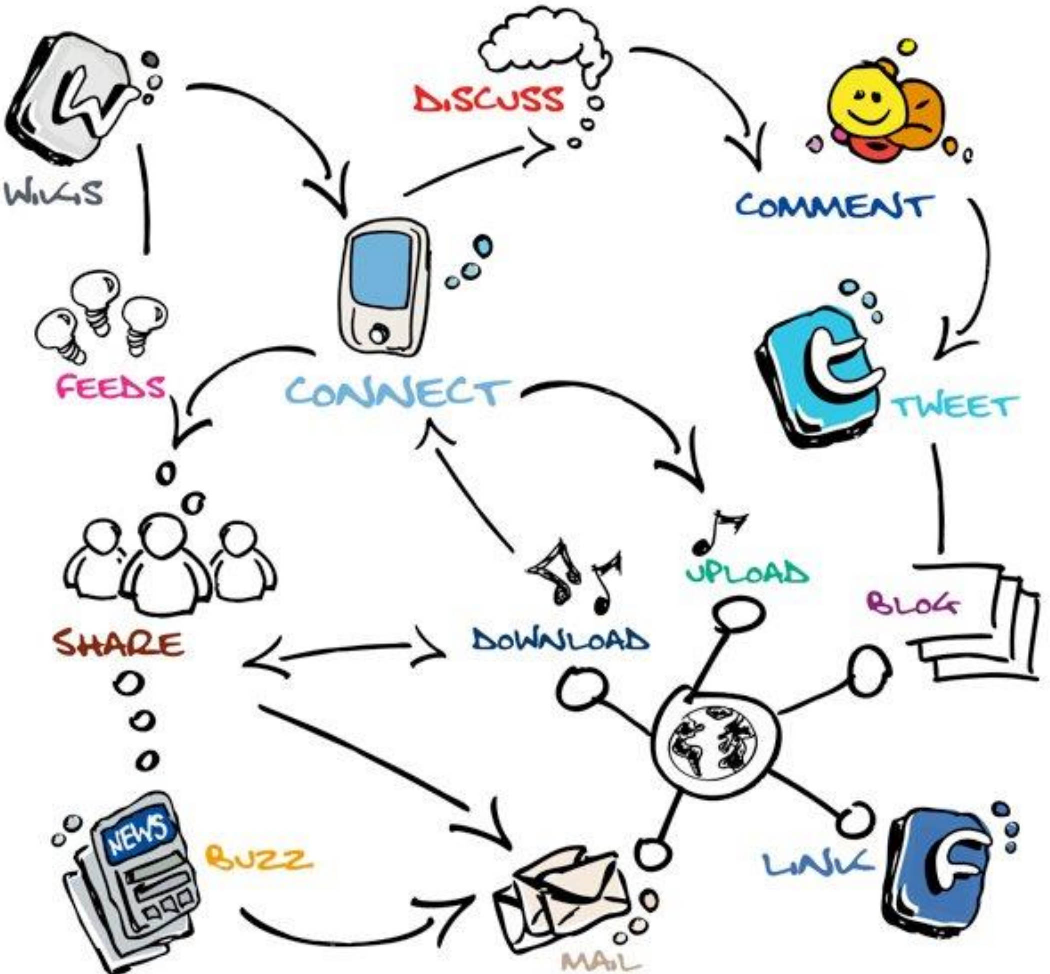


Network Formation Processes
Power-law degrees

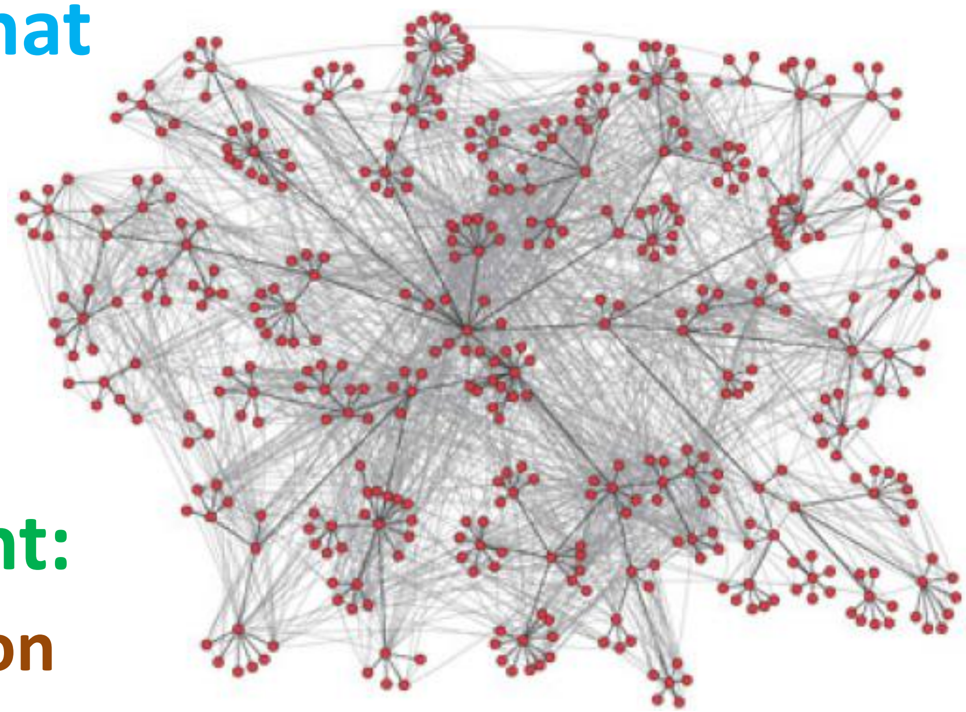
Ahmet Onur Durahim

How the Class Fits Together

Properties	Models	Algorithms
Small diameter, Edge clustering	Small-world model, Erdös-Renyi model	Decentralized search
Scale-free	Preferential attachment, Copying model	PageRank, Hubs and authorities
Strength of weak ties, Core-periphery	Kronecker Graphs	Community detection: Girvan-Newman, Modularity
Densification power law, Shrinking diameters	Microscopic model of evolving networks	Link prediction, Supervised random walks
Complex Graph Structure	Graph Neural Networks	Node Classification Graph Representation Learning
Information virality, Memetracking	Independent cascade model, Game theoretic model	Influence maximization, Outbreak detection, LIM

Networks Formation Processes

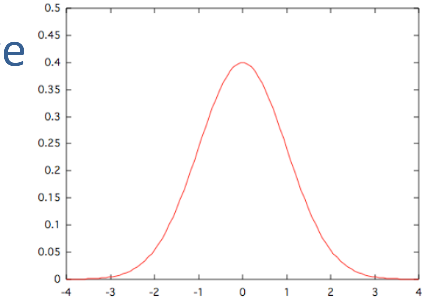
- **What do we observe that needs explaining**
- **Small-world model?**
  - Diameter
  - Clustering coefficient
- **Preferential Attachment:**
  - **Node degree distribution**
    - What fraction of nodes has degree $\boldsymbol{k}$ (as a function of $\boldsymbol{k}$)?
    - Prediction from simple random graph models: p($\boldsymbol{k}$)= exponential function of $\boldsymbol{k}$
    - **Observation:** ***Often a power-law –*** $\boldsymbol{p}(\boldsymbol{k})$ α $\boldsymbol{k}^{-\boldsymbol{\alpha}}$



Popularity on Web

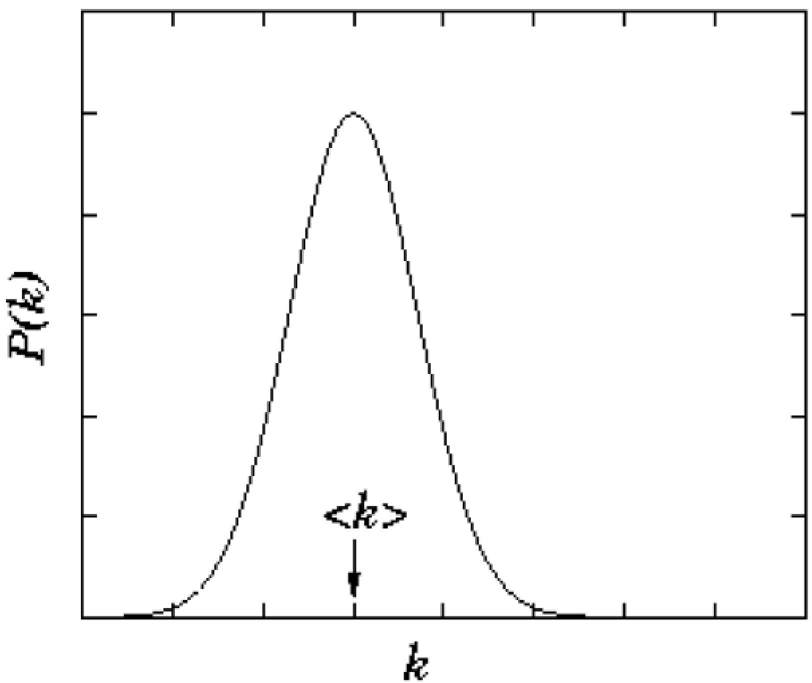
As a function of k, *what fraction* of pages on the Web have *k in-links*?

- ***A Simple Hypothesis: The Normal Distribution***
  - Central Limit Theorem says that if we take any sequence of small ***independent*** random quantities, then in the limit their sum (or average) will be distributed according to the normal distribution
- If we model the link structure of the Web by assuming that ***each page decides independently at random whether to link to any other given page***
  - the number of in-links to a given page is the sum of many independent random quantities
    - the presence or absence of a link from each other page
  - Hence we'd expect it to be normally distributed.

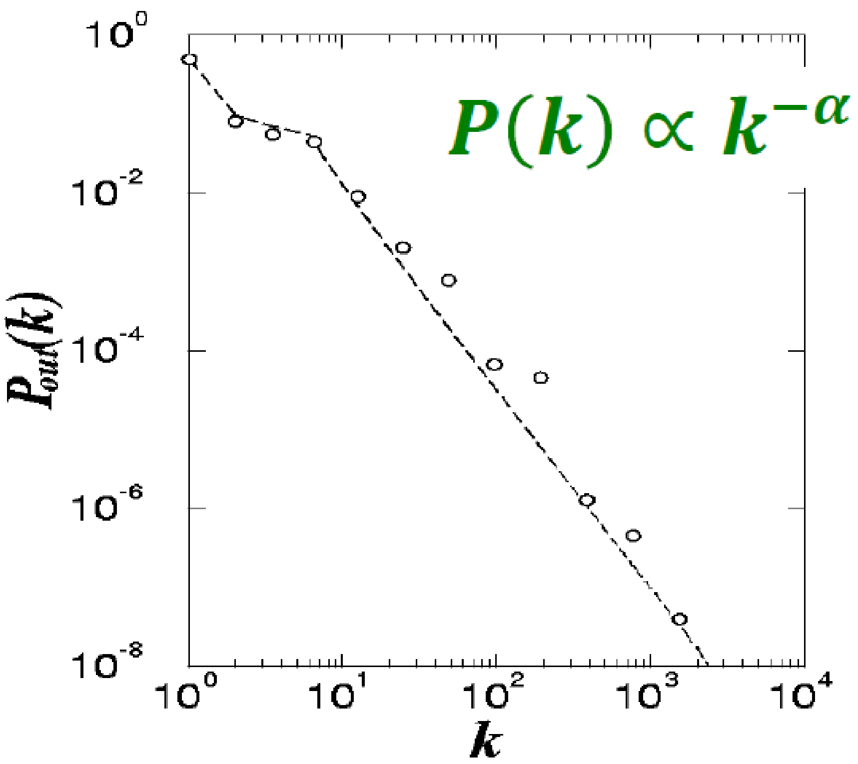


Degree Distributions

Expected based on G_{np}



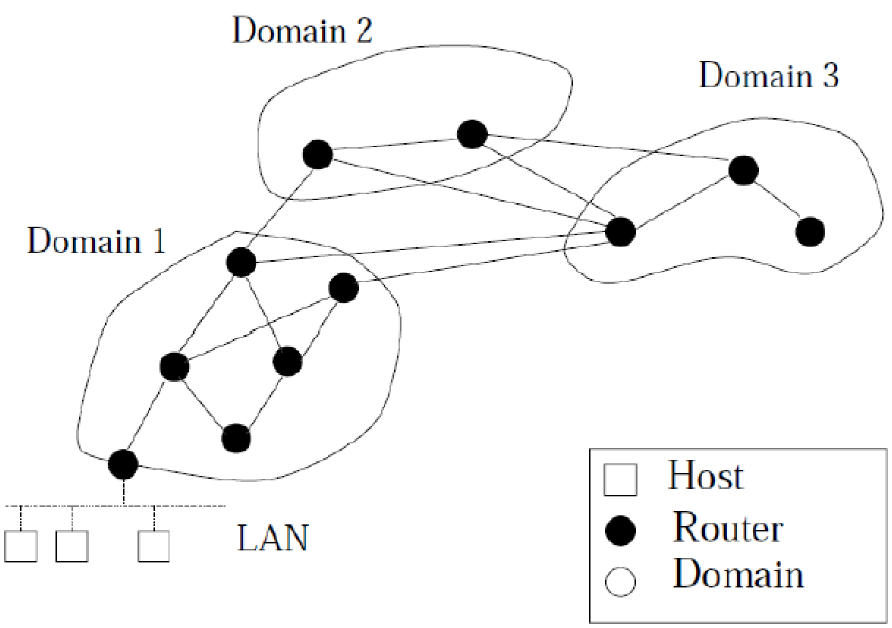
Found in data

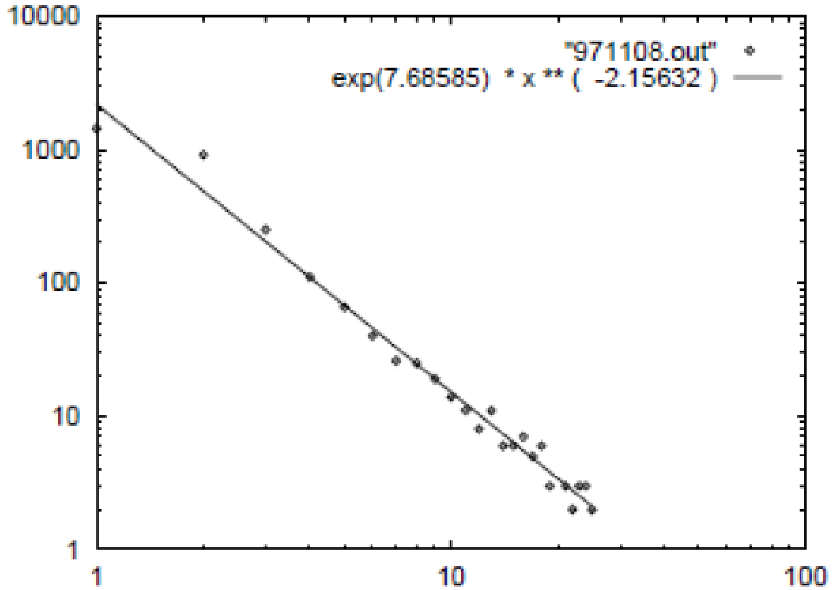


Node Degrees: AS

- **Internet Autonomous Systems**

  [Faloutsos, Faloutsos and Faloutsos, 1999]

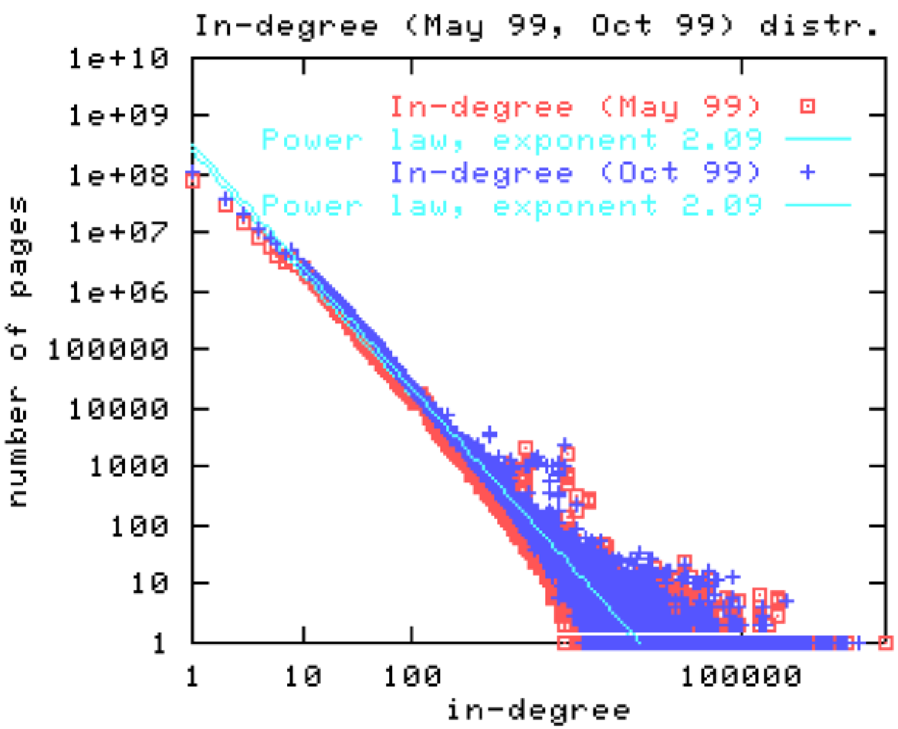


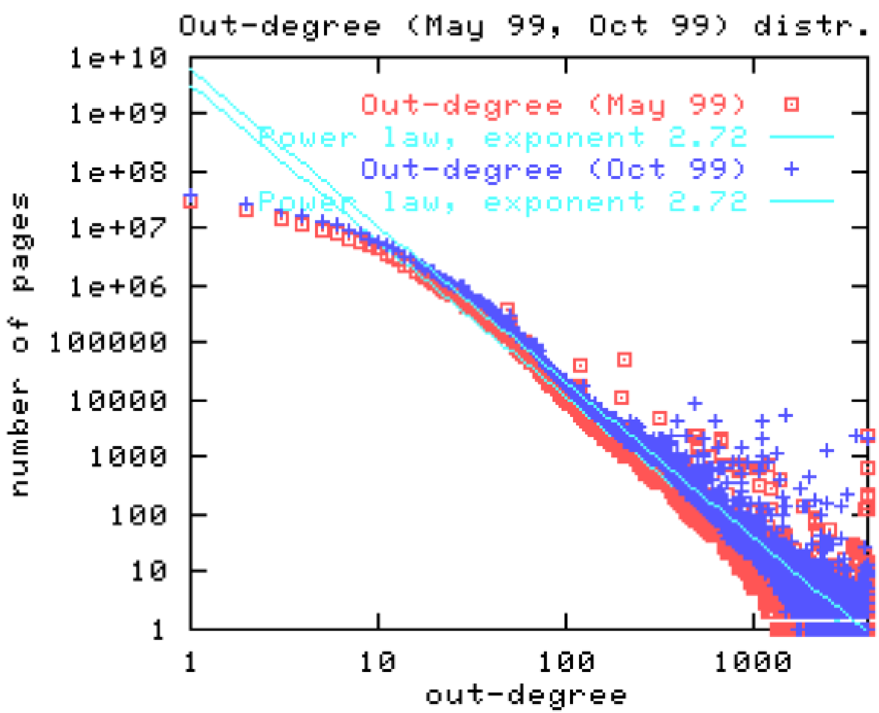


Internet domain topology

Node Degrees: Web

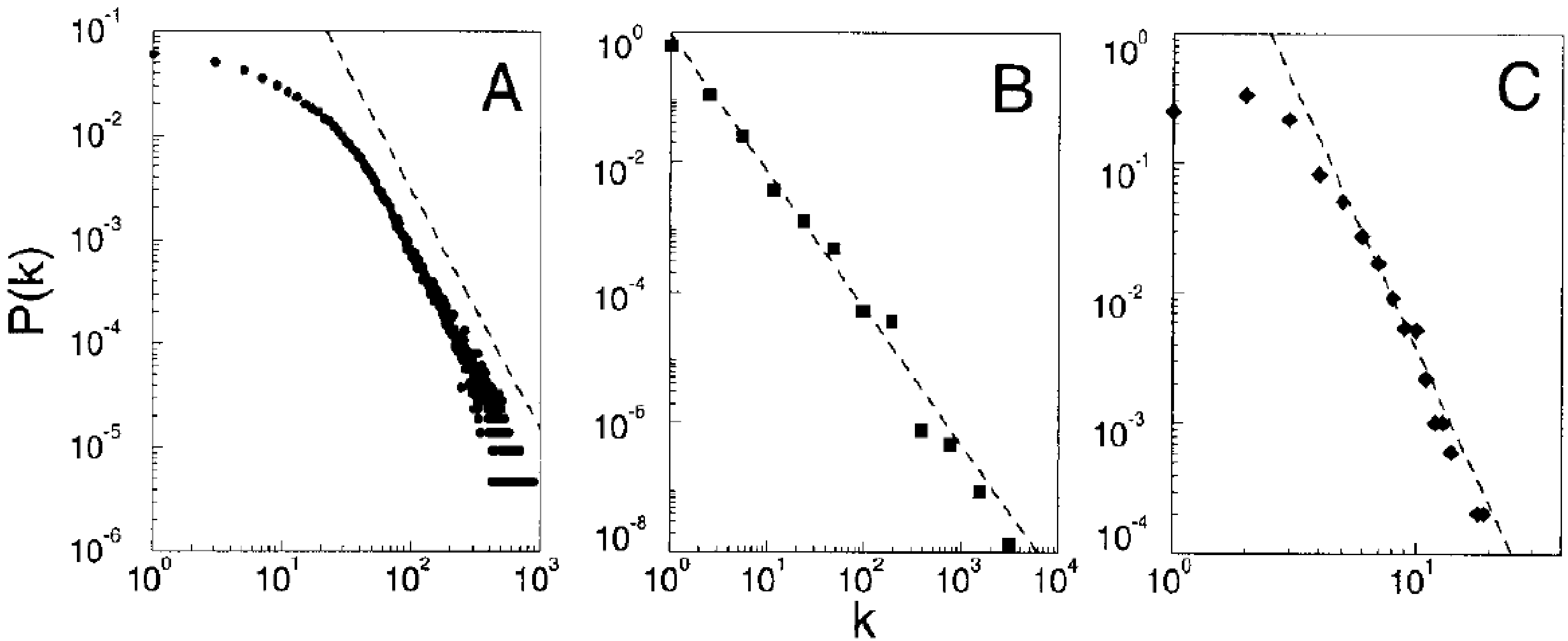
- **The World Wide Web** [Broder et al., 2000]





Node Degrees: Barabasi-Albert

- **Other Networks** [Barabasi-Albert, 1999]



Actor collaborations

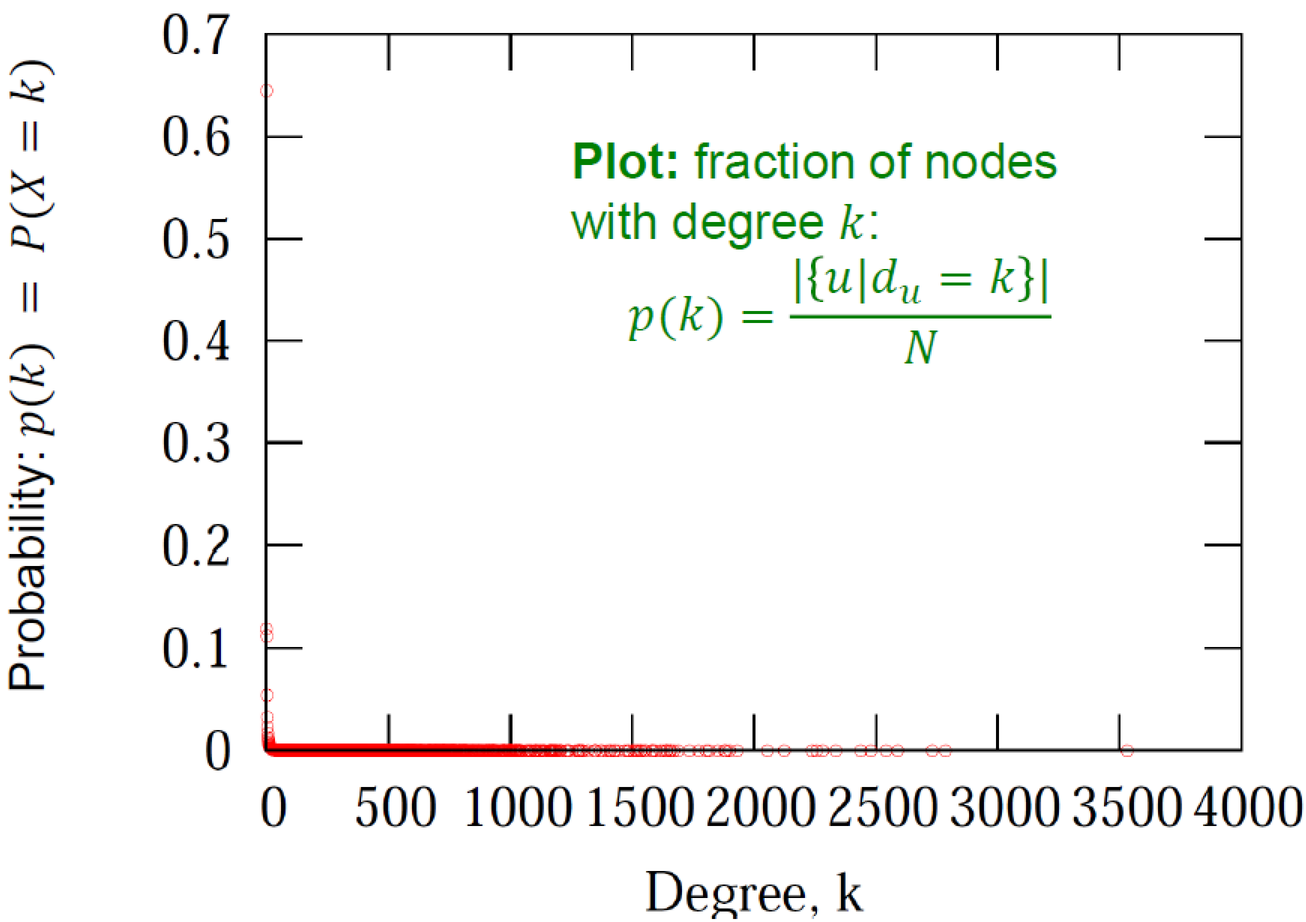
Web graph

Power-grid

[Leskovec et al. KDD '08]

Node Degrees in Networks

- **Take a network, plot a histogram of $P(k)$ vs. k**

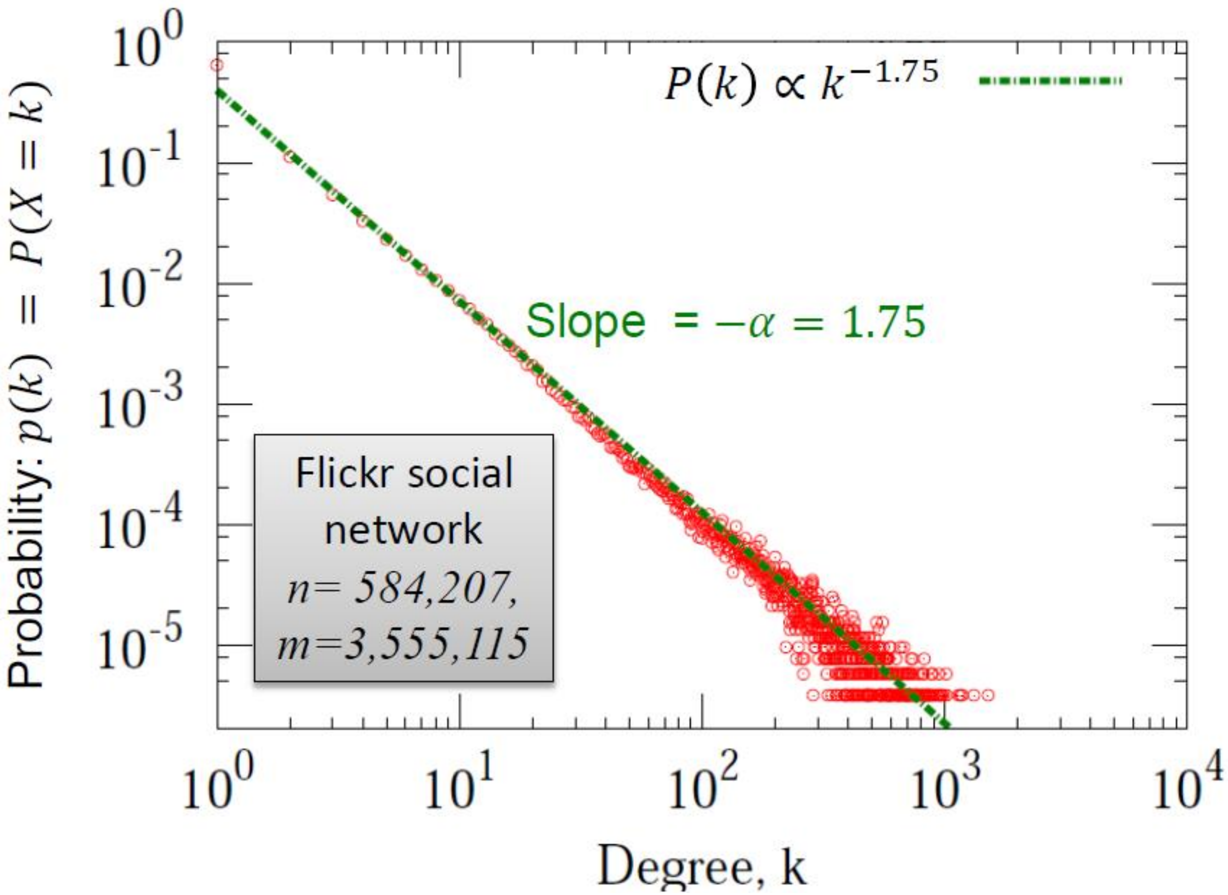


Flickr social network
$n = 584{,}207$,
$m = 3{,}555{,}115$

[Leskovec et al. KDD '08]

Node Degrees in Networks

- **Plot the same data on log-log scale:**

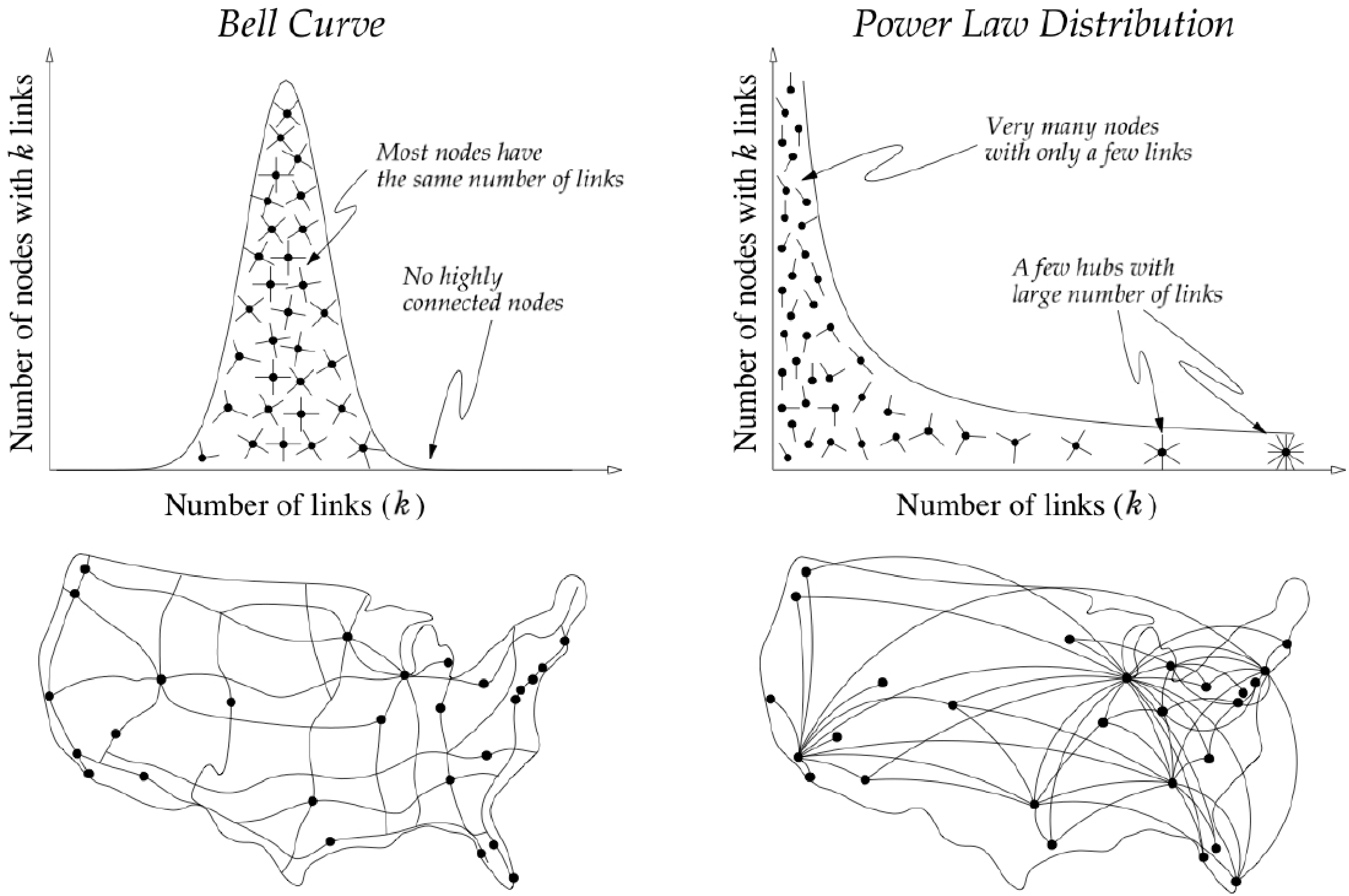


How to distinguish:
$P(k) \propto \exp(-k)$ **vs.**
$P(k) \propto k^{-\alpha}$ **?**

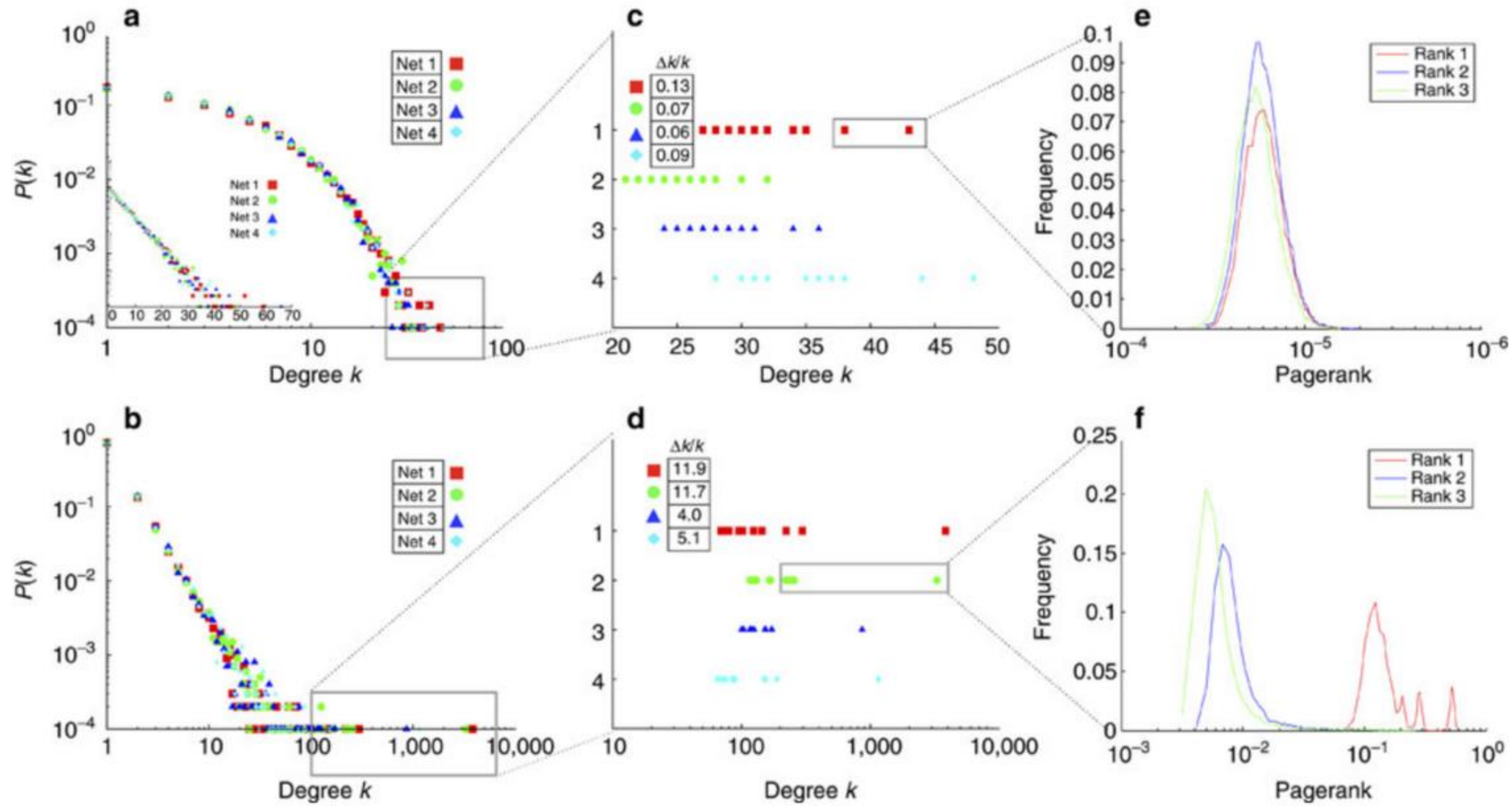
Take logarithms:
if $y = f(x) = e^{-x}$ then
$\log(y) = -x$
If $y = x^{-\alpha}$ then
$\log(y) = -\alpha \log(x)$

So, on log-log axis power-law looks like a straight line of slope $-\alpha$!

Exponential vs. Power-Law



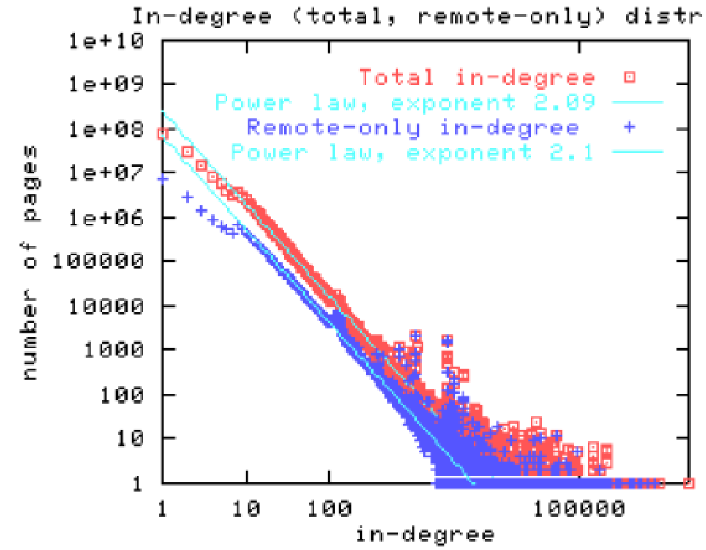
Exponential vs. Power-Law

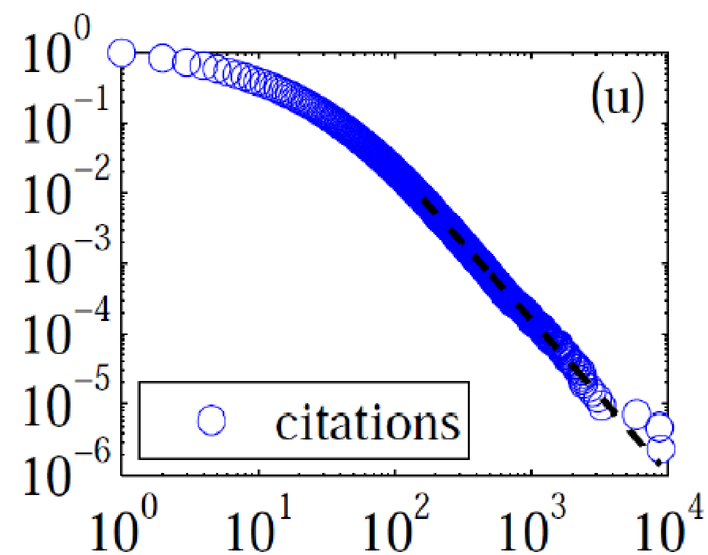


The degree distributions of (**a**) four networks with exponential degree distribution $P(k)\sim e^{-\lambda k}$ with λ=0.2 on a log–log scale (inset: log-linear) and (**b**) four networks with scale-free degree distribution $P(k)\sim k^{-\gamma}$ with γ=2.2, average degree $\langle k \rangle$=5 and size $N=10^4$. (**c**, **d**) The magnified area of a and b showing the gap between the largest degrees. The difference between the largest degrees in a scale-free network is consistently large compared with the largest values in an exponential network. The legend shows the relative difference in the top two degrees' $\Delta k = (k_{max} - k_{max-1})/k_{max-1}$) for both degree distributions. (**e**, **f**) The pagerank distributions of the top three nodes selected from each network type. For the power law distribution, the peak of each curve can be clearly distinguished in contrast with the exponential network where the top nodes' pagerank are practically indistinguishable.

Power-Law Degree Exponents

- **Power-law degree exponent is typically 2 < α < 3**
  - **Web graph:**
    - $\alpha_{in} = 2.1$, $\alpha_{out} = 2.7$ [Broder et al. 00]
  - **Autonomous systems:**
    - $\alpha = 2.4$ [Faloutsos^3, 99]
  - **Actor-collaborations:**
    - $\alpha = 2.3$ [Barabasi-Albert 00]
  - **Citations to papers:**
    - $\alpha \approx 3$ [Redner 98]
  - **Online social networks:**
    - $\alpha \approx 2$ [Leskovec et al. 07]





Rich-Get-Richer Models

A natural mechanism to generate power laws

- **Normal distributions** *arise from many* ***independent random decisions*** *averaging out*
- **Power laws** *arise from the feedback introduced by* ***correlated decisions*** *across a population*
  - build model ***not*** from
    - ***the internals of each person's decision-making process***
  - but from the observable consequences of
    - ***decision-making in the presence of cascades***
  - Assume that people have a tendency to
    - ***copy the decisions of people who act before them***

Scale-Free Networks

- **Definition:** **Networks with a power-law tail in their degree distribution are called "scale-free networks"**

If you zoom in then you see the same structure

- **Where does the name come from?**
  - **Scale invariance:** There is no characteristic scale
  - **Scale-free function:** $f(ax) = a^{\lambda} f(x)$
    - *Power-law function*: $f(ax) = a^{\lambda} x^{\lambda} = a^{\lambda} f(x)$

Log() or Exp() are not scale free!

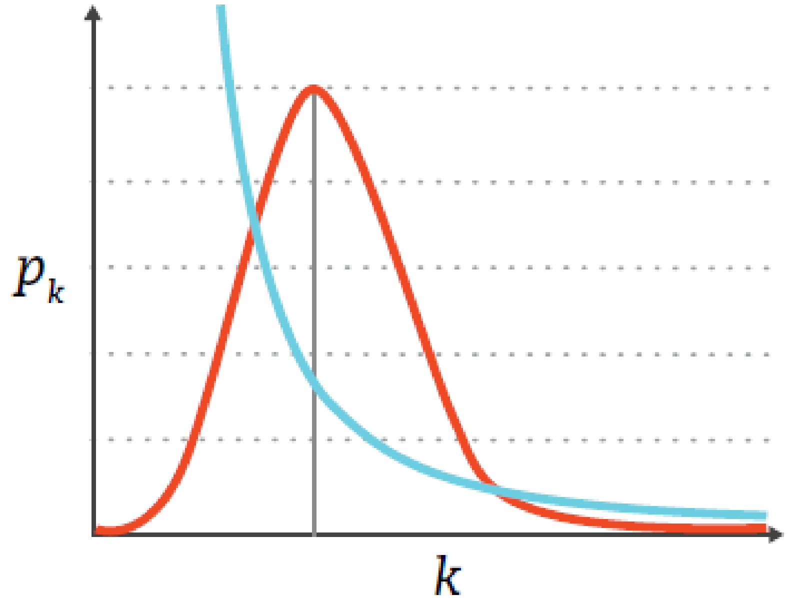
$f(ax) = \log(ax) = \log(a) + \log(x) = \log(a) + f(x)$

$f(ax) = \exp(ax) = \exp(x)^a = f(x)^a$

Where does "Scale-Free" name come from?

Scale-free networks lack an internal scale

- ***For any bounded distribution*** (e.g. a Poisson or a Gaussian distribution) the ***degree of a randomly chosen node*** will be **in the vicinity of ⟨*k*⟩**
  – Hence ⟨*k*⟩ serves as the **network's scale**
- ***In a scale-free network*** the second moment diverges, hence the ***degree of a randomly chosen node*** can be arbitrarily different from ⟨*k*⟩
  – As a scale-free network lacks an intrinsic scale, it is **scale-free**



Random network
Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$
Scale: $\langle k \rangle$

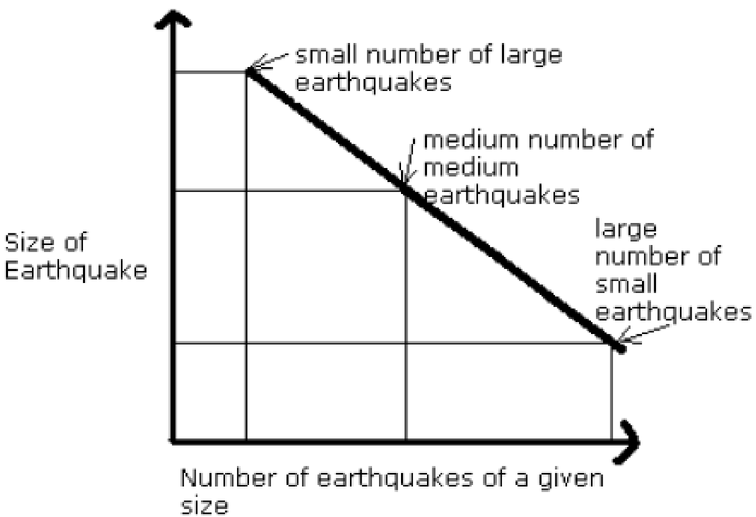
Scale-free network
Randomly chosen node: $k = \langle k \rangle \pm \infty$
$\langle k \rangle$ is meaningless as 'scale'

Scale Invariance

- Scaling of the density

$$x \to bx, \quad p(bx) = C(bx)^{-\alpha} = b^{-\alpha} C x^{-\alpha} \propto p(x)$$

- Scale invariance $\dfrac{p(100x)}{p(10x)} = \dfrac{p(10x)}{p(x)}$

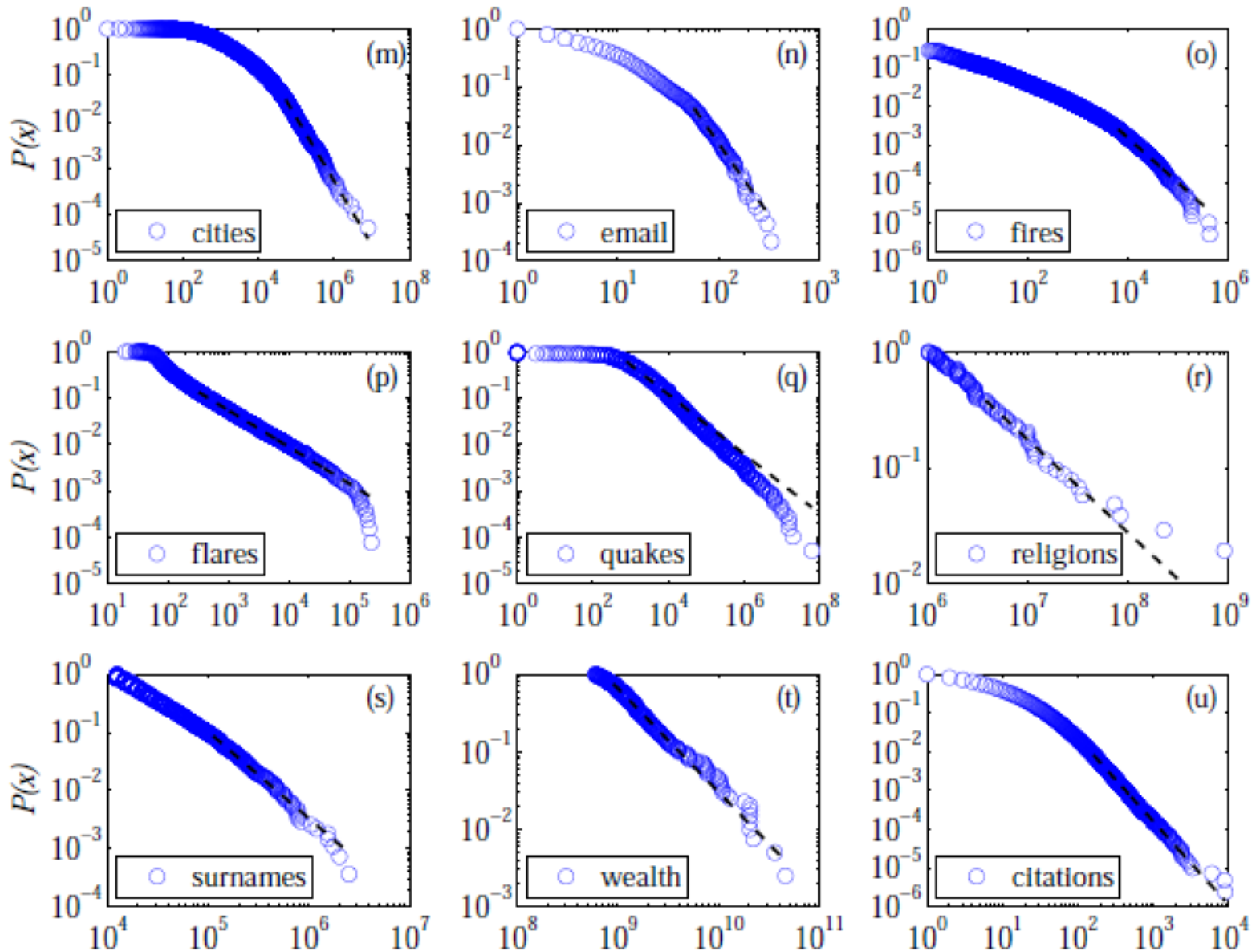


[Clauset-Shalizi-Newman 2007]

Power-Laws are Everywhere

- **In social systems – lots of power-laws:**
  - **Pareto, 1897** – Wealth distribution
  - **Lotka 1926** – Scientific output
  - **Yule 1920s** – Biological taxa and subtaxa
  - **Zipf 1940s** – Word frequency
  - **Simon 1950s** – City populations

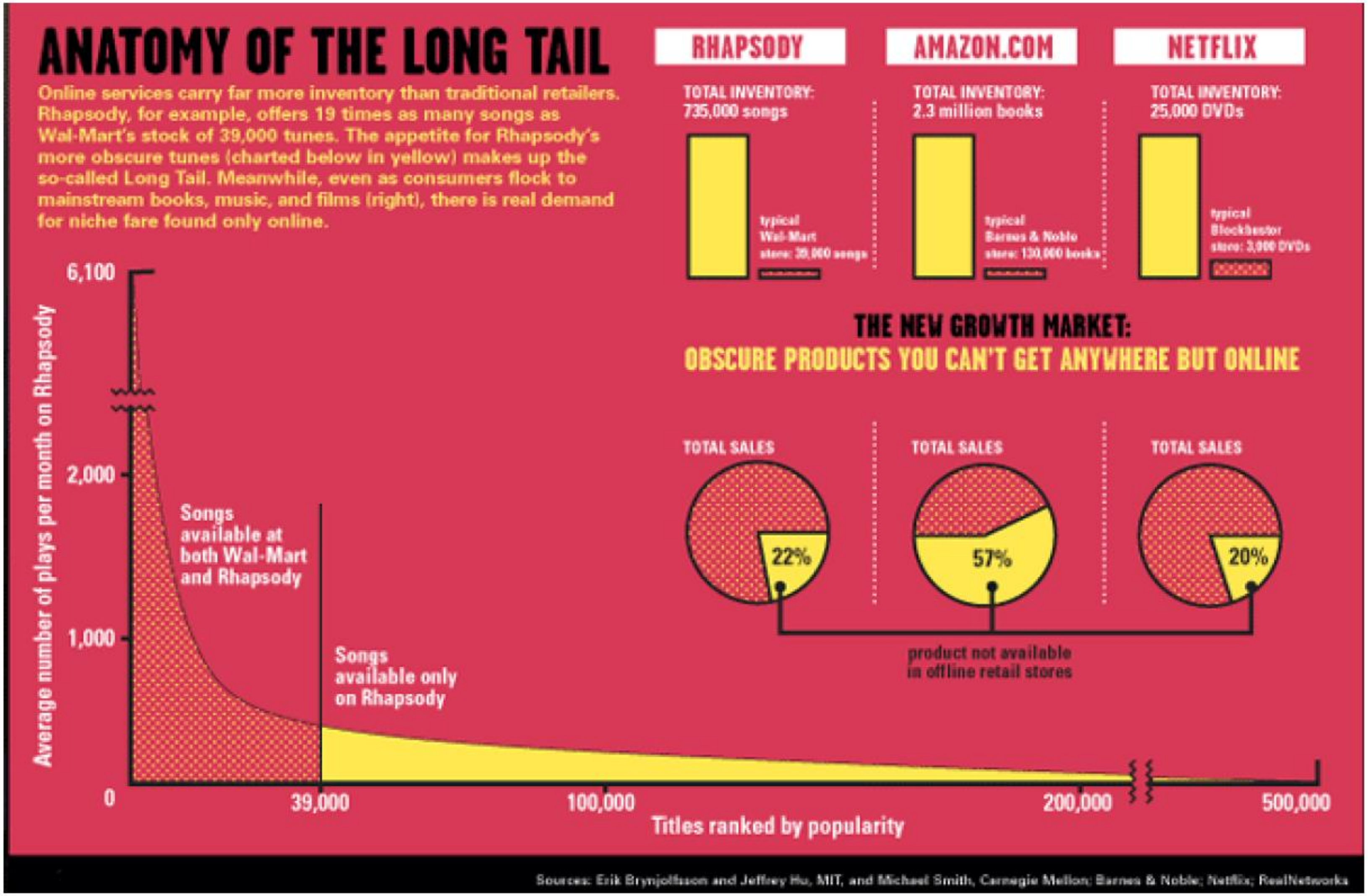
Power-Laws are Everywhere



Many other quantities follow heavy-tailed distributions

[Chris Anderson, Wired, 2004]

Anatomy of the Long Tail



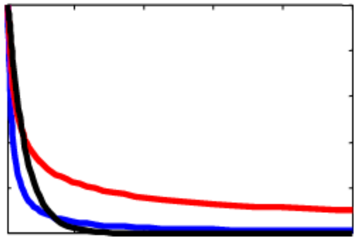
Mathematics of Power-Laws

Heavy Tailed Distributions

- **Degrees are heavily skewed:**

  Distribution $P(X > x)$ is **heavy tailed if:**

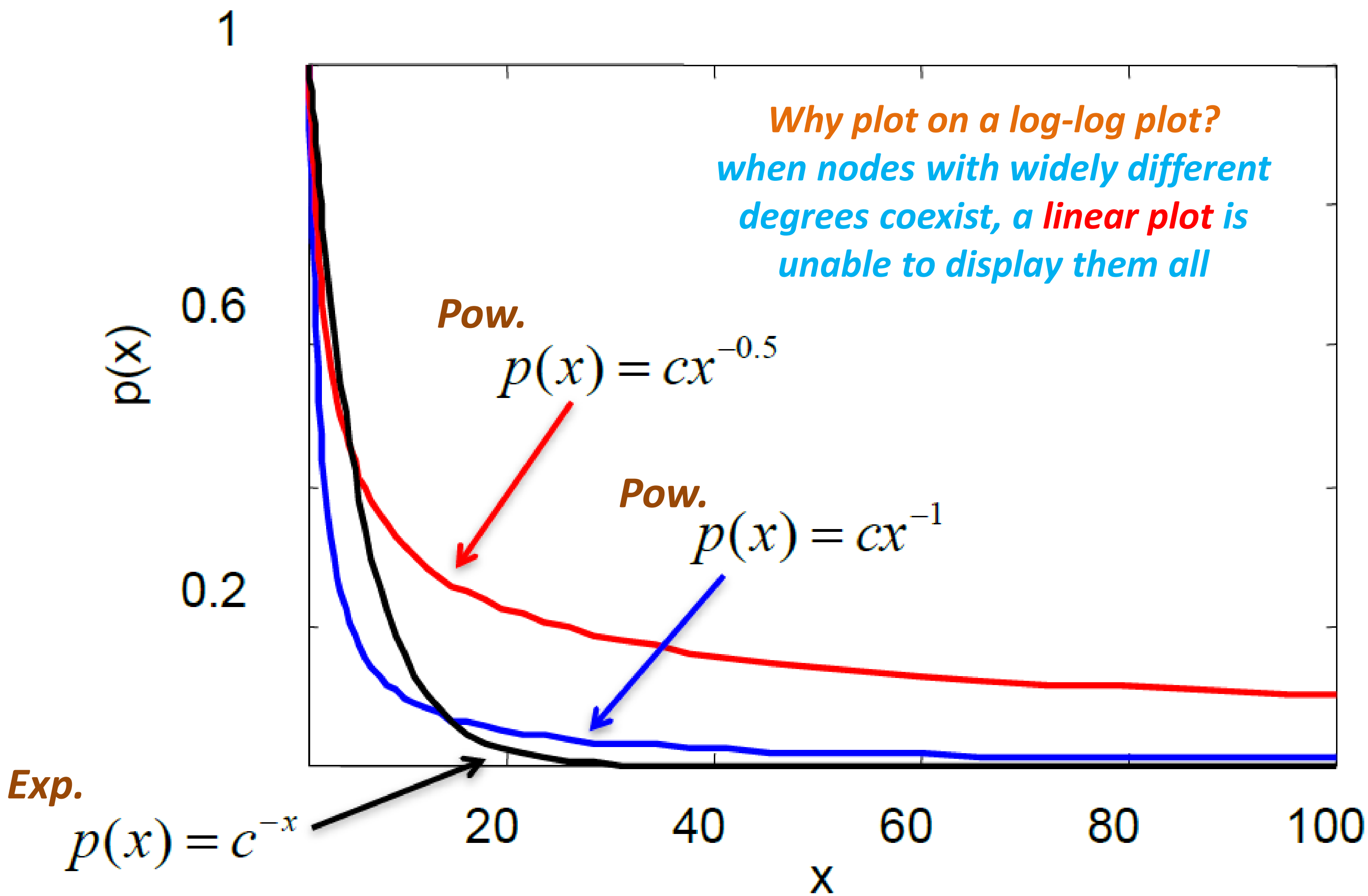
$$\lim_{x\to\infty} \frac{P(X > x)}{e^{-\lambda x}} = \infty$$



- **Note:**
  - **Normal PDF:** $p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - **Exponential PDF:** $p(x) = \lambda e^{-\lambda x}$
    - **then** $P(X > x) = 1 - P(X \leq x) = e^{-\lambda x}$

are not heavy tailed!

Exponential vs. Power-Law



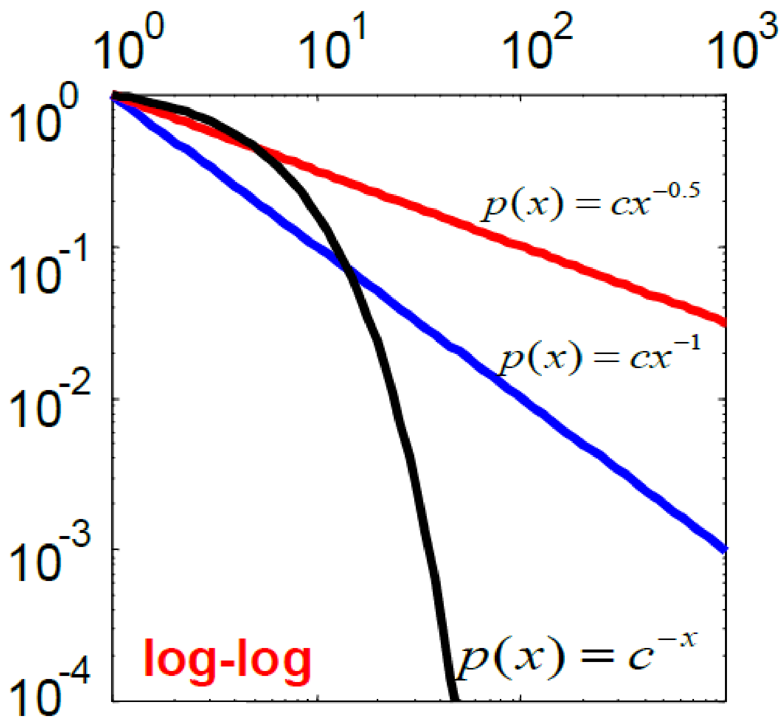
Above a certain x value, the power law is always higher than the exponential!

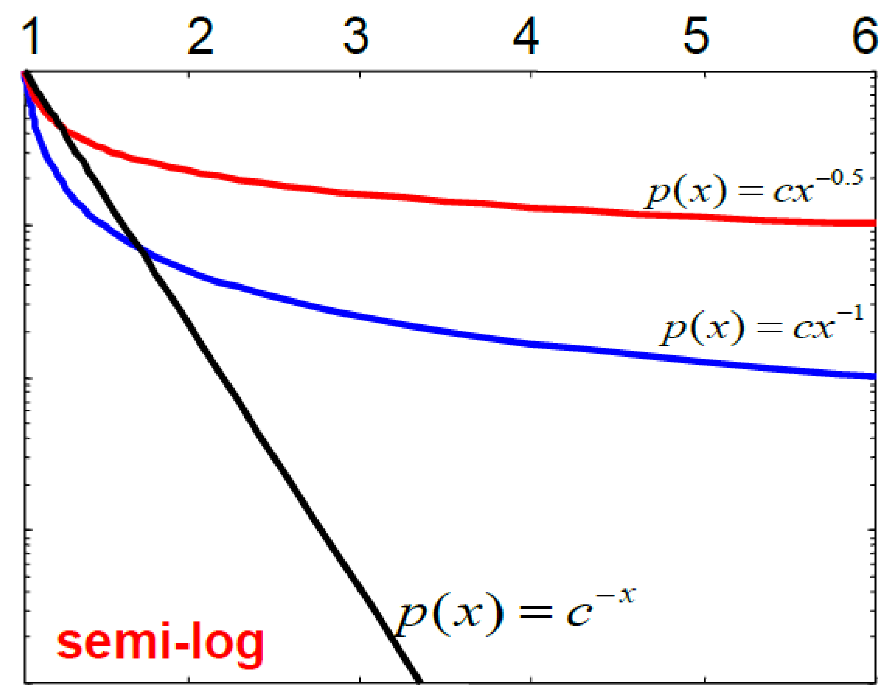
[Clauset-Shalizi-Newman 2007]

Exponential vs. Power-Law

Power-law vs. Exponential on log-log and semi-log (log-lin) scales

Exp. prob. dist. falls of a cliff and diminishes very quickly





x … logarithmic axis
y … logarithmic axis

x … linear
y … logarithmic

[Clauset-Shalizi-Newman 2007]

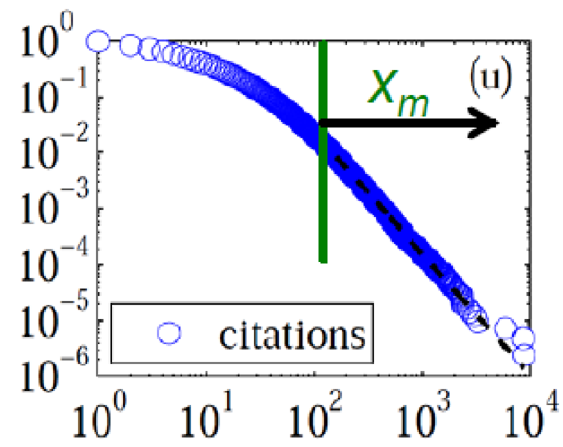
Heavy Tailed Distributions

- **Various names, kinds and forms:**
  - Long tail, Heavy tail, Zipf's law, Pareto's law
- **Heavy tailed distributions:**
  - **P(x) is proportional to:**

power law	$P(x) \propto x^{-\alpha}$
power law with cutoff	$x^{-\alpha} e^{-\lambda x}$
stretched exponential	$x^{\beta-1} e^{-\lambda x^{\beta}}$
log-normal	$\frac{1}{x} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right]$

NOT RESP.

Mathematics of Power-Laws



- **What is the normalizing constant?**

  $p(x) = Z\, x^{-\alpha}$ $\quad\quad Z = ?$

  - $p(x)$ is a distribution: $\int p(x)dx = 1$

Continuous approximation

$p(x)$ diverges as $x \to 0$ so x_m is the minimum value of the power-law distribution $x \in [x_m, \infty]$

  - $1 = \int_{x_m}^{\infty} p(x)dx = Z \int_{x_m}^{\infty} x^{-\alpha} dx$
  - $= -\frac{Z}{\alpha - 1}\left[x^{-\alpha+1}\right]_{x_m}^{\infty} = -\frac{Z}{\alpha-1}\left[\infty^{1-\alpha} - {x_m}^{1-\alpha}\right]$
  - $\Rightarrow Z = (\alpha - 1)x_m^{\alpha-1}$

Need: α > 1 !

$$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

Integral:

$\int (ax)^n = \frac{(ax)^{n+1}}{a(n+1)}$

Mathematics of Power-Laws

- **What's the expected value of a power-law random variable X?**
  - $E[X] = \int_{x_m}^{\infty} x\, p(x) dx = Z \int_{x_m}^{\infty} x^{-\alpha+1} dx$
  - $= \frac{Z}{2-\alpha} [x^{2-\alpha}]_{x_m}^{\infty} = \frac{(\alpha-1)x_m^{\alpha-1}}{-(\alpha-2)} [\infty^{2-\alpha} - x_m^{2-\alpha}]$

Need: $\alpha > 2$!

$$\Rightarrow E[X] = \frac{\alpha - 1}{\alpha - 2} x_m$$

Power-law density:

$$p(x) = \frac{\alpha - 1}{x_m} \left(\frac{x}{x_m}\right)^{-\alpha}$$

$$Z = \frac{\alpha - 1}{x_m^{1-\alpha}}$$

Moments of Degree Distributions

- The ***nth moment of the degree distribution*** is defined as:

$$\langle k^n \rangle = \sum_{k_{min}}^{\infty} k^n p_k = \int_{k_{min}}^{\infty} k^n p(k) dk$$

- **The lower moments have important interpretation:**
  - ***n=1:*** the first moment is the ***average degree***, $\langle k \rangle$
  - ***n=2:*** the second moment, $\langle k^2 \rangle$, is the ***variance*** $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$, measuring the ***spread*** in the degrees
    - its square root, σ, is the ***standard deviation***
  - ***n=3:*** the third moment, $\langle k^3 \rangle$, determines the ***skewness*** of a distribution, telling us how symmetric is p_k around the average $\langle k \rangle$. *Symmetric distributions have zero skewness*
- ***For a scale-free network*** the ***nth moment of the degree distribution*** is:

$$\langle k^n \rangle = \int_{k_{min}}^{k_{max}} k^n p(k) dk = C \frac{k_{max}^{n-\gamma+1} - k_{min}^{n-\gamma+1}}{n-\gamma+1}$$

Moments

- PDF

$$p(x) = \frac{C}{x^{\alpha}}, \; x \geq x_{min}$$

- First moment (mean value), α > 2

$$\langle x \rangle = \int_{x_{\min}}^{\infty} x p(x) dx = C \int_{x_{\min}}^{\infty} \frac{dx}{x^{\alpha-1}} = \frac{\alpha-1}{\alpha-2} x_{\min}$$

- Second moment, α > 3

$$\langle x^2 \rangle = \int_{x_{\min}}^{\infty} x^2 p(x) dx = C \int_{x_{\min}}^{\infty} \frac{dx}{x^{\alpha-2}} = \frac{\alpha-1}{\alpha-3} x_{\min}^2$$

- k^{th} moment, α > k +1

$$\langle x^k \rangle = \frac{\alpha-1}{\alpha-1-k} x_{\min}^k$$

Mathematics of P

$$\frac{(\alpha-1)x_m^{\alpha-1}}{-(\alpha-2)}\left[\infty^{2-\alpha} - x_m^{2-\alpha}\right]$$

- **Power-laws have *infinite moments*!**

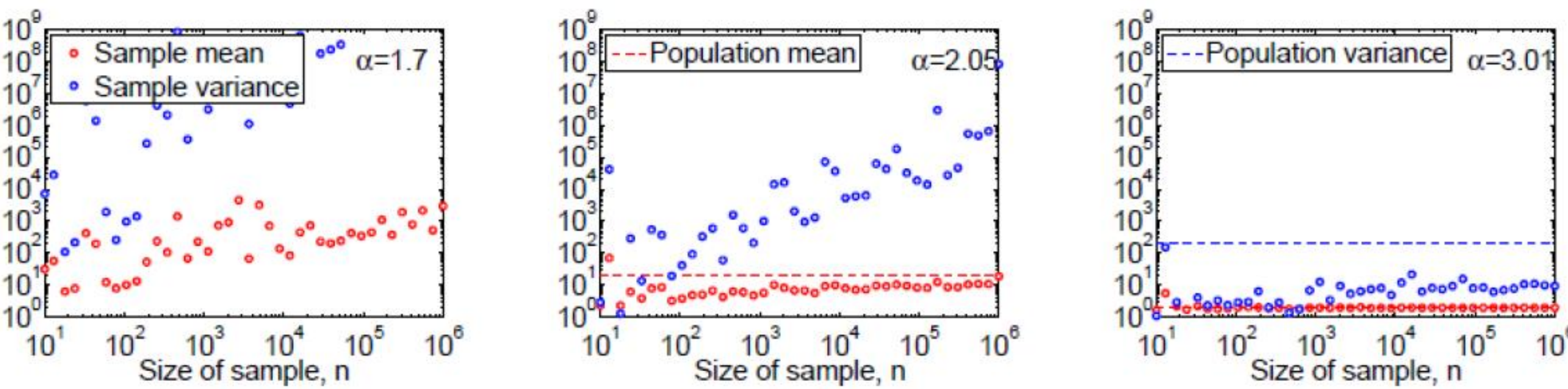
$$E[X] = \frac{\alpha - 1}{\alpha - 2} x_m$$

- If $\alpha \leq 2 : E[X] = \infty$
- If $\alpha \leq 3 : Var[X] = \infty$

In real networks
2 < α < 3 so:
E[X] = const
Var[X] = ∞

  - **Average is meaningless, as the variance is too high!**

- **<u>Consequence:</u> *Sample average* of n samples from a power-law with exponent α**



The Characteristics of Real Networks

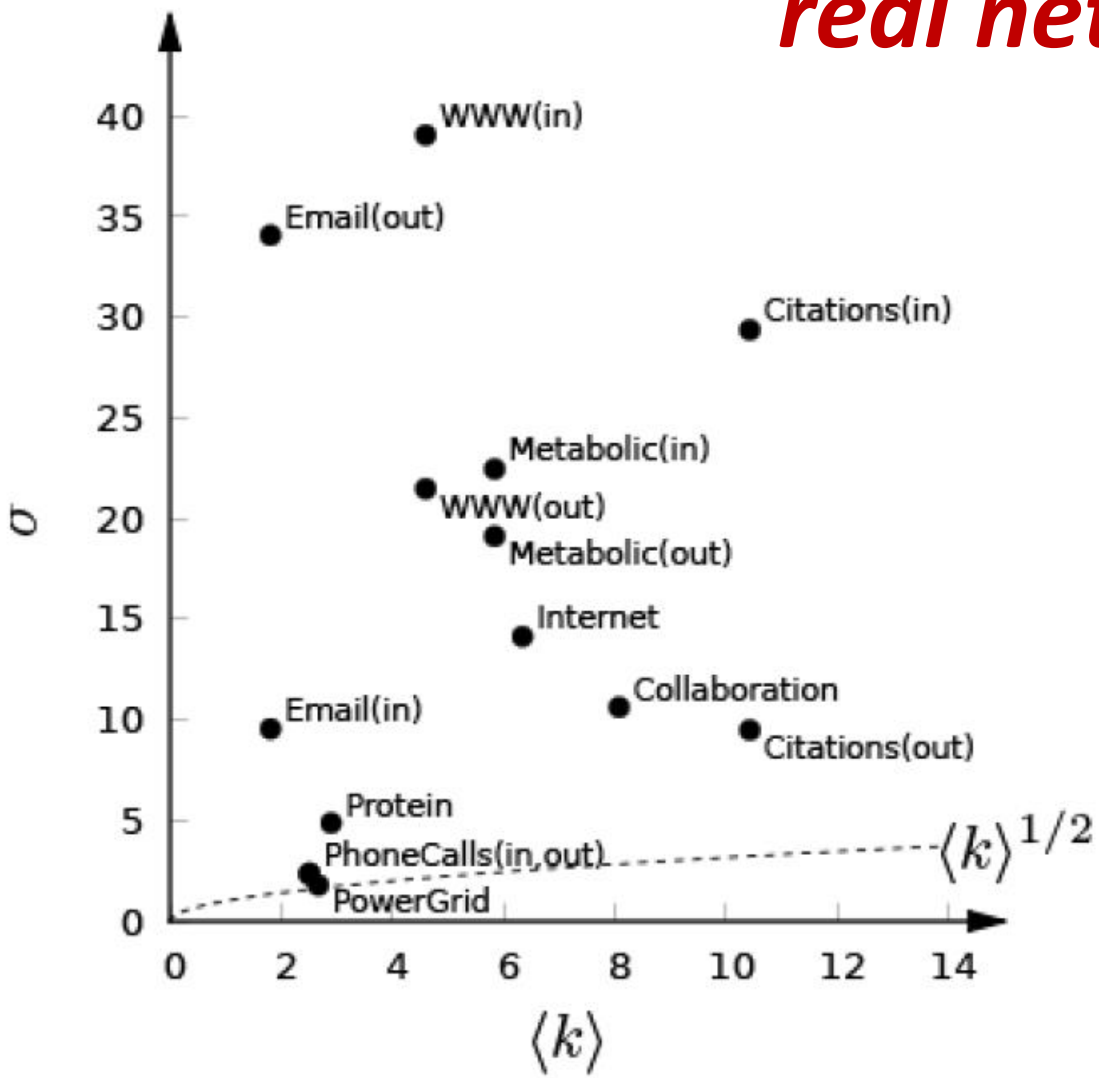
γ: estimated degree exponent

NETWORK	N	L	$\langle k \rangle$ $\langle k_{in} \rangle = \langle k_{out} \rangle$	σ_{in}	σ_{out}	σ	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	14.14	-	-	3.42*
WWW	325,729	1,497,134	4.60	39.05	21.48	-	2.31	2.00	-
Power Grid	4,941	6,594	2.67	-	-	1.79	-	-	Exp.
Mobile Phone Calls	36,595	91,826	2.51	2.39	2.32	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	9.56	34.07	-	3.43*	2.03	-
Science Collaboration	23,133	93,439	8.08	-	-	10.63	-	-	3.35
Actor Network	702,388	29,397,908	83.71	-	-	200.86	-	-	2.12
Citation Network	449,673	4,689,479	10.43	29.37	9.49	-	3.03**	4.00	-
E. Coli Metabolism	1,039	5,802	5.58	22.46	19.12	-	2.43	2.90	-
Yeast Protein Interactions	2,018	2,930	2.90	-	-	4.88	-	-	2.89*

$$\sigma = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$$

for most networks σ is much larger than ⟨k⟩
consequence of their scale-free nature

Standard deviation is large in ***real networks***



NOT RESP.

Mathematics of Power-Laws

Power-law density:

$$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

- **How to generate power-law random numbers?**
  - We want to generate $\boldsymbol{x}$ … a power-law distributed random number.

    Density: $$p(x) = \frac{\alpha-1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

  - But we have access to $\boldsymbol{r}$ … uniform random number

    Density: $p(r)$ =1 **if** $0 \le r \le 1$ **else** $p(r)$ =0

  - **We want to transform the densities!**
    - $\boldsymbol{P}(\boldsymbol{X} \le \boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{R} \le \boldsymbol{r})$

NOT RESP.

Mathematics of Power-Laws

- **Want to generate power-law random numbers!**
  - Equate the densities: $P(X \le x) = P(R \le r)$
  - $P(R \le r) = \int_0^r 1 dq = r$ *Prob. that R is ≤ r is r since R is a uniform random variable*
  - $P(X \le x) = \int_{x_m}^{x} \frac{\alpha-1}{x_m}\left(\frac{y}{x_m}\right)^{-\alpha} dy = \left[\frac{\alpha-1}{x_m} \cdot \frac{x_m}{1-\alpha}\left(\frac{y}{x_m}\right)^{1-\alpha}\right]_{x_m}^{x}$

$$= \left[-\left(\frac{y}{x_m}\right)^{1-\alpha}\right]_{x_m}^{x} = -\left(\frac{x}{x_m}\right)^{1-\alpha} + 1$$

  - Putting it all together: $r = 1 - \left(\frac{x}{x_m}\right)^{1-\alpha}$
  - Solving for x: $x = x_m(1-r)^{-1/(\alpha-1)}$

Integral:

$$\int (ax)^n = \frac{(ax)^{n+1}}{a(n+1)}$$

Power-law density:

$$p(x) = \frac{\alpha-1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

NOT RESP.

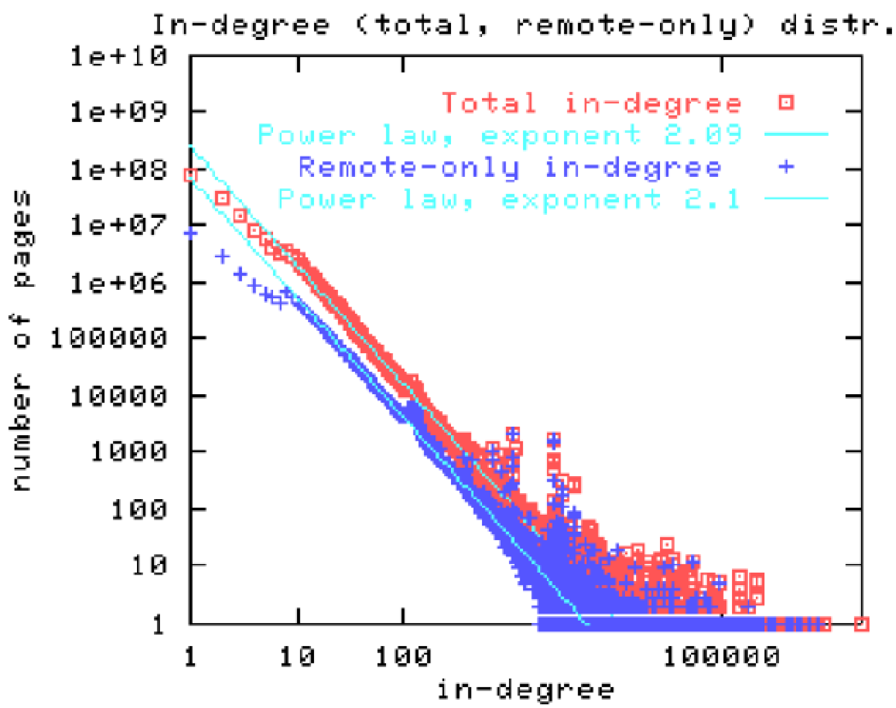
Estimating power-law exponent Alpha

NOT RESP.

Estimating Power-Law Exponent α

Estimating α from data:

- **(1)** Fit a line on log-log axis using least squares:
  - **Solve** $\arg\min_{\alpha}(\log(y) - \alpha\log(x) + b)^2$



BAD!

This approach can be affected by ***systematic biases***, resulting in an incorrect α (γ)

NOT RESP.

Systematic Fitting Issues

- ***A pure power law is really an idealized distribution that emerges in its form only in simple models***
  - In reality, a whole range of processes contribute to the topology of real networks
    - affecting the precise shape of the degree distribution
- The statistical tools used here to test the ***goodness of the fit*** rely on the ***Kolmogorov-Smirnov criteria***
  - measures the ***maximum distance*** between the *fitted model and the dataset*

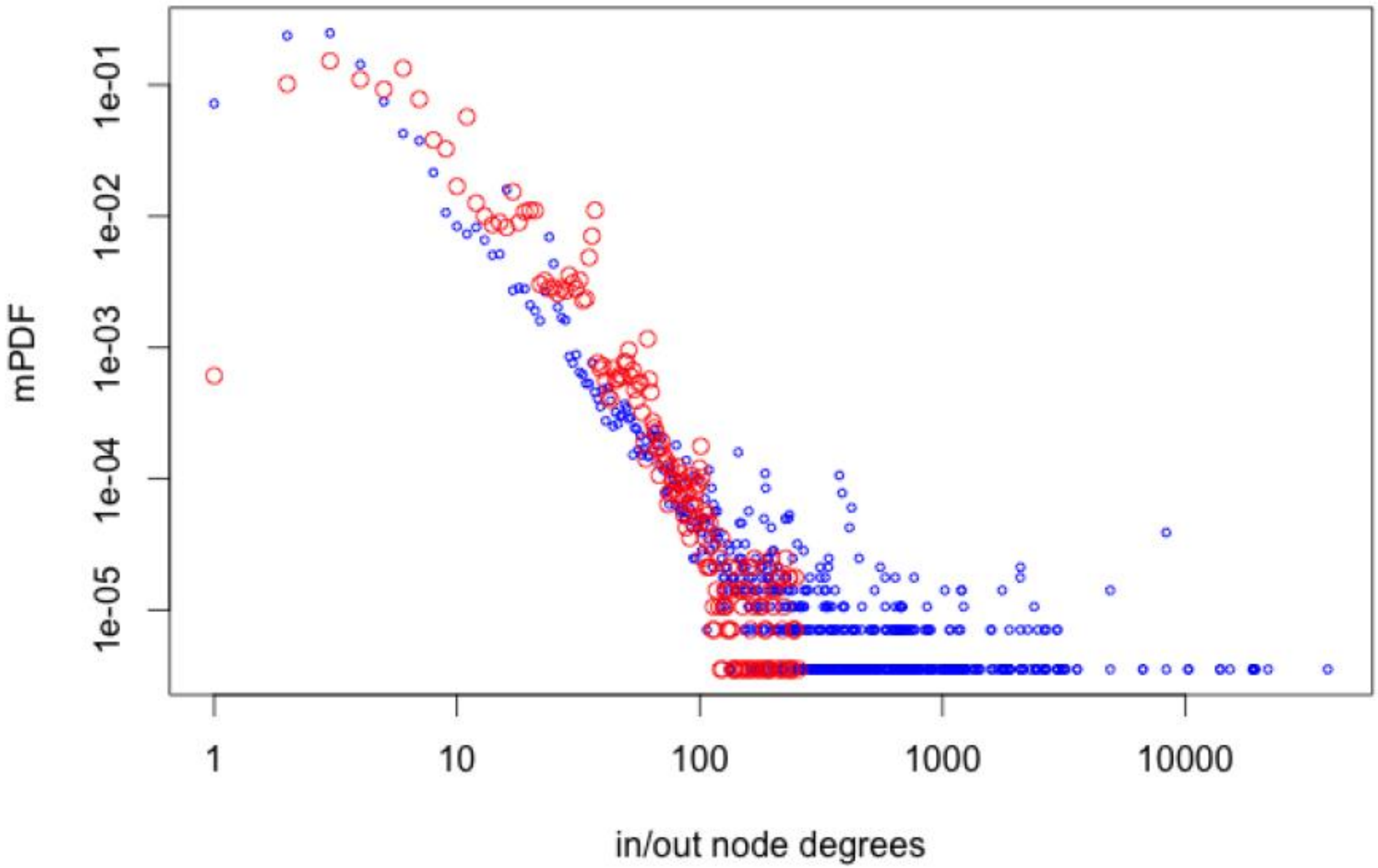
NOT RESP.

Systematic Fitting Issues

- If all data points follow a perfect power law, but ***a single point for some reason deviates from the curve***, we will ***lose the fit's statistical significance***
  - In real systems there are numerous reasons for such local deviations, that have little impact on the system's overall behavior
- The methodology described here often ***predicts a small scaling regime***
  - forcing us to remove a huge fraction of the nodes (often as many as 99%), to obtain a ***statistically significant fit***
  - Once plotted next to the original dataset, the obtained fit can be at times ridiculous, even if the method indicates statistical significance

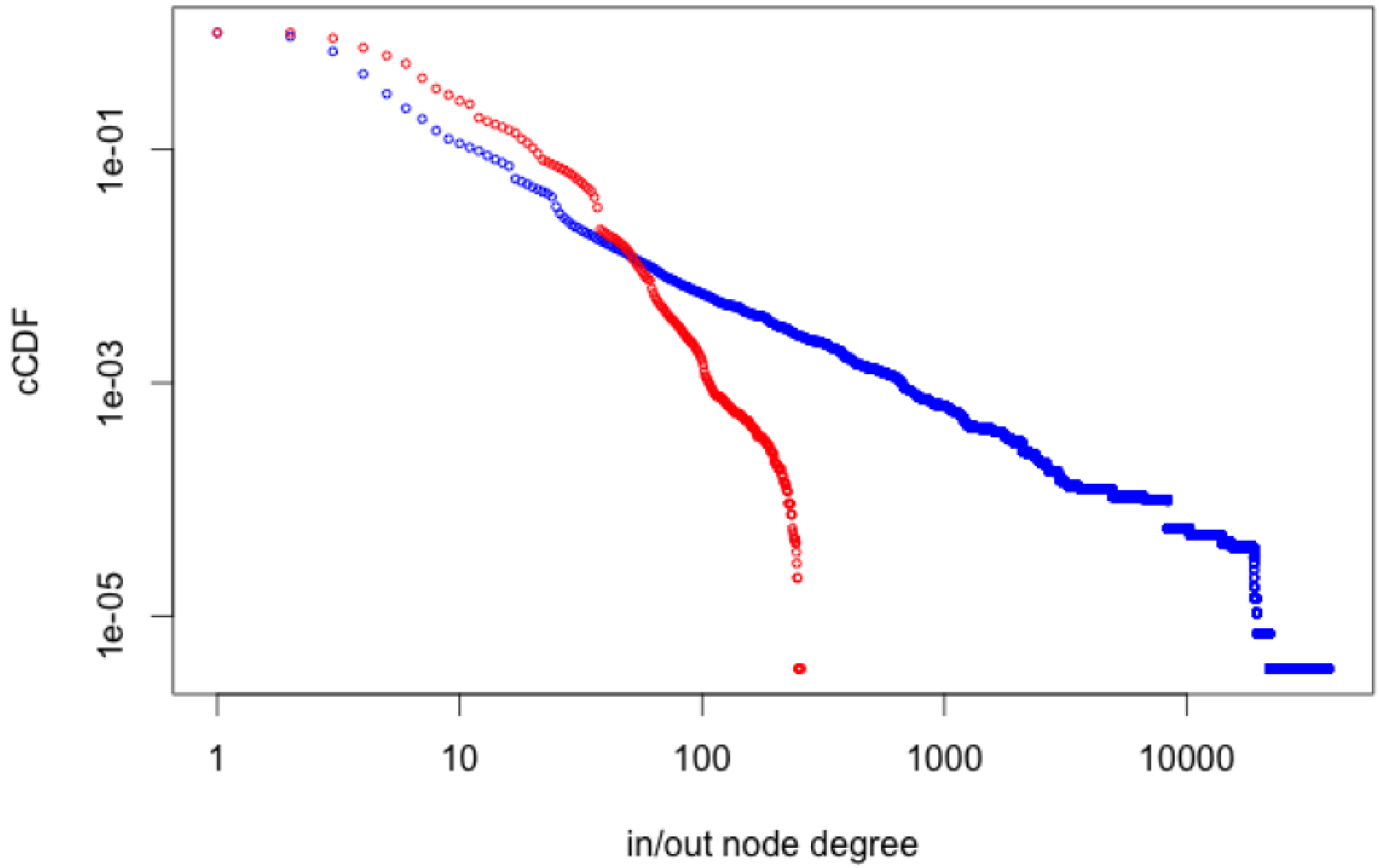
Power law networks

Probability mass function PMF/mPDF



Power law networks

Complementary cumulative distribution function cCDF



NOT RESP.

Estimating Power-Law Exponent α

OK!

Estimating α from data:

- Plot **Complementary CDF (CCDF)** $\boldsymbol{P(X \geq x)}$. Then the estimated $\boldsymbol{\alpha = 1 + \alpha'}$ where $\boldsymbol{\alpha'}$ is the slope of $\boldsymbol{P(X \geq x)}$.

- **Fact:** **If $\mathbf{p(x) = P(X = x)} \propto \boldsymbol{x^{-\alpha}}$ then $\mathbf{P(X} \geq \boldsymbol{x}\mathbf{)} \propto \boldsymbol{x^{-(\alpha-1)}}$**
  - $P(X \geq x) = \sum_{j=x}^{\infty} p(j) \approx \int_x^{\infty} Z\, y^{-\alpha} dy =$
  - $= \frac{Z}{1-\alpha}\left[y^{1-\alpha}\right]_x^{\infty} = \frac{Z}{1-\alpha} - x^{-(\alpha-1)}$

NOT RESP.

Estimating Power-Law Exponent α

OK!

Estimating α from data:

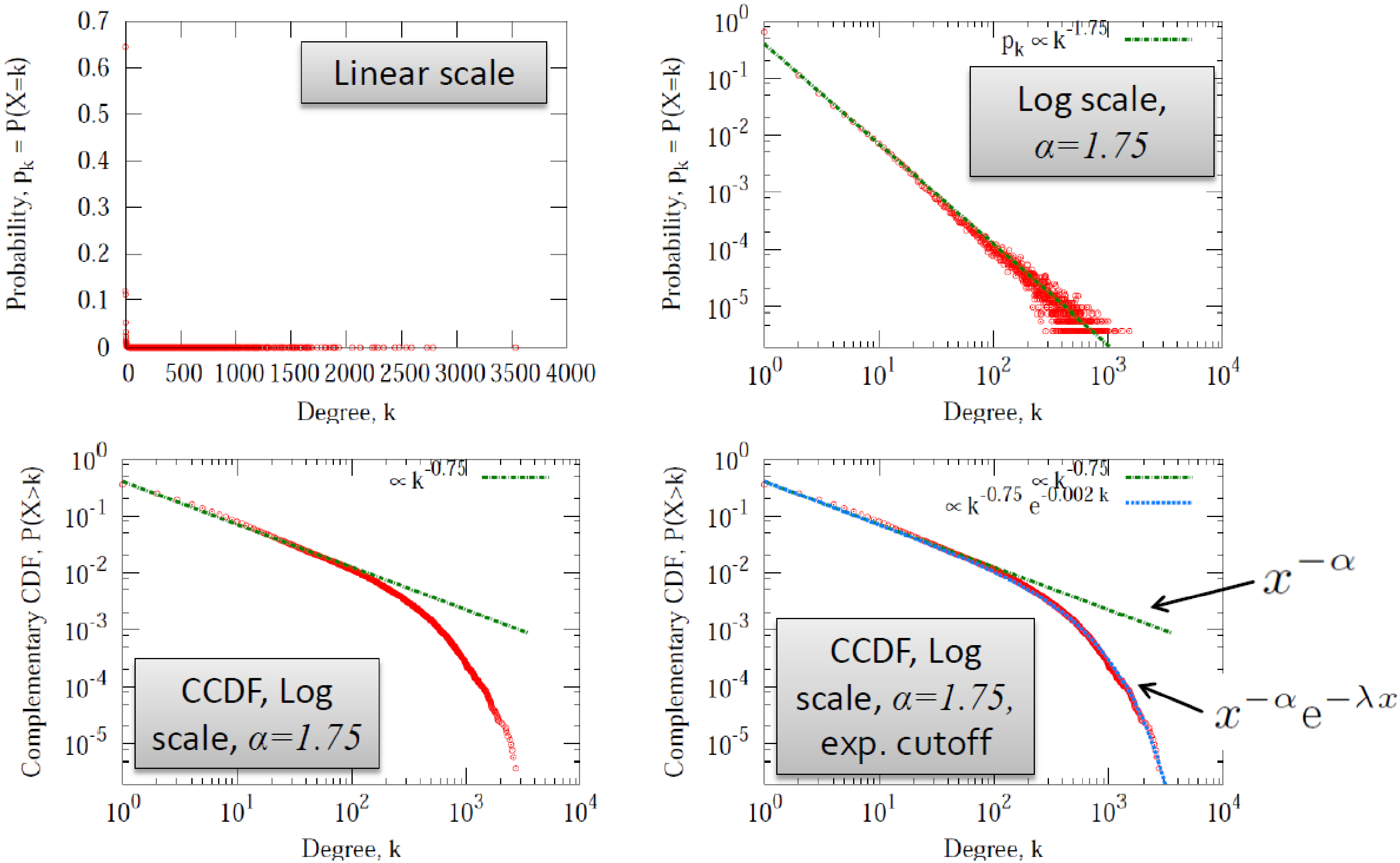
- **Use maximum likelihood approach:**
  - **The log-likelihood of observed data d_i:**
    - $L(\alpha) = \ln(\prod_i^n p(d_i)) = \sum_i^n \ln p(d_i)$
    - $= \sum_i^n \left(\ln(\alpha - 1) - \ln(x_m) - \alpha \ln\left(\frac{d_i}{x_m}\right)\right)$
  - **Want to find α that max $L(\alpha)$: Set** $\frac{\mathrm{dL}(\alpha)}{\mathrm{d}\alpha} = 0$
  - $\frac{\mathrm{dL}(\alpha)}{\mathrm{d}\alpha} = 0 \;\Rightarrow\; \frac{n}{\alpha - 1} - \sum_i^n \ln\left(\frac{d_i}{x_m}\right) = 0$
  - $\Rightarrow \boldsymbol{\hat{\alpha} = 1 + n\left[\sum_i^n \ln\left(\frac{d_i}{x_m}\right)\right]^{-1}}$

Power-law density:

$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$

NOT RESP.

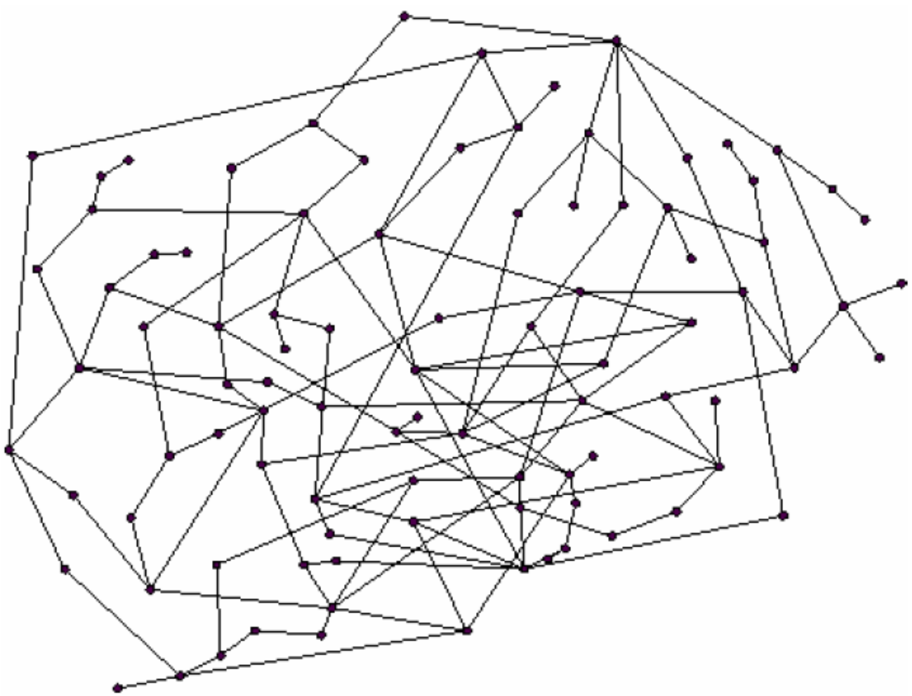
Flickr: Fitting Degree Exponent



Why are Power-Laws Surprising

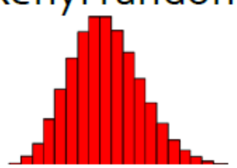
- **Not well characterized by the mean**
  - **Avg. US city size: 165k, StdDev: 410k**
  - **If human heights in US would be power-law**
    - Expect to have 60k as high as 2.72 m (world record), 10k people as high as giraffe and 1 person as high as Empire State building
- **Can not arise from sums of independent events!**
  - **Recall:** in $\boldsymbol{G_{np}}$ each pair of nodes in connected independently with prob. $\boldsymbol{p}$
    - $\boldsymbol{X}$… degree of node $\boldsymbol{v}$
    - $\boldsymbol{X_w}$ … event that $\boldsymbol{w}$ links to $\boldsymbol{v}$
    - $\boldsymbol{X = \sum_w X_w}$
    - $\boldsymbol{E[X] = \sum_w E[X_w] = (n-1)p}$
  - **Now, what is $\boldsymbol{P(X=k)}$? Central limit theorem!**
    - $\boldsymbol{X_1, \dots, X_n}$: random vars with mean $\boldsymbol{\mu}$, variance $\boldsymbol{\sigma^2}$
    - $\boldsymbol{S_n = \sum_i X_i}$: $E[S_n] = \boldsymbol{n\mu}$, $\mathrm{Var}[S_n] = \boldsymbol{n\sigma^2}$, $\mathrm{SD}[S_n] = \boldsymbol{\sigma\sqrt{n}}$
    - $\boldsymbol{P(S_n = E[S_n] + x \cdot \mathrm{SD}[S_n]) \sim \frac{1}{2\pi} e^{-\frac{x^2}{2}}}$

Random vs. Scale-free network

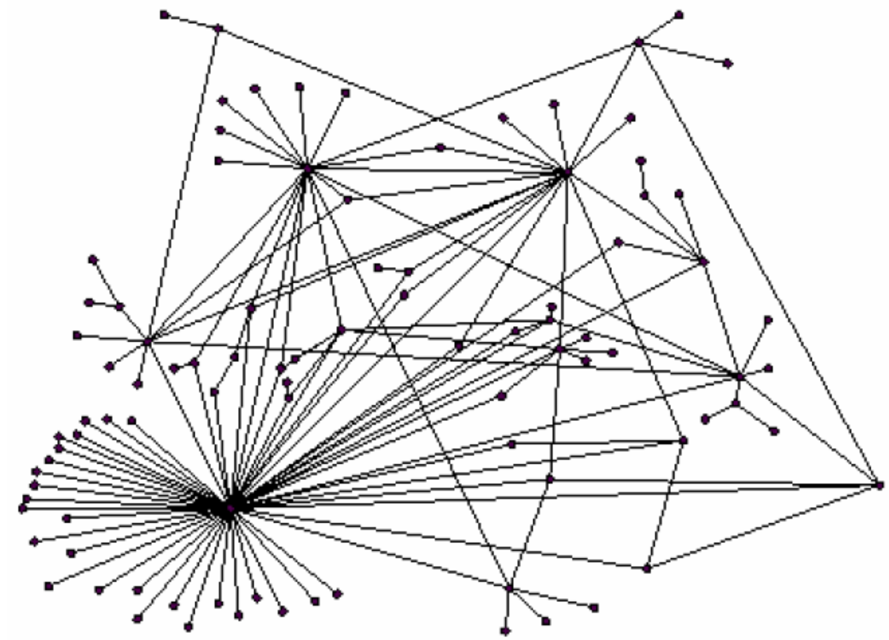


Random network

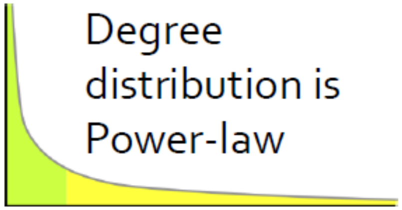
(Erdos-Renyi random graph)



Degree distribution is Binomial



Scale-free (power-law) network



Degree distribution is Power-law

Ultra Small Property

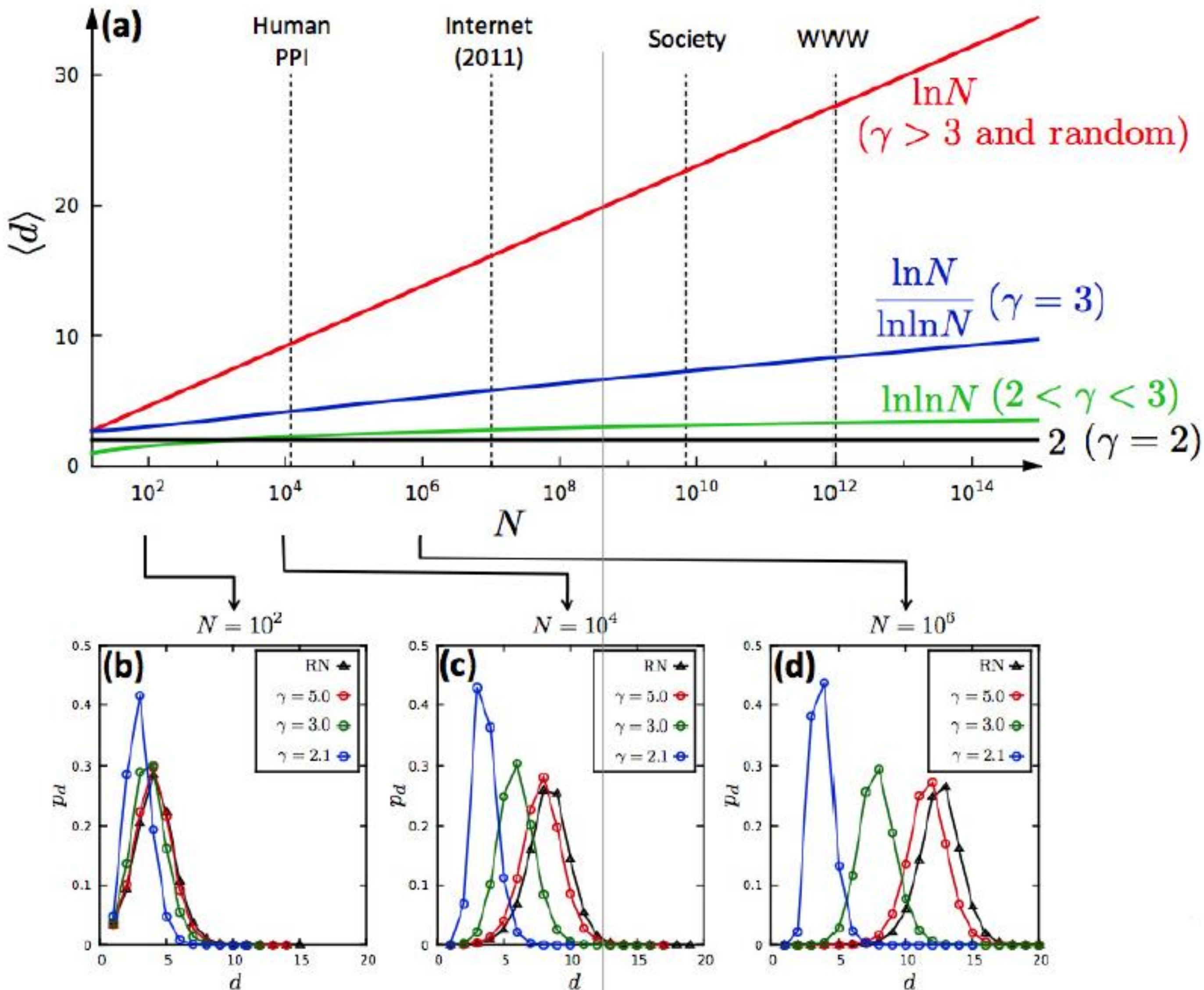
How do hubs affect the small world property?

- **Distances in a scale-free network** are either smaller or equal to the distances observed in an equivalent ***random network***

- The precise dependence of the average distance $\langle d \rangle$ on
  - the ***system size N***
  - the ***degree exponent γ (α)***

$$d \sim \begin{cases} \text{const.} & \text{if } \gamma = 2, \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & \text{if } 2 < \gamma < 3, \\ \dfrac{\ln N}{\ln \ln N} & \text{if } \gamma = 3, \\ \ln N & \text{if } \gamma > 3. \end{cases}$$

Distances in Scale-Free Networks



At a glance
Scale-free networks

DEGREE DISTRIBUTION

Discrete form:

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}.$$

Continuous form:

$$p(k) = (\gamma - 1)k_{min}^{\gamma-1} k^{-\gamma}.$$

SIZE OF THE LARGEST HUB

$$k_{max} \sim k_{min} N^{\frac{1}{\gamma-1}}.$$

MOMENTS OF p_k

$2 < \gamma < 3$: $\langle k \rangle$ finite, $\langle k^2 \rangle$ diverges when $N \to \infty$.

$\gamma > 3$: $\langle k \rangle$ and $\langle k^2 \rangle$ finite.

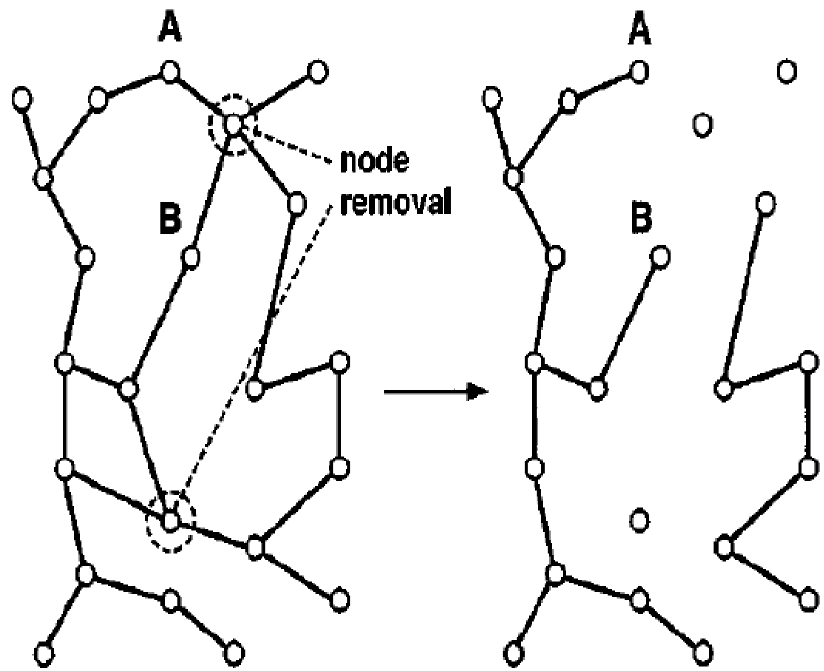
DISTANCES IN A SCALE-FREE NETWORK

$$d \sim \begin{cases} \text{const.} & \text{if } \gamma = 2, \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & \text{if } 2 < \gamma < 3, \\ \dfrac{\ln N}{\ln \ln N} & \text{if } \gamma = 3, \\ \ln N & \text{if } \gamma > 3. \end{cases}$$

Consequence of Power-Law Degrees

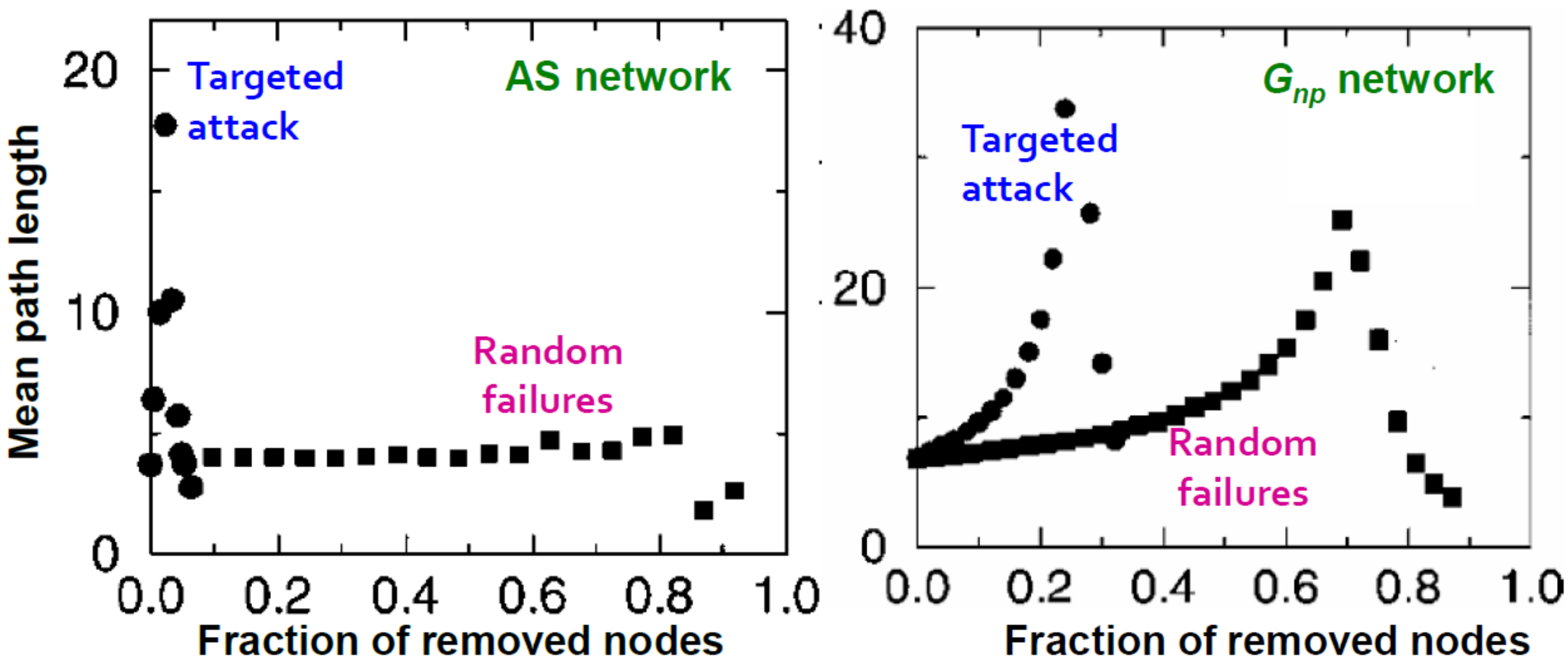
Consequence: Network Resilience

- **How does network connectivity change as nodes get removed?**
  [Albert et al. 00; Palmer et al. 01]
- **Nodes can be removed:**
  - **Random failure:**
    - Remove nodes uniformly at random
  - **Targeted attack:**
    - Remove nodes in order of decreasing degree



- This is important for ***robustness of the internet*** as well as ***epidemiology***
  - *Removal of vertices corresponds to vaccination*

Network Resilience



- ***Real networks*** are resilient to ***random failures***
- ***G_{np}*** has better resilience to ***targeted attacks***
  - Need to remove all pages of degree > 5 to disconnect the Web
  - But this is a very small fraction of all web pages