# Examples of Signals and the Fourier Series Representation
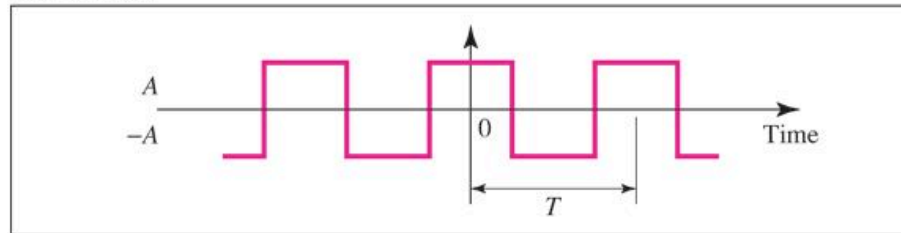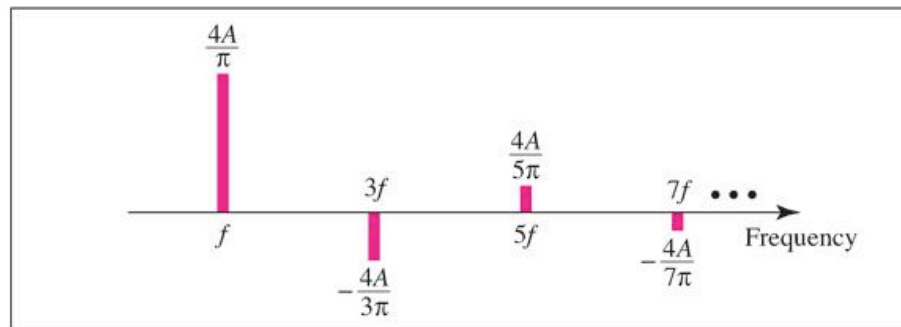
Time domain



$$A_0 = 0 \qquad A_n = \begin{bmatrix} \dfrac{4A}{n\pi} & \text{for } n = 1, 5, 9, \ldots \\ -\dfrac{4A}{n\pi} & \text{for } n = 3, 7, 11, \ldots \end{bmatrix} \qquad B_n = 0$$

$$s(t) = \frac{4A}{\pi} \cos(2\pi ft) - \frac{4A}{3\pi} \cos(2\pi 3ft) + \frac{4A}{5\pi} \cos(2\pi 5ft) - \frac{4A}{7\pi} \cos(2\pi 7ft) + \cdots$$
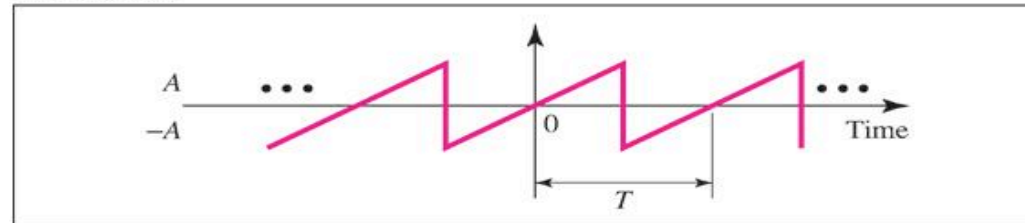


Frequency domain
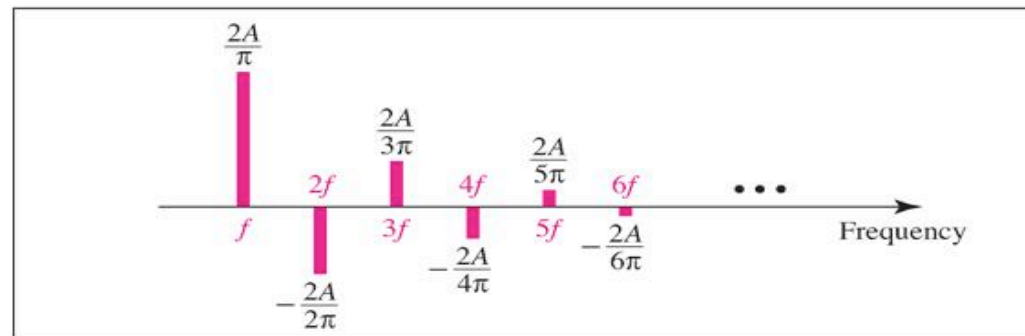
36  Home work

# Sawtooth Signal

Time domain

$$A_0 = 0 \qquad A_n = 0 \qquad B_n = \begin{cases} \dfrac{2A}{n\pi} & \text{for } n \text{ odd} \\ -\dfrac{2A}{n\pi} & \text{for } n \text{ even} \end{cases}$$

$$s(t) = \frac{2A}{\pi} \sin(2\pi ft) - \frac{2A}{2\pi} \sin(2\pi 2ft) + \frac{2A}{3\pi} \sin(2\pi 3ft) - \frac{2A}{4\pi} \sin(2\pi 4ft) + \bullet \bullet \bullet$$

Frequency domain

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# **Fourier Transform Pair** (คู่ผลการแปลงฟูเรียร์)

*Time domain → frequency domain*

Fourier Transform: *แปลงไป→Time domain*

*frequency domain*

$$\star \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

## Analysis

Inverse Fourier Transform: *แปลงกลับ → frequency domain*

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$
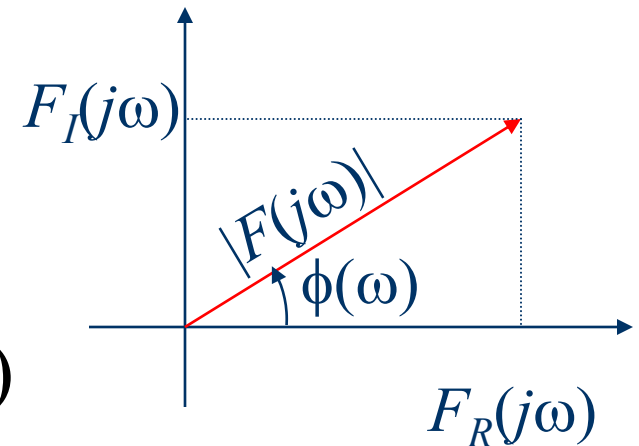
## Synthesis

# Continuous Spectra

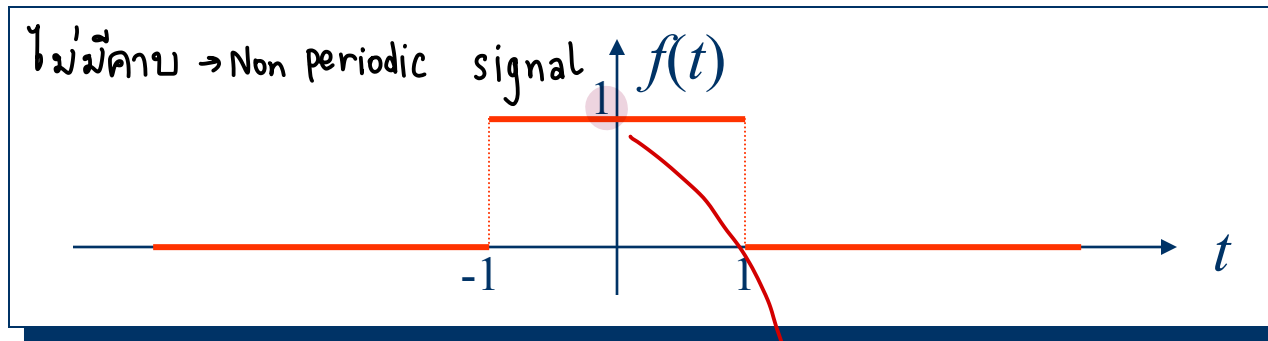$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= \underbrace{|F(j\omega)|}_{\text{Magnitude}}\, e^{j\underbrace{\phi(\omega)}_{\text{Phase}}}$$



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# Example

Time domain → ใช้ frequency
periodic → ใช้ Series
non-periodic → ใช้ Transform

ไม่มีคาบ → Non periodic signal $f(t)$



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^{1}$$

$$= \frac{j}{\omega}(e^{-j\omega} - e^{j\omega}) = \frac{2\sin\omega}{\omega}$$

$= 2\,sinc(\omega)$

$\downarrow$

$sinc \ function = \dfrac{sin\ x}{x}$

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$$z \quad \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^{1}$$

$$z \left( \frac{1}{-j\omega} e^{-j\omega(1)} \right) - \left( \frac{1}{-j\omega} e^{-j\omega(-1)} \right)$$

$$z \quad \frac{e^{-j\omega}}{-j\omega} - \frac{e^{j\omega}}{-j\omega} \qquad \boxed{j \; z \; i \; z \; \sqrt{-1}}$$

$$z \; \frac{j}{j} \cdot \frac{1}{-j\omega} \left( e^{-j\omega} - e^{j\omega} \right)$$

$$\frac{j}{-j^2\omega}$$

$$z \quad \frac{j}{\omega} \left( e^{-j\omega} - e^{j\omega} \right)$$

$$-(-1)^2 \omega$$

$$z \qquad z \; \frac{2j}{2j} \cdot \frac{j}{\omega} \left( e^{-j\omega} - e^{j\omega} \right) \qquad \frac{j}{\omega}$$

$$z \quad \frac{2j^2}{2\omega j} \left( e^{-j\omega} - e^{j\omega} \right)$$

$$z \quad \frac{-2}{2\omega j} \left( e^{-j\omega} - e^{j\omega} \right)$$

$$z \quad \frac{2}{\omega} \frac{\left( e^{j\omega} - e^{-j\omega} \right)}{2j} \leftarrow \quad \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$z \quad \frac{2 \sin \omega}{\omega}$$

# Example

$$\frac{2\sin\omega}{\omega} \xrightarrow{\text{diff}} 2\cos\omega$$

Sinc function    A (ขนาดสัญญาณ)
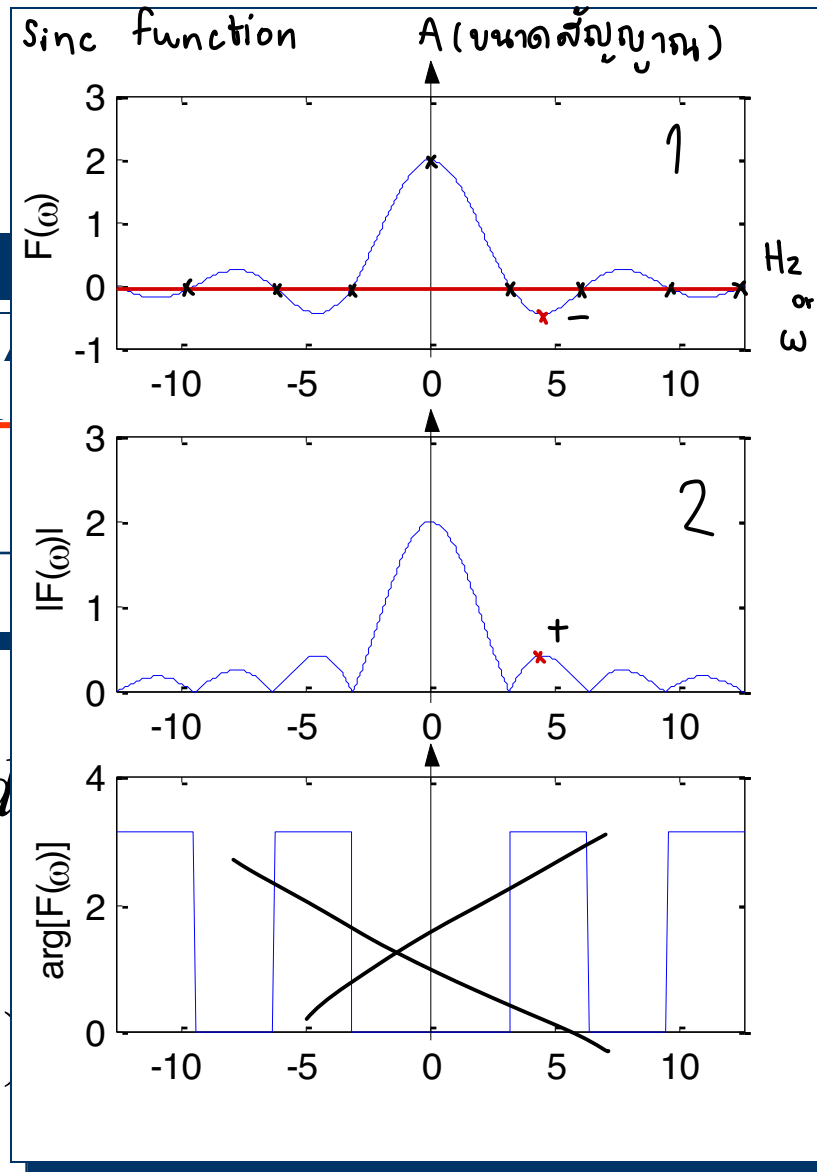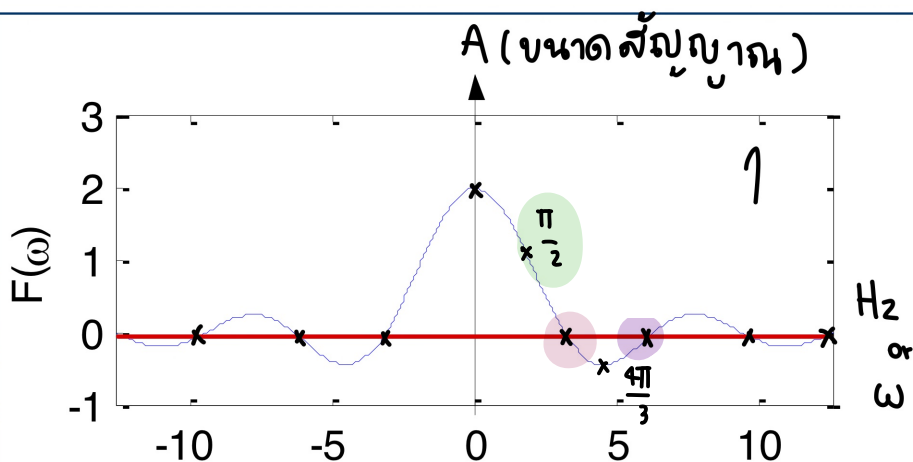


$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}d$$

$$= \frac{j}{\omega}(e^{-j\omega} - e^{j\omega})$$

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A (ขนาดสัญญาณ)

1

$H_2$

ω

$$\frac{2 \sin \omega}{\omega} \rightarrow 2 \cos \omega$$

$$\omega = 0 \rightarrow 2 \cos(0) \rightarrow 2$$

$\sin 0 \rightarrow \omega = \pi$

$\omega = 2\pi f$

$\pi = 2\pi f$

$f = \frac{1}{2}$

$\sin 0 \rightarrow \omega = 2\pi$

$f = 1$

↓

แกน x

$\frac{\pi}{2}$

$$F(\omega) = \frac{2 \sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$$

$$\frac{4}{\pi}$$

$$\omega = \frac{3\pi}{2} \rightarrow F(\omega) = \frac{4\pi}{3} \rightarrow -0.42$$

$$F(\omega) = 1.27$$

# Example: Multiplication by a Sinusoid

Time domain → $x(t) = p_\tau(t)\cos(\omega_0 t)$    sinusoidal burst

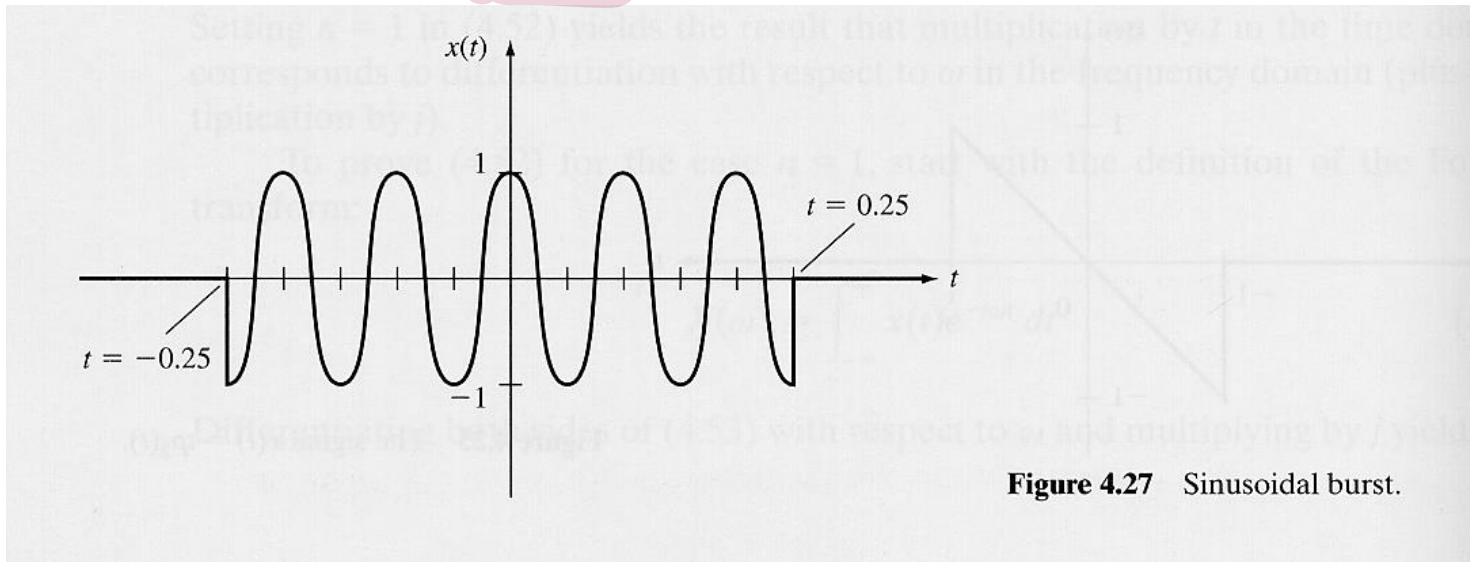

$x(t)$

$t = 0.25$

$t = -0.25$

**Figure 4.27** Sinusoidal burst.

Frequency domain

$$X(\omega) = \frac{1}{2}\left[ \tau \operatorname{sinc}\left( \frac{\tau(\omega + \omega_0)}{2} \right) + \tau \operatorname{sinc}\left( \frac{\tau(\omega - \omega_0)}{2} \right) \right]$$

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Sine wave → กราฟฟองนั่ง

Square wave → sinc function

Signature

# Example: Multiplication by a Sinusoid – Cont'd

$$X(\omega) = \frac{1}{2}\left[ \tau \operatorname{sinc}\left( \frac{\tau(\omega + \omega_0)}{2} \right) + \tau \operatorname{sinc}\left( \frac{\tau(\omega - \omega_0)}{2} \right) \right]$$
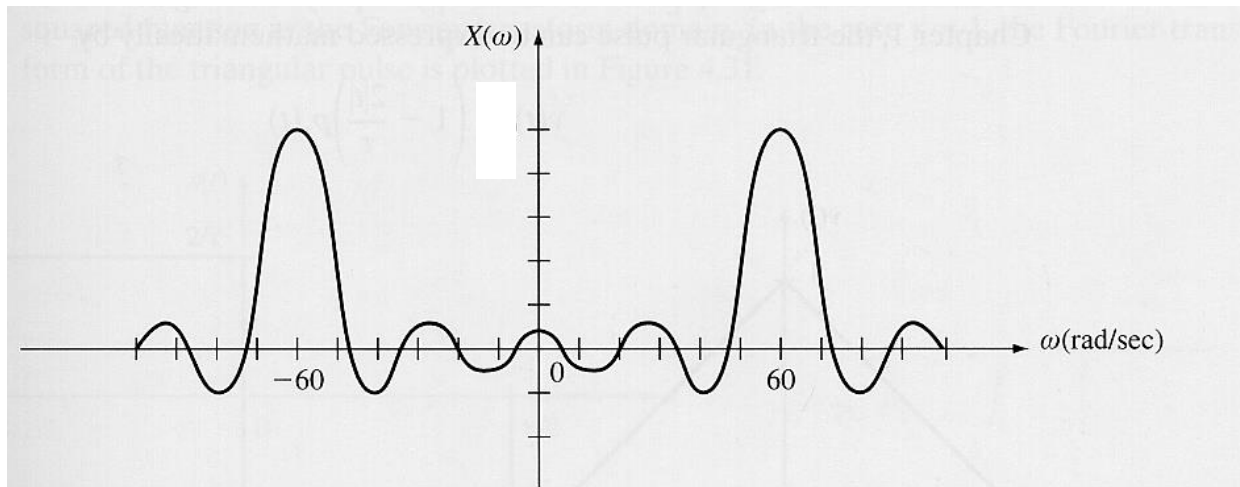


$$\begin{cases} \omega_0 = 60 \, rad/\sec \\ \tau = 0.5 \end{cases}$$

**Figure 4.28**  Fourier transform of the sinusoidal burst $x(t) = p_{0.5}(t) \cos 60t$.
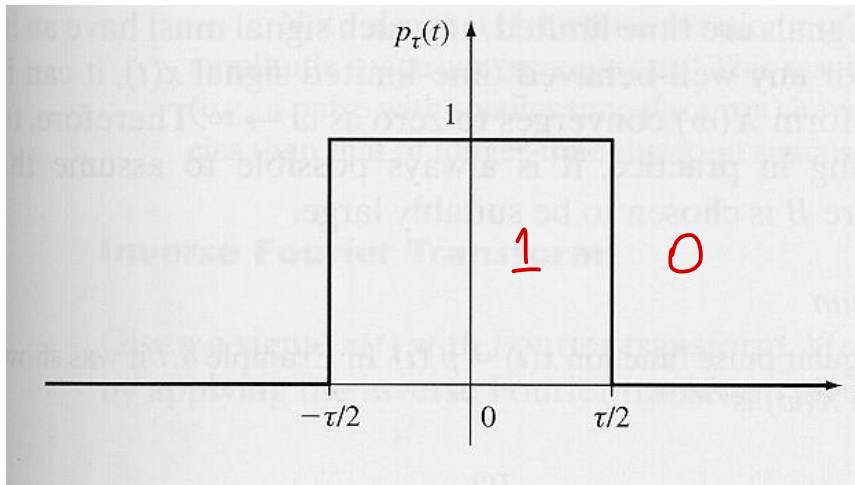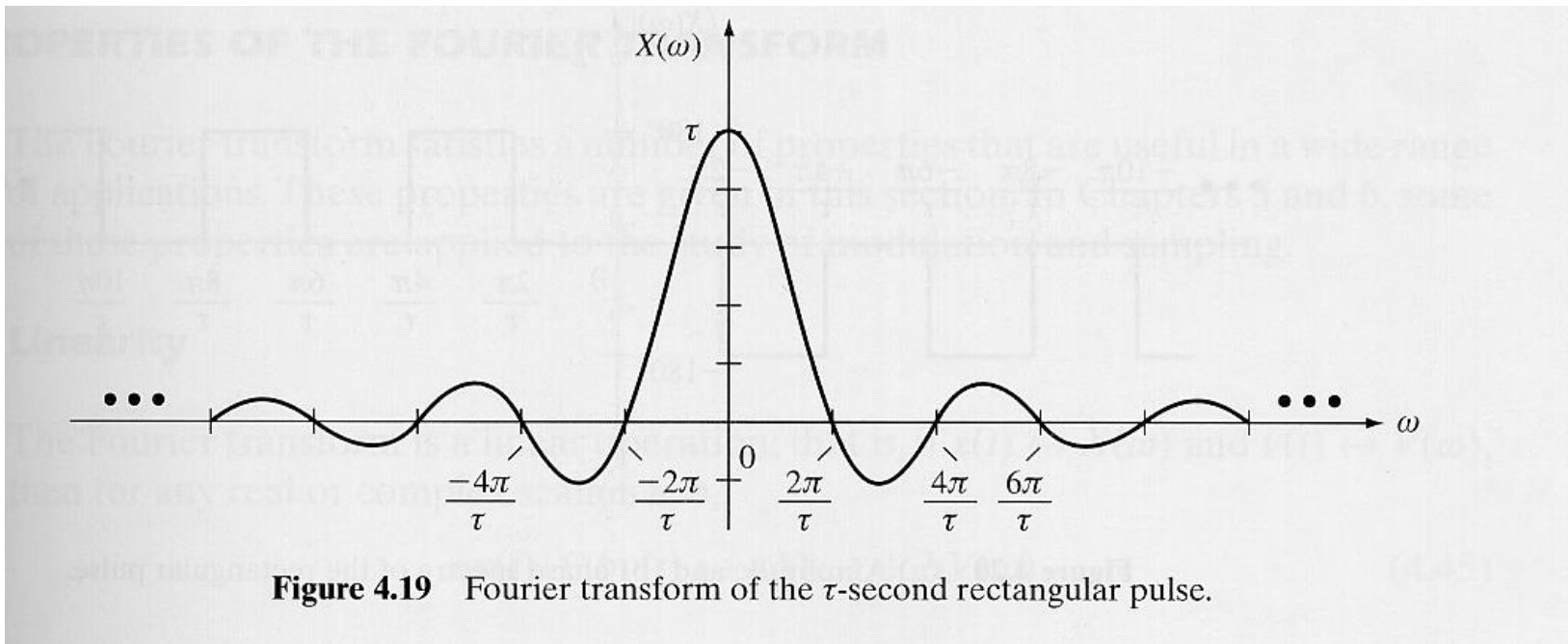
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# Example: Fourier Transform of the Rectangular Pulse



**Figure 4.18** Rectangular pulse of duration $\tau$ seconds.

$$X(\omega) = 2\int_0^{\tau/2} (1)\cos(\omega t)\,dt = \frac{2}{\omega}\left[\sin(\omega t)\right]_{t=0}^{t=\tau/2} = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

$$= \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

# Example: Fourier Transform of the Rectangular Pulse – Cont'd

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$



**Figure 4.19** Fourier transform of the $\tau$-second rectangular pulse.

$$\star \; \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \qquad \star \cos \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2}$$

$$V(\omega) = \int_{-\infty}^{\infty} v(t) \cos(\omega_c t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} v(t)(e^{j\omega_c t} + e^{-j\omega_c t}) e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} v(t) e^{-j(\omega + \omega_c)t}\, dt + \frac{1}{2} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} v(t) e^{-j(\omega - \omega_c)t}\, dt$$

$$= \frac{V_o}{2}\left[ \frac{e^{-j(\omega + \omega_c)t}}{-j(\omega + \omega_c)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \frac{V_o}{2}\left[ \frac{e^{-j(\omega - \omega_c)t}}{-j(\omega - \omega_c)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= \frac{V_o}{2}\left[\frac{e^{-j\frac{\tau}{2}(\omega+\omega_c)}- e^{j\frac{\tau}{2}(\omega+\omega_c)}}{-j(\omega+\omega_c)}\right] + \frac{V_o}{2}\left[\frac{e^{-j\frac{\tau}{2}(\omega-\omega_c)}- e^{j\frac{\tau}{2}(\omega-\omega_c)}}{-j(\omega-\omega_c)}\right]$$

แปลงกลับเป็น sin

$$= \frac{V_o}{\omega+\omega_c} sin\left[\frac{\tau(\omega+\omega_c)}{2}\right] + \frac{V_o}{\omega-\omega_c} sin\left[\frac{\tau(\omega-\omega_c)}{2}\right]$$

$$= \frac{\frac{V_o\tau}{2}}{\frac{\tau}{2}(\omega+\omega_c)} sin\left[\frac{\tau}{2}(\omega+\omega_c)\right] + \frac{\frac{V_o\tau}{2}}{\frac{\tau}{2}(\omega-\omega_c)} sin\left[\frac{\tau}{2}(\omega-\omega_c)\right]$$

$$= \frac{V_o\tau}{2}\left[sinc\{(\omega+\omega_c)\frac{\tau}{2}\} + sinc\{(\omega-\omega_c)\frac{\tau}{2}\}\right]$$