Optimization

master in fundamental principles of data science

Optimization Problem 1

Smallest Area problem

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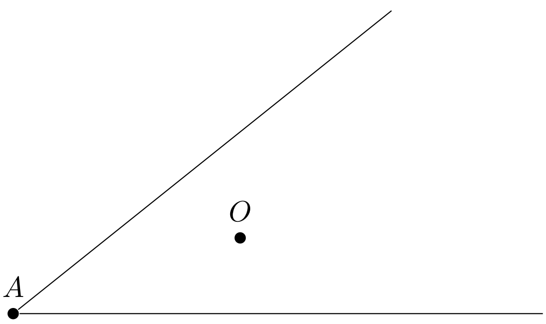
1 Problem description

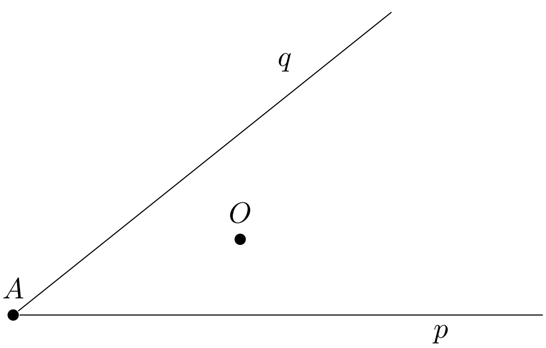
Given an angle with vertex A and a point O in its interior. Pass a line BC through the point O that cuts off from the angle a triangle of minimal area.



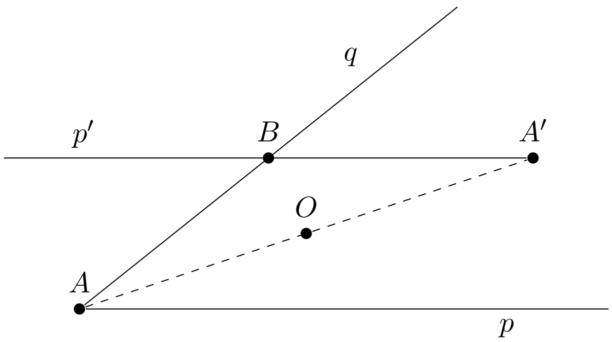
2 Solution

Let l be the line that passes through the point O and cuts off from the angle a triangle of minimal area. The intersection points of this line with the angle are going to be B and C. These points will form the segment BC where the midpoint is going to be O.

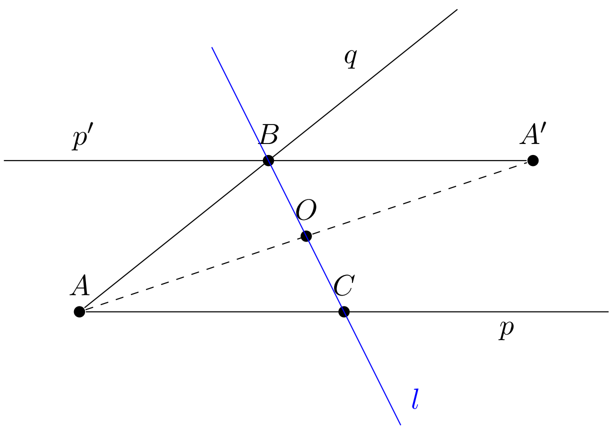
First, let's name the horizontal line as p and the other one as q so we have an easily followable naming convention. We obtain the following:



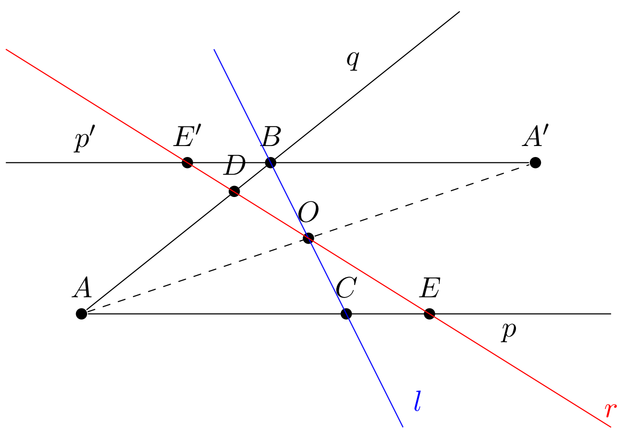
Now, let's try to prove our initial statement. Let s be a symmetry with respect to the point O. If we apply s to A, we obtain the point $A' = s(A)$. Now, if we apply s to the line p, we obtain the line $p' = s(p)$. Note that the line p' is parallel to p and intersects the line q at some point, which we will call B:



If we pass a line l through B and O, it will intersect the line p in a point named C. Note that $C = s(B)$, which means that the point O is the midpoint of the BC segment and implies that $|BO| = |OC|$:



Next, consider any random line, r, which passes through the point O. This line will intersect q and p at some points, D and E respectively:



Note that in this case $D \neq s(E)$ and instead $E' = s(E)$. This means that the point O is **not** the midpoint of the DE segment. Thus, we can easily see that $|DO| \neq |OE|$.

Now, we have two possible triangles: the triangle ABC and the triangle ADE:

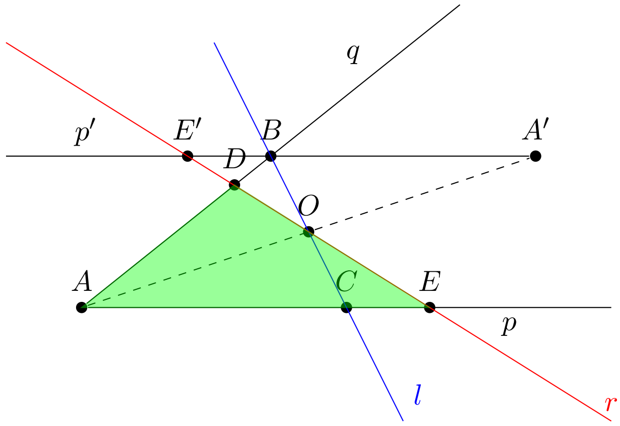


Figure 1: Representation of the ADE triangle.

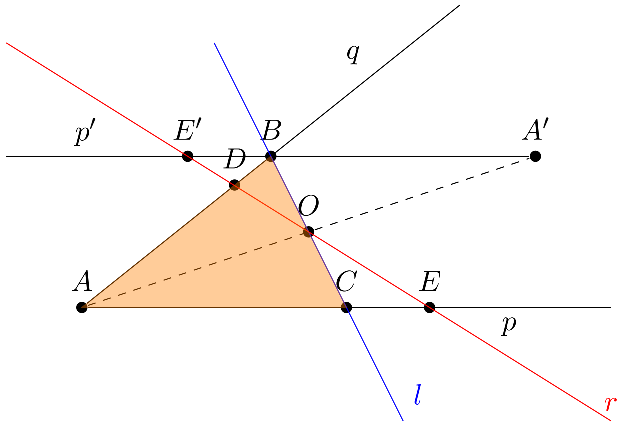


Figure 2: Representation of the ABC triangle.

We can easily see that:

$$[ADE] = [ADOC]+[OCE] = [ADOC]+[OBE'] > [ADOC]+[DBO] = [ABC]$$

which, in short, implies that $[ADE] > [ABC]$. We can clearly see that this is due to the fact that $[OCE] > [DBO]$.

Thus, we can conclude that a line l which intersects with the angle in the points

B and C and passes through an interior point O is going to span a triangle with minimal area iif B and C are equidistant from O, or which is the same, the point O is the midpoint of the BC segment.