Matrix multiplication relation with solving linear equations [Importance of matrix representation of linear eqns]

Matrix multiplication has been defined in such a way that it compactly represents a system of linear equations.

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots a_{nn}x_n &= b_n
\end{aligned}\right\}$$

System of n linear equations.

In matrix representation, we can write it as multiplication of two matrices:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{B}$$

$\Rightarrow AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$

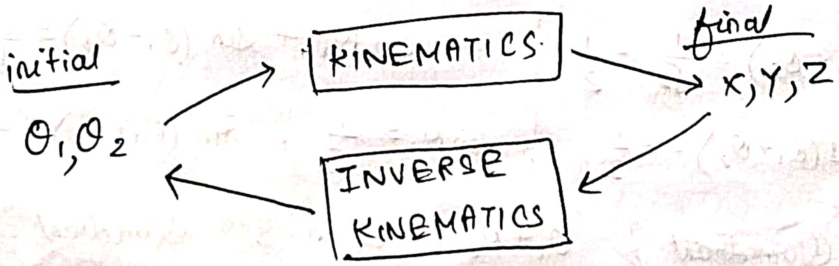
So if A is invertible and we can find A^{-1}, then we can get the values $x_1, x_2, \cdots x_n$, i.e., X matrix by multiplying A^{-1} & B.

Kinematics VS Inverse Kinematics:

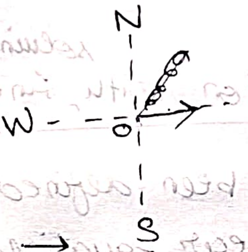
Kinematics $\rightarrow$ study of motion (without considering forces or torques).

↳ tells us where the robot's end-effector (x_1, x_2) will be if we know the joint angles (θ_1, θ_2)

Inverse kinematics $\rightarrow$ we know the desired position of the end effector (x, y) and we must find the joint angles (θ_1, θ_2) that will reach that point.

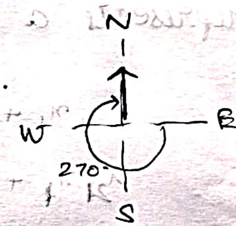


eg:- ① initial configuration.



Let the θ be the angle in the clockwise direction wrt to $\overrightarrow{OE}$.

If $\theta = 270^\circ$, then the final configuration will be $\overrightarrow{ON}$.



This is kinematics.

Now, if we were only the final configuration to be $\overrightarrow{ON}$, and were told to find θ, then $\theta = \left(2n\pi + \frac{3\pi}{2}\right)$, where $n = 0, 1, 2, \ldots$

and also $\theta = -\frac{\pi}{2} + 2n\pi$; $n = 0, 1, 2, \ldots$

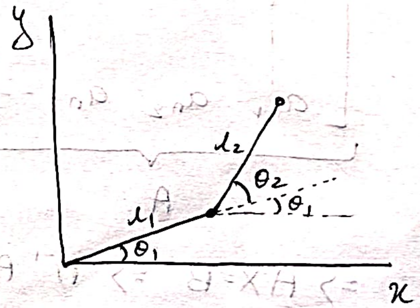
So we have more than one solution.

This is inverse kinematics.

② example of 2-arm, 2-joint Robotic arm.

Kinematics.

Given $\theta_1 = 30^\circ$ and $\theta_2 = 60^\circ$, find the final position of the end effector (x, y).



$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$

$= l_1 \cos 30^\circ + l_2 \cos 90^\circ$

$= \frac{\sqrt{3}}{2} l_1$

$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$

$= l_1 \sin 30^\circ + l_2 \sin 90^\circ = \frac{l_1}{2} + l_2$.

Inverse Kinematics.

Given $x, y = \left(\frac{2-\sqrt{2}}{2\sqrt{2}}, \frac{2-\sqrt{6}}{2\sqrt{2}}\right)$. Find the corresponding θ_1 & θ_2

Consider $l_1 = l_2 = 1$.

$\therefore x = \frac{2-\sqrt{2}}{2\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2}$. $\quad \therefore y = \frac{2-\sqrt{6}}{2\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}$.

$\Rightarrow \cos\theta_1 + \cos(\theta_1 + \theta_2) = \frac{1}{\sqrt{2}} - \frac{1}{2}$ $\quad \Rightarrow \sin\theta_1 + \sin(\theta_1 + \theta_2) = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}$

$\therefore \cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos(\theta_1 + \theta_2) = -\frac{1}{2}$ $\quad \sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin(\theta_1 + \theta_2) = -\frac{\sqrt{3}}{2}$

θ_1 lies in 1st Quadrant & $\theta_1 + \theta_2$ lies in 3rd Quadrant.

$\theta_1 = 45^\circ$ and $\theta_1 + \theta_2 = 240^\circ$.

$\therefore \theta_1 = 45^\circ$ & $\theta_2 = 195^\circ$.

Manipulators.

→ mechanical devices designed to interact with the environment by handling, grasping and ~~manipulation~~ manipulating objects.

→ combination of links and joints

→ Manipulators are made of actuators, that are devices that produce motion (linear / rotatory) by converting a control signal and an energy source into mechanical output. (eg:- DC, servo & stepper motors)

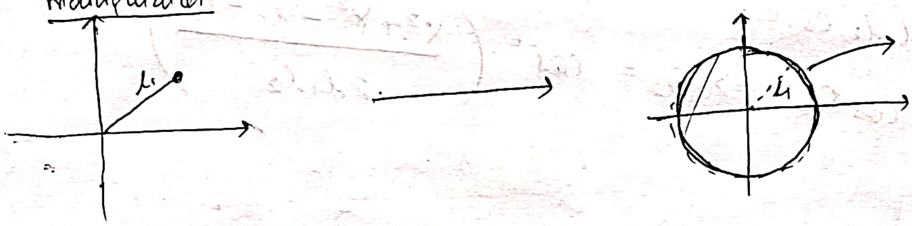
Work Space / Configuoration Space / Task Space.

→ area or volume where robot can actually work.

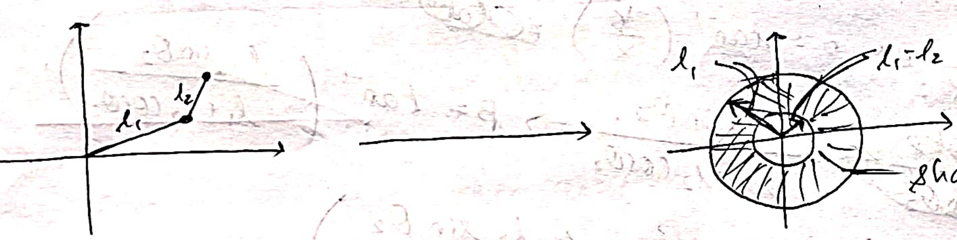
→ set of all points that the end-effector can reach, either positionally or orientationally, given the robot's joint limits and link lengths.

manipulator

Work space.



only the perimeter with radius l_1 is the work space.



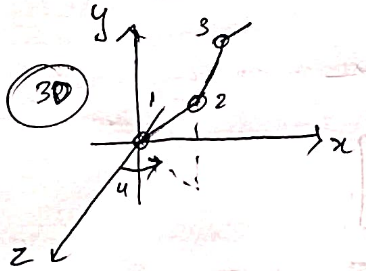
shaded region is the workspace.

Joint Space.

max region the joint can cover.

Degrees of freedom.

→ in robotics, it is the number of independent motors (in general, actuators)

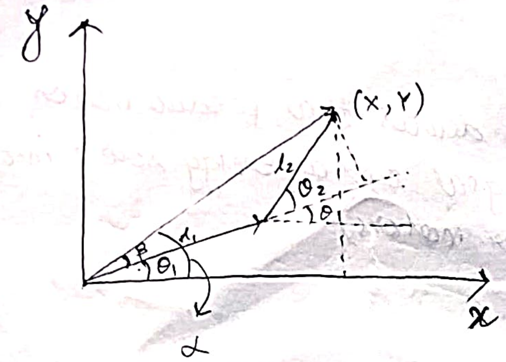


DOF = 4.

Safe Limit. → maximum allowable operating boundaries beyond which a robot's motion, force, or speed could become unsafe for itself, its surroundings. (eg:- speed limit, force limit, Joint torque limiters)

→ prevents mechanical damage.

Inverse Kinematics for a 2-dofs Manipulator



Find $\theta_1 = f_1(X, Y, l_1, l_2)$.
and $\theta_2 = f_2(X, Y, l_1, l_2)$.

$$l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) = X \quad -①$$
$$l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) = Y \quad -②.$$

Now $①^2 + ②^2$, yields.

$$X^2 + Y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \left[\cos\theta_1 \cos(\theta_1+\theta_2) + \sin\theta_1 \sin(\theta_1+\theta_2)\right].$$

$$\Rightarrow X^2 + Y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2.$$

$$\Rightarrow \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2} = \cos\theta_2 \Rightarrow \theta_2 = \cos^{-1}\left(\frac{X^2+Y^2-l_1^2-l_2^2}{2l_1 l_2}\right)$$

Now, $\theta_1 = \alpha - \beta$.

$$\tan\alpha = \frac{Y}{X} \Rightarrow \alpha = \tan^{-1}\left(\frac{Y}{X}\right)$$

$$\tan\beta = \frac{l_2 \sin\theta_2}{l_1 + l_2\cos\theta_2} \Rightarrow \beta = \tan^{-1}\left(\frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2}\right).$$

$$\text{so } \theta_1 = \alpha - \beta = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2}\right).$$

Alternative method.

Rearranging ① and ②, we can write as

$$\cos\theta_1 (l_1 + l_2\cos\theta_2) - \sin\theta_1 (l_2\sin\theta_2) = X$$
$$\sin\theta_1 (l_1 + l_2\cos\theta_2) + \cos\theta_1 (l_2\sin\theta_2) = Y$$

Representing this in the matrix form, we have,

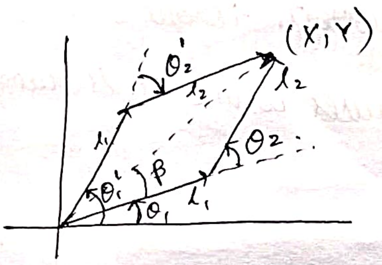
$$\underbrace{\begin{bmatrix} l_1 + l_2\cos\theta_2 & -l_2\sin\theta_2 \\ l_2\sin\theta_2 & l_1 + l_2\cos\theta_2 \end{bmatrix}}_{A \text{ (known too us as we know } l_1, l_2 \text{ and } \theta_2)} \underbrace{\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}}_{B \text{ (unknown)}} = \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{C \text{ (Known)}}.$$

$$\Rightarrow AB = C \Rightarrow B = A^{-1}C$$

We will get $\cos\theta_1$ and $\sin\theta_1$. From that we can get θ_1.

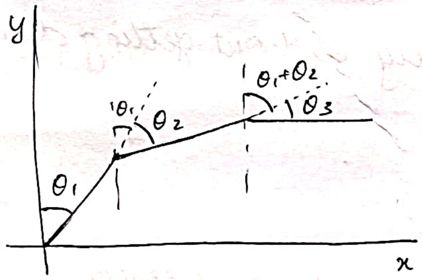
Some imp. points :-

- There can be two orientations for a manipulator arm to reach a particular point in space.



$\theta_1' = \theta_1 + 2\beta$.

$\theta_2' = -\theta_2$.

- 

We usually want the endeffector to be $\perp^r$ to the y-axis.

So we must have $\theta_1 + \theta_2 + \theta_3 = 90°$.

Major division of components in robotics.

① Sensors
② processors.
③ actuators.

Voltage logic is used in all of these. Sensors gather info from the environment or robot itself; processors receive this data, analyze it using algorithms or programmed logic, and make decision about what action robot should take; actuators carry out these decisions by converting electrical energy to mechanical work.

Operating voltage – specific voltage range for actuators to work efficiently and safely.

eg:- Servo motor (4.8 – 6V).

Static and Dynamic Output.

Static voltage – voltage at a stable state, i.e; without load or load not changing over time.

Static output – steady output signal at static voltage.

Dynamic voltage – Voltage while changing or switching between states, i.e; when the load changes with time.

Dynamic output – Output that changes with time at dynamic voltage.

Stall torque and operating torque.

Stall torque → max. torque a motor can produce at a particular voltage when the motor shaft is not rotating. At this point, the motor is stalled (0 RPM), draws maximum current, torque is at it's peak.

Operational torque – torque a motor or actuator produces while it is running under normal working conditions.

Power rating.

- maximum continuous power it can deliver safely without getting damaged at the rated voltage.

eg:- For a DC motor, rated voltage = 12 V.
Power rating = 50W.

then, at 12V → motor can deliver 50 W of mechanical power
at 10V – the output power will be lower (~42W)
at 15V – to get more power may overheat the motor.

eg:- Stall torque = 2Nm, Max speed = 3000 RPM, Power rating = 50W.
Task: lift a robotic arm that requires torque = 1Nm at 1500 RPM.

1st: $\omega = 2\pi \times \frac{1500}{60} \approx 157$ rad/s.

2nd: Mechanical power required: $P = \tau \cdot \omega = 1 Nm \times 157$ rad/s $\approx 157W$.

3rd: Motor power rating = 50 W.
Required power = 157 W.
} Required power exceeds the motor's power rating → the motor cannot safely lift the arm at this speed.

- Drawn Current — current drawn at a particular load and particular voltage.

Margin of Safe use.

"Safety buffer" between the components maximum ratings and it's actual operating conditions, ensuring reliability and durability.

$$\text{Safety margin} = \frac{\text{maximum output}}{\text{application output}} \geq 1.$$

Generally, Safety margin ~1.5.

Q1) Given:- Rover weight = 15 kg, Battery - 24V, $\mu = 1$.
How much should be capacity of battery (mAh) so that we can run for 40 mins at a speed of $1\,ms^{-1}$.

Soln: Force required to move the rover $F = \mu \overset{\text{Normal force}}{N} = 1 \times 15 \times 10 = 150\,N$

Power required, $P = F v = 150 \times 1\ N\,ms^{-1} = 150\ N\,m\,s^{-1}$.

Current to be drawn, $I = \frac{P}{V} = \frac{150}{24}\,A \approx 6.25\,A$.

Now total charge required $= 6.25 \times \frac{40}{60} \times 1000$ mA hr.

$= 4160$ mAh.

Note:- If we consider a safety margin of 1.5, then the capacity of the battery should be 4160×1.5 mAh $= 6250$ mAh.

* Capacity of battery required $\longrightarrow$ energy required by electrical components + all the mechanical processes.

C-rating.
→ How many times the battery's capacity can be discharged per hour.

Discharge current = C-rating × Battery Capacity.

eg:- Capacity = 2Ah
C-rating = 10C
∴ $I = 10 \times 2 = 20A$.

[Discharge current = $\frac{\text{Total charge (Ah)}}{\text{Discharge Time (h)}}$

∵ C-rating = no. of full discharge per hour:

Discharge Time = $\frac{1}{\text{C-rating}}$ hours.

∴ I = Crating × Battery Capacity (Ah)].

∴ Higher Crating ⇒ higher discharge current, I.

1C ⇒ battery will get discharged once in 1hr.
NC ⇒ battery will get discharged N times in 1hr or to discharge once it needs $\frac{1}{N}$ hours.

$C = \frac{60}{t}$, t is in mins. or $C = \frac{1}{t}$, if t is in hrs.

Note:- In (Q1), we needed the battery to discharge in 40 mins, so,

$$\text{C-rating} = \frac{60}{40} = \frac{3}{2}.$$

Q2> A Drone operates at 12V for 10 mins. Standard operational current → 30A.
Find Battery Capacity and C-rating.

Soln: $\text{C-rating} = \frac{60}{10} = 6.$

$$\text{Battery capacity} = 30 \times \frac{10}{60} \times 1000 = 5000 \text{ mAh}$$

Q3> In (Q2), if first 30 secs, the current required is 75 A and remaining time current required is 30A, then find the C-rating.

Soln: $\text{Capacity} = (75\text{A})(0.5 \text{ mins}) + (30\text{A})(9.5 \text{ mins})$
$= 322.5$ A mins. $= 5.375$ Ahr $= 5375$ mAh.

Now the discharge time, if the maximum current supplied by battery was 75 A, is, $t = \frac{5375}{75000}\text{ h} = 0.0717.$

$$\therefore \text{C-rating} = \frac{1}{0.0717} = 14.$$

Mechanical Advantage:

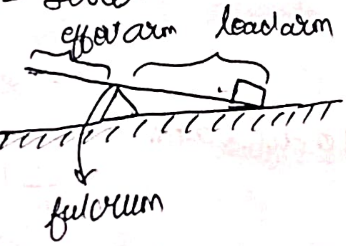
$$\text{Mechanical Advantage, MA} = \frac{\text{Output Force}}{\text{Input Force}} = \frac{\text{load}}{\text{effort}}.$$

→ how efficiently a machine helps lift or move a load using less effort.

If MA > 1 => machine amplifies force.

If MA < 1 => force is reduced.

Eg:- Lever.



Now, by principle of moments,

$$\text{Effort} \times \underbrace{\text{Effort Arm}}_{\text{distance from fulcrum where effort is applied}} = \text{Load} \times \underbrace{\text{Load arm}}_{\text{distance from fulcrum where load is applied.}}$$

$$\text{Now M.A.} = \frac{\text{Load}}{\text{effort}} = \frac{\text{effort arm}}{\text{load arm}}.$$

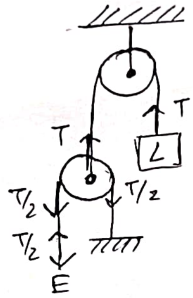
If effort arm > Load arm, then MA > 1

eg:- pulley.

$MA = \frac{L}{E}$.

For this case $L = E$

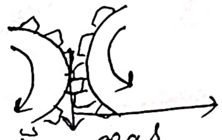
so $MA = 1$.



In this case $L = T$ and $E = T/2$

$\therefore M.A. = \frac{L}{E} = \frac{T}{T/2} = 2$.

eg:- gear, gear-ratios.



velocity at this interface should be same for both the gears.

For gears, mechanical advantage = gear ratio

$$\text{gear ratio} = \frac{\text{No. of teeth on driven gear}}{\text{No. of teeth on driver gear}} = \frac{\text{diameter of driven gear}}{\text{diameter of driver gear}}$$

for M.A., gear ratio > 1.

=> diameter of driven gear > diameter of driver gear.

=> Output torque > input torque

=> speed of driven gear < speed of driver gear.

$\boxed{\text{Planetary motors. \& Stepper motor}}$