Fundamentals of Robotic Kinematics and Power Systems

Prepared by

Arnav Bhayana

RoboTech Club, NISER

With reference to materials and inputs from:

Mayukh Ghosh , Sri Ram Nagulan , Gaurav Pal

For Teaching and Academic Use

November 2025

Contents

Chapter 1

Matrix Multiplication and Linear Equations

Learning Objectives

- Represent linear systems using matrices.
- Apply matrix inversion to solve equations.
- Connect matrix methods with robotic transformations.

1.1 Introduction to Matrix Representation

Matrix algebra forms the mathematical foundation of robotic motion, transformations, and control. All forward and inverse kinematic equations rely on matrix multiplication.

1.2 Linear System Representation

A linear system of equations can be elegantly represented using matrix notation:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \quad \Rightarrow \quad AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

If A is invertible (square and non-singular), then the solution is given by:

$$X = A^{-1}B$$

Key Concept

In robotics, matrix A often represents transformation matrices or Jacobian matrices that relate joint velocities to end-effector velocities. Matrix inversion is directly related to solving for joint angles given desired end-effector positions in inverse kinematics problems.

1.3 Application in Robotic Transformations

The power of matrix representation becomes evident in homogeneous transformation matrices used in robotics:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

where R is the rotation matrix and p is the position vector.

Chapter 2

Kinematics and Inverse Kinematics

Learning Objectives

- Derive forward and inverse kinematic equations for planar manipulators.
- Solve matrix-form inverse kinematics.
- Interpret multiple physical solutions and their implications.

2.1 Introduction to Robotic Manipulators

A **manipulator** is a mechanical linkage designed to move and orient objects in space, analogous to a human arm. Its key components are:

- **Base:** Structural support and mounting point.
- **Links:** Rigid members connecting joints.
- **Joints:** Provide relative motion between links, either rotational (revolute) or prismatic.
- **Actuators:** Motors that drive joint motion.
- **End-Effector:** The tool or gripper at the arm's end performing the desired task.

2.1.1 Workspace and Joint Space

The **workspace** (also called reachable space or working envelope) of a manipulator is the total volume or region in space that the robot's end-effector can reach. It defines the area or volume where the manipulator can actually operate.

The **joint space** of a robot manipulator is the mathematical space formed by all possible values of its joint variables.

2.1.2 Degrees of Freedom (DOF)

In robotics, the **degrees of freedom** refer to the number of independent movements or parameters that define the configuration or position of the manipulator in space.

2.1.3 Safe Limit

The **safe limit** defines the maximum permissible range of motion, speed, force, or torque ensuring the manipulator operates within mechanical and control constraints, keeping humans and equipment safe.

2.1.4 Major Divisions of Robotic Components

Voltaic logic is used across the following core components:

- **Sensors:** Detect physical quantities (e.g., position, force, temperature) and convert them to electrical signals for feedback control.
- **Processors:** Execute control algorithms, interpret sensor data, and coordinate actuator behavior.
- **Actuators:** Convert electrical signals into mechanical motion, enabling joint or end-effector movement.

2.2 Actuator Technologies in Robotics

2.2.1 Planetary Gear Motors

Planetary gear motors provide compact, high-torque output through their unique gear arrangement. They offer excellent efficiency and are common in robotics applications requiring high torque-to-size ratio.

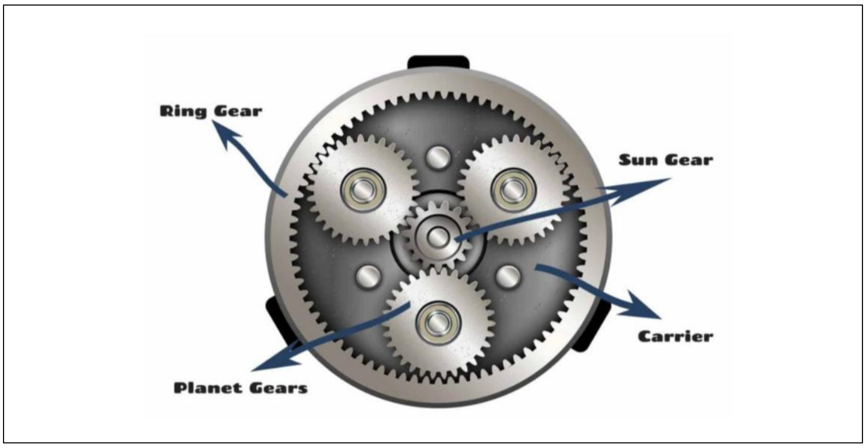

Figure 2.1: Placeholder – Diagram of a Planetary Gear Motor showing sun gear, planet gears, and ring gear arrangement.

2.2.2 Stepper Motors

Stepper motors are electromechanical devices that convert electrical pulses into discrete mechanical steps, enabling precise angular control without requiring feedback sensors for many applications.

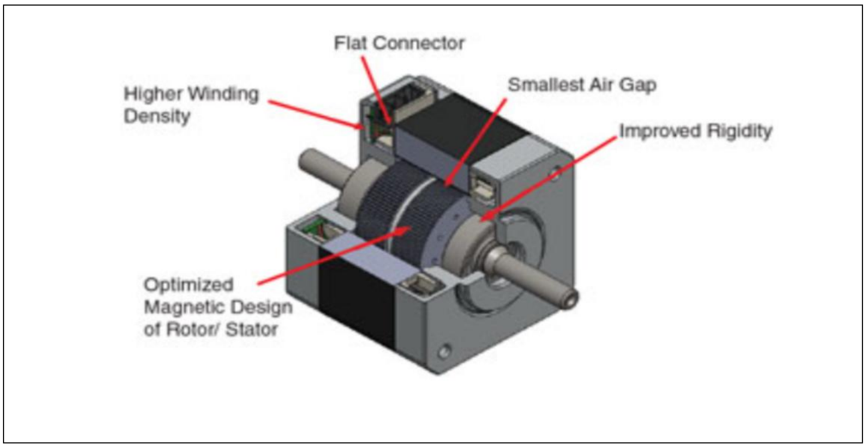

Figure 2.2: Placeholder – Diagram of a Stepper Motor showing stator coils and rotor teeth arrangement.

2.3 Forward Kinematics of a Two-Link Planar Arm

The forward kinematics problem determines the end-effector position given the joint angles:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2), \qquad y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

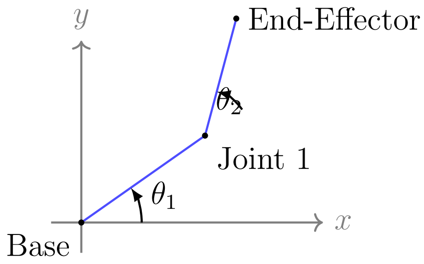

Figure 2.3: Two-link planar manipulator with joint angles θ_1 and θ_2, link lengths l_1 and l_2.

2.4 Inverse Kinematics: Problem Formulation

The inverse kinematics problem is more challenging: given desired end-effector position (x, y), find the joint angles (θ_1, θ_2) that achieve this position.

Example 2.1 – Comprehensive Inverse Kinematics of a Two-Link Planar Manipulator

Problem Statement

Given: A two-link planar manipulator with equal link lengths $l_1 = l_2 = 1$ and desired end-effector position:

$$(x, y) = \left(\frac{2 - \sqrt{2}}{2\sqrt{2}}, \frac{2 - \sqrt{6}}{2\sqrt{2}} \right) \Rightarrow (x, y) \approx (0.2071,\ -0.1589)$$

Goal: Determine all possible joint angle configurations (θ_1, θ_2) that achieve this position.

Step 1: Establish Kinematic Equations

From the forward kinematics equations:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) = \cos\theta_1 + \cos(\theta_1 + \theta_2) \tag{2.1}$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) = \sin\theta_1 + \sin(\theta_1 + \theta_2) \tag{2.2}$$

Step 2: Distance Constraint and Elbow Angle

Squaring and adding equations (1) and (2):

$$\begin{aligned} x^2 + y^2 &= [\cos\theta_1 + \cos(\theta_1 + \theta_2)]^2 + [\sin\theta_1 + \sin(\theta_1 + \theta_2)]^2 \\ &= \cos^2\theta_1 + 2\cos\theta_1\cos(\theta_1 + \theta_2) + \cos^2(\theta_1 + \theta_2) \\ &\quad + \sin^2\theta_1 + 2\sin\theta_1\sin(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2) \\ &= 2 + 2[\cos\theta_1\cos(\theta_1 + \theta_2) + \sin\theta_1\sin(\theta_1 + \theta_2)] \end{aligned}$$

Using trigonometric identity:

$$\cos A \cos B + \sin A \sin B = \cos(A - B) = \cos(-\theta_2) = \cos\theta_2$$

Therefore:

$$x^2 + y^2 = 2 + 2\cos\theta_2$$

Solving for $\cos\theta_2$:

$$\cos\theta_2 = \frac{x^2 + y^2 - 2}{2}$$

Substituting numerical values:

$$x^2 + y^2 \approx (0.2071)^2 + (-0.1589)^2 \approx 0.0429 + 0.0253 = 0.0682$$

$$\cos\theta_2 = \frac{0.0682 - 2}{2} \underset{8}{=} \frac{-1.9318}{2} = -0.9659$$

Example 2.1 – Continued (Page 2 of 3)

Step 3: Solve for θ_2

From $\cos\theta_2 = -0.9659$, we find:

$$\theta_2 = \cos^{-1}(-0.9659) = \pm 165^\circ \quad (\text{or} \quad \pm 2.8798 \text{ radians})$$

This gives two possible elbow configurations:

- $\theta_2 = +165^\circ$: **Elbow-down configuration**
- $\theta_2 = -165^\circ$: **Elbow-up configuration**

Step 4: Solve for θ_1 using Matrix Method

We can rewrite the original equations in matrix form. Let:

$$\alpha = \theta_1$$
$$\beta = \theta_1 + \theta_2$$

Then equations become:

$$\cos\alpha + \cos\beta = x$$
$$\sin\alpha + \sin\beta = y$$

Using sum-to-product identities:

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = x$$

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = y$$

Dividing the second by the first:

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{y}{x}$$

$$\frac{\alpha+\beta}{2} = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-0.1589}{0.2071}\right) \approx -37.5^\circ$$

Therefore:

$$\alpha + \beta = 2\theta_1 + \theta_2 = -75^\circ$$

Example 2.1 – Continued (Page 3 of 3)

Step 5: Find Complete Solutions

We now solve the system:

$$\begin{aligned} 2\theta_1 + \theta_2 &= -75^\circ \\ \theta_2 &= \pm 165^\circ \end{aligned}$$

Case 1: Elbow-down configuration ($\theta_2 = +165^\circ$)

$$2\theta_1 + 165^\circ = -75^\circ \Rightarrow 2\theta_1 = -240^\circ \Rightarrow \theta_1 = -120^\circ$$

$$\Rightarrow (\theta_1, \theta_2) = (-120^\circ, +165^\circ)$$

Case 2: Elbow-up configuration ($\theta_2 = -165^\circ$)

$$2\theta_1 - 165^\circ = -75^\circ \Rightarrow 2\theta_1 = 90^\circ \Rightarrow \theta_1 = 45^\circ$$

$$\Rightarrow (\theta_1, \theta_2) = (45^\circ, -165^\circ)$$

Step 6: Verification and Physical Interpretation

Let's verify both solutions using forward kinematics:
Solution 1: $(\theta_1, \theta_2) = (-120^\circ, +165^\circ)$

$$\begin{aligned} x &= \cos(-120^\circ) + \cos(-120^\circ + 165^\circ) = \cos(-120^\circ) + \cos(45^\circ) \\ &= -0.5 + 0.7071 = 0.2071 \quad \checkmark \\ y &= \sin(-120^\circ) + \sin(45^\circ) = -0.8660 + 0.7071 = -0.1589 \quad \checkmark \end{aligned}$$

Solution 2: $(\theta_1, \theta_2) = (45^\circ, -165^\circ)$

$$\begin{aligned} x &= \cos(45^\circ) + \cos(45^\circ - 165^\circ) = 0.7071 + \cos(-120^\circ) \\ &= 0.7071 - 0.5 = 0.2071 \quad \checkmark \\ y &= \sin(45^\circ) + \sin(-120^\circ) = 0.7071 - 0.8660 = -0.1589 \quad \checkmark \end{aligned}$$

Conclusion

The inverse kinematics problem for this two-link manipulator has **two distinct solutions**, representing different arm configurations that reach the same end-effector position. This multiplicity is characteristic of redundant manipulators and has important implications for path planning and obstacle avoidance in practical robotics applications.

Chapter 2 Summary

- **Forward kinematics** computes end-effector position (x, y) from given joint angles (θ_1, θ_2).
- **Inverse kinematics** determines joint angles (θ_1, θ_2) from desired end-effector position (x, y).
- Multiple configurations can reach the same position, leading to **multiple solutions**.
- The choice between solutions depends on practical constraints like joint limits, obstacle avoidance, and energy efficiency.

Chapter 3

Battery and Power Systems in Robotics

Learning Objectives

- Calculate power requirements for robotic systems.
- Select appropriate battery capacity based on operational needs.
- Understand C-rating and its implications for battery performance.
- Apply safety margins in power system design.

3.1 Power Fundamentals

The electrical power in a system is given by:

$$P = VI$$

where P is power in watts (W), V is voltage in volts (V), and I is current in amperes (A).

For mechanical systems, power relates force and velocity:

$$P = Fv$$

where F is force in newtons (N) and v is velocity in meters per second (m/s).

3.2 Battery Capacity Calculation

Example 3.1 – Mobile Robot Power Requirements

Scenario: A 15 kg robot moving at 1 m/s with friction coefficient $\mu = 1$, powered by a 24 V battery system.

Step 1: Calculate Required Force

$$F = \mu N = \mu m g = 1 \times 15 \times 9.8 = 147 \text{ N}$$

Step 2: Calculate Mechanical Power

$$P_{\text{mech}} = Fv = 147 \times 1 = 147 \text{ W}$$

Step 3: Calculate Electrical Current Assuming 100% efficiency for simplicity:

$$I = \frac{P}{V} = \frac{147}{24} \approx 6.13 \text{ A}$$

Step 4: Calculate Battery Capacity For 40 minutes of operation:

$$t = 40 \text{ minutes} = \frac{40}{60} \approx 0.667 \text{ hours}$$

$$Q = I \times t = 6.13 \times 0.667 \approx 4.09 \text{ Ah} = 4090 \text{ mAh}$$

Step 5: Apply Safety Margin With 1.5× safety margin:

$$Q_{\text{actual}} = 1.5 \times 4090 \approx 6135 \text{ mAh}$$

Recommendation: Choose a 24 V, 6200 mAh battery.

3.3 C-Rating and Discharge Characteristics

The C-rating defines how quickly a battery can be discharged safely:

$$I_{\text{max}} = C_r \times Q_{\text{Ah}}$$

where C_r is the C-rating and Q_{Ah} is the capacity in ampere-hours.

The discharge time can be estimated as:

$$t_{\text{discharge}} = \frac{1}{C_r} \text{ hours} \quad \text{or} \quad t_{\text{discharge}} = \frac{60}{C_r} \text{ minutes}$$

C-Rating Interpretation

- $C_r = 1$: Battery can be discharged over 1 hour
- $C_r = 10$: Battery can be discharged over 6 minutes
- $C_r = 0.5$: Battery can be discharged over 2 hours

Higher C-ratings indicate batteries capable of delivering higher peak currents.

3.4 Motor Performance Characteristics

3.4.1 Stall Torque vs. Operating Torque

- **Stall Torque:** Maximum torque at zero speed (motor stalled)
- **Operating Torque:** Torque under normal running conditions (typically 60–80% of stall torque)

3.4.2 Operating Voltage

The **operating voltage** is the electrical range required for correct functioning. Operation outside this range may cause instability or damage.

3.4.3 Static and Dynamic Output

- **Static voltage:** Constant voltage level in a circuit without fluctuation.
- **Static output:** Steady-state output under constant input conditions.
- **Dynamic voltage:** Voltage varying with time due to load or operating changes.
- **Dynamic output:** Time-varying system response before reaching equilibrium.

3.4.4 Margin of Safe Operation

$$\text{Margin} = \frac{\text{Maximum Rated Output}}{\text{Application Requirement}} \geq 1.5$$

A safety margin of 1.5–2.0 is recommended for reliable operation.

3.5 Mechanical Advantage in Robotic Systems

For gear systems and lever mechanisms:

$$\text{Mechanical Advantage} = \frac{\text{Output Force}}{\text{Input Force}} = \frac{\text{Load}}{\text{Effort}}$$

For lever systems specifically:

$$M.A. = \frac{\text{Effort Arm Length}}{\text{Load Arm Length}}$$

3.6 System Efficiency

Overall system efficiency accounts for losses in transmission and conversion:

$$\eta_{\text{total}} = \eta_{\text{motor}} \times \eta_{\text{gearbox}} \times \eta_{\text{transmission}} \times \cdots$$

$$\eta = \frac{\text{Useful Output Power}}{\text{Input Power}} \times 100\%$$

Typical efficiencies:

- DC Motors: 70–85%
- Planetary Gearboxes: 85–95%
- Belt Drives: 90–98%
- Lead Screws: 20–80% (depends on pitch and lubrication)

Chapter 3 Summary

- Always calculate both mechanical and electrical power requirements
- Select batteries with adequate capacity AND appropriate C-rating
- Apply safety margins of 1.5–2.0 for reliable operation
- Consider system efficiency when sizing components
- Planetary gear systems provide excellent torque multiplication in compact packages
- Stepper motors offer precise positioning without feedback systems