Optimization

master in fundamental principles of data science

Optimization Problem 0

The Fermat point of a set of points

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Given set of points $y_1, \ldots, y_m$ in the plane, find a point x^* whose sum of weighted distances to the given set of points is minimized. Mathematically, the problem is

$$\min \sum_{i=1}^{m} w_i \left\| x^* - y_i \right\|, \quad \text{subject to } x^* \in \mathbb{R}^2,$$

where $w_1, \ldots, w_m$ are given positive real numbers.

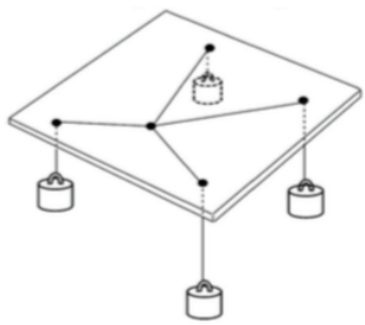

Figure 1: Example of a mechanical system.

1. Show that there exists a global minimum for this problem (that it can be realized by means of the mechanical model shown in the figure 1).

Consider the function f defined as $f(x) = \sum_{i=1}^{m} w_i \left\| x - y_i \right\|$. This function is nothing more and nothing less than the weighted sum of the norms $\|x - y_i\|$ for $i = 1, \ldots, m$. By definition, any vector norm is a convex function, and the weighted sum of convex functions is also a convex function provided that every w_i satisfies that $w_i > 0$ for $i = 1, \ldots, m$. This implies that f is convex, and by definition, a local minimum of f is also a global minimum. Thus, there must exist at least one x^* such that x^* is a local minimum of the function, and therefore, it is also a global minimum of the function f.

2. Is the optimal solution always unique?

In this case, the optimal solution is not always unique. Suppose the case in which we have two points y_1 and y_2 such that $y_1 \neq y_2$. Also, suppose that the weights are $w_1 = w_2 = 1$. Then the minimum is attained at all the points $x^* \in \{\lambda y_1 + (1 - \lambda) \mid \lambda \in [0, 1]\}$. We can see this as follows:

$$
\begin{aligned}
f(\lambda y_1 + (1-\lambda)) &= \|(\lambda-1)y_1 + (1-\lambda)y_2\| + \|\lambda y_1 - \lambda y_2\| = \\
&= \|(\lambda-1)(y_1 - y_2)\| + \|\lambda(y_1 - y_2)\| = \\
&= (\lambda-1)\|y_1 - y_2\| + \lambda\|y_1 - y_2\| = \\
&= \|y_1 - y_2\|
\end{aligned}
$$

Therefore, there exist multiple global minima, and the optimal solution is not unique.

3. Show that an optimal solution minimizes the potential energy of the mechanical model defined as $\sum_{i=1}^{m} w_i h_i$, where h_i is the height of the ith weight measured from some reference level.

Consider the model seen in figure 1. It consists of a platform with m different massless ropes attached to a point x^* (which in this case is the one between all of the other points). Each rope goes through one of the m, and each one has attached an object that weights w_i for $i = 1, \dots, m$. The length of each rope is given by l_i, where $l_i \in \mathbb{R}^+$.

Suppose that we set the reference level of the gravitational potential at the platform. Then, we have that given any attachment point x in the platform, the length of the rope is given by $l_i = \|x - y_i\| + h_i(x)$, for $i = 1, \dots, m$, where $h_i > 0$ and $-h_i$ is the height of the i-th weight measured from the platform.

The potential energy of the physical system can be written as follows:

$$
V(x) = -\sum_{i=1}^{m} w_i h_i(x) = -\sum_{i=1}^{m} w_i l_i + \sum_{i=1}^{m} w_i \|x - y_i\| = c + f(x) \tag{1}
$$

where c is a constant. Therefore, the only way to minimize the result of expression (1) is by minimizing the function f.