we get the maximum positive fluctuation (highest amount of energy, at time i above the predefined level),

if $c_i > 0$ then at time moment i production is realized at minimum assumed level,

if $d_i > 0$ then at time moment i production is realized at maximum assumed level.

The interpretation, in natural language, for the steps taken during the proof of Theorem 4.4.5 are:

At **Step 1** we compute, at a fix moment of time, the amount of energy produced and the associated limits of fluctuation, which have a logic (is possible to co-exist). For example, it is impossible to produce at time moment i, $i = \overline{1,n}$, at the minimum assumed level $(c_i > 0)$ and at the same time moment i, $i = \overline{1,n}$ to have a maximum positive fluctuation $(b_i > 0)$ – example corresponding to Scenario 9. Let's call the pair with the previous mentioned coordinates, an *energetic pair*.

At **Step 2** we evaluate the capacity of an energetic pair to satisfy the Kuhn-Tucker conditions referring to the gradient of Lagrangian. This mean that we evaluate which of the energetic pairs have the capacity to minimize the hypothetical result.

But is it mandatory to consider for each time moment the same energetic pair in order to compute the optimal solution? Is it not possible to consider different energetic pairs at different time moments? At **Step 3**, we check which of the energetic pairs, owning the capacity to minimize hypothetical result, are mutual exclusive (can not be combined).

> At **Step 4** we determine, the value of λ such that hypothetical result is minimized, either by considering the same energetic pair over the entire time period, or by considering different combinations of non-mutual exclusive energetic pairs. Thus, for the computed value of λ, we have an optimal solution which presumes that either the same energetic pair is repeated over the entire time period, or different non-mutual exclusive energetic pairs are combined over the time period. In case that at each time moment the same energetic pair is used, it does not mean that we will have a constant level of production, because the amount of energy to be produced at different time moments might be calculated based on price which can vary in time.

Returning to the initial minimax energy problem (4.2), Theorem 4.4.12 is computing the efficient frontier.

Theoretical results associated to minimax energy problem (4.2) were tested using real data for production and consumption of energy in Romania, provided by Transelectrica [148] and corresponding to the period 10th of November 2007, (time 22:18:00) to 4th of December 2015,(time 23:52:55) and price rates established by ANRE [129] and presented in Appendix A. Before starting the simulation, data were validated (anomalies were identified and eliminated using Box&Whisker and Grubbs tests).

Production plans provided by Theorem 4.4.5 were compared with real production and consumption curves from 4th of December 2015. According to their behavior, results were evaluated with excellent, good and weak, where

> *excellent* means that production plan is shaving the peak and following the production curve as flat as possible,
>
> *good* means that production plan is shaving the peak and flattering the production curve,
>
> *weak* means that production plan is far from real data.

For $\varepsilon = 3666$ MW, $\rho = 10808$ MW and $r = 6970$ MW as input data in the minimax energy problem (4.2), tests with solutions provided by Theorem 4.4.5 are generating two excellent results presented in Figure 4.3 and two good results presented in Figure 4.4. Excellent results were obtained for Plan III, while good results for Plan II. This conclusion is in line with our hypothesis that price is influencing the amount of energy to be produced and the optimization process.

The meaning of Kuhn-Tucker multipliers is shadow price. Detailed explanations regarding shadow price are available in [19]. For the particular solution 2 with Plan III of Theorem 4.4.5, which generates excellent result, we will provide an example of analyzing the shadow price. That solution is obtained if Scenario 3 is repeated for all time moments i, $i = \overline{1,n}$. Scenario 3 means that $a_i = 0$, $b_i > 0$, $c_i = 0$, $d_i = 0$.
Let's consider a fix time moment i, $i = \overline{1,n}$.
Because $b_i > 0$ it follows that

$$p_i x_i - p_i r = y_i$$

with the meaning that maximum positive fluctuation is attended at that time moment i, which implies that

$$p_i r - p_i x_i < y_i$$

with the meaning that maximum negative fluctuation can not be attended at that time moment i.
If we increase the value of the allowed fluctuation y_i with one unit and keep constant the minimum and maximum assumed level of energy to be delivered, then the hypothetical result will increase with b_i, meaning that one of the following holds:

fluctuation will increase with b_i,

turnover will decrease with b_i,

fluctuation will increase with a percentage from b_i and turnover will decrease with the remaining percentage from b_i.

Minimax energy model (4.2) proves outstanding performances, but is quite complex due to the predefined level which has to be determined before implementing the model. To mitigate this complexity we have created the index measure of fluctuation, defined as

$$\max_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}. \tag{5.1}$$

which skips the predefined level around which optimization is performed, but keeps the price.

Index measure is correlated with practice in production, of reporting loading to a predefined capacity (maximum in general). Measuring unit for index measure of fluctuation is money.

Index model (5.2), developed based on index measure of fluctuation (5.1) and turnover as indicator for economic performance, aims to shave the peak load by minimizing fluctuation of energy and maximizing turnover. Its efficient solution represents the amount of energy to be produced at each time moment, such that fluctuation is minimized and turnover maximized. Additionally, this solution owns the capacity to shave the peak load.

To solve index energy model (5.2) it was transformed, using Lemma 5.4.1, in the equivalent bi-criteria optimization problem (5.3). Based on Theorem 2.3.1 of Yu [160] and similar results of Bot et all [14] and Geoffrion [53], Lemma 5.4.2 was established and used to transform the equivalent bi-criteria optimization problem (5.3) in the parametric optimization problem (5.4).

Theorem 5.4.5, is computing an optimal solution for parametric optimization problem (5.4). Proof of this theorem uses the Kuhn-Tucker Theorem 2.3.4 and Theorem 4.4.11.
Proof of Theorem 5.4.5 is a four steps process, being similar to that for Theorem 4.4.5.

To be mentioned that for parametric problem (5.4) we have three sets of Kuhn-Tucker multipliers, meaning that for each time moment i, $i = \overline{1,n}$ we have three multipliers: a_i, b_i and c_i with the following interpretation:

if $a_i > 0$ then we get the maximum fluctuation,

if $b_i > 0$ then at time moment i production is realized at minimum assumed level,

if $c_i > 0$ then at time moment i production is realized at maximum assumed level.

The interpretation, in natural language, for the steps taken during proof of Theorem 5.4.5 is similar to the interpretation provided for Theorem 4.4.5.

Returning to the initial index energy problem (5.2), Theorem 5.4.6 is computing the efficient frontier.

Theoretical results associated to index energy problem (5.2) were tested using the same validated real data as for minimax energy problem (4.2) and the same three attributes (excellent, good and weak) were used to evaluate the performances of production plans provided by Theorem 5.4.5.

Two tests were performed. One of the solutions could not be tested due to the lack of price plans with four tariffs.

During the first test, the same inputs as for minimax model were used ($\varepsilon = 3666$ MW, $\rho = 10808$ MW). Several solution of Theorem 5.4.5 don't verify the existence and non-critical conditions. Two good results, presented in Figure 5.1, were generated and for one solution (11 with Plan III) we notice a potential for excellent.

The relative behavior of results obtained during test with $\varepsilon = 3666$ MW, $\rho = 10808$ MW as input data was an incentive to perform a second test in order to improve results. The new input data ($\varepsilon = 8500$ MW and $\rho = 26154$ MW) were chosen such that existence and non-critical conditions will be satisfied for more solution from Theorem 5.4.5. By choosing $\rho = 26154$ MW, the maximum installed capacity of Romania (24513 MW) is exceeded, but this does not represent a problem, as it was explained in Conclusions subsection of Chapter 4. Eight good results presented, in Figure 5.3, were generated and for the solution (11 with Plan III) we notice a potential for excellent.

During first test, the production curve associated to solution 11 with Plan III (Theorem 5.4.5) was to low, while during the second test it was to high. We have computed as necessary input data the following values $\varepsilon = 5000$ MW, $\rho = 15385$ MW and proceed to the third test, result being presented in Figure 5.6. The behavior of Solution 11 is improved, but at a certain time moment the amount of energy produced is not covering the demand, additional human interference being required.

We might conclude that both models, minimax model (4.2) and index model (5.2), provide outstanding results realizing the objective of our research and thus they can be validated. Inclusion of additional technical constraints, topology of the power grid, a better estimation for input data and a new approach for solving the transition from night period to day period might improve the accuracy of solution.

To define and implement additional technical constraints, like

the ones briefly presented in first section of Chapter 1 requires support from experts in energy production. Complexity of mathematical model might increase and the method developed to solve minimax and index models might not be efficient anymore. Chapter 3 of our thesis is presenting another method based on approximate problems. As approximate problems we have studied cases when components of objective function and constraints are replaced with their first or second order η- approximations.
We have studied conditions such that an efficient solution of initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

remains efficient for approximate problems $\left(P_k^{i,j}\right)$

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ G_t^2\left(x\right) \leq 0,\ t \in T \\ H_s^2\left(x\right) = 0,\ s \in S \end{cases}$$

where $(i, j) \in \{(1, 0),\ (1, 1), (2, 0),\ (2, 1),\ (2, 2)\}$, $k \in \{0, 1, 2\}$, and reciprocally. These conditions refer to invexity, incavity or avexity of functions involved. The new method might facilitate solving complex energy models by replacing them with approximate problem. Under certain conditions the efficient solution of the approximate problem will remain efficient for the initial problem.

Topology of the power grid is very important due to the First Law of Kirchhoff (at a junction of electrical circuit, the sum of currents which are going in is equal with the sum of currents which are going out) and due to losses which appear during transportation at long distances. In this context the location of power plants and consumers, with their related capacity and needs are playing an important role in determination of input data.
A better estimation for input data might use regression models and estimations. We might address our work [125], [77], [147], [91], [89] done in this area and extend it for energy optimization purposes.

Analyzing the morning and night transition paths for real production curve, we notice a trend similar to the first bisector and a rotation of the first bisector. First bisector is associated with fixed points. Resuming our work [122], [18], [20] performed in the field of fixed points might generate an improvement for bi-criteria energy models.

Our contributions to this thesis might be summarized as:

use of bi-criteria optimization problem for shaving peak load;

approximation theorems 3.3.1, 3.3.2, 3.3.5, 3.3.6, 3.3.9, 3.3.10, 3.3.11, 3.3.12, 3.4.3, 3.4.4, 3.4.5, 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.10, 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10, 3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6, 3.6.7, 3.6.8;

Examples related to the previous mentioned approximation theorems: 3.3.3, 3.3.7, 3.4.11, 3.4.12, 3.5.11, 3.5.12, 3.5.13;

definition of minimax measure of fluctuation (4.1) and index measure of fluctuation (5.1);

bi-criteria problems (4.2) and (5.2), used for shaving the peak load;

transformation of energy problems (4.2) and (5.2) in equivalent bi-criteria problems (4.3) and (5.3), using Lemma 4.4.1 and Lemma 5.4.1, which are also equivalent with parametric optimization problems (4.4) and (5.4), based on Lemma 4.4.2 and Lemma 5.4.2;

Theorem 4.4.11 which proves that order of scenarios does not change the solution;

Theorem 4.4.5 which computes the optimal solution for minimax parametric model (4.4);

Theorem 4.4.12 which computes the efficient solution for minimax model (4.2);

Theorem 5.4.5 which computes the optimal solution for parametric model (5.4);

Theorem 5.4.6 which computes the efficient solution for index energy model (5.2);

testing of minimax and index models using real data and some economic analysis performed for Kuhn-Tucker multipliers,

classification, using Definitions 4.4.6, 4.4.7 and 4.4.8, of Kuhn-Tucker multipliers based on to their capacity to generate feasible and optimal solutions for parametric optimization problem.

Results presented in this thesis were disseminated at two international conferences:

Bi-criteria problems for energy optimization, presented at Conference: International Conference on Approximation Theory and its Applications, organized in Sibiu, Romania during 26-29 May 2016.

Bi-criteria models for energy markets, presented at Conference: Management International Conference, organized in Monastier di Treviso, Italy, during 24-27 May 2017.

and in nine articles:

Minimax rule for energy optimization [98], published in Computers and Fluids, an ISI journal with an Impact Factor of 2.221 and a 5-years Impact Factor of 2.610.

Index model for peak-load shaving in energy production [92], submitted to Engineering Optimization, an ISI Journal with an Impact Factor of 1.728.

Bi-criteria models for peak-load shaving [90], accepted for publication by Journal of Academy of Business and Economics, a journal indexed in EBSCO, EconLit, Ulrich's, Index Copernicus, Research Bible.

Approximations of objective function in bi-criteria optimization problems [96], accepted for publication by European International Journal of Science and Technology, a journal indexed in Google Scholar, NewJour, Hochschulbibliothek Reutlingen, CrossRef.

Approximations of objective function and constraints in bi-criteria optimization problems [95], submitted to Journal of Numerical Analysis and Approximation Theory, a journal indexed in Mathematical Reviews, Zentralblatt MATH.

> *Approximations of bi-criteria optimization problem* [94], submitted to Studia Universitatis Babes-Bolyai Mathematica, a journal indexed in Mathematical Reviews, Zentralblatt MATH, EBSCO, ProQuest, Ulrichsweb.
>
> *Relations between η - approximation problems of a bi-criteria optimization problem* [97], submitted to Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, a journal indexed in Mathematical Reviews, Zentralblatt MATH, American Mathematical Society.
>
> *Bi-criteria problems for energy optimization* [39], publish in General Mathematics, a journal indexed in Zentralblatt MATH, EBSCO, Mathematical Reviews, Index Copernicus.
>
> *Portfolio optimization algorithms* [93], published in Studia Universitatis Babes-Bolyai Negotia, a journal indexed in EBSCO, Index Copernicus, ERIH PLUS.

and are supported by our previous work reflected in books, articles and conferences:

> *Matematici economice. Elemente de programare liniara si teoria probabilitatilor* [19], a book published by Presa Universitara Clujeana.
>
> *Strict fixed points results for multivalued contractions on gauge spaces* [122], published in Fixed Point Theory, an ISI journal with an Impact Factor of 1.030 in 2010.
>
> *Uniqueness algebraic conditions in the study of second order elliptic systems* [18], published in International Journal of Pure and Applied Mathematics, a journal indexed in Scopus.
>
> *Maximum principles for a class of second order parabolic systems in divergence form* [20], published in Journal of Nonlinear Functional Analysis and Differential Equations, a journal indexed in Scopus, Web of Science, Zentralblatt MATH.
>
> *Automotive industry and performances of US Economy* [125], published in International Journal of Finance and Economics, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Consumer's inflation expectations in Romania [147], published in International Journal of Business Research, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Comparative analysis of low-cost airlines websites [77], published in Proceedings of IABE - 2009 Las Vegas - Annual Conference, a volume indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Economic applications of dynamic optimization, presented at Conference: 10th International Symposium on Generalized Convexity and Monotonicity, organized in Cluj Napoca, Romania, during 22-27 August 2011.

A relation between transportation problems and profit, presented at Conference: Current Issues of Regional Development, organized in Sec, Czech Republic, during 26-27 June 2007.

Appendix A

Price of electricity in Romania

Price of low voltage energy, the day/night hours, respectively peak, gap and normal hours, established by Order 40 issued in 21st of June 2013 by ANRE are:

Plan I	Plan II		Plan III		
24h	day	night	peak	normal	gap
356.7	568.2	184.8	805.9	449.1	211.3

TABLE A.1: Energy prices

Plan II	
day	7-22
night	22-7

	Plan III		
	gap	normal	peak
winter	0-8; 21-0	9-21	8-9
summer	0-8; 22-0	10-19	8-10;19-22

TABLE A.2: Time periods for Plan II and Plan III

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