General Relativity

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Chapter 1

Special Relativity

Lecture 1: Minkowski spacetime

20 paź 2020 First we will formulate a geometry which is equivalent to basic relativistic symmetries known from the special relativity. It was actually the first step towards GR to understand special relativity as symmetries of a given geometry.

Definition 1 (Minkowski spacetime)**.** 4-dimensional affine space M with associated scalar product g (metric tensor) of signature $(-+++)$.

Definition 2 (Affine space)**.** We define,

- Vector space at each point
- Global parallelism (allows to identify vectors at different points)
- $m_0, m_1 \in M$ defines a vector at m_0 : $m_0 + \underbrace{m_1 - m_0}_{v'} = m_1$

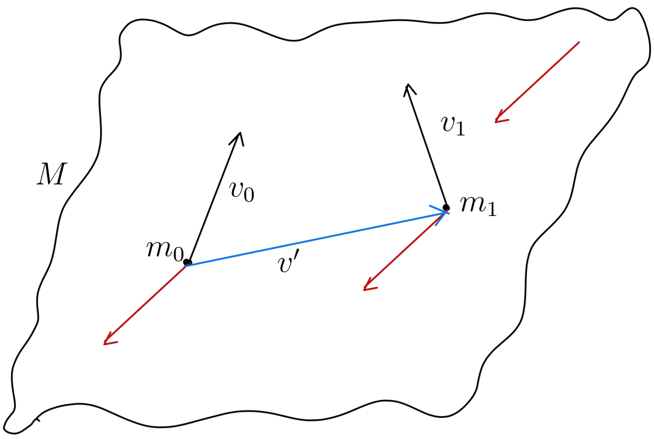

Figure 1.1: Affine space

In practice you can consider M as a 4D vector space upon a choice of an origin $m_0 \in M$. Let V be 4-dim vector space. We make $V \equiv M$, where M – affine space understood as 4D vector space up to translations.

Definition 3 (Metric tensor)**.**

$$
\begin{aligned}
g(u,v) &\in \mathbb{R} \\
g(u,v) &= g(w,u) \\
g(\alpha v + \beta u, w) &= \alpha g(v,w) + \beta g(u,w)
\end{aligned}
$$

We can see that $\forall g$, that is symmetric and bilinear, $\exists \{e_0, e_1, e_2, e_3\}$ such that:

$$
\begin{gathered}
g(e_0, e_0) = \pm 1 \\
\vdots \\
g(e_3, e_3) = \pm 1
\end{gathered}
$$

whenever we take product of two different basis vectors,

$$g\big(e_i, e_j\big) \overset{i \neq j}{=} 0$$

In other words,

$$g\big(e_i, e_j\big) = \pm \delta^i_j$$

Here we have an orthogonal basis. The number of $+$ and $-$ is invariant for a given g. In Minkowski spacetime it is $(- + ++)$.
This is the complete definition of Minkowski spacetime.

This scalar product is independent on choice of point $m \in M$ (because the global parallelism holds). In other words, the parallel transport preserves g.

Timelike, spacelike, null vectors

Scalar product distinguishes between different types of vectors that may emerge on M: timelike, spacelike and null.

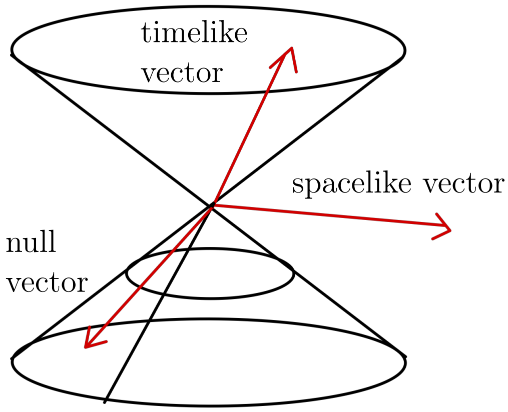

Figure 1.2: Classification of vectors on M

Definition 4.

$$g(v,v) = \begin{cases} < 0, & \text{timelike} \\ = 0, & \text{null} \\ > 0, & \text{spacelike} \end{cases}$$

Null vectors form a cone. Timelike vectors lay inside these cones, spacelike ones outside.

What is their physical interpretation?

Consider a curve $p\colon [\tau_0, \tau_1] \to M$. We can distinguish its tangent vectors by taking derivatives with respect to parameter:

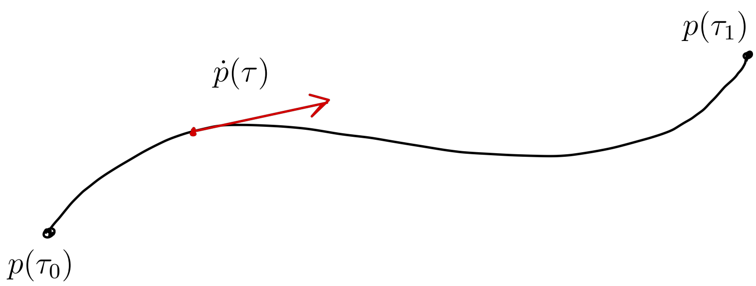

Figure 1.3: Tangent vector

If $\forall \tau$, all tangent vectors are of the same type, we define:

$$\begin{cases} g(\dot{p},\dot{p}) & < 0, \quad \text{timelike curve} \\ g(\dot{p},\dot{p}) & = 0, \quad \text{null curve} \\ g(\dot{p},\dot{p}) & > 0, \quad \text{spacelike curve} \end{cases}$$

A timelike curve is the worldline of a particle at speed $< c$.
A null curve represents particles with speed $= c$.

Proper time, proper distance

Definition 5 (Proper time/distance)**.** Let $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in M$ be timelike. Then, the proper time (as I started my clock and travelled along this curve) will be,

$$T \stackrel{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$$

For spacelike curves we will have a proper distance:

$$D \stackrel{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$$

T and D are reparametrisation invariant.

$$p'(\tau') = p\Big(\tau(\tau')\Big)$$

$$\int_{\tau_0'}^{\tau_1'} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau'}, \frac{\mathrm{d}p}{\mathrm{d}\tau'}\right)}\,\mathrm{d}\tau' = \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$$

Remark 1. For curves $p(\tau) = \tau e_0$ we can see that $T = (\tau_1 - \tau_0)$. If we assume that $\tau_0 = 0$ and that we calculate proper time at point $p(\tau)$, then $T = \tau$. So, with suitable assumptions our parameter τ represents the proper time along any curve being parametrized.

Inverse triangular inequality Consider straight timelike lines such that they form a triangle.

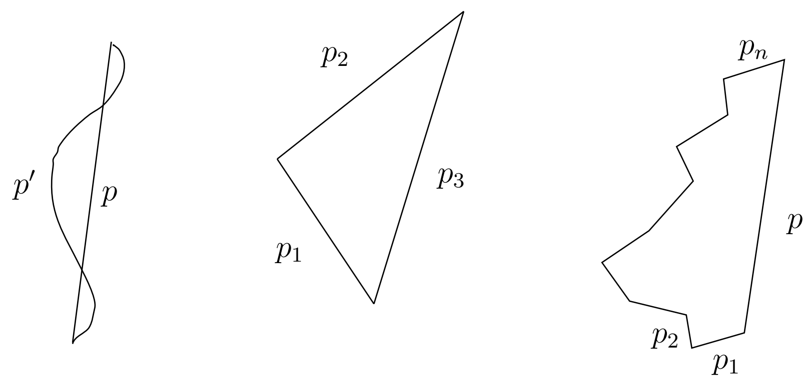

Figure 1.4: Inverse triangular inequality

Theorem 1.

$$\begin{aligned} T_1 + T_2 &< T_3 \\ T_1 + \cdots + T_n &< T \\ T' &\leq T \end{aligned}$$

Remark 2. The conclusion is that the timelike straight lines (points moving at constant speed $< c$) maximize time.

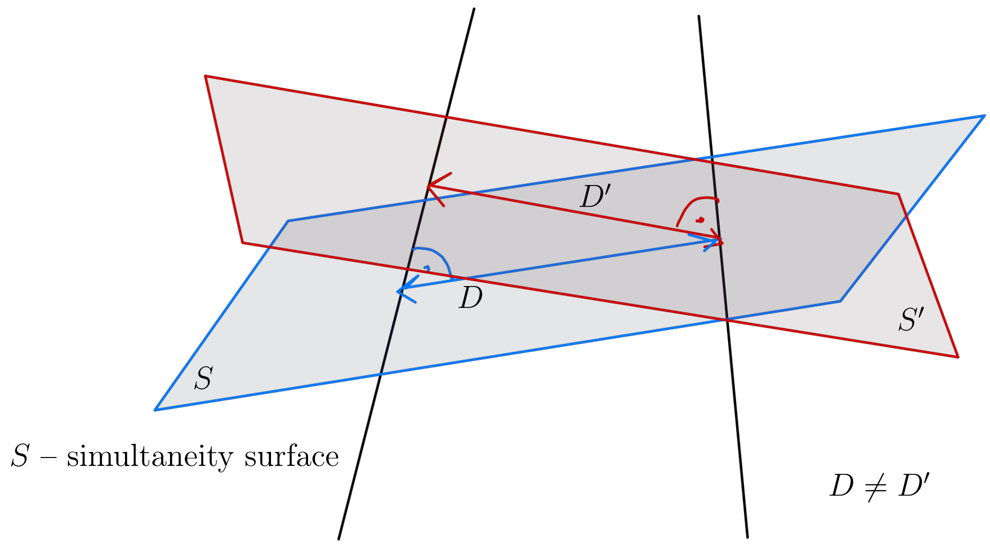

Figure 1.5: Simultaneity of spacetime events.

Remark 3 (Physical interpretation for spacelike curves)**.** D – distance between two events that we consider to happen simultaneously. Proper distance D measured along that S surface is distance between two events happening simultaneously.

Time and distances dilatation

Time dilatation Consider an observer p' moving at a constant speed with respect to observer p. We take $\tau \in [0, \tau_1]$. $p \in M$ comes with its notion of time T and any relatively moving observer p' comes with its T'. What is the relation between the times that they measure?

Let's calculate the events in spacetime at τ time.

$$\begin{aligned} p(\tau) &= \tau e_0 \\ p'(\tau) &= \tau(e_0 + \beta e_1) \end{aligned}$$

Now we calculate proper times,

$$\begin{aligned} T^2 &= -g(\tau_1 e_0, \tau_1 e_0) = \tau_1^2 \\ T'^2 &= -\tau_1^2 g(e_0 + \beta e_1, e_0 + \beta e_1) \\ &= -\tau_1^2 \left(-1 + \beta^2\right) = \tau_1^2 \left(1 - \beta^2\right) \\ T' &= \sqrt{1 - \beta^2} T \end{aligned}$$

Distances dilatation We have one observer and 2 other parallel observers (moving at the same speeds). Describe this from the point of view of the simultaneity surfaces.

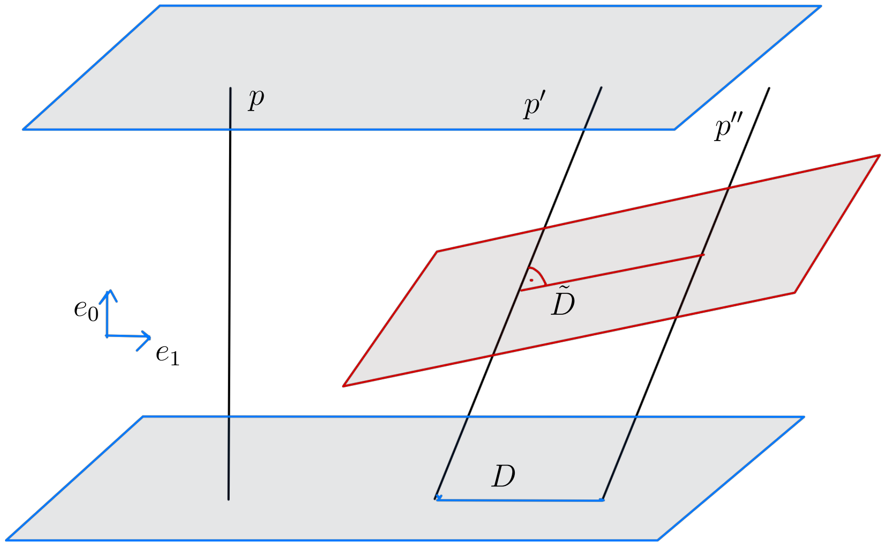

Figure 1.6: Distances dilatation.

Definition 6 (4-velocity)**.** Consider a timelike curve $p\colon [\tau_0, \tau_1] \to M$. We can always choose a parametrization such that the norm of tangent vectors is $= 1$. Suppose:

$$g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) \neq -1$$

Introduce new parameter $\mathrm{d}\tau' = \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$. τ' is just a proper time. If we take $p'(\tau') \stackrel{\text{def}}{=} p \circ \tau'(\tau)$ then, the 4-velocity is defined as:

$$u = \frac{\mathrm{d}p'}{\mathrm{d}\tau'} = \frac{\mathrm{d}p}{\mathrm{d}\tau} / \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}$$

It is easy to check, that

$$g(u, u) = -1$$

> **Definition 7** (4-acceleration)**.** Then, we can define the acceleration as:
> $$a \overset{\text{def}}{=} \frac{\mathrm{d}u}{\mathrm{d}\tau}, \quad \text{where} \quad g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) = -1$$

Now let's see some geometrical relation between u and a.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} g(u,u) &= g\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}, u\right) + g\left(u, \frac{\mathrm{d}u}{\mathrm{d}\tau}\right) = 2g(a,u) \\ 0 &= g(a,u)\end{aligned}$$

Problem class 1

Problem 1 Timelike circle

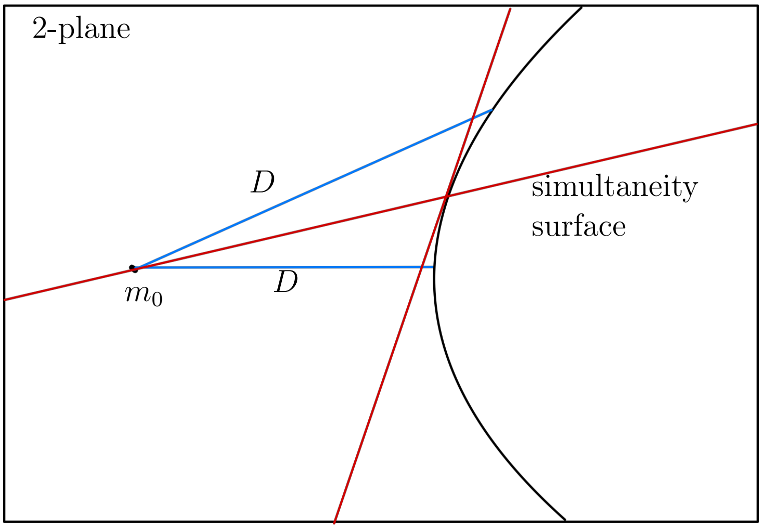

Figure 1.7: Constant proper distance curves.

We will show that the curve is a hyperbola. There is an isometry between any two tangent observers. The red one concludes there is some acceleration $a = \|\vec{a}\|$. Another one concludes (from symmetry) there is the same constant acceleration.

Let's consider a timelike curve $p(\tau) = x^0(\tau)e_0 + x^1(\tau)e_1$, parametrized by an arbitrary parameter τ. We want the whole curve to be timelike ($g(\dot{p}, \dot{p}) < 0$), so a vector $p(\tau)$ must be spacelike. We also require, that we have a constant proper distance from a given point $(0,0) = m_0 \in M$, so that:

$$\begin{aligned}g\big(p(\tau), p(\tau)\big) &= D^2 \\ -x^0(\tau)^2 + x^1(\tau)^2 &= D^2\end{aligned}$$

Now we see, it is the equation of a hyperbola. We can parametrize it with hyperbolic functions.

$$D^2\Big(-\sinh^2\tau + \cosh^2\tau\Big) = D^2$$

It means that our timelike circle can be expressed as:

$$p(\tau) = D(\sinh \tau e_0 + \cosh \tau e_1)$$

Now we can ask how to choose $\tau(\tau')$ so that τ' is proper time. We know that for a timelike curve parametrized with the proper time,

$$g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau'}, \frac{\mathrm{d}p}{\mathrm{d}\tau'}\right) = -1$$

No we calculate some derivatives.

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}\tau'} &= \frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} = \frac{\mathrm{d}\tau}{\mathrm{d}\tau'} D(\cosh \tau e_0 + \sinh \tau e_1) \\
-1 &= \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}\right)^2 D^2 \left(\sinh^2 \tau - \cosh^2 \tau\right) = -D^2 \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}\right)^2 \\
\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} &= \frac{1}{D} \implies \tau(\tau') = \frac{\tau'}{D} + C
\end{aligned}$$

We can assume that $\tau(0) = 0$, so that $C = 0$. Then,

$$p(\tau') = D\left(\sinh \frac{\tau'}{D} e_0 + \cosh \frac{\tau'}{D} e_1\right)$$

From now on, we will use τ as τ', ie. $\tau' \mapsto \tau$.

$$p(\tau) = D\left(\sinh \frac{\tau}{D} e_0 + \cosh \frac{\tau}{D} e_1\right)$$

Now let's calculate covariant velocity u and covariant acceleration a:

$$\begin{aligned}
u &= \frac{\mathrm{d}p}{\mathrm{d}\tau} = \cosh \frac{\tau}{D} e_0 + \sinh \frac{\tau}{D} e_1 \\
a &= \frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{1}{D}\left(\sinh \frac{\tau}{D} e_0 + \cosh \frac{\tau}{D} e_1\right) = \frac{p(\tau)}{D^2} \\
\|a\|^2 = g(a, a) &= \frac{1}{D^4} g\big(p(\tau), p(\tau)\big) = \frac{1}{D^2} \\
\|a\| &= \frac{1}{D} \equiv \frac{c^2}{D}
\end{aligned}$$

Remark 4. If we consider a timelike curve set in a two-plane by points which have same distance to the fixed point m_0, it corresponds to the constant covariant acceleration curve.

We can also calculate spatial velocity and acceleration along x^1 axis.

$$\begin{aligned}
v_x &= \frac{\mathrm{d}x^1}{\mathrm{d}x^0} = \frac{\mathrm{d}x}{\mathrm{d}t} \\
\mathrm{d}x &= \sinh \frac{\tau}{D} \,\mathrm{d}\tau \\
\mathrm{d}t &= \cosh \frac{\tau}{D} \,\mathrm{d}\tau \\
v_x &= \frac{\sinh \frac{\tau}{D} \,\mathrm{d}\tau}{\cosh \frac{\tau}{D} \,\mathrm{d}\tau} = \tanh \frac{\tau}{D} \xrightarrow{\tau \to \infty} 1
\end{aligned}$$

Of course we can express it in terms of $x^0 \stackrel{\text{def}}{=} t$.

$$
\begin{aligned}
t &= D \sinh \frac{\tau}{D} \\
t^2 + D^2 &= D^2 \cosh^2 \frac{\tau}{D} \\
1 + \frac{t^2}{D^2} &= \cosh^2 \frac{\tau}{D} \\
v_x &= \frac{t/D}{\sqrt{1 + t^2/D^2}} \xrightarrow{t \to \infty} 1
\end{aligned}
$$

It means, that any static observer will see the guy accelerating towards c. Let's calculate spatial acceleration.

$$
\begin{aligned}
a_x &= \frac{\mathrm{d}v_x}{\mathrm{d}x^0} \\
\mathrm{d}v_x &= \frac{1}{D} \frac{\cosh^2 \frac{\tau}{D} - \sinh^2 \frac{\tau}{D}}{\cosh^2 \frac{\tau}{D}} \mathrm{d}\tau = \frac{1}{D \cosh^2 \frac{\tau}{D}} \mathrm{d}\tau \\
a_x &= \frac{1}{D \cosh^3 \frac{\tau}{D}} = \frac{D^2}{(t^2 + D^2)^{3/2}}
\end{aligned}
$$

From the last equation it can be seen that relative to a static inertial observer, the guy will accelerate less and less rapidly as time flows.

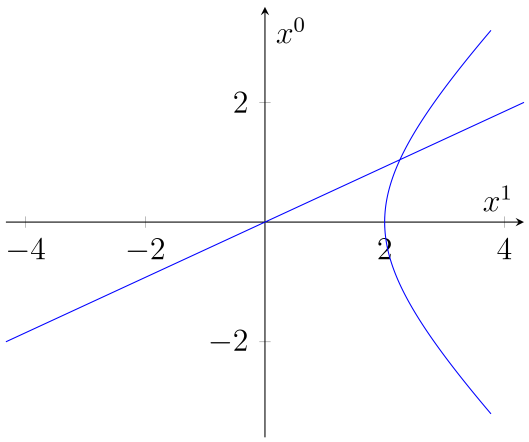

Figure 1.8: Example of constant acceleration curve with $D = 2$.

Now, let us see that simultaneity lines of our accelerating observers go through $(0, 0) \in M$. It will mean, that the proper distance between any such timelike circles would be a physical distance between them, therefore (from definition of these curves) the distance between any two constant covariant-accelerating observers would remain the same. Respectively to each other, they would stay at rest.

We can find the simultaneity line simply by searching for an orthogonal vector to the tangent vector of $p(\tau)$. We know that the tangent to any timelike curve is just its covariant velocity, and it was proven that the covariant acceleration is also orthogonal to u. Then, we know that:

$$
a = \frac{p(\tau)}{D^2}
$$

It is clear that for every τ, a is just scaled $p(\tau)$ so especially a line generated by the vector a goes through $(0,0)$.

Lecture 2: Coordinate systems in the Minkowski spacetime

27 paź 2020

Remark 5 (Remarks about the previous lecture)**.** To sum up some things:

1. We have a general formula for lengths on curves, given by an integral. When the curve happens to be a line (can be represented by a vector $\vec{v} = p(\tau_1) - p(\tau_0)$), it simplifies to $l = \sqrt{\pm g(\vec{v}, \vec{v})}$.
2. Suppose $p\colon [\tau_0, \tau_1] \to M$ and it is parametrised in such a clever way that $g(\dot{p}, \dot{p}) = -1$. It is equivalent to τ being the proper time.
3. If (2), then $\mathrm{d}p/\mathrm{d}\tau = u$ is 4-velocity (covariant velocity) and $\mathrm{d}u/\mathrm{d}\tau = a$, $g(u, a) = 0$ is 4-acceleration (covariant acceleration).
4. In a temporary frame at rest, we can define $e_0 \overset{\text{def}}{=} \dot{p}$ at $p(\tau_0)$ and just make sure that e_1, e_2, e_3 complete the orthonormal basis. Then at τ_0, $a = a^1 e_1 + a^2 e_2 + a^3 e_3$, where $a^i = \mathrm{d}^2(x^i) \big/ \mathrm{d}(x^0)^2$.

Inertial coordinate systems

Let $m_0 \in M$ and at this point we choose an orthonormal basis (e_μ). If we have a vector from point m_0 to any other $m \in M$, we can describe it as $m - m_0 = x^0 e_0 + \cdots + x^3 e_3$. In this way, we have assigned coordinates $\left(x^0, x^1, x^3, x^4\right)$ to point $m \in M$.
We can view x^i as functions on M. $x^i\colon M \to \mathbb{R}$.

Definition 8 (Time orientation)**.** Time orientation means that we choose one side of the spacetime cone to point in the future direction, and the opposite side of the cone pointing past (timelike vectors point directions). It is also a covariant structure, not a convention! Future is the upper side of a cone!
It means that we always use e_0 such that it is properly oriented.

Definition 9 (Spacetime orientation)**.** We admit only transformations from one frame to another frame which have only positive determinants. Given (e_0, e_1, e_2, e_3) we allow only (e'_0, e'_1, e'_2, e'_3) such that $e'_\mu = L^\nu{}_\mu e_\nu$, $\det L > 0$.

Definition 10 (Boost)**.** We choose another orthonormal frame. Suppose that $e_0 = Ae'_0 + Be'_1$, $e_1 = Ce'_0 + De'_1$, $e_2 = e'_2$, $e_3 = e'_3$ (e_0, e'_0 are future oriented $\iff A > 0$). From orthonormality,

$$\begin{aligned} -A^2 + B^2 &= -1 \\ -C^2 + D^2 &= 1 \\ -AC + BD &= 0, \quad A > 0 \end{aligned}$$

We get a solution:

$$A = \cosh\alpha, \quad B = \sinh\alpha, \quad \alpha \in \mathbb{R}$$

$$\begin{cases} e_0 = \cosh\alpha e'_0 + \sinh\alpha e'_1 \\ e_1 = \sinh\alpha e'_0 + \cosh\alpha e'_1 \end{cases}$$

Others remain unchanged. On the other hand, rotation is trivial.

Of course we still describe the same vectors on M, so:

$$\begin{aligned} x^0 e_0 + \cdots + x^3 e_3 &= x'^0 e'_0 + \cdots + x'^3 e'_3 \\ x^0 \cosh\alpha + x^1 \sinh\alpha &= x'^0 \\ x^0 \sinh\alpha + x^1 \cosh\alpha &= x'^1 \end{aligned}$$

We can define β, by:

$$\begin{aligned} \cosh\alpha &\stackrel{\text{def}}{=} \frac{1}{\sqrt{1-\beta^2}} \\ \sinh\alpha &\stackrel{\text{def}}{=} \frac{\beta}{\sqrt{1-\beta^2}} \\ \beta &= \frac{v}{c} \end{aligned}$$

In this approach, the Lorentz transformation appears as the secondary transformation!

Vector as a differential operator

Consider a point m and a vector $\vec{v}$ at that point. Let $M \subset \mathcal{U}$, where $\mathcal{U}$ is an open set.

$$\begin{aligned} f : \mathcal{U} &\to \mathbb{R} \\ \left.\frac{\mathrm{d}f(m + s\vec{v})}{\mathrm{d}s}\right|_{s=0} &= \vec{v}f \end{aligned}$$

We can think about a dual vector. Let's define the exterior derivative of f (co-vector):

$$\begin{aligned} \mathrm{d}f\big|_m : \vec{v}\big|_m &\to \mathbb{R} \\ \vec{v} \lrcorner \mathrm{d}f &\stackrel{\text{def}}{=} \vec{v}f \end{aligned}$$

We can see that exterior derivative is a linear operator.

If we have vectors and co-vectors defined by coordinates $\left(x^0, x^1, x^2, x^3\right)$,

$$\begin{aligned} e_\mu(f) &= \left.\frac{\mathrm{d}f\left(m + s e_\mu\right)}{\mathrm{d}s}\right|_{s=0} = \left.\frac{\mathrm{d}f\left(m + x^\mu e_\mu\right)}{\mathrm{d}x^\mu}\right|_{x^\mu=0} \\ &= \frac{\partial}{\partial x^\mu} f \end{aligned}$$

It means that

$$e_\mu \equiv \frac{\partial}{\partial x^\mu}$$

It means that we have a basis $\left(\partial_\mu\right)$ and co-basis $(\mathrm{d}x^\mu)$.

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \lrcorner\, \mathrm{d}x^\nu &= \frac{\partial}{\partial x^\mu}(x^\nu) = \delta^\nu{}_\mu \\ \vec{v} \lrcorner\, \mathrm{d}x^\nu &= \left(v^\mu \partial_\mu\right) \lrcorner\, \mathrm{d}x^\nu = v^\nu \end{aligned}$$

The metric tensor in terms of $(\mathrm{d}x^\mu)$

Let's define a tensor product as follows.

$$\begin{aligned} \mathrm{d}f \otimes \mathrm{d}h \, (\vec{v}, \vec{w}) &\stackrel{\text{def}}{=} \vec{v} \lrcorner\, \mathrm{d}f \cdot \vec{w} \lrcorner\, \mathrm{d}h \\ \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \, (\vec{v}, \vec{w}) &= \vec{v} \lrcorner\, \mathrm{d}x^\mu \cdot \vec{w} \lrcorner\, \mathrm{d}x^\nu = v^\mu w^\nu \end{aligned}$$

In particular,

$$\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \left(\partial_\alpha, \partial_\beta\right) = \delta^\mu{}_\alpha \delta^\nu{}_\beta$$

We want to do physics, not to write $\otimes$, so we define a notation for symmetrised tensor product:

$$\mathrm{d}f \, \mathrm{d}h \stackrel{\text{def}}{=} \frac{1}{2}(\mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f)$$

Especially,

$$\begin{aligned} \mathrm{d}f^2 &= \mathrm{d}f \, \mathrm{d}f = \mathrm{d}f \otimes \mathrm{d}f \\ \mathrm{d}x^0 \, \mathrm{d}x^1 \, (v, w) &= \frac{1}{2}\left(v^0 w^1 + v^1 w^0\right) \\ \mathrm{d}x^0 \, \mathrm{d}x^1 \, (\partial_0, \partial_1) &= \frac{1}{2} \end{aligned}$$

We can write the Minkowski metric tensor:

$$\begin{aligned} g &= -\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 \\ g(\partial_0, \partial_0) &= -\mathrm{d}x^0 \, \mathrm{d}x^0 \, (\partial_0, \partial_0) = -1 \end{aligned}$$

More generally, one could write

$$\begin{aligned} g(v, w) &= -\mathrm{d}x^0 \, \mathrm{d}x^0 \, (v, w) + \cdots + \mathrm{d}x^3 \, \mathrm{d}x^3 \, (v, w) \\ &= -v^0 w^0 + \cdots + v^3 w^3 \end{aligned}$$

what is completely consistent with definition of scalar product on Minkowski spacetime.

Lemma 1. If we consider two different, but orthonormal frames, from constructions above it follows that given $m \in M$ with $(e_0, \ldots, e_3)$ and $m' \in M$ with (e'_μ)

$$-\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 = -\left(\mathrm{d}x'^0\right)^2 + \left(\mathrm{d}x'^1\right)^2 + \left(\mathrm{d}x'^2\right)^2 + \left(\mathrm{d}x'^3\right)^2$$

However, we can also consider arbitrary, non-inertial coordinates. By non-inertial, we mean that they don't transform via Lorentz transformation/metric tensor doesn't look like written above.
Let's denote such coordinates by (y^μ). They can be functions of M.

$$\begin{aligned} y^0, y^1, y^2, y^3 &: \mathcal{U} \to \mathbb{R} \\ \left.\frac{\partial}{\partial y^0}\right|_m &= \left.\left(\frac{\partial x^0}{\partial y^0}\frac{\partial}{\partial x^0} + \cdots + \frac{\partial x^3}{\partial y^0}\frac{\partial}{\partial x^3}\right)\right|_m \\ \frac{\partial}{\partial y^\mu} \lrcorner\, \mathrm{d}y^\nu &= \delta^\nu{}_\mu \\ \mathrm{d}x^\mu &= \frac{\partial x^\mu}{\partial y^\nu}\,\mathrm{d}y^\nu \end{aligned}$$

Accelerated coordinates

We can introduce coordinates $\left(\rho, \tau, x^2, x^3\right)$, where $\rho = \sqrt{\left(x^1\right)^2 - \left(x^0\right)^2}$:

$$\begin{cases} x^0 = \rho \sinh \tau \\ x^1 = \rho \cosh \tau \\ x^2 = x^2 \\ x^3 = x^3 \end{cases}, \quad \rho > 0,\ \tau \in \mathbb{R}$$

These are called accelerated coordinates.

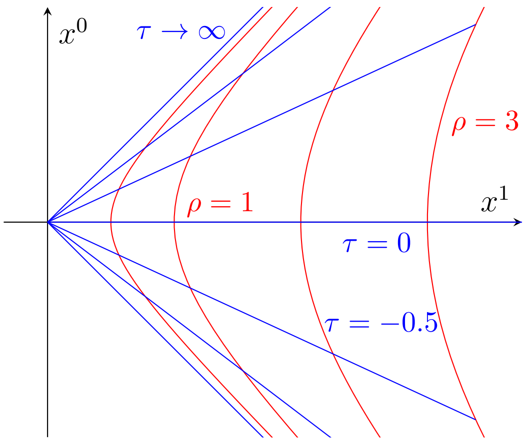

Figure 1.9: Accelerated coordinates with lines of constant t (blue) and constant ρ (red).

$$\begin{aligned} \mathrm{d}x^0 &= \sinh \tau\, \mathrm{d}\rho + \rho \cosh \tau\, \mathrm{d}\tau \\ \mathrm{d}x^1 &= \cosh \tau\, \mathrm{d}\rho + \rho \sinh \tau\, \mathrm{d}\tau \end{aligned}$$

Then,

$$\begin{aligned} g &= -(\sinh\tau\,\mathrm{d}\rho + \rho\cosh\tau\,\mathrm{d}\tau)^2 + (\cosh\tau\,\mathrm{d}\rho + \rho\sinh\tau\,\mathrm{d}\tau)^2 + \cdots \\ &= -\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 \end{aligned}$$

What is the meaning of this new form of a metric tensor? We can use it as a dictionary to understand the properties of any frame.

Let's drop x^2, x^3. Given any $\rho_1 > 0$, it defines a line of constant ρ_1. Similarly for τ_1.

1. The coefficient 1 before $\mathrm{d}\rho^2$ means that the distance between any $\rho_2 - \rho_1$ for any τ is constant.
2. The term $0\,\mathrm{d}\tau\,\mathrm{d}\rho$ means that lines of constant ρ and τ are orthogonal to each other (constant τ lines are simultaneity lines).
3. Meaning of $-\rho^2$ before $\mathrm{d}\tau^2$ is that if observers of constant ρ_1 and ρ_2 measure time, they measure different periods: $\Delta T_1/\Delta T_2 = \rho_1/\rho_2$. Even though the distance is the same between them, their clocks work differently.

Accelerating observers point of view: they see themselves as straight lines, they remain at the same distances, their clocks show different times. It is possible to set $\tau = 0$ simultaneously for all of them (we mention that, because in some coordinates it is not possible) – there exists a surface orthogonal to every observer at once.

Problem class 2 – Minkowski metric in non-flat coordinates

Problem 1 Uniformly accelerating coordinate system.

$$\phi\colon (\tau, \rho, y, z) \mapsto (\rho\sinh\tau, \rho\cosh\tau, y, z)$$

Now we calculate the metric tensor.

$$\begin{aligned} \mathrm{d}x^0 &= \mathrm{d}(\rho\sinh\tau) = \sinh\tau\,\mathrm{d}\rho + \rho\cosh\tau\,\mathrm{d}\tau \\ \mathrm{d}x^1 &= \mathrm{d}(\rho\cosh\tau) = \cosh\tau\,\mathrm{d}\rho + \rho\sinh\tau\,\mathrm{d}\tau \\ \mathrm{d}x^2 &= \mathrm{d}y \\ \mathrm{d}x^3 &= \mathrm{d}z \end{aligned}$$

Now we substitute these,

$$-\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 = -\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

What is the domain? We want for every variable to be well defined and its gradient $\neq 0$.

$$\rho = \sqrt{(x^1)^2 - (x^0)^2}, \quad \mathrm{d}\rho = \frac{x^1\,\mathrm{d}x^1 - x^0\,\mathrm{d}x^0}{\sqrt{(x^1 - x^0)(x^1 + x^0)}}$$

We have a problem if $x^1 = x^0$ or $x^1 = -x^0$. So we cannot cross these two lines. It is sometimes called a Rindler wedge. Without crossing any singularity, we can use these coordinates only in one of the wedges. Moreover, observers never can reach $\rho = 0$, because it would result in an infinite acceleration.

Right wedge observer and left wedge observer can never see each other (signals will not come), so $x^1 = x^0$ is their horizon! The cannot see through it.

$$\begin{aligned}
p(\tau) &= (\tau, \rho_1) \\
\dot{p}(\tau) &= \frac{\mathrm{d}\tau}{\mathrm{d}\tau}\partial_\tau + \frac{\mathrm{d}\rho_1}{\mathrm{d}\tau}\partial_\rho = \partial_\tau \\
\Delta T_1 &= \int_0^{\tau_0} \sqrt{-g(\dot{p},\dot{p})}\,\mathrm{d}\tau = \int_0^{\tau_0} \sqrt{-g(\partial_\tau,\partial_\tau)}\,\mathrm{d}\tau \\
&= \int_0^{\tau_0} \rho_1\,\mathrm{d}\tau = \rho_1\tau_0
\end{aligned}$$

Same thing happens for constant ρ_2 observer.

$$\begin{aligned}
\Delta T_2 &= \rho_2\tau_0 \\
\frac{\Delta T_1}{\Delta T_2} &= \frac{\rho_1}{\rho_2}
\end{aligned}$$

Just to be super clear, we will calculate metric tensor acting on two vectors.

$$g(\partial_\tau,\partial_\tau) = \Big(-\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2\Big)(\partial_\tau,\partial_\tau)$$

What it really means, is:

$$\big(-\rho^2\,\mathrm{d}\tau\otimes\mathrm{d}\tau + \mathrm{d}\rho\otimes\mathrm{d}\rho\big)(\partial_\tau,\partial_\tau) = -\rho^2$$

The rule was:

$$\begin{aligned}
\mathrm{d}\tau\otimes\mathrm{d}\tau\,(\partial_\tau,\partial_\tau) &= \partial_\tau\lrcorner\,\mathrm{d}\tau\cdot\partial_\tau\lrcorner\,\mathrm{d}\tau \\
\partial_\tau\lrcorner\,\mathrm{d}\tau &= \partial_\tau\tau = 1
\end{aligned}$$

Problem 2 Rotating frame.
Let's consider cylindrical coordinates.

$$\psi\colon\,\big(x^0, r, \phi', x^3\big) \mapsto \big(x^0, r\cos\phi', r\sin\phi', x^3\big)$$

We will calculate the metric tensor:

$$\begin{aligned}
g = &-\big(\mathrm{d}x^0\big)^2 + \big(\cos\phi'\,\mathrm{d}r - r\sin\phi'\,\mathrm{d}\phi'\big)^2 + \big(\sin\phi'\,\mathrm{d}r + r\cos\phi'\,\mathrm{d}\phi'\big)^2 + \big(\mathrm{d}x^3\big)^2 = \\
&= -\big(\mathrm{d}x^0\big)^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi'^2 + \big(\mathrm{d}x^3\big)^2
\end{aligned}$$

Attention! Metric tensor is symmetric so we didn't bother to be careful while evaluating and cancelling out similar coefficients before $\mathrm{d}x^i\,\mathrm{d}x^i$ and $\mathrm{d}x^j\,\mathrm{d}x^i$. If it was a non-symmetric tensor we would be more careful and use tensor products.

Now we can introduce another substitution: $\phi' = \phi + \omega x^0$.

$$\begin{aligned}
g &= -\big(\mathrm{d}x^0\big)^2 + \mathrm{d}r^2 + r^2\big(\mathrm{d}\phi + \omega\,\mathrm{d}x^0\big)^2 + \big(\mathrm{d}x^3\big)^2 \\
&= -\big(1 - r^2\omega^2\big)\left[\mathrm{d}x^0 - \frac{r^2\omega}{1-r^2\omega^2}\,\mathrm{d}\phi\right]^2 + \mathrm{d}r^2 + \frac{r^2}{1-r^2\omega^2}\,\mathrm{d}\phi^2 + \big(\mathrm{d}x^3\big)^2
\end{aligned}$$

Let us consider a circle $r =$ const. and $x^3 =$ const. We have two observers at ϕ_0, ϕ_1. They are in their frames at rest. How is measured the proper distance?

It can be checked that the vector $q = \partial_\phi + \dfrac{r^2\omega}{1 - r^2\omega^2}\partial_t$ is orthogonal to our constant ϕ, r, x^3 curves. This vector is of course tangent to the proper distance curve.
It is easy to check that this vector gives 0 when contracted with the first mixed term of our metric g. From the fully spatial term we get

$$g(q, q) = \frac{r^2}{r^2\omega^2 - 1}$$

We can calculate the circumference of such rotating circe,

$$L = \int_0^{2\pi} \frac{r}{\sqrt{1 - r^2\omega^2}} \, \mathrm{d}\phi = \frac{2\pi r}{\sqrt{1 - r^2\omega^2}} > 2\pi r$$

It means that in such rotating geometry, co-moving observers would measure that circles have stretched.

Lecture 3: Rotating observers in Minkowski spacetime

Let's carefully revisit the rotating coordinates. We introduce $t = x^0/c$ and ω – the angular velocity. Then, our metric works for every: 03 lis 2020

$$0 < r < \frac{c}{\omega}, \quad \phi \in [0, 2\pi), \quad z \in \mathbb{R}$$

Let's define a rotating observer:

$$p(t) = p^i(t)e_i = cte_0 + r\cos(\phi + \omega t)e_1 + r\sin(\phi + \omega t)e_2 + ze_3$$

We set z to be constant, ie. our observer moves in the xy plane.

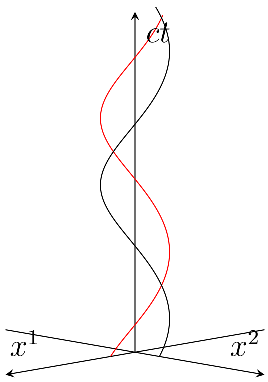

Figure 1.10: Two observers separated by $\Delta\phi = \pi/2$.

What is the observers' sense of time, distance and rotation?
Firstly, the proper time is given by:

$$\begin{aligned} c\,\mathrm{d}\tau &= \sqrt{-g\big(p'(t), p'(t)\big)}\,\mathrm{d}t \\ p'(t) &= ce_0 - r\omega\sin(\phi + \omega t)e_1 + r\omega\cos(\phi + \omega t)e_2 \\ &= ce_0 + \omega p^1 e_2 - \omega p^2 e_1 \end{aligned}$$

Rotating coordinates

We shall also introduce the rotating coordinates (as previously):

$$\psi\colon \begin{bmatrix} t \\ r \\ \phi \\ z \end{bmatrix} \mapsto \begin{bmatrix} ct \\ r\cos(\phi+\omega t) \\ r\sin(\phi+\omega t) \\ z \end{bmatrix}$$

$$g = -\left(c^2 - \omega^2 r^2\right) \mathrm{d}t^2 + 2r^2\omega \,\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

Now we consider a curve $p(t) = \big(p^t(t), p^r(t), p^\phi(t), p^z(t)\big) = (t, r, \phi, z)$, where by this notion I understand the coordinates in rotating system. To be more precise I should probably write something like $p(t) = \psi(t, r, \phi, z)$. Or I could say that in original Minkowski space we have a vector $v = (x^0, x^1, x^2, x^3)$ so that $(t, r, \phi, z) = \psi^* v$.
Anyway, we chose a curve for which r, ϕ, z are constants. Then,

$$\left.\frac{\partial}{\partial t}\right|_{r,\phi,z=\text{cst.}} = \tilde{\partial}_t, \quad \left.\frac{\partial}{\partial z}\right|_{r,\phi,t} = \tilde{\partial}_z$$

$$\left.\frac{\partial}{\partial r}\right|_{t,\phi,z} = \tilde{\partial}_r, \quad \left.\frac{\partial}{\partial \phi}\right|_{t,r,z} = \tilde{\partial}_\phi$$

Proper time We can calculate the proper time by,

$$c\,\mathrm{d}\tau = \sqrt{-g(\tilde{\partial}_t, \tilde{\partial}_t)}\,\mathrm{d}t = \sqrt{c^2 - \omega^2 r^2}\,\mathrm{d}t$$

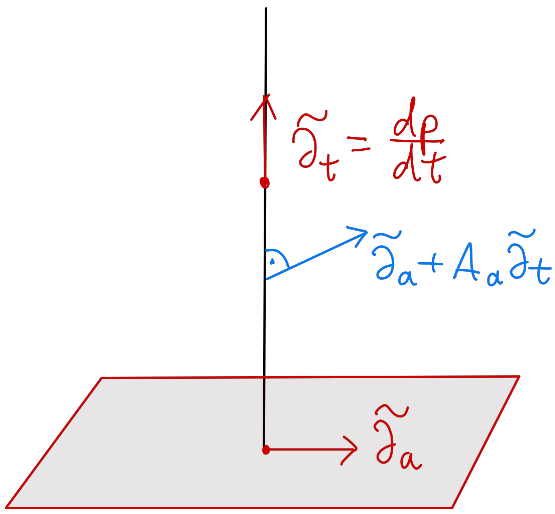

Figure 1.11: Simultaneity curves

Simultaneity curves Let a denote r, ϕ or z. Then we define:

$$\hat{\partial}_a \stackrel{\text{def}}{=} \tilde{\partial}_a + A_a \tilde{\partial}_t$$

We want $\hat{\partial}_a$ to be a tangent to simultaneity line, so

$$g(\tilde{\partial}_t, \hat{\partial}_a) = 0$$

$\hat{\partial}_a$ is essentially a base vector in a direction from the moving observer's perspective. It gives us the condition for A_a. We shall regroup our metric tensor.

$$\begin{aligned} g &= -\left(c^2 - \omega^2 r^2\right)\left(\mathrm{d}t - \frac{r^2\omega}{c^2 - r^2\omega^2}\,\mathrm{d}\phi\right)^2 + \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2 \\ &= -N^2\Big(\mathrm{d}t - \left[A_r\,\mathrm{d}r + A_\phi\,\mathrm{d}\phi + A_z\,\mathrm{d}z\right]\Big) + \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2 \end{aligned}$$

where $A_r = A_z = 0$. We will check, that these introduced A_i correspond to A_a.

$$\begin{aligned} \tilde{\partial}_a \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= -A_a \\ A_a \tilde{\partial}_t \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= +A_a \\ \implies \hat{\partial}_a \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= 0 \end{aligned}$$

Hence,

$$g(\tilde{\partial}_t, \tilde{\partial}_a + A_a \tilde{\partial}_t) = 0$$

because $\tilde{\partial}_t$ will give 0 when contracted with any 1-form other than $\mathrm{d}t$. In such way, we proved that there are several „basic" simultaneity curves:

$$\begin{aligned} \hat{\partial}_r &= \tilde{\partial}_r \\ \hat{\partial}_\phi &= \tilde{\partial}_\phi + \frac{r^2\omega}{c^2 - r^2\omega^2}\tilde{\partial}_t \\ \hat{\partial}_z &= \tilde{\partial}_z \end{aligned}$$

Now let's define:

$$g_{\text{obs}} = \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

We can see, that

$$g(\hat{\partial}_a, \hat{\partial}_b) = g_{\text{obs}}(\tilde{\partial}_a, \tilde{\partial}_b)$$

Conclusion

$$g = -\underbrace{\left(c^2 - \omega^2 r^2\right)}_{\substack{\text{proper time} \\ \text{part}}}\Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big)^2 + \underbrace{\mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2}_{g_{\text{obs}}}$$

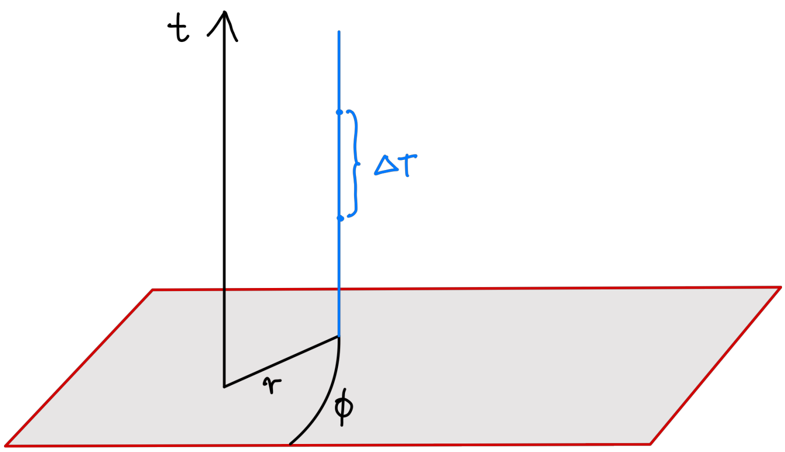

Figure 1.12: Observers proper time ΔT.

In these coordinates, observers moving with the angular velocity ω are „straight lines". Then the proper time is just simply

$$c\Delta T = N\Delta t = \sqrt{c^2 - \omega^2 r^2}\Delta t$$

where Δt is inertial observer's time (coordinate time).

The moving observers' 3D space is the 3D ball of points:

$$\left\{(r, \phi, z) \colon 0 < r < \frac{c}{\omega},\ \phi \in [0, 2\pi),\ -\infty < z < \infty\right\}$$

which is endowed with the g_{obs} part of the metric tensor.

For every spatial curve $[s_0, s_1] \ni s \mapsto \big(r(s), \phi(s), z(s)\big) = q(s)$, it means for $t = \text{const.}$ – a curve observed by an inertial observer in the middle point, the proper (physical) length is:

$$\Delta L = \int_{s_0}^{s_1} \left[g_{\text{obs}}(\dot{q}, \dot{q})\right]^{1/2} \mathrm{d}s = \int_{s_0}^{s_1} \left[\dot{r}^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\dot{\phi}^2 + \dot{z}^2\right]^{1/2} \mathrm{d}s$$

Note that the statement above is true, only because $g(\hat{\partial}_a, \hat{\partial}_a) = g_{\text{obs}}(\tilde{\partial}_a, \tilde{\partial}_a)$.

Now we consider any loop $[s_0, s_1] \ni s \mapsto \big(r(s), \phi(s), z(s)\big) = q(s)$, such that $q(s_0) = q(s_1)$. We want to check what is the discrepancy according to the inertial time. What we need is a curve $[s_0, s_1] \mapsto \hat{q}(s)$ contained in the worldsheet of the curve q that is such that:

$$\hat{q}^t(s) = ?,\ \hat{q}^r(s) = r(s),\ \hat{q}^\phi(s) = \phi(s),\ \hat{q}^z(s) = z(s)$$

and also orthogonal to the world line of every rotating observer. First we can easily find the tangent vector:

$$\begin{aligned}
\frac{\mathrm{d}\hat{q}^t}{\mathrm{d}s}\tilde{\partial}_t + \dot{r}\tilde{\partial}_r + \dot{\phi}\tilde{\partial}_\phi + \dot{z}\tilde{\partial}_z = \frac{\mathrm{d}\hat{q}}{\mathrm{d}s} &= \dot{r}\hat{\partial}_r + \dot{\phi}\hat{\partial}_\phi + \dot{z}\hat{\partial}_z \\
&= \dot{r}\tilde{\partial}_r + \dot{\phi}\Big(\tilde{\partial}_\phi + A_\phi\tilde{\partial}_t\Big) + \dot{z}\tilde{\partial}_z \\
&= \dot{\phi}A_\phi\tilde{\partial}_t + \dot{r}\tilde{\partial}_r + \dot{\phi}\tilde{\partial}_\phi + \dot{z}\tilde{\partial}_z
\end{aligned}$$

From this we can see, that

$$\frac{\mathrm{d}\hat{q}^t}{\mathrm{d}s} = \frac{\mathrm{d}\phi}{\mathrm{d}s} A_\phi$$

If we set $\hat{q}^t(s_0) = 0$, then

$$t(s) = \hat{q}^t(s) = \int_{s_0}^{s} A_\phi(s) \frac{\mathrm{d}\phi}{\mathrm{d}s}\,\mathrm{d}s = \int_{\phi(s_0)}^{\phi(s)} A_\phi \,\mathrm{d}\phi$$

Likewise, total discrepancy according to the inertial time would result from integrating along the whole loop, ie.

$$\left(\substack{\text{total}\\ \text{discrepancy}}\right) = \hat{q}^t(s_1) = \int_{\phi(s_0)}^{\phi(s_1)} A_\phi \,\mathrm{d}\phi = \int_{\phi(s_0)}^{\phi(s_0)+2\pi} A_\phi \,\mathrm{d}\phi$$

It can be also expressed as:

$$= \oint_{q(s)} A_a \,\mathrm{d}y^a$$

Non-zero total discrepancy over any closed curve leads to discontinuous notion of time. Rotating observers are unable to synchronize their clocks, because there does not exist a curve orthogonal to every observer on a ring at once – at any given coordinate time t_0.

Problem class 3

Problem 1 Let us consider coordinates, where one of the coordinates is:

$$\rho = \sqrt{(x^0)^2 - (x^i)^2}$$

where $\left|x^0\right| > \left|x^i\right|$. Then, curves of constant ρ are hyperboloids. The whole transformation would be as follows:

$$\begin{cases} x^0 = \rho \cosh\psi \\ x^1 = \rho \sinh\psi \sin\theta \cos\phi \\ x^2 = \rho \sinh\psi \sin\theta \sin\phi \\ x^3 = \rho \sinh\psi \cos\theta \end{cases}$$

Now we shall rewrite the metric tensor.

$$\begin{aligned} \mathrm{d}x^0 &= \cosh\psi \,\mathrm{d}\rho + \rho \sinh\psi \,\mathrm{d}\psi \\ \mathrm{d}x^1 &= \sinh\psi \sin\theta \cos\phi \,\mathrm{d}\rho + \rho\cosh\psi \sin\theta\cos\phi \,\mathrm{d}\psi + \rho \sinh\psi\cos\theta\cos\phi\,\mathrm{d}\theta \\ &\quad - \rho\sinh\psi\sin\theta\sin\phi\,\mathrm{d}\phi \\ \mathrm{d}x^2 &= \sinh\psi\sin\theta\sin\phi\,\mathrm{d}\rho + \rho\cosh\psi\sin\theta\sin\phi\,\mathrm{d}\psi + \rho\sinh\psi\cos\theta\sin\phi\,\mathrm{d}\theta \\ &\quad + \rho\sinh\psi\sin\theta\cos\phi\,\mathrm{d}\phi \\ \mathrm{d}x^3 &= \sinh\psi\cos\theta\,\mathrm{d}\rho + \rho\cosh\psi\cos\theta\,\mathrm{d}\psi - \rho\sinh\psi\sin\theta \end{aligned}$$

Now it is even more tedious work to combine them into a metric tensor.

$$g = -\,\mathrm{d}\rho^2 + \rho^2\left[\mathrm{d}\psi^2 + \sinh^2\psi\,\mathrm{d}\theta^2 + \sinh^2\psi\sin^2\theta\,\mathrm{d}\phi^2\right]$$

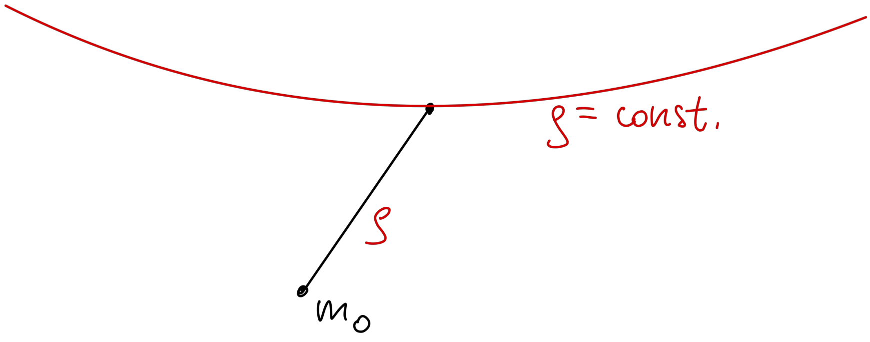

Figure 1.13: Curve of constant ρ.

All of the mixed terms get cancelled.
Let us consider an observer with constant ψ, θ, ϕ. Then we can parametrize any curve by the parameter s and say that:

$$\begin{aligned} p^\rho(s) &= s \\ p^\psi, p^\theta, p^\phi &= \mathrm{const}(s). \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= \dot{\rho}\partial_\rho + \dot{\psi}\partial_\psi + \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi = \partial_\rho \end{aligned}$$

In these coordinates spacetime looks like it was expanding, because of the term ρ^2 before the spatial geometry term.

Conformal compactification

Let us consider the standard spherical Minkowski metric,

$$g = -(\mathrm{d}x^0)^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

We introduce two new coordinates η, χ given by,

$$\begin{aligned} x^0 &= \frac{R_0}{2}\left[\tan\frac{\eta+\chi}{2} + \tan\frac{\eta-\chi}{2}\right] \\ r &= \frac{R_0}{2}\left[\tan\frac{\eta+\chi}{2} - \tan\frac{\eta-\chi}{2}\right] \\ &-\pi < \eta + \chi < \pi \\ &-\pi < \eta - \chi < \pi \end{aligned}$$

Now we want to rewrite the metric tensor.

$$\begin{aligned} \mathrm{d}x^0 &= \frac{R_0}{4}\left[\left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} + \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\eta + \left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} - \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\chi\right] \\ \mathrm{d}r &= \frac{R_0}{4}\left[\left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} - \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\eta + \left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} + \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\chi\right] \end{aligned}$$

After much trigonometry,

$$g = \frac{R_0^2}{\left(2\cos\frac{\eta+\chi}{2}\cos\frac{\eta-\chi}{2}\right)^2}\left(-\,\mathrm{d}\eta^2 + \mathrm{d}r^2 + \mathrm{d}\chi^2 + \sin\chi\big(\mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi\big)^2\right)$$

This is just a simple product of $\mathbb{R}\times S_3$ with some conformal factor.
It is called a compactification, because we managed to fit the whole infinite spacetime into a compact region – that metric in braces. It describes a compact region, because η,χ as well as θ,ϕ are bounded. Of course the whole metric still blows up somewhere, because of the conformal factor.

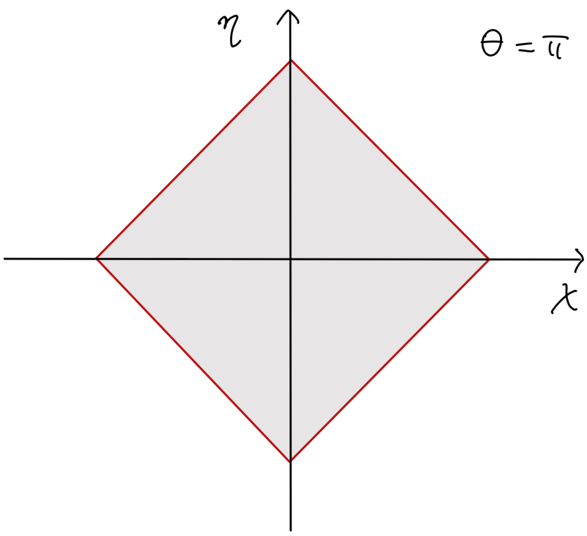

Figure 1.14: A slice of metric region with $\theta = \pi$.

Chapter 2

General Relativity framework

Lecture 4: Coordinate covariant differential calculus in $\mathbb{R}^n$

10 lis 2020

Some algebra

Vector spaces We consider V – a finite dimensional vector space, and its basis $(e_1, \ldots, e_n) \subset V$. Then $V \ni v = v^i e_i$. Let's consider the basis transformation,

$$e'_i = A^j{}_i e_j \implies v'^i = (A^{-1})^i{}_j v^j$$

Then, let V^* be the dual space to V, where $V^* \ni w\colon V \to \mathbb{R}$ (or $\mathbb{C}$).

$$w\colon V \ni v \mapsto v \lrcorner w = v^i w_i$$

In V^* there is a basis $(e^1, \ldots, e^n)$ dual to the basis of V.

$$\begin{aligned} e_i \lrcorner e^j &= \delta_i^{\,j} \\ v \lrcorner w &= v^i e_i \lrcorner w_j e^j = v^i w_j e_i \lrcorner e^j \\ &= v^i w_j \delta_i^{\,j} \end{aligned}$$

Tensor product First, we define a tensor product of two elements. Given $w, w' \in V^*$,

$$\begin{aligned} w \otimes w' \colon V \times V &\to \mathbb{R} \\ (v, v') &\mapsto v \lrcorner w \cdot v' \lrcorner w' \end{aligned}$$

Now, the tensor product of the dual space is by definition a span:

$$\begin{aligned} V^* \otimes V^* &= \operatorname{span}\big(w \otimes w' \colon w, w' \in V^*\big) \\ &= \Big\{ K_{ij} e^i \otimes e^j \colon K_{ij} \in \mathbb{R} \Big\} \end{aligned}$$

In other words, we can see the tensor product as functions of such space into reals:

$$V^* \otimes V^* \ni K, \quad K\colon V \times V \to \mathbb{R}$$

Now we want to generalize,

$$V^* \otimes \cdots \otimes V^* = \{K\colon V \times \cdots \times V \to \mathbb{R},\ \text{multilinear}\}$$

for K that are linear with respect to each slot.

$$e^{i_1} \otimes \cdots \otimes e^{i_k} \colon (v_1, \ldots, v_k) \mapsto v_1^{i_1} \cdots v_k^{i_k}$$
$$V^* \otimes \cdots \otimes V^* = \left\{ K_{i_1 \ldots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} \colon K_{i_1 \ldots i_k} \in \mathbb{R} \right\}$$

Analogically, we can define the tensor product of V,

$$V \otimes \cdots \otimes V = \{L \colon V^* \times \cdots \times V^* \to \mathbb{R},\ \text{multilinear}\}$$

Now, if $v_i \in V$ and $w_j \in V^*$,

$$v_1 \otimes \cdots \otimes v_k \colon (w_1, \ldots, w_k) \mapsto (v_1 \lrcorner w_1) \cdots (v_k \lrcorner w_k)$$
$$V \otimes \cdots \otimes V = \left\{ L^{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \colon L^{i_1 \ldots i_k} \in \mathbb{R} \right\}$$

Now we will combine these two types of a tensor product into one.

Definition 11. A general tensor product is defined as:

$$\underbrace{V \otimes \cdots \otimes V}_{l} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} = \left\{ T \colon V^{*l} \times V^k \to \mathbb{R},\ \text{multilinear} \right\}$$
$$= \left\{ T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l} \right\}$$

If we know what we mean, then we can write it in shorter notation, where vectors are X^i, covectors are Y_j and other tensors are $T^{i_1 \ldots}{}_{j_1 \ldots}$.

Wedge product For just two dual vectors $w, w' \in V^*$,

$$w \wedge w' \stackrel{\text{def}}{=} w \otimes w' - w' \otimes w$$

In general,

Definition 12. For $V^* \ni w^1, \ldots, w^k$

$$w^1 \wedge \cdots \wedge w^k \stackrel{\text{def}}{=} \sum_\sigma (-1)^{\operatorname{sgn} \sigma} w^{\sigma(1)} \otimes \cdots \otimes w^{\sigma(k)}$$

where we sum along all permutations.

We can consider a wedge product of spaces.

$$\underbrace{V^* \wedge \cdots \wedge V^*}_{k} = \operatorname{span}\left(w^1 \wedge \cdots \wedge w^k \colon w^i \in V^* \right)$$
$$= \left\{ \frac{1}{k!} W_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \right\}$$

A basis in this wedge space is:

$$\left\{ e^{i_1} \wedge \cdots \wedge e^{i_k} \right\} - \ \text{all possible, non-vanishing}$$

Wedge product is alternating.

$$e^{i_1} \wedge \cdots \wedge e^{j} \wedge \cdots \wedge e^{m} \wedge \cdots \wedge e^{i_k} = -e^{i_1} \wedge \cdots \wedge e^{m} \wedge \cdots \wedge e^{j} \wedge \cdots \wedge e^{i_k}$$

Let's consider an element, where $w_{ij} = -w_{ji}$:

$$\begin{aligned} V^* \wedge V^* = \left\{ \frac{1}{2} w_{ij} e^i \wedge e^j \right\} &= \left\{ \frac{1}{2} w_{ij} \Big(e^i \otimes e^j - e^j \otimes e^i \Big) \right\} \\ &= \Big\{ w_{ij} e^i \otimes e^j \Big\} \end{aligned}$$

Also, for any totally antisymmetric $w_{i_1 \ldots i_k}$

$$\frac{1}{k!} w_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} = w_{i_1 \ldots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$$

Example Suppose $\dim V = 3$. Basis of $V^* \wedge V^*$ is $\left\{ e^1 \wedge e^2,\, e^2 \wedge e^3,\, e^3 \wedge e^1 \right\}$. Basis of $V^* \wedge V^* \wedge V^*$ is $\left\{ e^1 \wedge e^2 \wedge e^3 \right\}$. Basis of any more wedged spaces of $\dim V = 3$ have the basis $\{0\}$.

Tangent and cotangent vectors to $\mathbb{R}^n$

Definition 13. X is called a vector tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$ if:

$$X \colon C^\infty(\mathbb{R}) \to \mathbb{R}$$

such that it is (i) linear and (ii) satisfies the local Leibniz identity: $X(fh) = f(x)X(h) + h(x)X(f)$.

Lemma 2. For every vector X tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$ there are $X^1, \ldots, X^n \in \mathbb{R}$ such that $\forall f \in C^\infty(\mathbb{R})$, $X(f) = X^i\, \partial_i f \,\big|_x$.

Definition 14 (Tangent space)**.** $\mathrm{T}_x\mathbb{R}^n$ is the space of vectors tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$. $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ set a basis in $\mathrm{T}_x\mathbb{R}^n$.

Definition 15. A covector at $x \in \mathbb{R}^n$ is a linear map

$$\begin{aligned} w \colon \mathrm{T}_x\mathbb{R}^n &\to \mathbb{R} \\ v^i \frac{\partial}{\partial x^i} &\mapsto w_i v^i \in \mathbb{R} \end{aligned}$$

where $(e^1, \ldots, e^n)$ is a basis in $\mathrm{T}^*_x\mathbb{R}^n$, dual to $(\partial_1, \ldots, \partial_n)$ and $w = w_i e^i$.

Lemma 3. Given $f \in C^\infty(\mathbb{R})$, $\mathrm{d}f$ is a map

$$\begin{gathered}\mathrm{d}f : \mathrm{T}_x\mathbb{R}^n \to \mathbb{R}\\ \forall v \in \mathrm{T}_x\mathbb{R}^n, \quad v \lrcorner \mathrm{d}f = v(f)\end{gathered}$$

In this sense, $\mathrm{d}f \in \mathrm{T}_x^*\mathbb{R}^n$.

In particular, consider $f = (x^1, \ldots, x^n)$, where $x^i \colon \mathbb{R}^n \to \mathbb{R}$.

$$\frac{\partial}{\partial x^j} \lrcorner \mathrm{d}x^i = \frac{\partial x^i}{\partial x^j} = \delta^i{}_j$$

hence $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ is a basis in $\mathrm{T}_x^*\mathbb{R}^n$.

Example Interpretation of a tangent vector at $x \in \mathbb{R}^n$. Consider a curve $\gamma \colon [s_0, s_2] \ni s \mapsto \big(x^1(s), \ldots, x^n(s)\big) \in \mathbb{R}^n$, where $\gamma(s_1) = x \in \mathbb{R}^n$. Then,

$$v\big|_x = \begin{pmatrix} \dfrac{\mathrm{d}x^1}{\mathrm{d}s} \\ \vdots \\ \dfrac{\mathrm{d}x^n}{\mathrm{d}s} \end{pmatrix}\Bigg|_{s_1}, \qquad \frac{\mathrm{d}}{\mathrm{d}s} f\big(x^1(s_1), \ldots, x^n(s_1)\big) = \frac{\mathrm{d}x^i}{\mathrm{d}s}\bigg|_{s_1} \frac{\partial}{\partial x^i} f\big(x(s_1)\big)$$

Vector fields on $\mathbb{R}^n$

Definition 16. If for every $\mathbb{R}^n \ni x \mapsto V_x \in \mathrm{T}_x\mathbb{R}^n$, $V_x = V_x^i\,\partial_i$, where $V_x^i \in \mathbb{C}^\infty(\mathbb{R}^n)$, we call it a vector field.

Lemma 4. Suppose V and W are vector fields on $\mathbb{R}^n$. Then, the commutator $[V, W] \colon C^\infty(\mathbb{R}^n) \ni f \mapsto V\big(W(f)\big) - W\big(V(f)\big)$ is also a vector field.

It also means, that we can define a vector field as an operation, not via coordinates.

Problem class 4

Problem 1 Consider $\mathbb{R}^n$, a point $x_0 \in \mathbb{R}^n$, a basis $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ in $\mathrm{T}_{x_0}\mathbb{R}^n$ and a dual basis $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ in $\mathrm{T}^*_{x_0}\mathbb{R}^n$. Consider a new coordinate system $(y^1, \ldots, y^n)$ and derive transformation between $(\partial/\partial y^i)$ and $(\partial/\partial x^i)$, and the same for dual basis.

Basis vectors are just differential operators on $\mathbb{R}^n$, so we can use chain rule/exterior derivative,

$$\begin{aligned}\frac{\partial}{\partial y^i} &= \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}\\ \mathrm{d}y^i &= \frac{\partial y^i}{\partial x^j}\,\mathrm{d}x^j\end{aligned}$$

Now, let's consider the transformation of a vector $v = v^i \, \partial_{x^i}$

$$v = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} = v'^j \frac{\partial}{\partial y^j}$$

And the same transformation for any covector:

$$\omega = \omega_i \, \mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial y^j} \, \mathrm{d}y^j = \omega'_j \, \mathrm{d}y^j$$

Problem 2 Given $V = V^i \, \partial_i$, $W = W^i \, \partial_i$, calculate the commutator $[V, W]^i \, \partial_i$.

Let f be an arbitrary function, then

$$\begin{aligned} [V, W](f) &= V\big(W(f)\big) - W\big(V(f)\big) \\ &= V^i \frac{\partial}{\partial x^i} \left(W^j \frac{\partial f}{\partial x^j} \right) - W^i \frac{\partial}{\partial x^i} \left(V^j \frac{\partial f}{\partial x^j} \right) \end{aligned}$$

Now we use Leibniz rule,

$$= V^i \left[\frac{\partial W^j}{\partial x^i} \frac{\partial f}{\partial x^j} + W^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right] - W^i \left[\frac{\partial V^j}{\partial x^i} \frac{\partial f}{\partial x^j} + V^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right]$$

We can notice, that 2nd order derivatives cancel out, because of the mixed partials equality.

$$\begin{aligned} &= V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j} - W^i V^j \frac{\partial^2 f}{\partial x^i \partial x^j} + V^i \frac{\partial W^j}{\partial x^i} \frac{\partial f}{\partial x^j} + W^i \frac{\partial V^j}{\partial x^i} \frac{\partial f}{\partial x^j} \\ &= \left[V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right] \frac{\partial f}{\partial x^j} = [V, W]^j \frac{\partial}{\partial x^j} f \end{aligned}$$

We have proven, that this is still a vector field, and found its components as well.

Problem 3 Show, that $[V, fW] = V(f)W + f[V, W]$.

$$\begin{aligned} [V, fW](h) &= V\big(fW(h)\big) - fW\big(V(h)\big) \\ &= V(f)W(h) + V\big(W(h)\big)f - fW\big(V(h)\big) \\ &= V(f)W(h) + f[V, W](h) \\ &= \big(V(f)W + f[V, W]\big)(h) \end{aligned}$$

Problem 4 Calculate the following commutators.

1. In $\mathbb{R}^3$, $\left[x^2 \, \partial_1 - x^1 \, \partial_2, x^3 \, \partial_2 - x^2 \, \partial_3\right]$

   We will use the result of the Problem 2. Let $V = (x^2, -x^1, 0)$, $W = (0, x^3, -x^2)$.

$$\begin{aligned} [V, W]^j &= V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \\ &= x^2 \frac{\partial W^j}{\partial x^1} - x^1 \frac{\partial W^j}{\partial x^2} - x^3 \frac{\partial V^j}{\partial x^2} + x^2 \frac{\partial V^j}{\partial x^3} \end{aligned}$$

Then, it is straightforward:

$$\begin{aligned}
[V, W]^1 &= -x^3 \frac{\partial x^2}{\partial x^2} + x^2 \frac{\partial x^2}{\partial x^3} = -x^3 \\
[V, W]^2 &= 0 \\
[V, W]^3 &= -x^1 \frac{\partial(-x^2)}{\partial x^2} = x^1
\end{aligned}$$

Finally,

$$[V, W] = -x^3\,\partial_1 + x^1\,\partial_3$$

We can also see, that the group generated by this commutator is SO(3). Commutator of rotation along ∂_3 with rotation along ∂_1 gave us rotation along ∂_2.

2. In $\mathbb{R}^4$, $\big[x^0\,\partial_1 + x^1\,\partial_0, x^0\,\partial_2 + x^2\,\partial_0\big]$

   Well, commutator is linear with respect to each entry (as long as vector field coefficients are just numbers), so we can consider 4 separate, simpler commutators.

$$\begin{aligned}
\big[x^0\,\partial_1, x^0\,\partial_2\big] &= 0 \\
\big[x^0\,\partial_1, x^2\,\partial_0\big] &= -\big[x^2\,\partial_0, x^0\,\partial_1\big]
\end{aligned}$$

   If we treat x^0 as f, we can use Problem 3:

$$\begin{aligned}
&= -\Big(x^2\,\partial_0(x^0)\,\partial_1 + x^0\big[x^2\,\partial_0, \partial_1\big]\Big) \\
&= -x^2\,\partial_1 + x^0\big[\partial_1, x^2\,\partial_0\big] \\
&= -x^2\,\partial_1 + x^0\,\partial_1(x^2)\,\partial_0 + x^0x^2\big[\partial_1, \partial_0\big] = -x^2\,\partial_1 \\
\big[x^1\,\partial_0, x^0\,\partial_2\big] &= x^1\,\partial_0(x^0)\,\partial_2 + x^0\big[x^1\,\partial_0, \partial_2\big] = x^1\,\partial_2 \\
\big[x^1\,\partial_0, x^2\,\partial_0\big] &= 0
\end{aligned}$$

   Now, we combine all terms,

$$\begin{aligned}
\big[x^0\,\partial_1 + x^1\,\partial_0, x^0\,\partial_2 + x^2\,\partial_0\big] &= 0 - x^2\,\partial_1 + x^1\,\partial_2 + 0 \\
&= -x^2\,\partial_1 + x^1\,\partial_2
\end{aligned}$$

   The group SO(1,3) (Lorentz group) is generated by that last commutator.

Lecture 5

17 lis 2020

Some properties of the commutator

Commutators satisfy the following identities:

$$0 = \big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]$$

Linearity, up to constants:

$$[X, aY + bZ] = a[X, Y] + b[X, Z], \quad a, b \in \mathbb{R}$$

Antisymmetry:

$$[X, Y] = -[Y, X]$$

Some variation of the Leibniz rule:

$$[X, fY] = X(f)Y + f[X, Y]$$

The flow of a vector field

Let $v = v^i \partial_i$ be a vector field in $\mathbb{R}^n$. An integral curve $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in \mathbb{R}^n$ of v is a curve, such that:

$$\frac{\mathrm{d}p^i(\tau)}{\mathrm{d}\tau} = v^i\big(p(\tau)\big)$$

at each τ, ie.

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = v_{p(\tau)}$$

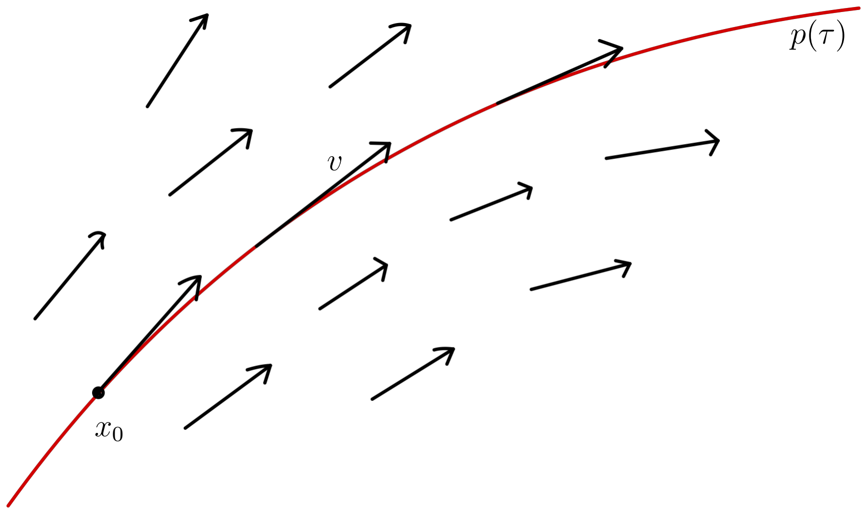

Figure 2.1: Integral curve.

Theorem 2. For every $x_0 \in \mathbb{R}^n$, $v_{x_0} \neq 0$ locally there is a unique integral curve p_{x_0} of v such that $p_{x_0}(0) = x_0$.

Definition 17 (The flow)**.** Given τ, we define a map in a neighbourhood of x:

$$x \mapsto p_x(\tau)$$

and call it the flow of a vector field.

Covector fields

Definition 18 (Covector field)**.** We call ω_x a covector field, if $\mathbb{R}^n \ni x \mapsto \omega_x \in \mathrm{T}^*\mathbb{R}^n$, where $\omega_x = \omega_i(x)\,\mathrm{d}x^i$ for $\omega_i \in C^m(\mathbb{R}^n)$.

Similarly, we can define a covariant tensor field by taking

$$\mathrm{T}^*\mathbb{R}^n \otimes \cdots \otimes \mathrm{T}^*\mathbb{R}^n \ni T = T_{i_1 \cdots i_k}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}$$

where $T_{i_1 \cdots i_k} \in C^m(\mathbb{R}^n)$. If $T_{i_1 \cdots i_k} = T_{[i_1 \cdots i_k]}$, then

$$T_{i_1 \cdots i_k}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} = \frac{1}{k!} T_{i_1 \cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

and we call it a differential k-form. There are also some useful properties!

Remark 6. Let $T_{i_1 \cdots i_k}$ be an arbitrary covariant tensor and $\omega \in \Omega^k(\mathbb{R}^k)$. Then,

$$\begin{aligned}\omega = \frac{1}{k!} T_{i_1 \cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{ik} &= \underbrace{T_{[\sigma(1) \cdots \sigma(k)]}\,\mathrm{d}x^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x^{\sigma(k)}}_{\text{arbitrary permutation}} \\ &= \frac{1}{k!} T_{[i_1 \cdots i_k]}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\end{aligned}$$

It also means, that

$$\frac{1}{k!} T_{i_1 \cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{ik} = T_{[i_1 \cdots i_k]}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}$$

If we had $\nu \in \Omega^k(\mathbb{R}^n)$, $n > k$ then everything would stay the same except the part with „arbitrary permutation", which would take a form of $\binom{n}{k}$ arbitrary permutations, each one from a multivalent k-element combination of $\{1, 2, \ldots, n\}$.

Exterior derivative

We will define this operation via its properties.

$$\begin{gathered}C^m(\mathbb{R}^n) \ni f \mapsto \mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \\ \mathrm{d}f \mapsto \mathrm{d}(\mathrm{d}f) = 0 \\ f\,\mathrm{d}h \mapsto \mathrm{d}(f\,\mathrm{d}h) = \mathrm{d}f \wedge \mathrm{d}h\end{gathered}$$

In general, assuming that $T_{i_1\cdots i_k} = T_{[i_1\cdots i_k]}$,

$$\begin{aligned}
\mathrm{d}\left(\frac{1}{k!}T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\right) &= \frac{1}{k!}\,\mathrm{d}T_{i_1\cdots i_k}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}\frac{\partial T_{i_1\cdots i_k}}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}T_{i_1\cdots i_k,j}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}T_{[i_1\cdots i_k,j]}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= (-1)^k\frac{1}{k!}T_{[i_1\cdots i_k,j]}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\wedge\mathrm{d}x^j
\end{aligned}$$

Let us consider a k-form ω and l-form ν.

$$\begin{aligned}
\mathrm{d}(\omega\wedge\nu) &= \mathrm{d}\omega\wedge\nu + (-1)^k\omega\wedge\mathrm{d}\nu\\
\mathrm{d}(f\omega) &= \mathrm{d}f\wedge\omega + f\,\mathrm{d}\omega\\
\mathrm{d}(a\omega+b\nu) &= a\,\mathrm{d}\omega + b\,\mathrm{d}\nu
\end{aligned}$$

Contraction with a vector

Let $T = T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}$ and $V = V^i\,\partial_i$.

Definition 19 (Contraction of a covariant tensor)**.** A contraction of a covariant tensor with a vector is defined as:

$$V\lrcorner T = V^{i_1}T_{i_1i_2\cdots i_k}\,\mathrm{d}x^{i_2}\otimes\cdots\otimes\mathrm{d}x^{i_k}$$

Moreover,

$$\begin{aligned}
V_k\lrcorner\big(V_{k-1}\lrcorner\cdots\lrcorner(V_1\lrcorner T)\big) &= V_k^{i_k}\cdots V_1^{i_1}T_{i_1\cdots i_k}\\
&\stackrel{\text{def}}{=} T(V_1,\ldots,V_k)
\end{aligned}$$

From this definition if simply follows, that:

$$V\lrcorner(T\otimes L) = (V\lrcorner T)\otimes L$$

Let us also consider a differential k-form ω.

$$V\lrcorner\omega = \frac{1}{(k-1)!}V^{i_1}\omega_{i_1i_2\cdots i_k}\,\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_k}$$

We have also other nice properties:

$$\begin{aligned}
V\lrcorner(\omega\wedge\nu) &= (V\lrcorner\omega)\wedge\nu + (-1)^k\omega\wedge(V\lrcorner\nu)\\
V\lrcorner(f\omega) &= fV\lrcorner\omega\\
V\lrcorner(\omega+\omega') &= V\lrcorner\omega + V\lrcorner\omega'
\end{aligned}$$

Remark 7. There are also useful relations between the contraction and the external derivative.

$$V \lrcorner \, \mathrm{d}f = V(f) = V^i \, \partial_i(f)$$

If W, V are any vector fields and $\omega \in \Omega^1(\mathbb{R}^n)$,

$$W \lrcorner (V \lrcorner \, \mathrm{d}\omega) \overset{\text{def}}{=} \mathrm{d}\omega \, (V, W) = V(W \lrcorner \omega) - W(V \lrcorner \omega) - [V, W] \lrcorner \omega$$

Proof.

$$\begin{aligned}
\omega &= \omega_i \, \mathrm{d}x^i \\
\mathrm{d}\omega &= \mathrm{d}\omega_i \wedge \mathrm{d}x^i = \omega_{i,j} \, \mathrm{d}x^j \wedge \mathrm{d}x^i = \omega_{i,j} (\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}x^j) \\
\mathrm{d}\omega \, (V, W) &= V^{i_1} W^{i_2} \, \mathrm{d}\omega \, (\partial_{i_1}, \partial_{i_2}) \\
&= V^{i_1} W^{i_2} \omega_{i,j} (\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}x^j)(\partial_{i_1}, \partial_{i_2}) \\
&= V^{i_1} W^{i_2} \omega_{i,j} ({\delta^j}_{i_1} {\delta^i}_{i_2} - {\delta^i}_{i_1} {\delta^j}_{i_2}) \\
&= V^{i_1} W^{i_2} (\omega_{i_2,i_1} - \omega_{i_1,i_2}) = W^{i_2} V(\omega_{i_2}) - V^{i_1} W(\omega_{i_1})
\end{aligned}$$

Notice, that we can use Leibniz rule: $V(W^i \omega_i) = V(W^i)\omega_i + W^i V(\omega_i)$, as $W^i, \omega_i \in C^k(\mathbb{R}^n)$

$$\begin{aligned}
&= V(W^{i_2} \omega_{i_2}) - V(W^{i_2}) \omega_{i_2} - W(V^{i_1} \omega_{i_1}) + W(V^{i_1}) \omega_{i_1} \\
&= V(W \lrcorner \omega) - W(V \lrcorner \omega) - [V, W] \lrcorner \omega
\end{aligned}$$

■

Application Suppose $e_1, \ldots, e_n$ are vector fields in $\mathbb{R}^n$ such that at each $x \in \mathbb{R}^n$, $(e_1, \ldots, e_n)$ is a basis and $\left[e_i, e_j\right] = {c_{ij}}^k e_k$ for ${c_{ij}}^k = \text{const}$. We also take such $e^1, \ldots, e^n$ dual 1-form fields that:

$$\mathrm{d}e^i = \frac{1}{2} {b^i}_{jk} e^j \wedge e^k, \quad {b^i}_{jk} = -{b^i}_{kj}$$

Find a relation between ${c_{ij}}^k$ and ${b^i}_{jk}$.

We shall start from applying two vector fields directly two our 2-form.

$$\begin{aligned}
\mathrm{d}e^i \, (e_{j_1}, e_{j_2}) &= \frac{1}{2} {b^i}_{jk} e^j \wedge e^k (e_{j_1}, e_{j_2}) \\
&= \frac{1}{2} {b^i}_{jk} \left({\delta^i}_{j_1} {\delta^k}_{j_2} - {\delta^j}_{j_2} {\delta^k}_{j_1} \right) = \frac{1}{2} \left({b^i}_{j_1 j_2} - {b^i}_{j_2 j_1} \right) \\
&= {b^i}_{j_1 j_2}
\end{aligned}$$

On the other hand we could use the identity derived above.

$$\begin{aligned}
\mathrm{d}e^i \, (e_{j_1}, e_{j_2}) &= e_{j_1}(e_{j_2} \lrcorner e^i) - e_{j_2}(e_{j_1} \lrcorner e^i) - \left[e_{j_1}, e_{j_2}\right] \lrcorner e^i \\
&= e_{j_1}({\delta^i}_{j_2}) - e_{j_2}({\delta^i}_{j_1}) - \left[e_{j_1}, e_{j_2}\right] \lrcorner e^i \\
&= -{c_{j_1 j_2}}^k e_k \lrcorner e^i = -{c_{j_1 j_2}}^i
\end{aligned}$$

We have proven something very interesting!

Remark 8. Let (e_i) be vector fields, and (e^j) be 1-form fields dual to them. If $\left[e_i, e_j\right] = c_{ij}{}^k e_k$, where $c_{ij}{}^k = \text{const.}$, then

$$\mathrm{d}e^i = -\frac{1}{2} c_{jk}{}^i e^j \wedge e^k$$

The Lie derivative

Definition 20 (Lie derivative)**.** Let V be a vector field in $\mathbb{R}^n$. We define the Lie derivative, by giving the sufficient amount of its properties.

$$\mathcal{L}_V : C^1(\mathbb{R}^n) \ni f \mapsto \mathcal{L}_V(f) \stackrel{\text{def}}{=} V(f)$$

Commutativity with d,

$$\mathcal{L}_V(\mathrm{d}f) \stackrel{\text{def}}{=} \mathrm{d}(\mathcal{L}_V f) = \mathrm{d}\big(V(f)\big)$$

Sort of Leibniz rule,

$$\begin{aligned} \mathcal{L}_V(h\,\mathrm{d}f) &\stackrel{\text{def}}{=} (\mathcal{L}_V h)\,\mathrm{d}f + h\,\mathrm{d}(\mathcal{L}_V f) \\ &= V(h)\,\mathrm{d}f + h\,\mathrm{d}\big(V(f)\big) \end{aligned}$$

When acting on tensors,

$$\begin{aligned} \mathcal{L}_V(T \otimes L) &= (\mathcal{L}_V T) \otimes L + T \otimes (\mathcal{L}_V L) \\ \mathcal{L}_V(T + L) &= \mathcal{L}_V T + \mathcal{L}_V L \\ \mathcal{L}_V(W \lrcorner T) &= (\mathcal{L}_V W) \lrcorner T + W \lrcorner (\mathcal{L}_V T) \end{aligned}$$

Exercise V, W are vector fields. Show, that $\mathcal{L}_V W = [V, W]$.

$$\begin{aligned} (\mathcal{L}_V W) \lrcorner\, \mathrm{d}x^i &= \mathcal{L}_V\Big(W \lrcorner\, \mathrm{d}x^i\Big) - W \lrcorner\, \mathrm{d}\Big(\mathcal{L}_V x^i\Big) \\ &= \mathcal{L}_V(W^i) - W \lrcorner\, \mathrm{d}V^i = V(W^i) - W \lrcorner V^i{}_{,j}\ \mathrm{d}x^j \\ &= V(W^i) - W^j V^i{}_{,j}\ = V(W^i) - W(V^i) = [V, W]^i \\ &= [V, W] \lrcorner\, \mathrm{d}x^i \end{aligned}$$

This formula works for an arbitrary $\mathrm{d}x^i$, so

$$\mathcal{L}_V W = [V, W]$$

Problem class 5

Problem 1 Consider a 1-form $A = q/r\,\mathrm{d}t$, where $q =$const., t, x, y, z are coordinates in $\mathbb{R}^n$ and $r = \sqrt{x^2 + y^2 + z^2}$. Calculate $\mathrm{d}A$.

$$\mathrm{d}A = q\,\mathrm{d}\Big((x^2 + y^2 + z^2)^{-1/2}\,\mathrm{d}t\Big)$$

$$= -\frac{x\,\mathrm{d}x \wedge \mathrm{d}t + y\,\mathrm{d}y \wedge \mathrm{d}t + z\,\mathrm{d}z \wedge \mathrm{d}t}{(x^2+y^2+z^2)^{-3/2}}$$

Problem 2 Calculate d of $x\,\mathrm{d}y$, $-y\,\mathrm{d}x$.
It's trivial.

Problem 3 Show that the following equality is true: $\mathcal{L}_V\omega = \mathrm{d}(V \lrcorner \omega) + V \lrcorner\, \mathrm{d}\omega$, where $\omega = \omega_i\,\mathrm{d}x^i$.

$$\mathcal{L}_V(\omega_i\,\mathrm{d}x^i) = V(\omega_i)\,\mathrm{d}x^i + \omega_i\,\mathrm{d}\Big(V(x^i)\Big)$$

Step I. Given $V = V^i\,\partial_i$, calculate $\mathcal{L}_V(\mathrm{d}x^i)$.

$$\mathcal{L}_V(\mathrm{d}x^i) = \mathrm{d}\Big(V(x^i)\Big) = \mathrm{d}V^i$$

Step II. Use Leibniz rule,

$$\begin{aligned}
\mathcal{L}_V(\omega_i\,\mathrm{d}x^i) &= V(\omega_i)\,\mathrm{d}x^i + \omega_i\mathcal{L}_V(\mathrm{d}x^i) \\
&= V^j\frac{\partial\omega_j}{\partial x^j}\,\mathrm{d}x^i + \omega_i\,\mathrm{d}V^i \\
&= V^j\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i + \omega_i\frac{\partial V^i}{\partial x^i}\,\mathrm{d}x^j \\
&= \left(V^j\frac{\partial\omega_i}{\partial x^j} + \omega_j\frac{\partial V^j}{\partial x^i}\right)\mathrm{d}x^i
\end{aligned}$$

Step III. $\mathrm{d}(V \lrcorner \omega)$

$$\mathrm{d}(V \lrcorner \omega) = \mathrm{d}(V^i\omega_i) = V^i\,\mathrm{d}\omega_i + \omega_i\,\mathrm{d}V^i$$

Step IV. $V \lrcorner\, \mathrm{d}\omega$

$$\begin{aligned}
V \lrcorner\, \mathrm{d}\omega = V \lrcorner(\mathrm{d}\omega_i \wedge \mathrm{d}x^i) &= V^j\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i - V^i\,\mathrm{d}\omega_i \\
\mathrm{d}(V \lrcorner \omega) + V \lrcorner\, \mathrm{d}\omega &= \omega_i\,\mathrm{d}V^i + V^i\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i \\
&= \left(\omega_i\frac{\partial V^i}{\partial x^j} + V^i\frac{\partial\omega_j}{\partial x^i}\right)\mathrm{d}x^j
\end{aligned}$$

We can see that they are exactly the same. This formula is also true for arbitrary $\omega = \dfrac{1}{k!}\omega_{i_1\ldots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ (any k-from).

Problem 4 (Application of 3) In $\mathbb{R}^3$ there are given 1-forms (e^1, e^2, e^3) that set a dual basis to vector fields (e_1, e_2, e_3) at each point $x \in \mathbb{R}^3$. Knowing that $\mathrm{d}e^1 = e^2 \wedge e^3$, $\mathrm{d}e^2 = e^3 \wedge e^1$, $\mathrm{d}e^3 = e^1 \wedge e^2$ calculate $\mathcal{L}_{e_i}e^j$.

We know, that

$$
\begin{aligned}
\mathcal{L}_{e_i} e^j &= \mathrm{d}(e_i \lrcorner e^j) + e_i \lrcorner \, \mathrm{d}e^j = 0 + e_i \lrcorner \left(\frac{1}{2} \varepsilon_{jlk} e^l \wedge e^k \right) \\
&= \frac{1}{2} \varepsilon_{jlk} (e_i \lrcorner e^l) \wedge e^k - \frac{1}{2} \varepsilon_{jlk} e^l \wedge (e_i \lrcorner e^k) \\
&= \frac{1}{2} \varepsilon_{jik} e^k - \frac{1}{2} \varepsilon_{jli} e^l = \frac{1}{2} \varepsilon_{jik} e^k + \frac{1}{2} \varepsilon_{jil} e^l \\
&= \varepsilon_{jik} e^k
\end{aligned}
$$

Problem 5 Consider $\mathbb{R}^2$ and calculate $\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $V = V^x \, \partial_x + V^y \, \partial_y$.

$$
\mathcal{L}_V(T \otimes L) = \mathcal{L}_V(T) \otimes L + T \otimes \mathcal{L}_V(L)
$$

Also we know, that

$$
\begin{aligned}
\mathrm{d}x^2 &= \mathrm{d}x \otimes \mathrm{d}x \\
\mathcal{L}_V(\mathrm{d}x^2) &= \mathcal{L}_V(\mathrm{d}x) \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathcal{L}_V(\mathrm{d}x) \\
\mathcal{L}_V(\mathrm{d}x) &= \mathrm{d}\big(V(x)\big) = \mathrm{d}V^x = \frac{\partial V^x}{\partial x} \, \mathrm{d}x + \frac{\partial V^x}{\partial y} \, \mathrm{d}y
\end{aligned}
$$

From this, we get:

$$
\begin{aligned}
\mathcal{L}_V(\mathrm{d}x^2) &= 2 \frac{\partial V^x}{\partial x} \, \mathrm{d}x \otimes \mathrm{d}x + \frac{\partial V^x}{\partial y} (\mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y) \\
\mathcal{L}_V(\mathrm{d}y^2) &= 2 \frac{\partial V^y}{\partial y} \, \mathrm{d}y \otimes \mathrm{d}y + \frac{\partial V^y}{\partial x} (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)
\end{aligned}
$$

Now it is straightforward,

$$
\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2) = 2 \frac{\partial V^x}{\partial x} \, \mathrm{d}x^2 + 2 \frac{\partial V^y}{\partial y} \, \mathrm{d}y^2 + \left(\frac{\partial V^x}{\partial y} + \frac{\partial V^y}{\partial x} \right) (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)
$$

Problem 6 (Continuation of 5) Find $V^x \, \partial_x + V^y \, \partial_y$ such that $\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2) = 0$.

We obtain a set of equations.

$$
\frac{\partial V^x}{\partial x} = 0, \quad \frac{\partial V^y}{\partial y} = 0, \quad \frac{\partial V^x}{\partial y} + \frac{\partial V^y}{\partial x} = 0
$$

From the first two equations we know that $V^x = V^x(y), V^y = V^y(x)$. Now we can guess the exact solutions having tee 3rd equation.

$$
V^x = y, \quad V^y = -x
$$

A general solution would be:

$$
V^x = Ay + B, \quad V^y = -Ax + C
$$

In overall, V turns out to be a rotation combined with a translation.

Lecture 6

LECTURE TO FILL UP 24 lis 2020

Problem class 6 – Stereographic projection

It is a map from a plane to a sphere. By applying the Tales theorem, we get

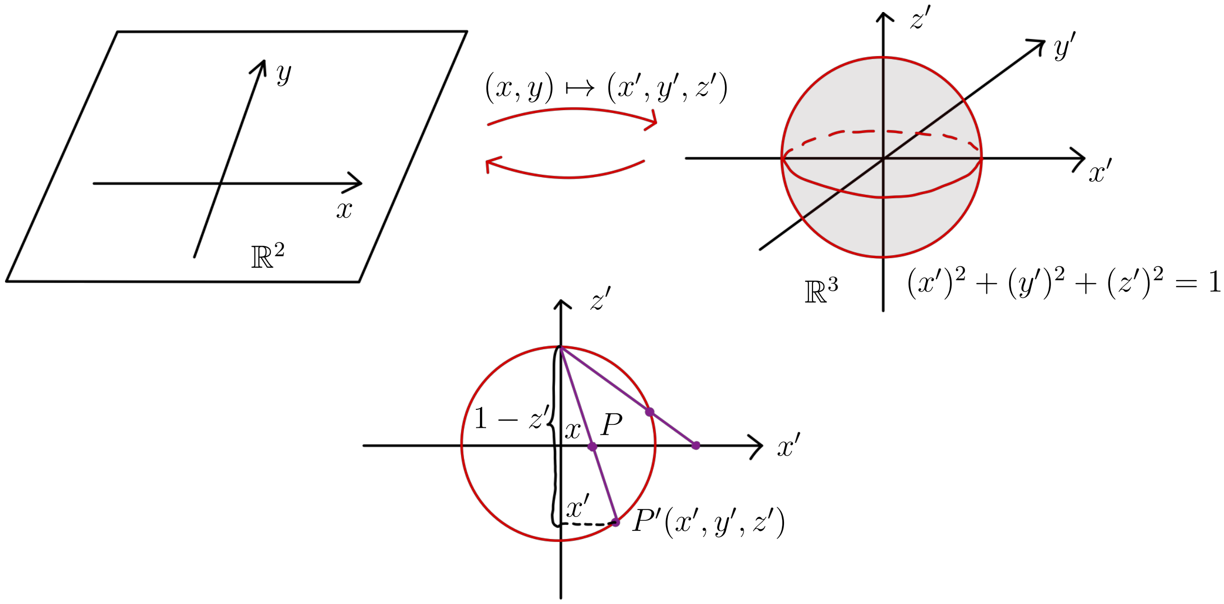

Figure 2.2: Stereographic projection.

$$\frac{x'}{1 - z'} = \frac{x}{1}$$

The same proportion is true for y,

$$\frac{y'}{1 - z'} = \frac{y}{1}$$

$$(x, y) = \left(\frac{x'}{1 - z'}, \frac{y'}{1 - z'} \right)$$

We want to invert this map.

$$x^2 + y^2 = \frac{(x')^2 + (y')^2}{(1 - z')^2} = \frac{1 - (z')^2}{(1 - z')^2}$$

$$= \frac{1 + z'}{1 - z'}$$

$$\frac{1 + x^2 + y^2}{2} = \frac{1}{2}\left(1 + \frac{1 + z'}{1 - z'}\right) = \frac{1 - z' + 1 + z'}{2(1 - z')}$$

$$= \frac{1}{1 - z'}$$

We conclude, that

$$
\begin{aligned}
x' &= x(1-z') = \frac{2x}{1+x^2+y^2} \\
y' &= \frac{2y}{1+x^2+y^2} \\
z' &= \frac{-1+x^2+y^2}{1+x^2+y^2}
\end{aligned}
$$

In overall,

$$
(x,y) \overset{\phi}{\mapsto} \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1)
$$

Is $(0,0,1)$ in the image? Well, S^2 isn't homeomorphic to $\mathbb{R}^2$, so it isn't. But there is a possibility to include $(0,0,1)$ as a point at infinity.

Problem 1 In $\mathbb{R}^3$, $g' = (\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2$. Calculate $\phi^* g'$ on $\mathbb{R}^2$.

$$
\begin{aligned}
\phi^* \mathrm{d}x' &= \mathrm{d}\left(\frac{2x}{1+x^2+y^2}\right) = \frac{2\,\mathrm{d}x\,(1+x^2+y^2) - 2x(2x\,\mathrm{d}x + 2y\,\mathrm{d}y)}{(1+x^2+y^2)^2} \\
&= \frac{2}{(1+x^2+y^2)^2}\big[(1-x^2+y^2)\,\mathrm{d}x - 2xy\,\mathrm{d}y\big] \\
\phi^* \mathrm{d}y' &= \frac{2}{(1+x^2+y^2)^2}\big[(1+x^2-y^2)\,\mathrm{d}y - 2xy\,\mathrm{d}x\big] \\
\phi^* \mathrm{d}z' &= \mathrm{d}\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right) \\
&= \frac{(2x\,\mathrm{d}x + 2y\,\mathrm{d}y)(x^2+y^2+1) - (2x\,\mathrm{d}x + 2y\,\mathrm{d}y)(x^2+y^2-1)}{(x^2+y^2+1)^2} \\
&= \frac{4}{(1+x^2+y^2)^2}(x\,\mathrm{d}x + y\,\mathrm{d}y)
\end{aligned}
$$

Now we should combine them into a metric tensor.

$$
\phi^* g' = (\phi^*\,\mathrm{d}x')^2 + (\phi^*\,\mathrm{d}y')^2 + (\phi^*\,\mathrm{d}z')^2
$$

Terms proportional to $\mathrm{d}x^2\,, \mathrm{d}y^2\,, \mathrm{d}x\,\mathrm{d}y$:

$$
\begin{aligned}
\sim \mathrm{d}x^2 &= \frac{4}{(1+x^2+y^2)^4}\big[(-x^2+y^2+1)^2 + 4x^2y^2 + 4x^2\big] \\
&= \frac{4}{(1+x^2+y^2)^2}(x^2+y^2+1)^2 = 4(1+x^2+y^2)^{-2} \\
\sim \mathrm{d}y^2 &= 4(1+y^2+x^2)^{-2} \\
\sim \mathrm{d}x\,\mathrm{d}y &= \frac{8}{(1+x^2+y^2)^2}\big[-(1+x^2+y^2)(2xy) - (1+x^2-y^2)(2xy) + 4xy\big] \\
&= \frac{8}{(1+x^2+y^2)^2}[-4xy + 4xy] = 0
\end{aligned}
$$

And finally,

$$\phi^* g' = \frac{4}{(1+x^2+y^2)^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$$

This is our metric of a round sphere in terms of stereographic coordinates. For example we could calculate the area of our sphere or the length of a line.

Let's calculate the length of the $y = 0$ line.

$$\begin{aligned} L &= \int_{-\infty}^{+\infty} \sqrt{\frac{4}{(1+x^2)^2}\,\mathrm{d}x^2} = \int_{-\infty}^{+\infty} \frac{2}{1+x^2}\,\mathrm{d}x \\ &= \left| \begin{matrix} x = \tan u \\ \mathrm{d}x = \mathrm{d}u \,/\, \cos^2(u) \end{matrix} \right| = \int_{-\pi/2}^{\pi/2} 2\,\mathrm{d}u = 2\pi \end{aligned}$$

Well, it works!

It is important to understand what we really did. We started from the euclidean metric, valid in the whole $\mathbb{R}^3$. At the moment, when we defined the parametrisation of S^2, we „cut" this metric to such sphere, so that g' measures the distances on S^2. When we embed any surface, the metric tensor is also „cut" to that surface. Because of that, the pullbacked tensor $\phi^* g'$ also measures distances on S^2, but in terms of different coordinates.