Optimization

master in fundamental principles of data science

Optimization Problem 3

Convex Function problem

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1 Problem description

Prove, without using the above theorem, that for any $a \in \mathbb{R}$***,*** $f(x) = e^{ax}$ ***is a convex function.***

2 Solution

To prove that the function $f(x) = e^{ax}$ is a convex function for any $a \in \mathbb{R}$, let's first suppose that we have two points, x and y, such that $x, y \in \mathbb{R}$ and $x \neq y$. For the function $f(x)$ to be convex, the following inequality has to be satisfied:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall \lambda \in [0,1] \tag{1}$$

Now, let us substitute f in expression (1) with our function:

$$e^{a(\lambda x + (1-\lambda)y)} \leq \lambda e^{ax} + (1-\lambda)e^{ay} \tag{2}$$

If we further operate, we obtain the following:

$$\begin{aligned}
e^{\lambda ax + ay - \lambda ay} &\leq \lambda e^{ax} + e^{ay} - \lambda e^{ay} \\
e^{\lambda ax} e^{ay} e^{-\lambda ay} &\leq \lambda e^{ax} + e^{ay} - \lambda e^{ay} \\
e^{\lambda ax} e^{-\lambda ay} &\leq \lambda \frac{e^{ax}}{e^{ay}} + 1 - \lambda \\
e^{\lambda ax - \lambda ay} &\leq \lambda e^{ax - ay} + 1 - \lambda \\
e^{\lambda a(x-y)} &\leq \lambda e^{a(x-y)} + 1 - \lambda
\end{aligned} \tag{3}$$

To simplify a little bit the previous expression, let us introduce a new variable, z, such that $z = a(x-y)$. If we substitute accordingly in the above result, we obtain the following:

$$e^{\lambda z} \leq \lambda e^{z} + 1 - \lambda \tag{4}$$

Knowing that e^x can be defined as a power series as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

we can substitute e^x in the expression (4). After doing that, we get:

$$
\begin{aligned}
1 + \lambda z + \frac{(\lambda z)^2}{2!} + \frac{(\lambda z)^3}{3!} + \dots &\leq \lambda\left(1 + z + \frac{(z)^2}{2!} + \frac{(z)^3}{3!} + \dots\right) + 1 - \lambda \\
\lambda + \lambda z + \frac{(\lambda z)^2}{2!} + \frac{(\lambda z)^3}{3!} + \dots &\leq \lambda\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) \\
\lambda\left(1 + z + \frac{\lambda z^2}{2!} + \frac{\lambda^2 z^3}{3!} + \dots\right) &\leq \lambda\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) \\
1 + z + \frac{\lambda z^2}{2!} + \frac{\lambda^2 z^3}{3!} + \dots &\leq 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \qquad (5)
\end{aligned}
$$

With the result seen in expression (5), we can distinguish 3 different cases:

- **Case** $a = 0$: It is easy to see that by substituting $a = 0$ in the above result, we get that $1 \leq 1$, which satisfies the inequality.
- **Case** $a > 0$: In this case, we can select x and y accordingly so that z is positive, which will be true when $x > y$. If we then select any value of λ such that $0 \leq \lambda \leq 1$, we can clearly see that the right expression will always be smaller than the left one because the addends on the left get scaled down by λ. Therefore, the inequality will be true.
- **Case** $a < 0$: Here we can also select the points x and y accordingly so that the value of z is positive. Since a is negative, the subtraction has to be negative, which will be true when $x < y$[1]. Since z is positive again, we get the same result as in the previous case. Thus, the inequality is satisfied again.

As we can see, the inequality defined in expression (5) is satisfied for any $a \in \mathbb{R}$. Thus, we can conclude that the function $f(x) = e^{ax}$ is convex for any value of a.

[1]Note that if we are using the same points as in the previous case, we are just swapping their values. Because of this, we cannot use the same value of λ as before. Therefore, we have to use λ', which is given by the expression $\lambda' = 1 - \lambda$.