i) $\forall x \in D(f): \partial f(x) \neq \emptyset \Rightarrow f$ is convex

$\Longleftrightarrow$

f is nonconvex $\Rightarrow \exists x \in D(f): \partial f(x) = \emptyset$

f - nonconvex $\Rightarrow \exists x, y \in D(f)\ \exists \alpha \in (0,1): f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha) f(y) \Rightarrow$

$\Rightarrow \exists$ fixed x, y, fixed α : $f(\alpha x + (1-\alpha)y) > f(x) \wedge f(\alpha x + (1-\alpha)y) > f(y)$

We prove: $\partial f(\alpha x + (1-\alpha)y) = \emptyset$ for fixed x, y, α

If $\partial f(\alpha x + (1-\alpha)y) \neq \emptyset$ then $\exists g: \forall u \in D(f)\ f(u) \geq f(\alpha x + (1-\alpha)y) + g^T(u - (\alpha x + (1-\alpha)y))$

We know $f(\alpha x + (1-\alpha)y) > f(x) \wedge f(\alpha x + (1-\alpha)y) > f(y)$

w.l.o.g: $x < \alpha x + (1-\alpha)y < y$

If $f(x) \geq f(\alpha x + (1-\alpha)y) + \underbrace{g^T(x - \alpha x - (1-\alpha)y)}_{g^T((1-\alpha)(x-y))}$ - have to be < 0

If $f(y) \geq f(\alpha x + (1-\alpha)y) + \underbrace{g^T(y - \alpha x + (1-\alpha)y)}_{g^T(-\alpha(x-y))}$ - have to be < 0

$g^T \overset{>0}{(1-\alpha)}(x-y) < 0 \qquad g^T(-\alpha)(x-y) < 0$

$g^T(x-y) < 0 \qquad g^T(x-y) > 0$

Contradiction $\Rightarrow g$ doesn't exist $\Rightarrow \partial f(\alpha x + (1-\alpha)y) = \emptyset \Rightarrow$

$\Rightarrow$ (f - is nonconvex $\Rightarrow \exists$ ~~$x, y \in D(f)\ \exists \alpha \in (0,1)$~~ $x \in D(f)$ $\partial f(x) = \emptyset$) $\Rightarrow$ ($\forall x \in D(f): \partial f(x) \neq \emptyset \Rightarrow f$ is convex)