1. Deriving the Roots

We start out with the implicit surface equation of a torus with radius r and width w. Here $\hat{x}, \hat{y}, \hat{z}$ correspond to some point X:

$$
\begin{aligned}
\left(r - \sqrt{\hat{x}^2 + \hat{y}^2}\right)^2 + \hat{z}^2 - w^2 = 0 \\
\left(r - \sqrt{\hat{x}^2 + \hat{y}^2}\right)^2 &= -\hat{z}^2 + w^2 \\
r^2 - 2r\sqrt{\hat{x}^2 + \hat{y}^2} + \hat{x}^2 + \hat{y}^2 &= -\hat{z}^2 + w^2 \\
-2r\sqrt{\hat{x}^2 + \hat{y}^2} &= -\hat{x}^2 - \hat{y}^2 - \hat{z}^2 - r^2 + w^2 \\
2r\sqrt{\hat{x}^2 + \hat{y}^2} &= \hat{x}^2 + \hat{y}^2 + \hat{z}^2 + r^2 - w^2 \\
4r^2\left(\hat{x}^2 + \hat{y}^2\right) &= \left(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 + r^2 - w^2\right)^2 \\
0 &= \left(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 + r^2 - w^2\right)^2 - 4r^2\left(\hat{x}^2 + \hat{y}^2\right)
\end{aligned}
$$

Now we substitute $X = (\hat{x}, \hat{y}, \hat{z})^T$ with a parametric ray $ray(t) = X_o + X_d t$, where $X_o = (a, b, c)^T, X_d = (x, y, z)^T$:

$$((a + xt)^2 + (b + yt)^2 + (c + zt)^2 + R^2 - w^2)^2 - 4r^2((a + xt)^2 + (b + yt)^2)$$

Expanding yields:

$$
\begin{aligned}
&a^4 + 4a^3tx + 2a^2b^2 + 4a^2bty + 2a^2c^2 + 4a^2ctz - 2a^2r^2 + 6a^2t^2x^2 \\
&+2a^2t^2y^2 + 2a^2t^2z^2 - 2a^2w^2 + 4ab^2tx + 8abt^2xy + 4ac^2tx + 8act^2xz \\
&-4ar^2tx + 4at^3x^3 + 4at^3xy^2 + 4at^3xz^2 - 4atw^2x + b^4 + 4b^3ty \\
&+2b^2c^2 + 4b^2ctz - 2b^2r^2 + 2b^2t^2x^2 + 6b^2t^2y^2 + 2b^2t^2z^2 - 2b^2w^2 \\
&+4bc^2ty + 8bct^2yz - 4br^2ty + 4bt^3x^2y + 4bt^3y^3 + 4bt^3yz^2 \\
&-4btw^2y + c^4 + 4c^3tz + 2c^2r^2 + 2c^2t^2x^2 + 2c^2t^2y^2 + 6c^2t^2z^2 \\
&-2c^2w^2 + 4cr^2tz + 4ct^3x^2z + 4ct^3y^2z + 4ct^3z^3 - 4ctw^2z + r^4 \\
&-2r^2t^2x^2 - 2r^2t^2y^2 + 2r^2t^2z^2 - 2r^2w^2 + t^4x^4 + 2t^4x^2y^2 + 2t^4x^2z^2 \\
&+t^4y^4 + 2t^4y^2z^2 + t^4z^4 - 2t^2w^2x^2 - 2t^2w^2y^2 - 2t^2w^2z^2 + w^4
\end{aligned}
$$

We collect the terms w.r.t to powers of t:

$$\begin{aligned}
&t^0\cdot(a^4 + 2a^2b^2 + b^4 + 2a^2c^2 + 2b^2c^2 + c^4 - 2a^2r^2 - 2b^2r^2\\
&\quad + 2c^2r^2 + r^4 - 2a^2w^2 - 2b^2w^2 - 2c^2w^2 - 2r^2w^2 + w^4)\\
&+t^1\cdot(4a^3x + 4ab^2x + 4ac^2x - 4ar^2x - 4aw^2x + 4a^2by\\
&\quad + 4b^3y + 4bc^2y - 4br^2y - 4bw^2y + 4a^2cz + 4b^2cz\\
&\quad + 4c^3z + 4cr^2z - 4cw^2z)\\
&+t^2\cdot(6a^2x^2 + 2b^2x^2 + 2c^2x^2 - 2r^2x^2 - 2w^2x^2 + 8abxy\\
&\quad + 2a^2y^2 + 6b^2y^2 + 2c^2y^2 - 2r^2y^2 - 2w^2y^2 + 8acxz\\
&\quad + 8bcyz + 2a^2z^2 + 2b^2z^2 + 6c^2z^2 + 2r^2z^2 - 2w^2z^2)\\
&+t^3\cdot(4ax^3 + 4bx^2y + 4axy^2 + 4by^3 + 4cx^2z + 4cy^2z\\
&\quad + 4axz^2 + 4byz^2 + 4cz^3)\\
&+t^4\cdot(x^4 + 2x^2y^2 + y^4 + 2x^2z^2 + 2y^2z^2 + z^4)
\end{aligned}$$

This can be a little bit simplified:

$$\begin{aligned}
&t^0\cdot(a^4 + 2a^2b^2 + b^4 + 2a^2c^2 + 2b^2c^2 + c^4 - 2a^2r^2 - 2b^2r^2\\
&\quad + 2c^2r^2 + r^4 - 2a^2w^2 - 2b^2w^2 - 2c^2w^2 - 2r^2w^2 + w^4)\\
&+t^1\cdot 4((a^2 + b^2 + c^2 - r^2 - w^2)(ax + by) + c(a^2 + b^2 + c^2 + r^2 - w^2)z)\\
&+t^2\cdot 2((c^2 - r^2 - w^2)(x^2 + y^2) + 4bcyz + (3c^2 + r^2 - w^2)z^2\\
&\quad + 4ax(by + cz) + a^2(3x^2 + y^2 + z^2) + b^2(x^2 + 3y^2 + z^2))\\
&+t^3\cdot 4(ax + by + cz)(x^2 + y^2 + z^2)\\
&+t^4\cdot(x^2 + y^2 + z^2)^2
\end{aligned}$$

And assuming that X_d is normalized, $X_d \perp X_o$ and $r = 1$:

$$
\begin{aligned}
& t^0 \cdot \left(a^4 + 2a^2\left(b^2 + c^2 - w^2 - 1\right) + b^4 + 2b^2\left(c^2 - w^2 - 1\right) + \left(c^2 - w^2 + 1\right)^2\right) \\
& + t^1 \cdot 8cz \\
& + t^2 \cdot 2\left(a^2 + b^2 + c^2 - w^2 + 2z^2 - 1\right) \\
& + t^4
\end{aligned}
$$

Now if we solve for t we get a pretty big formula, but a lot of the expressions are equal. So we substitute some of them for readability:

$$\alpha = a^2 + b^2 + c^2 - w^2 + 2z^2 - 1$$
$$\beta = a^4 + 2b^2a^2 + 2c^2a^2 - 2w^2a^2 - 2a^2 + b^4 + c^4 + w^4 - 2b^2 + 2b^2c^2 + 2c^2 - 2b^2w^2 - 2c^2w^2 - 2w^2 + 1$$
$$\gamma = 16\left(\alpha\right)^3 - 144\left(\beta\right)\left(\alpha\right) + 1728c^2z^2$$
$$\delta = 4\left(\alpha\right)^2 + 12\left(\beta\right)$$
$$\epsilon = \sqrt{\left(\gamma\right)^2 - 4\left(\delta\right)^3}$$
$$\zeta = \sqrt[3]{\gamma + \epsilon}$$

The 4 solutions:

$$t \to \frac{1}{2}\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)} - \frac{1}{2}\sqrt{-\frac{\zeta}{3\sqrt[3]{2}} - \frac{8}{3}\left(\alpha\right) - \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right) - \frac{16cz}{\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)}}}$$
$$t \to \frac{1}{2}\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)} + \frac{1}{2}\sqrt{-\frac{\zeta}{3\sqrt[3]{2}} - \frac{8}{3}\left(\alpha\right) - \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right) - \frac{16cz}{\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)}}}$$
$$t \to -\frac{1}{2}\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)} - \frac{1}{2}\sqrt{-\frac{\zeta}{3\sqrt[3]{2}} - \frac{8}{3}\left(\alpha\right) - \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right) + \frac{16cz}{\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)}}}$$
$$t \to -\frac{1}{2}\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)} + \frac{1}{2}\sqrt{-\frac{\zeta}{3\sqrt[3]{2}} - \frac{8}{3}\left(\alpha\right) - \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right) + \frac{16cz}{\sqrt{\frac{\zeta}{3\sqrt[3]{2}} - \frac{4}{3}\left(\alpha\right) + \frac{\sqrt[3]{2}}{3\zeta}\left(\delta\right)}}}$$