1. Porous medium heated from below

Regard a system of Darcy's equation for flow in a porous medium saturated with water and a convective heat equation. In addition to the simple formulations in the lecture, we include two additional effects:

- Heat expansion leading to a decrease of the water density $\rho(T)$ and therefore a tendency for less dense ("lighter") water to move up in the force field of gravity $\vec{g}$
- Convective heat transport: heat not only moves due to heat conduction, but also along with the moving water.

Here we provide the so-called Boussinesq approximation which keeps the temperature dependency of the density only in the gravity term:

$$
\begin{aligned}
\vec{q} &= -\rho_{ref} k(\nabla p - \rho(T)\vec{g}) \\
\partial_t \rho_{ref} + \nabla \cdot \vec{q} &= 0 \\
\partial_t (cT) - \nabla \cdot (\lambda \nabla T - c\vec{q}) &= 0
\end{aligned}
$$

The unknowns and parameters are as follows:

p		pressure
T		temperature
$\vec{q}$		water mass flux
$\rho(T)$	$\rho_{ref} - \alpha(T - T_{ref})$	density
k	100	permeability
$\vec{g}$	(0,-1)	gravity vector
c	0.001	heat capacity
α	0.01	heat expansion coefficient
λ	0.01	heat conductivity
ρ_{ref}	1	reference density
T_{ref}	0	reference temperature
P_{ref}	0	reference pressure

The values of the constants have been choosen as to ensure that the tasks posed below will work.

Regard the domain $\Omega = (0, 300) \times (0, 150)$ Let $\Gamma_{top} = (0, 300) \times 150$, $\Gamma_{bot} = (0, 300) \times 0$, We set Dirichlet boundary conditions

$$
\begin{aligned}
P &= 0 \quad \text{on} \Gamma_{top} \\
T &= 0 \quad \text{on} \Gamma_{top} \\
T &= 0 \quad \text{on} \Gamma_{bot}
\end{aligned}
$$

and homogeneous Neumann boundary conditions for all other boundaries.

Investigate two things:

- Stationary distribution under the above boundary conditions
- What happens if we set $T = T_{heat} > 0$ on Γ_{bot} ?

1.1. Implementation:

- Implement the finite volume discretization of the problem. Feel free to use the VoronoiFVM code. You can consult the (unfortunately unpublished) preprint https://dx.doi.org/10.20347/WIAS.PREPRINT.741. The discrete problem can be described using specific implementations for `storage`, `reaction` and `flux`, in VoronoiFVM.jl, and from the examples provided in the last lectures it should be possible to derive the implementation for the backward Euler time stepping scheme[1]
- Calculate the steady state for $T_{heat} = 0$

[1] I verified that this works.

- Calculate the transient solutions for $T_{heat} = 0.5$,$T_{heat} = 1$, $T_{heat} = 2$, $T_{heat} = 5$, $T_{heat} = 10$

  In order to watch the evolution, consider to use ‘scalarplot‘ from ‘GridVisualize.jl‘ with the ‘PlutoVista.jl‘ backend as used in the lecture notebooks.

1.2. Optionally:

- Use alternative timestepping methods from DifferentialEquations.jl (Please come back to me if you want to try this out – in fact the more sophisticated schemes allow for faster solution.)

1.3. Report:

- Introduce the problem and some information on the physical background
- Discuss the finite volume space discretization approach
- Discuss possibilities for the time discretization. Are there any obstacles for implementing the explicit Euler method ?
- Discuss possible solution methods for the discretized problem
- Present simulation results. Do you see qualitative differences in the transient behavior for growing T_{heat}?
- Discuss ways to improve performance
- Discuss any of the optional topics