# DA 507 - Optimization and Modeling

Lecture 2
Linear Programming: Examples
Integer Programming: Examples

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# Agenda

- **Linear Programming: Examples**
- **Integer Programming: Examples**
- **Project ideas discussion**
- Solving LPs: Graphical Solution Method
- Solving LPs: Simplex Algorithm
- Economic interpretation of Simplex algorithm
- Solving IPs: Branch-and-bound

# Introduction to Optimization Modeling

- Prescriptive models "prescribe" behavior for an organization that will enables it to best meet its goals. Components of this model include
  - objective function(s),
  - decision variables,
  - constraints.
- An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

# Introduction to Optimization Modeling (Eli Daisy)

- Eli Daisy produces the drug Wozac in huge batches by heating a chemical mixture in a pressurized container. Each time a batch is produced, a different amount of Wozac is produced. The amount produced is the process yield (measured in pounds).
- Daisy is interested in understanding the factors that influence the yield of Wozac production process.

We seek to maximize the yield for the production process. To maximize the process yield we need to find the values of V, P, T, A, B, and C that make the yield equation (below) as large as possible.

$$Yield = 300 + 0.8V + 0.01P + 0.06T + 0.001TP - 0.01T^2 – 0.001P^2$$
$$+11.7A + 9.4B + 16.4C + 19A * B + 11.4AC – 9.6BC$$

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- Objective function
- A function to be maximized or minimized
- Only certain values of the decision variables are possible; such restrictions on the decision variable values are called constraints

# Introduction to Optimization Modeling (Eli Daisy)

A descriptive model to determine the following factors influence yield:

- Container volume in liters (V)
- Container pressure in milliliters (P)
- Container pressure in degrees centigrade (T)
- Chemical composition of the processed mixture (A, B, C)

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- Chemical composition of the processed mixture (A, B, C)

- Volume must be between 1 and 5 liters ($1 \leq V \leq 5$),
- Pressure must be between 200 and 400 milliliters ($200 \leq P \leq 400$),
- Temperature must be between 100 and 200 degrees centigrade ($100 \leq T \leq 200$),
- Mixture must be made up entirely of A, B, and C ($A + B + C = 1$),
- For the drug to perform properly, only half the mixture at most can be product A ($A \leq 0.5$).

# Eli Daisy

Any specification of the decision variables (i.e. volume, pressure, etc..) that satisfies all the model's constraints is said to be in the *feasible region*. For example, a *feasible solution* is
V = 2, P = 300, T = 150, A = 0.4, B = 0.3, C = 0.3

Using some optimization software, it can be determined that the *optimal solution*, i.e. a specification of the decision variables that satisfy the constraints and produces the maximum yield, to its model is
V = 5, P = 200, T = 100, A = 0.294, B = 0, C = 0.706 with yield 209.384

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  - Deterministic: The coefficient of the cost and the constraints are known apriori (or are approximated)
- Some of these requirements can be relaxed/modeled by using integer programming (at the expense of computation time).

# What is Linear Programming (LP)?

- Linear programming (LP) is concerned with the optimization of a linear function over a set of linear constraints or restrictions.
- In 1947, George B. Dantzig worked on a logistical supply "program." Everything, in fact, started with Dantzig's work.
- However even before Dantzig, in 1939, Kantorovich worked on this problem and found a solution with organization and planning.
- Since Kantorovich's work is not known until 1959, the solution method proposed by Dantzig now enjoys a wide acceptance. This solution method is known as the simplex method.

# The Carpenter's Problem

- A carpenter tries to solve a weekly production planning problem to maximize profit. He solely makes tables and chairs, and sells all tables and chairs at a market place.
- Both chairs and tables require the same type of raw material. The raw material availability is only 50 units per week.
  - one chair requires 1 unit of the raw material;
  - one table requires 2 units of the raw material.
- The carpenter can work at most 40 hours every week. It takes 2 hours to make a chair while it takes only an hour to make a table.
- The cost of a chair is 100 TL and it sells for 120 TL; the cost of a table is 130 TL and it sells for 145 TL.

# The Carpenter's Problem

- Develop a profit maximizing mathematical model which takes into account both the raw material and labor hour constraints:
  - Define your decision variables.
  - Write down the objective function.
  - Write down the constraints and the domain of the decision variables.

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- $x_1$ = number of chairs produced/sold each week
- $x_2$ = number of table produced/sold each week

$$
\begin{aligned}
\textit{maximize } & (120-100)x_1 + (145-130)x_2, && && (1)\\
\text{subject to } & 2x_1 + x_2 \leq 40, && \text{(labor)} && (2)\\
& x_1 + 2x_2 \leq 50, && \text{(material)} && (3)\\
& x_1, x_2 \geq 0. && \text{(domain)} && (4)
\end{aligned}
$$

Consider the labor constraint only:
$2x_1 + x_2 \leq 40$

- How many chairs can you produce if you do not produce any tables?

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- How many tables can you produce if you do not produce any chairs?
    - 40
- How many tables can you produce if you produce 10 chairs?

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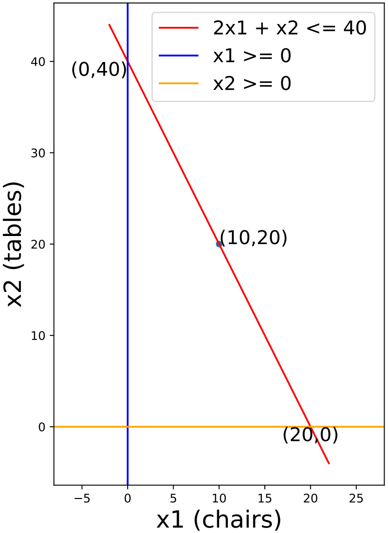

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  - 50
- How many tables can you produce if you do not produce any chairs?
  - 25
- How many tables can you produce if you produce 20 chairs?

Consider the material constraint only:

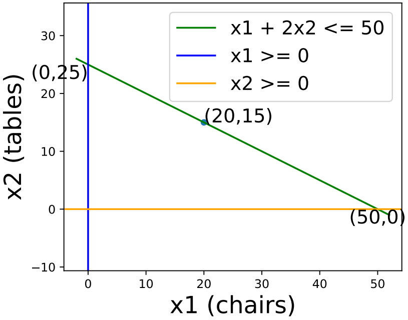
$x_1 + 2x_2 \leq 50$

- How many chairs can you produce if you do not produce any tables?
  - 50
- How many tables can you produce if you do not produce any chairs?
  - 25
- How many tables can you produce if you produce 20 chairs?
  - 15

Consider the material constraint only:

$x_1 + 2x_2 \leq 50$

- How many chairs can you produce if you do not produce any tables?
  - 50
- How many tables can you produce if you do not produce any chairs?
  - 25
- How many tables can you produce if you produce 20 chairs?
  - 15



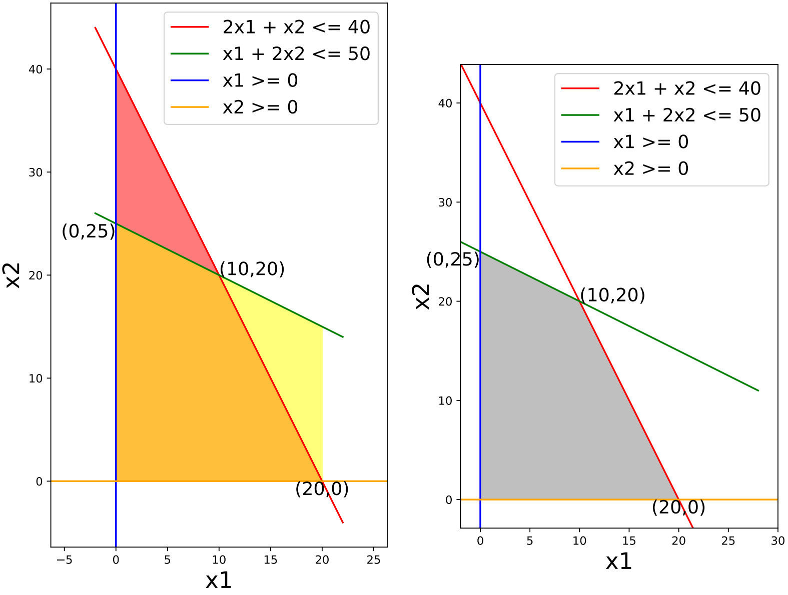
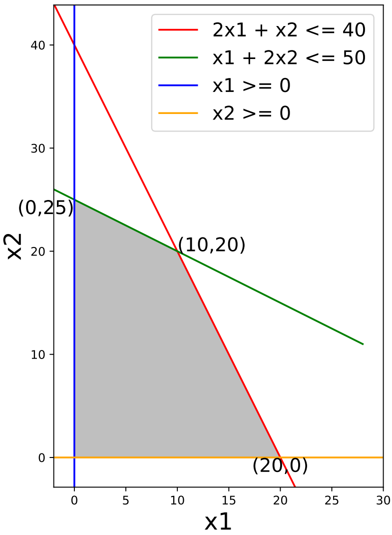



Figure: Feasible region.

Feasible region lies in the area between the points

- (0,0), i.e. $x_1 = 0, x_2 = 0$
- (20,0), $x_1 = 20, x_2 = 0$
- (10,20), $x_1 = 10, x_2 = 20$
- (0,25), $x_1 = 0, x_2 = 25$



The carpenter realizes that he needs a special glue to make the products and that every week he only has enough for 28 chairs or tables. He uses the same amount of glue for a chair and a table.

How do we modify the model?

- Decision variables

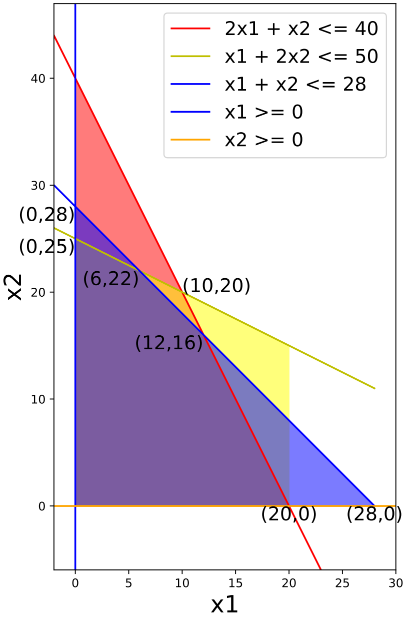
# The Carpenter's Problem (New Constraint)

The carpenter realizes that he needs a special glue to make the products and that every week he only has enough for 28 chairs or tables. He uses the same amount of glue for a chair and a table.
How do we modify the model?

- Decision variables
  - No change
- Constraints
  - $x_1 + x_2 \leq 28$
- Feasible region
  - Changes
- Profit
  - Changes

Feasible region lies in the area between the points

- (0,0), i.e. $x_1 = 0, x_2 = 0$
- (20,0), $x_1 = 20, x_2 = 0$
- (12,16), $x_1 = 12, x_2 = 16$
- (6,22), $x_1 = 6, x_2 = 22$
- (0,25), $x_1 = 0, x_2 = 25$



# The Carpenter's Problem (New Product)

The carpenter can also produce stools with a profit of 6\$. Each stool uses 0.5 hours of labor and 0.5 unit of material.

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- $x_3$ = number of stools produced/sold each week

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- $x_3$ = number of stools produced/sold each week

$$
\begin{aligned}
\textit{maximize } & 20x_1 + 15x_2 + 6x_3, && && (5)\\
\text{subject to } & 2x_1 + x_2 + 0.5x_3 \leq 40, && \text{(labor)} && (6)\\
& x_1 + 2x_2 + 0.5x_3 \leq 50, && \text{(material)} && (7)\\
& x_1, x_2, x_3 \geq 0. && \text{(domain)} && (8)
\end{aligned}
$$

# GUROBI

# The Dream Diet Meal Plan

- My dream diet requires that all the food I eat come from one of the four "basic food groups"
- At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola and strawberry cheesecake.
- Each brownie costs 50 cents, each scoop of icecream costs 20 cents ,each bottle of cola costs 30 cents and each piece of cheesecake costs 80 cents.
- Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar and 8 oz of fat. The nutritional content per unit of each food is shown in the table below. Formulate a linear programming model that can be used to satisfy my daily requirements at minimum cost.

# The Dream Diet Meal Plan

| Type of food | Calories | Chocolate | Sugar | Fat |
|---|---|---|---|---|
| Brownie | 400 | 3 | 2 | 2 |
| Ice cream (1 scoop) | 200 | 2 | 2 | 4 |
| Coke (1 bottle) | 150 | 0 | 4 | 1 |
| Cheesecake (1 piece) | 500 | 0 | 4 | 5 |

Decision variables:

- $x_1$ = brownie pieces
- $x_2$ = scoops of ice cream
- $x_3$ = bottles of coke
- $x_4$ = cheesecake pieces

# The Dream Diet Meal Plan

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$$
\begin{aligned}
\textit{minimize} \quad & 50x_1 + 20x_2 + 30x_3 + 80x_4, && && (9)\\
\text{subject to} \quad & 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500, && \text{(calories)} && (10)\\
& 3x_1 + 2x_2 \geq 6, && \text{(chocolate)} && (11)\\
& 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10, && \text{(sugar)} && (12)\\
& 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8, && \text{(fat)} && (13)\\
& x_1, x_2, x_3, x_4 \geq 0. && \text{(domain)} && (14)
\end{aligned}
$$

# A Multi-Plant Production Model

- There are two factories A and B. Each factory makes two products, standard and deluxe:
  - a unit of standard gives a profit contribution of 10 TL;
  - a unit of deluxe gives a profit contribution of 15 TL.
- Each factory uses two processes: polishing and grinding.
  - Factory A has a grinding capacity of 80 hours and polishing capacity for 60 hours.
  - Factory B has a grinding capacity of 60 hours and polishing capacity for 75 hours.
- Grinding a standard product in factory A takes 4 hours, while a deluxe product takes 2 hours. The same processing times in factory B are 5 and 3 hours, respectively.
- Polishing a standard product in factory A takes 2 hours, while a deluxe product takes 5 hours. The same processing times in factory B are 5 and 6 hours, respectively.

# A Multi-Plant Production Model

- Each unit of each product uses 4 kg. of a raw material. Of 120 kg raw material, the company has allocated
  - 75 kg to factory A, and
  - 45 kg to factory B.
- Formulate a mathematical model to determine the production amounts of each product in the factories such that the overall profit is maximized.

Decision variables:

- $x_{SA}$ = the amount of standard product produced at plant A
- $x_{DA}, x_{SB}, x_{DB}$ are defined similarly for products deluxe and plant B

# A Multi-Plant Production Model

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$$
\begin{aligned}
\textit{maximize} \quad & 10(x_{SA} + x_{SB}) + 15(x_{DA} + x_{DB}), && && (15)\\
\text{subject to} \quad & 4(x_{SA} + x_{DA}) \leq 75, && \text{(material (A))} && (16)\\
& 4x_{SA} + 2x_{DA} \leq 80, && \text{(grinding (A))} && (17)\\
& 2x_{SA} + 5x_{DA} \leq 60, && \text{(polishing (A))} && (18)\\
& 4(x_{SB} + x_{DB}) \leq 45, && \text{(material (B))} && (19)\\
& 5x_{SB} + 3x_{DB} \leq 60, && \text{(grinding (B))} && (20)\\
& 5x_{SB} + 6x_{DB} \leq 75, && \text{(polishing (B))} && (21)\\
& x_{SA}, x_{DA}, x_{SB}, x_{DB} \geq 0. && \text{(domain)} && (22)
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\end{aligned}$$

$$\begin{aligned}
\textit{maximize } & 10x_{SB} + 15x_{DB}, && && (28)\\
\text{subject to } & 4(x_{SB} + x_{DB}) \leq 45, && \text{(material (B))} && (29)\\
& 5x_{SB} + 3x_{DB} \leq 60, && \text{(grinding (B))} && (30)\\
& 5x_{SB} + 6x_{DB} \leq 75, && \text{(polishing (B))} && (31)\\
& x_{SB}, x_{DB} \geq 0. && \text{(domain)} && (32)
\end{aligned}$$

Table: Optimal solution for plants A and B.

| | Plant A | Plant B | Total |
|---|---|---|---|
| Standart | 11.25 | 0 | 11.25 |
| Deluxe | 7.5 | 11.25 | 18.75 |
| Profit | 225 | 168.75 | 393.75 |

# A Multi-Plant Production Model

- Each unit of each product uses 4 kg. of a raw material. **The company has 120 kg of raw material available.** ~~Of 120 kg raw material, the company has allocated~~
  - ~~75 kg to factory A, and~~
  - ~~45 kg to factory B.~~
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& 2x_{SA} + 5x_{DA} \le 60, && \text{(polishing (A))} && (35)\\
& 5x_{SB} + 3x_{DB} \le 60, && \text{(grinding (B))} && (36)\\
& 5x_{SB} + 6x_{DB} \le 75, && \text{(polishing (B))} && (37)\\
& 4x_{SA} + 4x_{DA} + 4x_{SB} + 4x_{DB} \le 120, && \text{(material)} && (38)\\
& x_{SA}, x_{DA}, x_{SB}, x_{DB} \ge 0. && \text{(domain)} && (39)
\end{aligned}
$$

# A Multi-Plant Production Model

Table: Optimal solutions for fixed and free allocation scenarios.

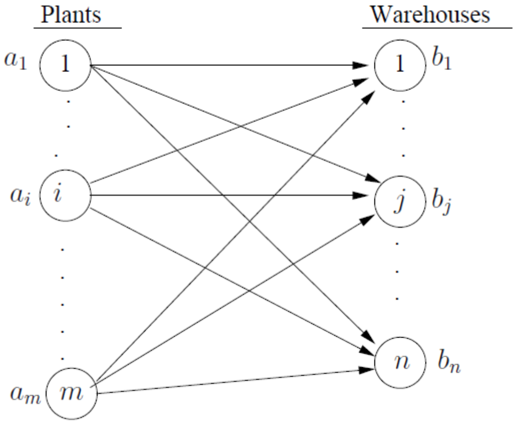
| | Fixed Allocation | | | | Free Allocation | | |
|---|---|---|---|---|---|---|---|
| | Plant A | Plant B | Total | | Plant A | Plant B | Total |
| Standart | 11.25 | 0 | 11.25 | vs. | 9.167 | 0 | 9.167 |
| Deluxe | 7.5 | 11.25 | 18.75 | | 8.33 | 12.5 | 20.83 |
| Profit | 225 | 168.75 | 393.75 | | | | 404.167 |

# GUROBI

# Transportation Problem

- The Brazilian coffee company processes coffee beans into coffee at 3 plants. The coffee is then shipped every week to 7 warehouses in major cities for retail, distribution, and exporting.
- It is desired to find the production-shipping pattern $x_{ij}$ from plant $i$ to warehouse $j$, $i = 1, \ldots, 3, j = 1, \ldots, 7$, that minimizes the overall shipping cost.
- Suppose that the unit shipping cost from plant $i$ to warehouse $j$ is $c_{ij}$. Further suppose that the production capacity at plant $i$ is $a_i$ and that the demand at warehouse $j$ is $b_j$.
- This the well-known transportation problem.

# Transportation Problem



$x_{ij}$ = the amount sent from plant $i$ to warehouse $j$.

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$$\text{minimize} \quad \sum_{i=1}^{3}\sum_{j=1}^{7} c_{ij}x_{ij}, \tag{40}$$

$$\text{subject to} \quad \sum_{j=1}^{7} x_{ij} \le a_i, \qquad \forall i \in \{1,2,3\}(\text{capacity}) \tag{41}$$

$$\sum_{i=1}^{3} x_{ij} \ge b_j, \qquad \forall j \in \{1,\ldots,7\}(\text{demand}) \tag{42}$$

$$x_{ij} \ge 0, \qquad \forall i \in \{1,2,3\}\forall j \in \{1,\ldots,7\}(\text{domain}) \tag{43}$$

# Transportation Problem

- A company ships their products from three different plants (one in LA, one in Atlanta, and one in New York City) to four regions of the United States (East, Midwest, South, West).
- Each plant has a capacity on how many products can be sent out, and each region has a demand of products they must receive.
- There is a different transportation cost between each plant, or each city, and each region.
- The company wants to determine how many products each plant should ship to each region in order to minimize the total transportation cost.

# Transportation Problem

- Decision variables:
  - The amount to ship from each plant to each region
- Constraints:
  - Demand: the total number of products received by a region (from each plant) is greater than or equal to its demand
  - Capacity: the total number of products shipped from a plant (to each region) is less than or equal to its capacity
- Objective function:
  - Minimize the total transportation costs

# Transportation Problem (Solver)

# Airplane Refueling (Adler, 1996)

- Trans Global Airlines (TGA) wants to optimize its purchases of jet fuel at the cities it serves around the world.
- Since the fuel efficiency of an airplane is related to its weight, an airplane carrying more fuel than needed to reach its destination will waste fuel.
- This fact suggests that a plane should take off with just enough fuel to reach its next destination.
- However, since fuel prices vary from city to city, a policy of minimal fuel purchases may be more costly than filling the plane to capacity at the inexpensive cities.
- You are tasked with minimizing the fuel related costs while respecting the safety.

# Airplane Refueling (Adler, 1996)

- To illustrate TGA's problem, consider an airplane that flies each day a so-called rotation that consists of the following four flight segments:
- New York → Los Angeles → San Francisco → Seattle
- Upon its arrival in New York, the airplane repeats the rotation.

| Departure City | Arrival City | A | B | C (gallons) | D (cents) |
|---|---|---|---|---|---|
| New York | Los Angeles | 23 | 33 | $2.90 + 0.40G$ | 82 |
| Los Angeles | San Francisco | 8 | 19 | $1.60 + 0.05G$ | 75 |
| San Francisco | Seattle | 19 | 33 | $4.75 + 0.25G$ | 77 |
| Seattle | New York | 25 | 33 | $1.75 + 0.45G$ | 89 |

A: Minimal fuel level at take-off

B: Maximal fuel level at take-off

C: Fuel consumption as a function of take-off weight $G$

D: Price per gallon fuel at the departure city

# Airplane Refueling (Adler, 1996)

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$x_i$: the amount of fuel purchased at city $i$ ($i = NY, LA, SF, SE$)
$a_i$: the amount of fuel the plane has at arrival to city $i$
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$$
\begin{aligned}
\textit{minimize} \; & 82x_{NY} + 75x_{LA} + 77x_{SF} + 89x_{SE}, && && (44)\\
\text{subject to} \; & 23 \leq d_{NY} \leq 33, && \text{(fuel level)} && (45)\\
& a_{NY} + x_{NY} = d_{NY}, && \text{(balance)} && (46)\\
& d_{NY} - 2.9 - 0.4d_{NY} = a_{LA}, && \text{(consumption)} && (47)\\
& x_i, a_i, d_i \geq 0, && \forall i \in \textit{cities} \text{ (domain)} && (48)
\end{aligned}
$$

Constraints (45)-(48) should be written for each city.


$x_{NY}$
$x_{LA}$
$a_{NY}$
$NY$
$d_{NY} \rightarrow$ Consumption $\rightarrow a_{LA}$
$LA$
$d_{LA}$

# Steel Production Planning (Adler, 1996)

- Consider the production planning problem of the National Steel Corporation (NSC), which produces a special-purpose steel that is used in the aircraft and aerospace industries. The marketing department of NSC has received orders for 2400, 2200, 2700, and 2500 tons of steel during each of the next four months: NSC can meet these demands by producing the steel by drawing from its inventory, or by any combination thereof.
- The production costs per ton of steel during each of the next four months are projected to be 7400,7500, 7600, and 7800. Because of these inflationary costs, it might be advantageous for NSC to produce more steel than it is needs in a given month and store the excess, although production capacity can never exceed 4000 tons in any month. All production takes place at the beginning of the month, and immediately thereafter the demand is met. The remaining steel is then stored in inventory at a cost of \$120/ton for each month that it remains there.

# Steel Production Planning (Adler, 1996)

- If the production level is increased or decreased from one month to the next, the company incurs a cost for implementing these changes. Specifically, for each ton of increased or decreased production over the previous month, the cost is \$50. The production of the first month, however, is exempt from this cost.
- The inventory at the beginning of the first month is 1000 tons of steel and the inventory level at the end of the fourth month should be at least 1500 tons.
- Formulate a production planning problem for NSC that will minimize the total cost over the next four months.

# Steel Production Planning (Adler, 1996)

$p_i$: steel production at period $i = 1, \ldots, 4$.
$i_i$: inventory at the end of period $i = 0, 1, \ldots, 4$.
$y_i$: increase in production at period $i = 2, 3, 4$.
$z_i$: decrease in production at period $i = 2, 3, 4$.

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$$Cost = 7400p_1 + 7500p_2 + 7600p_3 + 7800p_4 + 120\sum_{i=1}^{3} i_i + 50\sum_{i=2}^{4}(z_i + y_i)$$

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$$Cost = 7400p_1 + 7500p_2 + 7600p_3 + 7800p_4 + 120\sum_{i=1}^{3} i_i + 50\sum_{i=2}^{4}(z_i + y_i)$$

$$\begin{aligned}
&\textit{minimize} \;\; Cost, && && (49)\\
&\text{subject to } p_1 + i_0 \geq 2400, && \text{(demand)} && (50)\\
&p_2 + i_1 \geq 2200, && \text{(demand)} && (51)\\
&p_3 + i_2 \geq 2700, && \text{(demand)} && (52)\\
&p_4 + i_3 \geq 2500, && \text{(demand)} && (53)
\end{aligned}$$

# Steel Production Planning (Adler, 1996)

$$p_1 + i_0 - 2400 = i_1, \quad \text{(balance)} \quad (54)$$
$$p_2 + i_1 - 2200 = i_2, \quad \text{(balance)} \quad (55)$$
$$p_3 + i_2 - 2700 = i_3, \quad \text{(balance)} \quad (56)$$
$$p_4 + i_3 - 2500 = i_4, \quad \text{(balance)} \quad (57)$$
$$i_0 = 1000, i_4 \geq 1500, \quad \text{(inventory)} \quad (58)$$
$$p_i \leq 4000, \quad \forall i \in \{1, 2, 3, 4\} \text{(production)} \quad (59)$$
$$p_i, i_i \geq 0, \quad \text{(domain)} \quad (60)$$

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$$p_4 + i_3 - 2500 = i_4, \qquad \text{(balance)} \qquad (57)$$

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Demand constraints from the previous slide are not necessary as they are covered by the balance constraints and the domain constraints of the inventory variables.

Let us see how we can incorporate production level change constraints.

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$$y_2 \geq p_2 - p_1, z_2 \geq p_1 - p_2, \qquad \text{(change)} \qquad (61)$$

$$y_3 \geq p_3 - p_2, z_3 \geq p_2 - p_3, \qquad \text{(change)} \qquad (62)$$

$$y_4 \geq p_4 - p_3, z_4 \geq p_3 - p_4, \qquad \text{(change)} \qquad (63)$$

$$y_i, z_i \geq 0, \qquad \text{(domain)} \qquad (64)$$

# Steel Production Planning (Adler, 1996)

Let us see how we can incorporate production level change constraints. Allow $p_1 = 1000$, $p_2 = 1100$. We want $y_2 = 100$ as the production increases by 100 units. Together with nonnegativity constraints $y_2$ will be 100 (the smallest value it can take as larger values will cause more cost in the objective function) and $z_2$ will be 0.

$$\begin{aligned} y_2 &\geq 100 \\ z_2 &\geq -100 \end{aligned}$$

Allow $p_1 = 1100$, $p_2 = 1000$. We want $z_2 = 100$ as the production increases by 100 units. Together with nonnegativity constraints $z_2$ will be 100 (the smallest value it can take as larger values will cause more cost in the objective function) and $y_2$ will be 0.

$$\begin{aligned} y_2 &\geq -100 \\ z_2 &\geq 100 \end{aligned}$$

# Integer Programming

- Integer programming (also called integer linear programming, ILP) deals with a similar setting.

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- The objective function and the constraints are stated in the same fashion.
- The domain constraint are replaced by integrality constraints.
- Integer variables can be necessary to model real-life problems as you cannot sell half of something.
- Integer (binary) variables are required to model various kinds of constraints as well.

# The Carpenter's Problem

- A carpenter tries to solve a weekly production planning problem to maximize profit. He solely makes tables and chairs, and sells all tables and chairs at a market place.
- Both chairs and tables require the same type of raw material. The raw material availability is only 50 units per week.
  - one chair requires 1 unit of the raw material;
  - one table requires 2 units of the raw material.
- The carpenter can work at most 40 hours every week. It takes 2 hours to make a chair while it takes only an hour to make a table.
- The cost of a chair is 100 TL and it sells for 120 TL; the cost of a table is 130 TL and it sells for 145 TL.

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- The cost of a chair is 100 TL and it sells for 120 TL; the cost of a table is 130 TL and it sells for 145 TL.

What if we can only sell complete chairs or tables?

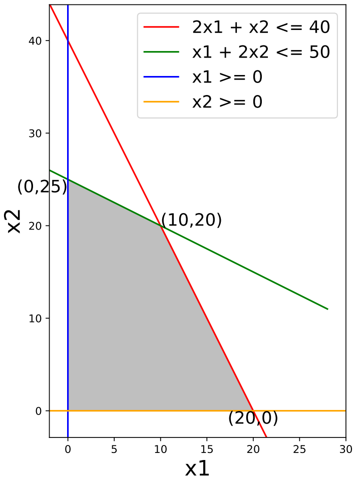
# The Carpenter's Problem

- $x_1$ = number of chairs produced/sold each week
- $x_2$ = number of table produced/sold each week

- $x_1$ = number of chairs produced/sold each week
- $x_2$ = number of table produced/sold each week

$$
\begin{aligned}
\textit{maximize } & (120-100)x_1 + (145-130)x_2, & & (65)\\
\text{subject to } & 2x_1 + x_2 \le 40, & \text{(labor)} & \quad (66)\\
& x_1 + 2x_2 \le 50, & \text{(material)} & \quad (67)\\
& \cancel{x_1, x_2 \ge 0}\; x_1, x_2 \ge 0 \text{ and integer.} & \text{(domain)} & \quad (68)
\end{aligned}
$$

From now on I will use $x_1, x_2 \in \mathbb{Z}$ as a short notation. For clarity $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.



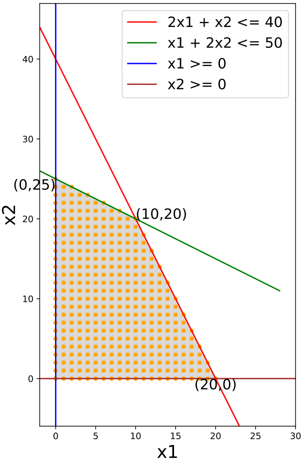




Figure: Feasible region for an IP consists of a set of points (rhs).

If the customer wants to buy a set (one table and two chairs), the selling price is 390 TL with a profit of 60 TL for the carpenter. You can only sell an integer number of sets (no such restriction on chairs or tables). Modify the mathematical model accordingly to reflect this new product option and the profit to be made from this option.

$x_3 =$ number of sets produced/sold each week

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$x_3$ = number of sets produced/sold each week

$$\begin{aligned}
\textit{maximize}\ & 20x_1 + 15x_2 + 60x_3, && && (69)\\
\text{subject to}\ & 2x_1 + x_2 + 5x_3 \leq 40, && \text{(labor)} && (70)\\
& x_1 + 2x_2 + 4x_3 \leq 50, && \text{(material)} && (71)\\
& x_1, x_2, x_3 \geq 0, x_3 \in \mathbb{Z}. && \text{(domain)} && (72)
\end{aligned}$$

You can mix different types of variables as long as the objective function and the constraints are linear.

# The Carpenter's Problem (Fixed Cost)

The carpenter has a fixed cost of 50 TL for producing chairs any week, and a fixed cost of 40 TL for producing tables. You do not incur the cost if you do not produce that item. Do you need additional decision variables? If so, how could you define the new variables? How does your model change?

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$$y_1 = \begin{cases} 1, & \text{if the carpenter produces chairs,} \\ 0, & \text{otherwise.} \end{cases}$$

$$y_2 = \begin{cases} 1, & \text{if the carpenter produces tables,} \\ 0, & \text{otherwise.} \end{cases}$$

The variables introduced here are called *binary* variables. They are useful to model such conditions in integer programming.

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$$
\begin{aligned}
\textit{maximize} \quad & 20x_1 + 15x_2 - 50y_1 - 40y_2, && && (73)\\
\text{subject to} \quad & 2x_1 + x_2 \le 40, && \text{(labor)} && (74)\\
& x_1 + 2x_2 \le 50, && \text{(material)} && (75)\\
& x_1 \le My_1, && \text{(fixed cost)} && (76)\\
& x_2 \le My_2, && \text{(fixed cost)} && (77)\\
& x_1, x_2 \ge 0, && \text{(domain)} && (78)\\
& y_1, y_2 \in \{0, 1\}. && \text{(domain)} && (79)
\end{aligned}
$$

$M$ is a very large number

# The Carpenter's Problem (Economies of Scale)

If the carpenter can produce at least 5 chairs (or 5 tables), then the labor requirement for the chair (or the table) is halved. This is only valid for the additional items. If the carpenter produces 5 chairs and 3 tables only the labor requirement for the chairs are halved.

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$x_1, x_3$ = number of chairs produced/sold each week at normal / reduced labor

$x_2, x_4$ = number of table produced/sold each week at normal / reduced labor

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$$y_1 = \begin{cases} 1, & \text{if the carpenter produces at least 5 chairs,} \\ 0, & \text{otherwise.} \end{cases}$$

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$$
\begin{aligned}
\textit{maximize } & 20(x_1 + x_3) + 15(x_2 + x_4), & & (80)\\
\text{subject to } & 2x_1 + x_2 + x_3 + 0.5x_4 \leq 40, & \text{(labor)} & \quad (81)\\
& x_1 + 2x_2 + x_3 + 2x_4 \leq 50, & \text{(material)} & \quad (82)\\
& x_3 \leq My_1, x1 \geq 5y_1 & \text{(econ. of scale)} & \quad (83)\\
& x_4 \leq My_2, x2 \geq 5y_2 & \text{(econ. of scale)} & \quad (84)\\
& x_1, x_2, x_3, x_4 \geq 0, & \text{(domain)} & \quad (85)\\
& y_1, y_2 \in \{0, 1\}. & \text{(domain)} & \quad (86)
\end{aligned}
$$

# Examples: Discrete Inputs and Outputs

- Inputs can only be added in increments of machine capacity/week.
- Workforce scheduling
- Fleet optimization
- Investment optimization: portfolio selection
- Knapsack problem
  - IP with a single constraint

# Examples: Problems with Logical Conditions

- For instance, in a product mix problem if product A is manufactured, then product B or C must be manufactured as well.
  - A mixed integer problem is obtained by introducing additional integer variables and constraints.
- Selecting a depot location from a number of candidate sites.
- Selecting a project from a number of alternatives.

# Examples: Combinatorial Problems

- Sequencing problems
  - Very often, operations need to be carried out in some sequence. Modeling precedence relationships requires use of integer variables.
    - Job shop scheduling problem.
    - Traveling salesman problem.
- Allocation problems
  - Transportation and assignment problems (special case).
  - Allocating tasks to machines (line balancing).
  - Quadratic assignment problem.
  - Airline crew scheduling.

# Do we need IP?

Consider the following problem

$$\begin{aligned} & maximize\ x_1 + x_2, && (87) \\ & \text{subject to } -2x_1 + 2x_2 \geq 1, && (88) \\ & \quad -8x_1 + 10x_2 \leq 13, && (89) \\ & \quad x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. && (90) \end{aligned}$$

- The LP optimal solution is x1=4 and x2=4.5.

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- What happens if we round continuous variables?
  - $x_2 = 4$ ?
  - $x_2 = 5$ ?

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- The variables represent liters of beer vs. number of airplanes.

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$$
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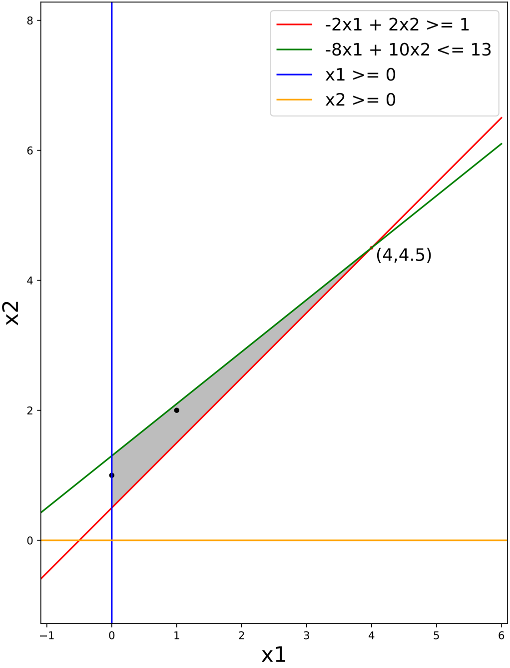
- The IP optimal solution is x1=1 and x2=2.
- The variables represent liters of beer vs. number of airplanes.
- In general, the round-off error is smaller when the variables take large values.


-2x1 + 2x2 >= 1
-8x1 + 10x2 <= 13
x1 >= 0
x2 >= 0
(4,4.5)
x2
x1
8
6
4
2
0
−1
0
1
2
3
4
5
6

# Solution methods

# Economic Interpretation