Elements of Advanced International Trade[1]

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Abstract

These notes are prepared for a Ph.D. level course in international trade.

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Chapter 1

An introduction into deductive reasoning

1.1 Inductive and deductive reasoning

Inductive or empirical reasoning is the type of reasoning that moves from specific observations to broader generalizations and theories. **Deductive** reasoning is the type of reasoning that moves from axioms to theorems and then applies the predictions of the theory to the specific observations.

Inductive reasoning has failed in several occasions in economics. According to Prescott (see Prescott (1998)) "The reason that these inductive attempts have failed ... is that the existence of policy invariant laws governing the evolution of an economic system is inconsistent with dynamic economic theory. This point is made forcefully in Lucas' famous critique of econometric policy evaluation."

Theories developed using deductive reasoning must give assertions that can be falsified by an observation or a physical experiment. The consensus is that if one cannot potentially find an observation that can falsify a theory then that theory is not scientific (Popper).

A general methodology of approaching a question using deductive reasoning is the following:

1) Observe a set of empirical (stylized) facts that your theory has to address and/or are relevant to the questions that you want to tackle,

2) Build a theory,

3) Test the theory with the data and then use your theory to answer the relevant questions,

4) Refine the theory, going through step 1

1.2 Employing and testing a model

A vague definition of two methodologies: calibration and estimation

Calibration is the process of picking the parameters of the model to obtain a match between the observed distributions of independent variables of the model and some key dimensions of the data. More formally, calibration is the process of establishing the relationship between a measuring device and the units of measure. In other words, if you think about the model as a "measuring device" calibrating it means to parameterize it to deliver sensible quantitative predictions.

Estimation is the process of picking the parameters of the model to minimize a function of the errors of the predictions of the model compared to some pre-specified targets. It is the approximate determination of the parameters of the model according to some pre-specified metric of differences between the model and the data to be explained.

It is generally considered a good practice to stick to the following principles (see Prescott (1998) and the discussion in Kydland and Prescott (1994)) when constructing quantitative models:

1. When modifying a standard model to address a question, the modification continues to display the key facts that the standard model was capturing.

2. The introduction of additional features in the model is supported by other evidence for these particular additional features.

3. The model is essentially a measurement instrument. Thus, simply estimating the magnitude of that instrument rather than calibrating the model can influence the ability of the model to be used as a measuring instrument. In addition the model's selection (or in particular, parametric specification) has to depend on the specific question to be addressed, rather than the answer we would like to derive. For example, "if the question is of the type, how much of fact X can be accounted for by Y, then choosing the parameter values in such a way as to make the amount accounted for as large as possible according to some metric makes no sense."

4. Researchers can challenge existing results by introducing new quantitatively relevant features in the model, that alter the predictions of the model in key dimensions.

1.3 International Trade: The Macro Facts

Chapter 2 of Eaton and Kortum (2011) manuscript

Chapter 2

An introduction to modeling

2.1 The Heckscher-Ohlin model

The Heckscher-Ohlin (H-O) model of international trade is a general equilibrium model that predicts that patterns of trade and production are based on the relative factor endowments of trading partners. It is a perfect competition model. In its benchmark version it assumes two countries with identical homothetic preferences and constant return to scale technologies (identical across countries) for two goods but different endowments for the two factors of production. The model's main prediction is that countries will export the good that uses intensively their relatively abundant factor and import the good that does not. We will present a very simple version of this model. Country i's representative consumer's problem is

$$
\begin{aligned}
&\max a_1 \log c_1^i + a_2 \log c_2^i \\
&s.t.\ p_1 c_1^i + p_2 c_2^i \leq r^i \bar{k}^i + w^i \bar{l}^i
\end{aligned}
$$

The production technologies of good ω in the two countries are identical and given by

$$y_\omega^i = z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{1-b_\omega}, i, \omega = 1, 2$$

and where $0 < b_2 < b_1 < 1$. This implies that good 1 is more capital intensive than good 2. Assume for simplicity that $\bar{k}^1/\bar{l}^1 > \bar{k}^2/\bar{l}^2$. This implies that country 1 is capital abundant relative to country 2. Finally, goods, labor, and capital markets clear. One of the common assumptions for the H-O model is that there is no factor intensity reversal which in our example is always the case given the Cobb-Douglas production function (one good is always more capital intensive than the other, with the capital intensity given by b_ω).

2.1.1 Autarky equilibrium

We first solve for the autarky equilibrium for country i. This is easy especially if we consider the social planner problem, but we will compute the competitive equilibrium instead. The Inada conditions for the consumer's utility function imply that both goods will be produced in equilibrium. Thus, we just have to take FOC for the consumer and look at cost minimization for the firm. For the consumer we have

$$\max a_1 \log c_1^i + a_2 \log c_2^i$$
$$s.t.\ p_1^i c_1^i + p_2^i c_2^i \leq r^i \bar{k}^i + w^i \bar{l}^i$$

which implies

$$a_1 = \lambda^i p_1^i c_1^i \tag{2.1}$$

$$a_2 = \lambda^i p_2^i c_2^i \tag{2.2}$$

$$p_1^i c_1^i + p_2^i c_2^i = r^i \bar{k}^i + w^i \bar{l}^i \tag{2.3}$$

This gives

$$p_2^i c_2^i = \frac{a_2}{a_1} p_1^i c_1^i \ . \tag{2.4}$$

The firm's cost minimization problem

$$\begin{gathered} \min r^i k_\omega^i + w^i l_\omega^i \\ \text{s.t. } y_\omega^i \leq z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{1-b_\omega} \end{gathered}$$

implies the following equation, under the assumption that both countries produce both goods,

$$\frac{b_\omega}{(1-b_\omega)} l_\omega^i w^i = r^i k_\omega^i \ . \tag{2.5}$$

We can also use the goods market clearing to obtain

$$\begin{aligned} c_\omega^i &= z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{1-b_\omega} \implies \\ c_\omega^i &= z_\omega l_\omega^i \left(\frac{b_\omega}{1-b_\omega} \frac{w^i}{r^i}\right)^{b_\omega} \ . \end{aligned} \tag{2.6}$$

Zero profits in equilibrium, $p_\omega^i z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{1-b_\omega} = r^i k_\omega^i + w^i l_\omega^i$, combined with (2.5), give us

$$\begin{aligned} p_\omega^i &= \frac{r^i k_\omega^i}{b_\omega z_\omega \frac{r^i k_\omega^i}{w^i}\frac{(1-b_\omega)}{b_\omega}\left(\frac{b_\omega}{(1-b_\omega)}\frac{w^i}{r^i}\right)^{b_\omega}} \\ &= \frac{\left(w^i\right)^{1-b_\omega}\left(r^i\right)^{b_\omega}}{z_\omega\left(1-b_\omega\right)^{1-b_\omega}\left(b_\omega\right)^{b_\omega}} \end{aligned}$$

We can also derive the labor used in each sector. From the consumer's FOCs, together with the expressions for p_ω^i and c_ω^i derived above, we obtain:

$$a_\omega = \lambda^i p_\omega^i c_\omega^i \implies$$

$$\left(1-b_\omega\right)\frac{a_\omega}{\lambda^i} = w^i l_\omega^i$$

this implies that

$$l_1^i \frac{\left(1-b_2\right)a_2}{\left(1-b_1\right)a_1} = l_2^i$$

We can use the labor market clearing condition and get

$$\begin{aligned} l_2^i + l_1^i &= \bar{l}^i \implies \\ l_1^i \frac{\left(1-b_2\right)a_2}{\left(1-b_1\right)a_1} + l_1^i &= \bar{l}^i \implies \\ l_1^i &= \frac{\left(1-b_1\right)a_1}{\left(1-b_2\right)a_2 + \left(1-b_1\right)a_1}\bar{l}^i \; . \end{aligned} \tag{2.7}$$

The results are similar for capital and thus,

$$k_1^i = \frac{b_1 a_1}{b_1 a_1 + b_2 a_2}\bar{k}^i \; ,$$

This implies

$$\frac{\bar{l}^i}{\bar{k}^i} = \frac{r^i}{w^i} \frac{\sum_\omega (1 - b_\omega) a_\omega}{\sum_\omega b_\omega a_\omega} \tag{2.8}$$

Thus, in a labor abundant country capital is relatively more expensive as we would expect. We can finally use the goods' market clearing conditions combined with the optimal choices for l^i_ω, k^i_ω to get the values for c^i_ω's as a function solely of parameters and endowments,

$$c^i_\omega = z_\omega \left(\frac{b_\omega a_\omega}{\sum_{\omega'} b_{\omega'} a_{\omega'}} \bar{k}^i \right)^{b_\omega} \left(\frac{(1 - b_\omega) a_\omega}{\sum_{\omega'} (1 - b_{\omega'}) a_{\omega'}} \bar{l}^i \right)^{1-b_\omega} . \tag{2.9}$$

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2.1.2 Free Trade Equilibrium

In the two country example, free trade implies that the price of each good is the same in both countries. Therefore, we will denote free trade prices without a country superscript. In the two country case it is important to distinguish among three conceptually different cases: in the first case both countries produce both goods, in the second case one country produces both goods and the other produces only one good, and in the last case each country produces only one good.

We first define the free trade equilibrium. A free trade equilibrium is a vector of allocations for consumers $\left(\hat{c}^i_\omega,\ i, \omega = 1, 2\right)$, allocations for the firm $\left(\hat{k}^i_\omega, \hat{l}^i_\omega,\ i, \omega = 1, 2\right)$, and prices $\left(\hat{w}^i_\omega, \hat{r}_\omega, \hat{p}_\omega,\ i, \omega = 1, 2\right)$ such that

1. Given prices consumer's allocation maximizes her utility for $i = 1, 2$

2. Given prices the allocations of the firms solve the cost minimization problem in

$i = 1, 2,$

$$b_\omega p_\omega z_\omega \left(k_\omega^i\right)^{b_\omega - 1} \left(l_\omega^i\right)^{1-b_\omega} \leq r^i \text{ , with equality if } y_\omega^i > 0$$
$$(1 - b_\omega) \, p_\omega z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{-b_\omega} \leq w^i \text{ , with equality if } y_\omega^i > 0$$

3. Markets clear

$$\sum_i \hat{c}_\omega^i = \sum_i \hat{y}_\omega^i, \ \omega = 1, 2$$
$$\sum_\omega \hat{k}_\omega^i = \bar{k}^i \text{ for each } i = 1, 2$$
$$\sum_\omega \hat{l}_\omega^i = \bar{l}^i \text{ for each } i = 1, 2 \text{ .}$$

2.1.3 No specialization

We analyze the three cases separately. First, let's think of the case in which both countries produce both goods.

$$\max a_1 \log c_1^i + a_2 \log c_2^i$$
$$s.t. \ p_1 c_1^i + p_2 c_2^i \leq r^i \bar{k}^i + w^i \bar{l}^i$$

$$a_1 = \lambda^i p_1 c_1^i \tag{2.10}$$
$$a_2 = \lambda^i p_2 c_2^i \tag{2.11}$$
$$p_1 c_1^i + p_2 c_2^i = r^i \bar{k}^i + w^i \bar{l}^i \tag{2.12}$$

This implies again that

$$p_2 c_2^i = \frac{a_2}{a_1} p_1 c_1^i \tag{2.13}$$

When both countries produce both goods the firms cost minimization problem implies the following two equalities,

$$\begin{aligned} b_\omega p_\omega z_\omega \left(k_\omega^i\right)^{b_\omega - 1} \left(l_\omega^i\right)^{1-b_\omega} &= r^i \ , \\ (1-b_\omega)\, p_\omega z_\omega \left(k_\omega^i\right)^{b_\omega} \left(l_\omega^i\right)^{-b_\omega} &= w^i \ , \end{aligned}$$

which in turn imply

$$\frac{b_\omega \left(l_\omega^i\right) w^i}{(1-b_\omega)\, r^i} = k_\omega^i \ . \tag{2.14}$$

Additionally, from zero profits,

$$p_\omega = \frac{\left(r^i\right)^{b_\omega} \left(w^i\right)^{1-b_\omega}}{z_\omega \left(b_\omega\right)^{b_\omega} \left(1-b_\omega\right)^{1-b_\omega}} \tag{2.15}$$

and, of course, technologies (by assumption) and prices (due to free trade) are the same in the two countries. Notice that the equality (2.15) is true for $i = 1, 2$ this implies that

$$\begin{aligned} \left(r^1\right)^{b_\omega} \left(w^1\right)^{1-b_\omega} &= \left(r^2\right)^{b_\omega} \left(w^2\right)^{1-b_\omega} \ \ \omega = 1, 2 \\ \left(\frac{r^1}{r^2}\right)^{b_\omega} &= \left(\frac{w^2}{w^1}\right)^{1-b_\omega} \ \ \omega = 1, 2 \end{aligned}$$

Noticing that the above expression holds for $\omega = 1, 2$ and replacing these two equations in one another we have

$$\left(\frac{w^2}{w^1}\right)^{\frac{(1-b_2)b_1}{b_2}-1+b_1} = 1 \implies$$
$$w^2 = w^1$$

and of course

$$r^2 = r^1 \ .$$

This shows that we have factor price equalization (FPE) in the free trade equilibrium.

From the cost minimization of the firm we have

$$\begin{aligned} b_\omega p^i_\omega z_\omega \left(k^i_\omega\right)^{b_\omega} \left(l^i_\omega\right)^{1-b_\omega} &= r^i k^i_\omega \implies \\ b_\omega p_\omega y^i_\omega &= r^i k^i_\omega \implies \\ p_\omega y^i_\omega &= \frac{r^i k^i_\omega}{b_\omega} \ . \end{aligned}$$

Summing up over i and using FPE we have

$$p_\omega \left(\sum_i y^i_\omega\right) = \frac{r}{b_\omega} \left(\sum_i k^i_\omega\right) \ . \tag{2.16}$$

The equations (2.10) and (2.11) imply

$$\frac{a_\omega}{\lambda^1} + \frac{a_\omega}{\lambda^2} = p_\omega \left(c^1_\omega + c^2_\omega\right) \tag{2.17}$$

Using goods market clearing, $\sum_i c_\omega^i = \sum_i y_\omega^i$, we have

$$\sum_i \frac{a_\omega}{\lambda^i} = p_\omega \left(c_\omega^1 + c_\omega^2\right) = p_\omega \sum_i y_\omega^i = \frac{r}{b_\omega} \sum_i k_\omega^i \implies$$

$$b_\omega a_\omega \sum_i \frac{1}{\lambda^i} = r \sum_i k_\omega^i \implies$$

$$\left(\sum_i \frac{1}{\lambda^i}\right) \sum_\omega b_\omega a_\omega = r \sum_\omega \sum_i k_\omega^i \implies$$

$$\sum_i \frac{1}{\lambda^i} = \frac{r\left(\bar{k}^1 + \bar{k}^2\right)}{\sum_\omega b_\omega a_\omega} \tag{2.18}$$

and in a similar manner

$$\sum_i \frac{1}{\lambda^i} = \frac{w\left(\bar{l}^1 + \bar{l}^2\right)}{\sum_\omega \left(1 - b_\omega\right) a_\omega} . \tag{2.19}$$

Using (2.18) and (2.19) we can determine the w/r ratio

$$\frac{\bar{l}^1 + \bar{l}^2}{\bar{k}^1 + \bar{k}^2} = \frac{r}{w} \frac{\sum_\omega \left(1 - b_\omega\right) a_\omega}{\sum_\omega b_\omega a_\omega} . \tag{2.20}$$

Assuming that one country is more capital abundant than the other (say $\bar{k}^1/\bar{l}^1 > \bar{k}^2/\bar{l}^2$), the equilibrium factor price ratio r/w under free trade lies in between the autarky factor prices of the two countries (determined in equation 2.8).

Using the relationships for the capital labor ratio (2.14) together with the above expression and factor market clearing conditions we can derive the equilibrium labor used from each country in each sector. Using the capital labor ratios for the second good and for both countries we get:

$$\frac{w}{r}\left[\left(\bar{l}^i - l_2^i\right)\frac{b_1}{(1-b_1)} + \left(l_2^i\right)\frac{b_2}{(1-b_2)}\right] = \bar{k}^i$$
$$\frac{l_2^i}{\bar{l}^i} = \frac{(1-b_2)(1-b_1)}{b_2 - b_1}\left(\frac{r}{w}\frac{\bar{k}^i}{\bar{l}^i} - \frac{b_1}{(1-b_1)}\right)$$
$$\frac{l_2^i}{\bar{l}^i} = \frac{(1-b_2)(1-b_1)}{b_1 - b_2}\left(\frac{b_1}{(1-b_1)} - \frac{\sum_\omega b_\omega a_\omega}{\sum_\omega (1-b_\omega) a_\omega}\frac{\bar{l}^1 + \bar{l}^2}{\bar{k}^1 + \bar{k}^2}\frac{\bar{k}^i}{\bar{l}^i}\right)$$

You may notice two things in this expression. First, if initial endowments of the two countries are inside a relative range, there is diversification since $l_j^i > 0$. If the endowments of a country for a given good are not in this range, then a country specializes in the other good (this range of endowments that implies diversification in production is commonly referred to as the cone of diversification). Second, conditional on diversification labor abundant countries use relatively more labor in the labor intensive sector.

What is the share of consumption for each country? We can use the FOC from the consumer's problem to obtain

$$p_1 c_1^i \left(1 + \frac{a_2}{a_1}\right) = r\bar{k}^i + w\bar{l}^i \implies$$
$$c_1^i = \frac{r\bar{k}^i + w\bar{l}^i}{p_1\left(1 + \frac{a_2}{a_1}\right)} \implies$$
$$c_1^i = \frac{1}{(1-b_\omega)}\frac{w\bar{l}^i}{p_1\left(1 + \frac{a_2}{a_1}\right)} \tag{2.21}$$

where in the last equivalence we used equation (2.14). Obtaining the rest of the allocations and prices is straightforward. In fact, you can show that if the production function exhibits CRS and the capital-labor ratio for both countries is fixed (in a given sector), total production can be represented by[1]

$$y_\omega = z_\omega \left(\sum_i k_\omega^i \right)^{b_\omega} \left(\sum_i l_\omega^i \right)^{1-b_\omega} .$$

We can determine $\sum_i k_\omega^i, \sum_i l_\omega^i$ by combining expression (2.18) with (2.16), (2.17) and using the market clearing condition. This gives

$$\frac{b_\omega a_\omega}{\sum_\omega b_\omega a_\omega} = \frac{\sum_i k_\omega^i}{\sum_i \bar{k}^i} , \tag{2.22}$$

and similarly for labor.

2.1.4 Specialization

[HW]

[1]Assume that $k^1/l^1 = k^2/l^2$. We only have to prove that given this assumption

$$A \left(k^1 + k^2\right)^b \left(l^1 + l^2\right)^{1-b} = A \left(k^1\right)^b \left(l^1\right)^{1-b} + A \left(k^2\right)^b \left(l^2\right)^{1-b} .$$

Using repeatedly the condition we have that

$$\begin{aligned}
A \left(\frac{k^1 + k^2}{l^1 + l^2}\right)^b = A \left(\frac{k^1}{l^1}\right)^b \frac{l^1}{l^1 + l^2} + A \left(\frac{k^2}{l^2}\right)^b \frac{l^2}{l^1 + l^2} \iff \\
\left(\frac{k^1 + k^2}{l^1 + l^2}\right)^b = \left(\frac{k^1}{l^1}\right)^b \iff \\
\left(\frac{\left(k^2 l^1\right)/l^2 + k^2}{l^1 + l^2}\right) = \left(\frac{k^1}{l^1}\right) \iff \\
\frac{k^2}{l^2} = \frac{k^1}{l^1} ,
\end{aligned}$$

which holds by assumption completing the proof.

2.1.5 The 4 big theorems.

In this final section for the H-O model we will state main theorems that hold in the benchmark model with two countries and two goods. Variants of these theorems hold under less or more restrictive assumptions. Our approach will still be as parsimonious as possible.[2]

Theorem 1. *Assume countries engage in free trade, there is* ***no*** *specialization (thus there is diversification) in equilibrium and there is no factor intensity-reversal, then factor prices equalize across countries.*

Proof. See main text □

Theorem 2. *(Rybczynski) Assume that the economies remain always incompletely specialized. An increase in the relative endowment of a factor will increase the ratio of production of the good that uses the factor intensively.*[3]

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Theorem 3. *(Stolper-Samuelson) Assume that the economies remain always incompletely specialized. An increase in the relative price of a good increases the real return to the factor used intensively in the production of that good and reduces the real return to the other factor.*

XXX

Theorem 4. *(Hekchser-Ohlin) Each country will produce the good which uses its abundant factor of production more intensively.*

[2]For a detailed treament you can look at the books of Feenstra (2003) and Bhagwati, Panagariya, and Srinivasan (1998).

[3]If prices were fixed a stronger version of the theorem can be proved.

Chapter 3

Models with Constant Elasticity Demand

Suppose there is a compact set S of countries. For now, we assume that S is discrete, although having a continuum of countries does not change things much. Whenever it is possible, we refer to an origin country as i and a destination country as j and order the subindices such that X_{ij} is the bilateral trade from i to j. We define as X_j the total spending of country j. We further denote by L_j the population of country j and let each consumer have a single labor unit that is inelastically supplied.

There are three common assumptions made about the market structure by trade theorists. The first is that markets in every country are perfectly competitive, so the price of a good is simply equal to its marginal cost. The second is that there is Bertrand competition so that the price of a good depends on the marginal cost of the least cost producer as well, potentially, on the cost of the second cheapest producer. The third is that production is monopolistically competitive so that the firm does not perceive any immediate competitor but it is affected by the overall level of competition. We will consider each below.

We also assume throughout that labor is the only factor of production (we will add intermediate inputs later on). We also assume that there are **iceberg trade costs** $\{\tau_{ij}\}_{i,j\in S}$. This means that in order from one unit of a good to arrive in destination j, destination i must ship τ_{ij} units. Iceberg trade costs are so called because a fraction $\tau_{ij}-1$ "melts" on its way from i to j, much as if you were towing an iceberg. We almost always assume that $\tau_{ij}\geq 1$ and usually assume that $\tau_{ii}=1$ for all $i\in S$, i.e. trade with oneself is costless. Furthermore, we sometimes assume that the following triangle inequality holds: for all $i,j,k\in S$: $\tau_{ij}\tau_{jk}\geq\tau_{ik}$. The triangle inequality says that it is never cheaper to ship a good via an intermediate location rather than sell directly to a destination.

3.1 Constant Elasticity Demand

We first introduce one of the most remarkably simple as well as versatile demand functions that will be the basis of our analysis for the next two chapters, the Constant Elasticity of Substitution (CES) demand function. Why do we do so? CES preferences have a number of attractive properties: (1) they are homothetic; (2) they nest a number of special demand systems (e.g. Cobb-Douglas); and (3) they are extremely tractable. Trade economists often do not believe that CES preferences are a good representation of actual preferences but are seduced into making frequent use of them due to their analytical convenience.

In particular, assume that the representative consumer in country j derives utility U_j from a set of varieties Ω the consumption of goods shipped from all countries $i\in S$:

$$U_j=\left(\sum_{\omega\in\Omega}a_{ij}\left(\omega\right)^{\frac{1}{\sigma}}q_{ij}\left(\omega\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \tag{3.1}$$

where $\sigma\geq 0$ is the elasticity of substitution and $a_{ij}\left(\omega\right)$ is an exogenous preference shifter.

A couple of things to note: first, $q_{ij}(\omega)$ is the *quantity* of a good shipped from i that *arrives* in j (the amount shipped is $\tau_{ij}q_{ij}(\omega)$); second, the fact that there is a representative consumer is not particularly important: we can always assume that workers (with identical preferences) are the ones consuming the goods and U_j can be interpreted as the total welfare of country j in this case (homotheticity is, of course, crucial for this property to be true).

We now solve the representative consumer's utility maximization problem. Given the importance of CES in the class we will proceed to do the full derivation for any given good $\omega \in \Omega$. Let the income of country j be denoted Y_j and let the price of a good (net of trade costs) from country i in country j be p_{ij}. Then the utility maximization problem is:

$$\max_{\left\{q_{ij}(\omega)\right\}_{\omega\in\Omega}} \left(\sum_{i\in\Omega} a_{ij}(\omega)^{\frac{1}{\sigma}} q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \text{ s.t. } \sum_{\omega\in\Omega} q_{ij}(\omega)\, p_{ij}(\omega) \leq X_j,$$

where I ignore the constraint that $q_{ij}(\omega) > 0$ (why is this okay?).

The Lagrangian is:

$$\mathcal{L} : \left(\sum_{\omega\in\Omega} a_{ij}(\omega)^{\frac{1}{\sigma}} q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - \lambda\left(\sum_{\omega\in\Omega} q_{ij}(\omega)\, p_{ij}(\omega) - X_j\right)$$

First order conditions (FOCs) are:

$$\frac{\partial\mathcal{L}}{\partial q_{ij}(\omega)} = 0 \iff \left(\sum_{\omega\in\Omega} a_{ij}(\omega)^{\frac{1}{\sigma}} q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} a_{ij}(\omega)^{\frac{1}{\sigma}} q_{ij}(\omega)^{-\frac{1}{\sigma}} = \lambda p_{ij}(\omega)$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0 \iff X_j = \sum_{\omega\in\Omega} q_{ij}(\omega)\, p_{ij}(\omega)$$

From the first FOC we have for any $i, i' \in S$:

$$\frac{a_{ij}(\omega)}{a_{ij}(\omega')} = \frac{p_{ij}^{\sigma}(\omega)}{p_{ij}^{\sigma}(\omega')} \frac{q_{ij}(\omega)}{q_{ij}(\omega')}$$

Rearranging and multiplying both sides by $p_{i'j}$ yields:

$$q_{ij}(\omega') \, p_{ij}(\omega') = \frac{1}{a_{ij}(\omega)} q_{ij}(\omega) \, p_{ij}(\omega)^{\sigma}(\omega) \, a_{ij}(\omega') \, p_{ij}^{1-\sigma}(\omega')$$

Summing over all $\omega' \in \Omega$ yields:

$$\begin{aligned}
\sum_{\omega' \in \Omega} q_{ij}(\omega') \, p_{ij}(\omega') = \frac{1}{a_{ij}(\omega)} q_{ij}(\omega) \, p_{ij}(\omega)^{\sigma} \sum_{\omega' \in \Omega} a_{ij}(\omega') \, p_{ij}(\omega')^{1-\sigma} \iff \\
X_j = \frac{1}{a_{ij}(\omega)} q_{ij}(\omega) \, p_{ij}(\omega)^{\sigma} P_j^{1-\sigma}
\end{aligned}$$

where the last line used the second FOC and $P_j \equiv \left(\sum_{\omega' \in \Omega} a_{ij}(\omega') \, p_{ij}(\omega')^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$ is known as the Dixit-Stiglitz price index. It is easy to show that $U_j = \frac{X_j}{P_j}$, i.e. dividing income by the price index gives the total welfare of country j. Rearranging the last line yields the **CES demand function**:

$$q_{ij}(\omega) = a_{ij}(\omega) \, p_{ij}^{-\sigma}(\omega) \, X_j P_j^{\sigma-1}, \tag{3.2}$$

Equation (3.2) implies that the quantity consumed in j of a good produced in i will be increasing with $j's$ preference for the good (a_{ij}), decreasing with the price of the good (p_{ij}), increasing with $j's$ spending (X_j), and increasing with $j's$ price index.

Note that the *value* of total trade is simply equal to the price times quantity. In what follows, let us denote the value of trade of good ω from country i to country j as $X_{ij}(\omega) \equiv$

$p_{ij}(\omega) q_{ij}(\omega)$. Then we have:

$$X_{ij}(\omega) = a_{ij}(\omega) p_{ij}^{1-\sigma}(\omega) X_j P_j^{\sigma-1}. \tag{3.3}$$

The only thing left to construct bilateral trade is to solve for the optimal price and aggregate across varieties, which we will do after a brief discussion of the gravity equation.

3.2 The gravity setup

It is helpful to provide a brief motivation of why we are interested in writing down a flexible model in the first place. "Classical" trade theories (Ricardo, Heckscher-Ohlin), while extremely useful in highlighting the economic forces behind trade, are very difficult to generalize to a set-up with many trading partners and bilateral trade costs. Because the real world clearly has both of these, the classical theories do not provide much guidance in doing empirical work. Because of this difficulty, those doing empirical work in trade began using a statistical (i.e. a-theoretic) model known as the **gravity equation** due to its similarity the Newton's law of gravitation. The gravity equation states that total trade flows from country i to country j, X_{ij}, are proportional to the product of the origin country's GDP Y_i and destination country's GDP Y_j and inversely proportional to the distance between the two countries, D_{ij}:[1]

$$X_{ij} = \alpha \frac{Y_i \times Y_j}{D_{ij}}. \tag{3.4}$$

For a variety of reasons (which we will go into later on in the course), this gravity equation is often estimated in a more general form, which we refer to as the **generalized gravity**

[1]This is actually in contrast to Newton's law of gravitation, where the force of gravity is inversely proportional to the *square* of the distance.

equation:

$$X_{ij} = K_{ij}\gamma_i\delta_j, \tag{3.5}$$

where K_{ij} is a measure of the resistance of trade between i and j, γ_i measures the origin "size" and δ_j measures the destination "size" (note that each country has two different measures of size)..

The gravity equation (3.4) and its generalization (3.5) have proven to be enormously successful at explaining a large fraction of the variation in observed bilateral trade flows; indeed, it is probably not too much of an exaggeration to say that the gravity equation is one of the most successful empirical relationships in all of economics. Because it was originally proposed as a statistical relationship, however, the absence of a theory justifying the relationship made it very difficult to ask any meaningful counterfactual questions; e.g. "what would happen to trade between i and j if the tariff was lowered between i and k?"

3.3 Armington model

The Armington model (Armington, 1969) is based on the premise that each country produces a different good and consumers would like to consume at least some of each country's goods. This assumption is of course ad hoc, and it completely ignores the "classical" trade forces such as increased specialization due to comparative advantage. However, as we will see, the model (when combined with Constant Elasticity of Substitution (CES) preferences as in (Anderson, 1979)) provides a nice characterization of trade flows between many countries.[2]

The Armington model (as formulated by (Anderson, 1979)) was important because

[2] Actually, in the main text, Anderson (1979) considers Cobb-Douglass preferences and writes that "there is little point in the exercise" of generalizing to CES preferences, doing so only in an appendix. Despite his reluctance to do so, the paper has been cited thousands of times as the example of an Armington model with CES preferences.

it provided the first theoretical foundation for the gravity relationship. It is also a great place to start our course, as one of the great surprises of the international trade literature over the past fifteen years has been how robust the results first present in the Armington model are across different quantitative trade models. By now, as we will discuss in this chapter, models that yield the gravity relationship (3.5) are ubiquitous and much of the rest of what follows will focus on analyzing their common properties.

3.3.1 The model

We now turn to the details of the Armington models and in particular to the supply side of this model, given CES demand.

The **Armington assumption** is that each country $i \in S$ produces a distinct variety of a good. Because countries map one-to-one to varieties, we index the varieties by their country names (this will not be true for Bertrand and monopolistic competition when we have to keep track both of varieties and countries).

Suppose that the market for each country/good is **perfectly competitive**, so that the price of a good is simply the marginal cost. Suppose each worker can produce A_i units of her country's good and let w_i be the wage of a worker. Then the marginal cost of production is simply $\frac{w_i}{A_i}$. This implies that the price at the factory door (i.e. without shipping costs) is $p_i = \frac{w_i}{A_i}$. What about with trade costs? Recall that with the iceberg formulation, $\tau_{ij} \geq 1$ units have to be shipped in order for one unit to arrive. This means that $\tau_{ij} \geq 1$ units have to be produced in country i in order for one unit to be consumed in country j. Hence the price in country j of consuming one unit from country i is:

$$p_{ij} = \tau_{ij} \frac{w_i}{A_i}. \tag{3.6}$$

Note that this implies that:

$$\frac{p_{ij}}{p_i} = \tau_{ij}, \tag{3.7}$$

i.e. the ratio of the price in any destination relative to the price at the factory door is simply equal to the iceberg trade cost. Equation (3.7) is called a **no-arbitrage equation**, as it means that there is no way for an individual to profit by buying a good in country i and sell in country j (or vice versa). Note, however, that there may still be profitable trading opportunities between triplets of countries even if equation (3.7) holds when the triangle inequality is not satisfied.

3.3.2 Gravity

Assuming that each country produces a different good ω, and substituting equation (3.6) into equation (3.3) yields a gravity equation for bilateral trade flows:

$$X_{ij} = a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{A_i}\right)^{1-\sigma} X_j P_j^{\sigma-1}. \tag{3.8}$$

To the extent that trade costs are increasing in distance, the value of bilateral trade flows will decline as long as $\sigma > 1$.

We can actually use equation to get a little close to the true gravity equation. The total income in a country is equal to its total sales:

$$Y_i = \sum_j X_{ij} = \sum_j a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{A_i}\right)^{1-\sigma} X_j P_j^{\sigma-1} \iff$$

$$\left(\frac{w_i}{A_i}\right)^{1-\sigma} = Y_i / \sum_j a_{ij}\tau_{ij}^{1-\sigma} X_j P_j^{\sigma-1}$$

Replacing this expression in the equation (3.8) yields:

$$X_{ij} = a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{Y_i}{\Pi_i^{1-\sigma}}\right)\left(\frac{X_j}{P_j^{1-\sigma}}\right), \tag{3.9}$$

where $\Pi_i^{1-\sigma} \equiv \sum_j a_{ij}\tau_{ij}^{1-\sigma}X_jP_j^{\sigma-1}$ bears a striking resemblance to the price index [insert foreshadowing here]. Equation (3.9) which shows that the bilateral trade spending is related to the product of the GDPs of the two countries (gravity!!), the distance/tradecost and a GE component.

Equation (3.9) is actually about as close as we will ever get to the original gravity equation. This is because all of our theories say that bilateral trade flows depend on more than just the bilateral trade costs and the incomes of the exporter and importer; what also matters is so-called “bilateral resistance”: intuitively, the greater the cost of exporting in general, the smaller the $\Pi_i^{1-\sigma}$; conversely, the greater the cost of importing in general, the smaller the $P_i^{1-\sigma}$. This means that trade between any two countries depends not only on the incomes of those two countries but also the “cost” of trading between those countries *relative to* trading with all other countries. This point was made in the enormously famous and influential paper “Gravity with Gravitas: A Solution of the Border Puzzle” (Anderson and Van Wincoop, 2003).

3.3.3 Welfare

We will now show that welfare in relationship to trade is given by a simple equation involving the trade to GDP ratio and parameters of the model (but no other equilibrium variables). We will be revisiting this relationship multiple times in these notes. To begin define λ_{ij} as the fraction of expenditure in j spent on goods arriving from location i:

$$\lambda_{ij} \equiv \frac{X_{ij}}{\sum_k X_{kj}}.$$

From equation (3.8) we have:

$$\lambda_{ij} = \frac{a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{A_i}\right)^{1-\sigma}}{\sum_k a_{kj}\tau_{kj}^{1-\sigma}\left(\frac{w_k}{A_k}\right)^{1-\sigma}} \iff$$

$$\lambda_{ij} = a_{ij}\tau_{ij}^{1-\sigma}A_i^{\sigma-1}\left(\frac{w_i}{P_j}\right)^{1-\sigma} \tag{3.10}$$

since $P_j^{1-\sigma} \equiv \sum_k a_{kj}\tau_{kj}^{1-\sigma}\left(\frac{w_k}{A_k}\right)^{1-\sigma}$. Remember from the CES derivations above that the utility of the representative agent is the real wage, i.e. $U_j = \frac{w_j}{P_j}$. Assume that $\tau_{jj} = 1$. Then by choosing $i = j$, equation (3.10) implies that welfare can be written as:

$$U_j = \lambda_{jj}^{\frac{1}{1-\sigma}}\alpha_{jj}^{\frac{1}{\sigma-1}}A_j, \tag{3.11}$$

i.e. welfare depends only on changes in the trade to GDP ratio, λ_{jj}, with an elasticity of $-1/(\sigma-1)$ which is the inverse of the trade elasticity.

3.4 Monopolistic Competition with Homogeneous Firms and CES demand

With the Armington model, we saw how we could justify the gravity relationship in trade using the ad-hoc assumption that every country produces a unique good as well as the assumption that consumers have a "love of variety" (i.e. they want to consume at least a little bit of every one of the goods). In this section, we will dispense of the first assumption by introducing firms into the model. However, we will continue to rely heavily on the second assumption by assuming that each firm produces a unique variety and consumers would like to consume at least a little bit of every variety.

The model considered today was introduced by Krugman (1980) and was an important part of the reason he won a Nobel prize. A key feature of the Krugman (1980) model is that there are **increasing returns to scale**, i.e. the average cost of production is lower the more that is produced. All else equal, this will lead to gains from trade, since by taking advantage of demand from multiple countries, firms can lower their average costs. To succinctly model the increasing returns from scale, we suppose that a firm has to incur a fixed entry cost f_i^e in order to produce. (The e might seem like unnecessary notation; however, we keep it here because in future models there will be both an entry cost and a fixed cost of serving a particular destination). We assume the fixed cost of entry (like the marginal cost) is paid to domestic workers so that f_i^e is the number of workers employed in the entry sector (think of them as the workers who build the firm). Some of the results toward the end of this section are based on the subsequent analysis of Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008).

3.4.1 Setup

The main departure from the perfect competition paradigm is that in monopolistic competition each differentiated variety is produced (potentially) by a different firm, where there is a measure M_i of firms in country i. This number of firms is determined in equilibrium by allowing firms to enter after incurring a fixed cost of entry in terms of domestic labor, f_i^e.

3.4.2 Demand

As in the Armington model, we assume that consumers have CES preferences over varieties. Hence a representative consumer in country $j \in S$ gets utility U_j from the consumption of goods shipped by all other firms in all other countries, where:

$$U_j = \left(\sum_{i \in S} \int_{\Omega_i} q_{ij} \left(\omega \right)^{\frac{\sigma - 1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma - 1}}, \tag{3.12}$$

where $q_j \left(\omega \right)$ is the quantity consumed in country j of variety ω. Note that for simplicity, I no longer include a preference shifter (although one could easily be incorporated) so that consumers treat all firms in all countries equally.

The consumer's utility maximization problem is very similar to the Armington model (which shouldn't be particularly surprising, given preferences are virtually the same). In particular, a consumer in country $j \in S$ optimal quantity demanded of good $\omega \in \Omega$ is:

$$q_{ij} \left(\omega \right) = p_{ij} \left(\omega \right)^{-\sigma} E_j P_j^{\sigma - 1},$$

where:

$$P_j \equiv \left(\sum_{i \in S} \int_{\Omega_i} p_{ij} \left(\omega \right)^{1 - \sigma} d\omega \right)^{\frac{1}{1 - \sigma}} \tag{3.13}$$

is the Dixit-Stiglitz price index.

The amount spent on variety ω is simply the product of the quantity and the price:

$$x_{ij} \left(\omega \right) = p_{ij} \left(\omega \right)^{1 - \sigma} E_j P_j^{\sigma - 1}. \tag{3.14}$$

Note that we derived a very similar expression in the Armington model, from which the gravity equation followed almost immediately. In this model however, this is the amount spent on the goods from a particular firm, so we now need to aggregate across all firms in country i to determine bilateral trade flows between i and j, i.e.:

$$X_{ij} \equiv \int_{\Omega_i} x_{ij}(\omega)\, d\omega = Y_j P_j^{\sigma-1} \int_{\Omega_i} p_{ij}(\omega)^{1-\sigma}\, d\omega. \tag{3.15}$$

3.4.3 Supply

All firms in country i have a common productivity, z_i, and produce one unit of the good using $\frac{1}{z_i}$ units of labor. The optimization problem faced by a firm ω from country i is:

$$\max_{\left\{q_j(\omega)\right\}_{j\in S}} \sum_{j\in S} \left(p_j(\omega)\, q_j(\omega) - w_i \frac{\tau_{ij}}{z_i} q_j(\omega) \right) - w_i f_i^e \text{ s.t. } q_j(\omega) = a_{ij} p_j(\omega)^{-\sigma} E_j P_j^{\sigma-1}$$

We can substitute the constraint into the maximand and write the equivalent unconstrained problem of choosing the price to sell to each location as:

$$\max_{\left\{p_j(\omega)\right\}_{j\in S}} \sum_{j\in S} \left(p_j^{1-\sigma}(\omega)\, E_j P_j^{\sigma-1} - w_i \frac{\tau_{ij}}{z_i} p_j^{-\sigma}(\omega)\, E_j P_j^{\sigma-1} \right) - w_i f_i^e$$

Note that the constant marginal cost assumption implies that the country can treat each destination as a separate optimization problem (this will come in helpful in models we will see later on).[3] Profit maximization implies that optimal pricing for a firm selling from country i to country j is

$$p_{ij}(z_i) = \frac{\sigma}{\sigma-1} \frac{\tau_{ij} w_i}{z_i}, \tag{3.16}$$

[3] Notice that here we haven't introduced fixed costs of exporting. Introducing these costs will change the analysis in that we may have countries for which all the firms chose not to export depending on values of the fixed costs and other variables. More extreme predictions can be delivered if the production cost f_i is only a cost to produce domestically and independent of the exporting cost. However, in order to create a true extensive margin of firms (i.e. more firms exporting when trade costs decrease) requires heterogeneity either in the productivities of firms (as we will do later on in the notes) or in the fixed costs of selling to a market (see Romer (1994)).

and since all firm decision will depend on parameter's and firm productivity we drop the ω notation from here on. We will make the notation a bit cumbersome by carrying around the z_i's in order to allow for direct comparison of our results with the heterogeneous firms example that will be studied later on.

3.4.4 Gravity

Because every firm is charging the same price, we can substitute the price equation (3.16) into the gravity equation (3.15) to yield:

$$X_{ij} = E_j P_j^{\sigma-1} \int_{\Omega_i} \left(\frac{\sigma}{\sigma-1} \frac{w_i}{z_i} \tau_{ij} \right)^{1-\sigma} d\omega \iff$$
$$X_{ij} = \left(\frac{\sigma}{\sigma-1} \right)^{1-\sigma} \tau_{ij}^{1-\sigma} \left(\frac{w_i}{z_i} \right)^{1-\sigma} M_i E_j P_j^{\sigma-1} \tag{3.17}$$

where $M_i \equiv \int_{\Omega_i} d\omega$ is the measure of firms producing in country i. Comparing this equation to the one derived for the Armington model with monopolistic, we see that the two expressions are nearly identical - the only difference here is that we have to keep track of the mass of firms M_i and all trade flows are smaller (if $\sigma > 1$) as a result of the markups.

3.4.5 Welfare

It turns out welfare can be written similarly to the Armington model. First, note that substituting equation (3.16) for the equilibrium price charged into the price index equation (3.13) yields:

$$P_j^{1-\sigma} \equiv \left(\frac{\sigma}{\sigma-1} \right)^{1-\sigma} \sum_k \tau_{kj}^{1-\sigma} \left(\frac{w_k}{z_k} \right)^{1-\sigma} M_k.$$

As above, define $\lambda_{ij} \equiv \frac{X_{ij}}{\sum_k X_{kj}}$ to be the fraction of expenditure of country j on goods sent from country i. Then using equation (3.17), we can write λ_{ij} as a function of the price index in j:

$$
\begin{aligned}
\lambda_{ij} &= \frac{\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{z_i}\right)^{1-\sigma}M_iE_jP_j^{\sigma-1}}{\sum_k\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\tau_{kj}^{1-\sigma}\left(\frac{w_k}{z_k}\right)^{1-\sigma}M_kE_jP_j^{\sigma-1}} \iff \\
\lambda_{ij} &= \frac{\tau_{ij}^{1-\sigma}\left(\frac{w_i}{z_i}\right)^{1-\sigma}M_i}{\sum_k\tau_{kj}^{1-\sigma}\left(\frac{w_k}{z_k}\right)^{1-\sigma}M_k} \iff \\
\lambda_{ij} &= \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\frac{\tau_{ij}^{1-\sigma}\left(\frac{w_i}{z_i}\right)^{1-\sigma}M_i}{P_j^{1-\sigma}} \iff \\
P_j &= \left(\frac{\sigma}{\sigma-1}\right)\tau_{ij}\left(\frac{w_i}{z_i}\right)M_i^{\frac{1}{1-\sigma}}\lambda_{ij}^{\frac{1}{\sigma-1}}. \qquad (3.18)
\end{aligned}
$$

Since equation (3.18) holds for any i and j, we can focus on the particular case where $i = j$. Then assuming $\tau_{jj} = 1$, we can write equation (3.18) as:

$$
\begin{aligned}
P_j &= \left(\frac{\sigma}{\sigma-1}\right)\left(\frac{w_j}{z_j}\right)M_j^{\frac{1}{1-\sigma}}\lambda_{jj}^{\frac{1}{\sigma-1}} \iff \\
\frac{w_j}{P_j} &= \left(\frac{\sigma-1}{\sigma}\right)z_jM_j^{\frac{1}{\sigma-1}}\lambda_{jj}^{\frac{1}{1-\sigma}}, \qquad (3.19)
\end{aligned}
$$

i.e. the real wage is declining in λ_{jj} or equivalently, increasing in trade openness. Note, however, that unlike the Armington model, firms are making positive profits, so that the real wage no longer captures the welfare of a location. To deal with this issue, we introduce a free entry condition.

3.4.6 Free entry

The final thing we have to do is determine the equilibrium number of firms that enter. In this model the mass of firms M_i is determined by the **free entry condition** which states that the profits of all firms must be equal to zero. The justification for this condition is that there is a large mass of potential firms (or equivalently, other differentiated products that could be produced), who choose not to enter. [Class question: why do they not enter?][4]

Hence, to determine the equilibrium mass of firms, we need to calculate the profits of any particular firm. Firms profits are:

$$\pi_i(\omega) \equiv \sum_j \left(p_{ij}(\omega) - \frac{w_i}{z_i}\tau_{ij} \right) q_{ij}(\omega) - w_i f_i^e \tag{3.20}$$

Substituting the consumer demand expression (3.1) and the price expression (3.16) into equation (3.20) yields:

$$\begin{aligned}
\pi_i(\omega) &= \sum_j \left(\frac{\sigma}{\sigma-1} c_i \tau_{ij} - \frac{w_i}{z_i}\tau_{ij} \right) \left(\frac{\sigma}{\sigma-1} \frac{w_i}{z_i}\tau_{ij} \right)^{-\sigma} E_j P_j^{\sigma-1} - w_i f_i^e \iff \\
\pi_i(\omega) &= \sum_j \left(\left(\frac{\sigma}{\sigma-1} - 1 \right) \right) \left(\frac{\sigma}{\sigma-1} \right)^{-\sigma} \left(\frac{w_i}{z_i}\tau_{ij} \right)^{1-\sigma} E_j P_j^{\sigma-1} - w_i f_i^e \iff \\
\pi_i(\omega) &= \sum_j \left(\frac{1}{\sigma-1} \right) \left(\frac{\sigma}{\sigma-1} \right)^{-\sigma} \left(\frac{w_i}{z_i}\tau_{ij} \right)^{1-\sigma} E_j P_j^{\sigma-1} - w_i f_i^e \iff \\
\pi_i(\omega) &= \frac{1}{\sigma} \left(\frac{\sigma}{\sigma-1} \right)^{1-\sigma} \sum_j \left(\frac{w_i}{z_i}\tau_{ij} \right)^{1-\sigma} E_j P_j^{\sigma-1} - w_i f_i^e
\end{aligned}$$

It turns out that in this framework, the profits of a firm have a simple relationship to the quantity the firm produces, which greatly simplifies the equilibrium. To see this, we first relate the profits a firm to its revenues. Note that from equation (10.1) and the price

[4]To see that the entry of additional firms pushes down the profits of any particular firm, note that combining expression (3.13) for the Dixit-Stiglitz price index with the price expression (3.16) from the producers optimization problem yields:

$$P_j \equiv \frac{\sigma}{\sigma-1} \left(\sum_{i\in S} M_i \left(\frac{w_i}{z_i}\tau_{ij} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Hence, an increase in the number of firms producing in any country reduces the price index, thereby decreasing profits. [Class question: what is the intuition for why more firms lowers the price index?]

expression (3.16) that the revenue a producer receives is:

$$r_i(\omega) \equiv \sum_{j \in S} p_{ij}(\omega) q_{ij}(\omega) = \left(\frac{\sigma}{\sigma - 1}\right)^{1-\sigma} \sum_{j \in S} \left(\frac{w_i}{z_i} \tau_{ij}\right)^{1-\sigma} E_j P_j^{\sigma-1}$$

so that variable profits are simply equal to revenue divided by the elasticity of substitution, i.e.:

$$\pi_i(\omega) + w_i f_i^e = \frac{1}{\sigma} r_i(\omega). \tag{3.21}$$

[Class question: what is the intuition of this result?].

From the price equation (3.16), if we assume that $\tau_{ii} = 1$, we can decompose the total revenue produced by a firm into the total quantity it produces and the price:

$$r_i(\omega) = p_i(\omega) q_i(\omega) = \frac{\sigma}{\sigma - 1} \left(\frac{w_i}{z_i}\right) q_i(\omega), \tag{3.22}$$

where I imposed the fact that the marginal cost $c_i = \frac{w_i}{A_i}$.

From the free entry condition, total profits of a firm are zero, i.e. $\pi_i(\omega) = 0$. Applying the free entry condition to equation (3.21) yields:

$$w_i f_i^e = \frac{1}{\sigma} r_i(\omega) \tag{3.23}$$

Substituting equation (3.22) into (3.23) then yields:

$$\begin{aligned} w_i f_i^e &= \frac{1}{\sigma - 1} \left(\frac{w_i}{z_i}\right) q_i(\omega) \iff \\ (\sigma - 1) f_i^e &= \frac{q_i(\omega)}{z_i} \end{aligned}$$

i.e. in equilibrium, the fixed cost of entry will be proportional $\frac{q_i(\omega)}{z_i}$, which is the amount of labor used in production. The last step is to note that the total labor used by all firms (for both entry and for production) has to equal the total number of workers in the country, L_i:

$$\begin{aligned} M_i \left(f_i^e + \frac{q_i(\omega)}{z_i} \right) = L_i \iff \\ M_i \left(f_i^e + (\sigma - 1) f_i^e \right) = L_i \iff \\ M_i = \frac{1}{\sigma} \frac{L_i}{f_i^e}. \end{aligned} \tag{3.24}$$

In equilibrium, the number of firms is proportional to the population of a country and inversely proportional to the entry costs and the elasticity of substitution. [Class question: what is the intuition for each of these comparative statics?].

Using the equilibrium number of firms given by expression (3.24) into the gravity equation given by (10.3) to yield:

$$X_{ij} = \frac{1}{\sigma} \left(\frac{\sigma}{\sigma - 1} \right)^{1-\sigma} \tau_{ij}^{1-\sigma} \left(\frac{w_i}{z_i} \right)^{1-\sigma} \frac{L_i}{f_i^e} E_j P_j^{\sigma-1} \tag{3.25}$$

and we can re-write the real wage equation (3.19) as:

$$\frac{w_j}{P_j} = \left(\frac{\sigma - 1}{\sigma} \right) \left(\frac{1}{\sigma} \right)^{\frac{1}{\sigma-1}} \left(\frac{L_j}{f_j^e} \right)^{\frac{1}{\sigma-1}} z_j \lambda_{jj}^{\frac{1}{1-\sigma}}. \tag{3.26}$$

Because firms earn zero profits with free entry, the real wage is now equal to the welfare of location j, so that equation (3.26) tells us that the welfare of a location is increasing with its openness. Intuitively, firms bid up the price of labor by using workers for the fixed cost of entry, which increases wages to the point that all profits accrue to wages.

Because the equilibrium number of firms is pinned down by exogenous model parameters, the Krugman (1980) gravity equation (3.25) can be formally shown to be isomorphic to Armington model discussed in the previous class. This means that with an appropriate transformation of model fundamentals both models will yield identical predictions for the equilibrium outcomes of the model. Hence, in a sense, making the Armington model more realistic by replacing the Armington assumption with firms did not end up changing anything that much.

3.5 Ricardian model

The Ricardian model is a model of perfect competition where countries produce the same goods using different technologies. The Ricardian model predicts that countries may specialize in the production of certain ranges of goods.

3.5.1 The two goods case

We consider the simple version of the model with two countries and two goods. In order to get as much intuition as possible we will first consider the case where both countries specialize in the production of one good.

The production technologies in the two countries $i = 1, 2$ are different for the two goods $\omega = 1, 2$ and given by

$$y^i(\omega) = z^i(\omega)\, l^i(\omega), i, \omega = 1, 2 \, .$$

Assume that country 1 has absolute advantage in the production of both goods

$$\begin{aligned} z^2(1) &< z^1(1) \ , \\ z^2(2) &< z^1(2) \ . \end{aligned}$$

Assume that country 1 has comparative advantage in the production of good 1 and country 2 in good 2

$$\frac{z^1(2)}{z^2(2)} < \frac{z^1(1)}{z^2(1)}\,. \tag{3.27}$$

Assume Cobb-Douglas preferences. The consumer's problem is

$$\max a(1)\log c^i(1) + a(2)\log c^i(2)$$
$$s.t.\ p(1)\,c^i(1) + p(2)\,c^i(2) \le w^i\bar{l}^i.$$

Consumer optimization implies that

$$p(2)\,c^i(2) = \frac{a(2)}{a(1)}p(1)\,c^i(1) \tag{3.28}$$

$$p(1)\,c^i(1) + p(2)\,c^i(2) = w^i\bar{l}^i \tag{3.29}$$

3.5.2 Autarky

Using firms cost minimization and the Inada conditions (that ensure that the consumer actually wants to consume both goods) from the consumer problem we directly obtain that

$$p^i(1)\,z^i(1) = w = p^i(2)\,z^i(2)\,.$$

Using the goods market clearing

$$c^i(\omega) = y^i(\omega) \text{ for } \omega = 1,2\,,$$

together with labor market clearing

$$l^i(\omega) = a(\omega)\,\bar{l}^i\,,$$

we get labor allocated to each good. Using the production function and goods market clearing we can obtain the rest of the allocations.

3.5.3 Free trade

Under free trade international prices equalize. Relative productivity patterns will determine specialization. There can be three possible specialization patterns, two where one country specializes and the other diversifies and one where both countries specialize.

1. [Specialization]Under the assumptions stated, at least one country specializes in the free trade equilibrium.

Proof. If not then the firm's cost minimization together with the consumer FOCs would imply

$$\frac{z^1(2)}{z^2(2)} = \frac{z^1(1)}{z^2(1)},$$

a contradiction. □

In the three different equilibria that can emerge the countries export what they have comparative advantage on (specialization into exporting). Under free trade this relative price has to be in the range (given the Inada conditions in consumption):

$$\frac{z^1(2)}{z^2(2)} \leq \frac{p(2)}{p(1)} \leq \frac{z^1(1)}{z^2(1)}$$

To consider an example of how the wages are determined notice that for the country that is under incomplete specialization equations cost minimization implies

$$\frac{p(1)\, z^i(1)}{p(2)\, z^i(2)} = \frac{w^i}{w^i} \implies \frac{p(1)}{p(2)} = \frac{z^i(2)}{z^i(1)},$$

i.e. this country sets the relative price of the two goods. Now assume that country 1 is incompletely specialized which means that country 2 specializes in good 2 and normalize $w^1 = 1$. Because of free trade and perfect competition it must be the case that the cost of producing good 2 in both countries is the same, i.e.

$$\frac{w^1}{z^1(2)} = \frac{w^2}{z^2(2)} \implies w^2 = \frac{z^2(2)}{z^1(2)} < 1 = w^1 .$$

Notice that using the wages and the zero profit conditions for country 1 we now get $p(1) z^1(1) = 1$ and $p(2) z^1(2) = 1$

$$\frac{z^1(1)}{z^1(2)} = \frac{p(2)}{p(1)} .$$

Finally using the budget constraints of the individual we can determine the levels of consumption and verify that the equilibrium is consistent with our initial assumption for the patterns of specialization (i.e. indeed country 2 exports good 2 and country 1 exports good 1)

3.6 Homeworks

1. *Dixit-Stiglitz Preferences*. Suppose that a consumer has wealth W, consumes from a set of differentiated varieties $\omega \in \Omega$, and solves the following CES maximization problem:

$$\max_{\{q(\omega)\}} U = \left(\int_{\Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}} \text{ s.t. } \int_{\Omega} p(\omega) q(\omega) \leq W, \tag{3.30}$$

where $\sigma > 0$, $q(\omega)$ is the quantity consumed of variety ω and $p(\omega)$ the price of variety ω.

(a) Find a price index P such that in equilibrium $U = \frac{W}{P}$.

(b) Derive the optimal $q(\omega)$ as a function of W, P and $p(\omega)$.

(c) Show that σ is the elasticity of substitution, i.e. for any $\omega, \omega' \in \Omega$, $\sigma = \frac{d \ln\left(\frac{q(\omega)}{q(\omega')}\right)}{d \ln\left(\frac{\partial U / \partial q(\omega')}{\partial U / \partial q(\omega)}\right)}$.

(d) What happens as $\sigma \to \infty$? $\sigma \to 1$? $\sigma \to 0$?

Chapter 4

Modeling with CES demand and production heterogeneity

The purpose of this chapter is to develop a general model for production heterogeneity in which different assumptions on technology and competition will give us different workhorse frameworks important for the quantitative analysis of trade. Our analysis of the general framework is based on the exposition of (Eaton and Kortum (2011)) and earlier results of (Kortum (1997)) and (Eaton and Kortum (2002)). We start with a simple extension of the Ricardian model with intra-sector heterogeneity.

4.1 Introduction to heterogeneity: The Ricardian model with a continuum of goods

The model of Dornbusch, Fischer, and Samuelson (1977) is based on the Ricardian model where trade and specialization patterns are determined by different productivities.[1] There

[1] The notes in this chapter are partially based on Eaton and Kortum (2011).

is absolute advantage due to higher productivity in producing certain goods, but also comparative advantage due to lower opportunity cost of producing some goods. The main drawback of the simple Ricardian model, similar to that of the Heckscher-Ohlin model, is in the complexity of solving for the patterns of specialization for a large number of industries.

Breakthrough: Dornbusch, Fischer, and Samuelson (1977) used a continuum of sectors. The characterization of the equilibrium ended up being very easy.

- Perfect competition
- 2 countries (H, F)
- Continuum of goods $\omega \in [0, 1]$
- CRS technology (labor only)
- Cobb-Douglas Preferences with equal share in each good
- Iceberg trade costs τ_{HF}, τ_{FH}

We normalize the domestic wage to 1. We want to characterize the set of goods produced and exported from each country. Without loss of generality we will characterize production and exporting for country F. We first need to compare the price of a good ω potentially offered by country H to country F to the corresponding price of the good produced by F in order to determine the set of goods produced by country F in equilibrium. For this purpose, we will order the goods in a decreasing order of domestic to foreign productivity and define $\underline{\omega}$ as the good with the lowest productivity produced in the foreign country. Thus, the foreign country produces goods $[\underline{\omega}, 1]$ while the domestic $[0, \bar{\omega}]$. When trade costs exist then the two sets will overlap, $\bar{\omega} > \underline{\omega}$, but if $\tau_{HF} = \tau_{FH}$ then $\bar{\omega} = \underline{\omega}$. A simple condition that determines which are the goods that will be produced by country F dictates that the price of these goods in country F has to be lower than the price of imported goods, i.e.

$$\frac{w_F}{z_F(\omega)} < \frac{\tau_{HF}}{z_H(\omega)} \implies A(\omega) \equiv \frac{z_H(\omega)}{z_F(\omega)} < \frac{\tau_{HF}}{w_F} .$$

Therefore, we can define

$$A(\underline{\omega}) = \frac{w_F}{\tau_{HF}}. \tag{4.1}$$

which determines that only products that will be produced by country F.

To find the set of goods that F will be exporting we need to determine set of goods produced by country H. Using a similar logic this simply entails finding the ω that satisfies

$$A(\underline{\omega}) = \tau_{FH} w_F \tag{4.2}$$

and $[0, \bar{\omega}]$ is the set of goods produced by the home country. Thus, F produces goods $[\underline{\omega}, 1]$ and exports $[\bar{\omega}, 1]$ since the domestic does not produce any of the goods in that last set.

In order to get sensible relationships from the model, DFS parametrize $\frac{z_F(\omega)}{z_H(\omega)}$ by using a monotonic function. In this last case we can invert A and get the exact range of goods produced by each country, i.e. effectively determine $\underline{\omega}$ and $\bar{\omega}$ as a function of parameters and w_F. Subsequently, we can solve for the equilibrium wage, using the labor market clearing

$$L_H = \underline{\omega}(w_F) w_F L_F + \bar{\omega}(w_F) L_H .$$

4.1.1 Where DFS stop and EK start

Eaton and Kortum (2002) (henceforth EK) treat productivities $z_i(\omega)$ as an independent realization of a random variable Z_i independently distributed according to the same distribution F_i for each good ω in country i. Given the continuum of goods (using a LLN argument) we can determine with certainty the fraction of goods produced by each country. This way EK are able to overcome the complications faced by the standard Ricardian framework and go much further in developing an analytical quantitative trade framework.

Assume that the random variable Z_i follows the Frechet distribution[2]:

$$\Pr\left(Z_i \leq z\right) = \exp\left[-A_i z^{-\theta}\right] .$$

The parameter $A_i > 0$ governs country's overall level of efficiency (absolute advantage) (with more productive countries having higher A_i's). The parameter $\theta > 1$ governs variation in productivity across different goods (comparative advantage) (higher θ less dispersed).

Now we will split the $[0,1]$ interval by thinking of $\bar{\omega}$ as the probability that the relative productivity of F to H is less than $\tilde{A}$, where $\tilde{A}$ can either be defined by (4.2). Therefore, in order to determine $\bar{\omega}$ which is defined as the share of goods that the domestic country produces we simply compute the probability that the domestic country is the cheapest provider of the good across all the range of productivities. For example using (4.2) for the

[2]See the appendix for the properties of the Frechet distribution and the next chapter for a derivation from first principles.

definition of $\tilde{A}$ we can derive

$$
\begin{aligned}
\bar{\omega} &= \lambda_{HH} \\
&= \Pr\left[\frac{z_F}{z_H} \leq \tilde{A}\right] \\
&= \Pr\left[z_F \leq \tilde{A} z_H\right] \\
&= \int_0^{+\infty} \underbrace{\exp\left[-A_F\left(\tilde{A}\right)^{-\theta}\right]}_{\Pr\left(z(\omega)\leq \tilde{A}_{zH}(\omega)\right)} \underbrace{dF_H\left(z\right)}_{\text{density of } z_H(\omega)} \\
&= \int_0^{+\infty} \exp\left[-A_F\left(\tilde{A}z\right)^{-\theta}\right]\theta A_H\left(z\right)^{-\theta-1}\exp\left[-A_H\left(z\right)^{-\theta}\right]dz \\
&= \frac{A_H}{A_H + A_F \tilde{A}^{-\theta}}
\end{aligned}
$$

Country H is spending $(1-\bar{\omega})\, w_H L_H$ on imports (given Cobb-Douglas) which implies

$$
X_{FH} = \frac{A_F\left(w_F \tau_{HF}\right)^{-\theta}}{A_H + A_F\left(w_F \tau_{HF}\right)^{-\theta}} w_H L_H
$$

Notice that this relationship is similar to the relationship (**??**) derived with the assumption of the Armington aggregator but with an exponent $-\theta$. A lower value of θ generates more heterogeneity. This means that the comparative advantage exerts a stronger force for trade against resistance impose by the geographic barrier τ_{in}. In other words with low θ there are many outliers that overcome differences in geographic barriers (and prices overall) so that changes in w's and τ's are not so important for determining trade.

4.2 A theory of technology starting from first principles

We start with a very general technological framework under the following assumptions. Time is continuous and there is a continuum of goods with measure $\mu\left(\Omega\right)$. Ideas for good

ω (ways to produce the same good with different efficiency) arrive at location i at date t at a Poisson rate with intensity

$$\bar{a} R_i(\omega, t)$$

where we think of $\bar{a}$ as research productivity and R as research effort. The quality of ideas is a realization from a random variable Q drawn independently from a Pareto distribution with $\theta > 1$, so that

$$\Pr[Q > q] = \left(q / \underline{q}\right)^{-\theta}, q \geq \underline{q}$$

where $\underline{q}$ is a lower bound of productivities. Note that the probability of an idea being bigger than q conditional on ideas being bigger than a threshold, is also Pareto (see appendix for the properties of the Pareto distribution).

The above assumptions together imply that the arrival rate of an idea of efficiency $Q \geq q$ is

$$\bar{a} R_i(\omega, t) \left(q / \underline{q}\right)^{-\theta}.$$

(normalize this with $\underline{q} \rightarrow 0$, $\bar{a} \rightarrow +\infty$ such that $\bar{a}\underline{q}^{-\theta} \rightarrow 1$ in order to consider all the ideas in $(0, +\infty)$). We also assume that there is no forgetting of ideas. Thus, we can summarize the history of ideas for good ω by

$$A_i(\omega, t) = \int_{-\infty}^{t} R_i(\omega, \tau) \, d\tau.$$

The number of ideas with efficiency $Q > q'$ is therefore distributed Poisson with a parameter $A(\omega, t)(q')^{-\theta}$ (using the previous normalization).

The unit cost for a location i of producing good ω with an efficiency of q is $c = w_i / q$. Given all the above, the expected number of techniques providing unit cost less than c is distributed Poisson with parameter

$$\Phi_i(\omega,t)\, c^{\theta}$$

where

$$\Phi_i(\omega,t) = A_i(\omega,t)\, w_i^{-\theta}.$$

But notice that this delivers back unit costs that are conditionally Pareto distributed

$$\begin{aligned}
\Pr\left[C \le c' | C \le c\right] &= \Pr\left[Q \ge \frac{w}{c'} = q' | Q \ge \frac{w}{c} = q\right] \\
&= \frac{A(\omega,t)\,(q')^{-\theta}}{A(\omega,t)\,(q)^{-\theta}} = \left(\frac{q}{q'}\right)^{\theta} = \left(\frac{c'}{c}\right)^{\theta}.
\end{aligned}$$

In what follows set

$$\Phi = \Phi_i(\omega,t).$$

4.2.1 Order Statistics and Various Moments

The generality of this approach still allow for a number of order statistics and key moments to be computed. We start by computing the distribution of the order statistics in the model.

2. $C^{(k)}$ is the *k'th* lowest unit cost technology for producing a particular good. Given this definition we have the main theorem for the joint distribution of the order statistics $C^{(k)}$

3. The joint density $C^{(k)}, C^{(k+1)}$ is

$$g\left(C^{(k)} = c_k, C^{(k+1)} = c_{k+1}\right) \equiv g_{k,k+1}\left(c_k, c_{k+1}\right)$$
$$= \frac{\theta^2}{(k-1)!}\Phi^{k+1} c_k^{\theta k-1} c_{k+1}^{\theta-1} \exp\left(-\Phi c_{k+1}^{\theta}\right)$$

for $0 < c_k \leq c_{k+1} < \infty$ while the marginal density of $C^{(k)}$ is:

$$g_k\left(c_k\right) = \frac{\theta}{(k-1)!}\Phi^k c_k^{\theta k-1} \exp\left(-\Phi c_k^{\theta}\right)$$

for $0 < c_k < +\infty$

Proof. We start by looking at costs $C \leq \bar{c}$. The distribution of a cost C conditional on $C \leq \bar{c}$ is:

$$F\left(c|\bar{c}\right) = \left(\frac{c}{\bar{c}}\right)^{\theta} \quad c \leq \bar{c}$$
$$F\left(c|\bar{c}\right) = 1 \qquad c > \bar{c}$$

The probability that a cost is less than c_k is $F\left(c_k|\bar{c}\right)$. Thus, if we have n techniques with unit cost less than $\bar{c}$, where $c_k \leq c_{k+1} \leq \bar{c}$, the probability that k are less than c_k while the remaining are greater than c_{k+1} is given by the multinomial:

$$\Pr\left[C^{(k)} \leq c_k, C^{(k+1)} \geq c_{k+1}|n\right] = \begin{pmatrix} n \\ k \end{pmatrix} F\left(c_k|\bar{c}\right)^k \left(1 - F\left(c_{k+1}|\bar{c}\right)\right)^{n-k}$$

Taking the negative of the cross derivative of this expression with respect to c_k, c_{k+1} gives

$$g_{k,k+1}\left(c_k, c_{k+1}|\bar{c}, n\right) = \frac{n! F\left(c_k|\bar{c}\right)^{k-1}\left[1 - F\left(c_{k+1}|\bar{c}\right)\right]^{n-k-1} F'\left(c_k|\bar{c}\right) F'\left(c_{k+1}|\bar{c}\right)}{(k-1)!\,(n-k-1)!}$$

for $c_{k+1} \geq c_k$ and $n \geq k+1$. For $n < k+1$ we can define $g_{k,k+1}(c_k, c_{k+1}|\bar{c}, n) = 0$. We also know that n is drawn from a Poisson distribution with parameter $\Phi\bar{c}^{\theta}$, the expectation of this joint distribution unconditional on n is:

$$
\begin{aligned}
g_{k,k+1}(c_k, c_{k+1}|\bar{c}) &= \sum_{n=0}^{\infty} \underbrace{\frac{\exp\left(-\Phi\bar{c}^{\theta}\right)\left(\Phi\bar{c}^{\theta}\right)^n}{n!}}_{\substack{\text{prob } n \text{ ideas arrived for} \\ \text{a particular good}}} \underbrace{g_{k,k+1}(c_k, c_{k+1}|\bar{c}, n)}_{\substack{\text{conditional on } n \text{ prob } C^{(k)}=c_k, \\ C^{(k+1)}=c_{k+1}}} = \\
&= \sum_{n=k+1}^{\infty} \frac{\exp\left(-\Phi\bar{c}^{\theta}\right)\left(\Phi\bar{c}^{\theta}\right)^n}{n!} \frac{n! F(c_k|\bar{c})^{k-1}\left[1 - F(c_{k+1}|\bar{c})\right]^{n-k-1} F'(c_k|\bar{c}) F'(c_{k+1}|\bar{c})}{(k-1)!(n-k-1)!} \\
&= \frac{\left(\Phi\bar{c}^{\theta}\right)^{k+1} \exp\left(-\Phi\bar{c}^{\theta} F(c_{k+1}|\bar{c})\right) F(c_k|\bar{c})^{k-1} F'(c_k|\bar{c}) F'(c_{k+1}|\bar{c})}{(k-1)!} \\
&\sum_{m=0}^{\infty} \exp\left(-\Phi\bar{c}^{\theta}\right)\left(\Phi\bar{c}^{\theta}\right)^m \frac{\exp\left(-\Phi\bar{c}^{\theta} F(c_{k+1}|\bar{c})\right)\left[1 - F(c_{k+}|\bar{c})\right]^m}{m!} \\
&= \frac{\left(\Phi\bar{c}^{\theta}\right)^{k+1} \exp\left(-\Phi\bar{c}^{\theta} F(c_{k+1}|\bar{c})\right) F(c_k|\bar{c})^{k-1} F'(c_k|\bar{c}) F'(c_{k+1}|\bar{c})}{(k-1)!} 1
\end{aligned}
$$

Substituting using the expression $F(c|\bar{c})$ we have that

$$
g_{k,k+1}(c_k, c_{k+1}|\bar{c}) = \frac{\theta^2}{(k-1)!} \Phi^{k+1} c_k^{\theta k-1} c_{k+1}^{\theta-1} \exp\left(-\Phi c_{k+1}^{\theta}\right)
$$

Now by letting $\bar{c} \to \infty$ we can integrate for the entire range of $c \geq c_k$ and derive the marginal density by making use of the above expression. We have that

$$
\begin{aligned}
g_k(c_k) &= \int_{c_k}^{\infty} g_{k,k+1}(c_k, c_{k+1}) dc_{k+1} \\
&= \frac{\theta^2}{(k-1)!} \Phi^{k+1} c_k^{\theta k-1} \int_{c_k}^{\infty} c_{k+1}^{\theta-1} e^{-\Phi c_{k+1}^{\theta}} dc_{k+1}.
\end{aligned}
$$

Now by making the substitution $u = c_{k+1}^{\theta}$,

$$\int_{c_k}^{\infty} c_{k+1}^{\theta-1} e^{-\Phi c_{k+1}^{\theta}} dc_{k+1} = \theta^{-1} \int_{c_k^{\theta}}^{\infty} e^{-\Phi u} du$$
$$= \theta^{-1} \Phi^{-1} e^{-\Phi c_k^{\theta}}.$$

Therefore

$$g_k(c_k) = \frac{\theta^2}{(k-1)!} \Phi^{k+1} c_k^{\theta k-1} \left(\theta^{-1} \Phi^{-1} e^{-\Phi c_k^{\theta}}\right)$$
$$= \frac{\theta}{(k-1)!} \Phi^k c_k^{\theta k-1} e^{-\Phi c_k^{\theta}}, \tag{4.3}$$

as asserted. $\square$

This result will be the base for a series of lemmas to be discussed later on. First, by noticing that $F_k'(c_k) = g_k(c_k)$ we can directly compute the probability $\Pr\left[C^{(k)} \leq \tilde{c}_k\right]$:

4. The distribution of the $k'th$ lowest cost $C^{(k)}$ is:

$$\Pr\left[C^{(k)} \leq \tilde{c}_k\right] = F_k(\tilde{c}_k) = 1 - \sum_{v=0}^{k-1} \frac{\left(\Phi \tilde{c}_k^{\theta}\right)^v}{v!} e^{-\Phi \tilde{c}_v^{\theta}} \tag{4.4}$$

This Lemma implies that the distribution of the lowest cost ($k = 1$) is the Frechet distribution

$$F_1(\tilde{c}_1) = 1 - \exp\left(-\Phi \tilde{c}_1^{\theta}\right)$$

Now in this context we will assume that ideas are randomly assigned to goods across the continuum. Given that there is a large number of goods (say of measure $\mu(\Omega)$) in the continuum we can drop the ω notation by simply denoting $A_i(\omega, t) = A_i(t) / \mu(\Omega)$ to be the average number of ideas available for a good, in location i at time t. Given the above, the measure of goods with cost less than c is $\Phi_i(t) / \mu(\Omega)\, c^{\theta}$ and the distribution of the lowest cost $C^{(1)}$ (the frontier idea) is

$$F_1(c_1) = 1 - \exp\left(-\left(\Phi_i(t)/\mu(\Omega)\right)\tilde{c}_1^{\theta}\right)$$

Thus a set of $\mu(\Omega) F_1(c_1)$ ideas can be produced at a cost less than c_1. We will proceed under this convention in the rest of this chapter.

Using the following proposition and the assumption of the CES demand we can directly derive the price index

5. For each order k, the b'th moment $(b > -\theta k)$ is

$$E\left[\left(C^{(k)}\right)^b\right] = \left(\Phi^{-1/\theta}\right)^b \frac{\Gamma\left[(\theta k + b)/\theta\right]}{(k-1)!},$$

where $\Gamma(\alpha) = \int_0^{+\infty} y^{\alpha-1} e^{-y} dy$.

Proof. First consider $k = 1$, where suppressing notation we denote by the marginal density of $C^{(k)}$,

$$g_k(c) = \frac{\theta}{(k-1)!}\Phi^k c_k^{\theta k-1}\exp\left[-\Phi c_k^{\theta}\right]$$

$$\begin{aligned} E\left[\left(C^{(1)}\right)^b\right] &= \int_0^{+\infty} c^b g_1(c)\, dc \\ &= \int_0^{+\infty} \Phi\theta c^{\theta+b-1}\exp\left[-\Phi c^{\theta}\right] dc \end{aligned}$$

changing the variable of integration to $v = \Phi c^{\theta}$ and applying the definition of the gamma function, we get

$$E\left[\left(C^{(1)}\right)^{b}\right] = \int_{0}^{+\infty} (v/\Phi)^{b/\theta} \exp\left[-v\right] dv$$
$$= (\Phi)^{-b/\theta} \Gamma\left[\frac{\theta+b}{\theta}\right] \tag{4.5}$$

well defined for $\theta + b > 0$. For general k we have

$$\begin{aligned} E\left[\left(C^{(k)}\right)^{b}\right] &= \int_{0}^{+\infty} c^{b} g_{k}(c)\, dc \\ &= \int_{0}^{+\infty} c^{b} \frac{\theta}{(k-1)!} \Phi^{k} c^{\theta k-1} \exp\left[-\Phi c^{\theta}\right] dc \\ &= \frac{\Phi^{k-1}}{(k-1)!} \int_{0}^{+\infty} c^{b+\theta k-\theta} \theta \Phi c^{\theta-1} \exp\left[-\Phi c^{\theta}\right] dc \\ &= \frac{\Phi^{k-1}}{(k-1)!} E\left[\left(C^{(1)}\right)^{b+\theta(k-1)}\right] \end{aligned} \tag{4.6}$$

□

Using the general technology framework we developed above and different assumptions on the competition structure we will be able to derive main quantitative models that are widely used in the recent international trade literature. We now provide a number of illustrations of this general framework that arise from explicilty specifying the competition and market structure.

4.3 Application I: Perfect competition (Eaton-Kortum)

A main factor inhibiting the use of “classical trade theory” in empirical gravity models was the perceived intractability of such a problem. The traditional Ricardian comparative advantage framework relied on two countries and two goods; many attempts to generalize the framework quickly led to a nightmare of corner solutions. While Dornbusch, Fischer, and Samuelson (1977) provided a tractable framework for a continuum of varieties with two countries, it was thought to be impossible to extend the framework to many countries and arbitrary trade costs (both which are necessary to deliver a gravity-like equation). (The closest generalization to many countries was the local comparative static analysis of Wilson (1980)).

This is where Eaton and Kortum (2002) enters. Using a model that bears a resemblance to a discrete-choice framework (a la **?**), they show how to derive gravity expressions for trade flows in a world with many countries, arbitrary trade costs (i.e. arbitrary geography), where trade is only driven by technological differences across countries (i.e. comparative advantage). The Eaton and Kortum (2002) framework not only shattered the age-old belief that there couldn't be a Ricardian gravity model, the model developed turns out to be remarkably elegant. It is also surprising that despite looking very different from the models we have considered thus far, the Eaton and Kortum (2002) trade expression remains formally isomorphic to those models.

4.3.1 Model Setup

Let us now turn to the set-up of the model.

The World

In this model, there a finite number of countries $i \in S \equiv \{1, ..., N\}$; (unlike previous models, there are technical difficulties in extending the model to a continuum of countries). There are a continuum of goods Ω. However, unlike in the Krugman (1980) and Melitz (2003) models which follow, *every country is able to produce every good*. Countries, however, vary (exogenously) in their productivity of each good; in particular, let $z_i(\omega)$ denote country i's efficiency at producing good $\omega \in \Omega$.

The Eaton and Kortum (2002) has no concept of a firm. Instead, it is assumed that all goods $\omega \in \Omega$ are produced using the same bundle of inputs with a constant returns to scale technology. Let the cost of a bundle of inputs in country $i \in S$ be c_i so that the cost of producing one unit of $\omega \in \Omega$ in country $i \in S$ is $\frac{c_i}{z_i(\omega)}$.

Finally, like the previous models we considered, suppose there is an iceberg trade cost $\tau_{ij} \geq 1$ of trading a good from $i \in S$ to $j \in S$.

Supply

Each good is assumed to be sold in perfectly competitive markets, so that the price a consumer in country $j \in S$ would pay *if* she were to purchase good $\omega \in \Omega$ from country $i \in S$ is:

$$p_{ij}(\omega) = \frac{c_i}{z_i(\omega)} \tau_{ij}. \tag{4.7}$$

However, consumers in country $j \in S$ are assumed to only purchase good $\omega \in \Omega$ from the country who can provide it at the lowest price, so the price a consumer in $j \in N$ actually pays for good $\omega \in \Omega$ is:

$$p_j(\omega) \equiv \min_{i \in S} p_{ij}(\omega) = \min_{i \in S} \frac{c_i}{z_i(\omega)} \tau_{ij}. \tag{4.8}$$

The basic idea behind the Eaton and Kortum (2002) is already present in equation (4.8): a country $j \in S$ will be more likely to purchase good $\omega \in \Omega$ from country $i \in S$ if (1) it has a lower unit cost c_i; (2) it has a higher good productivity $z_i(\omega)$; and/or (3) it has a lower trade cost τ_{ij}.

One of the major innovations of the Eaton and Kortum (2002) model is that the productivity $z_i(\omega)$ is treated as a random variable drawn independently and identically for each $\omega \in \Omega$. Define F_i to be the cumulative distribution function of the productivity in country $i \in S$. That is, for each $i \in S$, for all $\omega \in \Omega$:

$$F_i(z) \equiv \Pr\{z_i(\omega) \leq z\}$$

Eaton and Kortum (2002) assume that $F_i(z)$ is **Fréchet distributed** so that for all $z \geq 0$:

$$F_i(z) = \exp\left\{-T_i z^{-\theta}\right\}, \tag{4.9}$$

where $T_i > 0$ is a measure of the aggregate productivity of country i (note that a larger value of T_i decreases $F_i(z)$ for any $z \geq 0$, i.e. it increases the probability of larger values of z and $\theta > 1$ (which is assumed to be constant across countries) governs the distribution of productivities across goods within countries (as θ increases, the heterogeneity of productivity across goods declines).

Why make this particular distributional assumption for productivities? In a microfoundation that is related to the general technological framework we have discussed above Kortum (1997) showed that if the technology of producing goods is determined by the best "idea" of how to produce, then the limiting distribution is indeed Fréchet, where T_i reflects the country's stock of ideas. More generally, consider the random variable:

$$M_n = \max\{X_1, ..., X_n\},$$

where X_i are i.i.d. The Fisher–Tippett–Gnedenko theorem states that the only (normalized) distribution of M_n as $n \to \infty$ is an extreme value distribution, of which Fréchet is one of three types (Type II). Note that a conditional logit model assumes that the error term is Gumbel (Type I) extreme value distributed. If random variable x is Gumbel distributed, $\ln x$ is distributed Fréchet; hence, loosely speaking, the Fréchet distribution works better for models that are log linear (like the gravity equation), whereas the Gumbel distribution works better for models that are linear.

Demand

As in previous models, consumers have CES preferences so that the representative agent in country j has utility:

$$U_j = \left(\int_\Omega q_j(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}},$$

where $q_j(\omega)$ is the quantity that country j consumes of good ω. Note that unlike the Krugman (1980) model, not every good produced in every country will be sold to country j. Indeed, good $\omega \in \Omega$ will be produced by all countries but county j will only purchase it from one country. However, like the previous models considered, the CES preferences will yield a Dixit-Stiglitz price index:

$$P_j \equiv \left(\int_\Omega p_j(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}} \tag{4.10}$$

4.3.2 Equilibrium

We now consider the equilibrium of the model. Instead of relying on the CES demand equation as in the previous models, we use a probabilistic formulation in order to solve the model.

Prices

In perfect competition only the lowest cost producer of a good will supply that particular good. Thus, we want to derive the distribution of the minimum price over a set of prices offered by producers in different countries

$$p_j = \min \left\{ p_{1j}, ..., p_{Nj} \right\}$$

In order to find this distribution we take advantage of the properties of extreme value distributions for productivity z. Everything turns out to work beautifully!

First, let us consider the probability that country $i \in S$ is able to *offer* country $j \in S$ good $\omega \in \Omega$ for a price less than p. Because the technology is i.i.d across goods, this probability will be the same for all goods $\omega \in \Omega$. Define:

$$G_{ij}(p) \equiv \Pr \left\{ p_{ij}(\omega) \leq p \right\}$$

Using the perfect competition price equation (4.7) and the functional form of the Fréchet distribution (4.9), we have:

$$\begin{aligned}
G_{ij}(p) &\equiv \Pr \left\{ p_{ij}(\omega) \leq p \right\} \iff \\
G_{ij}(p) &= \Pr \left\{ \frac{c_i}{z_i(\omega)} \tau_{ij} \leq p \right\} \iff \\
G_{ij}(p) &= \Pr \left\{ \frac{c_i}{p} \tau_{ij} \leq z_i(\omega) \right\} \iff \\
G_{ij}(p) &= 1 - \Pr \left\{ z_i(\omega) \leq \frac{c_i}{p} \tau_{ij} \right\} \iff \\
G_{ij}(p) &= 1 - F_i \left(\frac{c_i}{p} \tau_{ij} \right) \iff \\
G_{ij}(p) &= 1 - \exp \left\{ -T_i \left(\frac{c_i}{p} \tau_{ij} \right)^{-\theta} \right\} \qquad (4.11)
\end{aligned}$$

Consider now the probability that country $j \in S$ *pays* a price less than p for good $\omega \in \Omega$. Again, because the technology is i.i.d across goods, this probability will be the same for all goods $\omega \in \Omega$. Define:

$$G_j\left(p\right) \equiv \Pr\left\{p_j\left(\omega\right) \leq p\right\}$$

Because country $j \in S$ buys from the least cost provider, using equation (4.8) and some basic tools of probability, we can write:

$$\begin{aligned} G_j\left(p\right) &= \Pr\left\{\min_{i \in S} p_{ij}\left(\omega\right) \leq p\right\} \iff \\ &= 1 - \Pr\left\{\min_{i \in S} p_{ij}\left(\omega\right) \geq p\right\} \iff \\ &= 1 - \Pr\left\{\cap_{i \in S}\left(p_{ij}\left(\omega\right) \geq p\right)\right\} \iff \\ &= 1 - \prod_{i \in S}\left(1 - G_{ij}\left(p\right)\right) \end{aligned} \tag{4.12}$$

Substituting equation (4.11) into equation (4.12) yields:

$$\begin{aligned} G_j\left(p\right) &= 1 - \prod_{i \in S}\left(1 - G_{ij}\left(p\right)\right) \iff \\ &= 1 - \prod_{i \in S} \exp\left\{-T_i\left(\frac{c_i}{p}\tau_{ij}\right)^{-\theta}\right\} \iff \\ &= 1 - \exp\left\{-p^{\theta}\sum_{i \in S} T_i\left(c_i\tau_{ij}\right)^{-\theta}\right\} \iff \\ &= 1 - \exp\left\{-p^{\theta}\Phi_j\right\}, \end{aligned} \tag{4.13}$$

where $\Phi_j \equiv \sum_{i \in S} T_i\left(c_i\tau_{ij}\right)^{-\theta}$. Equation (4.13) tells us what the distribution of prices will be across goods for country j. This, in turn, will allow us to calculate the price index in country j, P_j. Starting with the definition of the price index from equation (4.11), we have:

$$\begin{aligned}
P_j &\equiv \left(\int_\Omega p_j\left(\omega\right)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}} \iff \\
P_j^{1-\sigma} &= \int_0^\infty p^{1-\sigma} dG_j\left(p\right) \iff \\
P_j^{1-\sigma} &= \int_0^\infty p^{1-\sigma}\left(\frac{d}{dp}\left(1-\exp\left\{-p^\theta \Phi_j\right\}\right)\right) dp \iff \\
P_j^{1-\sigma} &= \theta\Phi_j \int_0^\infty p^{\theta-\sigma} \exp\left\{-p^\theta\Phi_j\right\} dp.
\end{aligned}$$

Define $x \equiv p^\theta \Phi_j$ so that with a change of variables we have:

$$\begin{aligned}
P_j^{1-\sigma} &= \int_0^\infty \left(\frac{x}{\Phi_j}\right)^{\frac{1-\sigma}{\theta}} \exp\left\{-x\right\} dx \iff \\
P_j^{1-\sigma} &= \Phi_j^{-\frac{1-\sigma}{\theta}} \int_0^\infty x^{\frac{1-\sigma}{\theta}} \exp\left\{-x\right\} dx \iff \\
P_j^{1-\sigma} &= \Phi_j^{-\frac{1-\sigma}{\theta}} \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right) \iff \\
P_j &= \Phi_j^{-\frac{1}{\theta}} \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)^{\frac{1}{1-\sigma}},
\end{aligned}$$

where $\Gamma\left(t\right) \equiv \int_0^\infty x^{t-1}e^{-x}dx$ is the Gamma function.

Hence, the equilibrium price index in country $j \in N$ can be written as:

$$P_j = C\left(\sum_{i\in S} T_i\left(c_i\tau_{ij}\right)^{-\theta}\right)^{-\frac{1}{\theta}}, \tag{4.14}$$

where $C \equiv \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)^{\frac{1}{1-\sigma}}$. [Class questions: What does this mean if trade is costless? When trade is infinitely costly?]

4.3.3 Gravity

Now suppose we are interested in determining the probability that $i \in S$ is the least cost provider of good $\omega \in \Omega$ to destination $j \in S$. Because all goods receive i.i.d. draws and there are a continuum of varieties, by the law of large numbers, this probability will be equal to the fraction of goods i sells to j. Define:

$$\begin{aligned}
\pi_{ij} &\equiv \Pr\left\{p_{ij}(\omega) \leq \min_{k \in S \setminus j} p_{kj}(\omega)\right\} \iff \\
&= \int_0^\infty \Pr\left\{\min_{k \in S \setminus j} p_{kj}(\omega) \geq p\right\} dG_{ij}(p) \iff \\
&= \int_0^\infty \Pr\left\{\cap_{k \in S \setminus j}\left(p_{kj}(\omega) \geq p\right)\right\} dG_{ij}(p) \iff \\
&= \int_0^\infty \prod_{k \in S \setminus j}\left(1 - G_{kj}(p)\right) dG_{ij}(p)
\end{aligned} \tag{4.15}$$

Substituting the distribution of price offers from equation (4.11) into equation (4.15) yields:

$$\begin{aligned}
\pi_{ij} &= \int_0^\infty \prod_{k \in S \setminus j}\left(1 - G_{kj}(p)\right) dG_{ij}(p) \iff \\
&= \int_0^\infty \prod_{k \in S \setminus j}\left(\exp\left\{-T_k\left(\frac{c_k}{p}\tau_{kj}\right)^{-\theta}\right\}\right)\left(\frac{d}{dp}\left(1 - \exp\left\{-T_i\left(\frac{c_i}{p}\tau_{ij}\right)^{-\theta}\right\}\right)\right) dp \iff \\
&= T_i\left(c_i\tau_{ij}\right)^{-\theta} \int_0^\infty \theta p^{\theta-1}\left(\exp\left\{-p^\theta \Phi_j\right\}\right) dp \iff \\
&= \frac{T_i\left(c_i\tau_{ij}\right)^{-\theta}}{\Phi_j}\left(-\exp\left\{-p^\theta\Phi_j\right\}|_0^\infty\right) \\
&= \frac{T_i\left(c_i\tau_{ij}\right)^{-\theta}}{\Phi_j}
\end{aligned} \tag{4.16}$$

Hence, the fraction of goods exported from i to j just depends on $i's$ share in $j's$ Φ_j. Note that more productive countries, countries with lower unit costs, and countries with lower bilateral trade costs (all relative to other countries) will comprise a larger fraction of the goods sold to j.

Note that π_{ij} is the fraction of *goods* that $j \in S$ purchases from $i \in S$; it may not be the fraction of $j's$ income that is spent on goods from country i. However, it turns out that with the Fréchet distribution, *the distribution of prices of goods that country j actually purchases from any country $i \in S$ will be the same.* To see this, note that the probability country $i \in S$ is able to offer good $\omega \in \Omega$ for a price lower than $\tilde{p}$ conditional on i having the lowest price is simply the product of inverse of the probability that i has the lowest cost good and the probability that j receives a price offer lower that $\tilde{p}$:

$$\begin{aligned}
\Pr\left\{p_{ij}(\omega) \leq \tilde{p} | p_{ij}(\omega) \leq \min_{k \in S \setminus i} p_{kj}(\omega)\right\} &= \frac{1}{\pi_{ij}} \int_0^{\tilde{p}} \Pr\left\{\min_{k \in S \setminus i} p_{kj}(\omega) \geq p\right\} dG_{ij}(p) \iff \\
&= \frac{1}{\pi_{ij}} \int_0^{\tilde{p}} \prod_{k \in S \setminus i} \left(1 - G_{kj}(p)\right) dG_{ij}(p) \iff \\
&= \frac{1}{\pi_{ij}} \frac{T_i \left(c_i \tau_{ij}\right)^{-\theta}}{\Phi_j} \left(-\exp\left\{-p^{\theta} \Phi_j\right\} |_0^{\tilde{p}}\right) \\
&= \frac{1}{\pi_{ij}} \frac{T_i \left(c_i \tau_{ij}\right)^{-\theta}}{\Phi_j} \left(1 - \exp\left\{-\tilde{p}^{\theta} \Phi_j\right\}\right) \\
&= G_j(\tilde{p}).
\end{aligned}$$

Intuitively, what is happening is that origins with better comparative advantage (lower trade costs, better productivity, etc.) in selling to j will exploit its advantage by selling a greater number of goods to j exactly up to the point where the distribution of prices it offers to j is the same as j's overall price distribution.

While this result depends heavily on the Fréchet distribution, it greatly simplifies the process of determining trade flows. Since the distribution of prices offered to an importing country $j \in S$ is *independent of the origin*, country $j's$ average expenditure per good does not depend on the source of the good. As a result, the fraction of goods purchased from a particular origin (π_{ij}) is equal to the fraction of $j's$ income spent on goods from country i, $\lambda_{ij} \equiv \frac{X_{ij}}{Y_j}$. This implies that the total expenditure of j on goods from country i is:

$$X_{ij} = \pi_{ij} E_j,$$

where from equation (4.16) we have:

$$X_{ij} = \frac{T_i \left(c_i \tau_{ij}\right)^{-\theta}}{\Phi_j} E_j \tag{4.17}$$

Supposing that $c_i = w_i$ and substituting in equation (4.14) for the price index yields:

$$X_{ij} = C^{-\theta} \tau_{ij}^{-\theta} w_i^{-\theta} T_i E_j P_j^{\theta}. \tag{4.18}$$

Hence, the Eaton and Kortum (2002) model yields a nearly identical gravity equation to the Armington model of Anderson (1979), except that the relevant elasticity is θ instead of $\sigma - 1$.

As in previous models, we can also push the gravity equation a little bit further. Note that in general equilibrium, the total income of a country will equal the amount it sells to all other countries:

$$Y_i = \sum_{j \in S} X_{ij} \tag{4.19}$$

Substituting gravity equation (10.2) into equation (4.19) yields:

$$\begin{aligned}
Y_i &= \sum_{j\in S} C^{-\theta}\tau_{ij}^{-\theta} w_i^{-\theta} T_i E_j P_j^{\theta} \iff \\
Y_i &= C^{-\theta} w_i^{-\theta} T_i \sum_{j\in S} \tau_{ij}^{-\theta} E_j P_j^{\theta} \iff \\
C^{-\theta} w_i^{-\theta} T_i &= \frac{Y_i}{\sum_{j\in S} \tau_{ij}^{-\theta} E_j P_j^{\theta}}
\end{aligned} \tag{4.20}$$

Now substituting equation (4.20) back into the gravity equation (10.2) yields:

$$\begin{aligned}
X_{ij} &= C^{-\theta}\tau_{ij}^{-\theta} w_i^{-\theta} T_i E_j P_j^{\theta} \iff \\
X_{ij} &= \tau_{ij}^{-\theta} \times \frac{Y_i}{\Pi_i^{-\theta}} \times \frac{E_j}{P_j^{-\theta}},
\end{aligned} \tag{4.21}$$

where

$$\Pi_i \equiv \left(\sum_{k\in S} \tau_{ik}^{-\theta} \frac{E_k}{P_k^{-\theta}} \right)^{-\frac{1}{\theta}}.$$

We will see soon that if trade costs are symmetric, $P_i = \Pi_i$.

Why is the trade elasticity different in this model? Recall that in the Armington and Krugman (1980) models, how responsive trade flows were to trade costs depended on how demand for a good was affected by the good's price, which was determined by consumer's elasticity of substitution. In this model, however, changes in trade costs affect the *extensive margin*, i.e. which goods an origin country trades with a destination country. As the bilateral trade costs rise, the origin country is the least cost provider in fewer goods; the greater the θ, the less heterogeneity in a country's productivity across different goods, so there are a greater number of goods for which it is no longer the least cost provider. Hence, the Eaton and Kortum (2002) model is similar to the Melitz (2003) model in that the elasticity of trade to trade costs ultimately depends on the density of producers/firms that are indifferent between exporting and not exporting: the greater the heterogeneity in productivity, the lower the density of these marginal producers.

4.3.4 Welfare

From the CES preferences, the welfare of a worker in country $i \in S$ can be written as:

$$W_i \equiv \frac{w_i}{P_i}. \tag{4.22}$$

Recall from above that $\lambda_{ij} \equiv \frac{X_{ij}}{E_j}$ is the fraction of $j's$ expenditure spent on i. From gravity equation (10.2) we then have that:

$$\lambda_{ij} = C^{-\theta} \tau_{ij}^{-\theta} w_i^{-\theta} T_i P_j^{\theta},$$

which, given $\tau_{ii} = 1$, implies:

$$\begin{aligned} \lambda_{ii} &= C^{-\theta} W_i^{-\theta} T_i \iff \\ W_i &= C \lambda_{ii}^{-\frac{1}{\theta}} T_i^{\frac{1}{\theta}}. \end{aligned} \tag{4.23}$$

Hence, as in the Krugman (1980) model, welfare can be expressed as a function of technology and the openness of a country. Indeed, as far as I am aware, Eaton and Kortum (2002) were the first to derive this expression, although the expression is usually known as the "ACR" equation after Arkolakis, Costinot, and Rodríguez-Clare (2012), who derived the conditions under which it holds more generally (we will see more of this in several weeks).

4.3.5 Key contributions of EK

The Eaton and Kortum model is a key defining point in the new trade literature not only because it has provided a simple framework to model comparative advantage but also because it has provided a set of probabilistic tools to think about the determination of trade and allocation of resources without having to worry about mathematical intricacies (e.g. corner solutions in the standard Ricardian setup etc.). These tools have been use henceforth for a variety of applications that we will discuss in many of the remaining chapters. Still, the structure we developed above can be useful with different forms of competition as is illustrated by Bernard, Eaton, Jensen, and Kortum (2003) and Melitz (2003) that model other forms of competition (Bertrand and monopolistic, respectively) and also work that brings models of trade closer to the trade data in many dimensions.

4.4 Application II: Bertrand competition (Bernand, Eaton, Jensen, Kortum)

- Consider the case where different producers have access to different technologies. If we assume Bertrand competition, the cost distribution will be given by the frontier producer ($k = 1$ in the expression 4.4) but prices are related to the distribution of the second lowest cost ($k = 2$). Since the lowest cost supplier is the one that will sell the good, the probability that a good is supplied from i to j is the same as in perfect competition, equation (**??**).

4.4.1 Supply side

The price of a good ω in market j is:

$$p_j(\omega) = \min\left\{C_{2j}(\omega), \bar{m}C_{1j}(\omega)\right\}$$

where we will define $C_{ij}(\omega)$ to be the cost of the i'th minimum cost producer of good ω in country j, and $\bar{m} = \sigma/(\sigma-1)$ is the optimal markup that a monopolist firm would charge (assuming CES preferences with a demand elasticity σ). Given heterogeneity among technological costs for firms we will derive the distribution of costs and markups in each given country.

Efficiency, markups, and measured productivity

Define again $C^{(k)}$ as the k'th lowest unit cost technology for producing a particular good. We have the following Lemma

6. The distribution of $C^{(k+1)}$ conditional on $C^{(k)} = c_k$ is:

$$\Pr\left[C^{(k+1)} \leq c_{k+1} | C^{(k)} = c_k\right] = 1 - \exp\left[-\Phi\left(c_{k+1}^{\theta} - c_k^{\theta}\right)\right], c_{k+1}^{\theta} \geq c_k^{\theta} \geq 0 \tag{4.24}$$

Proof. Using Bayes' rule we have

$$\begin{aligned}\Pr\left[C^{(k+1)} \leq c_{k+1} | C^{(k)} = c_k\right] &= \int_{c_k}^{c_{k+1}} \frac{g_{k,k+1}(c_k, c)}{g_k(c_k)} dc \\ &= \int_{c_k}^{c_{k+1}} \theta\Phi c^{\theta-1} \exp\left[-\Phi c^{\theta} + \Phi c_k^{\theta}\right] dc \\ &= 1 - \exp\left[-\Phi\left(c_{k+1}^{\theta} - c_k^{\theta}\right)\right].\end{aligned}$$

□

The above relationship also implies that (define $m' = c_{k+1}/c_k$)

$$\Pr\left[\frac{C^{(k+1)}}{c_k} \leq \frac{c_{k+1}}{c_k} | C^{(k)} = c_k\right] = \Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m' | C^{(k)} = c_k\right]$$
$$= 1 - \exp\left[-\Phi c_k^{\theta}\left(\left(m'\right)^{\theta} - 1\right)\right]$$

The distribution of the ratio $M' = C^{(2)}/C^{(1)}$ given $C^{(1)} = c_1$ is:

$$\Pr\left[M' \leq m' | C^{(1)} = c_1\right] = 1 - \exp\left[-\Phi c_1^{\theta}\left(\left(m'\right)^{\theta} - 1\right)\right] \ .$$

We have that the lower c_1, the more likely a high markup. Thus, in this context low-cost producers are more likely to charge a high markup and their measured (revenue) productivity is more likely to appear as higher. In this model, revenue productivity is associated one-to-one with the markup that the firm charges, since it equals

$$\underbrace{\left[M\left(\omega\right) w_i \tau_{ij} q\left(\omega\right) / z\left(\omega\right)\right]}_{\text{revenue}} / \underbrace{l\left(\omega\right)}_{\text{labor used}} = \left[M\left(\omega\right) w_i \tau_{ij} q\left(\omega\right) / z\left(\omega\right)\right] / \left[q\left(\omega\right) / z\left(\omega\right)\right] = M\left(\omega\right) w_i \tau_{ij}.$$

Notice that from this expression is straightforward that market structures/demand functions which imply costant markup imply no revenue productivity variation across firms.

Notice that the markup with Bertrand competition that BEJK consider is

$$M\left(\omega\right) = \min\left\{\frac{C^{(2)}\left(\omega\right)}{C^{(1)}\left(\omega\right)}, \bar{m}\right\} \tag{4.25}$$

We start by characterizing the distribution of the ratio $M' = C^{(2)}/C^{(1)}$. Conditional on the second lowest cost in market j being $C_i^{(2)} = c_2$, we have

$$\begin{aligned}
\Pr\left[M' \leq m' | C_j^{(2)} = c_2\right] &= \Pr\left[c_2/m' \leq C_j^{(1)} \leq c_2 | C_j^{(2)} = c_2\right] \\
&= \frac{\int_{c_2/m'}^{c_2} g_j\left(c_1, c_2\right) dc_1}{\int_0^{c_2} g_j\left(c_1, c_2\right) dc_1} \\
&= \frac{c_2^\theta - \left(c_2/m'\right)^\theta}{c_2^\theta} \\
&= 1 - \left(m'\right)^{-\theta}. \qquad (4.26)
\end{aligned}$$

This derivation implies that the distribution of this M' does not depend on c_2 and is also Pareto.[3] Thus, the unconditional distribution is also Pareto.[4] Given the markup function for the case of Betrand competition, equation (4.25), we have proved the following

[3]Using again the results of the theorem we can derive the distribution of m for each k. Notice that

$$\begin{aligned}
\Pr\left[C^{(k)} \leq c_k | C^{(k+1)} = c_{k+1}\right] &= \int_0^{c_k} g_{k,k+1}\left(c, c_{k+1}\right) / g_{k+1}\left(c_{k+1}\right) dc \\
&= \int_0^{c_k} \frac{\frac{\theta^2}{(k-1)!} \Phi^{k+1} c^{\theta k - 1} c_{k+1}^{\theta - 1} \exp\left(-\Phi c_{k+1}^\theta\right)}{\frac{\theta}{(k)!} \Phi^{k+1} c_{k+1}^{\theta(k+1)-1} e^{-\Phi c_{k+1}^\theta}} dc \\
&= \left(\frac{c_k}{c_{k+1}}\right)^{\theta k} \qquad (4.27)
\end{aligned}$$

and simply replacing for $c_k = \frac{c_{k+1}}{m}$ in expression (4.27) (and given that $C^{(k+1)} = c_{k+1}$) we can get

$$\begin{aligned}
\Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m | C^{(k+1)} = c_{k+1}\right] &= 1 - \Pr\left[C^{(k)} \leq \frac{c_{k+1}}{m} | C^{(k+1)} = c_{k+1}\right] \\
&= 1 - m^{-\theta k}
\end{aligned}$$

[4]An alternative derivation of the distribution of the markups can be obtained for $m < \bar{m}$. To compute the unconditional distribution of productivities for $m < \bar{m}$ we have that

$$\begin{aligned}
\Pr\left[M \leq m\right] &= \int_0^{+\infty} \left(1 - \exp\left[-\Phi c_1^\theta \left(m^\theta - 1\right)\right]\right) \theta c_1^{\theta - 1} \exp\left(-\Phi c_1^\theta\right) dc_1 \\
&= \exp\left(-\Phi c_1^\theta\right) - \left.\frac{\exp\left[-\Phi c_1^\theta m^\theta\right]}{m^\theta}\right|_0^{+\infty} = 1 - 1/m^\theta
\end{aligned}$$

proposition:

7. Under Bertrand competition the distribution of the markup M is:

$$\Pr\left[M \leq m\right] = F_M\left(m\right) = \begin{cases} 1 - m^{-\theta} \text{ if } m < \bar{m} \\ 1 \text{ if } m \geq \bar{m} \end{cases} .$$

With probability $\bar{m}^{-\theta}$ the markup is $\bar{m}$. The distribution of the markup is independent of $C^{(2)}$.

4.4.2 Gravity

Lengthy derivations can be used (see the online appendix of BEJK) to show that the joint distribution of the lowest and the second lowest cost of supplying a country, conditional on a certain country being the supplier, is independent of the country of origin, and is given by equation (4.27) for $k = 1$. Therefore, the market share of country i in j equals to the probability that country i is the supplier and thus

$$\lambda_{ij} = \frac{A_i\left(w_i\tau_{ij}\right)^{-\theta}}{\sum_{k=1}^{N} A_k\left(w_k\tau_{kj}\right)^{-\theta}} = \frac{\Phi_{ij}}{\Phi_j} \tag{4.28}$$

It is worth noting that the profits of firms depend on both their cost but also their competitor's cost. An interesting implication of the model is that the share of profit to aggregate revenue is constant and equals to $1/\left(\theta+1\right)$. The proof of this can be found in the online appendix of BEJK.

4.4.3 Welfare

Using the derivations for the distribution of markups, we can also derive a price index for this case. We have

$$
\begin{aligned}
P_j^{1-\sigma} &= \int_1^{\infty} E\left[p_j^{1-\sigma}|M=m\right]\theta m^{-\theta-1}dm \\
&= \underbrace{\int_1^{\bar{m}} E\left[\left(C_j^{(2)}\right)^{1-\sigma}\right]\theta m^{-(\theta+1)}dm}_{\text{marginal cost pricing}} + \underbrace{\int_{\bar{m}}^{+\infty} E\left[\left(\bar{m}C_j^{(2)}/m\right)^{1-\sigma}\right]\theta m^{-(\theta+1)}dm}_{\text{Dixit-Stiglitz pricing}} \\
&= E\left[\left(C_j^{(2)}\right)^{1-\sigma}\right]\left[\left(1-\bar{m}^{-\theta}\right)+\bar{m}^{-\theta}\frac{\theta}{1+\theta-\sigma}\right]
\end{aligned}
$$

where in the second equality we used the fact that the distribution of markups is independent of the second lowest cost, equation (4.26). We have already calculated $E\left[\left(C_i^{(2)}\right)^{1-\sigma}\right]$ in equation (4.6). Thus, the price index under Bertrand competition is

$$P_i = \gamma^{BC}\Phi_i^{-1/\theta}$$

$$\gamma^{BC} = \left[\left(1-\bar{m}^{-\theta}\right)+\bar{m}^{-\theta}\frac{\theta}{1+\theta-\sigma}\right]^{1/(1-\sigma)}\Gamma\left(\frac{2\theta+1-\sigma}{\theta}\right)^{1/(1-\sigma)}$$

and given the gravity expression the welfare as a function of trade is the same as in the case of Perfect competition

4.4.4 Key contributions of BEJK

- Develop a firm level model and explicitly tests its predictions with firm-level data.
- Model a framework where firms markups are variable and depending on competition. Alternative models of variable markups can be developed in monopolistic competition by allowing for a preference structure that departs from the CES aggre-

gator (see section 8.4).

- Develop a methodology of simulating an artificial economy with heterogeneous firms and finding the parameters of this economy that brings the predictions of the model closer to the data.

- Acknowledges the fact that "measured productivity" when measured as nominal output over employment is constant in models with constant markups. It developes a model that can deliver variable markups, thus measured productivity differentials.

4.5 Application III: Monopolistic competition with CES (Chaney-Melitz)

The Krugman (1980) model, while micro-founding the gravity equation based on a story of equilibrium firm entry, made the simplifying assumption that all firms were ex-ante identical. With the advent of digitized data on firm-level trading partners, however, it became clear that there existed an enormous heterogeneity in firm's exporting behavior Bernard, Jensen, Redding, and Schott (2007) provides an excellent overview of the empirical patterns concerning firms in international trade, of which we mention just a few. First, the vast majority of firms do not export; in the U.S. in 2000, only 4% of firms were exporters. Second, amongst those 4% of firms that did export, 96% of the value of exports came from just 10% of exporters. Third, comparing the firms that export to those that do not, the exporting firms tend to be larger, more productive, more skill- and capital-intensive, and to pay higher wages. These differences are apparent even before exporting begins, suggesting that more productive firms choose to export (rather than the act of exporting increasing the productivity of firms).

In response to these new empirical findings, Melitz (2003) developed an extension of the Krugman (1980) where firms varied (exogenously) in their productivities and self selected into exporting. This model has proven enormously successful for a number of reasons: first, it is able to capture many (but not all) of the empirical facts mentioned above, most notably that larger firms will be more likely to export; second, the model has proven incredibly flexible, generating a huge number of "extensions" to capture additional empirical patterns; and third, the model generates a new (potential) source for gains from trade: if falling trade costs leads higher productivity firms to grow and lower productivity firms to shrink, this reallocation of factors of production will increase the average productivity of a country. While there is some debate about whether this is actually an additional gain from trade (as we will see in a few weeks), the idea that greater trade can make a country more productive by increasing competition has made the (rare) leap from academic to political discourse; for example the U.S. trade representative web page lists as one of its major "benefits of trade" the fact that "trade expansion benefits families and businesses by supporting more productive, higher paying jobs in our export sectors."

4.5.1 Model Set-up

Let us now turn to the set-up of the model.

The world

As in the previous models, there is a compact set S of countries, where I will keep the notation that i is an origin country and a j is a destination country. Each country $i \in S$ is be populated by an exogenous measure L_i of workers/consumers where each worker supplies her unit of labor inelastically. Suppose that labor is the only factor of production.

Supply

As in the Krugman (1980) model, suppose that there is a continuum Ω of possible varieties that the world can produce, and suppose that every firm in the world produces a distinct variety $\omega \in \Omega$. Let the set of varieties produced by firms located in country $i \in S$ be denoted by $\Omega_i \subset \Omega$. (Note that Ω_i is an equilibrium object, as it will depend on the number of firms that are actively producing).

Instead of the fixed entry cost in Krugman (1980) model, suppose that there is a mass M_i of firms from country $i \in S$ and that firms must incur a fixed cost $f_{ij} > 0$ to export to each destination $j \in S$.[5]

The major innovation of the Melitz (2003) model is that firms are heterogeneous. To model this, we suppose that each firm in $i \in S$ has a productivity φ drawn from some cumulative distribution function $G_i(\varphi)$, i.e. it costs a firm with productivity φ exactly $\frac{1}{\varphi}$ units of labor to produce a single unit of its differentiated variety. In what follows, we will sometimes identify each firm by its productivity (since all firms with the same productivity within a particular country will act the same way) and sometimes identify each firm by its variety ω (since every firm produces a unique variety).

Finally, as in previous models, we suppose that all firms within a country are subject to iceberg trade costs $\{\tau_{ij}\}_{i,j \in S}$.

Demand As in the Krugman (1980) model, we assume that consumers have CES preferences over varieties. Hence a representative consumer in country $j \in S$ gets utility U_j from the consumption of goods shipped by all other firms in all other countries, where:

[5]In Melitz (2003), it was assumed that there was an additional entry cost f_i^e that determined the equilibrium mass of firms M_i. In the Chaney (2008) version of the model, M_i was assumed (for simplicity) to be proportional to the income in the origin. The Chaney (2008) version of the model has become more widely used because it allows for arbitrary bilateral trade costs (the original Melitz (2003) model imposed symmetry).

$$U_j = \left(\sum_{i \in S} \int_{\Omega_{ij}} \left(q_{ij} \left(\omega \right) \right)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}, \tag{4.29}$$

where $q_{ij}(\omega)$ is the quantity consumed in country j of variety ω.

4.5.2 Equilibrium

We now consider the equilibrium of the model.

Optimal demand

The consumer's utility maximization problem is identical to that of Krugman (1980): A consumer in country $j \in S$ optimal quantity demanded of good $\omega \in \Omega$ is:

$$q_{ij}(\omega) = p_{ij}(\omega)^{-\sigma} Y_j P_j^{\sigma-1}, \tag{4.30}$$

where:

$$P_j \equiv \left(\sum_{i \in S} \int_{\Omega_i} p_{ij}(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}} \tag{4.31}$$

is the Dixit-Stiglitz price index.

The amount spent on variety ω is simply the product of the quantity and the price:

$$x_{ij}(\omega) = p_{ij}(\omega)^{1-\sigma} Y_j P_j^{\sigma-1}. \tag{4.32}$$

To determine total trade flows, we need to aggregate across all firms in country i:

$$X_{ij} \equiv \int_{\Omega_i} x_{ij}(\omega) d\omega = Y_j P_j^{\sigma-1} \int_{\Omega_i} p_{ij}(\omega)^{1-\sigma} d\omega. \tag{4.33}$$

Unlike the Krugman (1980) model, firms with different productivities will charge different prices, so the integral in equation (4.33) becomes more complicated.

Optimal supply

We now determine the equilibrium prices that a firm with productivity φ sets (where we now identify firms by their productivity). The optimization problem is:

$$\max_{\left\{q_{ij}(\varphi)\right\}_{j\in S}} \sum_{j\in S} \left(p_{ij}\left(\varphi\right) q_{ij}\left(\varphi\right) - \frac{w_i}{\varphi}\tau_{ij} q_{ij}\left(\varphi\right) - f_{ij} \right) \text{ s.t. } q_{ij}\left(\varphi\right) = p_{ij}\left(\varphi\right)^{-\sigma} Y_j P_j^{\sigma-1}.$$

Substituting the constraint into the maximand yields:

$$\max_{\left\{q_{ij}(\varphi)\right\}_{j\in S}} \sum_{j\in S} \left(p_{ij}\left(\varphi\right)^{1-\sigma} Y_j P_j^{\sigma-1} - \frac{w_i}{\varphi}\tau_{ij} p_{ij}\left(\varphi\right)^{-\sigma} Y_j P_j^{\sigma-1} - f_{ij} \right)$$

The first order condition implies that a firm from $i \in S$ with productivity φ, conditional on selling to destination j, will charge a price:

$$p_{ij}\left(\varphi\right) = \frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij} \tag{4.34}$$

Combining the optimal price (equation (4.34)) and the optimal demand (equation (4.30)) gives the total revenue of a firm (conditional on exporting) to be:

$$x_{ij}\left(\varphi\right) \equiv p_{ij}\left(\varphi\right) q_{ij}\left(\varphi\right) = \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \tag{4.35}$$

and variable profits conditional on entering (note that we now define $\pi_{ij}(\varphi)$ as profits without the fixed costs):

$$\begin{aligned}
\pi_{ij}(\varphi) &\equiv \left(p_{ij}(\varphi) - \frac{w_i}{\varphi}\tau_{ij}\right) q_{ij}(\varphi) \\
&= \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij} - \frac{w_i}{\varphi}\tau_{ij}\right)\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{-\sigma} Y_j P_j^{\sigma-1} \\
&= \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\left(\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \\
&= \frac{1}{\sigma} x_{ij}(\varphi)
\end{aligned} \tag{4.36}$$

Note that both revenue and profits are increasing in a firm's productivity. [Class question: Why is this?]

Aggregation

We now discuss how to use the optimal behavior on the part of each firm to construct the aggregate variables necessary to generate a gravity equation. Let $\mu_{ij}(\varphi)$ be the (equilibrium) probability density function of the productivities of firms from country i that sell to country j and let M_{ij} be the (equilibrium) measure of firms exporting from i to j.

Then we can write the average prices charged by all firms in $i \in S$ selling to $j \in S$ as:

$$\begin{aligned}
\int_{\Omega_i} p_{ij}(\omega)^{1-\sigma} d\omega &= \int_0^\infty M_{ij}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} \mu_{ij}(\varphi)\, d\varphi \iff \\
&= \left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{1-\sigma} M_{ij} \int_0^\infty \varphi^{\sigma-1}\mu_{ij}(\varphi)\, d\varphi \iff \\
&= M_{ij}\left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{1-\sigma} \left(\tilde{\varphi}_{ij}\right)^{\sigma-1},
\end{aligned}$$

where $\tilde{\varphi}_{ij} \equiv \left(\int_0^\infty \varphi^{\sigma-1}\mu_{ij}(\varphi)\, d\varphi\right)^{\frac{1}{\sigma-1}}$ captures the "average" productivity of producers from i selling to j. This allows us to write the gravity equation (4.33) as:

$$X_{ij} = \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_{ij} \left(\tilde{\varphi}_{ij}\right)^{\sigma-1} Y_j P_j^{\sigma-1}. \tag{4.37}$$

Equation (4.37) resembles the gravity equation from Krugman (1980), except that we now have to keep track of both the number of firms selling to j (M_{ij}) and their average productivity $\tilde{\varphi}_{ij}$. Note that as the average productivity of entrants increases, the trade flows increase. [Class question: what is the intuition for this?].

Selection into exporting

In order to determine the equilibrium number of entrants M_{ij} and the average productivity of entrants $\tilde{\varphi}_{ij}$, we have to consider the export decisions of firms. A firm from country $i \in S$ with productivity φ conditional on producing will export to j if and only if:

$$\pi_{ij}\left(\varphi\right) \geq f_{ij}$$

From equations (4.35) and (4.36) we can write this as:

$$\begin{aligned} &\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \geq f_{ij} \iff \\ &\qquad \varphi \geq \varphi_{ij}^* \equiv \left(\frac{\sigma f_{ij}\left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}. \end{aligned} \tag{4.38}$$

Hence, only firms that are sufficiently productive will find it profitable to incur the fixed cost of exporting to destination j. This means that the model matches the empirical fact that larger and more productive firms select into exporting.

Together, equations (4.38) and (4.46) allow us to determine the "average" productivity of producers selling from i to j:

$$\tilde{\varphi}_{ij} = \left(\frac{1}{1 - G_i\left(\varphi^*_{ij}\right)} \int_{\varphi^*_{ij}}^{\infty} \varphi^{\sigma-1} dG_i\left(\varphi\right) \right)^{\frac{1}{\sigma-1}}. \tag{4.39}$$

and the density of firms selling from i to j:

$$M_{ij} = \left(1 - G_i\left(\varphi^*_{ij}\right)\right) M_i, \tag{4.40}$$

so that the gravity equation (4.37) becomes:

$$X_{ij} = \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_i \left(\int_{\varphi^*_{ij}}^{\infty} \varphi^{\sigma-1} dG_i\left(\varphi\right) \right) Y_j P_j^{\sigma-1}. \tag{4.41}$$

4.5.3 The Pareto Distribution

In this section, we show that when the distribution of firm productivities is a Pareto distribution, the model above simplifies nicely. This insight is due to Chaney (2008). The assumption that the distribution of firm's productivities is Pareto can actually be microfounded as follows: Let $\mu\left(\Omega\right) \subset \left(0, +\infty\right)$ is the set of available varieties. Let I_i the measure of ideas that fall randomly into goods. In some sense $I_i / \mu\left(\Omega\right)$ ideas correspond to each good. In the probabilistic context we described above, the monopolistic competition model arises in a very natural way. Let the distribution of the lowest cost for a good to be Frechet such that

$$F_1\left(c_1\right) = 1 - \exp\left(-\frac{I_i}{\mu\left(\Omega\right)} c_1^{\theta}\right).$$

The measure of firms with unit cost less than $C^{(1)} \leq c_1$, is $\mu\left(\Omega\right) F_1\left(c_1\right)$. Taking the limit of this expression for the number of potential varieties $\mu\left(\Omega\right) \to +\infty$ we can show that the distribution of the best producer's cost of a variety is Pareto. More details are given in appendix (14.1).

Suppose that $\varphi \in [1, \infty)$ and:

$$G_i(\varphi) = 1 - \varphi^{-\theta_i}, \tag{4.42}$$

where θ_i is the *shape parameter* of the distribution. We assume that $\theta_i > \sigma - 1$ (this parametric assumption is necessary in order for trade flows to be finite). Note that as θ_i increases, the probability that the productivity is below any given φ increases, i.e. the heterogeneity of producers is decreasing in θ_i.

If the productivities are Pareto distributed, then we can write:

$$\begin{aligned}
\int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi) &= \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} \left(\frac{d\left(1 - \varphi^{-\theta_i}\right)}{d\varphi} \right) d\varphi \iff \\
&= \theta_i \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-\theta_i-2} d\varphi \iff \\
&= \frac{\theta_i}{\theta_i + 1 - \sigma} \left(\varphi_{ij}^*\right)^{\sigma-\theta_i-1}
\end{aligned}$$

Recall from equation (4.38) above that we can write the export threshold φ_{ij}^* as a function of the fixed cost of export so that:

$$\begin{aligned}
\int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi) &= \frac{\theta_i}{\theta_i + 1 - \sigma} \left(\varphi_{ij}^*\right)^{\sigma-\theta_i-1} \iff \\
&= \frac{\theta_i}{\theta_i + 1 - \sigma} \left(\frac{\sigma f_{ij} \left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}} \right)^{\frac{\sigma-\theta_i-1}{\sigma-1}}
\end{aligned} \tag{4.43}$$

Substituting expression (4.43) into the gravity equation (4.41) above then yields:

$$X_{ij} = \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_i \left(\int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-1} dG_i(\varphi)\right) Y_j P_j^{\sigma-1} \iff$$

$$X_{ij} = \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} M_i \left(\frac{\theta_i}{\theta_i+1-\sigma}\left(\frac{\sigma f_{ij}\left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{\sigma-1}}{Y_j P_j^{\sigma-1}}\right)^{\frac{\sigma-\theta_i-1}{\sigma-1}}\right) Y_j P_j^{\sigma-1} \iff$$

$$X_{ij} = C_1 \left(\tau_{ij} w_i\right)^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i \left(Y_j P_j^{\sigma-1}\right)^{\frac{\theta_i}{\sigma-1}} \tag{4.44}$$

where $C_1 \equiv \sigma^{\frac{\sigma-\theta_i-1}{\sigma-1}} \left(\frac{\sigma}{\sigma-1}\right)^{-\theta_i} \left(\frac{\theta_i}{\theta_i+1-\sigma}\right)$.

4.5.4 Trade with firm heterogeneity

Armed with the gravity equation (4.44) we have calculated, we now turn to the implications of a trade model with heterogeneous firms.

Extensive and intensive margins of trade

Equation (4.44) bears a resemblance to the gravity equation derived by Krugman (1980), but the elasticity of trade flows with respect to variable trade costs is related to the Pareto shape parameter instead of the elasticity of substitution! Since we have assumed that $\theta_i > \sigma - 1$, this means that trade flows have become *more* responsive to changes in trade costs than in the Krugman (1980) model.

What gives? Intuitively, as trade costs fall two things happen: first, the firms already producing will export more (this is known as the **intensive margin**); second, smaller firms who were not exporting previously will begin to export (this is known as the **extensive margin**). Both of these effects will tend to increase trade; since the Krugman (1980) model only had the first effect, the model with heterogeneous firms will predict larger responses of trade flows to changes in trade costs.

We can actually determine the elasticity of both margins of trade separately to see the relative importance of both effects. (This was the central point of Chaney (2008)To do so, recall the Leibnez rule:

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f\left(x,z\right)dx = \int_{a(z)}^{b(z)} \frac{\partial f\left(x,z\right)}{\partial z}dx + f\left(b\left(z\right),z\right)\frac{\partial b\left(z\right)}{\partial z} - f\left(a\left(z\right),z\right)\frac{\partial a\left(z\right)}{\partial z}$$

Combining the original gravity equation (4.33) with the threshold exporting decision (4.38) we have:

$$X_{ij} = M_i \int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right)dG_i\left(\varphi\right)$$

Hence we can write the elasticity of trade flows with respect to variable trade costs as:

$$-\frac{\partial \ln X_{ij}}{\partial \ln \tau_{ij}} = -\frac{\partial X_{ij}}{\partial \tau_{ij}}\frac{\tau_{ij}}{X_{ij}} = -\frac{\int_{\varphi_{ij}^*}^{\infty} \frac{\partial}{\partial \tau_{ij}} x_{ij}\left(\varphi\right)\tau_{ij}dG_i\left(\varphi\right)}{\int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right)dG_i\left(\varphi\right)} + \frac{x_{ij}\left(\varphi_{ij}^*\right)\tau_{ij}\frac{\partial \varphi_{ij}^*}{\partial \tau_{ij}}dG_i\left(\varphi_{ij}^*\right)}{\int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right)dG_i\left(\varphi\right)},$$

where the first term reflects the effect of a change in trade costs on the intensive margin and the second term reflects the change on the extensive margin. From the revenue equation (4.35) we have:

$$\frac{\partial}{\partial \tau_{ij}} x_{ij}\left(\varphi\right) = \frac{\partial}{\partial \tau_{ij}}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} = \left(1-\sigma\right)\frac{x_{ij}\left(\varphi\right)}{\tau_{ij}}.$$

so that:

$$-\frac{\int_{\varphi_{ij}^*}^{\infty} \frac{\partial}{\partial \tau_{ij}} x_{ij}\left(\varphi\right)\tau_{ij}dG_i\left(\varphi\right)}{\int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right)dG_i\left(\varphi\right)} = \sigma - 1,$$

i.e. a decline in trade costs will cause all firms currently producing to increase their production with an elasticity of $\sigma - 1$ (this is the original Krugman (1980) effect). [Class question: Why is the intensive margin increasing with σ?]

From equation (4.38) governing the threshold productivity:

$$\frac{\partial \varphi_{ij}^*}{\partial \tau_{ij}} = \frac{\partial}{\partial \tau_{ij}} \left(\frac{\sigma f_{ij} \left(\frac{\sigma}{\sigma-1} w_i \tau_{ij} \right)^{\sigma-1}}{Y_j P_j^{\sigma-1}} \right)^{\frac{1}{\sigma-1}} = \frac{\varphi_{ij}^*}{\tau_{ij}}$$

so that:

$$\begin{aligned}
\frac{x_{ij}\left(\varphi_{ij}^*\right) \tau_{ij} \frac{\partial \varphi_{ij}^*}{\partial \tau_{ij}} dG_i\left(\varphi\right)}{\int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right) dG_i\left(\varphi\right)} &= \frac{x_{ij}\left(\varphi_{ij}^*\right) \varphi_{ij}^* dG_i\left(\varphi_{ij}^*\right)}{\int_{\varphi_{ij}^*}^{\infty} x_{ij}\left(\varphi\right) dG_i\left(\varphi\right)} \iff \\
&= \frac{\left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \left(\varphi_{ij}^*\right)^{\sigma-1-\theta_i}}{\left(\frac{\sigma}{\sigma-1} w_i \tau_{ij}\right)^{1-\sigma} Y_j P_j^{\sigma-1} \int_{\varphi_{ij}^*}^{\infty} \varphi^{\sigma-2-\theta_i} d\varphi} \iff \\
&= \frac{\left(\varphi_{ij}^*\right)^{\sigma-1-\theta_i}}{\frac{1}{\theta_i-\sigma+1}\left(\varphi_{ij}^*\right)^{\sigma-1-\theta_i}} \iff \\
&= \theta_i - \sigma + 1,
\end{aligned}$$

i.e. on the extensive margin, a decline in trade costs will induce less productive firms to enter the market. When the elasticity of substitution is low (i.e. σ is low), even the less productive firms will be able to capture relatively large market share, so that the difference in size between the entering firms and the existing firms is small, meaning that the effect on the extensive margin will be larger. With a Pareto distribution, the extensive margin dominates the intensive margin.

Free entry and the allocation of factors across firms

Up until now, we have taken the mass of producing firms M_i to be exogenous. We now consider what would happen if it was endogenously determined by a free entry condition (much as in Krugman (1980)). Suppose now that firms have to incur an entry cost $f_i^e > 0$

prior to learning their productivity. Then the free entry condition will require that the expected profits are equal to the entry cost:

$$f_i^e = E_\varphi \left[\sum_{j \in S} \max \left\{ \pi_{ij} \left(\varphi \right) - f_{ij}, 0 \right\} \right] \tag{4.45}$$

Since more productive firms are (weakly) more profitable in every market, this implies there will be an equilibrium productivity threshold φ_i^* where firms, upon drawing a productivity, will choose to produce only if their productivity exceeds φ_i^*. This implies that we can write re-write equation (4.45) as:

$$\begin{aligned} f_i^e &= \int_{\varphi^*}^{\infty} \sum_{j \in S} \max \left\{ \pi_{ij} \left(\varphi \right) - f_{ij}, 0 \right\} dG_i \left(\varphi \right) \iff \\ f_i^e &= \sum_{j \in S} \int_{\max\left\{ \varphi_i^*, \varphi_{ij}^* \right\}}^{\infty} \left(\pi_{ij} \left(\varphi \right) - f_{ij} \right) dG_i \left(\varphi \right), \end{aligned} \tag{4.46}$$

i.e. the fixed entry cost is simply equal to the sum across all destinations of the profits in those destinations for firms who are sufficiently productive to both pay the fixed entry and export costs.

What would happen to the profits of firms of different productivities if we were to lower the variable trade cost τ_{ij} for some $j \in S$? First, consider a firm whose productivity is greater than the threshold productivity necessary to export to j, i.e. $\varphi \geq \varphi_{ij}^*$. From equation (4.36), its profits are:

$$\pi_{ij} \left(\varphi \right) = \frac{1}{\sigma} \left(\frac{\sigma}{\sigma - 1} \frac{w_i}{\varphi} \tau_{ij} \right)^{1-\sigma} Y_j P_j^{\sigma-1} - f_{ij},$$

so that holding all else equal, its total profits must increase. Furthermore, the more productive this firm is, the greater the increase in its profits since $-\frac{\partial^2 \pi_{ij}(\varphi)}{\partial \tau_{ij} \partial \varphi} > 0$. If the total profits for all firms $\varphi \geq \varphi_{ij}^*$ are increasing, then the expected profits of entering the market will also increase; because of the free entry condition, this will induce a greater number of firms to enter the market, increasing the demand for local labor and driving up wages. As a result, the profits of firms with $\varphi < \varphi^*_{ij}$ will go down (as will the firms with productivities $\varphi < \varphi^*_{ij} + \varepsilon$), so that in equilibrium, only the most productive firms productivities will increase. In addition, as the wages increase, the minimum productivity required to produce anything at all (i.e. φ^*_i) will increase, forcing the least productive firms in the model to exit. Hence, the model implies that greater openness to trade will increase the average productivity of producing firms and will allocate labor toward the more productive firms.

4.5.5 Next steps

The Melitz (2003) model provides the backbone for many (most?) of the major trade papers written in the past ten years. While we will not have time to discuss its many extensions in detail, we should note a few. Helpman, Melitz, and Yeaple (2004) endogenizes a firm's decision whether to export or to pursue foreign direct investment (FDI). Melitz and Ottaviano (2008) derive a version of the model with linear demand (instead of CES) to analyze how mark-ups endogenously respond to trade liberalizations. Helpman, Melitz, and Rubinstein (2008) discuss how the model (with a bounded distribution of productivity) can be used to explain the zero trade flows observed in bilateral trade data and what it suggests for the estimation of empirical gravity models. Helpman, Itskhoki, and Redding (2010) incorporate labor market frictions into a Melitz (2003) framework. Arkolakis (2010) extends the Melitz (2003) framework to incorporate market penetration costs. Eaton, Kortum, and Kramarz (2011) use the Melitz (2003) framework to structurally estimate the exporting behavior of French firms.

4.6 Summary

- Established a macroeconomic framework where the concept of the firm had a meaning while the model was tractable and amenable to a variety of exercises. In this framework it is possible to think about trade liberalization and firms in GE.

- Explicitly modeled the importance of reallocation of production through the death of the least productive firms.

The Melitz (2003) model provides the backbone for many of the major trade papers written in the past years. Here we note just a few. Helpman, Melitz, and Yeaple (2004) endogenizes a firm's decision whether to export or to pursue foreign direct investment (FDI). Melitz and Ottaviano (2008) derive a version of the model with linear demand (instead of CES) to analyze how mark-ups endogenously respond to trade liberalizations. Helpman, Melitz, and Rubinstein (2008) discuss how the model (with a bounded distribution of productivity) can be used to explain the zero trade flows observed in bilateral trade data and what it suggests for the estimation of empirical gravity models. Helpman, Itskhoki, and Redding (2010) incorporate labor market frictions into a Melitz (2003) framework. Arkolakis (2010) extends the Melitz (2003) framework to incorporate market penetration costs. Eaton, Kortum, and Kramarz (2011) us

The Eaton and Kortum (2002) model remains the primary framework for the study of trade in perfectly competitive markets (especially agriculture). However, because of a lack of a real concept of a firm, it has proven less popular than the Melitz (2003) model for the study of firm-level data. In Bernard, Eaton, Jensen, and Kortum (2003), the authors did extend the framework to one where there is Bertrand competition between firms. The basic idea is straightforward: the price charged by the single firm that exports a variety is the marginal cost of the second best firm. While this allows for endogenous (non-constant) markups, the extension required somewhat more complicated probability tools and has turned out to be slightly less tractable than Melitz (2003) extensions such as Melitz and Ottaviano (2008), where there are endogenous markups due to non-CES preferences.

4.7 Homeworks

1. *General properties of gravity trade models.* Consider the model developed in section 4.3-4.5.

   (a) Argue that in all these models profits are a constant fraction of production.

   (b) Argue that in the model consider in Section (4.5) payments to fixed costs are a constant fraction of production.

2. *The Frechet distribution*. For all $n \in \{1, ..., N\}$, suppose that the random variable $z_n \geq 0$ is distributed according to the Frechet distribution, i.e.:

$$\Pr\{z_n \leq z\} = \exp\left(-T_n z^{-\theta}\right),$$

   where $T_i > 0$ and $\theta > 0$ are known parameters. Define the random variable $p_n = \frac{c_n}{z_n}$. Calculate:

$$\pi_{in} \equiv \Pr\left\{p_n \leq \min_{k \neq n} p_k\right\}$$

3. *The Pareto distribution.* Suppose that the random variable $z_n \in [b_n, \infty)$ is distributed according to the Pareto distribution, i.e.:

$$\Pr\{z_n \leq z\} = 1 - \left(\frac{b_n}{z}\right)^{-\theta},$$

   where $b_n > 0$ and $z_n > 0$ are known parameters. What is the distribution of z_n

conditional on being greater than $c_n > b_n$?

Chapter 5

Closing the model

In order to determine the solution of the endogenous variables of the models we constructed above in the general equilibrium. The individual goods markets are already assumed to clear since we replaced the consumer demand directly in the sales of the firm for each good.

In this section we will make use of three general assumption.Consider the

Aggregate profits are a constant share of revenues. Let Π_j denote country j's aggregate profits gross of entry costs (if any). The first macro-level restriction states that Π_j must be a constant share of country j's total revenues:

R1 *For any country j, $\Pi_j / Y_j = \bar{\pi}$ where $\zeta \geq 0$.*

Under perfect competition, R1 trivially holds since aggregate profits are equal to zero. Under monopolistic competition with homogeneous firms, R12 also necessarily holds because of Dixit-Stiglitz preferences; see Krugman (1980). In more general environments, however, R2 is a non-trivial restriction.

The second restriction

R2 *For any country pair,i, j,* the share of spending on fixed exporting cost to bilateral sales is constant$\gamma_{ij} = \bar{\gamma}$ *where* $\bar{\gamma} \geq 0$.

The third restriction is that the value of imports of goods must be equal to the value of exports of goods:

R3 *For any country j* $\sum_k X_{kj} = \sum_k X_{jk}$

In general, total income of the representative agent in country j may also depend on the wages paid to foreign workers by country j's firms as well as the wages paid by foreign firms to country j's workers. Thus, total expenditure in country j, $X_j \equiv \sum_i X_{ij}$, could be different from country j's total revenues, $Y_j \equiv \sum_{i=1}^{n} X_{ji}$. R1 rules out this possibility.

5.1 General Equilibrium

We now show how we can determine the wages, w_i, and spending, X_i that solve for the model's general equilibrium (income, y_i, can be written always as a straightforward function of spending in the cases we will analyze). It turns out that under the technological distributions and demand structures that we introduced above, we can first solve for wages and spending and all the rest of the variables can be written as simple functions of these two variables. To create a formal mapping to the data, where trade deficits are a commonplace, we can also allow for exogenous transfer payments to countries, D_i, (following Dekle, Eaton, and Kortum (2008)) which in a static model will imply an equal amount of trade deficit. Of course, these trade deficits have to sum up to zero across countries, $\sum_i D_i = 0$.

In principle, to solve for w_i, X_i we need to consider two sets of equations

i) the budget constraint of the representative consumer

$$\sum_k X_{ki} = \sum_k \lambda_{ki} X_i = X_i \equiv w_i L_i + \pi_i + D_i \,, \tag{5.1}$$

where π_i is total profits earned by firms from country i net of fixed marketing costs (if any),

ii) and the current account balance (there are no capital flows) that consists of exports, imports and related payment to labor for fixed marketing costs but can be equivalently written as total expenditure equals total income and transfers

$$\begin{aligned}
0 &= \underbrace{\sum_{k \neq i} X_{ik}}_{\text{exports}} - \underbrace{\sum_{k \neq i} X_{ki}}_{\text{imports}} + \underbrace{\sum_{k \neq i} \gamma_{ki} X_{ki} - \sum_{k \neq i} \gamma_{ik} X_{ik}}_{\text{net foreign income}} + D_i \implies \\
0 &= \sum_k X_{ik} - \sum_k X_{ki} + \sum_k \gamma_{ki} X_{ki} - \sum_k \gamma_{ik} X_{ik} + D_i \implies
\end{aligned}$$

$$X_i = \sum_k X_{ki} = \sum_k X_{ik} + \sum_k \gamma_{ki} X_{ki} - \sum_k \gamma_{ik} X_{ik} + D_i \,, \tag{5.2}$$

where γ_{ij} is the share of bilateral sales from i to j that accrues to labor for payments of fixed marketing costs, and trade flows X_{ij}.

These set of equations can be used to solve for w_i, X_i using an additional normalization. Notice that is straightforward to show that with the CES demand we assume budget balance is equivalent to the CES price index. In particular,

$$\begin{aligned}
\sum_k \int_{\Omega_k} p_{ki}^{1-\sigma}(\omega)\, d\omega = P_i^{1-\sigma} &\iff \\
\sum_k \frac{\int_{\Omega_k} p_{ki}^{1-\sigma}(\omega)\, d\omega}{P_i^{1-\sigma}} X_i = X_i &\iff \\
\sum_k \lambda_{ki} X_i = X_i.
\end{aligned}$$

5.2 Endogenous Entry

So far we assumed that N_i is given. Many papers assuming firm heterogeneity and monopolistic competition, including the original papers of Krugman (1980) and Melitz (2003), assume that new firms can freely choose to enter the economy and draw a productivity from a distribution $g_i(z)$ upon paying an entry cost f^e, in terms of labor units, where the distribution $g_i(z)$ could be degenerate so that all the mass is in a single point. Two restrictions are enough to guarantee that there is a simple solution for N_i.

We start from the free entry condtion, that indicates that the total profits of firms from j gross of entry fixed costs, Π_j, equal to fixed costs of entry

$$\frac{\Pi_j}{N_j} = w_j f_j^e.$$

Using R1 together with the above equation we have

$$\zeta Y_j = N_j w_j f_j^e,$$

for some constant ζ.

Now use R3 implying $Y_j = X_j$ where the last equals to total labor income from the free entry condition, $X_j = w_j L_j$. Therefore, we have

$$\begin{aligned} \zeta Y_j &= N_j w_j f_j^e \implies \\ \zeta w_j L_j &= N_j w_j f_j^e \implies \\ N_j &= \frac{L_j}{f_j^e \zeta}, \end{aligned} \tag{5.3}$$

i.e. entry is linear in population and does not depend on trade costs.

5.3 Solving for the Equilibrium when the Profit share is constant

From this general framework we need to specify the exact market structure to solve for the equilibrium wages. With perfect competition, as in Sections 3.3,4.3, there are no fixed marketing costs, $\gamma_{ij} = 0$. Thus, these two sets of equations can be thought as one set of equations with wages remaining to be solved as a non-linear equation on wages, given equation (**??**). Another simple case is that of monopolistic competition with homogeneous firms, Section 3.4. In that case there are again no fixed marketing costs and all income is ultimately accrued to labor due to free entry so that the budget constraint can be written as $X_i = w_i L_i + D_i$ and given this relationship and (**??**), equation (5.2) can be used to solve for all wages.

There are a number of cases where profits are not going to be zero in equilibrium and thus we have to solve for those. But in all the cases considered above, it turns out that profits are a constant fraction of country income, i.e. $\pi_i = \bar{\pi} / \sum_k X_{ik}$. Notice that in the case of Bertrand competition, Section 4.4, $\gamma_{ki} = 0$ as well and since profits are constant fraction of income, $\bar{\pi} = 1/ (\theta + 1)$ similarly wages can be solved using (4.28). To proceed to characterize 5.2 with monopolistic competition and firm heterogeneity notice that under the assumption of Pareto distributions and fixed marketing costs, it can be shown that $\gamma_{ij} = (\theta - \sigma + 1) / (\sigma\theta) \equiv \gamma$ and that π_i is a constant share of output $\bar{\pi} = (\sigma - 1) / (\sigma\theta)$ (independent of entry being endogenous or not).

With R1,R2 we can develop a very general framework to determine the equilibrium of the endogenous variables. Start by assuming that $\gamma_{ij} = \bar{\gamma}$, equation (5.2) can be written as[1]

$$\sum_k X_{ki} = X_i = \sum_k X_{ik} + \bar{\gamma} \sum_k X_{ki} - \bar{\gamma} \sum_k X_{ik} + D_i \tag{5.4}$$

[1]Notice that the following equation implies that R2 with $D_i = 0$ implies R3.

Now we can combine the above reformulation of the current account balance condition with the budget constraint of the consumer equation (5.1) (equivalently the labor market clearing), we obtain

$$w_i L_i + \pi_i - \bar{\gamma} \sum_k X_{ki} = (1 - \bar{\gamma}) \sum_k X_{ik}$$

and using again the budget constraint of the individual we can rewrite this equation as

$$\begin{aligned}
w_i L_i + \pi_i + D_i &= (1 - \bar{\gamma}) \sum_k X_{ik} + \gamma \left(\sum_k X_{ik} + \frac{D_i}{1-\gamma} \right) + D_i \implies \\
w_i L_i + \pi_i - \frac{\bar{\gamma}}{1-\bar{\gamma}} D_i &= \sum_k X_{ik}. \qquad (5.5)
\end{aligned}$$

Now note that with R2 we have

$$\pi_i = \bar{\pi} \sum_k X_{ik} = \bar{\pi} \sum_k \lambda_{ik} X_k \,. \qquad (5.6)$$

Combined with (5.5)

$$\begin{aligned}
w_i L_i + \pi_i - \frac{\bar{\gamma}}{1-\bar{\gamma}} D_i &= \frac{\pi_i}{\bar{\pi}} \implies \\
\frac{\bar{\pi}}{1-\bar{\pi}} \left[w_i L_i - \frac{\bar{\gamma}}{1-\bar{\gamma}} D_i \right] &= \pi_i \qquad (5.7)
\end{aligned}$$

and thus we can write

$$w_i L_i + \pi_i = \frac{1}{1-\bar{\pi}} w_i L_i - \frac{\bar{\pi}}{1-\bar{\pi}} \frac{\bar{\gamma}}{1-\bar{\gamma}} D_i.$$

We can thus summarize the equilibrium in all the models above as a set of wages that solves

$$w_i L_i + \pi_i - \frac{\gamma}{1-\gamma} D_i = \sum_k X_{ik} \implies$$

$$w_i L_i - \frac{\bar{\gamma}}{1-\bar{\gamma}} D_i = \sum_k \lambda_{ik} \left(w_k L_k + \left[1 - \bar{\pi} - \bar{\pi} \frac{\gamma}{1-\gamma} \right] D_k \right) \tag{5.8}$$

with λ_{ik} defined depending on the model using equations (**??**), (**??**) and is an explicit function of wages, alone. In this case the equilibrium can be solved using the set of equations defined by (5.8) and one normalization (due to the redudancy of one equation as a result of Walras law). In the case of the models of (**??**),(4.28) N_i is endogenously determined and we need to introduce the zero profit condition. Assuming R3 in this case, as we have illustrated , implies that N_i is linear in population and the same set of equations can be used to solved for wages. Finally, X_i can be found by utilizing the solution for wages, equation (5.7) which expresses profits as a function of wages, and the budget constraint of the representative consumer, equation (5.1).

5.4 Labor Mobility

To introduce economic geography in this general setup we assume that workers freely move across regions. In equilibrium this results to welfare equilization across regions. Because of the equivalence of the budget constraint with the price index we can simply consider the same two equations for equilibrium as above. For simplicity we will consider henceforth the case $D_i = 0$. Using equation 5.8 and R1, R3,

$$X_i = \sum_k X_{ik} \tag{5.9}$$

$$L_i w_i = \sum_k \int_{\Omega_k} \frac{p_{ik}(\omega)^{1-\sigma}}{P_k^{1-\sigma}} w_k L_k. \tag{5.10}$$

$$L_i w_i = \sum_k N_i \int_{z_{ik}^*} \frac{(c w_i \tau_{ik} / (z A_k))^{1-\sigma}}{P_k^{1-\sigma}} w_k L_k g(z)\, dz. \tag{5.11}$$

with Armington (XXXQuestion is how to we make a general statementXXX)

$$L_i w_i = \sum_k \frac{{w_i}^{1-\sigma}}{P_k^{1-\sigma}} {\tau_{ik}}^{1-\sigma} A_i^{\sigma-1} L_k w_k^{1-\sigma}.$$

$$L_i w_i^{\sigma} = \sum_k {W_i}^{1-\sigma} {\tau_{is}}^{1-\sigma} A_i^{\sigma-1} L_s w_s^{\sigma}.$$

The price index implies

$$P_i^{1-\sigma} = \sum_k p_{ki}\left(\omega\right)^{1-\sigma} d\omega \tag{5.12}$$

$$P_i^{1-\sigma} = \sum_k N_k \int_{z_{ik}^*} c^{1-\sigma} w_k^{1-\sigma} {\tau_{ki}}^{1-\sigma} z^{\sigma-1} A_k^{\sigma-1} g\left(z\right) dz \tag{5.13}$$

with Armington

$$w_i^{1-\sigma} = \sum_k W_i^{1-\sigma} T_{ki}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma}.$$

When welfare is equalized so that $W_i = W$ for all $i \in S$ equations (5.9) and (5.12) are linear operators whose eigenfunctions are $L_i w_i^{\sigma}$ and $w_i^{1-\sigma}$ and whose eigenvalues are $W^{\sigma-1}$, respectively. Note that the kernels of the two equations are transposes of each other. A spatial economy equilibrium is defined as w_i, L_i and W that solve equations (5.9), (5.12). These two results allow us to prove the following theorem:

8. Consider a geography model characterized by equations (5.9), (5.12). Then:

i) there exists a unique spatial equilibrium and this equilibrium is regular; and

ii) this equilibrium can be computed as the uniform limit of a simple iterative procedure.

Proof. See Allen and Arkolakis (2014). □

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5.5 Homeworks

1. Show that when bilateral trade costs a spatial equilibrium can be written as a single non-linear integral equation, which will allow to provide a simple characterization of the equilibrium system.

Chapter 6

Model Characterization

6.1 The Concept of a Model Isomorphism

It turns out that in the simple monopolistic competition framework with Pareto distribution of productivities of firms, the assumption of endogenous entry has little bite (see Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008)): the model with free entry is mathematically equivalent to one with a predetermined number of entrants (essentially the Chaney (2008) version of Melitz (2003)) where the number of entrants is proportionate to the population. The only difference between the two models is that all the profits are accrued to labor allocated for the production of the fixed cost of entry. In addition, the model delivers the same predictions for trade and welfare gains from trade as all the other gravity models we studied so far (Armington (1969), Eaton and Kortum (2002), Bernard, Eaton, Jensen, and Kortum (2003)). To understand these results we need to formally define an analyse the concept of a model isomorphism.

Consider two models, $\mathcal{E}_1 = (\alpha_1, \mathcal{V}_1)$ $\mathcal{E}_2 = (\alpha_2, \mathcal{V}_2)$ where $\oslash_i$ is a combination of model parameters (and fundamentals i.e preference and production structure) and $\mathcal{V}_i$ is a set of equilibrium outcomes. Formally, we define an isomorphism between two models as a bijective mapping $\mathcal{F}$ between the parameters $\text{ff}_1 \longleftrightarrow \text{ff}_2$ that leads to a bijective mapping to the model outcomes $(\mathcal{V}_1 \longleftrightarrow \mathcal{V}_2)$ such that given $\oslash_1 \xleftrightarrow{\mathcal{F}} \oslash_2$ we have that each element $v_1^i \in \mathcal{V}_1$, $v_2^i \in \mathcal{V}_2$ are such that $v_1^i = f^i\left(v_2^i\right)$ where $f \in \mathcal{F}$ up to a normalization, i.e. there exist parameter choices such that you can redefine the model outcomes in an equivalent way. The second requirement is very general in the sense that the variables of the one model are transformations of the variables of the other model. However, an isomorphism is not a mathematical formalism in this case for two reasons. First, models that are isomorphic, under certain environments yield the same policy prescriptions for a given change in policy (in our case, for example, that could be a percentage change in trade costs). Second, the mathematical and statistical apparatus that can be used for the solution of one model can be, as a result, used for the other model as well.

Below we illustrate two examples of isomorphisms that are key in our analysis. The first is a precise isomorphism, where $f^i = A^i$ i.e. the transformation involves only a constant. The second, is a more general isomorphism where the transformation potentially involves more than a constant transformation, but the policy prescriptions of the models are the same.

6.1.1 An Exact Isomorphism

We will use the concept of the isomorphism to illustrate that a model with exogenous entry (as in Chaney (2008)) is isomorphic to a model with endogenous entry (as in Melitz (2003), with the specification as done by Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008)). These models are summarized in the analysis of Section 4.5 without explicitly specifying the labor market clearing condition or the zero profit condition in the endogenous entry case. In the derivations below we assume that profits accrue to domestic consumers.

Assuming zero profits in expectation the expected profits of a firm must be equal to entry costs.[1] Using the free entry condition and a Pareto distribution with shape parameter $\theta > \sigma - 1$, c.d.f. $G(z; A_i) = 1 - \frac{A_i}{z^\theta}$, and support $[A_i^{1/\theta}, +\infty)$ we have[2]

$$\sum_k \pi_{ik}(z)\, dG(z; A_i) = w_i f^e \implies$$

$$\sum_k \int_{z^*_{ik}} \frac{\left(\frac{\sigma}{\sigma-1}\frac{\tau_{ik} w_i}{z}\right)^{1-\sigma}}{P_k^{1-\sigma}\sigma} w_k L_k \theta \frac{A_i^\theta}{z^{\theta+1}} dz - \sum_k \int_{z^*_{ik}} w_k f_{ik} \theta \frac{\left(z^*_{ik}\right)^\theta A_i^\theta}{z^{\theta+1}} dz = w_i f^e \implies$$

$$\sum_k w_k f_{ik} \frac{\theta}{\theta - \sigma + 1} \frac{A_i^\theta}{\left(z^*_{ik}\right)^\theta} - \sum_k w_k \frac{A_i^\theta}{\left(z^*_{ik}\right)^\theta} f_{ik} = w_i f^e \implies$$

$$\sum_k \frac{w_k}{w_i} f_{ik} \frac{A_i^\theta}{\left(z^*_{ik}\right)^\theta} \frac{\sigma - 1}{\theta - \sigma + 1} = f^e. \tag{6.1}$$

We now combine the free entry condition and a reformulation of the labor market clearing condition to compute the equilibrium of the model. Notice that the equilibrium number of entrants in country i, N_i, is determined by the following labor market clearing

[1] Essentially, we assume that there exists a perfect capital market, which requires firms to pay a fixed entry cost before drawing a productivity realization. Consequently, we multiply the LHS by $1 - G\left(z^*_{ii}, b_i\right)$, the probability of obtaining the average profit, since firms with profits below this average necessarily exit the market. Alternatively, we could have specified a more general case with intertemporal discounting, δ. In this case the expected profits from entry should equal the discounted entry cost in the equilibrium.

[2] An implication of free entry is that in the equilibrium all the profits are accrued to labor for the production of the entry cost.

condition:

$$N_i \left(\underbrace{\sum_k \int_{z_{ik}^*} \frac{\left(\frac{\sigma}{\sigma-1}\frac{\tau_{ik} w_i}{z}\right)^{-\sigma}}{P_k^{1-\sigma}} \frac{\tau_{ik}}{z} w_k L_k \theta \frac{\left(z_{ik}^*\right)^{\theta}}{z^{\theta+1}} \frac{A_i^\theta}{\left(z_{ik}^*\right)^{\theta}} dz}_{\text{labor used into production}} + f^e \right) + \underbrace{\sum_k N_k \frac{A_i^\theta}{\left(z_{ki}^*\right)^{\theta}} f_{ki}}_{\text{labor for fixed costs}} = L_i \implies \tag{6.2}$$

$$N_i \left(\sum_k (\sigma - 1) \frac{w_k}{w_i} f_{ik} \frac{A_i^\theta}{\left(z_{ik}^*\right)^{\theta}} \frac{\theta}{\theta - \sigma + 1} + f^e \right) + \sum_k N_k \frac{A_i^\theta}{\left(z_{ki}^*\right)^{\theta}} f_{ki} = L_i. \tag{6.3}$$

Substituting out equation (6.1), we obtain

$$N_i \left(\theta f^e + f^e\right) + \sum_k N_k A_i^\theta \left(z_{ki}^*\right)^{-\theta} f_{ki} = L_i,$$

which, together with the price index, and the definition of z_{ij}^* and $X_j = w_j L_j$,

$$P_i^{1-\sigma} = \sum_k N_k \int_{z_{ki}^*} \left(\frac{\sigma}{\sigma - 1} \frac{\tau_{ki} w_k}{z} \right)^{1-\sigma} \theta A_i^\theta z^{-\theta-1} dz \implies$$
$$w_i L_i = \frac{\theta \sigma}{\theta - \sigma + 1} \left(\sum_k N_k A_i^\theta \left(z_{ki}^*\right)^{-\theta} w_i f_{ki} \right),$$

implies that

$$N_i = \frac{\sigma - 1}{\theta \sigma f^e} L_i \,, \tag{6.4}$$

giving an explicit parametric form to equation 5.3.[3]

Notice that total export sales from country i to j are given by expression (**??**).[4] Define the fraction of total income of country j spent on goods from country i by λ_{ij}. Using the

[3] With a slightly altered proof the same results hold under the assumption that fixed costs are paid in terms of domestic labor.

[4] Average sales of firms from i conditional on operating in j are the same in the model with free entry and the one with a predetermined number of entrants.

definition of total sales from i to j and equations (**??**) and (6.4), we have

$$\lambda_{ij} = \frac{X_{ij}}{\sum_k X_{kj}},$$

which gives that

$$\lambda_{ij} = \frac{L_i A_i \left(\tau_{ij} w_i\right)^{-\theta} f_{ij}^{1-\theta/(\sigma-1)}}{\sum_k L_k A_k \left(\tau_{kj} w_k\right)^{-\theta} f_{kj}^{1-\theta/(\sigma-1)}}. \tag{6.5}$$

and the equilibrium wages can be simply determined using the labor market clearing condition (5.8) given the fact that fixed marketing costs are a constant proportion of bilateral sales, γ.

6.1.2 A Partial Isomorphism

Now consider all the models that we discussed above and assume that their parameters are such that the models yield the same level of domestic share of spending λ_{jj}. As we we will argue at a later point this can be done with different ways in different models. However, insofar as the model generate the welfare equation of the form $W_j = C_j \lambda_{jj}^{-1/\epsilon}$ where $C_j, \varepsilon > 0$ are model dependent constnats, the welfare gains from the expansion of trade are the same and equal to $\hat{W}_j = \hat{\lambda}_{jj}^{-1/\epsilon}$., as long as all the models are specified with the same parameter ε. In other words, the model may even give different implications for overall trade but they give isomorphism implications for the welfare gains from trade. This point is made in detail in Arkolakis, Costinot, and Rodríguez-Clare (2012).

6.2 General Equilibrium: Existence and Uniqueness

Because the various setups that we have studied share a common gravity form, their equilibrium analysis turns out to be simpler than it initially appears. The starting point is gravity models that yield the relationship between aggregate bilateral trade flows and model variables and parameters equation (3.5),

Condition 1. For any countries, i and $j \in S$ the value of aggregate bilateral flows is given by

$$X_{ij} = K_{ij}\gamma_i\delta_j, \tag{6.6}$$

To consider these models in general equilibrium two conditions have to hold: labor market clearing conditions and current account balance. The first condition implies that income generated in a country has to equal to total sales to all destinations,

Condition 2. For any location $i \in S$,

$$Y_i = \sum_j X_{ij}. \tag{6.7}$$

In addition, current account, if there are no capital flows or transfer implies trade balance,

Condition 3. For any location $i \in S$,

$$Y_i = \sum_j X_{ji}. \tag{6.8}$$

Notice, that in addition all these models have to satisfy Warlas law, so that one of our equations is redudandant.[5]For that reason we add a normalization that world income

[5]To see this note that summing these two equations over all $i \neq N$ and equating them we obtain $\sum_{i\neq N}\sum_j X_{ij} = \sum_{i\neq N}\sum_j X_{ji}$. By the definition of gross world income being total trade across all markets we obtain trade balance for the Nth location which implies Warlas' law.

equals to one:[6]

$$\sum_i Y_i = 1. \tag{6.9}$$

Inspecting the above equations, it is clear that they do not impose sufficient structure to solve the model since there is no restriction on the form that Y_i can essentially take. To these essential conditions of the model we add one more, that restricts furthermore the class of models that we focus on. This additional restriction differs across gravity trade and geography models.

Relationship between income and the shifters in gravity trade models. Our last condition for a trade model postulates a log-linear parametric relationship between gross income and the exporting and importing shifters:

Condition 4. For any location $i \in S$, $Y_i = B_i \gamma_i^{\alpha} \delta_i^{\beta}$, where we define $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ to be the *gravity constants* and $B_i > 0$ is an (exogenous) location specific shifter.

We now provide sufficient conditions for establishing existence and uniqueness in a general equilibrium gravity model. We start by formulating the equilibrium system implied by our assumptions. Using equations (6.7) and (6.8) and substituting equations X_{ij} and Y_i with (6.6) and C.4, respectively, yields:

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{6.10}$$

and

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{6.11}$$

[6]This is a valid normalization as long as $\alpha \neq \beta$. When $\alpha = \beta$, a suitable alternative normalization is $\sum_{i \in S} \gamma_i = 1$. None of the following results, unless explicitly noted, depend on the normalization chosen.

and using C.4, the normalization equation (6.6) becomes:

$$\sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} = 1. \tag{6.12}$$

Thus, given model fundamentals B_i, K_{ij} and gravity constants α, β, equilibrium is defined as γ_i and δ_i for all $i \in S$ such that equations (6.10), (6.11) and (6.12) are satisfied. In the special case where $\alpha = \beta = 1$, it is immediately evident from equations (6.10) that (6.11) have a solution only if the matrices with elements $\frac{K_{ij}}{B_i}$ and $\frac{K_{ji}}{B_i}$ both have a largest eigenvalue equal to one. Since this will not generally be true, in what follows we exclude this case.

Based on this formulation we can prove the following theorem:

Theorem 5. *Consider any general equilibrium gravity model. If $\alpha + \beta \neq 1$, then:*

i) The model has a positive solution and all possible solutions are positive;

ii) If $\alpha, \beta \leq 0$ or $\alpha, \beta \geq 1$, then the solution is unique.

Proof. See Allen, Arkolakis, and Takahashi (2014). □

Our approach can also be naturally extended to allow for labor mobility as in economic geography models. To do so, we slightly alter condition C.4 to allow for the gross income in a location to depend additionally on an endogenous constant λ, which can be interpreted as a monotonic transformation of welfare (which is equalized across locations in economic geography models). It is straightforward to show that the economic geography model of Allen and Arkolakis (2014) (which under certain parametric configurations is isomorphic to the economic geography models of Helpman (1998), Redding (2014), and Bartelme (2014)) satisfies the following condition

Condition 5. For any location $i \in S$, $Y_i = \frac{1}{\lambda} B_i \gamma_i^{\alpha} \delta_i^{\beta}$, where $\lambda > 0$ is an endogenous variable and all other variables are as above. Furthermore, we require that $\lambda^c = \sum_i C_i \gamma_i^d \delta_i^e$ for some $c, d, e \in \mathbb{R}$.

Given this alternative condition, we modify part (ii) of Theorem 5 slightly to prove the following Corollary:

Corollary 1. *Consider any economic geography model that satisfies conditions C.1, C.2, C.3, C.4, and C.5. Then (i) there exists a solution as long as $\alpha + \beta \neq 1$; and (ii) the equilibrium is unique if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$.*

Proof. See Allen, Arkolakis, and Takahashi (2014). □

Notice that the additional normalization is required to determine the level of the endogenous variable λ. In the economic geograhy example we consider in the previous section this endogenous variables corresponds to a monotonic transformation the overall welfare level.

6.3 Analytical Characterization of the Gravity Model

Thus far, we have provided various microeconomic foundations for the gravity trade model, defined the general equilibrium conditions, characterizes when an equilibrium exists and when it is unique, and discussed some general equilibrium properties of gravity trade models. Next, we are going to take our tools "out for a spin" and see what exactly the general equilibrium properties imply for the Armington model. Remember that it is a reasonably straightforward task to reinterpret the Armington model in other frameworks (i.e. there exist formal isomorphisms), so the choice of the Armington model is not particularly important. While most of this class will be applying the tools we have developed to a particular example, we think doing so both reinforces the power of the tools we have and provides new insights into the mechanisms at play.

It turns out that we can extend the range in which uniqueness is guaranteed if we constrain our analysis to a particular class of trade frictions which are the focus of a large empirical literature on estimating gravity trade models. We call these trade frictions quasi-symmetric.

Definition 1. Quasi Symmetry: We say the trade frictions matrix $\mathbf{K}$ is *quasi-symmetric* if there exists a symmetric $N \times N$ matrix $\tilde{\mathbf{K}}$ and $N \times 1$ vectors K^A and K^B such that for all $i, j \in S$ we have:

$$K_{ij} = \tilde{K}_{ij} K_i^A K_j^B, \text{ where } \tilde{K}_{ij} = \tilde{K}_{ji}$$

Loosely speaking, quasi-symmetric trade frictions are those that are reducible to a symmetric component and exporter- and importer-specific components. While restrictive, it is important to note that the vast majority of papers which estimate gravity equations assume that trade frictions are quasi-symmetric; for example Eaton and Kortum (2002) and Waugh (2010) assume that trade costs are composed by a symmetric component that depends on bilateral distance and on a destination or origin fixed effect.

When trade frictions are quasi-symmetric it can show that the system of equations (6.10) and (6.11) can be dramatically simplified, and the uniqueness more sharply characterized.

Theorem 6. *Consider any general equilibrium gravity model with quasi-symmetric trade costs. Then:*

i) The balanced trade condition is equivalent to the origin and destination shifters being equal up to scale, i.e.

$$\gamma_i K_i^A = \kappa \delta_i K_i^B \tag{6.13}$$

for some $\kappa > 0$ that is part of the solution of the equilibrium.

ii) If $\alpha + \beta \leq 0$ or $\alpha + \beta \geq 2$, the model has a unique positive solution.

Part (i) of the Theorem 6 is particularly useful since it allows to simplify the equilibrium system into a single non-linear equation:

$$\gamma_i^{\alpha+\beta-1} = \kappa^{\beta-1} \sum_j \widetilde{K}_{ij} B_i^{-1} \left(K_i^A\right)^{1-\beta} \left(K_i^B\right)^{\beta} \gamma_j. \tag{6.14}$$

For any gravity trade model where trade frictions are quasi symmetric, if trade is balanced, the goods market clearing condition holds, and the generalized labor market clearing condition holds, then the equilibrium origin fixed effects satisfy the following set of non-linear equations:

$$\tilde{\gamma}_i = \lambda \sum_{j \in S} F_{ij} \tilde{\gamma}_j^{\frac{1}{\alpha+\beta-1}}, \tag{6.15}$$

where $\tilde{\gamma}_i \equiv \gamma_i^{\alpha+\beta-1}$, $\lambda \equiv \kappa^{\beta-1} > 0$, and $F_{ij} \equiv K_{ij} \left(\frac{K_j^A}{K_j^B}\right) \left(\frac{K_i^A}{K_i^B}\right)^{-\beta} \frac{1}{B_i} > 0$. This implies that there will always exist a solution and the solution will be unique if $\alpha + \beta \geq 2$ or $\alpha + \beta \leq 0$.

Notice that the normalization XXX implies with quasisymmetry that

$$\begin{aligned}
\sum_i Y_i = 1 &\implies \\
\sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} = 1 &\implies \\
\sum_i B_i \gamma_i^{\alpha} \left(\frac{\gamma_i K_i^A}{\kappa K_i^B}\right)^{\beta} = 1 &\implies \\
\kappa^{-\beta} \sum_i B_i \left(\frac{K_i^A}{K_i^B}\right)^{\alpha} \gamma_i^{\alpha+\beta} = 1 &
\end{aligned}$$

so that the normalization can be used to pin down κ.

With economic geography this relationship holds but in addition, to determine the level of λ, we make use of the additional condition XXX.

Example: Armington model with quasi-symmetry Consider now an Armington model with intermediate inputs, but now assume that trade costs are quasi-symmetric. From part (i) of Theorem 6, we have $\gamma_i = \kappa\delta_i$, which implies:

$$\left(\frac{w_i^{\delta} P_i^{1-\delta}}{A_i}\right)^{1-\sigma} K_i^A = \kappa P_i^{\sigma-1} w_i L_i K_i^B,$$

or equivalently:

$$P_i = w_i^{\frac{1+(\sigma-1)\delta}{(1-\sigma)(2-\delta)}} \left(\kappa L_i A_i^{1-\sigma} \frac{K_i^B}{K_i^A}\right)^{\frac{1}{(1-\sigma)(2-\delta)}}. \tag{6.16}$$

Equation (6.16) provides some intuition for the uniqueness condition presented in Theorem 6: when $\sigma < \frac{1}{2}$, it is straightforward to show that the elasticity of the price index with respect to the wage is less than one. This implies that the wealth effect may dominate the substitution effect, so that the excess demand function need not be downward sloping.

In addition, combining equation (6.16) with equation (6.14), assuming $\delta = 1$, and noting that welfare $W_i = \frac{w_i}{P_i}$ yields the following equation:

$$\kappa W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}} = \sum_j K_{ij} A_i^{(\sigma-1)\tilde{\sigma}} A_j^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}} W_j^{-(\sigma-1)\tilde{\sigma}}, \tag{6.17}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$.[7] Equation (6.17) holds for both trade models (where labor is fixed) and economic geography models (where labor is mobile); in the former case, L_i is treated as exogenous parameter and W_i solved for; in the latter case L_i is treated as endogenous and

[7] When there are only two countries (so that trade costs are necessarily quasi-symmetric), we can use equation (6.17) to derive a single non-linear equation that yields the relative welfare in the two countries

$$K_{22}\left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} - K_{11}\left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + K_{21}\left(\frac{W_1}{W_2}\right)^{\tilde{\sigma}} = K_{12}.$$

Comparative statics for welfare with respect to changes in K_{ij} can be characterized using the implicit function theorem in this case.

W_i is assumed to be constant across locations. Hence, Theorem 6 highlights the fundamental similarity between trade and economic geography models.

We now discuss the equilibrium when there are just two countries. Note that when there are only two countries, all possible trade costs are quasi-symmetric. Define the kernel $F_{ij} \equiv K_{ij} \left(\frac{K_j^A}{K_j^B} \frac{K_i^B}{K_i^A} \right)^{\frac{\sigma}{2\sigma-1}} A_j^{\sigma\tilde{\sigma}} A_i^{(\sigma-1)\tilde{\sigma}} L_j^{\tilde{\sigma}} L_i^{-\tilde{\sigma}}$ (note that the kernel now includes all the exogenous variables in the model). Then the equilibrium conditions from equation (**??**) can then be written (for an arbitrary number of countries) as:

$$W_i^{\sigma\tilde{\sigma}} = \sum_{j \in S} F_{ij} W_j^{(1-\sigma)\tilde{\sigma}}.$$

With two countries, this becomes:

$$\begin{aligned} W_1^{\sigma\tilde{\sigma}} &= F_{11} W_1^{(1-\sigma)\tilde{\sigma}} + F_{12} W_2^{(1-\sigma)\tilde{\sigma}} \\ W_2^{\sigma\tilde{\sigma}} &= F_{22} W_2^{(1-\sigma)\tilde{\sigma}} + F_{21} W_1^{(1-\sigma)\tilde{\sigma}} \end{aligned}$$

Dividing the first equation by the second yields:

$$\begin{aligned} \left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} &= \frac{F_{11} W_1^{(1-\sigma)\tilde{\sigma}} + F_{12} W_2^{(1-\sigma)\tilde{\sigma}}}{F_{22} W_2^{(1-\sigma)\tilde{\sigma}} + F_{21} W_1^{(1-\sigma)\tilde{\sigma}}} \iff \\ \left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} \left(F_{22} W_2^{(1-\sigma)\tilde{\sigma}} + F_{21} W_1^{(1-\sigma)\tilde{\sigma}}\right) &= F_{11} W_1^{(1-\sigma)\tilde{\sigma}} + F_{12} W_2^{(1-\sigma)\tilde{\sigma}} \iff \\ \left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} \left(F_{22} + F_{21} \left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}}\right) &= F_{11} \left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + F_{12} \iff \\ F_{22} \left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} - F_{11} \left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + F_{21} \left(\frac{W_1}{W_2}\right)^{\tilde{\sigma}} &= F_{12} \qquad (6.18) \end{aligned}$$

Equation (6.18) shows that with two countries, the equilibrium relative welfare in the two regions is just the root of a polynomial equation! Furthermore, note that if if $\sigma > \frac{1}{2}$, then $\frac{\partial}{\partial(W_1/W_2)}\left(F_{22}\left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} - F_{11}\left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + F_{21}\left(\frac{W_1}{W_2}\right)^{\tilde{\sigma}} - F_{12}\right) > 0$ so that the implicit function theorem implies:

$$\frac{\partial}{\partial F_{11}}\left(\frac{W_1}{W_2}\right) > 0,\ \frac{\partial}{\partial F_{21}}\left(\frac{W_1}{W_2}\right) < 0,\ \frac{\partial}{\partial F_{12}}\left(\frac{W_1}{W_2}\right) > 0,\ \frac{\partial}{\partial F_{22}}\left(\frac{W_1}{W_2}\right) < 0.$$

[Class question: what is the intuition for these comparative statics?]

6.4 Computing the Equilibrium (Alvarez-Lucas)

Using the methodology of Alvarez and Lucas (2007) it can be proven that the model with this gravity structure has a unique equilibrium. To show existence Alvarez and Lucas (2007) define the analog of an excess demand function, which in our context and with zero exogenous deficits is given by,

$$f_i(\mathbf{w}) = \frac{1}{w_i}\left[\sum_j \frac{L_i A_i \left(\tau_{ij} w_i\right)^{-\theta} f_{ij}^{1-\theta/(\sigma-1)}}{\sum_k L_k A_k \left(\tau_{kj} w_k\right)^{-\theta} f_{kj}^{1-\theta/(\sigma-1)}} w_j L_j - w_i L_i\right],$$

where $\mathbf{w}$ is the vector of wages, and they show that it satisfies the standard properties of an excess demand function.[8] To show uniqueness, the gross substitute property has to be proven

$$\frac{\partial f_i(\mathbf{w})}{\partial w_k} > 0 \text{ for all } i \neq k$$
$$\frac{\partial f_i(\mathbf{w})}{\partial w_i} > 0 \text{ for all } i$$

and uniqueness follows from Propositon 17.F.3 of Mas-Colell, Whinston, and Green (1995).

[8]These properties are continuity, homogeneity of degree zero, Warlas' law, boundness from below and infinite excess demand if one wage is 0. See Mas-Colell, Whinston, and Green (1995), Chapter 17.

To compute the equilibrium, notice that for some $\kappa \in (0,1]$ we can define a mapping

$$T_i(\mathbf{w}) = w_i \left[1 + \kappa f_i(\mathbf{w}) / L_i\right] \tag{6.19}$$

Now if we start with wages that satisfy $\sum_i w_i L_i = 1$, we have

$$\begin{aligned}
\sum_i T_i(\mathbf{w}) L_i &= \sum_i w_i L_i + \sum_i w_i \kappa f(\mathbf{w}) \\
&= 1 + \kappa \sum_i \left[\sum_j \frac{L_i A_i \left(\tau_{ij} w_i\right)^{-\theta} f_{ij}^{1-\theta/(\sigma-1)}}{\sum_k L_k A_k \left(\tau_{kj} w_k\right)^{-\theta} f_{kj}^{1-\theta/(\sigma-1)}} w_j L_j - w_i L_i \right] \\
&= 1 - \kappa \sum_j w_j L_j + \kappa \sum_j w_j L_j = 1
\end{aligned}$$

so that $T_i(\mathbf{w})$ is mapping $\mathbf{w}$ such that it maps $\sum_i w_i L_i = 1$ to itself. By starting with an initial guess of the wages, and updating according to (6.19) the system is guaranteed to converge to the solution $T_i(\mathbf{w}) = w_i$ (see Alvarez and Lucas (2007)).

6.5 Conducting Counterfactuals: The Dekle-Eaton-Kortum Procedure

Dekle, Eaton, and Kortum (2008) have established a methodology for calculating counterfactual changes in the equilibrium variables with respect to changes in the iceberg costs or technology parameters. The merit of this approach is that it does not require prior information on the level of technology A_i and bilateral trade costs τ_{ij}, but rather only percentage changes in the magnitudes of these parameters. The idea is to use data for the endogenous variables λ_{ij}, y_j to calibrate the model in the initial equilibrium, and exploit the fact that the level of technology A_i and bilateral trade costs τ_{ij} are perfectly identified given the values for λ_{ij}, y_j.

The procedure can be applied to most of the frameworks above, and in fact delivers robust predictions for changes in trade and welfare as argued by Arkolakis, Costinot, and Rodríguez-Clare (2012), under the simple assumption that the elasticity of trade with respect to wages and trade costs is the same.

Denote the ratio of the variables in the new and the old equilibrium, e.g. $\hat{w}_j = w_j'/w_j$. We use labor in country j as our numeraire, $w_j = 1$. We will make crucial use of the fact that either profits are a constant fraction of income or that labor income is the only source of income in the models above so that we also obtain that $\hat{y}_i = \hat{w}_i$ for all $i = 1, ..., n$.

Under the assumption that the elasticity of trade with respect to wages and trade costs is the same, and equal to ε, the shares of expenditures on goods from country i in country j in the initial and new equilibrium, respectively, are given by

$$\lambda_{ij} = \frac{\chi_{ij} \cdot N_i \cdot \left(w_i \tau_{ij}\right)^{\varepsilon}}{\sum_{i'=1}^{n} \chi_{i'j} N_{i'} \cdot \left(w_{i'} \tau_{i'j}\right)^{\varepsilon}} \tag{6.20}$$

where χ_{ij} is some parameter of the model, other than τ_{ij} (e.g. bilateral fixed marketing costs). Thus, for example, $\varepsilon = -\theta$ in the Eaton and Kortum (2002) model whereas $\varepsilon = -(\sigma - 1)$ in the Armington (1969) setup. Notice that an essential simplifying assumption is that N_i is a constant and does not depend on technology or bilateral trade costs.

Combining this observation with the above two equations we obtain

$$\hat{\lambda}_{ij} = \frac{\left(\hat{w}_i \hat{\tau}_{ij}\right)^{\varepsilon}}{\sum_{i'=1}^{n} \lambda_{ij} \left(\hat{w}_{i'} \hat{\tau}_{i'j}\right)^{\varepsilon}} \tag{6.21}$$

From the previous expression and the fact that $\hat{w}_j \hat{\tau}_{jj} = 1$ by our choice of numeraire we have that

$$\hat{\lambda}_{jj} = \frac{1}{\sum_{i'=1}^{n} \lambda_{i'j} \left(\hat{w}_{i'} \hat{\tau}_{i'j}\right)^{\varepsilon}}.$$

For the models illustrated above, we can use the trade balance condition as argued by Arkolakis, Costinot, and Rodríguez-Clare (2012) so that in the new equilibrium:

$$w'_j L'_j = \sum_{j'=1}^{n} \lambda'_{ij} w'_j L'_j \implies$$

$$\hat{w}_j \hat{L}_j w_j L_j = \sum_{j'=1}^{n} \hat{w}_j \hat{L}_j \hat{\lambda}_{ij} \lambda_{ij} w_j L_j \tag{6.22}$$

If population is exogenous, equations (**??**) and (6.22) constitute a system on $\hat{w}_j$ with the additional normalization of one wage. The equilibrium changes in wages, w_i, and market shares, λ_{ij}, can be computed given expression (6.21) and (6.22), which completes the argument.

6.6 Homeworks

1. *Isomorphisms.* Define X_{ij} to be the value of trade flows from i to j. Consider the following generalized gravity equation:

$$X_{ij} = K_{ij} \gamma_i \delta_j, \tag{6.23}$$

where K_{ij} is a bilateral trade friction, γ_i is an origin fixed effect, and δ_j is a destination fixed effect.

   (a) For each of the following trade models, show how equilibrium trade flows can be expressed as equation (6.23). That is, write down the mapping between the generalize gravity equation and model fundamentals.

      i. Armington model (Anderson '79).

      ii. Monopolistic competition with free entry (Krugman '80).

iii. Perfect competition with heterogeneous technologies (Eaton and Kortum '02).

iv. Heterogeneous firms (with Pareto distribution) (Melitz '03 / Chaney '08).

(b) Suppose we only observe trade flows in the data. Can we empirically distinguish between the above models? If not, what other data would you need to observe in order to distinguish between the models?

Chapter 7

Gains from Trade

7.1 Trade Liberalization and Firm Heterogeneity

There is a common perception that the gains from trade are larger than what quantitative general-equilibrium models of trade can explain. A recurring goal in the trade literature has been to find new channels through which such models can generate larger gains. Recently, authors such as Melitz (2003) have postulated additional gains from the "selection" effect compared to the extensive margin effect already postulated by Romer (1994). Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008) show that some of the key quantitative frameworks in international trade deliver (Krugman, Eaton and Kortum, the Chaney version of Melitz and Arkolakis) welfare expressions that are closely comparable. Arkolakis, Costinot, and Rodríguez-Clare (2012) show that for a wide class of perfect and monopolistic competition models of trade welfare gains from trade can be written as a function of two sufficient statistics: the share of spending that goes to domestic goods, λ_{jj}, and the elasticity of trade parameter, ε. Their result imply that changes in welfare can be written as

$$\widehat{W}_j = \widehat{\lambda}_{jj}^{1/\varepsilon} \tag{7.1}$$

7.1.1 Trade Liberalization and Welfare gains (Arkolakis-Costinot-Rodriguez-Clare)

To understand the intuition for the main result of Arkolakis, Costinot, and Rodríguez-Clare (2012) we start the analysis from the simplest setup, the Armington model. The model is essentially identical to the model presented in section 3.3 assuming that the endowment is labor so that the price of the endowment is wage and that there are no preference shocks so that $\alpha_{ij} = 1$, $\forall i, j$.

We will obtain the result for the case of monopolistic competition where we assume that exporting and importing country wages matter for marketing fixed costs through a Cobb-Douglas function, $f_{ij} w_i^{\mu} w_j^{1-\mu}$, $\mu \in [0,1]$. Denoting z_{ij}^* the cutoff productivity determining the entry of firms from country i in country j; Ω_{ij} the set of goods that country j buys from country i can be written as

$$\Omega_{ij} = \left\{ \omega \in \Omega | z_{ij}(\omega) > z_{ij}^* \equiv \frac{\sigma^{\frac{\sigma}{\sigma-1}}}{\sigma - 1} \left(\frac{w_i \tau_{ij}}{P_j} \right) \left(\frac{f_{ij} w_i^{\mu} w_j^{1-\mu}}{X_j} \right)^{\frac{1}{\sigma-1}} \right\}. \tag{7.2}$$

where we assume that the production of fixed marketing costs f_{ij} is using a mix of domestic and foreign labor with respective shares $\mu, 1-\mu$. In what follows we assume $X_j = w_j L_j$ which is guaranteed under perfect competition and free entry.

Real wage is given by $W_j = w_j / P_j$, in that model where the price index is

$$P_j^{1-\sigma} = \sum_i N_i \int_{z_{ij}^*} \left(\frac{\tilde{\sigma} w_i \tau_{ij}}{z} \right)^{1-\sigma} g(z)\, dz.$$

Taking logs of the real wage and differentiating we obtain a formula for predicting welfare gains from trade after changes in trade costs,

$$d\ln W_j = d\ln w_j - \sum_i \lambda_{ij}\left[d\ln w_i + d\ln\tau_{ij} + \frac{d\ln N_i}{1-\sigma} + \frac{\gamma_{ij}}{1-\sigma}d\ln z_{ij}^*\right] \tag{7.3}$$

where

$$\gamma_{ij} = \frac{\left(z_{ij}^*\right)^{\sigma-1} g\left(z_{ij}^*\right)}{\int_{z_{ij}^*} z^{\sigma-1} g\left(z\right) dz}.$$

$$z_{ij}^* \equiv \frac{\sigma^{\frac{\sigma}{\sigma-1}}}{\sigma-1}\left(\frac{w_i\tau_{ij}}{P_j}\right)\left(\frac{f_{ij}w_i^{\mu}w_j^{1-\mu}}{X_j}\right)^{\frac{1}{\sigma-1}}$$

At this point it worths to pause to understand where the gains from trade are coming from. Notice that in the case of Armington preferences, $d\ln N_i = 0$ and also, effectively, $d\ln z_{ij}^* = 0$ so that

$$d\ln W_j = -\sum_{i=1}^{n}\lambda_{ij}\left[d\ln w_i + d\ln\tau_{ij}\right], \tag{7.4}$$

i.e. in the Armington model welfare gains from trade arise only because of improvement in terms of trade. Instead, in the monopolistic competition models, as expression 7.3 reveals there is an additional variety and entry effect. Do this extra terms imply larger gains from trade? It turns out that under some conditions, the asnwer is no, and we will investigate this below.

Before proceeding it is worth discussin in detail an important result from Atkeson and Burstein (2010). First notice that in the Armington model, expression 7.4 can be written under symmetry

$$d\ln W_j = -\left(1-\lambda_{jj}\right)d\ln\tau.$$

This expression is quite intuitive. Gains from trade, to a first-order, depend on the percentage change in trade costs, and the exposure of the country to trade. It turns out, that a similar condition can be derived in monopolistic competition under symmetry (see Atkeson and Burstein (2010)).[1] We have that

$$XXX$$

Using the wage normalization, $w_j = 1$, small changes in real income are now given by

$$d\ln W_j = -\sum_i \frac{\lambda_{ij}}{1-\sigma-\gamma_j}\left[\left(1-\sigma-\gamma_{ij}\right)\left(d\ln w_i + d\ln \tau_{ij}\right) + d\ln N_i + \frac{\gamma_{ij}}{1-\sigma}\left(\mu d\ln w_i\right)\right], \tag{7.5}$$

where $\gamma_j \equiv \sum_i \lambda_{ij}\gamma_{ij}$. where we choose wage of country j as the numeraire.

With Dixit-Stiglitz preferences, we get that market shares are given by

$$X_{ij} = \frac{f_{ij} w_i^{\mu} N_i \int_{z_{ij}^*}^{\infty} \left[w_i \tau_{ij}\right]^{1-\sigma} z^{\sigma-1} g_i(z)\, dz}{\sum_{i'=1}^{n} f_{i'j} w_{i'}^{\mu} N_{i'} \int_{z_{i'j}^*}^{\infty} \left[w_{i'}\tau_{i'j}\right]^{1-\sigma} z^{\sigma-1} g_{i'}(z)\, dz X_j} \tag{7.6}$$

where the density $g_i(z)$ of goods with productivity z in Ω_{ij} is simply given by the marginal density of g. Considering the ratio $\lambda_{ij}/\lambda_{jj} = X_{ij}/X_{jj}$ and differentiating and using the def-

[1] To see that, you need to differentiate the free entry condition

$$\sum_j w_i^{\mu} w_j^{1-\mu} f_{ij} \int_{z_{ij}^*} \left(\frac{z}{z_{ij}^*}\right)^{\sigma-1} g(z)\, dz - w_i^{\mu} w_j^{1-\mu} f_{ij} \sum_j \int_{z_{ij}^*} g(z)\, dz = w_i f^e.$$

This differentiation, under symmetry ($w_i = w$, $f_{ij} = f$, $\tau_{ij} = \tau$ for $i \neq j$) yieldsXXXX

$$(\sigma-1)\sum_j \left[\int_{z_{ij}^*} \left(\frac{z}{z_{ij}^*}\right)^{\sigma-1} g(z)\, dz\right] d\ln z_{ij}^* = 0 \implies \sum_j \left[\frac{\int_{z_{ij}^*} \left(\frac{z}{z_{ij}^*}\right)^{\sigma-1} g(z)\, dz}{\sum_j \int_{z_{ij}^*} \left(\frac{z}{z_{ij}^*}\right)^{\sigma-1} g(z)\, dz}\right] d\ln z_{ij}^* = 0$$

XXXXXX

inition of z_{ij}^*, equation (7.2),

$$d\ln\lambda_{ij} - d\ln\lambda_{jj} = d\ln N_i - d\ln N_j + (1-\sigma)\left(d\ln w_i + d\ln\tau_{ij}\right) + \mu d\ln w_i - \gamma_{ij} d\ln z_{ij}^* + \gamma_{jj} d\ln z_{jj}^* \tag{7.7}$$

$$= \left(1-\sigma-\gamma_{ij}\right) d\ln c_{ij} + \frac{\gamma_{ij}}{1-\sigma}\mu d\ln w_i - \left(\gamma_{ij}-\gamma_{jj}\right) d\ln z_{jj}^* + d\ln N_i - d\ln N_j. \tag{7.8}$$

Combining the expression(**??**) and (7.7) reveals that

$$d\ln W_j = -\left(\frac{\lambda_{ij}}{1-\sigma-\gamma_j}\right)\left[d\ln\lambda_{ij} - d\lambda_{jj} + \left(\gamma_{ij}-\gamma_{jj}\right) d\ln z_{jj}^* + d\ln N_i\right]$$

Three are the macro-level restrictions that are used to derive the result in a general perfect competition or monopolistically competitive setup, such as the ones studies in Chapters 3.4 and 4. We have already talked about R1 and R3. Below we present one more restriction.

The import demand system is CES. The last macro-level restriction is concerned with the partial equilibrium effects of variable trade costs on aggregate trade flows. Define the *import demand system* as the mapping from $(\boldsymbol{w}, \boldsymbol{N}, \boldsymbol{\tau})$ into $\boldsymbol{X} \equiv \left\{X_{ij}\right\}$, where $\boldsymbol{w} \equiv \{w_i\}$ is the vector of wages, $\boldsymbol{N} = \{N_i\}$ is the vector of measures of goods that can be produced in each country, and $\boldsymbol{\tau} \equiv \left\{\tau_{ij}\right\}$ is the matrix of variable trade costs. This mapping is determined by utility and profit maximization given preferences, technological constraints, and market structure. It excludes, however, labor market clearing conditions as well as free entry conditions (if any) which determine the equilibrium values of $\boldsymbol{w}$ and $\boldsymbol{N}$. The third macro-level restriction imposes restrictions on the partial elasticities, $\varepsilon_j^{ii'} \equiv \partial\ln\left(X_{ij}/X_{jj}\right)/\,\partial\ln\tau_{i'j}$, of that system:

R4 *The import demand system is such that for any importer j and any pair of exporters $i \neq j$ and*

$i' \neq j$, $\varepsilon_j^{ii'} = \varepsilon < 0$ if $i = i'$, and zero otherwise.

Each elasticity $\varepsilon_j^{ii'}$ captures the percentage change in the relative imports from country i in country j associated with a change in the variable trade costs between country i' and j holding wages and the measure of goods that can be produced in each country fixed.

Noting that $\partial \ln z_{ij}^* \Big/ \partial \ln \tau_{ij} = \partial \ln z_{jj}^* \Big/ \partial \ln \tau_{ij} - 1$ and $\partial \ln z_{ij}^* \Big/ \partial \ln \tau_{i'j} = \partial \ln z_{jj}^* \Big/ \partial \ln \tau_{i'j}$ if $i' \neq i$, we can define the import demand system as the following partial derivative,

$$\frac{\partial \ln \left(X_{ij} / X_{jj} \right)}{\partial \ln \tau_{i'j}} = \varepsilon_j^{ii'} = \begin{cases} 1 - \sigma - \gamma_{ij} - \left(\gamma_{ij} - \gamma_{jj} \right) \left(\frac{\partial \ln z_{jj}^*}{\partial \ln \tau_{ij}} \right) & \text{for} \quad i' = i \\ - \left(\gamma_{ij} - \gamma_{jj} \right) \left(\frac{\partial \ln z_{jj}^*}{\partial \ln \tau_{i'j}} \right) & \text{for} \quad i' \neq i \end{cases}, \tag{7.9}$$

where $\gamma_{ij} \equiv d \ln \int_{z_{i'j}^*}^{\infty} z^{1-\sigma} g_i \left(z \right) dz \Big/ d \ln z_{ij}^*$.

R4 implies $\gamma_{ij} = \gamma_{jj}$ and $1 - \sigma - \gamma_j = \varepsilon$ for all i, j, we obtain from 7.5 that $d \ln W_j = \left(d \ln \lambda_{jj} - d \ln N_j \right) / \varepsilon$, using the fact that $\sum_{i=1}^n \lambda_{ij} = 1 \implies \sum_{i=1}^n \lambda_{ij} d \ln \lambda_{ij} = 0$. To conclude, we simply note that free entry and R1 and R3, using the results of Section 5.2, imply $d \ln N_j = d \ln Y_j = 0$. Combining the two previous observations and integrating, we finally obtain expression (7.1) which is model invariant, as long as ϵ is chosen to be the same.

Going back to our various derivations in the previous chapters we note that all the models deliver similar expressions for welfare gains from trade as a function of λ_{ii}, and thus the trade share of GDP. In particular, the expressions for the Armington and Krugman models in Chapter 3.4 is similar to the one derived in other models with heterogeneous firms such as the ones of Eaton and Kortum (2002), the Chaney (2008) version of Melitz (2003) and Arkolakis (2010) in Chapter 4. The only difference is that in the latter cases $\sigma - 1$ is replaced by the parameter that determines the heterogeneity of the productivities of the firms or productivities of sectors.

7.2 Global Gains

We use expression (**??**). Set $\mu = 0$ and consider changes only in trade costs so that $d \ln f_j = 0$. Using R1, R3 as above, we have $d \ln N_i = 0$ and thus

$$\begin{aligned}\sum_j d \ln W_j &= \textstyle\sum_{j=1}^n \sum_{i=1}^n \lambda_{ji} d \ln w_j - \sum_{j=1}^n \sum_{i=1}^n \lambda_{ji} \left(d \ln w_i + d \ln \tau_{ij} \right) \\ &= \textstyle\sum_{j=1}^n \sum_{i=1}^n \left(\lambda_{ji} d \ln w_j - \lambda_{ij} d \ln w_i \right) - \displaystyle\sum_{j=1}^n \sum_{i=1}^n \lambda_{ij} d \ln \tau_{ij}\end{aligned} \tag{7.10}$$

The first double summation term in the RHS is zero by splitting it into two terms interchanging i and j. Thus, total welfare changes are given by

$$\sum_j d \ln W_j = - \sum_{j=1}^n \sum_{i=1}^n \lambda_{ij} d \ln \tau_{ij}.$$

This expression for global gains is derived by Atkeson and Burstein (2010); Fan, Lai, and Qi (2013). It straightforward to note that the starting expression for Armington and the Eaton Kortum model is expression (7.10) so that the same conclusions hold for perfect competition.

Chapter 8

Extensions: Modeling the Demand Side

We now discuss a number of ways that the simple model can be extended by with assumptions that have implications for the effective demand of the firm. We will discuss a nested CES structure, endogenous marketing costs, multi-product firms, and other preferences structure different than the constant elasticity demand.

8.1 Extension I: The Nested CES demand structure

We can consider a nested CES structure

$$\left(\int_{\Omega}\left(\sum_{k=1}^{N} x_k\left(\omega\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\sigma-1}{\sigma}\frac{\varepsilon}{\varepsilon-1}} d\omega\right)^{\frac{\sigma}{\sigma-1}}$$

that delivers the demand

$$x_i(\omega) = \left(\frac{p_i(\omega)}{P(\omega)}\right)^{-\varepsilon}\left(\frac{P(\omega)}{P}\right)^{-\sigma} X\,,$$

with

$$\begin{aligned} P(\omega) &= \left[\sum_{k=1}^{N} p_k(\omega)^{1-\varepsilon}\, d\omega\right]^{1/(1-\varepsilon)}\,, \\ P &= \left[\int_{\Omega} P(\omega)^{1-\sigma}\, d\omega\right]^{1/(1-\sigma)}\,. \end{aligned}$$

and X being the overall spending.

Serving a market incurs an entry cost

a) $\varepsilon \to \infty, F = 0$ PC EK02

b) $\varepsilon \to \infty, F = 0$ Bertrand BEJK

c) $\varepsilon = \sigma$, $F > 0$ monopolistic competition Melitz-Chaney

d) $\varepsilon > \sigma$, $F \geq 0$ (with either $F > 0$ or $\varepsilon \to \infty$) and Cournot, Atkeson and Burstein.

8.2 Extension II: Market penetration costs

The CES benchmark proved extremely useful for many applications. Its main weakness is in predicting the behavior of small firms-goods as Eaton, Kortum, and Kramarz (2011). These firms-goods tend to be a very large part of trade in a trade liberalization and as time evolves. To address this fact, a simple extension presented in Arkolakis (2010) does the job by modeling the fixed entry costs as cost of reaching individual consumers into individual destinations.

Each good is produced by at most a single firm and firms differ ex-ante only in their productivities z and their country of origin $i = 1, ..., N$. We denote the destination country by j. The preferences for consumer l are given by the standard symmetric constant elasticity of substitution (CES) objective function:

$$U^l = \left(\int_{\omega \in \Omega^l} x\left(\omega\right)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma-1}{\sigma}} ,$$

where $\sigma \in (1, +\infty)$ is the elasticity of substitution. When a good produced with a productivity z from country i is included in the choice set of consumer l, Ω^l, the demand of this consumer is given by,

$$x_{ij}\left(z\right) = y_j \frac{p_{ij}\left(z\right)^{-\sigma}}{P_j^{1-\sigma}} , \tag{8.1}$$

where $p_{ij}\left(z\right)$ is the price charged in country j, y_j the income per capita of the consumer, and P_j a price aggregate of the goods in the choice set of the consumer. An unrealistic assumption of the CES framework introduced by Dixit and Stiglitz is that all the consumers have access to the same set of goods Ω^l. This formulation departs from the standard formulation of trade models with CES preferences by proposing a formulation where Ω^l can be different for different consumers. In order to be able to fully characterize the general equilibrium of the model, we assume that consumers are reached independently by different firms and that each firm pays a cost to reach a fraction n of the consumers. In equilibrium, all firms z from country i will reach the same fraction of consumers in country j and thus their 'effective' sales will be:[1]

$$t_{ij}(z) = \underbrace{n_{ij}\left(z\right) L_j}_{\substack{\text{consumers} \\ \text{reached in } j}} \underbrace{y_j \frac{p_{ij}\left(z\right)^{1-\sigma}}{P_j^{1-\sigma}}}_{\text{sales per-consumer}}$$

[1]Given the existence of a continuum of firms and consumers I am making use of the Law of Large Numbers. This implies that $n_{ij}(z)$ from a probability becomes a fraction. The application of the Law of Large Numbers also implies that P_j is now a function of $n_{ij}(z)$'s and has a given value for all consumers.

where L_j is the measure of the population of country j. In order to give foundations to the market penetration cost function as an explicit function of $n_{ij}(z)$ we depart from the standard formulation where there is a uniform fixed marketing cost to enter the market and sell to all the consumers there. Instead, we consider an alternative formulation that intends to broadly capture the marketing costs incurred by the firm in order to increase their sales in a particular market. The marketing costs are modeled as increasing access costs that the firms pay in order to access an increasing number of customers in each given country. Due to market saturation, reaching additional consumers becomes increasingly difficult once a relatively large fraction of them has already been reached. Based on a derivation of a marketing technology from first principles the cost function of reaching a fraction n of a population of L consumers in Arkolakis (2010) is derived to be

$$f(n) = \begin{cases} \frac{L^{\alpha}}{\psi}\frac{1-(1-n)^{1-\beta}}{1-\beta} & \text{if } \beta \in [0,1) \cup (1,+\infty) \\ \frac{L^{\alpha}}{\psi}\log(1-n) & \text{if } \beta = 1 \end{cases} .$$

$1/\psi$ denotes the productivity of search effort and $a \in [0,1]$ regulates returns to scale of marketing costs with respect to the population size of the destination country. The parameter β determines how steeply the cost to reach additional consumers is rising. However, for any parametrization of β the marginal cost to reach the very first consumers in a given market j is always positive (the derivative is always bigger than zero). Thus, only firms with productivity above some threshold z_{ij}^{*} will have high enough revenues from the very first consumers to find it profitable to enter the market.[2] The case where $\beta = 1$ corresponds to the benchmark random search case of Butters (1977) and Grossman and Shapiro (1984). If $\beta = 0$ the function implies a linear cost to reach additional consumers, which in turn is isomorphic to the case of Melitz (2003)-Chaney (2008) given that firms

[2] With no additional heterogeneity across firms this implies a hierarchy of exporting destinations

reach either all the consumers in a market or none.

The production side of the firm is standard. Labor is the only factor of production. The firm z uses a production function that exhibits constant returns to scale and productivity z. It incurs an iceberg transportation cost τ_{ij} to ship a good from country i to country j. This implies that the optimal price of the firm is a constant markup $\sigma/(\sigma-1)$ over the unit cost of producing and shipping the good, $w_j\tau_{ij}/z$. The equilibrium of the model retains many of the desirable properties of the benchmark quantitative framework for considering bilateral trade flows develop by Eaton and Kortum (2002) and particularly the gravity structure. It also allows for endogenous decision of exporting and non-exporting of firms as in Melitz (2003).

How can this additional feature of endogenous market penetration costs help the model to address facts on exporters? The following version of the proposition proved in Arkolakis (2010) computes the responses of firm's sales in a trade liberalization episode:

9. [Elasticity of trade flows and firm size]
The partial elasticity of a firm's sales in market j with respect to variable trade costs, $\varepsilon_{ij}(z) = \left|\partial \ln t_{ij}(z) / \partial \ln \tau_{ij}\right|$, is decreasing with firm productivity, z, i.e. $d\varepsilon_{ij}(z)/dz < 0$ for all $z \geq z^*_{ij}$.

Proof. Compute the partial elasticity of trade flows $t_{ij}(z)$ with respect to a change in τ_{ij}, namely $\left|\partial \ln t_{ij}(z) / \partial \ln \tau_{ij}\right| = |\zeta(z)| \times \left|\partial \ln z^*_{ij} / \partial \ln \tau_{ij}\right|$, where

$$\zeta(z) = \underbrace{(\sigma-1)}_{\substack{\text{intensive margin}\\ \text{of per-consumer}\\ \text{sales elasticity}}} + \underbrace{\frac{\sigma-1}{\beta}\left[\left(\frac{z}{z^*_{ij}}\right)^{(\sigma-1)/\beta} - 1\right]^{-1}}_{\substack{\text{extensive margin of}\\ \text{consumers elasticity}}}.$$

Notice that $\zeta(z) \geq 0$ for $z \geq z^*_{ij}$. $\zeta(z)$ is also decreasing in z and thus decreasing in initial export sales. In fact, as $\beta \to 0$ then $\zeta(z) \to (\sigma - 1)$ for all $z \geq z_{ij}^*$. □

The proposition implies that trade liberalization benefits relatively more the smaller exporters in a market. The parameter β governs both the heterogeneity of exporters cross-sectional sales and also the heterogeneity of the growth rates of sales after a trade liberalization.

8.3 Extension III: Multiproduct firms

We now turn to an extension of the basic CES setup that can accomodate multiproduct firms. This extension is suggested by Arkolakis and Muendler (2010) and is modeling the idea of "core-competency" within the standard heterogeneous firms setup of Melitz (2003).

A conventional "variety" offered by a firm ω from source country i to destination j is the product composite

$$x_{ij}(\omega) \equiv \left(\sum_{g=1}^{G_{ij}(\omega)} x_{ijg}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where $G_{ij}(\omega)$ is the number of products that firm ω sells in country d and $x_{ijg}(\omega)$ is the quantity of product g that consumers consume. The consumer's utility at destination j is a CES aggregation over these bundles

$$U_j = \left(\sum_{i=1}^{N} \int_{\omega \in \boldsymbol{\Omega}_{ij}} x_{ij}(\omega)^{\frac{\sigma-1}{\sigma}} \, \mathrm{d}\omega \right)^{\frac{\sigma}{\sigma-1}} \quad \text{for } \sigma > 1, \tag{8.2}$$

where $\boldsymbol{\Omega}_{ij}$ is the set of firms that ship from source country i to destination j. For simplicity we assume that the elasticity of substitution across a firm's products is the same as the elasticity of substitution between varieties of different firms.[3]

The consumer's first-order conditions of utility maximization imply a product demand

$$x_{ijg}(\omega) = \frac{\left(p_{ijg}(\omega)\right)^{-\sigma}}{P_j^{1-\sigma}} X_j, \tag{8.3}$$

where p_{ijg} is the price of variety ω product g in market j and we denote by X_j the total spending of consumers in country j. The corresponding price index is defined as

$$P_j \equiv \left[\sum_{k=1}^{N} \int_{\omega \in \mathbf{\Omega}_{kj}} \sum_{g=1}^{G_{kj}(\omega)} p_{kjg}(\omega)^{-(\sigma-1)} \, d\omega \right]^{-\frac{1}{\sigma-1}}. \tag{8.4}$$

A firm of type z chooses the number of products $G_{ij}(z)$ to sell to a given market j. The firm makes each product $g \in \{1, 2, \ldots, G_{ij}(z)\}$ with a linear production technology, employing local labor with efficiency z_g. When exported, a product incurs a standard iceberg trade cost so that $\tau_{ij} > 1$ units must be shipped from i for one unit to arrive at destination j. We normalize $\tau_{ii} = 1$ for domestic sales. Note that this iceberg trade cost is common to all firms and to all firm-products shipping from i to j.

Without loss of generality we order each firm's products in terms of their efficiency so that $z_1 \geq z_2 \geq \ldots \geq z_{G_{ij}}$. A firm will enter export market j with the most efficient product first and then expand its scope moving up the marginal-cost ladder product by product. Under this convention we write the efficiency of the g-th product of a firm z as

$$z_g \equiv \frac{z}{h(g)} \qquad \text{with} \quad h'(g) > 0. \tag{8.5}$$

We normalize $h(1) = 1$ so that $z_1 = z$. We think of the function $h(g) : [0, +\infty) \to [1, +\infty)$

[3] Arkolakis and Muendler (2010) generalize the model to consumer preferences with two nests. The inner nest contains the products of a firm, which are substitutes with an elasticity of ε. The outer nest aggregates those firm-level product lines over firms and source countries, where the product lines are substitutes with a different elasticity $\sigma \neq \varepsilon$. general case of $\varepsilon \neq \sigma$ generates similar predictions at the firm-level and at the aggregate bilateral country level.

as a continuous and differentiable function but we will consider its values at discrete points $g = 1, 2, \ldots, G_{ij}$ as appropriate.

Related to the marginal-cost schedule $h(g)$ we define firm z's product efficiency index as

$$H(G_{ij}) \equiv \left(\sum_{g=1}^{G_{ij}} h(g)^{-(\sigma-1)} \right)^{-\frac{1}{\sigma-1}}. \tag{8.6}$$

This efficiency index will play an important role in the firm's optimality conditions for scope choice.

As the firm widens its exporter scope, it also faces a product-destination specific incremental local entry cost $f_{ij}(g)$ that is zero at zero scope and strictly positive otherwise:

$$f_{ij}(0) = 0 \quad \text{and} \quad f_{ij}(g) > 0 \quad \text{for all } g = 1, 2, \ldots, G_{ij}, \tag{8.7}$$

where $f_{ij}(g)$ is a continuous function in $[1, +\infty)$.

The incremental local entry cost $f_{ij}(g)$ accommodates fixed costs of marketing (e.g. with $0 < f_{ii}(g) < f_{ij}(g)$). In a market, the incremental local entry costs $f_{ij}(g)$ may increase or decrease with exporter scope. But a firm's local entry costs

$$F_{ij}\left(G_{ij}\right) = \sum_{g=1}^{G_{ij}} f_{ij}(g)$$

necessarily increase with exporter scope G_{ij} in country j because $f_{ij}(g) > 0$. We assume that the incremental local entry costs $f_{ij}(g)$ are paid in terms of importer (destination country) wages so that $F_{ij}(G_{ij})$ is homogeneous of degree one in w_j. Combined with the preceding varying firm-product efficiencies, this local entry cost structure allows us to endogenize the exporter scope choice at each destination j.

A firm with a productivity z from country i faces the following optimization problem for selling to destination market j

$$\pi_{ij}(z) = \max_{G_{ij}, p_{ijg}} \sum_{g=1}^{G_{ij}} \left(p_{ijg} - \tau_{ij} \, \frac{w_i}{z/h(g)} \right) \frac{(p_{ijg})^{-\sigma}}{P_j^{1-\sigma}} X_j \, - F_{ij}(G_{ij}).$$

The firm's first-order conditions with respect to individual prices p_{ijg} imply product prices

$$p_{ijg}(z) = \bar{m} \, \tau_{ij} \, w_i \, h(g)/z \tag{8.8}$$

with an identical markup over marginal cost $\bar{m} \equiv \sigma/(\sigma-1) > 1$ for $\sigma > 1$. A firm's choice of optimal prices implies optimal product sales for product g

$$p_{ijg}(z) x_{ijg}(z) = \left(\frac{P_j}{\bar{m} \, \tau_{ij} \, w_i} \, \frac{z}{h(g)} \right)^{\sigma-1} X_j. \tag{8.9}$$

Summing (8.9) over the firm's products at destination j, firm z's optimal total exports to destination j are

$$t_{ij}(z) = \sum_{g=1}^{G_{ij}(z)} p_{ijg}(z) \, x_{ijg}(z) = \left(\frac{P_j}{\bar{m} \, \tau_{ij} \, w_i} \, z \right)^{\sigma-1} X_j \, H(G_{ij}(z))^{-(\sigma-1)}, \tag{8.10}$$

where $H(G_{ij})$ is a firm's product efficiency index from (8.6). Expression (8.10) reveals that firm sales in country j are strictly increasing in productivity z given that the term $H(G_{ij}(z))^{-(\sigma-1)}$ weakly increases in $G_{ij}(z)$ and $G_{ij}(z)$ weakly increasing in z.

Given constant markups over marginal cost, profits at a destination j for a firm z selling G_{ij} are

$$\pi_{ij}(z) = \left(\frac{P_j}{\bar{m} \, \tau_{ij} \, w_i} \, z \right)^{\sigma-1} \frac{X_j}{\sigma} H\left(G_{ij}\right)^{-(\sigma-1)} \, - F_{ij}\left(G_{ij}\right).$$

The following assumption is required for the firm optimization to be well defined:

$$\tilde{f}'_{ij}(G) > 0 \tag{8.11}$$

where $\tilde{f}_{ij}(G) \equiv f_{ij}(G) \, h(G)^{\sigma-1}$

Under this assumption, the optimal choice for $G_{ij}(z)$ is the largest $G \in \{0, 1, \ldots\}$ such that operating profits from that product equal (or still exceed) the incremental local entry costs:

$$\left(\frac{P_j}{\bar{m}}\,\tau_{ij}\,w_i\,\frac{z}{h(G)}\right)^{\sigma-1}\frac{X_j}{\sigma} \geq f_{ij}(G) \quad \Longleftrightarrow$$
$$\pi_{ij}^{g=1}(z) \equiv \left(\frac{P_j\,z}{\bar{m}\,\tau_{ij}\,w_i}\right)^{\sigma-1}\frac{X_j}{\sigma} \geq f_{ij}(G)\,h(G)^{\sigma-1} \equiv \tilde{f}_{ij}(G). \tag{8.12}$$

Operating profits from the core product are $\pi_{ij}^{g=1}(z)$, and operating profits from each additional product g are $\pi_{ij}^{g=1}(z)/h(g)^{\sigma-1}$.

Assumption 8.11 is comparable to a second-order condition (for perfectly divisible scope in the continuum version of the model, Assumption 8.11 is equivalent to the second order condition). When Assumption 8.11 holds we will say that a firm faces *overall diseconomies of scope*.

We can express the condition for optimal scope more intuitively and evaluate the optimal scope of different firms. Firm z exports from i to j iff $\pi_{ij}(z) \geq 0$. At the break-even point $\pi_{ij}(z) = 0$, the firm is indifferent between selling its first product to market j and remaining absent. Equivalently, reformulating the break-even condition and using the above expression for minimum profitable scope, the productivity threshold z_{ij}^* for exporting from i to j is given by

$$\left(z_{ij}^*\right)^{\sigma-1} \equiv \frac{\sigma f_{ij}(1)}{X_j}\left(\frac{\bar{m}\,\tau_{ij}\,w_i}{P_j}\right)^{\sigma-1}. \tag{8.13}$$

In general, using (8.13), we can define the productivity threshold $z_{ij}^{*,G}$ such that firms with $z \geq z_{ij}^{*,G}$ sell at least G_{ij} products as

$$\left(z_{ij}^{*,G}\right)^{\sigma-1} = \frac{\tilde{f}_{ij}(G)}{f_{ij}(1)}\left(z_{ij}^*\right)^{\sigma-1}, \tag{8.14}$$

under the convention that $z^*_{ij} \equiv z^{*,1}_{ij}$. Note that if Assumption 8.11 holds then $z^*_{ij} < z^{*,2}_{ij} < z^{*,3}_{ij} < \ldots$ so that more productive firms introduce more products in a given market. So $G_{ij}(z)$ is a step-function that weakly increases in z.

Using the above definitions, we can rewrite individual product sales (8.9) and total sales (8.10) as

$$\begin{aligned} p_{ijg}(z)x_{ijg}(z) &= \sigma f_{ij}(1) \left(\frac{z}{z^*_{ij}}\right)^{\sigma-1} h(g)^{-(\sigma-1)} \\ &= \sigma \tilde{f}_{ij}\left[G_{ij}(z)\right] \left(\frac{z}{z^{*,G}_{ij}}\right)^{\sigma-1} h(g)^{-(\sigma-1)} \end{aligned} \tag{8.15}$$

and

$$t_{ij}(z) = \sigma f_{ij}(1) \left(\frac{z}{z^*_{ij}}\right)^{\sigma-1} H\left[G_{ij}(z)\right]^{-(\sigma-1)}. \tag{8.16}$$

The following proposition summarizes the findings.

10. If Assumption 8.11 holds, then for all $i, j \in \{1, \ldots, N\}$

- exporter scope $G_{ij}(z)$ is positive and weakly increases in z for $z \geq z^*_{ij}$;
- total firm exports $t_{ij}(z)$ are positive and strictly increase in z for $z \geq z^*_{ij}$.

Proof. The first statement follows directly from the discussion above. The second statement follows because $H(G_{ij}(z))^{-(\sigma-1)}$ strictly increases in $G_{ij}(z)$ and $G_{ij}(z)$ weakly increases in z so that $t_{ij}(z)$ strictly increases in z by (8.16). □

There are two key differences to the Melitz (2003) setup. The first is the term $H(G_{ij}(z))^{-(\sigma-1)}$ that reflects multi-product choice within the firm. Adding new products make this term higher, but with core-competency these new products are less and less important for overall sales. The second difference with the Melitz setup is the fixed cost term $F_{ij}\left(G_{ij}\right)$ that jointly with H determines the products optimization. These two features properly estimated from the data can be used to evaluate the prediction of this setup for a number of facts on multi-product exporters. We will come back to this point when we talk about the estimation of firm-level models.

8.3.1 Gravity and Welfare

The market shares in this model are given by

$$\lambda_{ij} = \frac{N_i A_i (w_i \tau_{ij})^{-\theta} f_{ij}(1)^{-\tilde{\theta}} \bar{F}_{ij}}{\sum_k N_k A_k (w_k \tau_{kj})^{-\theta} f_{kj}(1)^{-\tilde{\theta}} \bar{F}_{kj}}$$

where $f_{ij}(1)^{-\tilde{\theta}} \bar{F}_{ij} = \sum_{G=1}^{\infty} f_{ij}(G)^{-(\tilde{\theta}-1)} h(G)^{-\theta}$ and $\tilde{\theta} = \frac{\theta}{\sigma-1}$. The key new insight is that changes in the entry cost will have a different effect on overall trade than in the Melitz (2003) setup insofar they affect the entry costs for different products differently. Conditional on overall trade flows though, the welfare gains from trade are given by an expression that is similar to the Melitz (2003) setup. Thus, the difference is in the counterfactual predictions with respect to changes in trade costs.

8.4 Extension IV: General Symmetric Separable Utility Function

8.4.1 Monopolistic Competition with Homogeneous Firms (Krugman 79)

We now retain the monopolistic competition structure and all the notation from the previous section but consider a general symmetric separable utility function as in Krugman (1979).

8.4.1.1 Consumer's problem

The problem of the representative consumer from country j is

$$\max \left(\sum_{i=1}^{N} \int_{\Omega_i} u\left(x_j\left(\omega\right)\right) d\omega \right),$$

$$\text{s.t. } \sum_{i=1}^{N} \int_{\Omega_i} p_j\left(\omega\right) x_j\left(\omega\right) d\omega = w_j,$$

with $u' > 0$ and $u'' < 0$. We assume a particular regularity condition on the utility function that will allow us to focus on the empirically relevant cases, and in particular

$$-\frac{\partial \ln u'(x)}{\partial \ln x} = -\frac{x_j(\omega)\, u''\left(x_j(\omega)\right)}{u'\left(x_j(\omega)\right)} > 0\,. \tag{8.17}$$

This condition implies

$$u'\left(x_j(\omega)\right) = \lambda_j p_j(\omega) \tag{8.18}$$

where λ_j is the Lagrange multiplier of the consumer in country j. The demand function implied by this solution is given by

$$x_j\left(p_j(\omega)\right) = u'^{-1}\left(\lambda_j p_j(\omega)\right) \tag{8.19}$$

We focus on demand functions that have the choke price property, i.e. there exists a p_j^* so that given λ_j $x_j\left(p_j\right) = 0$ for all $p \geq p_j^*$. It is straightforward to show that this property requires $u'(0) < +\infty$ and that $p_j^* = u'(0)\,/\lambda_j$ and thus

$$x_j\left(p_j, p_j^*\right) = x_j\left(\frac{p_j(\omega)}{p_j^*}\right) = u'^{-1}\left(u'(0)\,\frac{p_j(\omega)}{p_j^*}\right).$$

8.4.1.2 Firm's Problem

We can now incorporate this demand as a constraint to the firm's problem. In particular, a firm from country i with productivity $z = z_i$ choses price in country j to maximize

$$\pi_{ij}(z) = \left(p(\omega) - \tau_{ij}\frac{w_i}{z}\right) u'^{-1}\left(\lambda_j p(\omega)\right) \tag{8.20}$$

The first order condition of this problem is given by (making use of the inverse function theorem and of expressions 8.18 and 8.19)

$$u'^{-1}\left(\lambda_j p_j(\omega)\right) + \left(p(\omega) - \tau_{ij}\frac{w_i}{z}\right) u'^{-1}\left(\lambda_j p(\omega)\right)' \lambda_j = 0 \implies$$
$$p = \frac{1}{\left(1 + \tilde{u}\left(x_j(\omega)\right)\right)} \frac{\tau_{ij} w_i}{z},$$

where

$$\tilde{u}\left(x_j(\omega)\right) = \frac{x_j(\omega)\, u''\left(x_j(\omega)\right)}{u'\left(x_j(\omega)\right)}$$

or alternatively we can simply write the price as a function of the elasticity of demand

$$p = \frac{\frac{d \ln x}{d \ln p}}{\frac{d \ln x}{d \ln p} - 1} \frac{\tau_{ij} w_i}{z}$$

and the markup can be written as

$$\mu\left(\frac{p_j(\omega)}{p_j^*}\right) = \frac{\frac{d \ln x}{d \ln p}}{\frac{d \ln x}{d \ln p} - 1} \tag{8.21}$$

A number of important points can be made for this expression.

First, notice that expression (8.21) depends on the demand elasticity. Thus, how the markup changes with firm size depends on how the demand elasticity changes with size. By simply inspecting the derivative of the markup function it is obvious that if demand is log-convave, i.e. $d \ln^2 x / (d \ln p)^2 < 0$, then markup increase with firm size and the opposite for log-convex demand.

Second, notice that the degree of pass-through depends on how markups change with marginal cost changes, for different levels of demand. This effectively requires taking one more derivative of the markup function and the result ultimately depends also on the third derivative of the demand function.

Finally, notice that the second order conditions of the optimization problem (8.20) are always satisfied if $\frac{d^2 \ln x}{d(\ln p)^2} < 0$. Now define $z_{ij}^* \equiv w_i \tau_{ij} / p_j^*$ then we can finally express

$$x_j = x\left(\frac{z}{z_{ij}^*}\right),$$

and

$$\mu_j = \mu\left(\frac{z}{z_{ij}^*}\right).$$

In this environment we can write bilateral trade shares as

$$\begin{aligned}
\lambda_{ij} &= \frac{N_i \mu\left(\frac{z_i}{z_{ij}^*}\right) \frac{\tau_{ij} w_i}{z_i} q\left(\frac{z_i}{z_{ij}^*}\right)}{\sum_k N_k \mu\left(\frac{z_k}{z_{kj}^*}\right) \frac{\tau_{kj} w_k}{z_k} q\left(\frac{z_k}{z_{kj}^*}\right)} \\
&= \frac{N_i \frac{\tau_{ij} w_i}{z_i} \mu\left(\frac{z_i}{z_{ij}^*}\right) q\left(\frac{z_i}{z_{ij}^*}\right)}{\sum_k N_k \frac{\tau_{kj} w_k}{z_k} \mu\left(\frac{z_k}{z_{kj}^*}\right) q\left(\frac{z_k}{z_{kj}^*}\right)}
\end{aligned}$$

Define v_{ij} as z/z_{ij}^*

$$\lambda_{ij} = \frac{N_i w_i \tau_{ij} \frac{e^{\mu\left(v_{ij}\right)}}{z_{ij}} q\left(\mu\left(v_{ij}\right) - v_{ij}\right)}{\sum_k N_k w_k \tau_{kj} \left(\frac{e^{\mu\left(v_{kj}\right)}}{z_{kj}} q\left(\mu\left(v_{kj}\right) - v_{kj}\right)\right)}$$

$$\lambda_{jj} = \frac{N_j p_j q\left(\mu\left(v_{jj}\right) - v_{jj}\right)}{\sum_k N_k w_k \tau_{kj} \left(\frac{e^{\mu\left(v_{kj}\right)}}{z_{kj}} q\left(\mu\left(v_{kj}\right) - v_{kj}\right)\right)}$$

Welfare (Neary-Mazrova)

Analytical expression for the welfare gains from trade are challenging to derive in this case. We will instead work in the case of two symmetric countries around the free trade equilibrium. The change in equivalent expenditure, e, required to keep the consumer is written as

$$V\left(N, N^{*}, p, p^{*}, \frac{I}{Y}\right) = f\left[Nu\left(\frac{Np}{(Np+N^{*}p^{*})}\frac{I}{Npe}\right) + N^{*}u\left(\frac{N^{*}p^{*}}{(Np+N^{*}p^{*})}\frac{I}{N^{*}p^{*}e}\right)\right]$$
$$= f\left[(N+N^{*})\,u\left(\frac{I}{(Np+N^{*}p^{*})\,e}\right)\right]$$

where f is some function p is the price of the domestic good and p^{*} the price of the foreign good and the two parts inside the indirect utility function represent the relative utility obtain from domestic and foreign goods. We can set this expression equal to a constant, the targeted level of utility, totally differentiate, divide by $N+N^{*}$ and solve for the equivalent expenditure to obtain[4]

$$\hat{N}\frac{N}{N+N^{*}} + \frac{Np}{Np+N^{*}p^{*}}\left[-\frac{u'\left(\frac{I}{(Np+N^{*}p^{*})Y}\right)\frac{I}{(Np+N^{*}p^{*})e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}\left(\hat{N}+\hat{p}\right)\right] +$$
$$\hat{N}^{*}\frac{N^{*}}{N+N^{*}} + \frac{N^{*}p^{*}}{Np+N^{*}p^{*}}\left[-\frac{u'\left(\frac{I}{(Np+N^{*}p^{*})e}\right)\frac{I}{(Np+N^{*}p^{*})e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}\left(\hat{N}^{*}+\hat{p}^{*}\right)\right] = \frac{u'\left(\frac{I}{(Np+N^{*}p^{*})e}\right)\frac{I}{(Np+N^{*}p^{*})e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}\hat{e}$$

$$\hat{N}\left(\frac{\lambda_N+\lambda_Y\xi}{\xi}\right) + \hat{N}^{*}\left(\frac{(1-\lambda_N)+(1-\lambda_Y)\,\xi}{\xi}\right) - \left\{\lambda_Y\left[\hat{p}\right] + (1-\lambda_Y)\left[\hat{p}^{*}\right]\right\} = \hat{e} \tag{8.22}$$

where

$$\xi = \frac{u'\left(\frac{I}{N^{*}p^{*}e}\right)\frac{I}{N^{*}p^{*}e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}$$

and

$$\lambda_Y = \frac{Np}{Np+N^{*}p^{*}}$$

4

$$\hat{N}N + \frac{Np}{Np+N^{*}p^{*}}\left[-(N+N^{*})\frac{u'\left(\frac{\lambda I}{(Np+N^{*}p^{*})e}\right)\frac{I}{(Np+N^{*}p^{*})e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}\left(\hat{N}+\hat{p}+\hat{e}\right)\right] +$$
$$\hat{N}^{*}N^{*} + \frac{N^{*}p^{*}}{Np+N^{*}p^{*}}\left[-(N+N^{*})\frac{u'\left(\frac{I}{N^{*}p^{*}e}\right)\frac{I}{N^{*}p^{*}e}}{u\left(\frac{I}{(Np+N^{*}p^{*})e}\right)}\left(\hat{N}^{*}+\hat{p}^{*}+\hat{e}\right)\right] = 0$$

Can we express this as a function of trade. Notice that domestic trade shares are given by

$$\lambda = \frac{Npq\,(p)}{\left(Npq\,(p) + N^* p^* q^*\,(p^*)\right)} \implies$$
$$\widehat{\lambda} = \left(\hat{N} + \hat{p} + \varepsilon\hat{p}\right) - \left[\left(\hat{N} + \hat{p} + \varepsilon\hat{p}\right)\lambda + \left(\hat{N}^* + \hat{p}^* + \varepsilon^*\hat{p}^*\right)(1-\lambda)\right] \implies$$
$$\widehat{\lambda} = \left(\hat{N} + \hat{p}\,(1+\varepsilon)\right)(1-\lambda) - \left(\hat{N}^* + \hat{p}^*\,(1+\varepsilon^*)\right)(1-\lambda)$$

where $\varepsilon = q'p/q$. Around free trade and symmetry $\lambda = 1-\lambda$, $\varepsilon = \varepsilon^*$

$$\widehat{\lambda} - \left(\hat{N} - \hat{N}^*\right)(1-\lambda) = (1-\lambda)(1+\varepsilon)\left(\hat{p} - \hat{p}^*\right) \implies$$
$$\frac{\widehat{\lambda}}{(1-\lambda)(1+\varepsilon)} - \frac{\left(\hat{N} - \hat{N}^*\right)}{(1+\varepsilon)} = \left(\hat{p} - \hat{p}^*\right)$$

and replacing in (8.22) (using also that around free trade $\lambda_Y = 1 - \lambda_Y$)

$$\begin{aligned}\hat{e} &= \left(\hat{N} + \hat{N}^*\right)\left(\frac{\lambda_N + \lambda_Y\xi}{\xi}\right) - \left\{\frac{(1-\lambda_Y)}{(1-\lambda)}\frac{\widehat{\lambda}}{(1+\varepsilon)} - \frac{(1-\lambda_Y)\left(\hat{N} - \hat{N}^*\right)}{(1+\varepsilon)}\right\}\\ &= \left(\hat{N} + \hat{N}^*\right)\left(\frac{\lambda_N + \lambda_Y\xi}{\xi} + \frac{1-\lambda_Y}{1+\varepsilon}\right) + \frac{\widehat{\lambda}}{(1+\varepsilon)} - \hat{p}\end{aligned}$$

and under initial free trade and symmetry it is also $\lambda_Y = \lambda$.

8.4.1.3 Monopolistic Competition with Heterogeneous Firms (Arkolakis-Costinot-Donaldson-Rodriguez Clare)

We now retain the monopolistic competition structure with separable utility function but we consider heterogeneous firms following Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012). The analysis here covers utility functions considered by Behrens and Murata (2009), Behrens, Mion, Murata, and Sudekum (2009), Saure (2009), Simonovska (2009), Dhingra and Morrow (2012) and Zhelobodko, Kokovin, Parenti, and Thisse (2011).

Demand can again be written as

$$x_{ij}(\omega) = u'^{-1}\left(\lambda_j p_{ij}(\omega)\right) \tag{8.23}$$

where λ_j is the Lagrange multiplier of the consumer. As long as $u'(0) < \infty$ we can define a "choke-up" price $p_j^* = u'(0)/\lambda_j$ so that if $p_{ij}(\omega) = p_{ij}^*$, $x_{ij}(p) = 0$. Given that the distribution we use is continuous and with unbounded support there will be always a firm from each i offering a price low enough to sell to all the markets. We maintain the restrictions made in Section 8.4.1.

8.4.1.4 Firm Problem

The firm problem is the same as in the homogeneity case. We have the price choice of the firm to be $p_{ij}(z) = \mu\left(p_{ij}(z)\right)\frac{\tau_{ij}w_i}{z}$ as indicated above. It can be shown that given the assumptions for u the markup is increasing in z and 0 for $z = z_{ij}^*$ where

$$z_{ij}^* = \frac{\tau_{ij}w_i}{p_j^*}$$

so that price can be expressed as

$$p_{ij}(z) = \mu\left(z/z_{ij}^*\right)\frac{\tau_{ij}w_i}{z}. \tag{8.24}$$

For the cross-section of firms we can offer a characterization of how the markup changes with changes in productivity using the properties discussed above. In particular, when demand is log-concave, $d\ln^2 x/\left(d\ln p\right)^2 < 0$, markups are increase on firm relative size.

Since all the papers discussed above consider the case of the log-concave demand we will proceed under this assumption as our benchmark in our analysis below.

8.4.1.5 Gravity

Using the expression for firm demand, equation (8.23), and firm prices (8.24), firms sales can now be written as

$$t_{ij}(z) = \mu\left(z/z_{ij}^*\right)\frac{\tau_{ij}w_i}{z}u'^{-1}\left(\mu\left(z/z_{ij}^*\right)z/z_{ij}^*\right). \tag{8.25}$$

Using this expression we can aggregrate across firms to compute average sales of firms from i in country j, given by

$$\bar{X}_{ij} = \int_{z_{ij}^*}\mu\left(z/z_{ij}^*\right)\frac{\tau_{ij}w_i}{z}u'^{-1}\left(\mu\left(z/z_{ij}^*\right)z_{ij}^*/z\right)\frac{\left(z_{ij}^*\right)^{\theta}}{z^{\theta+1}}dz \implies$$

$$= \int_{z_{ij}^*}\mu\left(z/z_{ij}^*\right)\frac{\tau_{ij}w_i}{z_{ij}^*}\frac{z_{ij}^*}{z}u'^{-1}\left(\mu\left(z/z_{ij}^*\right)z_{ij}^*/z\right)\frac{\left(z_{ij}^*\right)^{\theta}}{z^{\theta+1}}dz$$

and using a standard change of variables argument and the definition of z_{ij}^*,

$$\bar{X}_{ij} = p_j^*\int_1\frac{1}{\nu}u'^{-1}\left(\frac{\mu(\nu)}{\nu}\right)\left(\frac{1}{\nu}\right)^{\theta+1}d\nu\ , \tag{8.26}$$

i.e. average sales are source independent.

Therefore, trade shares are given by

$$\lambda_{ij} = \frac{N_i\frac{A_i}{\left(z_{ij}^*\right)^{\theta}}\bar{X}_{ij}}{\sum_k N_k\frac{A_k}{\left(z_{kj}^*\right)^{\theta}}\bar{X}_{kj}}$$

$$= \frac{N_i\left(w_i\tau_{ij}\right)^{-\theta}}{\sum_k N_k\left(w_k\tau_{kj}\right)^{-\theta}} \tag{8.27}$$

the standard formula for bilateral trade shares. N_i is the measure of entrants. Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012) that this number is independent of trade in this model, if there is a free entry condition, so in the interest of space, we will assume that $dN_i = 0$ in the derivations below.

8.4.1.6 Welfare

Because the demand is non-homothetic to characterize welfare we need to explicitly solve for the expenditure function rather than simply computing the real wage. Let $e_j \equiv e(\boldsymbol{p}_j, u_j)$ denote the expenditure function of a representative consumer in country j facing a vector of prices $\boldsymbol{p}_j$ and let u_j be the utility level of such a consumer at the initial equilibrium. By Shephard's lemma, we know that $de_j/dp_{\omega,j} = q(p_{\omega,j}, p_j^*, w_j)$ for all $\omega \in \Omega$. Since all price changes associated with a change from XXXXXX are infinitesimal, we can therefore express the associated change in expenditure as[5]

$$de_j = \int_{\omega \in \Omega} \left[q_{\omega,j} dp_{\omega,j} \right] d\omega,$$

where $dp_{\omega,j}$ is the change in the price of good ω in country j caused by the change from XXX. The previous expression can be rearranged in logs as

$$de_j = \int_{\omega \in \Omega} \left[\lambda(p_{\omega,j}, p_j^*, w_j) d \ln p_{\omega,j} \right] d\omega,$$

where $\lambda(p_{\omega,j}, p_j^*, w_j) \equiv p_{\omega,j} q_{\omega,j}/e_j$ is the share of expenditure on good ω in country j in the initial equilibrium. Using equation (8.25) and the fact that firms from country i only sell in country j if $z \geq z_{ij}^*$, we obtain

$$d \ln e_j = \sum_i \int_{z_{ij}^*}^{\infty} \lambda_{ij}(z) \left(d \ln w_i \tau_{ij} + d \ln \mu(z/z_{ij}^*) \right) dG_i(z)$$

[5] In principle, price changes may not be infinitesimal because of the creation of "new" goods or the destruction of "old" ones. This may happen for two reasons: (i) a change in the number of entrants N or (ii) a change in the productivity cut-off z^*. Since the number of entrants is independent of trade costs, as argued above, (i) is never an issue. Since the price of goods at the productivity cut-off is equal to the choke price, (ii) is never an issue either. This would not be true under CES utility functions. In this case, changes in productivity cut-offs are associated with non-infinitesimal changes in prices since goods at the margin go from a finite (selling) price to an (infinite) reservation price, or vice versa.

where

$$\lambda_i(z) = \frac{N_i \mu\left(z/z_{ij}^*\right) \frac{w_i \tau_{ij}}{z} q_{ij}\left(z/z_{ij}^*\right)}{\sum_i \int N_{i'} \mu\left(z/z_{i'j}^*\right) \frac{w_{i'} \tau_{i'j}}{z} q_{i'j}\left(z/z_{i'j}^*\right) dG_i(z)}$$

Combining equations (**??**) and (**??**) and equation (8.27), we obtain, after simplifications,XXXXX

$$d\ln e_j = \sum_i \int_{z_{ij}^*}^{\infty} \lambda_{ij} d\ln w_i \tau_{ij} - \sum_i \int_{z_{ij}^*}^{\infty} \frac{N_i \mu\left(z/z_{ij}^*\right) \frac{w_i \tau_{ij}}{z} q_{ij}\left(z/z_{ij}^*\right)}{\sum_i \int N_{i'} \mu\left(z/z_{i'j}^*\right) \frac{w_{i'} \tau_{i'j}}{z} q_{i'j}\left(z/z_{i'j}^*\right) dG_i(z)} \frac{d\ln \mu(z/z_{ij}^*)}{d\left(z_{ij}^*\right)} dz_{ij}^* dG_i(z)$$

or

$$d\ln e_j = \sum_i \int_{z_{ij}^*}^{\infty} \lambda_{ij} d\ln w_i \tau_{ij} - \sum_i \rho d\ln z_{ij}^* dG_i(z)$$

where

$$\rho = \frac{d\ln \mu(v)}{d\ln(v)} \frac{N_i \mu(v) \frac{w_i \tau_{ij}}{z} q_{ij}(v) v^{-\theta-1}}{\sum_i \int N_{i'} \mu(v) \frac{w_{i'} \tau_{i'j}}{z} q_{i'j}(v) v^{-\theta-1} dv}$$

is a weighted average of the markup elasticities $\mu'(v)$ across all firms, where $v = z/z_{ij}^*$.XXXXa nd the definition of the productivity cut-off $z_{ij}^* \equiv w_i \tau_{ij}/p_j^*$, we can rearrange the expression above asXXXXXX

Notice that the first term is a standard term and represents gains from trade from marginal costs reductions, but movements in markups have direct effects (negative impact to welfare from exporters raising markups) and GE implications (potentially positive effects on welfare because domestic producers lower their markups).

Finally, we can use the labor market clearing and the expression for total bilateral sales to obtain and differentiating

$$d\ln e_j = \sum_i \lambda_{ij} d\ln c_{ij} - \rho \sum_i \lambda_{ij} d\ln c_{ij} + \rho d\ln p_j^*$$

$$d\ln p_j^* = \frac{\theta}{1+\theta} \sum_i \lambda_{ij} d\ln\left(w_i \tau_{ij}\right) \tag{8.28}$$

At this point, notice that from the gravity equation, XXXXX is equal to $d\ln \lambda_{jj}/\theta$.

Putting everything together we have

$$d \ln e_j = \sum_i \lambda_{ij} d \ln c_{ij} - \rho \sum_i \lambda_{ij} d \ln c_{ij} + \rho d \ln p_j^*$$

These derivations imply that under standard restrictions on consumer demand and the distribution of firm productivity, gains from trade liberalization are weakly *lower* than those predicted by the models with constant markups considered in ACR.

Chapter 9

Modeling Vertical Production Linkages

In this section we will present some straightforward ways of introducing intermediate inputs into the heterogeneous firms models. We will comment on the different ways in which the production theory developed above can be used.

9.1 Each good is both final and intermediate

In their heterogeneous sectors framework Eaton and Kortum (2002) have used the intermediate inputs structure initially proposed by Krugman and Venables (1995). The idea is that the production of each good requires labor and intermediate inputs, with labor having a constant share ι. Intermediates comprise the full set of goods that are also used as finals and they are combined according to the same CES aggregator. Therefore, the overall price index in country i, P_i (derived in previous sections), becomes the appropriate index of intermediate goods prices in this case. The cost of an input bundle in country i is thus

$$c_i = w_i^{\iota} P_i^{1-\iota} \tag{9.1}$$

The overall changes in the predictions of the model are small, but the main effect is that trade shares are now affected by ι and thus

$$\lambda_{ij} = \frac{A_i \tau_{ij}^{-\theta} \left(w_i^{\iota} P_i^{1-\iota} \right)^{-\theta}}{\sum_{k=1}^{N} A_k \tau_{kj}^{-\theta} \left(w_k^{\iota} P_k^{1-\iota} \right)^{-\theta}} .$$

9.2 Each good has a single specialized intermediate input

Yi (2003) develops a model where endogenous vertical specialization into different stages of production is allowed. The output $y^2(\omega)$ for a final good $\omega \in \Omega$ is produced using input from a uniquely specialized intermediate good $y^1(\omega)$. The corresponding production functions are

$$\begin{aligned} y_i^2(\omega) &= z_i^2(\omega)\, l_i^2(\omega)^{\iota}\, y_i^1(\omega)^{1-\iota} \ , i = 1,2 \\ y_i^1(\omega) &= z_i^1(\omega)\, l_i^1(\omega) \ , i = 1,2 \end{aligned}$$

where the output of each one of the stages can be produced by either countries and $1-\iota$ is the share of intermediates into production. The model is essentially a two stages Dornbusch, Fischer, and Samuelson (1977) model with Perfect Competition in all the markets. The interesting feature of the Yi (2003) model is that the degree of specialization in either stage of production for a given country is endogenous and depends on trade barriers and the comparative advantage of the two countries.[1] When for a given good both stages of the production are performed abroad, trade of that good is more sensitive to trade cost changes. Yi (2003) uses this feature of the model to offer an explanation of the rapid growth of world trade during the past decades.

[1] While in the Eaton and Kortum (2002) intermediates good framework there are two possible production patterns for the good that is sold in a given market (either home or foreign is the producer of the sold good) in the model of Yi (2003) there are 4 for the two stages of a given variety. These are (HH) Home (country) produces stages 1 and 2, (FF) Foreign produces stages 1 and 2, (HF) Home produces stage 1, Foreign produces stage 2 and (FH) Foreign produces stage 1, Home produces stage 2.

The main drawback of his approach is that calibration is constrained by the usage of the Dornbusch, Fischer, and Samuelson (1977) framework. Thus, Yi (2003) can use general monotonic functions for the relative productivity of one of the stages of production between the two countries but not of both. Of course, this setup is very difficult to be generalized in more than two countries.

9.3 Each good uses a continuum of inputs

Arkolakis and Ramanarayanan (2008) propose a different intermediate inputs structure by merging and generalizing the two approaches described above. Goods are produced in two stages with the second stage of production (production of "final goods") using goods produced in the first stage ("intermediate goods"). Production is vertically specialized to the extent that one country uses imported intermediate goods to produce output that is exported. There is a continuum of measure one of goods in the first stage of production, and in the second stage of production. We index both intermediate and final goods by ω, although they are distinct commodities.

Each first-stage intermediate input ω can be produced with a CRS labor only technology given by

$$y_i^1(\omega) = z_i^1(\omega)\, l_i^1(\omega) \ , \tag{9.2}$$

with efficiency denoted by $z_i^1(\omega)$. The technology for producing output of final good ω is:[2]

$$y_2^i(\omega) = z_i^2(\omega)\left(l_i^2\right)^{\iota}\left(\int m_i\left(\omega,\omega'\right)^{\frac{\sigma-1}{\sigma}} d\omega'\right)^{\frac{(1-\iota)\sigma}{\sigma-1}} \ , \tag{9.3}$$

where $m^i(\omega,\omega')$ is the use of intermediate good ω' in the production of final good ω. The parameter σ is the elasticity of substitution between different intermediate inputs.

We use the probabilistic representation of Eaton and Kortum (2002) for good-specific

[2]Unless otherwise noted, integration is over the entire set of goods in the relevant stage of production.

efficiencies. For each country i and stage s, z_s^i in (9.2) and (9.3) is drawn from a Fréchet distribution characterized by the cumulative distribution function

$$F_i^s(z) = e^{-A_i^s z^{-\theta}},$$

for $s = 1, 2$ and $i = 1, 2$, where $A_i^s > 0$ and $\theta > 1$. Efficiency draws are independent across goods, stages, and countries. The probability that a particular stage-s good ω can be produced in country i with efficiency less than or equal to z_i^s is given by $F_i^s(z_i^s)$. Since draws are independent across the continuum of goods, $F_i^s(z_i^s)$ also denotes the fraction of stage-s goods that country i is able to produce with efficiency at most z_i^s.

Following Eaton and Kortum (2002), it is straightforward to show that the distribution of prices of stage-1 goods that country i offers to country j equals

$$G_{ij}^s(p) = 1 - e^{-A_s^i \left(q_s^{ij}\right)^{-\theta} p^\theta},$$

where q_s^{ij} is the unit cost of producing and shipping the good. This means that the overall distribution of prices of stage-s goods available in country j is

$$G_s^j(p) = 1 - e^{-\Phi_s^j p^\theta}, \tag{9.4}$$

where

$$\Phi_s^j \equiv \sum_k A_s^k \left(q_s^{kj}\right)^{-\theta}. \tag{9.5}$$

The probability that country j buys a certain good from country i, as Eaton and Kortum (2002) show, equals

$$\lambda_s^{ij} = \frac{A_s^i \left(q_s^{ij}\right)^{-\theta}}{\Phi_s^j}. \tag{9.6}$$

As in Eaton and Kortum (2002), it is also true that, because the distribution of stage-s goods actually purchased by country j from country i is equal to the overall price distribution G_s^j, the fraction λ_s^{ij} of goods purchased from country i also equals to the fraction of country j's total expenditures on stage-s goods that it spends on goods from country i.

The interesting feature of this intermediate inputs structure is that the specialization patterns introduced by Yi (2003) still hold. However, the model is much easier to calibrate given that the function that determined comparative advantage can be easily linked to observable trade shares for each stage of production.

Chapter 10

The gravity estimator

We finally turn to the empirical side of modern international trade and discuss how one would take a gravity equation to the data. We will discuss four different ways of estimating the gravity equation and recovering the underlying structural parameters; all four methods use a regression framework, although they vary in the underlying assumptions and in how well they adhere to the theory behind the gravity equation.

Given the importance of the gravity equation in empirical trade, we should also mention that there exists two excellent and extensive reviews of the empirical gravity location in Baldwin and Taglioni (2006), XXAndersonX, and Head and Mayer (2013).

10.1 A structural gravity equation

For our purposes let us begin by considering any trade model that yields the following **gravity equation**:

$$X_{ij} = K_{ij}\gamma_i\delta_j. \tag{10.1}$$

Models liein the previous chapters XXXXX. Let us suppose in equilibrium that the **goods market clears**:

$$Y_i = \sum_{j \in S} X_{ij},$$

and **trade is balanced**:

$$Y_j = \sum_{i \in S} X_{ij}$$

By combining the gravity equation with the balanced trade condition, we can write the destination fixed effect δ_j as a function of its income Y_j and the origin fixed effects in all other countries:

$$\begin{aligned} Y_j &= \sum_{i \in S} X_{ij} \iff \\ Y_j &= \sum_{i \in S} K_{ij} \gamma_i \delta_j \iff \\ \delta_j &= \frac{Y_j}{\sum_{i \in S} K_{ij} \gamma_i} \end{aligned}$$

Substituting this expression back into the gravity equation yields an expression for bilateral trade flows that depends only on the origin fixed effect:

$$\begin{aligned} X_{ij} &= K_{ij} \gamma_i \delta_j \iff \\ X_{ij} &= \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} Y_j. \end{aligned} \tag{10.2}$$

Substituting equation (10.2) into the goods market clearing condition allows us to write the origin fixed effect γ_i as a function of the origin income Y_i and the origin fixed effects in all other countries:

$$
\begin{aligned}
Y_i &= \sum_{j \in S} X_{ij} \iff \\
Y_i &= \sum_{j \in S} \frac{K_{ij} \gamma_i}{\sum_{j \in S} K_{kj} \gamma_k} Y_j \iff \\
\gamma_i &= \frac{Y_i}{\sum_{j \in S} \frac{K_{ij}}{\sum_{k \in S} K_{kj} \gamma_k} Y_j}.
\end{aligned}
$$

Finally, substituting this expression back into the gravity equation (10.2) allows us to write bilateral trade flows as a function of the (exogenous) bilateral frictions K_{ij}, the income in the origin and destination, and measures of the "bilateral resistance":

$$
\begin{aligned}
X_{ij} &= \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} Y_j \iff \\
X_{ij} &= K_{ij} \times \frac{Y_i}{\left(\sum_{j \in S} \frac{K_{ij}}{\sum_{k \in S} K_{kj} \gamma_k} Y_j \right)} \times \frac{Y_j}{\left(\sum_{k \in S} K_{kj} \gamma_k \right)} \iff \\
X_{ij} &= K_{ij} \times \frac{Y_i}{\left(\sum_{k \in S} K_{ik} \frac{Y_k}{\Pi_k} \right)} \times \frac{Y_j}{\Pi_j},
\end{aligned}
\tag{10.3}
$$

where we define $\Pi_j \equiv \sum_{k \in S} K_{kj} \gamma_k$. Let us call equation (10.3) the **structural gravity equation**. As an aside, note that when $\mathbf{K} \equiv \left\{ K_{ij} \right\}$ is quasi-symmetric, we have shown that $K_i^A \gamma_i = \kappa K_i^B \delta_i$, which implies that:

$$
\begin{aligned}
K_i^A \frac{Y_i}{\left(\sum_{k \in S} K_{ik} \frac{Y_k}{\Pi_k} \right)} &= \kappa K_i^B \frac{Y_i}{\Pi_i} \iff \\
\Pi_i &= \kappa \frac{K_i^B}{K_i^A} \sum_{k \in S} K_{ik} \frac{Y_k}{\Pi_k}.
\end{aligned}
$$

However, for the time being, we will work with an arbitrary $\mathbf{K}$.

10.2 The traditional gravity estimator

Until about a decade ago, almost all estimation procedures based on the gravity equation assumed that trade frictions K_{ij} were a linear function of observed bilateral covariates (e.g. distance, common language, shared border, etc.) $\mathbf{T}_{ij}$, i.e.:

$$K_{ij} = \mathbf{T}_{ij}\beta + \varepsilon_{ij}$$

and estimated β by running the following regression:

$$\ln X_{ij} = \mathbf{T}_{ij}\beta + \ln Y_i + \ln Y_j. \tag{10.4}$$

Call equation (10.4) the **traditional gravity estimator**. Comparing the traditional estimating gravity equation to the structural gravity equation (10.3), it is immediately obvious that the traditional estimating gravity equation is missing (i.e. not controlling for) Π_i or $\sum_{k\in S} K_{ik}\frac{Y_k}{\Pi_k}$. Since we can write the structural gravity equation as:

$$X_{ij} = \frac{K_{ij}\frac{Y_j}{\Pi_j}}{\sum_{k\in S} K_{ik}\frac{Y_k}{\Pi_k}} \times Y_i,$$

we can see that the structural gravity equation implies that the share of trade flows from i to j depends on how large $K_{ij}\frac{Y_j}{\Pi_j}$ is to $K_{ik}\frac{Y_k}{\Pi_k}$ in all other countries. Since $\Pi_j \equiv \sum_{k\in S} K_{kj}\gamma_k$, destinations that are more economically remote (in terms of having lower K_{kj} than average) will tend to have lower Π_j, which will cause country $i \in S$ to export a greater share of its total trade to those destinations. Intuitively, this is because more remote countries will have higher price indices, and hence will be willing to pay more for any given good. This is what Anderson and Van Wincoop (2003) refer to as "multilateral resistance."

Because Π_j will (generically) varies across destinations, the traditional estimating gravity equation will suffer from omitted variable bias. Furthermore, because Π_j depends on the average trade friction between j and the rest of the world, it will be correlated with K_{ij}, which will result in biased estimates of β. This means that you should *never use the traditional estimating gravity equation to estimate trade costs*. Indeed, Baldwin and Taglioni (2006) award papers doing this with the "gold medal error" of estimating gravity equations.

10.3 The fixed effects gravity estimator

An alternative to the traditional estimator is to take logs of the gravity equation (10.1):

$$\ln X_{ij} = \ln K_{ij} + \ln \gamma_i + \ln \delta_j.$$

If we assume that $\ln K_{ij} = \mathbf{T}_{ij}\beta + \varepsilon_{ij}$, this equation becomes:

$$\ln X_{ij} = \mathbf{T}_{ij}\beta + \ln \gamma_i + \ln \delta_j + \varepsilon_{ij}. \tag{10.5}$$

Call equation (10.5) the **fixed effects gravity estimator**. Since $\mathbf{T}_{ij}$ are observed and $\ln \gamma_i$ and $\ln \delta_j$ can be estimated by including dummy variables for each origin country and each destination country (note: there are two dummy variables for country, a point to which we will return below), β can be estimated consistently by applying the fixed effects estimator to equation (10.5) and γ_i and δ_j can be consistently estimated (to scale) from the coefficients on the dummy variables.

Given the estimates from equation (10.5), the fixed effects gravity estimator allows us to recover the multilateral resistance terms (to scale) in the structural gravity equation as follows.By taking exponents of the estimates, we can back out predicted values of the origin fixed effect and the bilateral trade costs:

$$\widetilde{\gamma}_i \equiv \exp\left(\widetilde{\ln \gamma_i}\right)$$
$$\tilde{K}_{ij} \equiv \exp\left(\mathbf{T}_{ij}\tilde{\beta}\right)$$

Since $\Pi_j \equiv \sum_{k \in S} K_{kj} \gamma_k$, we then can construct an estimate of the destination multilateral resistance term:

$$\tilde{\Pi}_j \equiv \sum_{k \in S} \tilde{K}_{kj} \widetilde{\gamma}_k = \sum_{k \in S} \exp\left(\mathbf{T}_{kj} \tilde{\beta} + \ln \tilde{\gamma}_k\right)$$

which then allows to construct the origin multilateral resistance term $\sum_{k \in S} \tilde{K}_{ik} \frac{Y_k}{\tilde{\Pi}_k}$. Note that nowhere in these derivations did we use the estimated destination fixed effect, which suggests that the destination fixed effects are "nuisance parameters," i.e. they are unnecessary (given the equilibrium conditions) to fully derive the gravity equation. This is because, as we saw above, balanced trade implies that the destination fixed effect is pinned down by the origin fixed effects:

$$\delta_j = \frac{Y_j}{\sum_{i \in S} K_{ij} \gamma_i},$$

a restriction that is not made in equation (10.4). This is the major drawback of this estimation procedure: by relying just on the gravity structure of the model, the fixed effects estimator imposes no equilibrium conditions on the estimation, and as such, the resulting estimates will (generically) not ensure that the goods market clears or that trade is balanced. The other major drawback of the fixed effect estimation procedure is that there may be computational difficulties to including so many dummy variables, especially if one is interested in estimating γ_i and δ_j. (However, if one is simply interested in estimating β, there exist new ways of doing so without having to invert the large dependent variable matrix, see **?**).

That being said, the fixed effects gravity estimator is probably the most common estimator of gravity equations today, as it is simple to implement and model consistent (with the caveat above). It is also straightforward to extend the fixed effects gravity estimator to include multiple years (by adding origin-country-year and destination-country-year dummies), multiple industries (by adding origin-country-industry and destination-country-industry dummies), etc.

10.4 The ratio gravity estimator

An alternative approach popularized by XXEKXXX is to consider as the dependent variable the (log) trade *shares* rather than the (log) trade *levels*. From gravity equation (10.1) we have:

$$\frac{X_{ij}}{X_{jj}} = \frac{K_{ij}\gamma_i\delta_j}{K_{jj}\gamma_j\delta_j} = \frac{K_{ij}\gamma_i}{K_{jj}\gamma_j} \implies$$
$$\ln\left(\frac{X_{ij}}{X_{jj}}\right) = \ln\left(\frac{K_{ij}}{K_{jj}}\right) + \ln\gamma_i - \ln\gamma_j.$$

If we assume that $\ln\left(\frac{K_{ij}}{K_{jj}}\right) = \mathbf{T}_{ij}\beta + \varepsilon_{ij}$, then this equation becomes:

$$\ln\left(\frac{X_{ij}}{X_{jj}}\right) = \mathbf{T}_{ij}\beta + \ln\gamma_i - \ln\gamma_j + \varepsilon_{ij}. \tag{10.6}$$

Following Head and Mayer (2013), we call the (10.6) the **ratio gravity estimator**. Because the destination fixed effect is constrained to be the negative of the origin fixed effect, the ratio gravity estimator no longer has any nuisance parameters in the estimation, which makes the estimation easier to implement using dummy variables. However, while the ratio gravity estimator has N (where N is the number of countries) fewer parameters to estimate than the fixed effect estimator, it also has N fewer observations since any time $i = j$ equation (10.6) simplifies to the trivial equation $0 = \varepsilon_{ii}$; hence the degrees of freedom remains unchanged. In addition, as with the fixed effect estimator, the ratio gravity estimator is based only on the gravity equation (10.1), so it too does not impose that the general equilibrium conditions hold.

Furthermore, since unlike the fixed effects estimator, the ratio gravity estimator only identifies the relative trade frictions rather than the absolute trade frictions. That is, taking exponents of $\ln\left(\frac{K_{ij}}{K_{jj}}\right) = \mathbf{T}_{ij}\beta + \varepsilon_{ij}$ yields:

$$\hat{K}_{ij} = \exp\left(\mathbf{T}_{ij}\hat{\beta}\right)\hat{K}_{jj}.$$

This means that we are unable to recover the multilateral resistance term Π_j since:

$$\sum_{k \in S} \exp\left(\mathbf{T}_{kj}\tilde{\beta} + \ln\tilde{\gamma}_k\right) = \frac{1}{\tilde{K}_{jj}} \sum_{k \in S} \tilde{K}_{kj}\hat{\gamma}_k = \frac{\tilde{\Pi}_j}{\tilde{K}_{jj}}.$$

In order to call this expression the multilateral resistance, one would have to assume (as is often done) that $K_{jj} = 1$, i.e. internal trade is costless.

10.5 The general equilibrium gravity estimator

The major disadvantage of the traditional fixed effects estimator is that it treats the origin and destination fixed effects – which capture the general equilibrium effects of the gravity model – as nuisance parameters to be controlled for. We now develop a "general equilibrium estimator" that directly accounts for these general equilibrium effects, which, as we will see, allows the econometrician to exploit the network structure of trade to overcome some common econometric issues.

This estimator can be implemented in changes since in that case the (hatted) origin and destination fixed effects are functions of the entire matrix of (hatted) bilateral trade frictions, i.e. for all $i \in S$ and $j \in S$, we can write $\hat{\gamma}_i\left(\hat{\mathbf{T}}\mu\right)$ and $\hat{\delta}_j\left(\hat{\mathbf{T}}\mu\right)$, where $\hat{\mathbf{T}}\mu$ is an $N \times N$ matrix whose $\langle i, j\rangle$ element is $\hat{T}'_{ij}\mu$.XXXWe need to talk earlier in the paper about the inversionXXX The general equilibrium estimator μ^*_{GE} minimizes the squared deviation from observed (hatted) bilateral trade flows *while accounting for the effect of μ on the equilibrium (hatted) origin and destination fixed effects*:XXXIs this a matrix of coeffficients

$$\mu^*_{GE} \equiv \arg\min_{\mu \in \mathbb{R}^S}\left(\sum_i \sum_j \left(\ln \hat{X}^o_{ij} - \hat{T}'_{ij}\mu - \ln\hat{\gamma}_i\left(\hat{\mathbf{T}}\mu\right) - \ln\hat{\delta}_j\left(\hat{\mathbf{T}}\mu\right)\right)^2\right).$$

By taking first order conditions we can derive an implicit equation for μ^*_{GE}. In principal, the general equilibrium estimator could then be calculated through an iterative procedure or through a non-linear least squares routine as in Anderson and Van Wincoop (2003). However, it turns out that we can do better. Consider the following first order approximations of the log change in the origin and destination fixed effects:

$$\ln \hat{\gamma}_i \left(\hat{\mathbf{T}}\mu\right) \approx \sum_k \sum_l \frac{\partial \ln \gamma_i}{\partial \ln K_{kl}} \hat{T}'_{kl}\mu \text{ and } \ln \hat{\delta}_j \left(\hat{\mathbf{T}}\mu\right) \approx \sum_k \sum_l \frac{\partial \ln \delta_j}{\partial \ln K_{kl}} \hat{T}'_{kl}\mu. \tag{10.7}$$

By taking first order conditions and applying these first order approximations, we can derive a straightforward closed form solution for the general equilibrium estimator (once we turn the $N \times N$ matrices into $N^2 \times 1$ vectors).

First order conditions for μ^*:

$$0 = -2 \sum_i \sum_j \left(\hat{T}_{i\ldots} \right.$$

$$0 = \sum_i \sum_j \left(\hat{T}_{ij} + \ldots \right.$$

$$\sum_i \sum_j \left(\hat{T}_{ij} + \left(\sum_k \sum_l \hat{T}_{kl} \left(\frac{\partial \ln \hat{\gamma}_i}{\partial \ln K_{kl}} + \frac{\partial \ln \hat{\delta}_j}{\partial \ln K_{kl}} \right) \right) \right) \left(\hat{T}'_{ij} + \sum_k \sum_l \left(\frac{\partial \ln \gamma_i}{\partial \ln K_{kl}} + \frac{\partial \ln \delta_j}{\partial \ln K_{kl}} \right) \hat{T}'_{kl} \right) \mu^*_{GE} = \sum_i \sum_j \left(\hat{T}_{ij} + \ldots \right.$$

$$\mu^*_{GE} = \left(\sum_i \sum_j \left(\hat{T}_{ij} + \ldots \right. \right.$$

$$\mu^* = \left(\left((I + D) \ldots \right. \right.$$

where I is the $N^2 \times N^2$ identify matrix. Note that:

$$(I + D)\,\hat{T} = \hat{T} + D\hat{T} = \left\{ \hat{T}'_n + \sum_{m=1}^{N^2} D_{nm} \hat{T}'_m \right\}$$

It was in the paper:

$$\mu^*_{GE} = \left(\sum_i \sum_j \left(\hat{T}_{ij} + \left(\sum_k \sum_l \hat{T}_{kl} \left(\frac{\partial \ln \hat{\gamma}_i}{\partial \ln K_{kl}} + \frac{\partial \ln \hat{\delta}_j}{\partial \ln K_{kl}}\right)\right)\right) \left(\hat{T}'_{ij} + \sum_k \sum_l \left(\frac{\partial \ln \gamma_i}{\partial \ln K_{kl}} + \frac{\partial \ln \delta_j}{\partial \ln K_{kl}}\right) \hat{T}'_{kl}\right)\right)^{-1} \left(\sum_i \sum_j\right.$$

Then the general equilibrium gravity estimator is:

$$\mu^*_{GE} = \left(\left(\mathbf{D}\hat{\mathbf{T}}\right)' \left(\mathbf{D}\hat{\mathbf{T}}\right)\right)^{-1} \left(\mathbf{D}\hat{\mathbf{T}}\right) \hat{\mathbf{y}}. \tag{10.8}$$

where $\hat{\mathbf{T}}$ denotes the $N^2 \times S$ vector whose $\langle i + j(N-1) \rangle$ row is the $1 \times S$ vector $\hat{T}'_{ij}$, $\mathbf{D}$ denote the $N^2 \times N^2$ matrix whose $\langle i + j(N-1), k + l(N-1) \rangle$ element is $\frac{\partial \ln X_{ij}}{\partial \ln K_{kl}}$, and $\hat{\mathbf{y}}$ denote the $N^2 \times 1$ vector whose $\langle i + j(N-1) \rangle$ row is $\ln \hat{X}^o_{ij}$.

Equation (10.8) says that, to a first order, the general equilibrium estimator is the coefficient one gets from of an ordinary squares regression of the observed hatted variables on a "general equilibrium transformed" explanatory variable $\hat{T}^{GE}_{ij}$:

$$\ln \hat{X}^o_{ij} = \left(\hat{T}^{GE}_{ij}\right)' \mu_{GE} + \varepsilon_{ij},$$

where:

$$\hat{T}^{GE}_{ij} \equiv \sum_k \sum_l \frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} \hat{T}_{kl}.$$

Chapter 11

Estimating trade costs

11.1 A note on estimating trade costs

You might be concerned that for the general equilibrium estimator, we had to assume that $K_i^A = K_i^B = 1$. If we did not make this assumption, then from equation (**??**), it is immediately apparent that we would not be able to separately identify $\{\gamma_i\}$ from K_i^A. This is actually a concern for all of the estimators we have considered thus far, as throughout we have (implicitly) assumed that $\mathbf{T}_{ij}$ comprises only observables.

More generally, however, the true trade frictions might depend both on observables and unobservables. Suppose for example that:

$$\ln K_{ij} = \mathbf{T}_{ij}\beta + \ln K_i^A + \ln K_j^B + \varepsilon_{ij},$$

where K_i^A and K_j^B are unobserved. Then the fixed effects estimator (10.5) would become:

$$\ln X_{ij} = \mathbf{T}_{ij}\beta + \ln \gamma_i K_i^A + \ln \delta_j K_j^B + \varepsilon_{ij}.$$

In this case, the fixed effects only estimate the composite of $\ln \gamma_i K_i^A$ and $\ln \delta_j K_j^B$, which prevents us from using the fixed effects estimates to identify the multilateral resistance terms.

Indeed, Eaton and Kortum (2002), in addition to assuming that $K_{jj} = 1$ assume that $K_i^A = 1$ so that $K_{ij} = \left(\mathbf{T}_{ij}\beta\right) K_j^B$. This changes the ratio gravity estimator they use to:

$$\ln\left(\frac{X_{ij}}{X_{jj}}\right) = \mathbf{T}_{ij}\beta + \ln\gamma_i - \ln\gamma_j + \ln K_j^B + \varepsilon_{ij},$$

i.e. the origin and destination fixed effects are no longer the same because of the unobserved destination fixed effect in the trade cost. Waugh (2010), instead, assumes that $K_j^B = 1$ so that $K_{ij} = \left(\mathbf{T}_{ij}\beta\right) K_i^A$, which changes the ratio gravity estimator to:

$$\ln\left(\frac{X_{ij}}{X_{jj}}\right) = \mathbf{T}_{ij}\beta + \ln\gamma_i - \ln\gamma_j + \ln K_i^A + \varepsilon_{ij},$$

i.e. the origin and destination fixed effects are no longer the same because of the unobserved origin fixed effect in the trade cost. Of course, both these assumptions fit the trade data exactly as well; all the assumptions do is change the interpretation of the fixed effects!

A final note on trade costs. Suppose we are willing to assume $K_i^A = K_j^B = 1$ so that trade costs are strictly symmetric and we assume that $K_{ii} = 1$ for all $i \in S$. Then one can identify trade frictions simply from the following expression (known as the Head-Ries index):

$$\sqrt{\frac{X_{ij}X_{ji}}{X_{ii}X_{jj}}} = \sqrt{\frac{\left(K_{ij}\gamma_i\delta_j\right)\left(K_{ji}\gamma_j\delta_i\right)}{\left(K_{ii}\gamma_i\delta_i\right)\left(K_{jj}\gamma_j\delta_j\right)}} = K_{ij}$$

The bottom line is that in all these estimation procedures, in order to recover structural parameters of interest, it is important to place assumptions on the trade costs or to observe the trade costs directly. Because of this, we will be spending some time in the next few lectures on the various methods to estimate the trade costs using methods that do not entirely rely on the observed trade flows.

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In the previous class, we discussed how one can identify a unique (to-scale) set of origin fixed effects, destination fixed effects, and bilateral trade frictions that are consistent with any observed bilateral trade flows as well as the general equilibrium conditions. Recall, however, that the bilateral trade frictions we identified were the composite of both the underlying trade costs and the elasticity of trade flows to those trade costs. Next, we discuss how one can separate the trade costs from the elasticity of trade flows to trade costs using data on differences in prices across space.

11.2 Identifying the elasticity of trade to variable trade costs

As we have in the past, consider any gravity trade model:

$$X_{ij} = K_{ij}\gamma_i\delta_j \tag{11.1}$$

where the generalized labor market clearing condition holds:

$$Y_i = \bar{B}_i\gamma_i^{\alpha},$$

where $\alpha < 0$. As we have seen in previous lectures, the parameter α is very important, as (1) it is necessary to identify the model parameters given the data; (2) it is necessary to calculate the model equilibrium given model parameters; and (3) it plays a crucial role in the calculation of welfare. Hence, we would like to have a way of estimating it.

With a few more assumptions, estimation becomes. Suppose too that the (non-general) labor market clearing condition holds as well, i.e. $Y_i = w_iL_i$, so that:

$$\begin{gathered} w_iL_i = \bar{B}_i\gamma_i^{\alpha} \iff \\ \gamma_i = \left(\frac{L_i}{\bar{B}_i}\right)^{\frac{1}{\alpha}} w_i^{\frac{1}{\alpha}} \end{gathered}$$

Finally, suppose that (1) the marginal cost of production is proportional to w_i and (2) the only bilateral frictions are iceberg trade costs $\{\tau_{ij}\}$, where $\tau_{ii} = 1$ for all $i \in S$ (which implies that the marginal cost of producing a good in i and selling it in j is proportional to

$w_i\tau_{ij}$); (3) prices are proportional to marginal costs; and (4) as in Arkolakis, Costinot, and Rodríguez-Clare (2012), the import demand system is "CES." These assumptions [I think] imply that we can write the bilateral trade cost function as:

$$K_{ij} = \tau_{ij}^{\frac{1}{\alpha}} K_i^A K_j^B. \tag{11.2}$$

Substituting equation (11.2) into the gravity equation (11.1) and taking logs then yields:

$$\ln X_{ij} = \frac{1}{\alpha} \ln \tau_{ij} + \ln \left(K_i^A \gamma_i \right) + \ln \left(K_j^B \delta_j \right). \tag{11.3}$$

If τ_{ij} we observed, we could estimate α by regressing (log) bilateral trade flows equation on the (log) iceberg costs and conditioning on origin and destination fixed effects. Hence, in order to estimate α, we first need to estimate the bilateral iceberg trade costs. Unfortunately, we have seen that the trade flows alone will only allow us to identify the total bilateral trade frictions $\{K_{ij}\}$, so we need to rely additional data to recover the iceberg trade costs. The most popular method of doing so is to rely on price data and the no-arbitrage condition.

11.3 The no arbitrage condition

If prices are proportional to marginal costs and there are iceberg trade costs then for any product $\omega \in \Omega$ produced in any origin $i \in S$ and for any destinations $j \in S$ and $k \in S$:

$$\frac{p_{ij}(\omega)}{p_{ik}(\omega)} = \frac{\tau_{ij}}{\tau_{ik}}.$$

If we assume that $\tau_{ii} = 1$ for all $i \in S$ then setting $k = i$ yields the **no-arbitrage condition**:

$$\frac{p_{ij}(\omega)}{p_{ii}(\omega)} = \tau_{ij}. \tag{11.4}$$

The no-arbitrage condition provides an exceedingly simply and surprisingly powerful way of identifying the iceberg trade costs. The simplicity of the identification is self-evident: if an origin sells a good to itself and sells it to another destination, then the iceberg trade costs is simply equal to the ratio of the destination price to the origin price.

Why is the result surprisingly powerful? It is because no-arbitrage condition should hold regardless of the model. To see this, suppose that the no-arbitrage condition did not hold and instead that $\frac{p_{ij}(\omega)}{p_{ii}(\omega)} > \tau_{ij}$. This should not be an equilibrium, because any self-interested arbitrageur could purchase the good in $i \in S$, resell the good in $j \in S$, and make a profit. Conversely, suppose that $\frac{p_{ij}(\omega)}{p_{ii}(\omega)} < \tau_{ij}$. This implies that whoever was selling the good from i to j ought to have just sold locally. In the first case, money was being left on the table, while in the latter case, money was being thrown away, both of which tend to make us economists nervous.[1]

Despite the simplicity and power of using the no-arbitrage condition to identify the iceberg trade costs, there are three major difficulties in empirically implementing the estimation strategy:

1. The observed prices in both the origin and destination have to be for the same good ω. Observing prices of identical goods is especially difficult for differentiated varieties; for example, prices of t-shirts across locations may vary because of the quality of the t-shirts rather than because of trade costs.

2. Even if the goods are identical, we need to know that the good was produced in $i \in S$ and sold to $j \in S$. If $j \in S$ purchased a good from another location (or produced it locally), there is no reason that the no-arbitrage equation must hold with location i. Note the inherent tension between the first difficulty and this difficulty: if one is

[1]In my job market paper, I argue that it can be the case that $\frac{p_{ij}}{p_{ii}} > \tau_{ij}$ if it is costly for arbitrageurs to discover what the price is in other locations. I believe that such "information frictions" are quantitatively important in real world markets.

able to find a good that is truly identical across locations (e.g. a commodity), there is a high likelihood that it is produced in many locations.

3. The no-arbitrage condition only holds if the price of the good is proportional to the marginal cost of production. This assumption would be violated, for example, if producers had market power and were able to charge variable mark-ups in different destinations. Note in the case of CES, producers do not charge variable mark-ups, but this result is particular to the CES case (and likely unrealistic).

Let us now discuss some of the approaches taken in the trade literature that have attempted (more or less successfully) to navigate these three difficulties.

11.4 Estimating the no-arbitrage conditions

We now consider four methods used to estimate the no-arbitrage conditions.

11.4.1 The Eaton and Kortum (2002) approach

Eaton and Kortum (2002) observe 50 manufactured products across the 19 countries in their data set. They note that if a product $\omega \in \Omega$ is not traded between the two countries, then it must be the case that producers of ω found it more profitable to sell domestically, i.e.:

$$p_i(\omega) > \frac{p_j(\omega)}{\tau_{ij}} \iff \tau_{ij} > \frac{p_j(\omega)}{p_i(\omega)},$$

i.e. the iceberg trade costs exceed the price gap. Conversely, if a product is traded, then the no arbitrage equation holds with equality. These two facts imply that the price ratio of all products is bounded above by the iceberg trade cost.

Since they do not observe which of the 50 manufactured products are actually traded between any pair of countries, they employ a "brute force" method of estimating the trade cost by taking the maximum price ratio observed across all products as their measure of the bilateral iceberg trade costs:

$$\hat{\tau}_{ij}^{EK} \equiv \max_{\omega \in \Omega} \frac{p_j(\omega)}{p_i(\omega)}. \tag{11.5}$$

Equation (11.5) is a valid estimator of the true iceberg trade costs if at least one of the observed products is traded, prices are measured perfectly, the products observed are identical, and prices are proportional to marginal cost.[2]

Using this estimator, Eaton and Kortum (2002) find a trade elasticity (i.e. $\frac{1}{\alpha}$) of roughly eight; i.e. a 10 percent increase in trade costs is associated with an 80% decline in trade flows. More recently, **?** have argued that because it is possible for none of the observed products to have actually been traded, an estimator based on equation (11.5) will be biased downwards. Because observed trade flows can be rationalized equally well with a higher trade elasticity and lower trade costs or a lower trade elasticity and higher trade costs, if the estimated trade costs are biased downwards, the implied elasticity of trade will be biased upwards. They develop a simulated method of moments estimator that corrects this error, and find an elasticity of trade of approximately four, which currently is the standard in the trade literature.

11.4.2 The Donaldson (2012) approach

In Donaldson (2012) (which we will see in detail in a few lectures), the author had a clever solution to the three difficulties mentioned above of estimating the no-arbitrage condition. He found a homogeneous good where the unique location of production was known: salt! As he writes:

"Throughout Northern India, several different types of salt were consumed,

[2] Recognizing that prices likely are measured with error, Eaton and Kortum (2002) actually use the second highest observed price ratio as their preferred estimator of the iceberg trade cost.

each of which was regarded as homogenous and each of which was only capable of being made at one unique location."

In the simplest case, having such a good would allow one to construct bilateral trade costs immediately from the no-arbitrage equation, as $\tau_{ij} = \frac{p_{ij}(\omega)}{p_{ii}(\omega)}$. However, even with "perfect" good for which to apply the no-arbitrage condition, Donaldson (2012) faced two additional difficulties. First, it turned out that he did not observe the price of a variety $\omega \in \Omega$ of salt at the origin. Second, since not every location $i \in S$ produced its own unique variety of salt, at best, he could only apply the no-arbitrage condition to find a subset of the bilateral iceberg trade costs.

To solve both problems, Donaldson (2012) made a parametric assumption that $\ln \tau_{ij} = \mathbf{T}_{ij}\beta + \varepsilon_{ij}$. With this assumption, the no arbitrage condition becomes:

$$\begin{aligned} \tau_{ij} &= \frac{p_{ij}(\omega)}{p_{ii}(\omega)} \iff \\ \ln p_{ij}(\omega) &= \ln p_{ii}(\omega) + \ln \tau_{ij} \iff \\ \ln p_{ij}(\omega) &= \ln p_{ii}(\omega) + \mathbf{T}_{ij}\beta + \varepsilon_{ij}. \end{aligned} \tag{11.6}$$

By including a salt-variety ω fixed effect, β can be estimated using just the observed variation in prices of a particular variety across destinations. Once β is estimated, the trade costs between any origin and destination can be imputed from the parametric assumption. Furthermore, α can be estimated by regressing bilateral trade flows on $\mathbf{T}_{ij}\hat{\beta}$ using the gravity regression in equation (11.3). Donaldson (2012) estimates the elasticity of trade flows for each commodity in his data set separately and finds a mean of roughly four, consistent with **?**.

The Donaldson (2012) approach assumes that salt is traded in perfectly competitive markets, which, because salt is a commodity, seems reasonable.

11.4.3 The Allen (2012) approach

In Allen (2012), I use the spatial dispersion in prices of agricultural commodities (which, unlike Donaldson (2012), were produced in many regions) in order to infer the size of trade costs.

The insight of the approach in this paper is to note that even when two countries do not trade, the no arbitrage condition provides information about the size of the trade costs. The intuition is the same as in the Eaton and Kortum (2002) above: if a particular commodity is not observed to be traded between two locations, then this must mean that the trade cost exceeded the price gap between the two locations, i.e. $X_{ij}(\omega) = 0 \implies \tau_{ij} > \frac{p_j(\omega)}{p_i(\omega)}$, whereas when trade does occur between the two locations, then this must mean that the no-arbitrage equation holds, i.e. $X_{ij}(\omega) > 0 \implies \tau_{ij} = \frac{p_j(\omega)}{p_i(\omega)}$.

Suppose we observe the price of a particular commodity $\omega \in \Omega$ in each location $i \in S$ in each period $t \in \{1, ..., T\}$, i.e. $p_{it}(\omega)$. Furthermore, suppose for any pair of origin $i \in S$ and destination $j \in S$, we observe whether or not trade flows occur in each period $t \in \{1, ..., T\}$, i.e. $\mathbf{1}\{X_{ijt}(\omega) > 0\}$. Finally, suppose that the bilateral trade cost of commodity ω in time t depend on a time invariant bilateral trade cost and an idiosyncratic error that is i.i.d. across time periods:

$$\ln \tau_{ijt}(\omega) = \ln \tau_{ij}(\omega) + \varepsilon_{ijt}(\omega),$$

where $\varepsilon_{ijt}(\omega) \sim N(0, \sigma^2)$.

We can then estimate $\ln \tau_{ij}(\omega)$ using a maximum likelihood routine. The log likelihood function can be written as:

$$l\left(\ln\tau_{ij},\sigma\right)=\sum_{t=1}^{T}(\mathbf{1}\left\{X_{ijt}\left(\omega\right)>0\right\}\ln\phi\left(\frac{1}{\sigma}\left(\ln\frac{p_{jt}\left(\omega\right)}{p_{it}\left(\omega\right)}-\ln\tau_{ij}\right)\right)+,$$
$$\mathbf{1}\left\{X_{ijt}\left(\omega\right)=0\right\}\ln\left(1-\Phi\left(\frac{1}{\sigma}\left(\ln\frac{p_{jt}\left(\omega\right)}{p_{it}\left(\omega\right)}-\ln\tau_{ij}\right)\right)\right))$$

which bears a very close resemblance to a Tobit estimator. Using this estimator to identify trade costs and then regressing trade flows on these trade costs to identify the trade elasticity yields an elasticity of a little bit more than two in the context of agricultural trade flows between islands in the Philippines.

In the presence of information frictions where positive trade flows merely indicate that the price ratio exceeds the bilateral trade costs (i.e. $X_{ij}\left(\omega\right)>0\implies\tau_{ij}<\frac{p_j(\omega)}{p_i(\omega)}$), the the log likelihood function can be written as:

$$l\left(\ln\tau_{ij},\sigma\right)=\sum_{t=1}^{T}(\mathbf{1}\left\{X_{ijt}\left(\omega\right)>0\right\}\ln\Phi\left(\frac{1}{\sigma}\left(\ln\frac{p_{jt}\left(\omega\right)}{p_{it}\left(\omega\right)}-\ln\tau_{ij}\right)\right)+,$$
$$\mathbf{1}\left\{X_{ijt}\left(\omega\right)=0\right\}\ln\left(1-\Phi\left(\frac{1}{\sigma}\left(\ln\frac{p_{jt}\left(\omega\right)}{p_{it}\left(\omega\right)}-\ln\tau_{ij}\right)\right)\right)).$$

In this case, the log likelihood function is identical to the following Probit regression:

$$\mathbf{1}\left\{X_{ijt}\left(\omega\right)>0\right\}=\beta\ln\frac{p_{jt}\left(\omega\right)}{p_{it}\left(\omega\right)}+\alpha_{ij},$$

where $\beta=\frac{1}{\sigma}$ and $\alpha_{ij}=-\frac{1}{\sigma}\ln\tau_{ij}\left(\omega\right)$. Hence, identifying the iceberg trade cost is straightforward: you regress whether or not trade flows occurred on the observed log price ratio and a constant. The coefficient on the price ratio identifies the variance of the distribution of measurement error in the trade costs and the variance, combined with the constant, identifies the iceberg trade cost. Intuitively, as the measurement error goes to zero, any increase of the log price ratio above the threshold α_{ij} will induce trade with probability one, so that β will approach infinity.

The advantage of this approach relative to Eaton and Kortum (2002) is that it explicitly allows for measurement error in trade costs; the advantage of this approach relative to Donaldson (2012) is that one does not need to observe exactly where a product was produced. The disadvantage relative to both approaches is that requires knowing whether or not trade flows occurred at the product level.

Like in the Donaldson (2012) approach, an assumption in the Allen (2012) approach is that prices are proportional to marginal costs, which because the focus is on agricultural commodities, seems reasonable.

11.4.4 The ? approach

While Allen (2012) and Donaldson (2012) have overcome the problem of variable mark-ups by focusing on goods for which the assumption of perfect competition seems reasonable, **?** take very seriously the possibility of mark-ups.

Prior to discussing how **?** deal with variable mark-ups, let us briefly mention how they overcome the first two problems (of knowing that goods are identical knowing that trade flows actually occurred). They overcome both these problems using extremely detailed price data (collected at the bar code level) and for which they know the exact location for where the good was produced. The bar-code level data assures us that the products are indeed identical, while knowing the location where the good was produced, as in Donaldson (2012) allows us to infer the origin of each product. It should be emphasized that this was a massive data collection process!

The basic intuition for how **?** deal with variable mark-ups is that they show that much of what is needed in order to identify variable mark-ups can be inferred from how shocks to the origin price pass-through to the destination price. In what follows, we consider a much simplified version of their set-up. Suppose that the price differences between an destination $j \in S$ and the origin $i \in S$ depend on both the (exogenous) iceberg trade frictions and the (endogenous) mark-up:

$$\ln p_j - \ln p_i = \ln \tau_{ij} + \ln \mu \left(p_i \tau_{ij} \right), \tag{11.7}$$

where μ is the mark-up over marginal cost, which may depend on the marginal cost $p_i\tau_{ij}$. As in Donaldson (2012), **?** parameterize $\ln \tau_{ij}$ by assuming that it can be written as a function f of a vector of observables

$$\ln \tau_{ij} = \mathbf{T}_{ij}\beta,$$

so that equation (11.7) can be written as:

$$\ln p_j - \ln p_i = \mathbf{T}_{ij}\beta + \ln \mu \left(p_i \exp \left\{ \mathbf{T}_{ij}\beta \right\} \right). \tag{11.8}$$

Fully differentiating equation (11.8) with respect to any element of $\mathbf{T}_{ij}$ yields:

$$\begin{aligned} \frac{d}{d\mathbf{T}_{ij}} \left(\ln p_j - \ln p_i \right) &= \beta + \frac{\mu'(\cdot)}{\mu \left(p_i \tau_{ij} \right)} p_i \tau_{ij} \beta \iff \\ \frac{d}{d\mathbf{T}_{ij}} \left(\ln p_j - \ln p_i \right) &= \rho\beta, \end{aligned} \tag{11.9}$$

where $\rho \equiv \left(1 + \frac{\partial \ln \mu}{\partial c_{ij}} \right)$ which I call the "pass-through rate" captures how a change in the marginal cost passes-through to a change in prices, holding constant the market competitiveness and the demand. Equation (11.9) says that a change in bilateral trade costs will have two effects on the spatial price gap: first, it will increase the marginal cost of selling from i to j; second, it will affect the mark-up.

The key thing to note is that we can identify the pass-through rate ρ_j by seeing how the destination price is affected by changes to the origin price; intuitively, changes to the origin price affect the (endogenously-determined) mark-up just as a change to the bilateral trade cost would. To see this, note that the effect of a change in the origin price on the destination price can be related back to the pass-through rate ρ:

$$\frac{d\ln p_j}{d\ln p_i} = \frac{d\ln\mu\left(p_i\tau_{ij}\right)}{d\ln p_i} = \frac{\mu'\left(\cdot\right)}{\mu\left(p_i\tau_{ij}\right)} p_i\tau_{ij} = \rho - 1$$

That is, if we observed (exogenous) variation the origin price $\ln p_{it}$, a simple two-step estimation procedure allows us to identify β. First, we regress the log of the destination price on the log of the origin price to identify ρ:

$$\ln p_{jt} = (\rho - 1)\ln p_{it} + \varepsilon_{it}^{A}. \tag{11.10}$$

Once ρ is identified from equation (11.10), we then regress the difference in log prices on the observables determining bilateral trade costs to identify β:

$$\ln p_{jt} - \ln p_{it} = \left(\mathbf{T}_{ij}\hat{\rho}\right)\beta + \varepsilon_{it}^{B}. \tag{11.11}$$

In **?**, they also show how to control for spatial differences in market competitiveness and demand, but the key insight of using estimated price pass-through to control for endogenous mark-ups remains.

11.5 Conclusion and next steps

This lecture concludes the the part of the course focusing on methods! In the next three classes, I will be presenting papers that I feel do a good job bringing structural gravity models to the data. Next class, I will present Allen and Arkolakis (2014), where we use a gravity model to ask economic geography questions (we have already seen a little bit of the theory in the paper, but none of the empirics). Following that class, I will present Donaldson (2012), who uses a gravity model to assess the effect of the construction of railroads. Finally, I will present Ahlfeldt, Redding, Sturm, and Wolf (2012), who use a gravity model to assess the reallocation of economic resources resulting from the creation (and destruction) of the Berlin wall.

Chapter 12

Some facts on disaggregated trade flows

12.1 Firm heterogeneity

- Firms appear to have huge differences in sales and measured productivities (Bernard, Eaton, Jensen, and Kortum (2003)–BEJK–)

- In fact, only a tiny fraction of firms export to at least one market and an even smaller fraction export to multiple destinations (only 16% of French firms sells to at least one destination other than France, 3.3% sell to at least 10 destinations and a mere .05% to 100 or more! See figure 15.1 drawn from Eaton, Kortum, and Kramarz (2011)). Moreover, exporters typically earn a small fraction of their total revenues from their exporting sales (BEJK).

- Exporters have a size advantage over non-exporters. In fact, exporters that sell to many countries sell more in total and in the domestic market than exporters that sell to few destinations or firms that sell only domestically (Eaton, Kortum, and Kramarz (2004), Eaton, Kortum, and Kramarz (2011) –EKK–). This fact is illustrated in Figure 15.1 given that the slope of the line in the plot is far less than 1 (around

.35): including firms less successful in exporting means less than linear increase in total sales in France.

- The number of exporters entering a market, their average size and the total number of products sold increases with the size of the market, with an elasticity that is roughly constant. (Klenow and Rodríguez-Clare (1997), Hummels and Klenow (2005) EKK, Arkolakis and Muendler (2010) –AM–). The elasticity of entry for French exporters can be seen in Figure **??**.

- The distribution of sales of firms in a country, conditional on selling to that country, is robust across countries. It features a Pareto tail when looking at the large firms, and large deviations from Pareto when looking at the small firms: there are too many "too" small guys selling to each destination. Figure 15.3 illustrates the distribution of size of firms in different destinations, grouping destinations in three categories depending on the overall sales of French firms there.

- Firms that sell more goods sell more per good (Bernard, Redding, and Schott (2011) and AM). This feature is true across destinations as Figure 15.2 indicates (AM). In fact the distribution of goods is also robust across destinations (AM).

- At a more disaggregated level, AM document that the most successful products of a firm (the metric being the rank of the product in the most popular market) are systematically more likely to be sold in other markets and conditional on being sold are systematically more likely to sell more than other less successful products. Table 15.4 summarizes the findings of AM.

The above facts suggests the existence of important trade barriers, that only relatively productive firms can overcome. In addition, the facts suggest that the costs of market penetration have similar characteristics across markets and that the same driving forces govern the behavior of firms.

12.2 Trade liberalization

- There is a substantial response of trade flows to price changes induced by changes in tariffs during trade liberalizations (see for example Romalis (2007)). This response is much larger than the response of trade flows to price changes over the business cycle frequency –2-3 years–. The elasticity to changes in tariffs has been estimated in the range of 8-10 while the one for short run adjustments around 1.5 to 2 (See Ruhl (2009) for a review).
- A large number of new firms engage in trade after trade liberalization (see discussion in Arkolakis (2010)). Also a large number of new products are traded after a trade liberalization (**?**, Arkolakis (2010)). New goods typically come with very small sales (Arkolakis (2010)).
- Goods with little trade before a liberalization have higher growth rates of their trade flows after trade liberalization. (see figure 15.5 and Arkolakis (2010)).
- Trade liberalization forces the least productive firms to exit the market. (Bernard and Jensen (1999), Pavcnik (2002), Bernard, Jensen, and Schott (2003))

The above facts on trade liberalization suggest that firms respond to short run (e.g. exchange rate movements) changes differently than they respond to permanent changes (e.g tariff reductions). Their response to permanent changes depends also on their initial size. Whatever the explanation for this behavior, ultimately it should also be consistent with the previous facts on exporting behavior of heterogeneous firms.

12.3 Trade dynamics

- A large number of firms do not export continuously to a given destination (more than 40%). In addition a large number of new firms start exporting every year at a given destination. These new firms and the firms that die typically have tiny sales (Eaton, Eslava, Kugler, and Tybout (2008)).

- The growth rate of small exporters to a given destination is higher than the growth rate of larger exporters (Eaton, Eslava, Kugler, and Tybout (2008), Arkolakis (2011)).

- The variance of the growth rate of small exporters to a given destination is larger than the variance of growth of large exporters (Expected to be true: see Arkolakis (2011) and the facts presented by Sutton (2002)).

Chapter 13

Estimating Models of Trade

Anderson and Van Wincoop (2003) developed a framework that delivers structural relationships for trade among countries (or regions) based on the model analyzed in section (3.3). This model is useful to identify parameters related to the cost of distance and the border. As we showed in the previous chapters, and as elaborated in Anderson and Van Wincoop (2004) and Arkolakis, Costinot, and Rodríguez-Clare (2012), that basic setup has very similar properties in terms of bilateral aggregate trade and welfare to richer models of trade and heterogeneity. New, heterogeneous-firm models generate a number of predictions at the firm-level which can also be used to obtain key parameters of the model. In this chapter we will discuss the identification of key parameters of these models determining aggregated but also disaggregated trade. Alternative ways of estimating gravity equations are summarized in a survey by Anderson and Van Wincoop (2004)

13.1 The Anderson and van Wincoop procedure

Anderson and Van Wincoop (2003) develop a general equilibrium methodology to obtain estimates of the costs of trade in the model as a function of distance proxies.

Using the equation (**??**) we have

$$\frac{X_i}{X^W} = w_i^{1-\sigma} \sum_j \alpha_{ij} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \frac{X_j}{X^W}$$

where X^W is total world spending (income). Using the bilateral demand

$$X_{ij} = p_{ij} x_{ij} = \alpha_{ij} w_i^{1-\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_j$$

we then have

$$X_{ij} = \frac{X_i X_j}{X^W} \alpha_{ij} \left(\frac{\tau_{ij}}{\left(\sum_k \alpha_{ik} \left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} \frac{X_k}{X^W}\right)^{1/(1-\sigma)} P_j} \right)^{1-\sigma}$$

while by summing up over i's we can compute the price index,

$$\sum_{k'} X_{k'j} = \sum_{k'} \frac{X_{k'} X_j}{X^W} \alpha_{k'j} \left(\frac{\tau_{k'j}}{\left(\sum_k \alpha_{ik} \left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} \frac{X_k}{X^W}\right)^{1/(1-\sigma)} P_j} \right)^{1-\sigma} \implies$$

$$P_j = \left[\sum_{k'} \frac{\alpha_{k'j} \left(\tau_{k'j}\right)^{1-\sigma} \frac{X_{k'}}{X^W}}{\sum_k \alpha_{ik} \left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} \frac{X_k}{X^W}} \right]^{1/(1-\sigma)}$$

where we used the fact that balanced trade implies $X_j = \sum_k X_{kj}$.

If we define

$$\Xi_k^{1-\sigma} = \sum_j \alpha_{kj} \left(\frac{\tau_{kj}}{P_j}\right)^{1-\sigma} \frac{X_j}{X^W}$$

then

$$P_j = \left[\sum_{k=1} \alpha_{kj} \frac{\left(\tau_{kj}\right)^{1-\sigma}}{\Xi_k^{1-\sigma}} \frac{X_k}{X^W} \right]^{1/(1-\sigma)}$$

under symmetric trade barriers, $\tau_{ij} = \tau_{ji}$, $\alpha_{ij} = \alpha_{ij}$, from the last equations it turns out that

$\Xi_j = P_j$, so that

$$X_{ij} = \frac{X_i X_j}{X^W} \left(\frac{\tau_{ij}}{P_i P_j} \right)^{1-\sigma}$$

Anderson and Van Wincoop (2003) estimate the stochastic form of the equation

$$\ln \left(\frac{X_{ij}}{X_i X_j} \right) = k + a_1 \ln \tilde{\tau}_{ij} - a_2 D_{ij} - \ln P_i^{1-\sigma} - \ln P_j^{1-\sigma} + \varepsilon_{ij} \tag{13.1}$$

where D_{ij} is a dummy variable related to borders and $a_1 = (1-\sigma)\, \tilde{a}_1, \tilde{\tau}_{ij} = \tau_{ij}^{(1-\sigma)}$. The innovation of Anderson and vanWincoop was to perform this estimation expressing P_i, P_j as an explicit function of the model parameters, σ and $\tilde{a}_1$, a_2 as well as (observable) multilateral resistance terms. The authors cannot separately estimate σ since its effect on distance cannot be separately identified from $\tilde{a}_1$ with their methodology. Nevertheless, their method delivers much more sensible effects for the coefficient on borders. Estimation without considering P_i, P_j as a function of the parameters to be estimated overstates the effect of distance of trade. The intuition is that smaller countries are likely to have higher price indices since they impose trade barriers to larger countries.

13.2 The Head and Ries procedure

The Head and Ries (2001) procedure is another method of estimating the parameters on distance that dispenses of the need of computing the equilibrium of the model. If one looks at the relationship

$$\frac{X_{ij} X_{ji}}{X_{ii} X_{jj}} = \left(\tau_{ij} \tau_{ji} \right)^{1-\sigma} \tag{13.2}$$

then this relationship is an adjustment that takes care of the critique of Anderson and vanWincoop of neglecting the impact of parameters on general equilibrium variables. Parameters can be estimated through a linear regression.

13.3 The Eaton and Kortum procedure

Another approach that gives an unbiased estimate of parameter a_1 is to replace the inward and outward multilateral resistance indices and production variables, $X_i - \ln P_i^{1-\sigma}$ and $X_j - \ln P_j^{1-\sigma}$, with inward and outward region specific dummies. This approach is adopted by a series of papers (e.g. Eaton and Kortum (2002)).

Eaton and Kortum also provide a variety of different methods to estimate the parameter that governs the elasticity of trade. In the Eaton and Kortum (2002) model this is the parameter of the Frechet distribution that governs productivity heterogeneity, θ (whereas in the Armington model it is $\sigma - 1$). Using a relationship similar to (**??**) and specifying intermediate inputs as in equation (9.1), they can derive a relationship of the form

$$\ln \frac{X'_{ij}}{X'_{jj}} = -\theta \ln \tau_{ij} + S_i - S_j \tag{13.3}$$

where $S_i = A_i / (1-\iota) - \theta \ln w_i$, S_j are destination fixed effects and $X'_{ij} = X_{ij} - [(1-\iota)/\iota] \ln (X_{ii}/X_i)$ with $1 - \iota$ the share of intermediates in manufacturing production.[1] They also use proxies for distance, border effects etc. for the first term in order to estimate $\theta \ln \tau_{ij}$ but while they can distinguish the effect of the components (proxies) of that term they cannot distinguish that effect from the effect of the multiplicative term θ. To address that problem and using their estimates from the previous stage for S_i they estimate

$$S_i = \frac{1}{1-\iota} \ln A_i - \theta \ln w_i$$

[1] Eaton and Kortum (2002) estimate

$$\ln \tau_{ij} = f + m_j + \delta_{ij}$$

where f includes distance and other geographic barrier fixed effects, m_j a destination fixed effect and δ_{ij} an error term. To capture potential reciprocity in geographic barriers, they assume that the error term δ_{ij} consists of two components: $\delta_{ij} = \delta^1_{ij} + \delta^2_{ij}$. The country-pair specific component δ^2_{ij} (with variance σ^2_2) affects two-way trade, so that $\delta^2_{ij} = \delta^2_{ji}$, while δ^1_{ij} (with variance σ^2_1) affects one-way trade. This error structure implies that the variance-covariance matrix of δ diagonal elements $E\left(\delta_{ij}\delta_{ij}\right) = \sigma^1_1 + \sigma^2_2$ and certain nonzero off-diagonal elements $E\left(\delta_{ij}\delta_{ji}\right) = \sigma^2_2$.

using technology and education fundamentals to be the proxies for A_i and data for wages adjusted for education. Using a 2SLS estimation they get $\theta = 3.6$.

The second alternative is to estimate the bilateral trade equation (13.3) using their proxy of $\ln(P_i d_{ij}/P_j)$, instead of the geography terms along with source and destination effects. The proxy for d_{ij} is constructed by looking at the (second) highest ratio of prices of homogeneous products across different destinations and the proxy for P_i/P_j as the average of these price ratios. Using a 2SLS and geography variables to instrument for the proxy of $\ln(P_i d_{ij}/P_j)$ their estimate for this procedure is a $\theta = 12.86$.

The favorite estimate of the Eaton and Kortum (2002) is the derivation of the θ using the trade shares equation in terms of prices

$$\frac{X_{ij}/X_j}{X_{ii}/X_i} = \left(\frac{P_i d_{ij}}{P_j}\right)^{-\theta} .$$

With simple method of moments, $-\theta$ is simply the ratio of the mean of $\ln \frac{X_{ij}/X_j}{X_{ii}/X_i}$ and their proxies of $\ln \frac{P_i d_{ij}}{P_j}$. Simonovska and Waugh (2009) propose an alternative estimation of the Eaton and Kortum (2002) by using the above equation and a simulated method of moments approach adapted from Eaton, Kortum, and Kramarz (2011).

13.4 Calibration of a firm-level model of trade

Parameters Determining Firm Sales Advantage We now turn to techniques developed in determining deeper structural parameters of these models, that determine the micro behavior of the firms. An example of these parameters is the marketing parameter β and the ratio of the Pareto parameter and the elasticity of sales, $\tilde{\theta} = \theta/(\sigma - 1)$, used in Arkolakis (2010). Both these parameters determine the distribution of sales of firms and can be calibrated by looking at the size advantage of prolific exporters, i.e. the size advantage of firms that are able to penetrate more markets.

This advantage can be uncovered by looking at the following two structural relationships of the model i) normalized average sales of firms from France, F, conditional on selling to market j,

$$\frac{\bar{X}_{FF|j}}{\bar{X}_{FF}} = \frac{\frac{\left(\frac{M_{Fj}}{M_{FF}}\right)^{-1/\tilde{\theta}}}{1-1/\tilde{\theta}} - \frac{\left(\frac{M_{Fj}}{M_{FF}}\right)^{-1/\left(\tilde{\beta}\tilde{\theta}\right)}}{1-1/\left(\tilde{\theta}\tilde{\beta}\right)}}{\frac{1}{1-1/\tilde{\theta}} - \frac{1}{1-1/\left(\tilde{\theta}\tilde{\beta}\right)}} \tag{13.4}$$

and ii) exporting intensity of firms in percentile Pr_{Fj} in market j,

$$\frac{t_{Fj}\left(\text{Pr}_{Fj}\right)}{\bar{X}_{Fj}} / \frac{t_{FF}\left(\text{Pr}_{FF}\right)}{\bar{X}_{FF}} = \frac{1-\left(1-\text{Pr}_{Fj}\right)^{1/\left(\tilde{\theta}\beta\right)}}{\left(\frac{M_{Fj}}{M_{FF}}\right)^{-1/\tilde{\theta}} - \left(1-\text{Pr}_{Fj}\right)^{1/\left(\tilde{\theta}\beta\right)}\left(\frac{M_{Fj}}{M_{FF}}\right)^{-1/\left(\tilde{\theta}\tilde{\beta}\right)}} \tag{13.5}$$

Notice that parameters θ and σ affect equations (13.4) and (13.5) only insofar they affect $\tilde{\theta}$. Higher θ implies less heterogeneity in firm productivities (and thus in firm sales), whereas higher σ translates the same heterogeneity in productivities to larger dispersion in sales.

For the calibration, Arkolakis (2010) uses a simple method of moments estimate. In particular, β and $\tilde{\theta}$ are picked so that the mean of the left-hand side is equal to the mean of the right-hand side for both equation (13.4) and equation (13.5) evaluated at the median percentile in each market j. The solution delivers $\beta = .915$ and $\tilde{\theta} = 1.65$. Notice that using equation (13.4), a method of moments estimate for the fixed model with $\beta = 0$ gives a $\tilde{\theta} = 1.49$.

To complete the calibration of the model, we need to assign magnitudes to σ and θ. Broda and Weinstein (2006) estimate the elasticity of substitution for disaggregated categories. The average and median elasticity for SITC 5-digit goods is 7.5 and 2.8, respectively (see their table IV). A value of $\sigma = 6$ falls in the range of estimates of Broda and Weinstein (2006) and yields a markup of around 1.2, which is consistent with those values reported in the data (see Martins, Scarpetta, and Pilat (1996)). In addition, $\tilde{\theta} = 1.65$ and $\sigma = 6$ imply that the marketing costs to GDP ratio in the model is around 6.6% within the range of marketing costs to GDP ratios reported in the data. Finally, this parameterization implies that $\theta = 8.25$ for the endogenous cost model which is very close to the main estimate of Eaton and Kortum (2002) (8.28) and within the range of estimates of Romalis (2007) ($6.2 - 10.9$) and the ones reported in the review of Anderson and Van Wincoop (2004) ($5 - 10$). Since the model retains the aggregate predictions of the Melitz-Chaney framework if θ is the same I will calibrate the two models to have $\theta = 8.25$. For the fixed cost model, given the calibrated $\tilde{\theta} = 1.49$, it implies a $\sigma = 6.57$.

Calibration for a multi-product firms model

Parametrizing a multi-products firm model requires to dig deeper into establishing predictions at the within-firm level. We will now briefly go over the calibration procedure of Arkolakis and Muendler (2010) for their model described in 8.3. Guided by various log-linear relationships observed in their data (see, for example Figure 15.2) they specify the following functional relationships

$$\begin{aligned} f_{ij}(g) = &\ f_{ij} \cdot g^{\delta} \quad \text{for } \delta \in (-\infty, +\infty), \\ h(g) = &\ g^{\alpha} \qquad\quad \text{for } \alpha \in [0, +\infty). \end{aligned} \tag{13.6}$$

This specification gives product level sales for the g-th ranked product of the firm as

$$p_{ijg}(z)x_{ijg}(z) = \sigma f_{ij}(1)\, G_{ij}(z)^{\delta+\alpha(\sigma-1)} \left(\frac{z}{z_{ij}^{*,G}}\right)^{\sigma-1} g^{-\alpha(\sigma-1)} \,.$$

Using the logarithm of this structural relationship, a regression of the sales of the firm on a constant, a firm fixed effect and the number of the products of the firm obtains $\alpha\,(\sigma - 1) = 2.66$ and $\delta \simeq -1.38$.

13.5 Estimation of a firm-level model

We present here the framework of Eaton, Kortum, and Kramarz (2011) that is the first work that estimates a multi-country firm-level model of trade making use of the firm-level data. The idea is to identify a set of micro facts on exporters and to develop a consistent modeling framework that would explain these micro observations using model relationships. Then the authors estimate the fundamental parameters of the model using the micro data. In this respect the paper of Eaton, Kortum, and Kramarz (2011) is parallel to the Eaton and Kortum (2002) framework.

13.5.1 The model

Sales of the firm are given by

$$t_{ij}(\omega) = a_j(\omega)\, n_{ij} \left(\frac{p_{ij}}{P_j}\right)^{1-\sigma},$$

derived by asymmetric CES utility function with preference for each good affected by $a_j(\omega)$ (these could be interpreted as Armington type bias in a particular good). The term $a_j(\omega)$ reflects an exogenous demand shock specific to good ω in market j. The term P_j is the CES price index that will be analyzed in a moment.

Producers are heterogeneous and the unit cost for a producer from i in producing a good and shipping it to country j is

$$c_{ij}(\omega) = \frac{w_i \tau_{ij}}{z_i(\omega)}$$

where τ_{ij} is an iceberg cost. The measure of potential producers who can produce their good with efficiency at least z is

$$\mu_i(z) = A_i z^{-\theta}.$$

Given the unit cost this implies that the measure of goods that can be delivered to country

j from anywhere in the world at unit cost c or less in j is

$$\begin{aligned}\mu_j(c) &= \sum_{k=1}^{N} \mu_{kj}(c) \\ &= \sum_{k=1}^{N} A_k \left(w_k \tau_{kj}\right)^{-\theta} c^{\theta} \\ &\equiv \sum_{k=1}^{N} \Phi_{kj} c^{\theta} \\ &\equiv \Phi_j c^{\theta}\end{aligned}$$

Conditional on selling in a market the producer makes the profit from producer from i in j

$$\pi_{ij}(\omega) = \max_{p,n} \left(1 - \frac{c_j(\omega)}{p}\right) a_j(\omega)\, n \left(\frac{p}{P_j}\right)^{1-\sigma} X_j - \varepsilon_j(\omega)\, f_j \frac{1-(1-n)^{1-\beta}}{1-\beta},$$

where $c_j(\omega)$ is the unit production cost, $\varepsilon_j(\omega)$ an entry cost and $f_j > 0$. Producer charges a constant markup

$$p_{ij} = \bar{m} c_j(\omega)\ , \bar{m} = \sigma / (\sigma - 1)$$

Define

$$\eta_j(\omega) = \frac{a_j(\omega)}{\varepsilon_j(\omega)}.$$

Thus, we can describe seller's behavior in market j in terms of its cost draws $c_j(\omega) = c$, the demand shock $a_j(\omega) = a$, and the redefined entry shock $\eta_j(\omega) = \eta$. It can be shown using the results of section 8.2 combined with this framework that a firm will enter a market j iff its cost draw $c \geq \bar{c}_j(\omega)$

$$\bar{c}_j(\eta) = \left(\eta \frac{X_j}{\sigma f_j}\right)^{1/(\sigma-1)} \frac{P_j}{\bar{m}}. \tag{13.7}$$

Notice that the entry threshold depends on a only through η. For the firms with $c \geq \bar{c}_j(\omega)$

the fraction of buyers reached in a market will be (for $\beta > 0$)

$$n_{ij}(\eta, c) = 1 - \left(\frac{c}{\bar{c}_j(\eta)}\right)^{\frac{(\sigma-1)}{\beta}}$$

You can rewrite sales as

$$t_{ij}(\eta) = \varepsilon_j \left[1 - \left(\frac{c}{\bar{c}_j(\eta)}\right)^{\frac{(\sigma-1)}{\beta}}\right] \left(\frac{c}{\bar{c}_j(\eta)}\right)^{-(\sigma-1)} \sigma f_j$$

Notice that even though Eaton, Kortum, and Kramarz (2011) add these 3 levels of firm heterogeneity they can determine easily all the aggregate variables of the model. First, the price index is given by the following integration

$$\begin{aligned} P_j &= \left[\int\int \left(\int_0^{\bar{c}_j(\eta)} \alpha n_{ij}(\eta, c)\, \bar{m}^{1-\sigma} c^{1-\sigma} d\mu_j(c)\right) g(\alpha, \eta)\, d\alpha d\eta\right]^{-1/(\sigma-1)} \\ &= \bar{m}\left[\Phi_j \left(\frac{\theta}{\theta - \sigma + 1} - \frac{\theta}{\theta + (\sigma-1)\frac{\beta-1}{\beta}}\right) \int\int \alpha \bar{c}_j(\eta)^{\theta-(\sigma-1)} g(\alpha, \eta)\, d\alpha d\eta\right] \end{aligned}$$

which substituting for the entry hurdle (13.7) gives

$$P_j = \bar{m}\left(\kappa_1 \Phi_j\right)^{-1/\theta} \left(\frac{X_j}{\sigma f_j}\right)^{(1/\theta)-1/(\sigma-1)},$$

where

$$\kappa_1 = \left[\frac{\theta}{\theta - \sigma + 1} - \frac{\theta}{\theta + (\sigma-1)\frac{\beta-1}{\beta}}\right] \int\int \alpha \eta^{\frac{\theta-(\sigma-1)}{\sigma-1}} g(\alpha, \eta)\, d\alpha d\eta\,,$$

and $g(\alpha, \eta)$ is the joint density of the realizations of producer-specific costs. Second, from the model we can get a series of relationships directly related to observables. The measure of entrants in market j is

$$\begin{aligned} M_j &= \int \bar{c}_j(\eta)\, g \\ &= \frac{\kappa_2}{\kappa_1} \frac{X_j}{\sigma f_j} \end{aligned}$$

where

$$\kappa_2 = \int \eta^{\theta/(\sigma-1)} g_2(\eta)\, d\eta$$

Number of firms selling from i to j

$$M_{ij} = \frac{\kappa_2}{\kappa_1} \frac{\lambda_{ij} X_j}{\sigma f_j},$$

where

$$\lambda_{ij} = \frac{\Phi_{ij}}{\Phi_j}$$

being the observed market share, which exactly the same as in the monopolistic competition model with productivity as the only source of variation. Finally, average sales are given by

$$\bar{X}_{ij} = \frac{\kappa_1}{\kappa_2} \sigma f_j$$

It also turns out that the distribution of sales in a market, and hence mean sales, is invariant to the location of the supplier.

Notice that all these relationships are derived independently of the actual distribution of demand and entry shocks. This separability allows for a very simple and generic solution of the model that retains the forces of the previous structure while allowing for additional levels of heterogeneity that brings the model closer to the data.

13.5.2 Estimation, simulated method of moments

There are particular steps in the estimation procedure proposed by the authors. They match 4 sets of moments (each set of moments is denoted as m)

a) The distribution of exporting sales in individual destinations by different percentiles in these destinations,

b) the sales of french firms in France of firms that sells in individual destinations by different percentiles in France,

c) normalized export intensity of firms by market by different percentiles in France,

d) the fraction of firms selling to each possible combination of the top seven exporting destinations.

These 4 set of moments contribute to the objective function

$$Q\left(m\right)=\sum_{k=1}^{\#m}w^{k}\left(m\right)\left(\hat{p}^{k}\left(m\right)-p^{k}\left(m\right)\right)^{2}$$

where $\hat{p}^{k}\left(m\right)$ are the simulated observations for each moment and $p^{k}\left(m\right)$ the ones related to the data. The authors use the following weights

$$w^{k}\left(m\right)=N/p^{k}\left(m\right)$$

where N is the number of firms in the data sample. With these weights each $Q\left(m\right)$ is a chi-square statistic with degrees of freedom given by the number of moments to be matched ($\#m$). Chi square is the limiting distribution of $Q\left(m\right)$ (for N large) under the null that the sampling error is the only source of error and, thus, observed sales follow a multinomial distribution with the actual probabilities as parameters. Hence, the means of the $Q\left(m\right)$'s equal their degrees of freedom and their variances twice their means.

The paper has a set of important contributions

- It identifies a set of statistics in the data that will be a rigorous test for all future trade theories.
- It develops a model that is consistent with these facts and can account for different levels of heterogeneity. In particular, it shows how the model can motivate research to interpret and "read" the data in a way consistent to the model.
- It develops an internally consistent methodology for estimating firm-level models.

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Chapter 14

Appendix

14.1 Distributions

This appendix explains the details of the two main distributions used in these notes[1].

14.1.1 The Fréchet Distribution

The type II extreme value distribution, also called the Fréchet distribution, is one of three distributions that can arise as the limiting distribution of the maximum of a sequence of independent random variables. The distribution function for the Fréchet distribution is

$$F(x) = \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{-\theta}\right\},$$

for $x > \mu$, where $\theta > 0$ is a shape parameter, $\sigma > 0$ is a scale parameter and $\mu \in \mathbb{R}$ is a location parameter. The density of the Fréchet distribution is

$$f(x) = \frac{\theta}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-\theta-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{-\theta}\right\},$$

[1] Many thanks to Alex Torgovitsky for the preparation of this appendix

for $x > \mu$. If X is a Fréchet-distributed random variable then

$$\begin{aligned}
E(X) &= \int_{\mu}^{\infty} x\frac{\theta}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-\theta-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{-\theta}\right\}dx \\
&= \sigma\int_{0}^{\infty} y^{-\frac{1}{\theta}}e^{-y}dy + \mu\int_{0}^{\infty} e^{-y}dy \\
&= \sigma\Gamma\left(\frac{\theta-1}{\theta}\right) + \mu,
\end{aligned}$$

where $y := \left(\frac{x-\mu}{\sigma}\right)^{-\theta}$ and

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1}e^{-t}dt$$

is the Gamma function. Now assume that $\mu = 0$, take $T := \sigma^{\theta}$ and rewrite the distribution function as

$$F(x) = e^{-Ax^{-\theta}},$$

so that the Fréchet distribution is now parameterized by θ, A. Notice that for any given θ and A is increasing in the scale parameter, σ. Figure 15.6, shows how θ and A affect the Fréchet distribution.

The Pareto Distribution

The Pareto distribution is parameterized by a shape parameter, $\theta > 0$, a scale parameter $m > 0$ and has support $[m, \infty)$ with distribution function

$$F(x) = 1 - \left(\frac{m}{x}\right)^{\theta}.$$

The density function is

$$f(x) = \frac{\theta m^{\theta}}{x^{\theta+1}}.$$

The n^{th} moment of a Pareto distributed random variable can easily be calculated as

$$E(X^n) = \int_m^\infty x^n \theta m^\theta x^{-\theta-1} dx = \begin{cases} \frac{\theta m^n}{\theta - n}, & \text{if } \theta > n \\ +\infty, & \text{if } \theta \leq n \end{cases}$$

which shows that the shape parameter controls the number of existent moments. Direct computation yields

$$E(X) = \frac{\theta m}{\theta - 1}, \quad \text{if } \theta > 1,$$
$$Var(X) = \frac{\theta m^2}{(\theta - 1)^2 (\theta - 2)}, \quad \text{if } \theta > 2.$$

The Pareto distribution is an example of a power law distribution, which can be seen by observing that

$$\Pr[X \geq x] = \left(\frac{m}{x}\right)^\theta.$$

This implies that

$$\log(\Pr[X \geq x]) = \theta \log(m) - \theta \log(x),$$

so that the log of the mass of the upper tail past x is linear in $\log(x)$. For example, if the number of employees in a randomly sampled firm, X, is Pareto distributed, then the proportion of firms in the population that have more than x employees is linear with the number of employees on a log-log scale. This is related to a useful self-replicating feature of the Pareto distribution, which is that the distribution of X conditional on the event $[X \geq \bar{x}]$, where $\bar{x} \geq m$, is given by

$$\Pr[X \geq x | X \geq \bar{x}] = \frac{\Pr[X \geq x]}{\Pr[X \geq \bar{x}]} = \left(\frac{\bar{x}}{x}\right)^\theta,$$

for $x \geq \bar{x}$. That is, truncating the Pareto distribution on the left produces another Pareto distribution with the same shape parameter! Figure 15.6, shows how θ and the initial point m affect the Pareto distribution.

Chapter 15

Figures and Tables

AverageSalesDestinations.wmf XXX need to replace XXX

Figure 15.1: Sales in France from firms grouped in terms of the minimum number of destinations they sell to. Source: Eaton, Kortum, and Kramarz (2011).

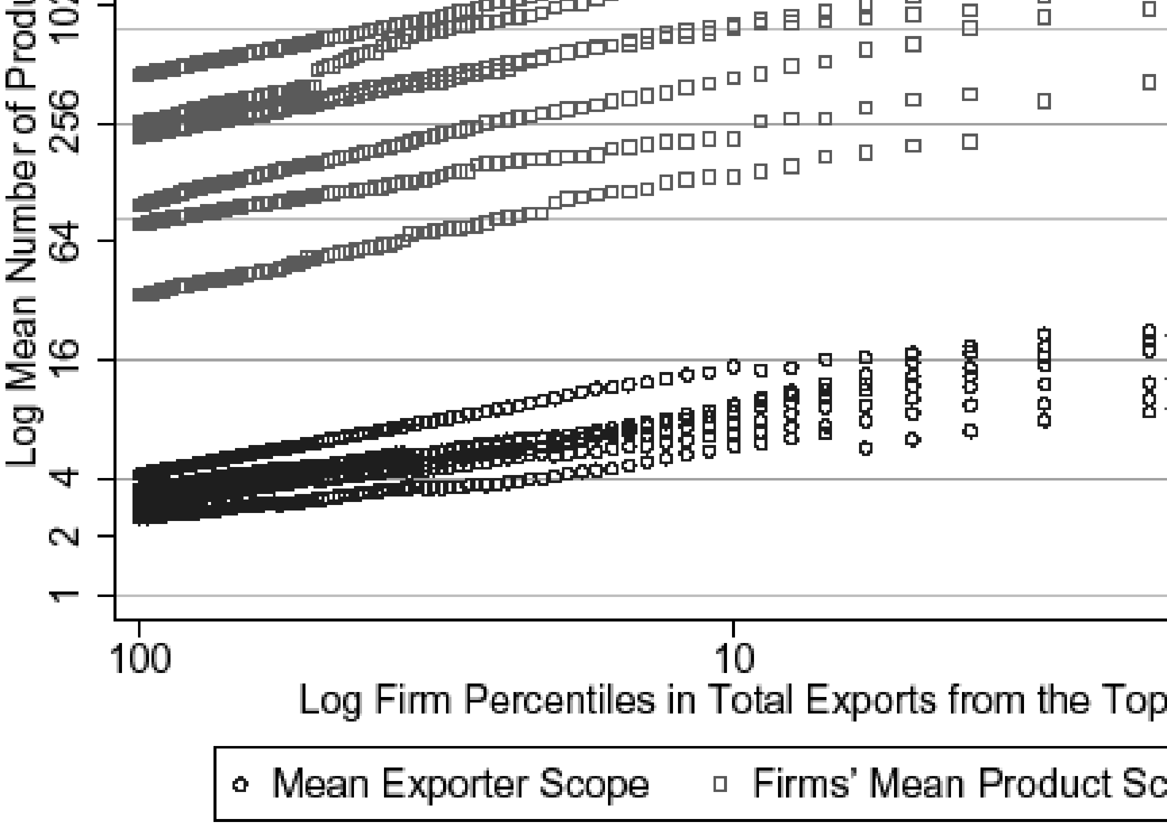

Figure 15.2: Distributions of average sales per good and average number of goods sold. Means taker over all firms larger or equal than the percentile considered in the graph. Source: Arkolakis and Muendler (2010). Products at the Harmonized-System 6-digit level. Destinations ranked by total exports.

DistributionsData.wmf XXX need to replace XXX

Figure 15.3: Distribution of sales for Portugal and means of other destinations group in terciles depending on total sales of French firms there. Each box is the mean over each size group for a given percentile and the solid dots are the sales distribution in Portugal. *Source Eaton, Kortum, and Kramarz (2011).*

tableproducts.wmf XXX need to replace XXX

Figure 15.4: Product Rank, Product Entry and Product Sales for Brazilian Exporters

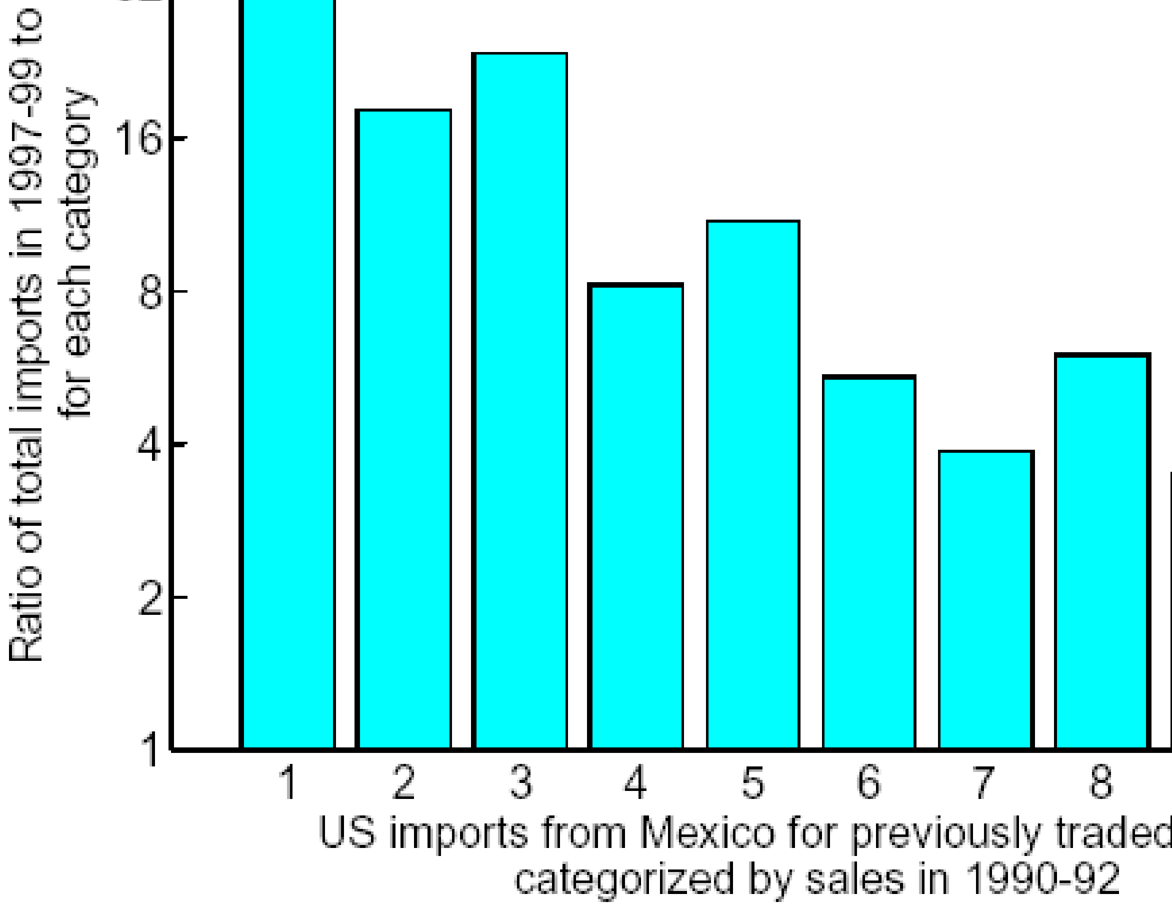

Figure 15.5: Increases in trade and initial trade. Source: Arkolakis (2010). Products at the Harmonized-System 6-digit level. Data are from www.sourceoecd.org

Frechet.wmf XXX need to replace XXX

Figure 15.6: Frechet Distribution

Paretopic.wmf XXX need to replace XXX

Figure 15.7: Pareto Distribution