Optimization

master in fundamental principles of data science

Optimization problem 2

Concrete mixing problem

Author
Vladislav Nikolov Vasilev

Faculty of Mathematics and Computer Science

Academic year 2021-2022

1 Problem description

Suppose that we are mixing concrete and are using n different gravel sizes $s_1, \ldots, s_n$.

The ideal mixture is given by $\boldsymbol{c} = (c_1, \ldots, c_n)$, where c_i $(0 \leq c_i \leq 1)$ is the fraction of size s_i in the mix, and $\sum_{i=1}^{n} c_i = 1$.

Gravel mixtures come from m different mines. The gravel composition at each mine $j = 1, \ldots, m$ is given by $C_j = (c_{1j}, \ldots, c_{nj})$, where $0 \leq c_{ij} \leq 1$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} c_{ij} = 1$.

Let $\boldsymbol{x} = (x_1, \ldots, x_m)$ be the vector that represents the fraction of gravel of the mines in the mixture, where $0 \leq x_j \leq 1$ for all $j = 1, \ldots, m$ and $\sum_{j=1}^{m} x_j = 1$.

Find the best possible approximation $\boldsymbol{x} = (x_1, \ldots, x_m)$ of the ideal mixture, $\boldsymbol{c} = (c_1, \ldots, c_n)$, by using the material from the m mines.

2 Solution

We have that $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times m}$, where the matrix $C = (C_1, \ldots, C_m)$ has C_j as columns.

Finding the best possible vector $\boldsymbol{x}$ means finding a vector such that

$$C\boldsymbol{x} = \boldsymbol{c} \tag{1}$$

This means that the vector result of the matrix-vector product $C\boldsymbol{x}$ has to be as similar as possible to $\boldsymbol{c}$. With this in mind, we can rewrite expression (1) as the following optimization problem:

$$\boldsymbol{x}^\star = \min_{x \in \mathbb{R}^m} \|C\boldsymbol{x} - \boldsymbol{c}\|^2 \tag{2a}$$

$$\text{subjet to} \sum_{j=1}^{m} x_j = 1, \text{ and } x_j \geq 0 \tag{2b}$$

where $\boldsymbol{x}^\star$ is the best possible solution out of all the feasible ones. Note that minimizing $\|C\boldsymbol{x} - \boldsymbol{c}\|^2$ is the same as minimizing $\|C\boldsymbol{x} - \boldsymbol{c}\|$.

Now, knowing that $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$, the distance found in expression (2a) can be rewritten as follows:

$$
\begin{aligned}
\|C\boldsymbol{x} - \boldsymbol{c}\|^2 &= (C\boldsymbol{x} - \boldsymbol{c})^T(C\boldsymbol{x} - \boldsymbol{c}) \\
&= \boldsymbol{x}^T C^T C\boldsymbol{x} - \boldsymbol{x}^T C^T \boldsymbol{c} - \boldsymbol{c}^T C\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{c} \\
&= \boldsymbol{x}^T C^T C\boldsymbol{x} - (\boldsymbol{x}^T C^T \boldsymbol{c})^T - \boldsymbol{c}^T C\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{c} \\
&= \boldsymbol{x}^T C^T C\boldsymbol{x} - \boldsymbol{c}^T C\boldsymbol{x} - \boldsymbol{c}^T C\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{c} \\
&= \boldsymbol{x}^T C^T C\boldsymbol{x} - 2\boldsymbol{c}^T C\boldsymbol{x} + \|\boldsymbol{c}\|^2
\end{aligned}
\tag{3}
$$

Thanks to the last expression in (3), we can clearly see that this is a **Quadratic Optimization** problem because the objective function is quadratic and all the restrictions are linear.

Notice that if $C \in \mathbb{R}^{n\times m}$, then $C^T C \in \mathbb{R}^{n\times n}$ and $\boldsymbol{c}^T C \in \mathbb{R}^m$. Let $z = C\boldsymbol{x}$, then:

$$
\boldsymbol{x}^T C^T C\boldsymbol{x} = \boldsymbol{z}^T\boldsymbol{z} = \|\boldsymbol{z}\|^2 \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^m \tag{4}
$$

From the inequality in expression (4) we know that the Euclidean norm is only going to be 0 if $\boldsymbol{x} = 0$. Taking also into account the problem restrictions that can be seen in (2b), we know that $\sum_{j=1}^m x_j = 1$, which means that $\boldsymbol{x} \neq 0$. This implies that $C^T C$ is **positive definite** and that an optimal value can be found.

Thus, we can conclude that a non-zero value of $\boldsymbol{x}$ can be found so that the distance $\|C\boldsymbol{x} - \boldsymbol{c}\|^2$ is as close to 0 as possible, which means that the found value of $\boldsymbol{x}$ is going to be the optimal solution to this minimization problem.