Optimization

master in fundamental principles of data science

Optimization Problem 6

Conjugate gradient method problem II

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Academic year 2021-2022

1 Problem description

Consider the conjugate gradient method applied to the minimization of

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$$

where A is a positive definite and symmetric matrix.

Show that the iterate $\mathbf{x}_k$ minimizes f over

$$\mathbf{x}_0 + < \mathbf{v}_0, A\mathbf{v}_0, \dots, A_{k-1}\mathbf{v}_0 >$$

where $\mathbf{v}_0 = \nabla f(\mathbf{x}_0)$, and $< \mathbf{v}_0, A\mathbf{v}_0, \dots, A_{k-1}\mathbf{v}_0 >$ is the subspace generated by $\mathbf{v}_0, A\mathbf{v}_0, \dots, A_{k-1}\mathbf{v}_0$.

2 Solution

Let $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k \in \mathbb{R}^n$ with $k \leq n$ be k mutually conjugate directions with respect to a given matrix A, which is symmetric and positive definite. Then, $\mathbf{x}_k$, which is a point defined from this directions, will minimize the function f over the affine set $\mathbf{x}_0 + < \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k >$. Basically, what we want to prove is that $< \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k > = < \mathbf{v}_0, A\mathbf{v}_0, \dots, A_{k-1}\mathbf{v}_0 >$.

- If $k = 1$, we have that $\mathbf{z}_1 = -\mathbf{v}_0 = -\nabla f(\mathbf{x})$.