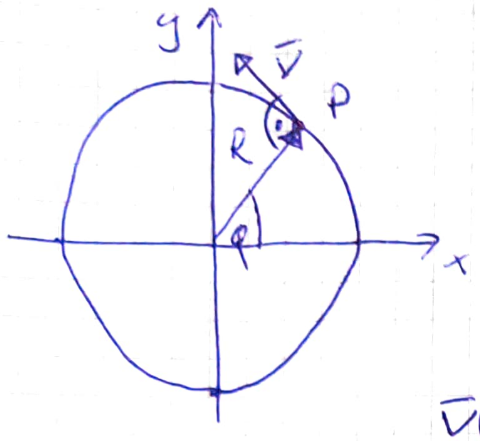
Zad. 1



$$\varphi(t) = \omega t = \frac{v}{R} t$$

$$x(t) = R\cos\varphi = R\cos\omega t$$

$$y(t) = R\sin\varphi = R\sin\omega t$$

$$\bar{r}(t) = [x, y] = R[\cos\omega t, \sin\omega t]$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right] =$$

$$= R[-\omega\sin\omega t, \omega\cos\omega t] =$$

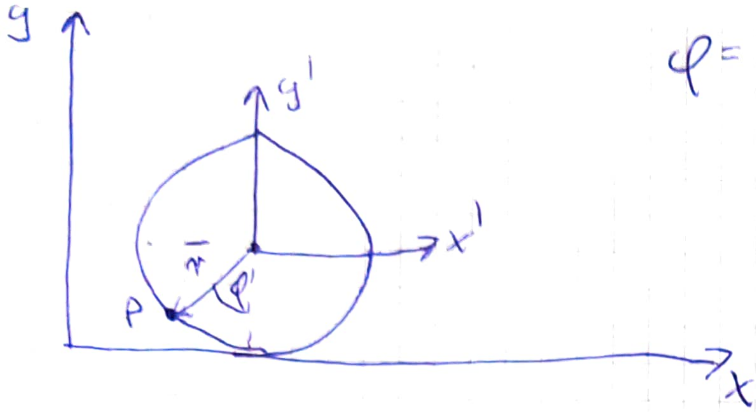
$$= R\omega[-\sin\omega t, \cos\omega t]$$

$$\bar{r} \cdot \bar{v} = R[\cos\omega t, \sin\omega t] \cdot R\omega[-\sin\omega t, \cos\omega t] =$$

$$= R^2\omega(-\sin\omega t\cos\omega t + \sin\omega t\cos\omega t) = 0$$

$$\bar{r} \cdot \bar{v} = 0 \Rightarrow \bar{v} \perp \bar{r}$$

Zad. 2



$$\varphi = \frac{v\,t}{R} = \left\| \omega = \frac{v}{R} \right\| = \omega t$$

a) układ związany ze środkiem okręgu (x', y')

$$x'(t) = -R\sin\varphi = -R\sin\omega t$$
$$y'(t) = -R\cos\varphi = -R\cos\omega t \Rightarrow \bar{r}'(t) = -R(\sin\omega t, \cos\omega t)$$

$$\bar{v}'(t) = \frac{d\bar{r}'}{dt} = -R(\omega\cos\omega t, -\omega\sin\omega t) = -R\omega(\cos\omega t, -\sin\omega t)$$

$$\bar{a}'(t) = \frac{d\bar{v}'}{dt} = -R\omega(-\omega\sin\omega t, -\omega\cos\omega t) =$$
$$= R\omega^2(\sin\omega t, \cos\omega t)$$

$$\bar{a}_t = (\bar{a}\cdot\hat{e}_v)\,\hat{e}_v = \left\| \hat{e}_v = \frac{\bar{v}}{|\bar{v}|} \right\| =$$

$$= (\bar{a}\cdot\bar{v})\frac{\hat{e}_v}{|\bar{v}|} = (\bar{a}\cdot\bar{v})\frac{\bar{v}}{|\bar{v}|^2} =$$

$$= R\omega^2(-R\omega)\big(\sin\omega t\cos\omega t + \cos\omega t(-\sin\omega t)\big)\frac{\bar{v}}{|\bar{v}|^2}$$

$$= 0\,\frac{\bar{v}}{|\bar{v}|^2} = \bar{0}$$

$$\bar{a} = \bar{a}_t + \bar{a}_n = \bar{0} + \bar{a}_n \Rightarrow \bar{a}_n = \bar{a}$$

$$|\bar{a}_n| = R\omega^2 (\sin^2 \omega t + \cos^2 \omega t)^{1/2} = R\omega^2$$

b) układ związany z ziemią: (x, y).
Transformacja $(x,y) \leftrightarrow (x', y')$ to transformacja Galileusza:

$$\bar{r}(t) = \bar{r}'(t) + \bar{R}(t)$$

Położenie środka układu O' w układzie O.

$$\bar{R}(t) = (vt, R)$$

$$x(t) = x'(t) + vt = -R\sin\omega t + vt = \| v = R\omega \| = R(\omega t - \sin \omega t)$$

$$y(t) = y'(t) + R = -R\cos\omega t + R = R(1 - \cos\omega t)$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = R(\omega - \omega\cos\omega t, \omega\sin\omega t) = R\omega(1 - \cos\omega t, \sin\omega t)$$

$$\bar{a}(t) = \frac{d\bar{v}}{dt} = R\omega(\omega\sin\omega t, \omega\cos\omega t) = R\omega^2(\sin\omega t, \cos\omega t)$$

$$\bar{a}_t = (\bar{a}\cdot\hat{e}_v)\,\hat{e}_v = \left(\frac{\bar{a}\cdot\bar{v}}{|\bar{v}|}\right)\hat{e}_v \qquad |\bar{a}_t| = \frac{\bar{a}\cdot\bar{v}}{v}$$

$$\bar{a}\cdot\bar{v} = R\omega^2\cdot R\omega\left(\sin\omega t\cdot(1-\cos\omega t) + \cos\omega t\cdot\sin\omega t\right) =$$

$$= R^2\omega^3\sin\omega t$$

$$|\bar{v}| = R\omega\left((1-\cos\omega t)^2 + \sin^2\omega t\right)^{1/2} =$$

$$= R\omega\left(1 - 2\cos\omega t + \cos^2\omega t + \sin^2\omega t\right)^{1/2} =$$

$$= R\omega\left(2 - 2\cos\omega t\right)^{1/2} = \sqrt{2}\,R\omega\sqrt{1-\cos\omega t} =$$

$$= \sqrt{2}\,R\omega\,\sqrt{2}\sqrt{\frac{1-\cos\omega t}{2}} = \left\|\sqrt{\frac{1-\cos\omega t}{2}} = \left|\sin\frac{\omega t}{2}\right|\right\| =$$

$$= 2R\omega\left|\sin\frac{\omega t}{2}\right|$$

$$\frac{\bar{a}\cdot\bar{v}}{|\bar{v}|} = \frac{R^2\omega^3 2\sin\frac{\omega t}{2}\cos\frac{\omega t}{2}}{2R\omega\left|\sin\frac{\omega t}{2}\right|} = R\omega^2\cos\frac{\omega t}{2}\,\mathrm{sgn}\left(\sin\frac{\omega t}{2}\right)$$

$$= R\omega^2\cos\frac{\omega t}{2}\cdot\begin{cases} +1 & \frac{\omega t}{2}\in[0,\pi]+2k\pi \\ -1 & \frac{\omega t}{2}\in[\pi,2\pi]+2k\pi\end{cases}$$

Przyspieszenie styczne zmienia wartość prędkości, więc $a_t = \dot{v}$

$$\dot{v} = R\omega^2\left|\cos\frac{\omega t}{2}\right|$$

$$\bar{a} = \bar{a}_t + \bar{a}_n \Rightarrow |\bar{a}|^2 = a_t^2 + a_n^2 \Rightarrow a_n = \sqrt{a^2 - a_t^2}$$

$$a_n = R\omega^2\sqrt{1-\cos^2\frac{\omega t}{2}} = R\omega^2\left|\sin\frac{\omega t}{2}\right|$$

Zad. 3

Dany jest tor ruchu : $r(t) = \begin{pmatrix} A\cos\omega t \\ A\sin\omega t \\ Bt \end{pmatrix}$

A, B, ω - const.

- $ds = \sqrt{dx^2 + dy^2 + dz^2} =$

$$= \left\| \begin{array}{l} dx = \frac{dx}{dt} \cdot dt \\ \uparrow \\ \text{różniczka } x, \\ \text{mała zmiana } x \end{array} \right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= |\bar{v}|\, dt$$

$$\bar{v} = \begin{pmatrix} -A\omega\sin\omega t \\ A\omega\cos\omega t \\ B \end{pmatrix} \Rightarrow |\bar{v}|^2 = (-A\omega\sin\omega t)^2 + (A\omega\cos\omega t)^2 + B^2 = (A\omega)^2 + B^2$$

$$ds = |\bar{v}|\, dt = \sqrt{(A\omega)^2 + B^2}\, dt$$

- $$S(t_1, t_2) = \int_{t_1}^{t_2} \frac{ds}{dt}\, dt = \left\| \frac{ds}{dt} = |\bar{v}| \right\| = \int_{t_1}^{t_2} |\bar{v}|\, dt =$$

$$= \int_{t_1}^{t_2} \sqrt{A\omega^2 + B^2}\, dt = \left\| A, B, \omega - \text{const.} \right\| =$$

$$= \sqrt{A^2\omega^2 + B^2} \int_{t_1}^{t_2} dt = \sqrt{A^2\omega^2 + B^2}\,(t_2 - t_1)$$

$$\bullet\ \bar{t} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt} \cdot \frac{dt}{ds} = \left\| \underbrace{\frac{dt}{ds}}_{=\frac{1}{|\bar{v}|}} = \frac{1}{\frac{ds}{dt}} \right\| = \frac{\bar{v}}{|\bar{v}|} = \hat{e}_v$$

$$\bar{t} = \left(A^2\omega^2 + B^2\right)^{-1/2} \begin{pmatrix} -A\omega \sin \omega t \\ A\omega \cos \omega t \\ B \end{pmatrix}$$

$$\bullet\ \bar{n} = \frac{d\bar{t}}{ds} \Big/ \left|\frac{d\bar{t}}{ds}\right|$$

$$\frac{d\bar{t}}{ds} = \frac{d\bar{t}}{dt} \cdot \frac{dt}{ds} = \frac{d\bar{t}}{dt} \cdot \frac{1}{|\bar{v}|} =$$

$$= \left(A^2\omega^2 + B^2\right)^{-1/2} \begin{pmatrix} -A\omega^2 \cos \omega t \\ -A\omega^2 \sin \omega t \\ 0 \end{pmatrix} \left(A^2\omega^2 + B^2\right)^{-1/2} =$$

$$= \left(A^2\omega^2 + B^2\right)^{-1} \begin{pmatrix} -A\omega^2 \cos \omega t \\ -A\omega^2 \sin \omega t \\ 0 \end{pmatrix} = \frac{-A\omega^2}{\left(A^2\omega^2 + B^2\right)} \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix}$$

$$\left|\frac{d\bar{t}}{ds}\right| = \frac{A\omega^2}{A^2\omega^2 + B^2} \Rightarrow \bar{n} = -\begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix}$$

- $\rho = \left|\frac{d\bar{t}}{ds}\right|^{-1} = \dfrac{A^2\omega^2 + B^2}{A\omega^2}$

- $\bar{a} = \dfrac{d\bar{v}}{dt} = \begin{pmatrix} -A\omega^2 \cos\omega t \\ -A\omega^2 \sin\omega t \\ 0 \end{pmatrix} \Rightarrow |\bar{a}| = A\omega^2$

- $a_t = \bar{a}\cdot\bar{t} = -A\omega^2 \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} \begin{pmatrix} -A\omega \sin\omega t \\ A\omega \cos\omega t \\ B \end{pmatrix} \left(A^2\omega^2 + B^2\right)^{-1/2}$

$$\frac{-A\omega^2}{\sqrt{A^2\omega^2+B^2}}\Big[\cos\omega t\cdot(-A\omega)\cdot\sin\omega t + \sin\omega t\cdot(A\omega)\cdot\cos\omega t\Big] =$$

$$= 0$$

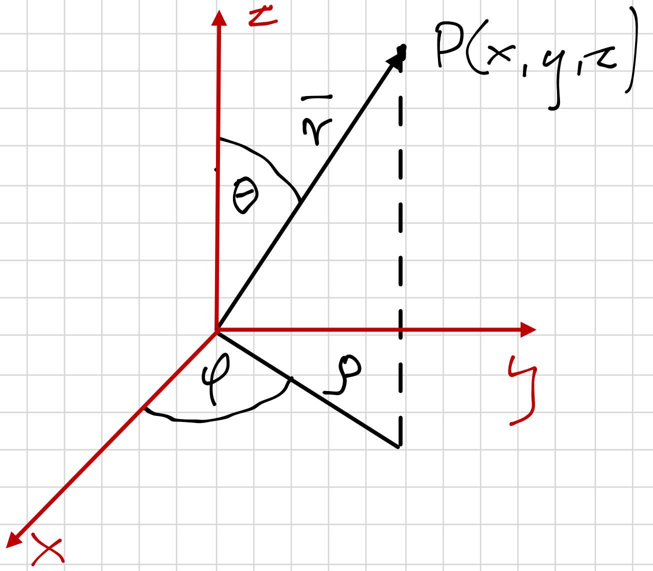
- $a_n = \bar{a}\cdot\bar{n} = \sqrt{|\bar{a}|^2 - a_t^2} = \Big\| a_t = 0 \Big\| = |\bar{a}|$

$$= A\omega^2$$

$\bar{a} = 0\cdot\bar{t} + a\cdot\bar{n}$

to oznacza, że długości vektora prędkości jest stała: $\dfrac{d|\bar{v}|}{dt} = 0$

Zad. 4



$P(x, y, z)$ - punkt o współrzędnych x, y, z

$\bar{r} = x \cdot \hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z$

- wektor wodzący punktu P

a) układ walcowy: $P(\rho, \varphi, z)$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\bar{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$$

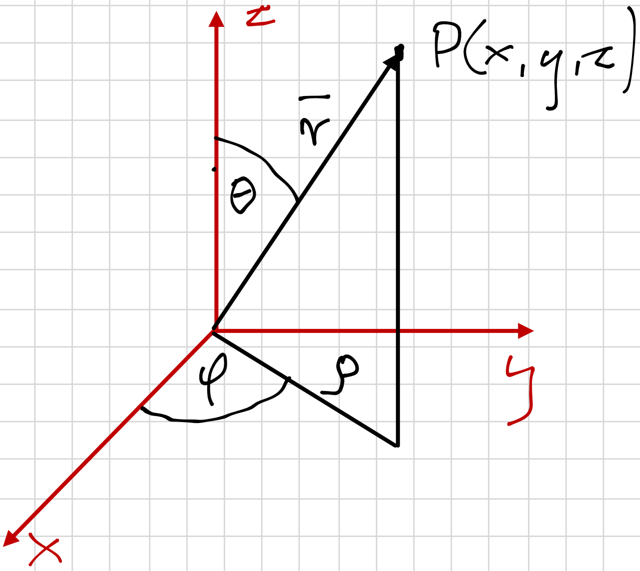
b) układ sferyczny: $P(r, \theta, \varphi)$

$\rho = r\sin\theta$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases}$$

$$\bar{r} = r\,\hat{e}_r$$

Zad. 5



$\hat{e}_q$ – wersor w kierunku współrzędnej q:

$$\hat{e}_q = \frac{\partial \bar{r}}{\partial q} \Big/ \left|\frac{\partial \bar{r}}{\partial q}\right|$$

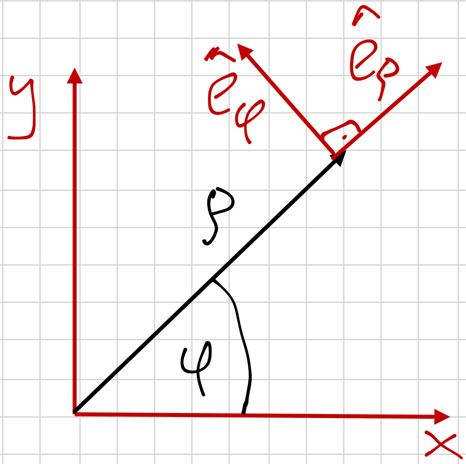
$$\bar{r} = x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z =$$

$$= \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$$

- $\left.\begin{array}{l} \frac{\partial \bar{r}}{\partial \rho} = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y \\ \left|\frac{\partial \bar{r}}{\partial \rho}\right|^2 = \cos^2\varphi + \sin^2\varphi = 1 \end{array}\right\}$ $\hat{e}_\rho = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y$

- $\left.\begin{array}{l} \frac{\partial \bar{r}}{\partial \varphi} = -\rho\sin\varphi\,\hat{e}_x + \rho\cos\varphi\,\hat{e}_z \\ \left|\frac{\partial \bar{r}}{\partial \varphi}\right| = \rho \end{array}\right\}$ $\hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y$

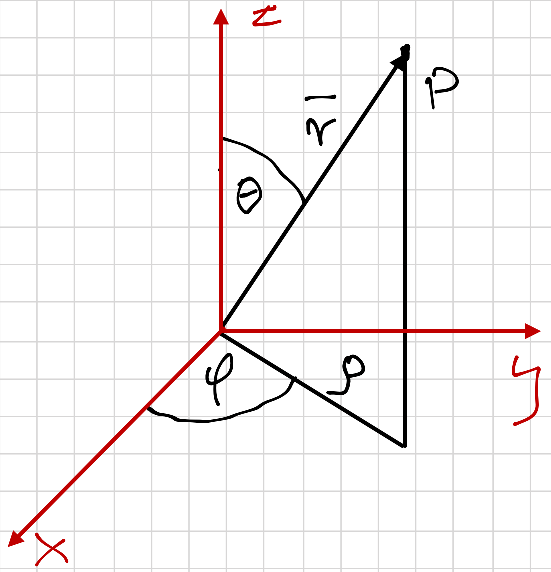
- $\left.\frac{\partial \bar{r}}{\partial z} = \hat{e}_z\right\}$ $\hat{e}_z = \hat{e}_z$



$$\hat{e}_\rho = \cos\varphi\ \hat{e}_x + \sin\varphi\ \hat{e}_y$$
$$\hat{e}_\varphi = -\sin\varphi\ \hat{e}_x + \cos\varphi\ \hat{e}_y$$
$$\hat{e}_z = \hat{e}_z$$
$$\varphi = \varphi(t)$$

- $$\dot{\hat{e}}_\rho = -\sin\varphi \cdot \dot{\varphi}\ \hat{e}_x + \cos\varphi \cdot \dot{\varphi}\ \hat{e}_y =$$
  $$= \dot{\varphi}\left(-\sin\varphi\ \hat{e}_x + \cos\varphi\ \hat{e}_y\right) =$$
  $$= \dot{\varphi}\ \hat{e}_\varphi$$
- $$\dot{\hat{e}}_\varphi = -\cos\varphi\ \dot{\varphi}\ \hat{e}_x - \sin\varphi \cdot \dot{\varphi} \cdot \hat{e}_y =$$
  $$= -\dot{\varphi}\left(\cos\varphi\ \hat{e}_x + \sin\varphi\ \hat{e}_y\right) =$$
  $$= -\dot{\varphi}\ \hat{e}_\rho$$
- $$\dot{\hat{e}}_z = 0$$

Zad. 6



- Punkt P jest określony przez współrzędne kartezjańskie:
$$P = (\sqrt{2}, \sqrt{2}, 2\sqrt{3})_{xyz}$$

- pole wektorowe – każdemu punktowi przestrzeni jest przypisany wektor.

Tutaj $\vec{A}(P) = \hat{e}_x - 2\hat{e}_y + 3\hat{e}_z =$
$= (1, -2, 3)_{xyz}$

Jeśli w problemie występują różne układy współrzędnych warto zaznaczyć w jakim układzie są podane składowe.

- współrzędne punktu P w układzie sferycznym

$$r = \sqrt{x^2 + y^2 + z^2} =$$
$$= \sqrt{2 + 2 + 12} = \sqrt{16} = 4$$

$$\cos\theta = \frac{z}{r} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow$$
$$\theta = \frac{\pi}{6}$$

$$\operatorname{tg}\varphi = \frac{y}{x} = \frac{\sqrt{2}}{\sqrt{2}} = 1,\ y > 0 \Rightarrow \varphi = \frac{\pi}{4}$$

$$P = \left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)_{r\theta\varphi}$$

Uwaga!

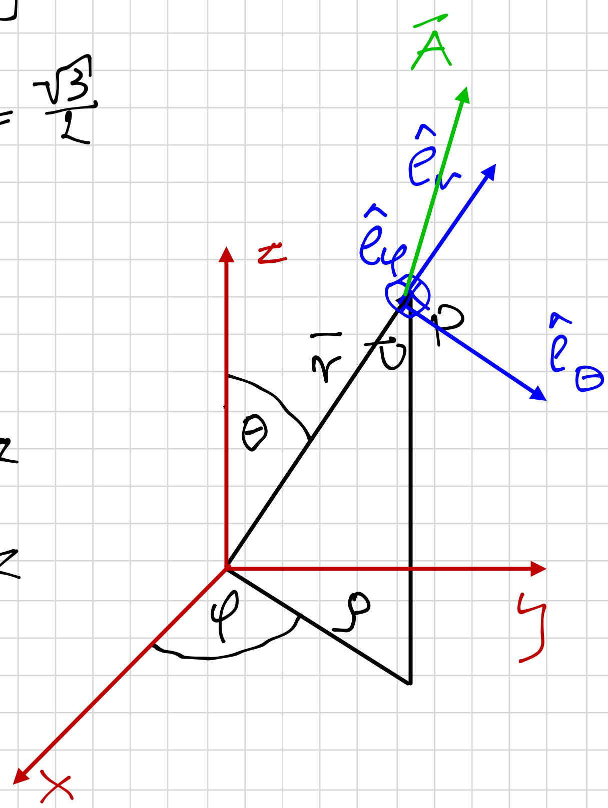
Wektor wodzący punktu P w układzie sferycznym to $\bar{r}_P = 4 \cdot \hat{e}_r$

Wersory układu sferycznego:

$$\begin{cases} \hat{e}_r = \sin\theta\cos\varphi\ \hat{e}_x + \sin\theta\sin\varphi\ \hat{e}_y + \cos\theta\ \hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\varphi\ \hat{e}_x + \cos\theta\sin\varphi\ \hat{e}_y - \sin\theta\ \hat{e}_z \\ \hat{e}_\varphi = -\sin\varphi\ \hat{e}_x + \cos\varphi\ \hat{e}_y \end{cases}$$

$\theta = \frac{\pi}{6} \Rightarrow \sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\varphi = \frac{\pi}{4} \Rightarrow \sin\varphi = \cos\varphi = \frac{\sqrt{2}}{2}$



$$\hat{e}_r = \frac{\sqrt{2}}{4}\hat{e}_x + \frac{\sqrt{2}}{4}\hat{e}_y + \frac{\sqrt{3}}{2}\hat{e}_z$$

$$\hat{e}_\theta = \frac{\sqrt{6}}{4}\hat{e}_x + \frac{\sqrt{6}}{4}\hat{e}_y - \frac{1}{2}\hat{e}_z$$

$$\hat{e}_\varphi = -\frac{\sqrt{2}}{2}\hat{e}_x + \frac{\sqrt{2}}{2}\hat{e}_y$$

- $A_r = \bar{A}\cdot\hat{e}_r = (1,-2,3)_{xyz}\cdot\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{4}\right)_{xyz} =$

  $= \frac{\sqrt{2}}{4} - \frac{2\sqrt{2}}{4} + \frac{3\sqrt{3}}{2} = \frac{6\sqrt{3}-\sqrt{2}}{4}$

- $A_\theta = \bar{A}\cdot\hat{e}_\theta = (1,-2,3)_{xyz}\cdot\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, -\frac{1}{2}\right)_{xyz} =$

  $= \frac{\sqrt{6}}{4} - \frac{2\sqrt{6}}{4} - \frac{3}{2} = \frac{-\sqrt{6}-6}{4}$

- $A_\varphi = \bar{A}\cdot\hat{e}_\varphi = (1,-2,3)_{xyz}\cdot\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)_{xyz} =$

  $= -\frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$

$$\bar{A} = (1,-2,3)_{xyz} = \left(\frac{6\sqrt{3}-\sqrt{2}}{4}, -\frac{\sqrt{6}+6}{4}, -\frac{3\sqrt{2}}{2}\right)_{r\theta\varphi}$$

$$|\bar{A}|^2 = 1+(-2)^2+3^2 = 14$$

$$= \left(\frac{6\sqrt{3}-\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{6}+6}{4}\right)^2 + \left(-\frac{3\sqrt{2}}{2}\right)^2 =$$

$$= \frac{36\cdot 3 - 2\cdot 6\sqrt{6} + 2}{16} + \frac{6 - 2\cdot 6\cdot \sqrt{6} + 36}{16} + \frac{9\cdot 2}{4} =$$

$$= \frac{36\cdot 3 + 2 + 6 + 36 + 9\cdot 8}{16} = 14$$

Zad. 7

- układ kartezjański: $\dot{\hat{e}}_x=0$, $\dot{\hat{e}}_y=0$, $\dot{e}_z=0$

$$\bar{r}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z=(x,y,z)_{xyz}$$

$$\bar{v}=\dot{\bar{r}}=\dot{x}\hat{e}_x+\dot{y}\hat{e}_y+\dot{z}\hat{e}_z=(\dot{x},\dot{y},\dot{z})_{xyz}$$

$$\bar{a}=\dot{\bar{v}}=\ddot{\bar{r}}=(\ddot{x},\ddot{y},\ddot{z})_{xyz}$$

- układ walcowy: $\dot{\hat{e}}_\rho=\dot{\varphi}\hat{e}_\varphi$, $\dot{\hat{e}}_\varphi=-\dot{\varphi}\hat{e}_\rho$, $\dot{\hat{e}}_z=0$

$$\bar{r}=\rho\hat{e}_\rho+z\hat{e}_z$$

$$\bar{v}=\dot{\bar{r}}=\dot{\rho}\hat{e}_\rho+\rho\dot{\hat{e}}_\rho+\dot{z}\hat{e}_z+z\dot{\hat{e}}_z=$$

$$=\dot{\rho}\hat{e}_\rho+\rho\dot{\varphi}\hat{e}_\varphi+\dot{z}\hat{e}_z=$$

$$=(\dot{\rho},\rho\dot{\varphi},\dot{z})_{\rho\varphi z}$$

$$\bar{a}=\dot{\bar{v}}=\ddot{\rho}\hat{e}_\rho+\dot{\rho}\dot{\hat{e}}_\rho+\dot{\rho}\dot{\varphi}\hat{e}_\varphi+\rho\ddot{\varphi}\hat{e}_\varphi+$$

$$+\rho\dot{\varphi}\dot{\hat{e}}_\varphi+\ddot{z}\hat{e}_z+\dot{z}\dot{\hat{e}}_z=$$

$$=\ddot{\rho}\hat{e}_\rho+\dot{\rho}\dot{\varphi}\hat{e}_\varphi+\dot{\rho}\dot{\varphi}\hat{e}_\varphi+\rho\ddot{\varphi}\hat{e}_\varphi+$$

$$-\rho\dot{\varphi}^2\hat{e}_\rho+\ddot{z}\hat{e}_z=$$

$$=\hat{e}_\rho(\ddot{\rho}-\rho\dot{\varphi}^2)+\hat{e}_\varphi(2\dot{\rho}\dot{\varphi}+\rho\ddot{\varphi})+\ddot{z}\hat{e}_z=$$

$$=(\ddot{\rho}-\rho\dot{\varphi}^2,\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi},\ddot{z})_{\rho\varphi z}$$

• układ sferyczny: $\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta + \dot{\varphi}\sin\theta\,\hat{e}_\varphi$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r + \dot{\varphi}\cos\theta\,\hat{e}_\varphi$$

$$\dot{\hat{e}}_\varphi = -\dot{\varphi}\sin\theta\,\hat{e}_r - \dot{\varphi}\cos\theta\,\hat{e}_\theta$$

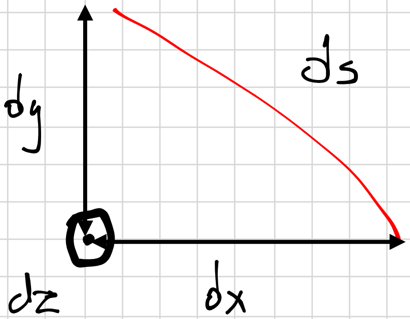
$$\bar{r} = r\,\hat{e}_r$$

$$\bar{v} = \dot{\bar{r}} = \left(\dot{r},\ r\dot{\theta},\ r\dot{\varphi}\sin\theta\right)_{r\theta\varphi}$$

$$\bar{a} = \dot{\bar{v}} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta, \\ 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta \end{pmatrix}_{r\theta\varphi}$$

Zad. 8

a) układ kartezjański



$$ds = \sqrt{dx^2 + dy^2 + dz^2} =$$

wyrażamy różniczki jako funkcje jednego parametru, np. t

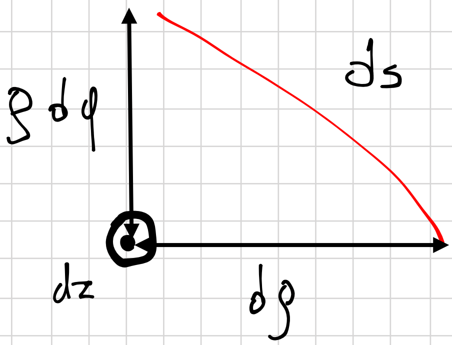
$dx = \frac{dx}{dt}\,dt$, itd.

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \sqrt{v_x^2 + v_y^2 + v_z^2}\,dt$$

$$= |\vec{v}|\,dt$$

b) układ walcowy

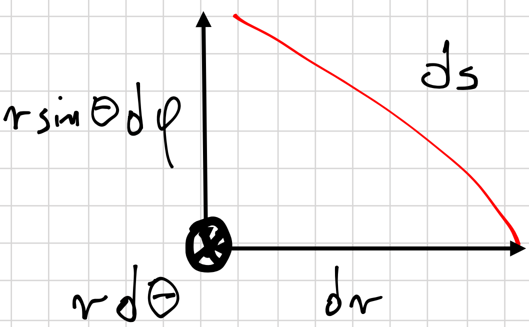


$$ds = \sqrt{(d\rho)^2 + (\rho\,d\varphi)^2 + (dz)^2} =$$

$$= \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\varphi}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt =$$

$$= \sqrt{v_\rho^2 + v_\varphi^2 + v_z^2}\,dt$$

c) układ sferyczny

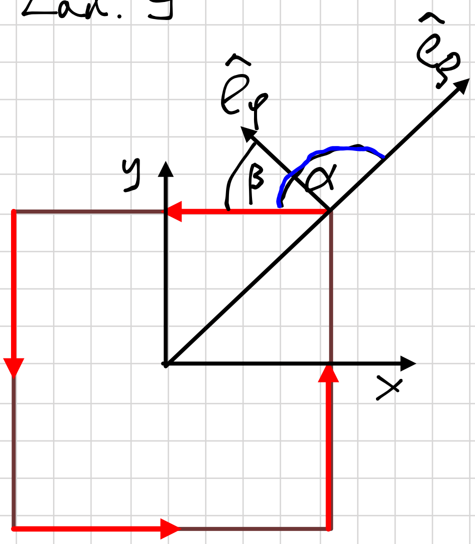


$$ds = \sqrt{(dr)^2 + (r\,d\Theta)^2 + (r\sin\Theta\,d\varphi)^2}$$

$$= \sqrt{\left(\frac{dr}{dt}\right)^2 + r\left(\frac{d\Theta}{dt}\right)^2 + \left(r\sin\Theta\frac{d\varphi}{dt}\right)^2} =$$

$$= \sqrt{(v_r)^2 + (v_\Theta)^2 + (v_\varphi)^2}$$

Zad. 9



- $|\bar{v}| = \text{const.} = v_0$
- α - kąt między wektorem prędkości a promieniem wodzącym
  $\alpha = \text{const.} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Wybieramy układ walcowy i płaszczyznę $z=0$

$$\bar{v} = (v_\rho, v_\varphi, v_z) = \left\| v_z = 0 \right\| = v_\rho \hat{e}_\rho + v_\varphi \hat{e}_\varphi$$

$$v_\rho = \bar{v} \cdot \hat{e}_\rho = |\bar{v}| \cdot \cos\alpha = v_0 \cos\left(\frac{3\pi}{4}\right) = -\frac{v_0}{\sqrt{2}}$$

$$v_\varphi = \bar{v} \cdot \hat{e}_\varphi = |\bar{v}| \cos\beta = \left\| \begin{matrix} \beta = \alpha - \frac{\pi}{2} \\ \cos(\alpha - \frac{\pi}{2}) = \sin\alpha \end{matrix} \right\| = \frac{v_0}{\sqrt{2}}$$

Z drugiej strony w układzie walcowym mamy: $\bar{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\varphi}\hat{e}_\varphi + \dot{z}\hat{e}_z$

Stąd:

$$\dot{\rho} = -\frac{v_0}{\sqrt{2}} \qquad \rho\dot{\varphi} = \frac{v_0}{\sqrt{2}}$$

a) poszukujemy $r(t)$ i $\varphi(t)$

całkujemy:

- $\frac{d\rho}{dt} = -\frac{v_0}{\sqrt{2}} \Rightarrow \quad \rho(t) = \int \frac{d\rho}{dt} \cdot dt = \int -\frac{v_0}{\sqrt{2}} dt$

$$= -\frac{v_0}{\sqrt{2}} t + C$$

$$\rho(t=0) = \frac{a\sqrt{2}}{2} \Rightarrow C = \frac{a\sqrt{2}}{2}$$

$$\rho(t) = \frac{a\sqrt{2}}{2} - \frac{v_0 t}{\sqrt{2}} = \frac{\sqrt{2}}{2}\left(a - v_0 t\right)$$

- $$\rho \frac{d\varphi}{dt} = \frac{v_0}{\sqrt{2}} \Rightarrow \frac{d\varphi}{dt} = \frac{1}{\rho}\frac{v_0}{\sqrt{2}} = \sqrt{2}\left(a - v_0 t\right)^{-1} \cdot \frac{v_0}{\sqrt{2}}$$

$$= \frac{v_0}{a - v_0 t}$$

$$\varphi(t) = \int \frac{d\varphi}{dt}\, dt = \int \frac{v_0}{a - v_0 t}\, dt = \frac{v_0}{a}\int \frac{dt}{1 - \frac{v_0}{a}t} =$$

$$= \frac{v_0}{a} \ln\left|1 - \frac{v_0}{a}t\right| \left(-\frac{v_0}{a}\right)^{-1} + C =$$

$$= -\ln\left|1 - \frac{v_0}{a}t\right| + C$$

$$\varphi(0) = -\ln|1| + C \Rightarrow C = 0$$

$$\varphi(t) = -\ln\left|1 - \frac{v_0}{a}t\right|$$

b) czas do spotkania („konsumpcji”)

Spotkanie nastąpi w środku kwadratu czyli dla $\rho = 0$

$$\rho(T) = 0 = \frac{\sqrt{2}}{2}\left(a - v_0 T\right) \Rightarrow T = \frac{a}{v_0}$$

c) poszukujemy postaci parametrycznej $\rho(\varphi)$

$$\varphi(t) = -\ln\left|\frac{a-v_0t}{a}\right| = \left\|\ \frac{a-v_0t}{a} = \frac{\sqrt{2}}{a}\rho(t)\ \right\|$$

$$= -\ln\left|\frac{\sqrt{2}}{a}\rho\right| \Rightarrow e^{-\varphi} = \frac{\sqrt{2}}{a}\rho$$

$$\rho(\varphi) = \frac{a}{\sqrt{2}}e^{-\varphi}$$

d) składowe przyspieszenia, promień krzywizny i całkowita droga

- przyspieszenie w układzie walcowym:

$$\bar{a} = (\ddot{\rho} - \rho\dot{\varphi}^2,\ 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi},\ \ddot{z})$$

$$\ddot{\rho} = \frac{d}{dt}v_\rho = \frac{d}{dt}\left(-\frac{v_0}{\sqrt{2}}\right) = 0$$

$$\ddot{\varphi} = \frac{d}{dt}\left(\frac{1}{\rho}v_\varphi\right) = \frac{d}{dt}\left(\frac{1}{\rho}\frac{v_0}{\sqrt{2}}\right) = -\frac{1}{\rho^2}\frac{v_0}{\sqrt{2}}\dot{\rho} =$$

$$= -\frac{v_0}{\rho^2\sqrt{2}}\left(-\frac{v_0}{\sqrt{2}}\right) = \frac{v_0}{2\rho^2}$$

$$\bar{a} = \left(-\rho\left(\frac{v_0}{\sqrt{2}\rho}\right)^2,\ 2\left(-\frac{v_0}{\sqrt{2}}\right)\left(\frac{v_0}{\sqrt{2}\rho}\right) + \rho\frac{v_0}{2\rho^2},\ 0\right) =$$

$$= \left(\underbrace{-\frac{v_0^2}{2\rho}}_{a_\rho},\ \underbrace{-\frac{v_0^2}{2\rho}}_{a_\varphi},\ 0\right)_{\rho\varphi z}$$

$$a_t = \bar{a} \cdot \bar{t} = \left\| \bar{t} = \frac{\bar{v}}{|\bar{v}|} = \left(-\frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}}, 0\right)\frac{1}{v_0} \right\|$$

$$= \frac{v_0^2}{2\rho}(-1,-1) \cdot \frac{1}{\sqrt{2}}(-1, 1) = 0$$

Inaczej:

$$a_t = \frac{d}{dt}|\bar{v}| = \frac{d}{dt}(v_0) = 0$$

$$a_n^2 = |\bar{a}|^2 - a_t^2 = \| a_t = 0 \| = |\bar{a}|$$

$$|a| = \frac{v_0^2}{2\rho}\sqrt{1^2+1^2} = \frac{v_0^2}{\sqrt{2}\rho} = a_n$$

Z drugiej strony $a_n = \frac{v^2}{\rho}$ stąd

$$\rho = \frac{\bar{v}^2}{a_n} = v_0^2 \cdot \frac{\sqrt{2}\rho}{v_0^2} = \sqrt{2}\rho(t)$$

promień krzywizny

współrzędna układu walcowego

Przebyta droga S:

$$S = \int ds = \| ds = |\bar{v}| dt \| = \int |\bar{v}| dt =$$

$$= \int_0^T v_0 \, dt = v_0 T = v_0 \cdot \frac{a}{v_0} = a$$