General Relativity

Lecturer:
prof. Jerzy Lewandowski

Scribe:
Szymon Cedrowski

© I 2020

Contents

Chapter 1

Special Relativity

Lecture 1: Minkowski spacetime

First we will formulate a geometry which is equivalent to basic relativistic symmetries known from the special relativity. It was actually the first step towards GR to understand special relativity as symmetries of a given geometry. 20 paź 2020

Definition 1 (Minkowski spacetime)**.** 4-dimensional affine space M with associated scalar product g (metric tensor) of signature $(- + ++)$.

Definition 2 (Affine space)**.** We define,

- Vector space at each point
- Global parallelism (allows to identify vectors at different points)
- $m_0, m_1 \in M$ defines a vector at m_0 : $m_0 + \underbrace{m_1 - m_0}_{v'} = m_1$

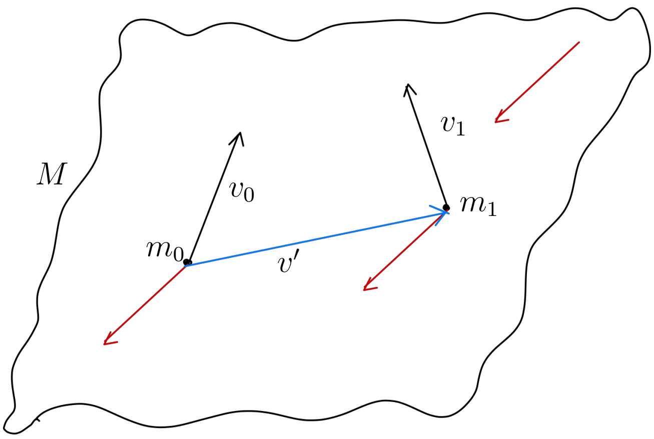

Figure 1.1: Affine space

In practice you can consider M as a 4D vector space upon a choice of an origin $m_0 \in M$. Let V be 4-dim vector space. We make $V \equiv M$, where M – affine space understood as 4D vector space up to translations.

Definition 3 (Metric tensor)**.**

$$g(u, v) \in \mathbb{R}$$
$$g(u, v) = g(w, u)$$
$$g(\alpha v + \beta u, w) = \alpha g(v, w) + \beta g(u, w)$$

We can see that $\forall g$, that is symmetric and bilinear, $\exists\{e_0, e_1, e_2, e_3\}$ such that:

$$g(e_0, e_0) = \pm 1$$
$$\vdots$$
$$g(e_3, e_3) = \pm 1$$

whenever we take product of two different basis vectors,

$$g\big(e_i, e_j\big) \overset{i \neq j}{=} 0$$

In other words,

$$g\big(e_i, e_j\big) = \pm \delta^i_j$$

Here we have an orthogonal basis. The number of $+$ and $-$ is invariant for a given g. In Minkowski spacetime it is $(- + ++)$.
This is the complete definition of Minkowski spacetime.

This scalar product is independent on choice of point $m \in M$ (because the global parallelism holds). In other words, the parallel transport preserves g.

Timelike, spacelike, null vectors

Scalar product distinguishes between different types of vectors that may emerge on M: timelike, spacelike and null.

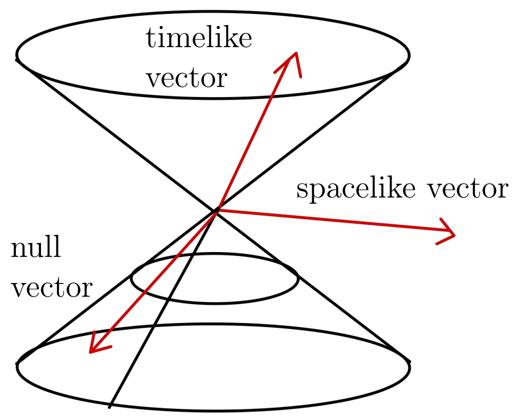

Figure 1.2: Classification of vectors on M

Definition 4.

$$g(v,v) = \begin{cases} < 0, & \text{timelike} \\ = 0, & \text{null} \\ > 0, & \text{spacelike} \end{cases}$$

Null vectors form a cone. Timelike vectors lay inside these cones, spacelike ones outside.

What is their physical interpretation?

Consider a curve $p\colon [\tau_0, \tau_1] \to M$. We can distinguish its tangent vectors by taking derivatives with respect to parameter:

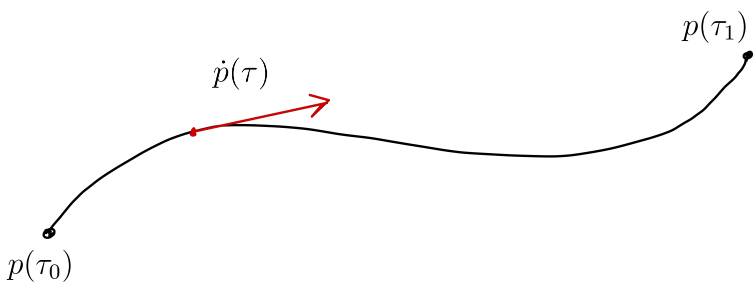

Figure 1.3: Tangent vector

If $\forall \tau$, all tangent vectors are of the same type, we define:

$$\begin{cases} g(\dot{p},\dot{p}) & < 0, \quad \text{timelike curve} \\ g(\dot{p},\dot{p}) & = 0, \quad \text{null curve} \\ g(\dot{p},\dot{p}) & > 0, \quad \text{spacelike curve} \end{cases}$$

A timelike curve is the worldline of a particle at speed $< c$.
A null curve represents particles with speed $= c$.

Proper time, proper distance

Definition 5 (Proper time/distance)**.** Let $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in M$ be timelike. Then, the proper time (as I started my clock and travelled along this curve) will be,

$$T \overset{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

For spacelike curves we will have a proper distance:

$$D \overset{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

T and D are reparametrisation invariant.

$$p'(\tau') = p\big(\tau(\tau')\big)$$
$$\int_{\tau_0'}^{\tau_1'} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau'}, \frac{\mathrm{d}p}{\mathrm{d}\tau'}\right)}\, \mathrm{d}\tau' = \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

Remark 1. For curves $p(\tau) = \tau e_0$ we can see that $T = (\tau_1 - \tau_0)$. If we assume that $\tau_0 = 0$ and that we calculate proper time at point $p(\tau)$, then $T = \tau$. So, with suitable assumptions our parameter τ represents the proper time along any curve being parametrized.

Inverse triangular inequality Consider straight timelike lines such that they form a triangle.

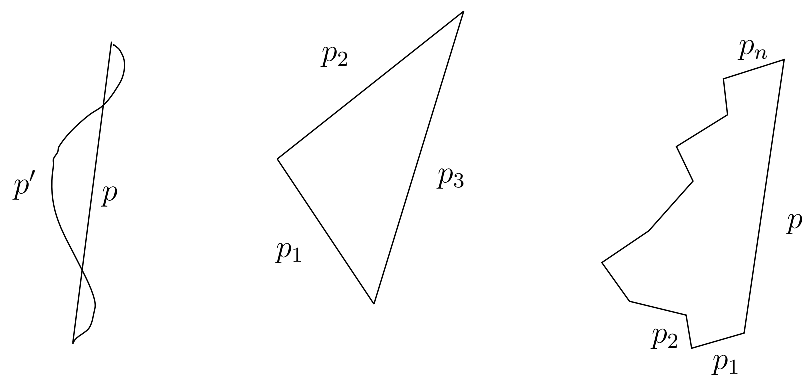

Figure 1.4: Inverse triangular inequality

Theorem 1.

$$T_1 + T_2 < T_3$$
$$T_1 + \cdots + T_n < T$$
$$T' \leq T$$

Remark 2. The conclusion is that the timelike straight lines (points moving at constant speed $< c$) maximize time.

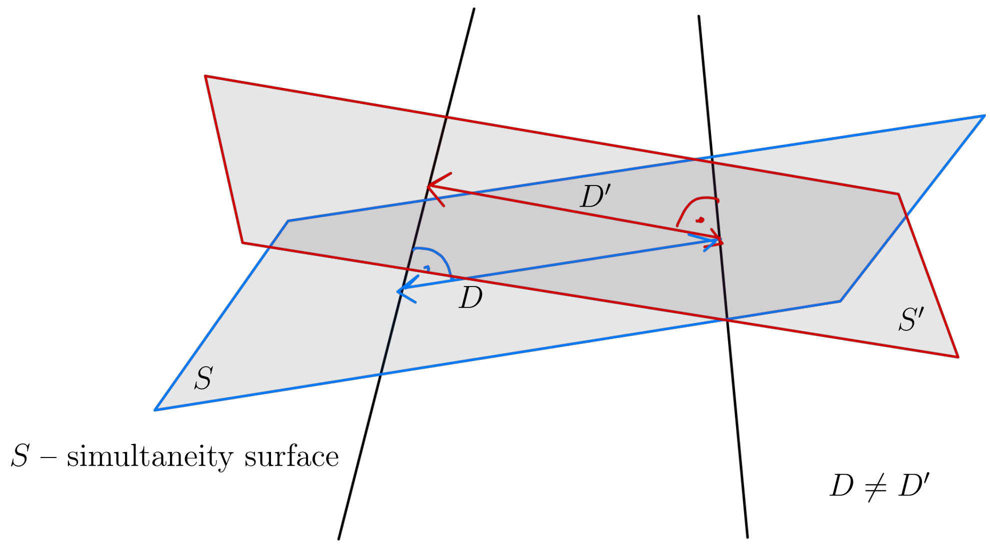

Figure 1.5: Simultaneity of spacetime events.

Remark 3 (Physical interpretation for spacelike curves)**.** D – distance between two events that we consider to happen simultaneously. Proper distance D measured along that S surface is distance between two events happening simultaneously.

Time and distances dilatation

Time dilatation Consider an observer p' moving at a constant speed with respect to observer p. We take $\tau \in [0, \tau_1]$. $p \in M$ comes with its notion of time T and any relatively moving observer p' comes with its T'. What is the relation between the times that they measure?

Let's calculate the events in spacetime at τ time.

$$p(\tau) = \tau e_0$$
$$p'(\tau) = \tau(e_0 + \beta e_1)$$

Now we calculate proper times,

$$\begin{aligned}T^2 &= -g(\tau_1 e_0, \tau_1 e_0) = \tau_1^2 \\ T'^2 &= -\tau_1^2 g(e_0 + \beta e_1, e_0 + \beta e_1) \\ &= -\tau_1^2 \left(-1 + \beta^2\right) = \tau_1^2 \left(1 - \beta^2\right) \\ T' &= \sqrt{1-\beta^2} T\end{aligned}$$

Distances dilatation We have one observer and 2 other parallel observers (moving at the same speeds). Describe this from the point of view of the simultaneity surfaces.

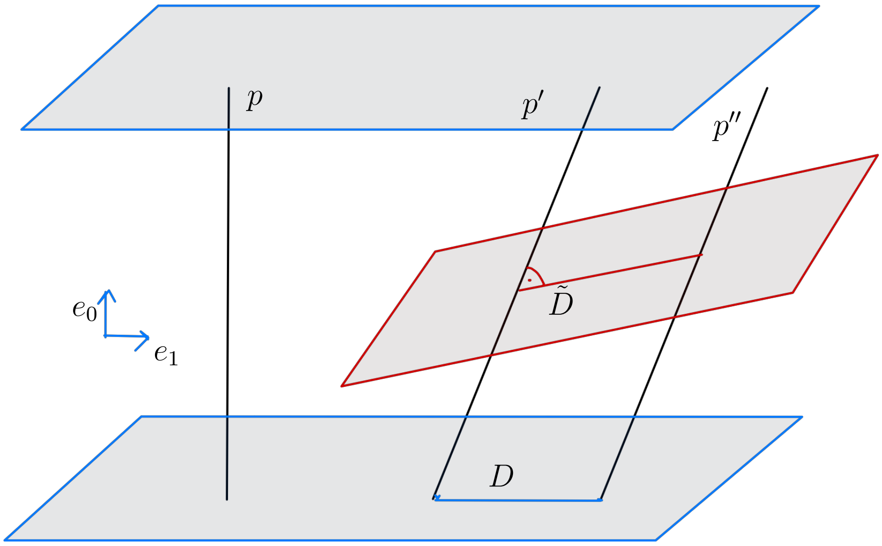

Figure 1.6: Distances dilatation.

Definition 6 (4-velocity)**.** Consider a timelike curve $p\colon [\tau_0, \tau_1] \to M$. We can always choose a parametrization such that the norm of tangent vectors is $= 1$. Suppose:

$$g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) \neq -1$$

Introduce new parameter $\mathrm{d}\tau' = \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$. τ' is just a proper time. If we take $p'(\tau') \overset{\text{def}}{=} p \circ \tau'(\tau)$ then, the 4-velocity is defined as:

$$u = \frac{\mathrm{d}p'}{\mathrm{d}\tau'} = \frac{\mathrm{d}p}{\mathrm{d}\tau} / \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}$$

It is easy to check, that

$$g(u, u) = -1$$

> **Definition 7** (4-acceleration)**.** Then, we can define the acceleration as:
>
> $$a \stackrel{\text{def}}{=} \frac{\mathrm{d}u}{\mathrm{d}\tau}, \quad \text{where} \quad g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) = -1$$

Now let's see some geometrical relation between u and a.

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g(u,u) = g\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}, u\right) + g\left(u, \frac{\mathrm{d}u}{\mathrm{d}\tau}\right) = 2g(a,u)$$
$$0 = g(a,u)$$

≻ Problem class 1

Problem 1 Timelike circle

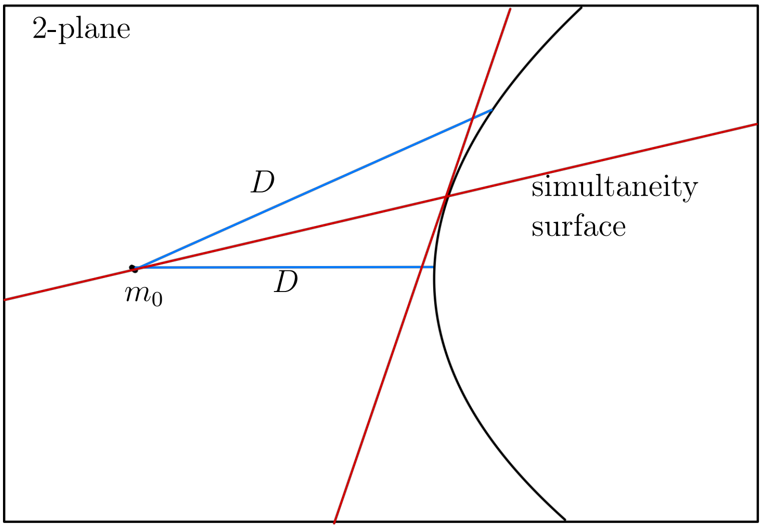

Figure 1.7: Constant proper distance curves.

We will show that the curve is a hyperbola. There is an isometry between any two tangent observers. The red one concludes there is some acceleration $a = \|\vec{a}\|$. Another one concludes (from symmetry) there is the same constant acceleration.

Let's consider a timelike curve $p(\tau) = x^0(\tau)e_0 + x^1(\tau)e_1$, parametrized by an arbitrary parameter τ. We want the whole curve to be timelike ($g(\dot{p}, \dot{p}) < 0$), so a vector $p(\tau)$ must be spacelike. We also require, that we have a constant proper distance from a given point $(0,0) = m_0 \in M$, so that:

$$g\big(p(\tau), p(\tau)\big) = D^2$$
$$-x^0(\tau)^2 + x^1(\tau)^2 = D^2$$

Now we see, it is the equation of a hyperbola. We can parametrize it with hyperbolic functions.

$$D^2\Big(-\sinh^2\tau + \cosh^2\tau\Big) = D^2$$

It means that our timelike circle can be expressed as:

$$p(\tau) = D(\sinh\tau e_0 + \cosh\tau e_1)$$

Now we can ask how to choose $\tau(\tau')$ so that τ' is proper time. We know that for a timelike curve parametrized with the proper time,

$$g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau'}, \frac{\mathrm{d}p}{\mathrm{d}\tau'}\right) = -1$$

No we calculate some derivatives.

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}\tau'} &= \frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} = \frac{\mathrm{d}\tau}{\mathrm{d}\tau'}D(\cosh\tau e_0 + \sinh\tau e_1) \\
-1 &= \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}\right)^2 D^2\left(\sinh^2\tau - \cosh^2\tau\right) = -D^2\left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}\right)^2 \\
\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} &= \frac{1}{D} \implies \tau(\tau') = \frac{\tau'}{D} + C
\end{aligned}$$

We can assume that $\tau(0) = 0$, so that $C = 0$. Then,

$$p(\tau') = D\left(\sinh\frac{\tau'}{D}e_0 + \cosh\frac{\tau'}{D}e_1\right)$$

From now on, we will use τ as τ', ie. $\tau' \mapsto \tau$.

$$p(\tau) = D\left(\sinh\frac{\tau}{D}e_0 + \cosh\frac{\tau}{D}e_1\right)$$

Now let's calculate covariant velocity u and covariant acceleration a:

$$\begin{aligned}
u &= \frac{\mathrm{d}p}{\mathrm{d}\tau} = \cosh\frac{\tau}{D}e_0 + \sinh\frac{\tau}{D}e_1 \\
a &= \frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{1}{D}\left(\sinh\frac{\tau}{D}e_0 + \cosh\frac{\tau}{D}e_1\right) = \frac{p(\tau)}{D^2} \\
\|a\|^2 = g(a,a) &= \frac{1}{D^4}g\big(p(\tau), p(\tau)\big) = \frac{1}{D^2} \\
\|a\| &= \frac{1}{D} \equiv \frac{c^2}{D}
\end{aligned}$$

> **Remark 4.** If we consider a timelike curve set in a two-plane by points which have same distance to the fixed point m_0, it corresponds to the constant covariant acceleration curve.

We can also calculate spatial velocity and acceleration along x^1 axis.

$$\begin{aligned}
v_x &= \frac{\mathrm{d}x^1}{\mathrm{d}x^0} = \frac{\mathrm{d}x}{\mathrm{d}t} \\
\mathrm{d}x &= \sinh\frac{\tau}{D}\,\mathrm{d}\tau \\
\mathrm{d}t &= \cosh\frac{\tau}{D}\,\mathrm{d}\tau \\
v_x &= \frac{\sinh\frac{\tau}{D}\,\mathrm{d}\tau}{\cosh\frac{\tau}{D}\,\mathrm{d}\tau} = \tanh\frac{\tau}{D} \xrightarrow{\tau\to\infty} 1
\end{aligned}$$

Of course we can express it in terms of $x^0 \stackrel{\text{def}}{=} t$.

$$
\begin{aligned}
t &= D \sinh \frac{\tau}{D} \\
t^2 + D^2 &= D^2 \cosh^2 \frac{\tau}{D} \\
1 + \frac{t^2}{D^2} &= \cosh^2 \frac{\tau}{D} \\
v_x &= \frac{t/D}{\sqrt{1 + t^2/D^2}} \xrightarrow{t \to \infty} 1
\end{aligned}
$$

It means, that any static observer will see the guy accelerating towards c. Let's calculate spatial acceleration.

$$
\begin{aligned}
a_x &= \frac{\mathrm{d}v_x}{\mathrm{d}x^0} \\
\mathrm{d}v_x &= \frac{1}{D} \frac{\cosh^2 \frac{\tau}{D} - \sinh^2 \frac{\tau}{D}}{\cosh^2 \frac{\tau}{D}} \mathrm{d}\tau = \frac{1}{D \cosh^2 \frac{\tau}{D}} \mathrm{d}\tau \\
a_x &= \frac{1}{D \cosh^3 \frac{\tau}{D}} = \frac{D^2}{(t^2 + D^2)^{3/2}}
\end{aligned}
$$

From the last equation it can be seen that relative to a static inertial observer, the guy will accelerate less and less rapidly as time flows.

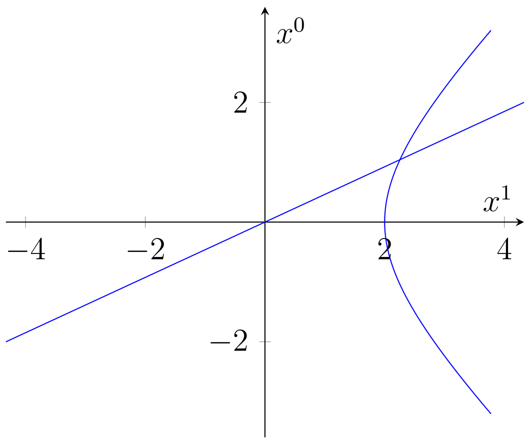

Figure 1.8: Example of constant acceleration curve with $D = 2$.

Now, let us see that simultaneity lines of our accelerating observers go through $(0, 0) \in M$. It will mean, that the proper distance between any such timelike circles would be a physical distance between them, therefore (from definition of these curves) the distance between any two constant covariant-accelerating observers would remain the same. Respectively to each other, they would stay at rest.

We can find the simultaneity line simply by searching for an orthogonal vector to the tangent vector of $p(\tau)$. We know that the tangent to any timelike curve is just its covariant velocity, and it was proven that the covariant acceleration is also orthogonal to u. Then, we know that:

$$
a = \frac{p(\tau)}{D^2}
$$

It is clear that for every τ, a is just scaled $p(\tau)$ so especially a line generated by the vector a goes through $(0, 0)$.

Lecture 2: Coordinate systems in the Minkowski spacetime

27 paź 2020

Remark 5 (Remarks about the previous lecture)**.** To sum up some things:

1. We have a general formula for lengths on curves, given by an integral. When the curve happens to be a line (can be represented by a vector $\vec{v} = p(\tau_1) - p(\tau_0)$), it simplifies to $l = \sqrt{\pm g(\vec{v}, \vec{v})}$.
2. Suppose $p \colon [\tau_0, \tau_1] \to M$ and it is parametrised in such a clever way that $g(\dot{p}, \dot{p}) = -1$. It is equivalent to τ being the proper time.
3. If (2), then $\mathrm{d}p/\mathrm{d}\tau = u$ is 4-velocity (covariant velocity) and $\mathrm{d}u/\mathrm{d}\tau = a$, $g(u, a) = 0$ is 4-acceleration (covariant acceleration).
4. In a temporary frame at rest, we can define $e_0 \overset{\text{def}}{=} \dot{p}$ at $p(\tau_0)$ and just make sure that e_1, e_2, e_3 complete the orthonormal basis. Then at τ_0, $a = a^1 e_1 + a^2 e_2 + a^3 e_3$, where $a^i = \mathrm{d}^2(x^i) \big/ \mathrm{d}(x^0)^2$.

Inertial coordinate systems

Let $m_0 \in M$ and at this point we choose an orthonormal basis (e_μ). If we have a vector from point m_0 to any other $m \in M$, we can describe it as $m - m_0 = x^0 e_0 + \cdots + x^3 e_3$. In this way, we have assigned coordinates $\left(x^0, x^1, x^3, x^4\right)$ to point $m \in M$.
We can view x^i as functions on M. $x^i \colon M \to \mathbb{R}$.

Definition 8 (Time orientation)**.** Time orientation means that we choose one side of the spacetime cone to point in the future direction, and the opposite side of the cone pointing past (timelike vectors point directions). It is also a covariant structure, not a convention! Future is the upper side of a cone!
It means that we always use e_0 such that it is properly oriented.

Definition 9 (Spacetime orientation)**.** We admit only transformations from one frame to another frame which have only positive determinants. Given (e_0, e_1, e_2, e_3) we allow only (e'_0, e'_1, e'_2, e'_3) such that $e'_\mu = L^\nu{}_\mu e_\nu$, $\det L > 0$.

Definition 10 (Boost)**.** We choose another orthonormal frame. Suppose that $e_0 = Ae'_0 + Be'_1$, $e_1 = Ce'_0 + De'_1$, $e_2 = e'_2$, $e_3 = e'_3$ (e_0, e'_0 are future oriented $\iff A > 0$). From orthonormality,

$$\begin{aligned} -A^2 + B^2 &= -1 \\ -C^2 + D^2 &= 1 \\ -AC + BD &= 0, \quad A > 0 \end{aligned}$$

We get a solution:

$$A = \cosh\alpha, \quad B = \sinh\alpha, \quad \alpha \in \mathbb{R}$$

$$\begin{cases} e_0 = \cosh\alpha e'_0 + \sinh\alpha e'_1 \\ e_1 = \sinh\alpha e'_0 + \cosh\alpha e'_1 \end{cases}$$

Others remain unchanged. On the other hand, rotation is trivial.

Of course we still describe the same vectors on M, so:

$$\begin{aligned} x^0 e_0 + \cdots + x^3 e_3 &= x'^0 e'_0 + \cdots + x'^3 e'_3 \\ x^0 \cosh\alpha + x^1 \sinh\alpha &= x'^0 \\ x^0 \sinh\alpha + x^1 \cosh\alpha &= x'^1 \end{aligned}$$

We can define β, by:

$$\begin{aligned} \cosh\alpha &\overset{\text{def}}{=} \frac{1}{\sqrt{1-\beta^2}} \\ \sinh\alpha &\overset{\text{def}}{=} \frac{\beta}{\sqrt{1-\beta^2}} \\ \beta &= \frac{v}{c} \end{aligned}$$

In this approach, the Lorentz transformation appears as the secondary transformation!

Vector as a differential operator

Consider a point m and a vector $\vec{v}$ at that point. Let $M \subset \mathcal{U}$, where $\mathcal{U}$ is an open set.

$$\begin{aligned} f : \mathcal{U} &\to \mathbb{R} \\ \left.\frac{\mathrm{d}f(m + s\vec{v})}{\mathrm{d}s}\right|_{s=0} &= \vec{v}f \end{aligned}$$

We can think about a dual vector. Let's define the exterior derivative of f (co-vector):

$$\begin{aligned} \mathrm{d}f\,\big|_m : \vec{v}\,\big|_m &\to \mathbb{R} \\ \vec{v} \lrcorner \mathrm{d}f &\overset{\text{def}}{=} \vec{v}f \end{aligned}$$

We can see that exterior derivative is a linear operator.

If we have vectors and co-vectors defined by coordinates $\left(x^0, x^1, x^2, x^3\right)$,

$$\begin{aligned} e_\mu(f) &= \left.\frac{\mathrm{d}f\left(m + se_\mu\right)}{\mathrm{d}s}\right|_{s=0} = \left.\frac{\mathrm{d}f\left(m + x^\mu e_\mu\right)}{\mathrm{d}x^\mu}\right|_{x^\mu=0} \\ &= \frac{\partial}{\partial x^\mu} f \end{aligned}$$

It means that

$$e_\mu \equiv \frac{\partial}{\partial x^\mu}$$

It means that we have a basis $\left(\partial_\mu\right)$ and co-basis $(\mathrm{d}x^\mu)$.

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \lrcorner \, \mathrm{d}x^\nu &= \frac{\partial}{\partial x^\mu}(x^\nu) = \delta^\nu{}_\mu \\ \vec{v} \lrcorner \, \mathrm{d}x^\nu &= \left(v^\mu \partial_\mu\right) \lrcorner \, \mathrm{d}x^\nu = v^\nu \end{aligned}$$

The metric tensor in terms of $(\mathrm{d}x^\mu)$

Let's define a tensor product as follows.

$$\begin{aligned} \mathrm{d}f \otimes \mathrm{d}h \, (\vec{v}, \vec{w}) &\overset{\text{def}}{=} \vec{v} \lrcorner \, \mathrm{d}f \cdot \vec{w} \lrcorner \, \mathrm{d}h \\ \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \, (\vec{v}, \vec{w}) &= \vec{v} \lrcorner \, \mathrm{d}x^\mu \cdot \vec{w} \lrcorner \, \mathrm{d}x^\nu = v^\mu w^\nu \end{aligned}$$

In particular,

$$\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \left(\partial_\alpha, \partial_\beta\right) = \delta^\mu{}_\alpha \delta^\nu{}_\beta$$

We want to do physics, not to write $\otimes$, so we define a notation for symmetrised tensor product:

$$\mathrm{d}f \, \mathrm{d}h \overset{\text{def}}{=} \frac{1}{2}(\mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f)$$

Especially,

$$\begin{aligned} \mathrm{d}f^2 &= \mathrm{d}f \, \mathrm{d}f = \mathrm{d}f \otimes \mathrm{d}f \\ \mathrm{d}x^0 \, \mathrm{d}x^1 \, (v, w) &= \frac{1}{2}\left(v^0 w^1 + v^1 w^0\right) \\ \mathrm{d}x^0 \, \mathrm{d}x^1 \, (\partial_0, \partial_1) &= \frac{1}{2} \end{aligned}$$

We can write the Minkowski metric tensor:

$$\begin{aligned} g &= -\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 \\ g(\partial_0, \partial_0) &= -\mathrm{d}x^0 \, \mathrm{d}x^0 \, (\partial_0, \partial_0) = -1 \end{aligned}$$

More generally, one could write

$$\begin{aligned} g(v, w) &= -\mathrm{d}x^0 \, \mathrm{d}x^0 \, (v, w) + \cdots + \mathrm{d}x^3 \, \mathrm{d}x^3 \, (v, w) \\ &= -v^0 w^0 + \cdots + v^3 w^3 \end{aligned}$$

what is completely consistent with definition of scalar product on Minkowski spacetime.

Lemma 1. If we consider two different, but orthonormal frames, from constructions above it follows that given $m \in M$ with $(e_0, \dots, e_3)$ and $m' \in M$ with (e'_μ)

$$-\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 = -\left(\mathrm{d}x'^0\right)^2 + \left(\mathrm{d}x'^1\right)^2 + \left(\mathrm{d}x'^2\right)^2 + \left(\mathrm{d}x'^3\right)^2$$

However, we can also consider arbitrary, non-inertial coordinates. By non-inertial, we mean that they don't transform via Lorentz transformation/metric tensor doesn't look like written above.
Let's denote such coordinates by (y^μ). They can be functions of M.

$$\begin{aligned}
y^0, y^1, y^2, y^3 &: \mathcal{U} \to \mathbb{R} \\
\left.\frac{\partial}{\partial y^0}\right|_m &= \left.\left(\frac{\partial x^0}{\partial y^0}\frac{\partial}{\partial x^0} + \dots + \frac{\partial x^3}{\partial y^0}\frac{\partial}{\partial x^3}\right)\right|_m \\
\frac{\partial}{\partial y^\mu} \lrcorner \, \mathrm{d}y^\nu &= \delta^\nu{}_\mu \\
\mathrm{d}x^\mu &= \frac{\partial x^\mu}{\partial y^\nu}\,\mathrm{d}y^\nu
\end{aligned}$$

Accelerated coordinates

We can introduce coordinates $\left(\rho, \tau, x^2, x^3\right)$, where $\rho = \sqrt{\left(x^1\right)^2 - \left(x^0\right)^2}$:

$$\begin{cases} x^0 = \rho \sinh \tau \\ x^1 = \rho \cosh \tau \\ x^2 = x^2 \\ x^3 = x^3 \end{cases}, \quad \rho > 0,\ \tau \in \mathbb{R}$$

These are called accelerated coordinates.

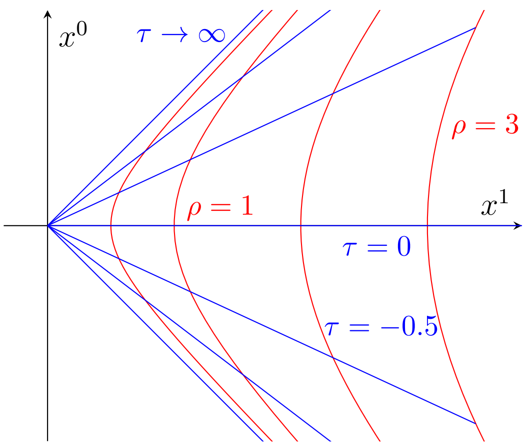

Figure 1.9: Accelerated coordinates with lines of constant t (blue) and constant ρ (red).

$$\begin{aligned}
\mathrm{d}x^0 &= \sinh \tau \,\mathrm{d}\rho + \rho \cosh \tau \,\mathrm{d}\tau \\
\mathrm{d}x^1 &= \cosh \tau \,\mathrm{d}\rho + \rho \sinh \tau \,\mathrm{d}\tau
\end{aligned}$$

Then,

$$g = -(\sinh\tau\,\mathrm{d}\rho + \rho\cosh\tau\,\mathrm{d}\tau)^2 + (\cosh\tau\,\mathrm{d}\rho + \rho\sinh\tau\,\mathrm{d}\tau)^2 + \cdots$$
$$= -\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2$$

What is the meaning of this new form of a metric tensor? We can use it as a dictionary to understand the properties of any frame.

Let's drop x^2, x^3. Given any $\rho_1 > 0$, it defines a line of constant ρ_1. Similarly for τ_1.

1. The coefficient 1 before $\mathrm{d}\rho^2$ means that the distance between any $\rho_2 - \rho_1$ for any τ is constant.
2. The term $0\,\mathrm{d}\tau\,\mathrm{d}\rho$ means that lines of constant ρ and τ are orthogonal to each other (constant τ lines are simultaneity lines).
3. Meaning of $-\rho^2$ before $\mathrm{d}\tau^2$ is that if observers of constant ρ_1 and ρ_2 measure time, they measure different periods: $\Delta T_1/\Delta T_2 = \rho_1/\rho_2$. Even though the distance is the same between them, their clocks work differently.

Accelerating observers point of view: they see themselves as straight lines, they remain at the same distances, their clocks show different times. It is possible to set $\tau = 0$ simultaneously for all of them (we mention that, because in some coordinates it is not possible) – there exists a surface orthogonal to every observer at once.

≻ Problem class 2 – Minkowski metric in non-flat coordinates

Problem 1 Uniformly accelerating coordinate system.

$$\phi\colon (\tau, \rho, y, z) \mapsto (\rho\sinh\tau, \rho\cosh\tau, y, z)$$

Now we calculate the metric tensor.

$$\begin{aligned}
\mathrm{d}x^0 &= \mathrm{d}(\rho\sinh\tau) = \sinh\tau\,\mathrm{d}\rho + \rho\cosh\tau\,\mathrm{d}\tau\\
\mathrm{d}x^1 &= \mathrm{d}(\rho\cosh\tau) = \cosh\tau\,\mathrm{d}\rho + \rho\sinh\tau\,\mathrm{d}\tau\\
\mathrm{d}x^2 &= \mathrm{d}y\\
\mathrm{d}x^3 &= \mathrm{d}z
\end{aligned}$$

Now we substitute these,

$$-\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 = -\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

What is the domain? We want for every variable to be well defined and its gradient $\neq 0$.

$$\rho = \sqrt{(x^1)^2 - (x^0)^2}, \quad \mathrm{d}\rho = \frac{x^1\,\mathrm{d}x^1 - x^0\,\mathrm{d}x^0}{\sqrt{(x^1 - x^0)(x^1 + x^0)}}$$

We have a problem if $x^1 = x^0$ or $x^1 = -x^0$. So we cannot cross these two lines. It is sometimes called a Rindler wedge. Without crossing any singularity, we can use these coordinates only in one of the wedges. Moreover, observers never can reach $\rho = 0$, because it would result in an infinite acceleration.

Right wedge observer and left wedge observer can never see each other (signals will not come), so $x^1 = x^0$ is their horizon! The cannot see through it.

$$\begin{aligned}
p(\tau) &= (\tau, \rho_1) \\
\dot{p}(\tau) &= \frac{\mathrm{d}\tau}{\mathrm{d}\tau}\partial_\tau + \frac{\mathrm{d}\rho_1}{\mathrm{d}\tau}\partial_\rho = \partial_\tau \\
\Delta T_1 &= \int_0^{\tau_0} \sqrt{-g(\dot{p},\dot{p})}\,\mathrm{d}\tau = \int_0^{\tau_0} \sqrt{-g(\partial_\tau,\partial_\tau)}\,\mathrm{d}\tau \\
&= \int_0^{\tau_0} \rho_1\,\mathrm{d}\tau = \rho_1\tau_0
\end{aligned}$$

Same thing happens for constant ρ_2 observer.

$$\begin{aligned}
\Delta T_2 &= \rho_2\tau_0 \\
\frac{\Delta T_1}{\Delta T_2} &= \frac{\rho_1}{\rho_2}
\end{aligned}$$

Just to be super clear, we will calculate metric tensor acting on two vectors.

$$g(\partial_\tau,\partial_\tau) = \Big(-\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2\Big)(\partial_\tau,\partial_\tau)$$

What it really means, is:

$$\big(-\rho^2\,\mathrm{d}\tau\otimes\mathrm{d}\tau + \mathrm{d}\rho\otimes\mathrm{d}\rho\big)(\partial_\tau,\partial_\tau) = -\rho^2$$

The rule was:

$$\begin{aligned}
\mathrm{d}\tau\otimes\mathrm{d}\tau\,(\partial_\tau,\partial_\tau) &= \partial_\tau\lrcorner\,\mathrm{d}\tau\cdot\partial_\tau\lrcorner\,\mathrm{d}\tau \\
\partial_\tau\lrcorner\,\mathrm{d}\tau &= \partial_\tau\tau = 1
\end{aligned}$$

Problem 2 Rotating frame.
Let's consider cylindrical coordinates.

$$\psi\colon \big(x^0, r, \phi', x^3\big) \mapsto \big(x^0, r\cos\phi', r\sin\phi', x^3\big)$$

We will calculate the metric tensor:

$$\begin{aligned}
g &= -\big(\mathrm{d}x^0\big)^2 + \big(\cos\phi'\,\mathrm{d}r - r\sin\phi'\,\mathrm{d}\phi'\big)^2 + \big(\sin\phi'\,\mathrm{d}r + r\cos\phi'\,\mathrm{d}\phi'\big)^2 + \big(\mathrm{d}x^3\big)^2 = \\
&= -\big(\mathrm{d}x^0\big)^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi'^2 + \big(\mathrm{d}x^3\big)^2
\end{aligned}$$

Attention! Metric tensor is symmetric so we didn't bother to be careful while evaluating and cancelling out similar coefficients before $\mathrm{d}x^i\,\mathrm{d}x^i$ and $\mathrm{d}x^j\,\mathrm{d}x^i$. If it was a non-symmetric tensor we would be more careful and use tensor products.

Now we can introduce another substitution: $\phi' = \phi + \omega x^0$.

$$\begin{aligned}
g &= -\big(\mathrm{d}x^0\big)^2 + \mathrm{d}r^2 + r^2\big(\mathrm{d}\phi + \omega\,\mathrm{d}x^0\big)^2 + \big(\mathrm{d}x^3\big)^2 \\
&= -\big(1 - r^2\omega^2\big)\left[\mathrm{d}x^0 - \frac{r^2\omega}{1-r^2\omega^2}\,\mathrm{d}\phi\right]^2 + \mathrm{d}r^2 + \frac{r^2}{1-r^2\omega^2}\,\mathrm{d}\phi^2 + \big(\mathrm{d}x^3\big)^2
\end{aligned}$$

Let us consider a circle $r = \text{const.}$ and $x^3 = \text{const.}$ We have two observers at ϕ_0, ϕ_1. They are in their frames at rest. How is measured the proper distance?

It can be checked that the vector $q = \partial_\phi + \dfrac{r^2\omega}{1 - r^2\omega^2}\partial_t$ is orthogonal to our constant ϕ, r, x^3 curves. This vector is of course tangent to the proper distance curve.
It is easy to check that this vector gives 0 when contracted with the first mixed term of our metric g. From the fully spatial term we get

$$g(q, q) = \frac{r^2}{r^2\omega^2 - 1}$$

We can calculate the circumference of such rotating circe,

$$L = \int_0^{2\pi} \frac{r}{\sqrt{1 - r^2\omega^2}}\,\mathrm{d}\phi = \frac{2\pi r}{\sqrt{1 - r^2\omega^2}} > 2\pi r$$

It means that in such rotating geometry, co-moving observers would measure that circles have stretched.

Lecture 3: Rotating observers in Minkowski spacetime

03 lis 2020 Let's carefully revisit the rotating coordinates. We introduce $t = x^0/c$ and ω – the angular velocity. Then, our metric works for every:

$$0 < r < \frac{c}{\omega}, \quad \phi \in [0, 2\pi), \quad z \in \mathbb{R}$$

Let's define a rotating observer:

$$p(t) = p^i(t)e_i = cte_0 + r\cos(\phi + \omega t)e_1 + r\sin(\phi + \omega t)e_2 + ze_3$$

We set z to be constant, ie. our observer moves in the xy plane.

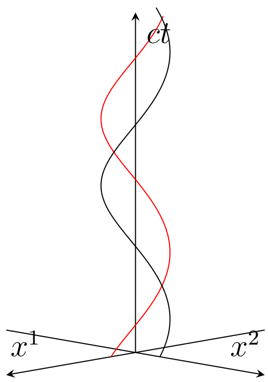

Figure 1.10: Two observers separated by $\Delta\phi = \pi/2$.

What is the observers' sense of time, distance and rotation?
Firstly, the proper time is given by:

$$\begin{aligned} c\,\mathrm{d}\tau &= \sqrt{-g\big(p'(t), p'(t)\big)}\,\mathrm{d}t \\ p'(t) &= ce_0 - r\omega\sin(\phi + \omega t)e_1 + r\omega\cos(\phi + \omega t)e_2 \\ &= ce_0 + \omega p^1 e_2 - \omega p^2 e_1 \end{aligned}$$

Rotating coordinates

We shall also introduce the rotating coordinates (as previously):

$$\psi\colon \begin{bmatrix} t \\ r \\ \phi \\ z \end{bmatrix} \mapsto \begin{bmatrix} ct \\ r\cos(\phi+\omega t) \\ r\sin(\phi+\omega t) \\ z \end{bmatrix}$$

$$g = -\left(c^2 - \omega^2 r^2\right) \mathrm{d}t^2 + 2r^2\omega\, \mathrm{d}t\, \mathrm{d}\phi + \mathrm{d}r^2 + r^2\, \mathrm{d}\phi^2 + \mathrm{d}z^2$$

Now we consider a curve $p(t) = \left(p^t(t), p^r(t), p^\phi(t), p^z(t)\right) = (t, r, \phi, z)$, where by this notion I understand the coordinates in rotating system. To be more precise I should probably write something like $p(t) = \psi(t, r, \phi, z)$. Or I could say that in original Minkowski space we have a vector $v = (x^0, x^1, x^2, x^3)$ so that $(t, r, \phi, z) = \psi^* v$.
Anyway, we chose a curve for which r, ϕ, z are constants. Then,

$$\left.\frac{\partial}{\partial t}\right|_{r,\phi,z=\text{cst.}} = \tilde{\partial}_t, \qquad \left.\frac{\partial}{\partial z}\right|_{r,\phi,t} = \tilde{\partial}_z$$
$$\left.\frac{\partial}{\partial r}\right|_{t,\phi,z} = \tilde{\partial}_r, \qquad \left.\frac{\partial}{\partial \phi}\right|_{t,r,z} = \tilde{\partial}_\phi$$

Proper time We can calculate the proper time by,

$$c\,\mathrm{d}\tau = \sqrt{-g(\tilde{\partial}_t, \tilde{\partial}_t)}\,\mathrm{d}t = \sqrt{c^2 - \omega^2 r^2}\,\mathrm{d}t$$

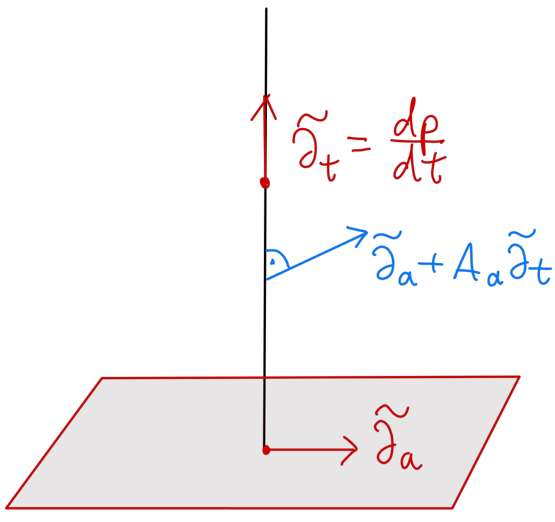

Figure 1.11: Simultaneity curves

Simultaneity curves Let a denote r, ϕ or z. Then we define:

$$\hat{\partial}_a \overset{\text{def}}{=} \tilde{\partial}_a + A_a \tilde{\partial}_t$$

We want $\hat{\partial}_a$ to be a tangent to simultaneity line, so

$$g(\tilde{\partial}_t, \hat{\partial}_a) = 0$$

$\hat{\partial}_a$ is essentially a base vector in a direction from the moving observer's perspective. It gives us the condition for A_a. We shall regroup our metric tensor.

$$\begin{aligned} g &= -\left(c^2 - \omega^2 r^2\right)\left(\mathrm{d}t - \frac{r^2\omega}{c^2 - r^2\omega^2}\,\mathrm{d}\phi\right)^2 + \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2 \\ &= -N^2\Big(\mathrm{d}t - \big[A_r\,\mathrm{d}r + A_\phi\,\mathrm{d}\phi + A_z\,\mathrm{d}z\big]\Big) + \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2 \end{aligned}$$

where $A_r = A_z = 0$. We will check, that these introduced A_i correspond to A_a.

$$\begin{aligned} \tilde{\partial}_a \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= -A_a \\ A_a \tilde{\partial}_t \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= +A_a \\ \implies \hat{\partial}_a \lrcorner \Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big) &= 0 \end{aligned}$$

Hence,

$$g(\tilde{\partial}_t, \tilde{\partial}_a + A_a \tilde{\partial}_t) = 0$$

because $\tilde{\partial}_t$ will give 0 when contracted with any 1-form other than $\mathrm{d}t$. In such way, we proved that there are several „basic" simultaneity curves:

$$\begin{aligned} \hat{\partial}_r &= \tilde{\partial}_r \\ \hat{\partial}_\phi &= \tilde{\partial}_\phi + \frac{r^2\omega}{c^2 - r^2\omega^2}\tilde{\partial}_t \\ \hat{\partial}_z &= \tilde{\partial}_z \end{aligned}$$

Now let's define:

$$g_{\text{obs}} = \mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

We can see, that

$$g(\hat{\partial}_a, \hat{\partial}_b) = g_{\text{obs}}(\tilde{\partial}_a, \tilde{\partial}_b)$$

Conclusion

$$g = -\underbrace{\left(c^2 - \omega^2 r^2\right)}_{\substack{\text{proper time}\\\text{part}}}\Big(\mathrm{d}t - A_b\,\mathrm{d}y^b\Big)^2 + \underbrace{\mathrm{d}r^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\,\mathrm{d}\phi^2 + \mathrm{d}z^2}_{g_{\text{obs}}}$$

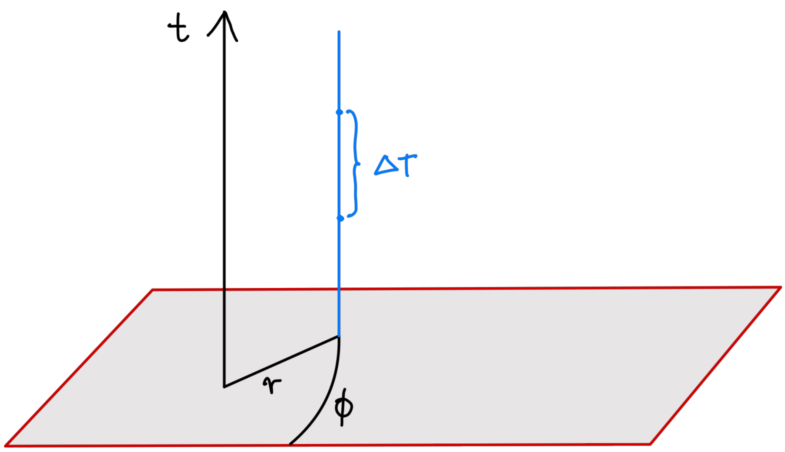

Figure 1.12: Observers proper time ΔT.

In these coordinates, observers moving with the angular velocity ω are „straight lines". Then the proper time is just simply

$$c\Delta T = N\Delta t = \sqrt{c^2 - \omega^2 r^2}\Delta t$$

where Δt is inertial observer's time (coordinate time).

The moving observers' 3D space is the 3D ball of points:

$$\left\{(r, \phi, z) \colon 0 < r < \frac{c}{\omega},\ \phi \in [0, 2\pi),\ -\infty < z < \infty\right\}$$

which is endowed with the g_{obs} part of the metric tensor.

For every spatial curve $[s_0, s_1] \ni s \mapsto \big(r(s), \phi(s), z(s)\big) = q(s)$, it means for $t = \mathrm{const.}$ – a curve observed by an inertial observer in the middle point, the proper (physical) length is:

$$\Delta L = \int_{s_0}^{s_1} \left[g_{\mathrm{obs}}(\dot{q}, \dot{q})\right]^{1/2} \mathrm{d}s = \int_{s_0}^{s_1} \left[\dot{r}^2 + \frac{r^2}{1 - \frac{\omega^2}{c^2} r^2}\dot{\phi}^2 + \dot{z}^2\right]^{1/2} \mathrm{d}s$$

Note that the statement above is true, only because $g(\hat{\partial}_a, \hat{\partial}_a) = g_{\mathrm{obs}}(\tilde{\partial}_a, \tilde{\partial}_a)$.

Now we consider any loop $[s_0, s_1] \ni s \mapsto \big(r(s), \phi(s), z(s)\big) = q(s)$, such that $q(s_0) = q(s_1)$. We want to check what is the discrepancy according to the inertial time. What we need is a curve $[s_0, s_1] \mapsto \hat{q}(s)$ contained in the worldsheet of the curve q that is such that:

$$\hat{q}^t(s) = ?,\ \hat{q}^r(s) = r(s),\ \hat{q}^\phi(s) = \phi(s),\ \hat{q}^z(s) = z(s)$$

and also orthogonal to the world line of every rotating observer. First we can easily find the tangent vector:

$$\begin{aligned}
\frac{\mathrm{d}\hat{q}^t}{\mathrm{d}s}\tilde{\partial}_t + \dot{r}\tilde{\partial}_r + \dot{\phi}\tilde{\partial}_\phi + \dot{z}\tilde{\partial}_z = \frac{\mathrm{d}\hat{q}}{\mathrm{d}s} &= \dot{r}\hat{\partial}_r + \dot{\phi}\hat{\partial}_\phi + \dot{z}\hat{\partial}_z \\
&= \dot{r}\tilde{\partial}_r + \dot{\phi}\Big(\tilde{\partial}_\phi + A_\phi\tilde{\partial}_t\Big) + \dot{z}\tilde{\partial}_z \\
&= \dot{\phi}A_\phi\tilde{\partial}_t + \dot{r}\tilde{\partial}_r + \dot{\phi}\tilde{\partial}_\phi + \dot{z}\tilde{\partial}_z
\end{aligned}$$

From this we can see, that

$$\frac{\mathrm{d}\hat{q}^t}{\mathrm{d}s} = \frac{\mathrm{d}\phi}{\mathrm{d}s} A_\phi$$

If we set $\hat{q}^t(s_0) = 0$, then

$$t(s) = \hat{q}^t(s) = \int_{s_0}^{s} A_\phi(s) \frac{\mathrm{d}\phi}{\mathrm{d}s}\,\mathrm{d}s = \int_{\phi(s_0)}^{\phi(s)} A_\phi \,\mathrm{d}\phi$$

Likewise, total discrepancy according to the inertial time would result from integrating along the whole loop, ie.

$$\left(\substack{\text{total}\\ \text{discrepancy}}\right) = \hat{q}^t(s_1) = \int_{\phi(s_0)}^{\phi(s_1)} A_\phi \,\mathrm{d}\phi = \int_{\phi(s_0)}^{\phi(s_0)+2\pi} A_\phi \,\mathrm{d}\phi$$

It can be also expressed as:

$$= \oint_{q(s)} A_a \,\mathrm{d}y^a$$

Non-zero total discrepancy over any closed curve leads to discontinuous notion of time. Rotating observers are unable to synchronize their clocks, because there does not exist a curve orthogonal to every observer on a ring at once – at any given coordinate time t_0.

≻ Problem class 3

Problem 1 Let us consider coordinates, where one of the coordinates is:

$$\rho = \sqrt{(x^0)^2 - (x^i)^2}$$

where $\left|x^0\right| > \left|x^i\right|$. Then, curves of constant ρ are hyperboloids. The whole transformation would be as follows:

$$\begin{cases} x^0 = \rho \cosh\psi \\ x^1 = \rho \sinh\psi \sin\theta \cos\phi \\ x^2 = \rho \sinh\psi \sin\theta \sin\phi \\ x^3 = \rho \sinh\psi \cos\theta \end{cases}$$

Now we shall rewrite the metric tensor.

$$\begin{aligned}
\mathrm{d}x^0 &= \cosh\psi \,\mathrm{d}\rho + \rho \sinh\psi \,\mathrm{d}\psi \\
\mathrm{d}x^1 &= \sinh\psi \sin\theta \cos\phi \,\mathrm{d}\rho + \rho\cosh\psi \sin\theta\cos\phi \,\mathrm{d}\psi + \rho \sinh\psi \cos\theta\cos\phi \,\mathrm{d}\theta \\
&\quad - \rho \sinh\psi \sin\theta \sin\phi \,\mathrm{d}\phi \\
\mathrm{d}x^2 &= \sinh\psi \sin\theta \sin\phi \,\mathrm{d}\rho + \rho\cosh\psi \sin\theta\sin\phi \,\mathrm{d}\psi + \rho \sinh\psi \cos\theta\sin\phi \,\mathrm{d}\theta \\
&\quad + \rho \sinh\psi \sin\theta \cos\phi \,\mathrm{d}\phi \\
\mathrm{d}x^3 &= \sinh\psi \cos\theta \,\mathrm{d}\rho + \rho\cosh\psi\cos\theta \,\mathrm{d}\psi - \rho\sinh\psi\sin\theta
\end{aligned}$$

Now it is even more tedious work to combine them into a metric tensor.

$$g = -\,\mathrm{d}\rho^2 + \rho^2\Big[\mathrm{d}\psi^2 + \sinh^2\psi \,\mathrm{d}\theta^2 + \sinh^2\psi \sin^2\theta \,\mathrm{d}\phi^2\Big]$$

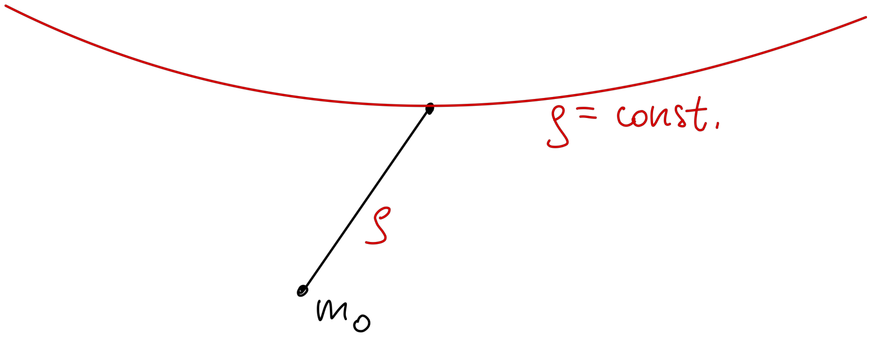

Figure 1.13: Curve of constant ρ.

All of the mixed terms get cancelled.
Let us consider an observer with constant ψ, θ, ϕ. Then we can parametrize any curve by the parameter s and say that:

$$\begin{aligned} p^\rho(s) &= s \\ p^\psi, p^\theta, p^\phi &= \text{const}(s). \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= \dot\rho\partial_\rho + \dot\psi\partial_\psi + \dot\theta\partial_\theta + \dot\phi\partial_\phi = \partial_\rho \end{aligned}$$

In these coordinates spacetime looks like it was expanding, because of the term ρ^2 before the spatial geometry term.

Conformal compactification

Let us consider the standard spherical Minkowski metric,

$$g = -(\mathrm{d}x^0)^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

We introduce two new coordinates η, χ given by,

$$\begin{aligned} x^0 &= \frac{R_0}{2}\left[\tan\frac{\eta+\chi}{2} + \tan\frac{\eta-\chi}{2}\right] \\ r &= \frac{R_0}{2}\left[\tan\frac{\eta+\chi}{2} - \tan\frac{\eta-\chi}{2}\right] \\ &-\pi < \eta + \chi < \pi \\ &-\pi < \eta - \chi < \pi \end{aligned}$$

Now we want to rewrite the metric tensor.

$$\begin{aligned} \mathrm{d}x^0 &= \frac{R_0}{4}\left[\left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} + \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\eta + \left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} - \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\chi\right] \\ \mathrm{d}r &= \frac{R_0}{4}\left[\left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} - \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\eta + \left(\frac{1}{\cos^2\frac{\eta+\chi}{2}} + \frac{1}{\cos^2\frac{\eta-\chi}{2}}\right)\mathrm{d}\chi\right] \end{aligned}$$

After much trigonometry,

$$g = \frac{R_0^2}{\left(2\cos\frac{\eta+\chi}{2}\cos\frac{\eta-\chi}{2}\right)^2}\Big(-\,\mathrm{d}\eta^2 + \mathrm{d}r^2 + \mathrm{d}\chi^2 + \sin\chi\big(\mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi\big)^2\Big)$$

This is just a simple product of $\mathbb{R}\times S_3$ with some conformal factor.
It is called a compactification, because we managed to fit the whole infinite spacetime into a compact region – that metric in braces. It describes a compact region, because η, χ as well as θ, ϕ are bounded. Of course the whole metric still blows up somewhere, because of the conformal factor.

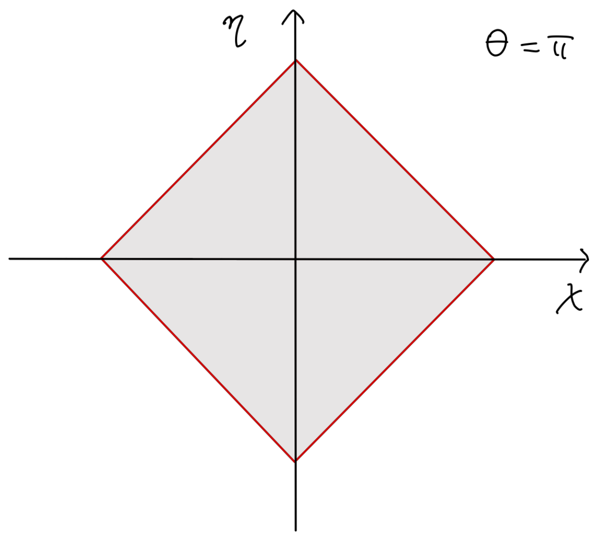

Figure 1.14: A slice of metric region with $\theta = \pi$.

Chapter 2

General Relativity framework

Lecture 4: Coordinate covariant differential calculus in $\mathbb{R}^n$

10 lis 2020

Some algebra

Vector spaces We consider V – a finite dimensional vector space, and its basis $(e_1, \ldots, e_n) \subset V$. Then $V \ni v = v^i e_i$. Let's consider the basis transformation,

$$e'_i = A^j{}_i e_j \implies v'^i = (A^{-1})^i{}_j v^j$$

Then, let V^* be the dual space to V, where $V^* \ni w \colon V \to \mathbb{R}$ (or $\mathbb{C}$).

$$w \colon V \ni v \mapsto v \lrcorner w = v^i w_i$$

In V^* there is a basis $(e^1, \ldots, e^n)$ dual to the basis of V.

$$\begin{aligned} e_i \lrcorner e^j &= \delta_i^{\ j} \\ v \lrcorner w &= v^i e_i \lrcorner w_j e^j = v^i w_j e_i \lrcorner e^j \\ &= v^i w_j \delta_i^{\ j} \end{aligned}$$

Tensor product First, we define a tensor product of two elements. Given $w, w' \in V^*$,

$$\begin{aligned} w \otimes w' \colon V \times V &\to \mathbb{R} \\ (v, v') &\mapsto v \lrcorner w \cdot v' \lrcorner w' \end{aligned}$$

Now, the tensor product of the dual space is by definition a span:

$$\begin{aligned} V^* \otimes V^* &= \operatorname{span}\bigl(w \otimes w' \colon w, w' \in V^*\bigr) \\ &= \Bigl\{ K_{ij} e^i \otimes e^j \colon K_{ij} \in \mathbb{R} \Bigr\} \end{aligned}$$

In other words, we can see the tensor product as functions of such space into reals:

$$V^* \otimes V^* \ni K, \quad K \colon V \times V \to \mathbb{R}$$

Now we want to generalize,

$$V^* \otimes \cdots \otimes V^* = \{K \colon V \times \cdots \times V \to \mathbb{R}, \text{ multilinear}\}$$

for K that are linear with respect to each slot.

$$e^{i_1} \otimes \cdots \otimes e^{i_k} \colon (v_1, \ldots, v_k) \mapsto v_1^{i_1} \cdots v_k^{i_k}$$
$$V^* \otimes \cdots \otimes V^* = \left\{ K_{i_1 \ldots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} \colon K_{i_1 \ldots i_k} \in \mathbb{R} \right\}$$

Analogically, we can define the tensor product of V,

$$V \otimes \cdots \otimes V = \{ L \colon V^* \times \cdots \times V^* \to \mathbb{R},\ \text{multilinear} \}$$

Now, if $v_i \in V$ and $w_j \in V^*$,

$$v_1 \otimes \cdots \otimes v_k \colon (w_1, \ldots, w_k) \mapsto (v_1 \lrcorner w_1) \cdots (v_k \lrcorner w_k)$$
$$V \otimes \cdots \otimes V = \left\{ L^{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \colon L^{i_1 \ldots i_k} \in \mathbb{R} \right\}$$

Now we will combine these two types of a tensor product into one.

Definition 11. A general tensor product is defined as:

$$\underbrace{V \otimes \cdots \otimes V}_{l} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} = \left\{ T \colon V^{*l} \times V^k \to \mathbb{R},\ \text{multilinear} \right\}$$
$$= \left\{ T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l} \right\}$$

If we know what we mean, then we can write it in shorter notation, where vectors are X^i, covectors are Y_j and other tensors are $T^{i_1 \ldots}{}_{j_1 \ldots}$.

Wedge product For just two dual vectors $w, w' \in V^*$,

$$w \wedge w' \overset{\text{def}}{=} w \otimes w' - w' \otimes w$$

In general,

Definition 12. For $V^* \ni w^1, \ldots, w^k$

$$w^1 \wedge \cdots \wedge w^k \overset{\text{def}}{=} \sum_\sigma (-1)^{\operatorname{sgn} \sigma} w^{\sigma(1)} \otimes \cdots \otimes w^{\sigma(k)}$$

where we sum along all permutations.

We can consider a wedge product of spaces.

$$\underbrace{V^* \wedge \cdots \wedge V^*}_{k} = \operatorname{span}\left(w^1 \wedge \cdots \wedge w^k \colon w^i \in V^* \right)$$
$$= \left\{ \frac{1}{k!} W_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \right\}$$

A basis in this wedge space is:

$$\left\{ e^{i_1} \wedge \cdots \wedge e^{i_k} \right\} - \ \text{all possible, non-vanishing}$$

Wedge product is alternating.

$$e^{i_1} \wedge \cdots \wedge e^j \wedge \cdots \wedge e^m \wedge \cdots \wedge e^{i_k} = -e^{i_1} \wedge \cdots \wedge e^m \wedge \cdots \wedge e^j \wedge \cdots \wedge e^{i_k}$$

Let's consider an element, where $w_{ij} = -w_{ji}$:

$$\begin{aligned} V^* \wedge V^* = \left\{ \frac{1}{2} w_{ij} e^i \wedge e^j \right\} &= \left\{ \frac{1}{2} w_{ij} \Big(e^i \otimes e^j - e^j \otimes e^i \Big) \right\} \\ &= \left\{ w_{ij} e^i \otimes e^j \right\} \end{aligned}$$

Also, for any totally antisymmetric $w_{i_1 \dots i_k}$

$$\frac{1}{k!} w_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} = w_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$$

Example Suppose $\dim V = 3$. Basis of $V^* \wedge V^*$ is $\left\{ e^1 \wedge e^2,\, e^2 \wedge e^3,\, e^3 \wedge e^1 \right\}$. Basis of $V^* \wedge V^* \wedge V^*$ is $\left\{ e^1 \wedge e^2 \wedge e^3 \right\}$. Basis of any more wedged spaces of $\dim V = 3$ have the basis $\{0\}$.

Tangent and cotangent vectors to $\mathbb{R}^n$

Definition 13. X is called a vector tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$ if:

$$X \colon C^\infty(\mathbb{R}) \to \mathbb{R}$$

such that it is (i) linear and (ii) satisfies the local Leibniz identity:
$X(fh) = f(x)X(h) + h(x)X(f)$.

Lemma 2. For every vector X tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$ there are $X^1, \dots, X^n \in \mathbb{R}$ such that $\forall f \in C^\infty(\mathbb{R})$, $X(f) = X^i \, \partial_i f \,\big|_x$.

Definition 14 (Tangent space)**.** $\mathrm{T}_x\mathbb{R}^n$ is the space of vectors tangent to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$. $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ set a basis in $\mathrm{T}_x\mathbb{R}^n$.

Definition 15. A covector at $x \in \mathbb{R}^n$ is a linear map

$$\begin{aligned} w \colon \mathrm{T}_x\mathbb{R}^n &\to \mathbb{R} \\ v^i \frac{\partial}{\partial x^i} &\mapsto w_i v^i \in \mathbb{R} \end{aligned}$$

where $(e^1, \dots, e^n)$ is a basis in $\mathrm{T}_x^*\mathbb{R}^n$, dual to $(\partial_1, \dots, \partial_n)$ and $w = w_i e^i$.

Lemma 3. Given $f \in C^\infty(\mathbb{R})$, $\mathrm{d}f$ is a map

$$\begin{gathered}\mathrm{d}f : \mathrm{T}_x\mathbb{R}^n \to \mathbb{R} \\ \forall v \in \mathrm{T}_x\mathbb{R}^n, \quad v \lrcorner \mathrm{d}f = v(f)\end{gathered}$$

In this sense, $\mathrm{d}f \in \mathrm{T}^*_x\mathbb{R}^n$.

In particular, consider $f = (x^1, \ldots, x^n)$, where $x^i : \mathbb{R}^n \to \mathbb{R}$.

$$\frac{\partial}{\partial x^j} \lrcorner \mathrm{d}x^i = \frac{\partial x^i}{\partial x^j} = \delta^i{}_j$$

hence $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ is a basis in $\mathrm{T}^*_x\mathbb{R}^n$.

Example Interpretation of a tangent vector at $x \in \mathbb{R}^n$. Consider a curve $\gamma : [s_0, s_2] \ni s \mapsto \big(x^1(s), \ldots, x^n(s)\big) \in \mathbb{R}^n$, where $\gamma(s_1) = x \in \mathbb{R}^n$. Then,

$$v\Big|_x = \begin{pmatrix} \dfrac{\mathrm{d}x^1}{\mathrm{d}s} \\ \vdots \\ \dfrac{\mathrm{d}x^n}{\mathrm{d}s} \end{pmatrix}\Bigg|_{s_1}, \qquad \frac{\mathrm{d}}{\mathrm{d}s} f\big(x^1(s_1), \ldots, x^n(s_1)\big) = \frac{\mathrm{d}x^i}{\mathrm{d}s}\bigg|_{s_1} \frac{\partial}{\partial x^i} f\big(x(s_1)\big)$$

Vector fields on $\mathbb{R}^n$

Definition 16. If for every $\mathbb{R}^n \ni x \mapsto V_x \in \mathrm{T}_x\mathbb{R}^n$, $V_x = V^i_x\,\partial_i$, where $V^i_x \in \mathbb{C}^\infty(\mathbb{R}^n)$, we call it a vector field.

Lemma 4. Suppose V and W are vector fields on $\mathbb{R}^n$. Then, the commutator $[V, W] : C^\infty(\mathbb{R}^n) \ni f \mapsto V\big(W(f)\big) - W\big(V(f)\big)$ is also a vector field.

It also means, that we can define a vector field as an operation, not via coordinates.

≻ Problem class 4

Problem 1 Consider $\mathbb{R}^n$, a point $x_0 \in \mathbb{R}^n$, a basis $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ in $\mathrm{T}_{x_0}\mathbb{R}^n$ and a dual basis $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ in $\mathrm{T}^*_{x_0}\mathbb{R}^n$. Consider a new coordinate system $(y^1, \ldots, y^n)$ and derive transformation between $(\partial/\partial y^i)$ and $(\partial/\partial x^i)$, and the same for dual basis.

Basis vectors are just differential operators on $\mathbb{R}^n$, so we can use chain rule/exterior derivative,

$$\begin{aligned}\frac{\partial}{\partial y^i} &= \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j} \\ \mathrm{d}y^i &= \frac{\partial y^i}{\partial x^j}\,\mathrm{d}x^j\end{aligned}$$

Now, let's consider the transformation of a vector $v = v^i\,\partial_{x^i}$

$$v = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} = v'^j \frac{\partial}{\partial y^j}$$

And the same transformation for any covector:

$$\omega = \omega_i \,\mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial y^j} \,\mathrm{d}y^j = \omega'_j \,\mathrm{d}y^j$$

Problem 2 Given $V = V^i\,\partial_i$, $W = W^i\,\partial_i$, calculate the commutator $[V, W]^i\,\partial_i$.

Let f be an arbitrary function, then

$$\begin{aligned}
[V, W](f) &= V\big(W(f)\big) - W\big(V(f)\big) \\
&= V^i \frac{\partial}{\partial x^i}\left(W^j \frac{\partial f}{\partial x^j}\right) - W^i \frac{\partial}{\partial x^i}\left(V^j \frac{\partial f}{\partial x^j}\right)
\end{aligned}$$

Now we use Leibniz rule,

$$= V^i \left[\frac{\partial W^j}{\partial x^i}\frac{\partial f}{\partial x^j} + W^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right] - W^i \left[\frac{\partial V^j}{\partial x^i}\frac{\partial f}{\partial x^j} + V^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right]$$

We can notice, that 2nd order derivatives cancel out, because of the mixed partials equality.

$$\begin{aligned}
&= V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j} - W^i V^j \frac{\partial^2 f}{\partial x^i \partial x^j} + V^i \frac{\partial W^j}{\partial x^i}\frac{\partial f}{\partial x^j} + W^i \frac{\partial V^j}{\partial x^i}\frac{\partial f}{\partial x^j} \\
&= \left[V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}\right]\frac{\partial f}{\partial x^j} = [V, W]^j \frac{\partial}{\partial x^j} f
\end{aligned}$$

We have proven, that this is still a vector field, and found its components as well.

Problem 3 Show, that $[V, fW] = V(f)W + f[V, W]$.

$$\begin{aligned}
[V, fW](h) &= V\big(fW(h)\big) - fW\big(V(h)\big) \\
&= V(f)W(h) + V\big(W(h)\big)f - fW\big(V(h)\big) \\
&= V(f)W(h) + f[V, W](h) \\
&= \big(V(f)W + f[V, W]\big)(h)
\end{aligned}$$

Problem 4 Calculate the following commutators.

1. In $\mathbb{R}^3$, $\left[x^2\,\partial_1 - x^1\,\partial_2, x^3\,\partial_2 - x^2\,\partial_3\right]$

   We will use the result of the Problem 2. Let $V = (x^2, -x^1, 0)$, $W = (0, x^3, -x^2)$.

$$\begin{aligned}
[V, W]^j &= V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \\
&= x^2 \frac{\partial W^j}{\partial x^1} - x^1 \frac{\partial W^j}{\partial x^2} - x^3 \frac{\partial V^j}{\partial x^2} + x^2 \frac{\partial V^j}{\partial x^3}
\end{aligned}$$

Then, it is straightforward:

$$\begin{aligned}
[V,W]^1 &= -x^3\frac{\partial x^2}{\partial x^2} + x^2\frac{\partial x^2}{\partial x^3} = -x^3 \\
[V,W]^2 &= 0 \\
[V,W]^3 &= -x^1\frac{\partial(-x^2)}{\partial x^2} = x^1
\end{aligned}$$

Finally,

$$[V,W] = -x^3\,\partial_1 + x^1\,\partial_3$$

We can also see, that the group generated by this commutator is SO(3). Commutator of rotation along ∂_3 with rotation along ∂_1 gave us rotation along ∂_2.

2. In $\mathbb{R}^4$, $\big[x^0\,\partial_1 + x^1\,\partial_0, x^0\,\partial_2 + x^2\,\partial_0\big]$

   Well, commutator is linear with respect to each entry (as long as vector field coefficients are just numbers), so we can consider 4 separate, simpler commutators.

$$\begin{aligned}
\big[x^0\,\partial_1, x^0\,\partial_2\big] &= 0 \\
\big[x^0\,\partial_1, x^2\,\partial_0\big] &= -\big[x^2\,\partial_0, x^0\,\partial_1\big]
\end{aligned}$$

   If we treat x^0 as f, we can use Problem 3:

$$\begin{aligned}
&= -\Big(x^2\,\partial_0(x^0)\,\partial_1 + x^0\big[x^2\,\partial_0, \partial_1\big]\Big) \\
&= -x^2\,\partial_1 + x^0\big[\partial_1, x^2\,\partial_0\big] \\
&= -x^2\,\partial_1 + x^0\,\partial_1(x^2)\,\partial_0 + x^0x^2\big[\partial_1, \partial_0\big] = -x^2\,\partial_1 \\
\big[x^1\,\partial_0, x^0\,\partial_2\big] &= x^1\,\partial_0(x^0)\,\partial_2 + x^0\big[x^1\,\partial_0, \partial_2\big] = x^1\,\partial_2 \\
\big[x^1\,\partial_0, x^2\,\partial_0\big] &= 0
\end{aligned}$$

   Now, we combine all terms,

$$\begin{aligned}
\big[x^0\,\partial_1 + x^1\,\partial_0, x^0\,\partial_2 + x^2\,\partial_0\big] &= 0 - x^2\,\partial_1 + x^1\,\partial_2 + 0 \\
&= -x^2\,\partial_1 + x^1\,\partial_2
\end{aligned}$$

   The group $\mathrm{SO}(1,3)$ (Lorentz group) is generated by that last commutator.

Lecture 5

17 lis 2020

Some properties of the commutator

Commutators satisfy the following identities:

$$0 = \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big]$$

Linearity, up to constants:

$$[X, aY + bZ] = a[X, Y] + b[X, Z], \quad a, b \in \mathbb{R}$$

Antisymmetry:

$$[X, Y] = -[Y, X]$$

Some variation of the Leibniz rule:

$$[X, fY] = X(f)Y + f[X, Y]$$

The flow of a vector field

Let $v = v^i \partial_i$ be a vector field in $\mathbb{R}^n$. An integral curve $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in \mathbb{R}^n$ of v is a curve, such that:

$$\frac{\mathrm{d}p^i(\tau)}{\mathrm{d}\tau} = v^i\big(p(\tau)\big)$$

at each τ, ie.

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = v_{p(\tau)}$$

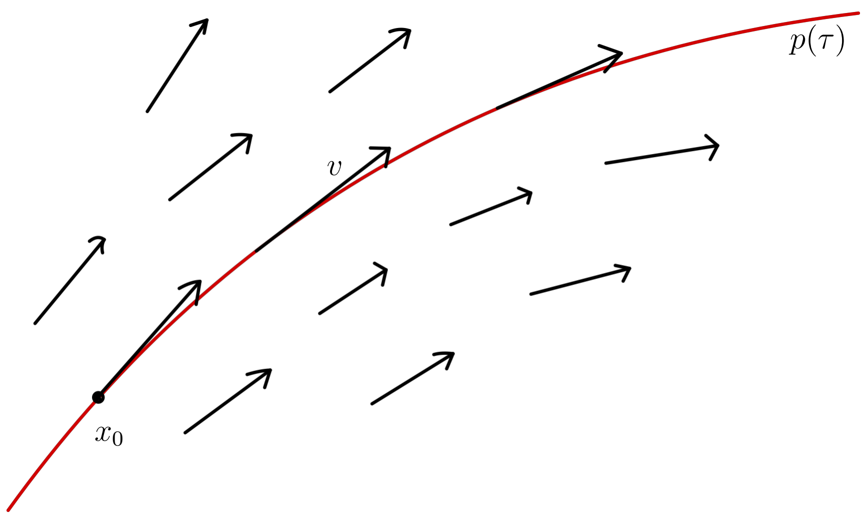

Figure 2.1: Integral curve.

Theorem 2. For every $x_0 \in \mathbb{R}^n$, $v_{x_0} \neq 0$ locally there is a unique integral curve p_{x_0} of v such that $p_{x_0}(0) = x_0$.

Definition 17 (The flow)**.** Given τ, we define a map in a neighbourhood of x:

$$x \mapsto p_x(\tau)$$

and call it the flow of a vector field.

Covector fields

Definition 18 (Covector field)**.** We call ω_x a covector field, if $\mathbb{R}^n \ni x \mapsto \omega_x \in \mathrm{T}^*\mathbb{R}^n$, where $\omega_x = \omega_i(x)\,\mathrm{d}x^i$ for $\omega_i \in C^m(\mathbb{R}^n)$.

Similarly, we can define a covariant tensor field by taking

$$\mathrm{T}^*\mathbb{R}^n \otimes \cdots \otimes \mathrm{T}^*\mathbb{R}^n \ni T = T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}$$

where $T_{i_1\cdots i_k} \in C^m(\mathbb{R}^n)$. If $T_{i_1\cdots i_k} = T_{[i_1\cdots i_k]}$, then

$$T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} = \frac{1}{k!} T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

and we call it a differential k-form. There are also some useful properties!

Remark 6. Let $T_{i_1\cdots i_k}$ be an arbitrary covariant tensor and $\omega \in \Omega^k(\mathbb{R}^k)$. Then,

$$\begin{aligned}\omega = \frac{1}{k!} T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{ik} &= \underbrace{T_{\left[\sigma(1)\cdots\sigma(k)\right]}\,\mathrm{d}x^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x^{\sigma(k)}}_{\text{arbitrary permutation}} \\ &= \frac{1}{k!} T_{[i_1\cdots i_k]}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\end{aligned}$$

It also means, that

$$\frac{1}{k!} T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{ik} = T_{[i_1\cdots i_k]}\,\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}$$

If we had $\nu \in \Omega^k(\mathbb{R}^n)$, $n > k$ then everything would stay the same except the part with „arbitrary permutation", which would take a form of $\binom{n}{k}$ arbitrary permutations, each one from a multivalent k-element combination of $\{1, 2, \ldots, n\}$.

Exterior derivative

We will define this operation via its properties.

$$\begin{gathered}C^m(\mathbb{R}^n) \ni f \mapsto \mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \\ \mathrm{d}f \mapsto \mathrm{d}(\mathrm{d}f) = 0 \\ f\,\mathrm{d}h \mapsto \mathrm{d}(f\,\mathrm{d}h) = \mathrm{d}f \wedge \mathrm{d}h\end{gathered}$$

In general, assuming that $T_{i_1\cdots i_k} = T_{[i_1\cdots i_k]}$,

$$\begin{aligned}
\mathrm{d}\left(\frac{1}{k!}T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\right) &= \frac{1}{k!}\,\mathrm{d}T_{i_1\cdots i_k}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}\frac{\partial T_{i_1\cdots i_k}}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}T_{i_1\cdots i_k,j}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= \frac{1}{k!}T_{[i_1\cdots i_k,j]}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
&= (-1)^k\frac{1}{k!}T_{[i_1\cdots i_k,j]}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\wedge\mathrm{d}x^j
\end{aligned}$$

Let us consider a k-form ω and l-form ν.

$$\begin{aligned}
\mathrm{d}(\omega\wedge\nu) &= \mathrm{d}\omega\wedge\nu + (-1)^k\omega\wedge\mathrm{d}\nu\\
\mathrm{d}(f\omega) &= \mathrm{d}f\wedge\omega + f\,\mathrm{d}\omega\\
\mathrm{d}(a\omega+b\nu) &= a\,\mathrm{d}\omega + b\,\mathrm{d}\nu
\end{aligned}$$

Contraction with a vector

Let $T = T_{i_1\cdots i_k}\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}$ and $V = V^i\,\partial_i$.

Definition 19 (Contraction of a covariant tensor)**.** A contraction of a covariant tensor with a vector is defined as:

$$V\lrcorner T = V^{i_1}T_{i_1 i_2\cdots i_k}\,\mathrm{d}x^{i_2}\otimes\cdots\otimes\mathrm{d}x^{i_k}$$

Moreover,

$$\begin{aligned}
V_k\lrcorner\big(V_{k-1}\lrcorner\cdots\lrcorner(V_1\lrcorner T)\big) &= V_k^{i_k}\cdots V_1^{i_1}T_{i_1\cdots i_k}\\
&\stackrel{\text{def}}{=} T(V_1,\ldots,V_k)
\end{aligned}$$

From this definition if simply follows, that:

$$V\lrcorner(T\otimes L) = (V\lrcorner T)\otimes L$$

Let us also consider a differential k-form ω.

$$V\lrcorner\omega = \frac{1}{(k-1)!}V^{i_1}\omega_{i_1 i_2\cdots i_k}\,\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_k}$$

We have also other nice properties:

$$\begin{aligned}
V\lrcorner(\omega\wedge\nu) &= (V\lrcorner\omega)\wedge\nu + (-1)^k\omega\wedge(V\lrcorner\nu)\\
V\lrcorner(f\omega) &= fV\lrcorner\omega\\
V\lrcorner(\omega+\omega') &= V\lrcorner\omega + V\lrcorner\omega'
\end{aligned}$$

Remark 7. There are also useful relations between the contraction and the external derivative.

$$V \lrcorner \, \mathrm{d}f = V(f) = V^i \, \partial_i(f)$$

If W, V are any vector fields and $\omega \in \Omega^1(\mathbb{R}^n)$,

$$W \lrcorner (V \lrcorner \, \mathrm{d}\omega) \overset{\text{def}}{=} \mathrm{d}\omega \, (V, W) = V(W \lrcorner \omega) - W(V \lrcorner \omega) - [V, W] \lrcorner \omega$$

Proof.

$$\begin{aligned}
\omega &= \omega_i \, \mathrm{d}x^i \\
\mathrm{d}\omega &= \mathrm{d}\omega_i \wedge \mathrm{d}x^i = \omega_{i,j} \, \mathrm{d}x^j \wedge \mathrm{d}x^i = \omega_{i,j}(\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}x^j) \\
\mathrm{d}\omega \, (V, W) &= V^{i_1} W^{i_2} \, \mathrm{d}\omega \, (\partial_{i_1}, \partial_{i_2}) \\
&= V^{i_1} W^{i_2} \omega_{i,j} (\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}x^j)(\partial_{i_1}, \partial_{i_2}) \\
&= V^{i_1} W^{i_2} \omega_{i,j} ({\delta^j}_{i_1} {\delta^i}_{i_2} - {\delta^i}_{i_1} {\delta^j}_{i_2}) \\
&= V^{i_1} W^{i_2} (\omega_{i_2,i_1} - \omega_{i_1,i_2}) = W^{i_2} V(\omega_{i_2}) - V^{i_1} W(\omega_{i_1})
\end{aligned}$$

Notice, that we can use Leibniz rule: $V(W^i \omega_i) = V(W^i)\omega_i + W^i V(\omega_i)$, as $W^i, \omega_i \in C^k(\mathbb{R}^n)$

$$\begin{aligned}
&= V(W^{i_2} \omega_{i_2}) - V(W^{i_2})\omega_{i_2} - W(V^{i_1} \omega_{i_1}) + W(V^{i_1})\omega_{i_1} \\
&= V(W \lrcorner \omega) - W(V \lrcorner \omega) - [V, W] \lrcorner \omega
\end{aligned}$$

■

Application Suppose $e_1, \ldots, e_n$ are vector fields in $\mathbb{R}^n$ such that at each $x \in \mathbb{R}^n$, $(e_1, \ldots, e_n)$ is a basis and $\left[e_i, e_j\right] = {c_{ij}}^k e_k$ for ${c_{ij}}^k = \text{const}$. We also take such $e^1, \ldots, e^n$ dual 1-form fields that:

$$\mathrm{d}e^i = \frac{1}{2} {b^i}_{jk} e^j \wedge e^k, \quad {b^i}_{jk} = -{b^i}_{kj}$$

Find a relation between ${c_{ij}}^k$ and ${b^i}_{jk}$.

We shall start from applying two vector fields directly two our 2-form.

$$\begin{aligned}
\mathrm{d}e^i \, (e_{j_1}, e_{j_2}) &= \frac{1}{2} {b^i}_{jk} e^j \wedge e^k (e_{j_1}, e_{j_2}) \\
&= \frac{1}{2} {b^i}_{jk} \left({\delta^i}_{j_1} {\delta^k}_{j_2} - {\delta^j}_{j_2} {\delta^k}_{j_1} \right) = \frac{1}{2} \left({b^i}_{j_1 j_2} - {b^i}_{j_2 j_1} \right) \\
&= {b^i}_{j_1 j_2}
\end{aligned}$$

On the other hand we could use the identity derived above.

$$\begin{aligned}
\mathrm{d}e^i \, (e_{j_1}, e_{j_2}) &= e_{j_1}(e_{j_2} \lrcorner e^i) - e_{j_2}(e_{j_1} \lrcorner e^i) - \left[e_{j_1}, e_{j_2}\right] \lrcorner e^i \\
&= e_{j_1}({\delta^i}_{j_2}) - e_{j_2}({\delta^i}_{j_1}) - \left[e_{j_1}, e_{j_2}\right] \lrcorner e^i \\
&= -{c_{j_1 j_2}}^k e_k \lrcorner e^i = -{c_{j_1 j_2}}^i
\end{aligned}$$

We have proven something very interesting!

Remark 8. Let (e_i) be vector fields, and (e^j) be 1-form fields dual to them. If $\big[e_i, e_j\big] = c_{ij}{}^k e_k$, where $c_{ij}{}^k = \text{const.}$, then

$$\mathrm{d}e^i = -\frac{1}{2} c_{jk}{}^i e^j \wedge e^k$$

The Lie derivative

Definition 20 (Lie derivative)**.** Let V be a vector field in $\mathbb{R}^n$. We define the Lie derivative, by giving the sufficient amount of its properties.

$$\mathcal{L}_V : C^1(\mathbb{R}^n) \ni f \mapsto \mathcal{L}_V(f) \overset{\text{def}}{=} V(f)$$

Commutativity with d,

$$\mathcal{L}_V(\mathrm{d}f) \overset{\text{def}}{=} \mathrm{d}(\mathcal{L}_V f) = \mathrm{d}\big(V(f)\big)$$

Sort of Leibniz rule,

$$\begin{aligned}\mathcal{L}_V(h\,\mathrm{d}f) &\overset{\text{def}}{=} (\mathcal{L}_V h)\,\mathrm{d}f + h\,\mathrm{d}(\mathcal{L}_V f)\\ &= V(h)\,\mathrm{d}f + h\,\mathrm{d}\big(V(f)\big)\end{aligned}$$

When acting on tensors,

$$\begin{aligned}\mathcal{L}_V(T \otimes L) &= (\mathcal{L}_V T) \otimes L + T \otimes (\mathcal{L}_V L)\\ \mathcal{L}_V(T + L) &= \mathcal{L}_V T + \mathcal{L}_V L\\ \mathcal{L}_V(W \lrcorner T) &= (\mathcal{L}_V W) \lrcorner T + W \lrcorner (\mathcal{L}_V T)\end{aligned}$$

Exercise V, W are vector fields. Show, that $\mathcal{L}_V W = [V, W]$.

$$\begin{aligned}(\mathcal{L}_V W) \lrcorner\, \mathrm{d}x^i &= \mathcal{L}_V\Big(W \lrcorner\, \mathrm{d}x^i\Big) - W \lrcorner\, \mathrm{d}\Big(\mathcal{L}_V x^i\Big)\\ &= \mathcal{L}_V(W^i) - W \lrcorner\, \mathrm{d}V^i = V(W^i) - W \lrcorner V^i{}_{,j}\ \mathrm{d}x^j\\ &= V(W^i) - W^j V^i{}_{,j}\ = V(W^i) - W(V^i) = [V, W]^i\\ &= [V, W] \lrcorner\, \mathrm{d}x^i\end{aligned}$$

This formula works for an arbitrary $\mathrm{d}x^i$, so

$$\mathcal{L}_V W = [V, W]$$

≻ Problem class 5

Problem 1 Consider a 1-form $A = q/r\,\mathrm{d}t$, where $q = $const., t, x, y, z are coordinates in $\mathbb{R}^n$ and $r = \sqrt{x^2 + y^2 + z^2}$. Calculate $\mathrm{d}A$.

$$\mathrm{d}A = q\,\mathrm{d}\Big((x^2 + y^2 + z^2)^{-1/2}\,\mathrm{d}t\Big)$$

$$= -\frac{x\,\mathrm{d}x \wedge \mathrm{d}t + y\,\mathrm{d}y \wedge \mathrm{d}t + z\,\mathrm{d}z \wedge \mathrm{d}t}{(x^2+y^2+z^2)^{-3/2}}$$

Problem 2 Calculate d of $x\,\mathrm{d}y$, $-y\,\mathrm{d}x$.
It's trivial.

Problem 3 Show that the following equality is true: $\mathcal{L}_V\omega = \mathrm{d}(V\lrcorner\omega) + V\lrcorner\,\mathrm{d}\omega$, where $\omega = \omega_i\,\mathrm{d}x^i$.

$$\mathcal{L}_V(\omega_i\,\mathrm{d}x^i) = V(\omega_i)\,\mathrm{d}x^i + \omega_i\,\mathrm{d}\Big(V(x^i)\Big)$$

Step I. Given $V = V^i\,\partial_i$, calculate $\mathcal{L}_V(\mathrm{d}x^i)$.

$$\mathcal{L}_V(\mathrm{d}x^i) = \mathrm{d}\Big(V(x^i)\Big) = \mathrm{d}V^i$$

Step II. Use Leibniz rule,

$$\begin{aligned}
\mathcal{L}_V(\omega_i\,\mathrm{d}x^i) &= V(\omega_i)\,\mathrm{d}x^i + \omega_i\mathcal{L}_V(\mathrm{d}x^i) \\
&= V^j\frac{\partial\omega_j}{\partial x^j}\,\mathrm{d}x^i + \omega_i\,\mathrm{d}V^i \\
&= V^j\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i + \omega_i\frac{\partial V^i}{\partial x^i}\,\mathrm{d}x^j \\
&= \left(V^j\frac{\partial\omega_i}{\partial x^j} + \omega_j\frac{\partial V^j}{\partial x^i}\right)\mathrm{d}x^i
\end{aligned}$$

Step III. $\mathrm{d}(V\lrcorner\omega)$

$$\mathrm{d}(V\lrcorner\omega) = \mathrm{d}(V^i\omega_i) = V^i\,\mathrm{d}\omega_i + \omega_i\,\mathrm{d}V^i$$

Step IV. $V\lrcorner\,\mathrm{d}\omega$

$$\begin{aligned}
V\lrcorner\,\mathrm{d}\omega = V\lrcorner(\mathrm{d}\omega_i \wedge \mathrm{d}x^i) &= V^j\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i - V^i\,\mathrm{d}\omega_i \\
\mathrm{d}(V\lrcorner\omega) + V\lrcorner\,\mathrm{d}\omega &= \omega_i\,\mathrm{d}V^i + V^i\frac{\partial\omega_i}{\partial x^j}\,\mathrm{d}x^i \\
&= \left(\omega_i\frac{\partial V^i}{\partial x^j} + V^i\frac{\partial\omega_j}{\partial x^i}\right)\mathrm{d}x^j
\end{aligned}$$

We can see that they are exactly the same. This formula is also true for arbitrary $\omega = \frac{1}{k!}\omega_{i_1\ldots i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}$ (any k-from).

Problem 4 (Application of 3) In $\mathbb{R}^3$ there are given 1-forms (e^1, e^2, e^3) that set a dual basis to vector fields (e_1, e_2, e_3) at each point $x \in \mathbb{R}^3$. Knowing that $\mathrm{d}e^1 = e^2 \wedge e^3$, $\mathrm{d}e^2 = e^3 \wedge e^1$, $\mathrm{d}e^3 = e^1 \wedge e^2$ calculate $\mathcal{L}_{e_i}e^j$.

We know, that

$$\begin{aligned}
\mathcal{L}_{e_i} e^j &= \mathrm{d}(e_i \lrcorner e^j) + e_i \lrcorner \, \mathrm{d}e^j = 0 + e_i \lrcorner \left(\frac{1}{2} \varepsilon_{jlk} e^l \wedge e^k \right) \\
&= \frac{1}{2} \varepsilon_{jlk} (e_i \lrcorner e^l) \wedge e^k - \frac{1}{2} \varepsilon_{jlk} e^l \wedge (e_i \lrcorner e^k) \\
&= \frac{1}{2} \varepsilon_{jik} e^k - \frac{1}{2} \varepsilon_{jli} e^l = \frac{1}{2} \varepsilon_{jik} e^k + \frac{1}{2} \varepsilon_{jil} e^l \\
&= \varepsilon_{jik} e^k
\end{aligned}$$

Problem 5 Consider $\mathbb{R}^2$ and calculate $\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $V = V^x \, \partial_x + V^y \, \partial_y$.

$$\mathcal{L}_V(T \otimes L) = \mathcal{L}_V(T) \otimes L + T \otimes \mathcal{L}_V(L)$$

Also we know, that

$$\begin{aligned}
\mathrm{d}x^2 &= \mathrm{d}x \otimes \mathrm{d}x \\
\mathcal{L}_V(\mathrm{d}x^2) &= \mathcal{L}_V(\mathrm{d}x) \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathcal{L}_V(\mathrm{d}x) \\
\mathcal{L}_V(\mathrm{d}x) &= \mathrm{d}\big(V(x)\big) = \mathrm{d}V^x = \frac{\partial V^x}{\partial x} \, \mathrm{d}x + \frac{\partial V^x}{\partial y} \, \mathrm{d}y
\end{aligned}$$

From this, we get:

$$\begin{aligned}
\mathcal{L}_V(\mathrm{d}x^2) &= 2 \frac{\partial V^x}{\partial x} \, \mathrm{d}x \otimes \mathrm{d}x + \frac{\partial V^x}{\partial y} (\mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y) \\
\mathcal{L}_V(\mathrm{d}y^2) &= 2 \frac{\partial V^y}{\partial y} \, \mathrm{d}y \otimes \mathrm{d}y + \frac{\partial V^y}{\partial x} (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)
\end{aligned}$$

Now it is straightforward,

$$\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2) = 2 \frac{\partial V^x}{\partial x} \, \mathrm{d}x^2 + 2 \frac{\partial V^y}{\partial y} \, \mathrm{d}y^2 + \left(\frac{\partial V^x}{\partial y} + \frac{\partial V^y}{\partial x} \right) (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)$$

Problem 6 (Continuation of 5) Find $V^x \, \partial_x + V^y \, \partial_y$ such that $\mathcal{L}_V(\mathrm{d}x^2 + \mathrm{d}y^2) = 0$.

We obtain a set of equations.

$$\frac{\partial V^x}{\partial x} = 0, \quad \frac{\partial V^y}{\partial y} = 0, \quad \frac{\partial V^x}{\partial y} + \frac{\partial V^y}{\partial x} = 0$$

From the first two equations we know that $V^x = V^x(y), V^y = V^y(x)$. Now we can guess the exact solutions having tee 3rd equation.

$$V^x = y, \quad V^y = -x$$

A general solution would be:

$$V^x = Ay + B, \quad V^y = -Ax + C$$

In overall, V turns out to be a rotation combined with a translation.

Lecture 6

24 lis 2020

Integrating differential forms

Given a curve $p\colon [\tau_0, \tau_1] \to \mathbb{R}^n$ and a 1-form $\omega = \omega_a \, \mathrm{d}x^a$ we define

$$\int_p \omega \overset{\text{def}}{=} \int_{\tau_0}^{\tau_1} \omega_a\big(p(\tau)\big) \frac{\mathrm{d}p^a}{\mathrm{d}\tau} \, \mathrm{d}\tau$$

which is invariant with respect to change of parametrization, preserving the orientation.

$$\begin{aligned} &\big[\tau_0', \tau_1'\big] \to [\tau_0, \tau_1] \xrightarrow{p} \mathbb{R}^n \\ &\int_{p'} \omega = \int_{\tau_0'}^{\tau_1'} \omega_a\Big(p\big(\tau(\tau')\big)\Big) \frac{\mathrm{d}p^a}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}\tau'} \, \mathrm{d}\tau' \\ &\qquad = \operatorname{sgn}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}\right) \int_p \omega \end{aligned}$$

Given a curve $p\colon [\tau_0, \tau_1] \to \mathbb{R}^n$ and 1-forms $\omega^1 = \omega^1_a \, \mathrm{d}x^a, \dots, \omega^k = \omega^k_a \, \mathrm{d}x^a$,

$$\int_p \sqrt{(\omega^1)^2 + \cdots + (\omega^k)^2} \overset{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{\left(\omega^1_a \frac{\mathrm{d}p^a}{\mathrm{d}\tau}\right)^2 + \cdots + \left(\omega^k_a \frac{\mathrm{d}p^a}{\mathrm{d}\tau}\right)^2} \, \mathrm{d}\tau$$

Such integral is reparametrization and orientation invariant.
Now let us consider a surface $S\colon \big[\sigma^1_0, \sigma^1_1\big] \times \big[\sigma^2_0, \sigma^2_1\big] \to \mathbb{R}^n$ and a 2-form $\omega = \frac{1}{2}\omega_{ab} \, \mathrm{d}x^a \wedge \mathrm{d}x^b$. Then,

$$\int_S \omega \overset{\text{def}}{=} \int_{\sigma^2_0}^{\sigma^2_1} \int_{\sigma^1_0}^{\sigma^1_1} \omega_{ab}\big(S(\sigma^1, \sigma^2)\big) \frac{\partial S^a}{\partial \sigma^1} \frac{\partial S^b}{\partial \sigma^2} \, \mathrm{d}\sigma^1 \, \mathrm{d}\sigma^2$$

Similarly if we have two 2-forms,

$$\int_S \sqrt{(\omega^1)^2 + (\omega^2)^2} \overset{\text{def}}{=} \int_{\sigma^2_0}^{\sigma^2_1} \int_{\sigma^1_0}^{\sigma^1_1} \sqrt{\left(\omega^1_{ab} \frac{\partial S^a}{\partial \sigma^1} \frac{\partial S^b}{\partial \sigma^2}\right)^2 + \left(\omega^2_{ab} \frac{\partial S^a}{\partial \sigma^1} \frac{\partial S^b}{\partial \sigma^2}\right)^2} \, \mathrm{d}\sigma^1 \, \mathrm{d}\sigma^2$$

The integral $\displaystyle\int_S \omega$ is invariant with respect to orientation for reparametrisations such that $\det\left(\dfrac{\partial(\sigma)}{\partial(\sigma')}\right) > 0$. The other integrals with square roots are invariant even after relaxing this condition.

Finally, we consider a k-form $\omega = \dfrac{1}{k!}\omega_{a_1 \cdots a_k} \, \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_k}$ as well as $S\colon \big[\sigma^1_0, \sigma^1_1\big] \times \cdots \times \big[\sigma^k_0, \sigma^k_1\big] \to \mathbb{R}^n$. We generalise,

$$\int_S \omega \overset{\text{def}}{=} \int_{\sigma^k_0}^{\sigma^k_1} \cdots \int_{\sigma^1_0}^{\sigma^1_1} \mathrm{d}\sigma^1 \cdots \mathrm{d}\sigma^k \, \frac{\partial S^{a_1}}{\partial \sigma^1} \cdots \frac{\partial S^{a_k}}{\partial \sigma^k}$$

As, well such integral is reparametrisation invariant provided that

$$\det\left(\frac{\partial \sigma^A}{\partial \sigma'^B}\right)^{A=1,\dots,k}_{B=1,\dots,k} > 0$$

Pullback

Before, we have introduced the following operations: $\otimes, \wedge, \lrcorner, \mathrm{d}\,, \mathcal{L}, \int$. We want to investigate their behaviour under diffeomorphisms on $\mathbb{R}^n \to \mathbb{R}^m$.
Consider a map $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ and a function $f\colon \mathbb{R}^m \to \mathbb{R}$. We introduce a new operation, the pullback:

$$\phi^* f\colon C^k(\mathbb{R}^m) \to C^k(\mathbb{R})$$
$$(\phi^* f)(x) \stackrel{\text{def}}{=} f\big(\phi(x)\big) = (f \circ \phi)(x)$$

For a 1-form:

$$\phi^*(h\,\mathrm{d}f) = (\phi^* h)\,\mathrm{d}(\phi^* f)$$

By linearity,

$$\phi^*(h_1\,\mathrm{d}f_1 + \cdots + h_n\,\mathrm{d}f_n) = (\phi^* h_1)\,\mathrm{d}(\phi^* f_1) + \cdots + (\phi^* h_n)\,\mathrm{d}(\phi^* f_n)$$
$$\phi^*(\omega_a\,\mathrm{d}y^a) = (\phi^*\omega_a)\,\mathrm{d}(\phi^* y^a)$$

Now we generalise it on covariant tensors,

$$\phi^*\Big(T_{a_1\cdots a_k}\,\mathrm{d}x^a \otimes \cdots \otimes \mathrm{d}x^{a_k}\Big) = (\phi^* T_{a_1\cdots a_k})\,\mathrm{d}(\phi^* x^{a_1}) \otimes \cdots \otimes \mathrm{d}(\phi^* x^{a_k})$$

Example Let's take $\phi\colon \mathbb{R} \to \mathbb{R}^2$ such that $x \mapsto (2x, x^2)$ and $f(y^1, y^2) = y^1 + y^2$.

$$\phi^* f = y^1(x) + y^2(x) = 2x + x^2 \in \mathbb{R}$$

Pushforward

Let us consider again a map $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$. Given $v \in \mathrm{T}_x\mathbb{R}^n$ and $p\colon [\tau_0, \tau_1] \to \mathbb{R}^n$ such that $\mathrm{d}p/\mathrm{d}\tau\ |_{\tau=\tau_0}= v$. Then,

$$\phi_* v \stackrel{\text{def}}{=} \left.\frac{\mathrm{d}\phi\big(p(\tau)\big)}{\mathrm{d}\tau}\right|_{\phi(\tau_0)}$$

But there may be some difficulties, dependent on ϕ, when mapping arbitrary vector fields. For example, if ϕ wasn't injective, we could map vectors at different points, to a point $\phi(x) = \phi(x')$. To be safe, we assume that $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism: $\phi' \in C^0(\mathbb{R}^n)$ exists, as well as ϕ^{-1} with $(\phi^{-1})' \in C^0(\mathbb{R}^n)$.
Then, the every vector field v on $\mathbb{R}^n$ is pushed forward into a vector field $\phi_* v$ in $\mathbb{R}^n$.

There is also the consistency between the pullback and the pushforward. If we take $v \in \mathrm{T}_x\mathbb{R}^n$ and $\omega = \omega_a\,\mathrm{d}y^a$ at $\phi(x)$, ie. $\omega \in \mathrm{T}^*_{\phi(x)}\mathbb{R}^m$, then:

Remark 9. Invariance of tensor operations under diffeomorphisms.

$$v \lrcorner (\phi^*\omega) = (\phi_* v) \lrcorner \omega$$
$$(\phi^{-1})^*(v \lrcorner \omega) = (\phi_* v) \lrcorner (\phi^{-1*}\omega)$$
$$\mathrm{d}(\phi^*\omega) = \phi^*\,\mathrm{d}\omega$$
$$\phi^*(\mathcal{L}_v\omega) = \mathcal{L}_{\phi_*^{-1} v}\phi^*\omega$$
$$\int_S \phi^*\omega = \int_{\phi\circ S}\omega$$

$\succ$ **Problem class 6 – Stereographic projection**

It is a map from a plane to a sphere. By applying the Tales theorem, we get

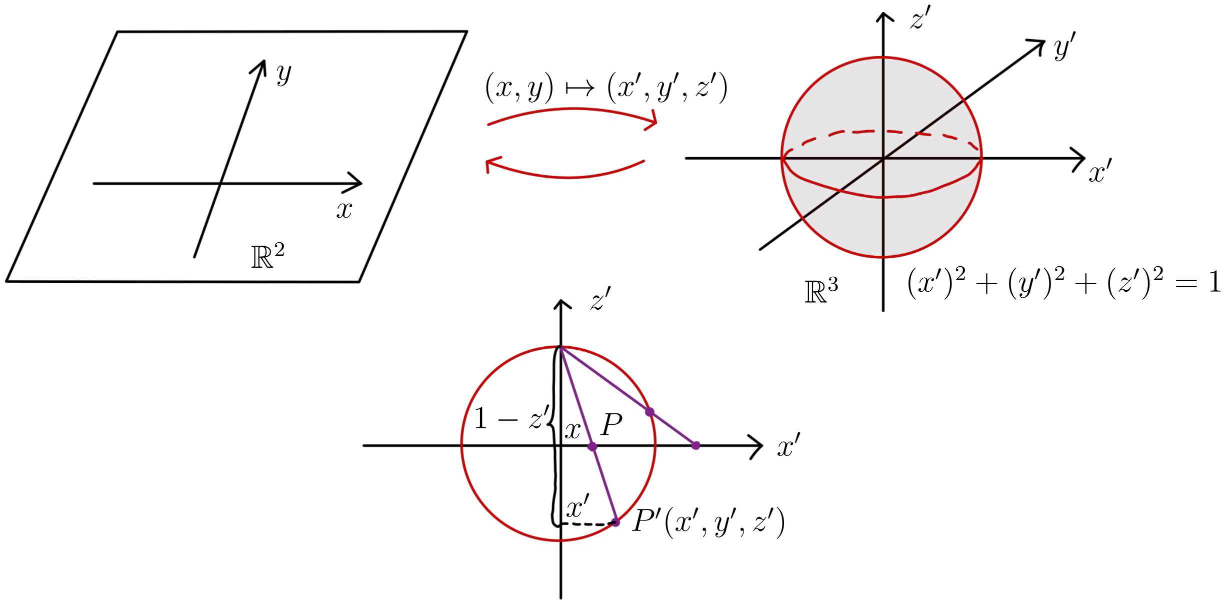

Figure 2.2: Stereographic projection.

$$\frac{x'}{1 - z'} = \frac{x}{1}$$

The same proportion is true for y,

$$\frac{y'}{1 - z'} = \frac{y}{1}$$

$$(x, y) = \left(\frac{x'}{1 - z'}, \frac{y'}{1 - z'} \right)$$

We want to invert this map.

$$\begin{aligned}
x^2 + y^2 &= \frac{(x')^2 + (y')^2}{(1 - z')^2} = \frac{1 - (z')^2}{(1 - z')^2} \\
&= \frac{1 + z'}{1 - z'} \\
\frac{1 + x^2 + y^2}{2} &= \frac{1}{2}\left(1 + \frac{1 + z'}{1 - z'} \right) = \frac{1 - z' + 1 + z'}{2(1 - z')} \\
&= \frac{1}{1 - z'}
\end{aligned}$$

We conclude, that

$$\begin{aligned}
x' &= x(1 - z') = \frac{2x}{1 + x^2 + y^2} \\
y' &= \frac{2y}{1 + x^2 + y^2} \\
z' &= \frac{-1 + x^2 + y^2}{1 + x^2 + y^2}
\end{aligned}$$

In overall,

$$(x,y) \overset{\phi}{\mapsto} \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1)$$

Is $(0,0,1)$ in the image? Well, S^2 isn't homeomorphic to $\mathbb{R}^2$, so it isn't. But there is a possibility to include $(0,0,1)$ as a point at infinity.

Problem 1 In $\mathbb{R}^3$, $g' = (\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2$. Calculate $\phi^* g'$ on $\mathbb{R}^2$.

$$\begin{aligned}
\phi^* \mathrm{d}x' &= \mathrm{d}\left(\frac{2x}{1+x^2+y^2}\right) = \frac{2\,\mathrm{d}x\,(1+x^2+y^2) - 2x(2x\,\mathrm{d}x + 2y\,\mathrm{d}y)}{(1+x^2+y^2)^2} \\
&= \frac{2}{(1+x^2+y^2)^2}\left[(1-x^2+y^2)\,\mathrm{d}x - 2xy\,\mathrm{d}y\right] \\
\phi^* \mathrm{d}y' &= \frac{2}{(1+x^2+y^2)^2}\left[(1+x^2-y^2)\,\mathrm{d}y - 2xy\,\mathrm{d}x\right] \\
\phi^* \mathrm{d}z' &= \mathrm{d}\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right) \\
&= \frac{(2x\,\mathrm{d}x + 2y\,\mathrm{d}y)(x^2+y^2+1) - (2x\,\mathrm{d}x + 2y\,\mathrm{d}y)(x^2+y^2-1)}{(x^2+y^2+1)^2} \\
&= \frac{4}{(1+x^2+y^2)^2}(x\,\mathrm{d}x + y\,\mathrm{d}y)
\end{aligned}$$

Now we should combine them into a metric tensor.

$$\phi^* g' = (\phi^* \mathrm{d}x')^2 + (\phi^* \mathrm{d}y')^2 + (\phi^* \mathrm{d}z')^2$$

Terms proportional to $\mathrm{d}x^2, \mathrm{d}y^2, \mathrm{d}x\,\mathrm{d}y$:

$$\begin{aligned}
\sim \mathrm{d}x^2 &= \frac{4}{(1+x^2+y^2)^4}\left[(-x^2+y^2+1)^2 + 4x^2y^2 + 4x^2\right] \\
&= \frac{4}{(1+x^2+y^2)^2}(x^2+y^2+1)^2 = 4(1+x^2+y^2)^{-2} \\
\sim \mathrm{d}y^2 &= 4(1+y^2+x^2)^{-2} \\
\sim \mathrm{d}x\,\mathrm{d}y &= \frac{8}{(1+x^2+y^2)^2}\left[-(1+x^2+y^2)(2xy) - (1+x^2-y^2)(2xy) + 4xy\right] \\
&= \frac{8}{(1+x^2+y^2)^2}[-4xy + 4xy] = 0
\end{aligned}$$

And finally,

$$\phi^* g' = \frac{4}{(1+x^2+y^2)^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$$

This is our metric of a round sphere in terms of stereographic coordinates. For example we could calculate the area of our sphere or the length of a line.

Let's calculate the length of the $y = 0$ line.

$$
\begin{aligned}
L &= \int_{-\infty}^{+\infty} \sqrt{\frac{4}{(1+x^2)^2}\,\mathrm{d}x^2} = \int_{-\infty}^{+\infty} \frac{2}{1+x^2}\,\mathrm{d}x \\
&= \left| \begin{matrix} x = \tan u \\ \mathrm{d}x = \mathrm{d}u \,/\, \cos^2(u) \end{matrix} \right| = \int_{-\pi/2}^{\pi/2} 2\,\mathrm{d}u = 2\pi
\end{aligned}
$$

Well, it works!

It is important to understand what we really did. We started from the euclidean metric, valid in the whole $\mathbb{R}^3$. At the moment, when we defined the parametrisation of S^2, we „cut" this metric to such sphere, so that g' measures the distances on S^2. When we embed any surface, the metric tensor is also „cut" to that surface. Because of that, the pullbacked tensor $\phi^* g'$ also measures distances on S^2, but in terms of different coordinates.

Lecture 7: Manifolds, natural operations on tensor fields

01 gru 2020

Definition 21 (Manifold)**.** n-dimensional manifold is a set M with a family of maps (called an atlas):

$$
M \supset U_I \xrightarrow{\psi_I} \mathbb{R}^n, \quad I \in \mathcal{J}
$$

such that the following conditions are satisfied:

(1) $\bigcup_{I \in \mathcal{J}} U_I = M$

(2) ψ_I is injective $\forall I \in J$

(3) $\psi_I(U_I)$ is open in $\mathbb{R}^n$, $\forall I \in \mathcal{J}$

(4) $\psi_I(U_I \cap U_J)$ is open $\forall I, J \in \mathcal{J}$

(5) $\psi_I \circ \psi_J^{-1} \colon \psi_J(U_I \cap U_J) \to \psi_I(U_I \cap U_J)$ is continuous $\forall I, J \in \mathcal{J}$

(6) The topology induced on M by $\left\{\psi_I^{-1}(V) \colon V \text{ open in } \mathbb{R}^n\right\}$ is Hausdorff and paracompact.

Definition 22 (Paracompact space)**.** A topological space is called to be paracompact iff every open cover has an open refinement that is locally finite. In other words, for each point there exists an open neighbourhood that intersects only finite number of such sets in an open refinement.

Paracompactness is necessary to ensure that there is a partition of unity.
We can require more from our charts.

Definition 23. M is called a $C^{(k)}$ manifold if all the maps $\psi_J \circ \psi_I^{-1}$ are $C^{(k)}$, where $k = 0, 1, \ldots$.

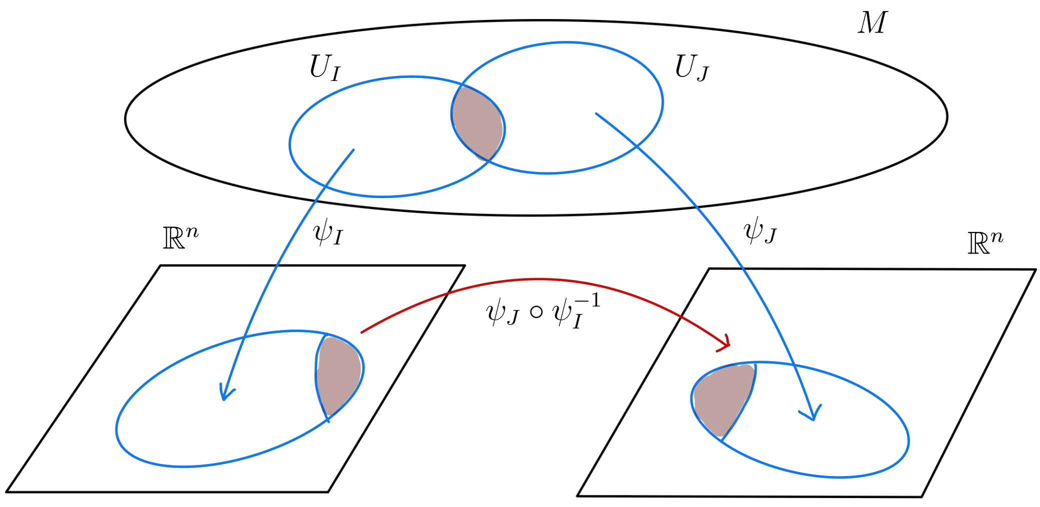

Figure 2.3: General manifold.

Remark 10. Given M and two atlases $\{\psi_I\}_{I\in\mathcal{J}}$, $\{\psi_{I'}\}_{I'\in\mathcal{J}'}$, they are considered equivalent if $\{\psi''_{I''}\}_{I''\in\mathcal{J}''} = \{\psi_I\}_{I\in\mathcal{J}} \cup \{\psi_{I'}\}_{I'\in\mathcal{J}'}$ is also a n-dimensional, C^k manifold.

Definition 24. Given a C^k manifold M, a function $f\colon M \to \mathbb{R}$ is called of class C^l for $0 \le l \le k$ if $f \circ \psi_I^{-1}$ is of class C^l for every $I \in \mathcal{J}$.
Eventually, everything is being reduced to the $\mathbb{R}^n$ framework.

Definition 25 (Tangent vector)**.** A vector tangent to M at $m \in M$ is

$$v\colon C^1(M) \to \mathbb{R}$$
$$v(fh) = v(f)h(m) + f(m)v(h)$$

for any $f, h\colon M \to \mathbb{R}$

If we take a chart $\psi_I = (x^1, \dots, x^n)$, then we can say that:

$$v(f) = v^a \frac{\partial}{\partial x^a}\bigg|_{\psi_I(m)} (f \circ \psi_I^{-1})$$

After all, we still work on $\mathbb{R}^n$ this way, by simply identifying v at $m \in M$ with $v = v^a\,\partial_a$ at $\psi_I(m) \in \mathbb{R}^n$.

Definition 26 (Tangent space)**.** $\mathrm{T}_m M = \{v\colon \text{tangent vector to } M \text{ at } m\}$ is the n-dimensional linear space, that consists of vectors defined by using a chart ψ_I.

Definition 27 (Dual tangent space)**.** The algebraic dual to $\mathrm{T}_m M$ is denoted $\mathrm{T}^*_m M$ and is called the co-tangent space. Elements of $\mathrm{T}^*_m M$ are called co-vectors.

Definition 28 (Tensor space)**.** Tensor space at $m \in M$ is simply:

$$\mathrm{T}_m M \otimes \cdots \otimes \mathrm{T}_m M \otimes \mathrm{T}^*_m M \otimes \cdots \otimes \mathrm{T}^*_m M$$

In a similar way we construct the wedge product, vector, co-vector fields, tensor fields, differential forms etc.
For instance, we can see that each of this object is indeed geometric being as it obeys some transformation rules and its nature is invariant under choice of coordinates. It is guaranteed by commutativity of pullback. Let's consider a differential form ω defined on some intersection of charts on manifold M. ω corresponds to $\omega_a \,\mathrm{d}x^a$ when we introduce the chart $\psi_I = (x^1, \ldots, x^n)$ and similarly in the another chart $\psi_J = (x'^1, \ldots, x'^n)$ we have $\omega'_a \,\mathrm{d}x'^a$. We can see, that

$$\begin{aligned} \omega_a \,\mathrm{d}x^a &= (\psi_I \circ \psi_J^{-1})^* (\omega'_a \,\mathrm{d}x'^a) \\ \mathrm{d}(\omega_a \,\mathrm{d}x^a) &= \mathrm{d}\big[(\psi_I \circ \psi_J^{-1})^* (\omega'_a \,\mathrm{d}x'^a)\big] \\ &= (\psi_I \circ \psi_J^{-1})^* \,\mathrm{d}(\omega'_a \,\mathrm{d}x'^a) \end{aligned}$$

It means, that $\mathrm{d}\omega$ is also a geometric object, not just a local, purely coordinate-based operation.

Coordinate independent definitions of the operations: $\lrcorner, \wedge, \mathrm{d}\,, \mathcal{L}$

Let $v \in \mathrm{T}_m M$, $\omega \in \mathrm{T}^*_m M$.

$$v \lrcorner w \in \mathbb{R}$$

has already an invariant definition. Let $\omega^{(1)}, \omega^{(2)} \in \mathrm{T}^*_m M$.

$$\omega^{(1)} \wedge \omega^{(2)} \overset{\text{def}}{=} \omega^{(1)} \otimes \omega^{(2)} - \omega^{(2)} \otimes \omega^{(1)}$$

$$(\omega^{(1)} \wedge \cdots \wedge \omega^{(k)}) \wedge (\nu^{(1)} \wedge \cdots \wedge \nu^{(l)}) \overset{\text{def}}{=} \omega^{(1)} \wedge \cdots \wedge \nu^{(l)}$$

with the linearity condition. A contraction of a form constructed from 1-forms:

$$v \lrcorner (\omega^{(1)} \wedge \cdots \wedge \omega^{(k)}) \overset{\text{def}}{=} (v \lrcorner \omega^{(1)}) \omega^{(2)} \wedge \cdots \wedge \omega^{(k)} - (v \lrcorner \omega^{(2)}) \omega^{(1)} \wedge \omega^{(3)} \wedge \cdots \wedge \omega^{(k)} + \cdots$$

Then, a commutator:

$$[v, w](f) \overset{\text{def}}{=} v\big(w(f)\big) - w\big(v(f)\big)$$

which is coordinate invariant purely from the definition of a vector at $m \in M$. Then, an exterior derivative d. Given $f \colon M \to \mathbb{R}^n$ and $v \in \mathrm{T}_m M$, we define $\mathrm{d}f \in \mathrm{T}^*_m M$ such that

$$v \lrcorner \,\mathrm{d}f = v(f)$$

which is a coordinate independent definition. We extend it to arbitrary differential forms by imposing the following rules: linearity, $\mathrm{d}(\omega \wedge \nu) = \mathrm{d}\omega \wedge \nu + (-1)^l \omega \wedge \mathrm{d}\nu$, where ω is a l-form.
Then, the Lie derivative defined during the previous lecture is also coordinate independent as we used coordinate independent operations to define it.

Integration of differential forms on manifolds

The subtlety that emerges in integrating over manifolds is that we have multiple charts that overlap and our forms are defined differently in each one of them. In general we have to use the partition of unity (which existence is ensured by the paracompactness of a manifold).

Definition 29 (Partition of unity)**.** There exists a partition of unity: $\rho_I \colon M \to \mathbb{R}$ such that for every $m \in M$ only finite number of ρ's is non-zero, $\sum \rho_I(m) = 1$ and such that every ρ_I either satisfies $\rho_I(m) = 0$ or $\mathrm{supp}(\rho_I) \subset \mathrm{dom}(\psi_I)$ (lies in a domain of some chart).
It follows that each $\rho_I\omega$ has a support in one of the local charts.

Then,

$$\omega = 1 \cdot \omega = \sum_I \rho_I \omega$$

$$\int_S \omega = \sum_I \int_S \rho_I \omega = \sum_I \int_{S \cap \mathrm{dom}(\psi_I)} \rho_I \omega$$

Now we are left with simple integration over subsets of $\mathbb{R}^n$, as discussed previously.

Conclusion The operations $\lrcorner, \wedge, \mathrm{d}\,, \mathcal{L}, \int$ pass to a manifold, because they are pushforward and pullback invariant.

$\succ$ Problem class 7

Problem 1 The manifold structure of a circle.

Consider a circle. We parametrize it by an angle $\phi \in [0, 2\pi]$. But there is a problem with $0, 2\pi$. We have to define two charts, covering that circle and define transition functions. We can split the circle into two intervals and parametrize them by angles.

$$\psi_1 = \phi_1, \quad \phi_1 \in (0, 2\pi)$$
$$\psi_2 = \phi_2, \quad \phi_2 \in (-\pi, \pi)$$

Though, for $\psi \in (0, 2\pi)$ we can write is as:

$$\psi_1(\phi) = \phi$$
$$\psi_2(\phi) = \phi - \pi$$

The domain od the transition function has to cover $(0, \pi)$ and $(\pi, 2\pi)$.

$$(\psi_1 \circ \psi_2^{-1})(x) = \begin{cases} x & x \in (0, \pi) \\ x + 2\pi & x \in (-\pi, 0) \end{cases}$$

Problem 2 Define atlas and transition functions for S^2. Use the stereographic projection.

We consider a sphere $x'^2 + y'^2 + z'^2 = 1$. Let ψ_1 be a stereographic projection such that it maps the southern pole to $(0,0)$, and ψ_2 maps the northern pole.

$$\begin{aligned}
\psi_1\colon (x', y', z') &\mapsto \left(\frac{x'}{1-z'}, \frac{y'}{1-z'}\right) = (x, y) \\
\psi_2\colon (x', y, z') &\mapsto \left(\frac{x'}{1+z'}, \frac{y'}{1+z'}\right) = (x'', y'') \\
\psi_1 \circ \psi_2^{-1}\colon (x'', y'') &\to (x, y) \\
\psi_1 \circ \psi_2^{-1}\colon (x'', y'') &\mapsto \left(\frac{x''}{x''^2 + y''^2}, \frac{y''}{x''^2 + y''^2}\right)
\end{aligned}$$

And it is defined for $(x'', y'') \neq (0,0)$. It does make sense, because $(0,0)$ was the point, where the poles could go, so the ψ_1 and ψ_2 cover simultaneously each point on the sphere, except these two poles.

We can also see that $\psi_1 \circ \psi_2^{-1}$ describes the inversion!

Problem 3 Consider the following tensor: $g = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}}$ in a part of $\mathbb{R}^2\colon \left\{(t, r)\colon ct \in \mathbb{R},\ (0 < r < 2m \vee 2m < r)\right\}$. Find a new coordinate $u(r, t)$, such that in the coordinates (u, r) the tensor is well defined at $r = 2m$.

Let introduce the new, intermediate variable.

$$\begin{aligned}
\mathrm{d}\hat{r} &= \frac{\mathrm{d}r}{1 - \frac{2m}{r}} \\
\hat{r} &= \int \frac{\mathrm{d}r}{1 - \frac{2m}{r}}
\end{aligned}$$

And another one,

$$\begin{aligned}
\mathrm{d}u &= \mathrm{d}t + \mathrm{d}\hat{r} \\
\mathrm{d}t &= \mathrm{d}u - \frac{\mathrm{d}r}{1 - \frac{2m}{r}}
\end{aligned}$$

Now we can rewrite the metric in terms of (u, r).

$$\begin{aligned}
g &= -\left(1 - \frac{2m}{r}\right)\left(\mathrm{d}u^2 - \frac{2\,\mathrm{d}u\,\mathrm{d}r}{1 - \frac{2m}{r}} + \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r}\right)^2}\right) + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} \\
&= -\left(1 - \frac{2m}{r}\right)\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r
\end{aligned}$$

Now this metric is well defined even for $r = 2m$.

Lecture 8: Connections, covariant derivative

First we shall see some motivation for this topic, by postulating the naive (and wrong) definition of a derivative operator that should be covariant. Let us consider two vector fields $v = v^a(x)\,\partial_a$ $w = w^a(x)\,\partial_a$. Thinking too simply, we would consider 08 gru 2020

$$"\nabla_v w = v^a \frac{\partial}{\partial x^a}(w^b)\frac{\partial}{\partial x^b}$$

As it turns out such operation isn't covariant as we would want it to be (we want to consider theories that are not based on parametrization). Let us change coordinates:

$$v = v^a \frac{\partial x'^c}{\partial x^a}\frac{\partial}{\partial x'^c}$$
$$w = w^b \frac{\partial x'^d}{\partial x^b}\frac{\partial}{\partial x'^d}$$

Then,

$$\begin{aligned}
"\nabla_v w &= v^a \frac{\partial x'^c}{\partial x^a}\frac{\partial}{\partial x'^c}(w^b)\frac{\partial x'^d}{\partial x^b}\frac{\partial}{\partial x'^d} \\
&= v^a \frac{\partial x'^c}{\partial x^a}\frac{\partial}{\partial x'^c}\left(w^b \frac{\partial x'^d}{\partial x^b}\right)\frac{\partial}{\partial x'^d} - v^a \frac{\partial x'^c}{\partial x^a}\frac{\partial^2 x'^d}{\partial x'^c \partial x^b} w^b \frac{\partial}{\partial x'^d} \\
&= \left(v'^c \frac{\partial}{\partial x'^c}(w'^d) - v'^c w'^e \frac{\partial x^b}{\partial x'^e}\frac{\partial^2 x'^d}{\partial x'^c \partial x^b}\right)\frac{\partial}{\partial x'^d}
\end{aligned}$$

Now it is clear, that such definition is not covariant, as not always the second partial derivative cancels out.

Connection 1-form on a tangent frame bundle

Definition 30. The tangent frame bundle of a manifold M is:

$$\mathrm{T}(M) \overset{\text{def}}{=} \{(m, v) \colon m \in M,\, v \in \mathrm{T}_m M\} = \bigcup_{m \in M} \{m\} \times \mathrm{T}_m M$$

and $(e_1, \ldots, e_n)$ is a basis of $\mathrm{T}_m M$.

Definition 31. A connection 1-form is a matrix of 1-forms $(\Gamma^i{}_j)_{j=1,\ldots,n}^{i=1,\ldots,n}$ defined on $U \subset M$ given n-vector fields $e_1, \ldots, e_n$ on U such that $(e_1, \ldots, e_n)$ is a basis of $\mathrm{T}_m M$ for every $m \in U$. What makes the connection is the following transformation law:

$$\begin{aligned}
e'_i &= A^j{}_i e_j \\
\Gamma'^i{}_j &= (A^{-1})^i{}_k \Gamma^k{}_l A^l{}_j + (A^{-1})^i{}_k \,\mathrm{d}A^k{}_j
\end{aligned}$$

where A is a basis transformation matrix (Jacobian).

Remark 11. Now everything can vary from point to point, so ${A^j}_i$ is a function, not a constant linear transformation. We define the connection coefficient ${\Gamma^i}_{jk}$ (its components are functions) as follows:

$$ {\Gamma^i}_j = {\Gamma^i}_{jk} e^k $$

Note that it transforms like a tensor only under linear coordinate transformations, so it is not a real tensor (therefore we call it „coefficient") under general diffeomorphisms.

If we consider the metric, torsion free connection, then ${\Gamma^i}_{jk}$ is called the Christoffel symbol.

Example Let's take the following coordinates: $(e_1, \ldots, e_n) = (\partial/\partial x^1, \ldots, \partial/\partial x^n)$ and $(e'_1, \ldots, e'_n) = (\partial/\partial x'^1, \ldots, \partial/\partial x'^n)$. Then,

$$ \begin{aligned} e'_i &= \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} \\ {A^j}_i &= \frac{\partial x^j}{\partial x'^i} \end{aligned} $$

Curvature 2-form of a connection 1-form

Definition 32. The curvature 2-form, denoted by ${\mathcal{R}^i}_j$, is defined as:

$$ {\mathcal{R}^i}_j \overset{\text{def}}{=} \mathrm{d}{\Gamma^i}_j + {\Gamma^i}_k \wedge {\Gamma^k}_j $$

The transformation rule follows from the definition:

$$ {\mathcal{R}'^i}_j = {(A^{-1})^i}_k {\mathcal{R}^k}_l {A^l}_j $$

The conclusion is that this is a tensor field (because it transforms like tensors) and we can express it as:

$$ {\mathcal{R}^i}_j = \frac{1}{2} {R^i}_{jkl} e^k \wedge e^l $$

From these, the 1st Bianchi identity emerges:

Theorem 3 (1st Bianchi identity)**.**

$$ \mathrm{d}{\mathcal{R}^i}_j + {\Gamma^i}_k \wedge {\mathcal{R}^k}_j - {\mathcal{R}^i}_k \wedge {\Gamma^k}_j = 0 $$

Proof.

$$ \begin{aligned} &\mathrm{d}{\mathcal{R}^i}_j + {\Gamma^i}_k \wedge {\mathcal{R}^k}_j - {\mathcal{R}^i}_k \wedge {\Gamma^k}_j = \\ &= (\mathrm{d}^2{\Gamma^i}_j + \mathrm{d}{\Gamma^i}_k \wedge {\Gamma^k}_j - {\Gamma^i}_k \wedge \mathrm{d}{\Gamma^k}_j) + ({\Gamma^i}_k \wedge \mathrm{d}{\Gamma^k}_j + {\Gamma^i}_k \wedge {\Gamma^k}_l \wedge {\Gamma^l}_j) \\ &\quad - (\mathrm{d}{\Gamma^i}_k \wedge {\Gamma^k}_j + {\Gamma^i}_l \wedge {\Gamma^l}_j \wedge {\Gamma^k}_j) \\ &= \mathrm{d}{\Gamma^i}_k \wedge ({\Gamma^k}_j - {\Gamma^k}_j) + ({\Gamma^i}_k - {\Gamma^i}_k) \wedge \mathrm{d}{\Gamma^k}_j = 0 \end{aligned} $$

■

Covariant derivative

The covariant derivative is defined via the connection form. Let $f \in C^1(M)$.

$$\nabla f \stackrel{\text{def}}{=} \mathrm{d}f$$

Let $v = v^i e_i$ be a vector field.

$$\nabla v \stackrel{\text{def}}{=} \Big(\mathrm{d}v^i + \Gamma^i{}_j v^j\Big) \otimes e_i$$

Let $\omega = \omega_i e^i$ be a 1-form field.

$$\nabla \omega \stackrel{\text{def}}{=} \Big(\mathrm{d}\omega_i - \Gamma^j{}_i \omega_j\Big) \otimes e^i$$

Now we can easily generalise to arbitrary tensor fields.

Definition 33 (Covariant derivative)**.** Let $T = T^{i\cdots}{}_{\cdots j}\, e_i \otimes \cdots \otimes e^j$. Then,

$$\nabla T \stackrel{\text{def}}{=} \Big(\mathrm{d}T^{i\cdots}{}_{\cdots j} + \Gamma^i{}_k T^{k\cdots}{}_{\cdots j} + \cdots - \Gamma^k{}_j T^{i\cdots}{}_{\cdots k}\Big) \otimes e_i \otimes \cdots \otimes e^j$$

There's a catch!

$$\nabla(T \otimes L) \neq \nabla T \otimes L + T \otimes \nabla L$$

because of the ordering obstacle.

Definition 34. Now let's consider the covariant derivative with respect to a vector field.

$$\nabla_Y T \stackrel{\text{def}}{=} Y \lrcorner \nabla T$$

If $Y \in \mathrm{T}_m M$, $v = v^i e_i$, $\omega = \omega_i e^i$,

$$\begin{aligned}
\nabla_Y f &= Y \lrcorner\, \mathrm{d}f = Y(f) \\
\nabla_Y v &= Y \lrcorner (\mathrm{d}v^i + \Gamma^i{}_j v^j) \otimes e_i = \Big(Y(v^i) + Y^k \Gamma^i{}_{jk} v^j\Big) e_i \\
&= Y^k \Big(e_k(v^i) + \Gamma^i{}_{jk} v^j\Big) e_i
\end{aligned}$$

And we denote it by

$$= Y^k (\nabla_k v^i) e_i$$

For a 1-form it is similar,

$$\nabla_Y \omega = Y^k \Big(e_k(\omega_i) - \Gamma^j{}_{ik} \omega_j\Big) e^i = Y^k (\nabla_k \omega_i) e^i$$

And for an arbitrary tensor field:

$$\begin{aligned}
\nabla_Y T &= \Big(Y(T^{i\cdots}{}_{\cdots j}) + \Gamma^i{}_{lk} Y^k T^{l\cdots}{}_{\cdots j} + \cdots - \Gamma^l{}_{jk} Y^k T^{i\cdots}{}_{\cdots l}\Big) e_i \otimes \cdots \otimes e^j \\
&= Y^k \Big(e_k(T^{i\cdots}{}_{\cdots j}) + \Gamma^i{}_{lk} T^{l\cdots}{}_{\cdots j} + \cdots - \Gamma^l{}_{jk} T^{i\cdots}{}_{\cdots l}\Big) e_i \otimes \cdots \otimes e^j \\
&= Y^k (\nabla_k T^{i\cdots}{}_{\cdots j}) e_i \otimes \cdots \otimes e^j
\end{aligned}$$

We have introduced the following short-hand notation:

$$\begin{aligned}\nabla_k v^i &= e_k(v^i) + \Gamma^i{}_{jk} v^j\\ \nabla_k \omega_i &= e_k(\omega_i) - \Gamma^j{}_{ik}\omega_j\\ \nabla_k T^{i\cdots}{}_{\cdots j} &= e_k(T^{i\cdots}{}_{\cdots j}) + \Gamma^i{}_{lk} T^{l\cdots}{}_{\cdots j} + \cdots - \Gamma^l{}_{jk} T^{i\cdots}{}_{\cdots l}\end{aligned}$$

There is an important property – covariant derivative commutes with contraction.

$$\begin{aligned}\nabla_Y(v^i\omega_i) &= Y(v^i\omega_i) = Y(v^i)\omega_i + v^i Y(\omega_i)\\ &= (\nabla_Y v^i)\omega_i + v^i(\nabla_Y \omega_i)\\ &= (\nabla_Y v)^i\omega_i + v^i(\nabla_Y\omega)_i\end{aligned}$$

Remark 12. Summary of ∇_Y properties defined at $m \in M$ for $Y \in \mathrm{T}_m M$, a tensor field T.

$$\begin{aligned}\nabla_{\alpha Y + \beta Z} T &= \alpha \nabla_Y T + \beta \nabla_Z T\\ \nabla_Y f &= Y(f)\\ \nabla_Y(T \otimes L) &= \nabla_Y T \otimes L + T \otimes \nabla_Y L\\ \nabla_Y(T + L) &= \nabla_Y T + \nabla_Y L\\ \nabla_i(T^{\cdots k}{}_{\cdots} L^{\cdots}{}_{k\cdots}) &= (\nabla_i T^{\cdots k}{}_{\cdots}) L^{\cdots}{}_{k\cdots} + T^{\cdots k}{}_{\cdots}(\nabla_i L^{\cdots}{}_{k\cdots})\end{aligned}$$

Comparison of ∇_Y with $\mathcal{L}_Y$ To define $\mathcal{L}_Y$ we need a vector field Y, when for ∇_Y we need just a vector $Y \in \mathrm{T}_m M$. The main point is that given $Y = Y^i e_i$, $\nabla_Y = Y^i \nabla_{e_i}$. On the other hand the Lie derivative depends on $Y^i(m), e_j(Y^i)(m)$.

The key property is that ∇T is independent on a choice of frame field $(e_1, \ldots, e_n)$ at each point $m \in U \subset M$.

Remark 13. ∇T is independent of a choice of frame $(e_1, \ldots, e_n)$ at each point $m \in U \subset M$.

Exterior covariant derivative

Consider a vector valued k-form:

$$\omega = \omega^i \otimes e_i, \quad \omega^i = \frac{1}{k!}\omega^i{}_{j_1\cdots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k}$$

Then,

$$\mathrm{D}\omega \stackrel{\text{def}}{=} \left(\mathrm{d}\omega^i + \Gamma^i{}_k \wedge \omega^k\right) \otimes e_i$$

Consider a co-vector valued k-form:

$$\omega = \omega_i \otimes e^i, \quad \omega_i = \frac{1}{k!}\omega_{ij_1\cdots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k}$$

$$\mathrm{D}\omega \stackrel{\text{def}}{=} \left(\mathrm{d}\omega_i - \Gamma^k{}_i \wedge \omega_k\right) \otimes e^i$$

Definition 35 (Exterior covariant derivative)**.** Consider a tensor valued k-form

$$\omega = \omega^{i\cdots}{}_{\cdots j} \otimes e_i \otimes \cdots \otimes e^j, \quad \omega^{i\cdots}{}_{\cdots j} = \frac{1}{k!}\omega^{i\cdots}{}_{\cdots j j_1 \cdots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k}$$

$$\mathrm{D}\omega \overset{\text{def}}{=} \Big(\mathrm{d}\omega^{i\cdots}{}_{\cdots j} + \Gamma^i{}_l \wedge \omega^{l\cdots}{}_{\cdots j} + \cdots - \Gamma^l{}_j \wedge \omega^{i\cdots}{}_{\cdots l}\Big) \otimes e_i \otimes \cdots \otimes e^j$$

If $\omega = \omega^i \otimes e_i$ is a vector valued k-form,

$$\begin{aligned}
\mathrm{D}^2\omega &= \mathrm{D}(\mathrm{D}\omega) = \mathrm{D}\Big[(\mathrm{d}\omega^i + \Gamma^i{}_k \wedge \omega^k) \otimes e_i\Big] \\
&= \Big(\mathrm{d}^2\omega^i + \Gamma^i{}_l \wedge \mathrm{d}\omega^l + \mathrm{d}(\Gamma^i{}_k) \wedge \omega^k - \Gamma^i{}_k \wedge \mathrm{d}\omega^k + \Gamma^i{}_l \wedge \Gamma^l{}_k \wedge \omega^k\Big) \otimes e_i \\
&= \Big(\mathrm{d}(\Gamma^i{}_k) \wedge \omega^k + (\Gamma^i{}_l \wedge \Gamma^l{}_k) \wedge \omega^k\Big) \otimes e_i = (\mathcal{R}^i{}_k \wedge \omega^k) \otimes e_i \\
&= (\mathcal{R}^i{}_j \wedge \omega^j) \otimes e_i
\end{aligned}$$

If $\omega = \omega_i \otimes e^i$ i a co-vector valued k-form,

$$\mathrm{D}^2\omega = \mathrm{D}(\mathrm{D}\omega) = -(\mathcal{R}^j{}_i \wedge \omega_j) \otimes e^i$$

Suppose that we have a wedge product of two such vector/co-vector valued forms, where μ is k-form, and evaluate exterior covariant derivative on just components:

$$\begin{aligned}
\mathrm{D}(\mu^i \wedge \nu_j) &= \mathrm{D}\mu^i \wedge \nu_j + (-1)^k \mu^i \wedge \mathrm{D}\nu_j \\
\mathrm{D}^2(\mu^i \wedge \nu_j) &= \mathrm{D}^2\mu^i \wedge \nu_j + (-1)^{k+1}\mathrm{D}\mu^i \wedge \mathrm{D}\nu_j \\
&\quad + (-1)^k \mathrm{D}\mu^i \wedge \mathrm{D}\nu_j + (-1)^{2k}\mu^i \wedge \mathrm{D}^2\nu_j \\
&= \mathrm{D}^2\mu^i \wedge \nu_j + \mu^i \wedge \mathrm{D}^2\nu_j \\
&= (\mathcal{R}^i{}_k \wedge \mu^k) \wedge \nu_j - \mu^i \wedge (\mathcal{R}^k{}_j \wedge \nu_k) \\
&= \mathcal{R}^i{}_k \wedge (\mu^k \wedge \nu_j) - \mathcal{R}^k{}_j \wedge (\mu^i \wedge \nu_k)
\end{aligned}$$

Now we can make it more compact. Let's take a $(1,1)$-tensor valued k-form $\omega = \omega^i{}_j \otimes e_i \otimes e^j$, where $\omega^i{}_j = \frac{1}{k!}\omega^i{}_{j j_1 \cdots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k}$. Then,

$$\mathrm{D}^2\omega = (\mathcal{R}^i{}_k \wedge \omega^k{}_j - \mathcal{R}^k{}_j \wedge \omega^i{}_k) \otimes e_i \otimes e^j$$

Theorem 4 (2nd Bianchi identity)**.**

$$\mathrm{D}(\mathcal{R}^i{}_j \otimes e_i \otimes e^j) = 0$$

Proof. $\mathcal{R}^i{}_j \otimes e_i \otimes e^j$ is a $(1,1)$-tensor valued 2-form, so

$$\begin{aligned}
\mathrm{D}(\mathcal{R}^i{}_j \otimes e_i \otimes e^j) &= \Big(\mathrm{d}\mathcal{R}^i{}_j + \Gamma^i{}_l \wedge \mathcal{R}^l{}_j - \Gamma^l{}_j \wedge \mathcal{R}^i{}_l\Big) \otimes e_i \otimes e^j \\
&= \Big(\mathrm{d}\mathcal{R}^i{}_j + \Gamma^i{}_l \wedge \mathcal{R}^l{}_j - (-1)^{1\cdot 2}\mathcal{R}^i{}_l \wedge \Gamma^l{}_j\Big) \otimes e_i \otimes e^j
\end{aligned}$$

Now the part in the bracket is exactly the 1st Bianchi identity, so

$$= 0$$

■

Torsion of a connection

Let's define the Kronecker $\delta^i{}_j$ tensor as:

$$
\begin{aligned}
K &\stackrel{\text{def}}{=} \delta^i{}_j e^j \otimes e_i = e^j \otimes e_j = e'^j \otimes e'_j \\
\nabla K &= \left(\mathrm{d}\delta^i{}_j + \Gamma^i{}_l \delta^l{}_j - \Gamma^l{}_j \delta^i{}_l\right) \otimes e^j \otimes e_i \\
&= \left(0 + \Gamma^i{}_j - \Gamma^i{}_j\right) \otimes e^j \otimes e_i = 0
\end{aligned}
$$

This is a covariantly constant tensor (as we would expect).
Tensor K can be also viewed as a vector valued 1-form $K = \omega^i \otimes e_i$, where $\omega^i = e^i$. Thus,

$$
\mathrm{D}K = \left(\mathrm{d}e^i + \Gamma^i{}_k \wedge e^k\right) \otimes e_i
$$

We define,

Definition 36 (Torsion)**.** A connection $(\Gamma^i{}_j)^{i=1,\dots,n}_{j=1,\dots,n}$ has an associated torsion tensor:

$$
\text{Torsion} = \mathrm{D}K = \left(\mathrm{d}e^j + \Gamma^j{}_k \wedge e^k\right) \otimes e_j
$$

A connection Γ is torsion free if $\mathrm{D}K = 0$, ie. $\mathrm{d}e^j + \Gamma^j{}_k \wedge e^k = 0$.

Now let's consider a torsion free connection.

$$
\begin{aligned}
0 = \mathrm{D}^2 K &= (\mathcal{R}^i{}_j \wedge e^j) \otimes e_i \\
\mathcal{R}^i{}_j \wedge e^j &= 0 = \frac{1}{2} R^i{}_{jkl} e^k \wedge e^l \wedge e^j
\end{aligned}
$$

It is equivalent of saying, that

$$
\begin{aligned}
3\frac{1}{3!} R^i{}_{jkl} e^j \wedge e^k \wedge e^l &= 0 = 3R^i{}_{[jkl]} e^j \otimes e^k \otimes e^l \\
R^i{}_{[jkl]} &= 0
\end{aligned}
$$

Torsion free, metric connection

Definition 37 (A spacetime metric tensor)**.** On a manifold M consider a tensor field

$$
M \ni m \mapsto g \in \mathrm{T}^*_m M \otimes \mathrm{T}^*_m M
$$

such that it is symmetric:

$$
\begin{aligned}
Y \lrcorner (X \lrcorner g) &= X \lrcorner (Y \lrcorner g) \\
g(X, Y) &= g(Y, X)
\end{aligned}
$$

and has signature $(- + \cdots +)$. We call it a metric tensor.

Theorem 5. Given a metric tensor g, there is a unique connection on the tangent frame bundle, such that $\mathrm{D}K = 0$ and $\nabla g = 0$.
We also call it the Levi-Civita connection.

Proof. Let $(x^1, \ldots, x^n)$ be local coordinates in some open $U \subset M$. Consider a basis $(e_1, \ldots, e_n) = (\partial/\partial x^1, \ldots, \partial/\partial x^n)$ and a dual basis $(e^1, \ldots, e^n) = (\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$.
Suppose $(\Gamma^i{}_j)$ is a connection 1-form. Let's check the condition of being torsion free.

$$\begin{aligned}
\text{Torsion} = \mathrm{D}K &= \left(\mathrm{d}e^i + \Gamma^i{}_k \wedge e^k\right) \otimes e_i \\
&= \left(\mathrm{d}(\mathrm{d}x^i) + \Gamma^i{}_k \wedge \mathrm{d}x^k\right) \otimes \frac{\partial}{\partial x^i} \\
&= \Gamma^i{}_k \wedge \mathrm{d}x^k \otimes \frac{\partial}{\partial x^i} \\
&= \Gamma^i{}_{kl}\, \mathrm{d}x^l \wedge \mathrm{d}x^k \otimes \frac{\partial}{\partial x^i} \\
&= -2\Gamma^i{}_{[kl]}\, \mathrm{d}x^k \otimes \mathrm{d}x^l \otimes \frac{\partial}{\partial x^i}
\end{aligned}$$

From our condition,

$$\mathrm{D}K = 0 \iff \Gamma^i{}_{[kl]} = 0$$

It means, that:

$$\begin{aligned}
2\Gamma^i{}_{[kl]} &= \Gamma^i{}_{kl} - \Gamma^i{}_{lk} = 0 \\
\Gamma^i{}_{kl} &= \Gamma^i{}_{lk}
\end{aligned}$$

Now we investigate the metric tensor.

$$\begin{aligned}
g &= g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \\
\nabla g &= \left(\mathrm{d}g_{ij} - \Gamma^l{}_i g_{lj} - \Gamma^l{}_j g_{il}\right) \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j = 0 \\
0 &= g_{ij,k}\, \mathrm{d}x^k - \Gamma^l{}_{ik} g_{lj}\, \mathrm{d}x^k - \Gamma^l{}_{jk} g_{il}\, \mathrm{d}x^k
\end{aligned}$$

Now we just lower the indices and get system of two tensor equations to solve:

$$\begin{cases} g_{ij,k} - \Gamma_{jik} - \Gamma_{ijk} = 0 \\ \Gamma_{ijk} = \Gamma_{ikj} \end{cases}$$

The idea is to solve it given g_{ij} with respect to the unknown Γ_{ijk}.

$$\begin{cases} g_{ij,k} - \Gamma_{jik} - \Gamma_{ijk} = 0 \\ g_{jk,i} - \Gamma_{kji} - \Gamma_{jki} = 0 \\ g_{ki,j} - \Gamma_{ikj} - \Gamma_{kij} = 0 \end{cases}$$

Let's add the first two rows and subtract the third.

$$\begin{aligned}
2\Gamma_{jki} &= g_{ij,k} + g_{jk,i} - g_{ki,j} \\
\Gamma^l{}_{ki} &= \frac{1}{2} g^{lj} (g_{ij,k} + g_{jk,i} - g_{ki,j}) \\
\Gamma^i{}_{jk} &= \frac{1}{2} g^{im} \left(\frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right)
\end{aligned}$$

This identity is fairy easy too remember: i stays on the top, we have to exchange (j,k) between numerator and denominator, when there must be some one index more (dummy m). And the only term with „−" is this one, where we differentiate over dummy index. Easy!

This way we have given the exact formula for ${\Gamma^i}_{jk}$, so there indeed is a unique connection. ■

≻ Problem class 8

Problem 1 Consider a 2-dimensional manifold and a (local) frame $e^1 = \mathrm{d}x$, $e^2 = \mathrm{d}y$. Knowing, that $({\Gamma^i}_j)^{i=x,y}_{j=x,y} = 0$, suppose that $x = r\cos\phi$, $y = r\sin\phi$ and ${e'}_1 = \partial_r$, ${e'}_2 = \partial_\phi$. Find ${\Gamma'^i}_j$.

The general formula is:

$$\begin{aligned} {e'}_i &= {A^j}_i e_j \quad (e^j = {A^j}_i e'^i) \\ {\Gamma'^i}_j &= {(A^{-1})^i}_n {\Gamma^n}_m {A^m}_j + {(A^{-1})^i}_n \,\mathrm{d}{A^n}_j \end{aligned}$$

As ${\Gamma^i}_j = 0$, we can write in matrix notation as follows:

$$\Gamma' = A^{-1}\,\mathrm{d}A$$

We treat upper indices as the rows of a matrix and lower indices as its columns. It means that A_i is represented by $1 \times n$ matrix, A^i by $n \times 1$ matrix and ${A^i}_j$ by $n \times n$ matrix. Thus,

$$\begin{aligned} \partial_r &= \cos\phi\,\partial_x + \sin\phi\,\partial_y \\ \partial_\phi &= -r\sin\phi\,\partial_x + r\cos\phi\,\partial_y \\ {e'}_i &= {A^j}_i e_j = e_j {A^j}_i = EA \\ &= \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \end{aligned}$$

Then, the inverse and exterior derivative:

$$\begin{aligned} A^{-1} &= \frac{1}{r}\begin{pmatrix} r\cos\phi & r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \\ \mathrm{d}A &= \begin{pmatrix} -\sin\phi\,\mathrm{d}\phi & -\sin\phi\,\mathrm{d}r - r\cos\phi\,\mathrm{d}\phi \\ \cos\phi\,\mathrm{d}\phi & \cos\phi\,\mathrm{d}r - r\sin\phi\,\mathrm{d}\phi \end{pmatrix} \end{aligned}$$

Now we can calculate the connections:

$$\begin{aligned} \Gamma' = A^{-1}\,\mathrm{d}A &= \frac{1}{r}\begin{pmatrix} r\cos\phi & r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} -\sin\phi\,\mathrm{d}\phi & -\sin\phi\,\mathrm{d}r - r\cos\phi\,\mathrm{d}\phi \\ \cos\phi\,\mathrm{d}\phi & \cos\phi\,\mathrm{d}r - r\sin\phi\,\mathrm{d}\phi \end{pmatrix} \\ &= \frac{1}{r}\begin{pmatrix} 0 & -r^2\,\mathrm{d}\phi \\ \mathrm{d}\phi & \mathrm{d}r \end{pmatrix} \end{aligned}$$

We shall also calculate the curvature.

$$\begin{aligned}
\mathcal{R}^i{}_j &= \mathrm{d}\Gamma^i{}_j = \Gamma^i{}_k \wedge \Gamma^k{}_j \\
\mathcal{R} &= \mathrm{d}\Gamma + \Gamma \wedge \Gamma
\end{aligned}$$

We expect, that $\mathcal{R}' = 0$.

$$\begin{aligned}
\mathrm{d}\Gamma' &= \begin{pmatrix} 0 & -\,\mathrm{d}r \wedge \mathrm{d}\phi \\ -\frac{1}{r^2}\,\mathrm{d}r \wedge \mathrm{d}\phi & 0 \end{pmatrix} \\
\Gamma' \wedge \Gamma' &= \frac{1}{r^2}\begin{pmatrix} 0 & -r^2\,\mathrm{d}\phi \\ \mathrm{d}\phi & \mathrm{d}r \end{pmatrix} \wedge \begin{pmatrix} 0 & -r^2\,\mathrm{d}\phi \\ \mathrm{d}\phi & \mathrm{d}r \end{pmatrix} \\
&= \frac{1}{r^2}\begin{pmatrix} 0 & r^2\,\mathrm{d}r \wedge \mathrm{d}\phi \\ \mathrm{d}r \wedge \mathrm{d}\phi & 0 \end{pmatrix} = -\,\mathrm{d}\Gamma'
\end{aligned}$$

Indeed, $\mathcal{R}' = 0$.

Problem 2 Completed in the lecture notes.

Problem 3 Calculate the Levi-Civita connection $\Gamma^i{}_{jk}$ for $g = -f(r)\,\mathrm{d}t^2 + g(r)\,\mathrm{d}r^2$.

We have $g_{tt} = -f(r)$, $g_{rr} = g(r)$, $g_{tr} = g_{rt} = 0$ and $g^{tt} = -1/f(r)$, $g^{rr} = 1/g(r)$. Then,

$$g_{tt,r} = -\frac{\partial f}{\partial r}, \quad g_{rr,r} = \frac{\partial g}{\partial r}$$

These are the only non-zero derivatives. The calculation rule is as follows:

$$\begin{aligned}
\Gamma^i{}_{jk} &= \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}) \\
\Gamma^r{}_{rr} &= \frac{1}{2} g^{rr} g_{rr,r} + \frac{1}{2} g^{rt}(2g_{rt,r} - g_{rr,t}) = \frac{1}{2}\frac{g'(r)}{g(r)} \\
\Gamma^t{}_{rr} &= \frac{1}{2} g^{tt}(2g_{rt,r} - g_{rr,t}) = 0 \\
\Gamma^r{}_{tt} &= \frac{1}{2} g^{rr}(2g_{tr,t} - g_{tt,r}) = \frac{1}{2}\frac{f'(r)}{g(r)} \\
\Gamma^t{}_{tt} &= \frac{1}{2} g^{tt} g_{tt,t} = 0
\end{aligned}$$

Now let's see what about the mixed terms.

$$\begin{aligned}
\Gamma^t{}_{rt} = \Gamma^t{}_{tr} &= \frac{1}{2} g^{tt}(g_{rt,t} + g_{tt,r} - g_{rt,t}) = \frac{1}{2}\frac{f'(r)}{f(r)} \\
\Gamma^r{}_{rt} = \Gamma^r{}_{tr} &= \frac{1}{2} g^{rr}(g_{rr,t} + g_{tr,r} - g_{rt,r}) = 0
\end{aligned}$$

These are all $2^3 = 8$ Christoffel symbols.

$$\Gamma^r{}_{ij} = \begin{pmatrix} \frac{1}{2}\frac{g'(r)}{g(r)} & 0 \\ 0 & \frac{1}{2}\frac{f'(r)}{g(r)} \end{pmatrix}$$

$$\Gamma^t{}_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\frac{f'(r)}{f(r)} \\ \frac{1}{2}\frac{f'(r)}{f(r)} & 0 \end{pmatrix}$$

Problem 4 Calculate the torsion free and metric connection $(\Gamma^i{}_j)^{i=1,2,3}_{j=1,2,3}$ for the metric $g = (e^1)^2 + (e^2)^2 + (e^3)^2$ where $\mathrm{d}e^1 = e^2 \wedge e^3$, $\mathrm{d}e^2 = e^3 \wedge e^1$, $\mathrm{d}e^3 = e^1 \wedge e^2$.
Use: $0 = \text{Torsion} = \mathrm{d}e^i + \Gamma^i{}_j \wedge e^j$

$$\nabla g = \mathrm{d}g_{ij} + \Gamma^n{}_i g_{nj} + \Gamma^n{}_j g_{in} = 0$$

Note, that $g_{ii} = 1$, so $\mathrm{d}g_{ij} = 0$. This equation (if we lower the indices) transforms to:

$$\begin{aligned} \Gamma_{ji} + \Gamma_{ij} &= 0 \\ \Gamma_{ji} &= -\Gamma_{ij} \end{aligned}$$

Actually, $g_{ij} = \delta_{ij}$, so

$$\Gamma^j{}_i = -\Gamma^i{}_j$$

Also,

$$0 = \mathrm{d}e^i + \Gamma^i{}_j \wedge e^j$$

What can be turned into system of equations:

$$\begin{cases} 0 = e^2 \wedge e^3 + \Gamma^1{}_2 \wedge e^2 + \Gamma^1{}_3 \wedge e^3 \\ 0 = e^3 \wedge e^1 + \Gamma^2{}_1 \wedge e^1 + \Gamma^2{}_3 \wedge e^3 \\ 0 = e^1 \wedge e^2 + \Gamma^3{}_1 \wedge e^1 + \Gamma^3{}_2 \wedge e^2 \end{cases}$$

Now we add them together and use the antisymmetry of Γ,

$$e^2 \wedge e^3 + e^3 \wedge e^1 + e^1 \wedge e^2 + \Gamma^1{}_2 \wedge (e^2 - e^1) + \Gamma^1{}_3 \wedge (e^3 - e^1) + \Gamma^2{}_3 \wedge (e^3 - e^2) = 0$$

Now maybe it could be solved xD. Anyway, because of the symmetries we have a system of 3 equations with 3 unknown 1-forms. The solution is $\Gamma^i{}_{jk} = \frac{1}{2}\varepsilon^i{}_{jk}$. The wider context and proof is in my solutions to Problem set 4.

Lecture 9: Spacetime geometry, curvature

15 gru 2020

Properties of a covariant derivative

Let $(e_1, \ldots, e_n)$ be a frame, $\Gamma^i{}_j = \Gamma^i{}_{jk} e^k$ the connection form and $T = T^{i\cdots}{}_{\cdots j}\, e_i \otimes \cdots \otimes e^j$. Then,

$$\begin{aligned} \nabla T &= \nabla_k T^{i\cdots}{}_{\cdots j}\, e^k \otimes e_i \otimes \cdots \otimes e^j \\ \nabla_k T^{i\cdots}{}_{\cdots j} &= e_k(T^{i\cdots}{}_{\cdots j}) + \Gamma^i{}_{lk} T^{l\cdots}{}_{\cdots j} + \cdots - \Gamma^l{}_{jk} T^{i\cdots}{}_{\cdots l} \end{aligned}$$

Then,

$$(\nabla_X T)^{i\cdots}{}_{\cdots j} = X^k \nabla_k T^{i\cdots}{}_{\cdots j}$$

Let's consider the covariant derivative, acting two times on a tensor.

$$\nabla\nabla T = \nabla_m (\nabla_n T^{i\cdots}{}_{\cdots j}) e^m \otimes e^n \otimes e_i \otimes \cdots \otimes e^j$$

Now it is important to know what you contract and how, when considering derivatives along vector fields:

$$\begin{aligned} \nabla_X(\nabla_Y T) &= X^m \nabla_m (Y^n \nabla_n T) \\ &= X^m Y^n \nabla_m \nabla_n T + (\nabla_X Y)^n \nabla_n T \end{aligned}$$

Then we can consider a vector Z^i,

$$\begin{aligned} \nabla_m \nabla_n Z^i - \nabla_n \nabla_m Z^i &= R^i{}_{jmn} Z^j \\ \big((\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z\big)^i &= R^i{}_{jmn} Z^j X^m Y^n + \big(\nabla_{\nabla_X Y - \nabla_Y X} Z\big)^i \end{aligned}$$

Achtung! Be careful, in Wald there is another convention of defining R, which is equivalent in the metric, torsion free case but not in general. Similarly, we proceed with a 1-form ω_i,

$$\begin{aligned} \nabla_m \nabla_n \omega_i - \nabla_n \nabla_m \omega_i &= -R^j{}_{imn} \omega_j \\ \big((\nabla_X \nabla_Y - \nabla_Y \nabla_X) \omega\big)_i &= -R^j{}_{imn} \omega_j X^m Y^n + \big(\nabla_{\nabla_X Y - \nabla_Y X} \omega\big)_i \end{aligned}$$

Now let's consider the same directional derivatives acting on a function and use the properties introduced above.

$$\begin{aligned} (\nabla_X \nabla_Y - \nabla_Y \nabla_X) f &= X\big(Y(f)\big) - Y\big(X(f)\big) = [X, Y](f) \\ &= X^m \nabla_m (Y^n \nabla_n f) - Y^n \nabla_n (X^m \nabla_m f) \\ &= X^m Y^n (\nabla_m \nabla_n f - \nabla_n \nabla_m f) + (\nabla_X Y)^n \nabla_n f - (\nabla_Y X)^m \nabla_m f \\ &= X^m Y^n (\nabla_m \nabla_n - \nabla_n \nabla_m) f + \nabla_{\nabla_X Y - \nabla_Y X} f \end{aligned}$$

To conclude,

Remark 14.

$$\begin{aligned} X^m Y^n (\nabla_m \nabla_n - \nabla_n \nabla_m) f &= \big([X, Y] - \nabla_{\nabla_X Y - \nabla_Y X}\big) f \\ &= \big([X, Y] - \nabla_X Y + \nabla_Y X\big) f \end{aligned}$$

Now let's make a step down and constrain ourselves to a torsion free connection, ie. $\mathrm{d}e^i + \Gamma^i{}_j \wedge e^j = 0$.

Lemma 5. For a torsion free connection, the following statements are true:

$$\nabla_X Y - \nabla_Y X = [X, Y]$$
$$\nabla_m \nabla_n f = \nabla_n \nabla_m f$$

Proof. I will prove it quickly in a holonomic base.

$$\begin{aligned}
\nabla_X Y &= X^n \nabla_n Y = X^n \Big(e_n(Y^i) + \Gamma^i{}_{jn} Y^j \Big) e_j \\
\nabla_Y X &= Y^n \nabla_n X = Y^n \Big(e_n(X^i) + \Gamma^i{}_{jn} X^j \Big) e_i \\
(\nabla_X Y - \nabla_Y X)^i &= X^n e_n(Y^i) - Y^n e_n(X^i) + \Gamma^i{}_{jn}(X^n Y^j - Y^n X^j) \\
&= [X, Y]^i + \Gamma^i{}_{jn}(X^n Y^j - Y^n X^j)
\end{aligned}$$

Now, in a holonomic base, $\mathrm{D}K = 0 \implies \Gamma^i{}_{jk} = \Gamma^i{}_{kj}$, so

$$= [X, Y]^i + \Gamma^i{}_{jn} X^n Y^j - \Gamma^i{}_{nj} Y^n X^j$$

After renaming dummy indices,

$$= [X, Y]^i + \Gamma^i{}_{jn} Y^j X^n - \Gamma^i{}_{jn} Y^j X^n = [X, Y]^i$$

The second implication is obtained by, substituting the first one to the Remark (14). ∎

Consider a differential k-form:

$$\begin{aligned}
\omega &= \frac{1}{k!} \omega_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \\
\mathrm{d}\omega &= \frac{1}{k!} e_j(\omega_{i_1 \cdots i_k}) e^j \wedge e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= \frac{1}{k!} e_{[j}(\omega_{i_1 \cdots i_k]}) e^j \wedge e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= \frac{1}{(k+1)!}(k+1) e_{[j}(\omega_{i_1 \cdots i_k]}) e^j \wedge e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= \frac{1}{(k+1)!}(k+1) \nabla_{[j} \omega_{i_1 \cdots i_k]} e^j \wedge e^{i_1} \wedge \cdots \wedge e^{i_k}
\end{aligned}$$

Therefore, we can see „d" as a differential operator, that assigns a $(k+1)$-form to a k-form in a such way, that:

$$\mathrm{d}\colon \omega_{i_1 \cdots i_k} \mapsto (k+1) \nabla_{[j} \omega_{i_1 \cdots i_k]}$$

Curvature of a torsion free connection

$$\mathcal{R}^i{}_j = \frac{1}{2} R^i{}_{jkl} e^k \wedge e^l = \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$$

$R^i{}_{jkl} e_i \otimes \cdots \otimes e^l$ turns out to be a tensor. Now, let's consider a torsion free connection.

$$\begin{aligned}
0 &= \mathrm{d}e^i + \Gamma^i{}_j \wedge e^j \\
0 &= \mathrm{d}^2 e^i + \mathrm{d}\Gamma^i{}_j \wedge e^j - \Gamma^i{}_j \wedge \mathrm{d}e^j \\
0 &= \mathrm{d}\Gamma^i{}_j \wedge e^j + \Gamma^i{}_j \wedge \Gamma^j{}_k \wedge e^k \\
0 &= (\mathrm{d}\Gamma^i{}_k + \Gamma^i{}_j \wedge \Gamma^j{}_k) \wedge e^k = \mathcal{R}^i{}_k \wedge e^k
\end{aligned}$$

Now, we get the following identity,

$$0 = \mathcal{R}^i{}_j \wedge e^j = \frac{1}{2} R^i{}_{jkl} e^k \wedge e^l \wedge e^j = \frac{1}{2} R^i{}_{jkl} e^j \wedge e^k \wedge e^l$$

Therefore,

Remark 15.

$$R^i{}_{[jkl]} = 0$$

By using the obvious symmetry $R^i{}_{jkl} = -R^i{}_{jlk}$,

$$R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$$

This is called The 1st algebraic Bianchi identity. Also, the following 2nd differential Bianchi identity holds:

$$\nabla_m R^i{}_{jkl} + \nabla_k R^i{}_{jlm} + \nabla_l R^i{}_{jmk} = 0$$
$$\nabla_{[m|} R^i{}_{j|kl]} = R^i{}_{j[kl;m]} = 0$$

Symmetries of $\Gamma^i{}_{jk}$, $R^i{}_{jkl}$

Given a metric tensor g let's use a co-tangent orthonormal frame (e^i) such that:

$$g = g_{ij} e^i \otimes e^j, \quad g_{ij} = \text{const.}$$

We assume metric compatible connection, ie. $\nabla g = 0$:

$$\begin{aligned} 0 &= \mathrm{d}g_{ij} + \Gamma^l{}_i g_{lj} + \Gamma^l{}_j g_{il} \\ &= \Gamma_{ij} + \Gamma_{jl} \end{aligned}$$

From that,

$$\Gamma_{ij} = -\Gamma_{ij}$$

or equivalently,

$$\Gamma_{ijk} = -\Gamma_{jik}$$

In means that Γ defines the Lie algebra that is a subalgebra of general linear transformations.
Let's consider $\mathrm{d}e^i = c^i{}_{jk} e^j \wedge e^k$, $c^i{}_{jk} = -c^i{}_{kj}$ are some functions. We seek for a metric, torsion free connection. As I wrote in Problem set 4,

$$\Gamma_{ijk} = c_{ijk} + c_{jki} - c_{kij}$$

Now let's focus on the Riemann tensor. From the metricity of connection ($\nabla g = 0$) follows the anti-symmetry in the first two indices.

$$R_{ijkl} = -R_{jikl}$$

Because this is true in an orthonormal frame and R is a tensor, it continues to be true in any frame.

Remark 16 (Symmetries of the Riemann tensor)**.** We assume a metric, torsion free connection.

$$R_{ijkl} = -R_{ijkl} = -R_{jikl}$$

By using the algebraic Bianchi identity, we get a symmetry:

$$R_{ijkl} = R_{klij}$$

And because of that, we can rearrange the differential Bianchi identity:

$$\nabla_{[i}R_{jk]lm} = 0 = \nabla_{[i|}R_{lm|jk]}$$

Definition 38 (Ricci tensor, Ricci scalar)**.** Ricci tensor is defined as contraction of the Riemann tensor.

$$R_{jk} \overset{\text{def}}{=} R^{i}{}_{jik}$$

From metricity and torsion freeness follows

$$R_{jk} = R_{kj}$$

Now, the Ricci scalar is defined as contraction of Ricci tensor with the metric tensor.

$$R \overset{\text{def}}{=} g^{jk}R_{jk}$$

To sum up, for any connection we define a curvature $\mathcal{R}^{i}{}_{j}$. Then, $R^{i}{}_{jkl}$ is curvature tensor (defined for any connection). The second pair of its indices is always antisymmetric, the first pair is only antisymmetric, when we apply metricity condition.

Now we will act like a sneaky person and try to do something with the differential Bianchi identity.

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

Metric covariant derivative commutes with the metric tensor, so let's contract it with g^{jl},

$$\nabla_i R_{km} + g^{jl}\nabla_j R_{kilm} - \nabla_k R_{im} = 0$$

Now we contract with g^{im},

$$g^{im}\nabla_i R_{km} + g^{jl}\nabla_j R_{kl} - \nabla_k R = 0$$

By renaming the dummy indices,

$$g^{im}\nabla_i R_{km} - \frac{1}{2}\nabla_k R = 0$$
$$g^{im}\nabla_i R_{km} - \frac{1}{2}\delta^{i}{}_{k}\nabla_i R = 0$$

But $\delta^i{}_k = g^i{}_k$, and g commutes,

$$\nabla_i \left(R^i{}_k - \frac{1}{2} R g^i{}_k \right) = 0$$

By raising the index, we finally get:

$$\nabla_i \left(R^{ik} - \frac{1}{2} R g^{ik} \right) = 0$$
$$\nabla_i G^{ik} = 0$$

The guy G is called the Einstein tensor.

Definition 39 (Einstein tensor)**.**

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

From definition, it is obvious that Einstein tensor is symmetric $G_{ij} = G_{ji}$. We have proven, that G is divergenceless:

$$g^{ki} \nabla_k G_{ij} = 0$$

Relation between the Lie derivative and the metric, torsion free covariant derivative

Consider a vector field X and its flow $\phi_t \colon M \to M$ such that

$$\left. \frac{\mathrm{d}\phi_t(m)}{\mathrm{d}t} \right|_{t=0} = X$$

For a function f,

$$\mathcal{L}_X f = X(f) = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\phi_\varepsilon^* f) \right|_{\varepsilon=0}$$

For a k-form ω,

$$\mathcal{L}_X \omega = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\phi_\varepsilon^* \omega) \right|_{\varepsilon=0}$$

For a vector field Y, if we define $\phi^* Y = (\phi^{-1})_* Y$

$$\mathcal{L}_X Y = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\phi_\varepsilon^* Y) \right|_{\varepsilon=0}$$

and for a tensor T:

$$\mathcal{L}_X T = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\phi_\varepsilon^* T) \right|_{\varepsilon=0}$$

Now we shall look for some relations to the covariant derivative.
For functions they appear to be the same:

$$\nabla_X f = X(f) = \mathcal{L}_X f$$

For a torsion free connection and vector fields X and Y,

$$\nabla_X Y - \nabla_Y X = [X, Y] = \mathcal{L}_X Y$$

For a co-vector $\omega = \omega_i e^i$,

$$\begin{aligned} \mathcal{L}_X \omega &= X \lrcorner \, \mathrm{d}\omega + \mathrm{d}(X \lrcorner \omega) \\ (\mathcal{L}_X \omega)_i &= (\nabla_X \omega)_i + (\nabla_i X)^j \omega_j \end{aligned}$$

The Lie derivative obeys Leibniz rule with respect to contraction. A suitable test for that property is:

$$\begin{aligned} \mathcal{L}_X (Y^i \omega_i) = \mathcal{L}_X (Y \lrcorner \omega) &= Y \lrcorner \mathcal{L}_X \omega + (\mathcal{L}_X Y) \lrcorner \omega \\ &= (\mathcal{L}_X Y^i) \omega_i + Y^i (\mathcal{L}_X \omega)_i \\ &= \omega_i (\nabla_X Y^i) + Y^i (\mathcal{L}_X \omega)_i \end{aligned}$$

The general decomposition in terms of covariant derivative is as follows:

Remark 17. In the case of zero torsion, we have:

$$\begin{aligned} (\mathcal{L}_X T)^{i \cdots j}{}_{k \cdots l} = X^\nu \nabla_\nu T^{i \cdots j}{}_{k \cdots l} & \\ - T^{\nu \cdots j}{}_{k \cdots l} \nabla_\nu X^i - \cdots - T^{i \cdots \nu}{}_{k \cdots l} \nabla_\nu X^j & \\ + T^{i \cdots j}{}_{\nu \cdots l} \nabla_k X^\nu + \cdots + T^{i \cdots j}{}_{k \cdots \nu} \nabla_l X^\nu & \end{aligned}$$

In particular, we can take the Lie derivative of a metric tensor:

$$(\mathcal{L}_X g)_{ij} = X^k \nabla_k g_{ij} + g_{lj} \nabla_i X^l + g_{il} \nabla_j X^l$$

The 1st part is zero, because the metric tensor kills connection. For the same reason, we can also act with the metric tensors inside covariant derivative, thus

$$= \nabla_i (X_j) + \nabla_j (X_i)$$

Induced metric, parallel transport, geodesics

Definition 40 (Volume tensor)**.** Let us define the volume tensor on a manifold M such that $\dim M = n$.

$$\begin{aligned} V &\overset{\text{def}}{=} \sqrt{\left|\det(g_{ij})\right|} e^1 \wedge \cdots e^n \\ &= \frac{1}{n!} \sqrt{|\det g|} \varepsilon_{i_1 \cdots i_n} e^{i_1} \wedge \cdots \wedge e^{i_n} \\ &= V_{i_1 \cdots i_n} \, e^{i_1} \otimes \cdots \otimes e^{i_n} \end{aligned}$$

where ε is just the Levi-Civita tensor.

$$V_{i_1 \cdots i_n} = \sqrt{|\det g|} \varepsilon_{i_1 \cdots i_2}$$

It really is a tensor, meaning it is invariant of choice of the basis.

Definition 41 (Induced metric)**.** Consider a surface $S \subset M$ and vectors X, Y tangent to S ie. $X, Y \in \mathrm{T}S \subset \mathrm{T}M$. We introduce the induced metric $g^{(S)}$ such that:

$$g^{(S)}(X, Y) = g(X, Y)$$

Given this metric, we can consider induced volume forms.

Example Metric tensor induced on a curve. Given a curve $p\colon [\tau_0, \tau_1] \to M$.

$$\dot{p} = \frac{\mathrm{d}p^i}{\mathrm{d}\tau} \frac{\partial}{\partial x^i}$$

For sure, $\dot{p} \in \mathrm{T}p$,

$$\begin{aligned} g^{(p)}(\dot{p}, \dot{p}) &\stackrel{\text{def}}{=} g(\dot{p}, \dot{p}) = g(\dot{p}^i\, \partial_i, \dot{p}^j\, \partial_j) \\ &= g_{ij}\dot{p}^i\dot{p}^j = g^{(p)}_{\tau\tau} \\ \det g^{(p)}_{\tau\tau} &= g^{(p)}_{\tau\tau} \end{aligned}$$

Thus, the induced volume will be:

$$V^{(p)} = \left|g^{(p)}_{\tau\tau}\right|^{1/2} \mathrm{d}\tau = \left|g_{ij}\dot{p}^i\dot{p}^j\right|^{1/2} \mathrm{d}\tau$$

Note, that this way we calculate the length of this curve:

$$l_p = \int_p V^{(p)} = \int_{\tau_0}^{\tau_1} \left|g_{ij}\dot{p}^i\dot{p}^j\right|^{1/2} \mathrm{d}\tau$$

Definition 42 (Parallel transport)**.** Consider a curve $p\colon [\tau_0, \tau_1] \to M$. We say that a vector Y is parallely transported along p if for every τ and $p(\tau)$, if

$$\nabla_{\dot{p}} Y = 0$$

Definition 43 (Geodesic)**.** A curve p in M is geodesic if it is parallely transported along itself, ie.

$$\nabla_{\dot{p}} \dot{p} = 0$$

There is an important characteristic of geodesics – metric tensor is constant on them.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} g(\dot{p}, \dot{p}) &= \nabla_{\dot{p}}\big(g(\dot{p}, \dot{p})\big) \\ &= (\nabla_{\dot{p}} g)(\dot{p}, \dot{p}) + g\big(\dot{p}, \nabla_{\dot{p}}\dot{p}\big) + g\big(\nabla_{\dot{p}}\dot{p}, \dot{p}\big) = 0 \\ g(\dot{p}, \dot{p}) &= \text{const.} \end{aligned}$$

It allows us to introduce some classification of geodesics. When the signature of g is $(- + \cdots +)$ then are three possibilities:

$$g(\dot{p}, \dot{p}) = \begin{cases} > 0 & \text{spacelike geodesic} \\ = 0 & \text{null geodesic} \\ < 0 & \text{timelike geodesic} \end{cases}$$

≻ Problem class 9

Problem 1 Consider a metric g on M and the corresponding covariant derivative (metric, torsion free). Suppose a function $f\colon M \to \mathbb{R}$ satisfies $g^{ij}e_i(f)e_j(f) = \text{const.}$ and $\mathrm{d}f \neq 0$. Show that $\nabla_X X = 0$, where $X = g^{ij}e_j(f)e_i$.

Let's admit, that $\nabla_j f = e_j(f)$. Therefore,

$$\begin{aligned} X &= g^{ij}(\nabla_j f)e_i \\ \nabla_X X^i &= \nabla_X(g^{ij}\nabla_j f) = g^{ij}\nabla_X \nabla_j f = g^{ij}X^l \nabla_l \nabla_j f \\ &= g^{ij}g^{lk}(\nabla_k f)\nabla_l \nabla_j f \end{aligned}$$

Moreover,

$$g^{lk}\nabla_j(\nabla_l f \nabla_k f) = g^{lk}(\nabla_k f)\nabla_j \nabla_l f + g^{lk}(\nabla_l f)\nabla_j \nabla_k f = (*)$$

l, k are just dummy indices and g is symmetric, so we can write:

$$= 2g^{lk}\nabla_k f \nabla_j \nabla_l f$$

Therefore,

$$(*) = \frac{1}{2}g^{ij}\nabla_j\left(g^{lk}\nabla_l f \nabla_k f\right) = \frac{1}{2}g^{ij}\nabla_j c = 0$$

Problem 2 There is a metric g and ∇ (metric and torsion free). Suppose X is a Killing vector, meaning $\mathcal{L}_X g = 0$ and it is timelike. Calculate the acceleration of the family of observers following the integral curves of X.

Let's make an observation:

$$\begin{aligned} \nabla_X\big(g(X,X)\big) &= \mathcal{L}_X\big(g(X,X)\big) \\ &= (\mathcal{L}_X g)(X,X) + 2g(\mathcal{L}_X X, X) \\ &= 0 + 2g\big([X,X], X\big) \end{aligned}$$

As $[X, X] = 0$,

$$\nabla_X\big(g(X,X)\big) = 0$$

Also, we have shown that:

$$\begin{aligned} (\mathcal{L}_X g)_{ij} &= \nabla_i X_j + \nabla_j X_i \\ \nabla_i X_j &= -\nabla_j X_i \end{aligned}$$

Now, we know that that 3-velocity u is normalized from definition, so:

$$
\begin{aligned}
u &= \frac{X}{\sqrt{-g(X,X)}} \\
a^i &\overset{\text{def}}{=} \nabla_u u^i = \frac{1}{\left|g(X,X)\right|}\nabla_X X^i + \frac{X^i}{\sqrt{-g(X,X)}}\nabla_X\left(\frac{1}{\sqrt{-g(X,X)}}\right) \\
&= \frac{1}{\left|g(X,X)\right|}\nabla_X X^i
\end{aligned}
$$

After lowering the index,

$$
a_k = \nabla_u u_k = \frac{X^i \nabla_i X_k}{\left|g(X,X)\right|} = -\frac{X^i \nabla_k X_i}{\left|g(X,X)\right|} = -\frac{g^{ij} X_i \nabla_k X_j}{\left|g(X,X)\right|}
$$

As g is symmetric, after evaluating the Leibniz rule we get,

$$
= -\frac{1}{2}\frac{\nabla_k (g^{ij} X_i X_j)}{\left|g(X,X)\right|} = -\frac{1}{2}\frac{\nabla_k \big(g(X,X)\big)}{\left|g(X,X)\right|}
$$

Lecture 10: Metric, spacetime, geometry

We consider n-dimensional manifold and a metric tensor g of arbitrary signature (p,q), where $p+q=n$. If the signature is $-+\dots+$, then we call (M,g) spacetime and timelike vectors are related to point observers or point particles moving in M. 22 gru 2020
We assume that connection is metric and torsion free.

Definition 44 (Weyl Tensor)**.** For $n \geq 3$ we can define the Weyl C_{ijkl} tensor such that:

$$
R_{ijkl} = C_{ijkl} + \frac{2}{n-2}\Big(g_{i[k}R_{l]j} - g_{j[k}R_{l]i}\Big) - \frac{2}{(n-1)(n-2)} R g_{i[k} g_{l]j}
$$

It is interesting, because it obeys the same symmetries as the Riemann tensor:

$$
C_{ijkl} = -C_{jikl} = -C_{ijlk} = C_{klij}
$$

But upon a contraction, is cancels out:

$$
C^i{}_{jil} = 0
$$

If $\dim M = 3$, then $C_{ijkl} = 0$.

Last time we introduced geodesics as curves p such that $\nabla_{\dot p}\dot p = 0$. In a frame $(e_1,\dots,e_n)$ this equation read as:

$$
\dot p^i \nabla_i \dot p^j = \dot p^i \Big(e_i(\dot p^j) + \Gamma^j{}_{li}\dot p^l\Big) = 0
$$

Or equivalently,

$$\frac{\mathrm{d}\dot{p}^j}{\mathrm{d}\tau} + \dot{p}^i\dot{p}^l\Gamma^j{}_{il} = 0$$

If we take a holonomic basis,

$$\frac{\mathrm{d}^2 p^i}{\mathrm{d}\tau^2} + \Gamma^i{}_{jk}\frac{\mathrm{d}p^j}{\mathrm{d}\tau}\frac{\mathrm{d}p^k}{\mathrm{d}\tau} = 0$$

Now, consider a family of curves $p_\varepsilon\colon [\tau_0,\tau_1] \to M$, such that $p_\varepsilon(\tau_0) = p(\tau_0)$, $p_\varepsilon(\tau_1) = p(\tau_1)$, $p_0(\tau) = p(\tau)$.

Lemma 6. $\forall \tau \in [\tau_0,\tau_1]$, $\forall p_\varepsilon\colon \varepsilon \in [-\varepsilon_0,\varepsilon_0]$,

$$\nabla_{\dot{p}}\dot{p} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_{\tau_0}^{\tau_1} g(\dot{p}_\varepsilon,\dot{p}_\varepsilon)\,\mathrm{d}\tau = 0$$

What if we consider reparametrizations? If we take $\tilde{p}\colon [s_0,s_1] \to [\tau_0,\tau_1] \xrightarrow{p} M$,

$$\dot{p} = \frac{\mathrm{d}p}{\mathrm{d}\tau} = \frac{\mathrm{d}\tilde{p}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{\mathrm{d}s}{\mathrm{d}\tau}\dot{\tilde{p}}$$

Then,

$$\begin{aligned} 0 = \nabla_{\dot{p}}\dot{p} &= \nabla_{\frac{\mathrm{d}s}{\mathrm{d}\tau}\dot{\tilde{p}}}\frac{\mathrm{d}s}{\mathrm{d}\tau}\dot{\tilde{p}} = \frac{\mathrm{d}s}{\mathrm{d}\tau}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\nabla_{\dot{\tilde{p}}}\dot{\tilde{p}} + \dot{\tilde{p}}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\dot{\tilde{p}}\right) \\ &= \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2\left(\nabla_{\dot{\tilde{p}}}\dot{\tilde{p}} - \kappa\dot{\tilde{p}}\right) \end{aligned}$$

Therefore, we can see, that the following equation:

$$\nabla_{\dot{\tilde{p}}}\dot{\tilde{p}} = \kappa\dot{\tilde{p}}$$

is reparametrization invariant, as there exists a reparametrization $s \mapsto \tau(s)$, which satisfies the original definition of a geodesic. Also, $\mathrm{d}s/\mathrm{d}\tau = \text{const.}$ preserves $\nabla_{\dot{\tilde{p}}}\dot{\tilde{p}} = 0$.

Then, if $g(\dot{p},\dot{p}) \neq 0$, we can choose τ such that $g(\dot{p},\dot{p}) = \pm 1$. Then, τ is the proper length/time.

Lemma 7. We also have a weaker condition. $\forall \tau \in [\tau_0,\tau_1]$, $\kappa\colon [\tau_0,\tau_1] \to \mathbb{R}$,

$$\nabla_{\dot{p}}\dot{p} = \kappa\dot{p} \iff \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_{\tau_0}^{\tau_1}\sqrt{\left|g(\dot{p},\dot{p})\right|}\,\mathrm{d}\tau = 0$$

Careful! Geodesics are the curves, locally minimizing length but not conversely. That's why this is a weaker condition. But we can see, that if $\mathrm{d}s/\mathrm{d}\tau = \text{const.}$, ie. when such length-minizing curve is parametrized proportional to its length, then it is a geodesic.

Definition 45 (Variation)**.**

$$\delta p \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} p_\varepsilon, \quad \delta p_\varepsilon(\tau_0) = 0 = \delta p_\varepsilon(\tau_1)$$

$$\delta g(\dot p, \dot p) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} g(\dot p_\varepsilon, \dot p_\varepsilon) = 2\bigg(\frac{\mathrm{d}}{\mathrm{d}\tau}\big(g(\delta p, \dot p)\big) - g\Big(\delta p, \nabla_{\dot p}\dot p\Big)\bigg)$$

Definition 46 (Conformal equivalence)**.** Two metrics $\tilde g$ and g are called to be conformally equivalent if $\tilde g_{ij} = a^2 g_{ij}$, where a is a smooth, strictly positive function. We call $\tilde g$ to arise from g via a conformal transformation.

Now we will show conformal invariance of $\nabla_{\dot p}\dot p = \kappa \dot p$, $g(\dot p, \dot p) = 0$. Let $g = a^2 \tilde g$. Then,

$$\nabla_{\dot p}\dot p = \tilde\nabla_{\dot p}\dot p + 2\frac{\mathrm{d}a}{\mathrm{d}\tau}\dot p$$

Hence,

$$\tilde\nabla_{\dot p}\dot p = \tilde\kappa \dot p, \quad \tilde\kappa = \kappa - 2\frac{\mathrm{d}a}{\mathrm{d}\tau}$$

Suppose that:

$$X \overset{\text{def}}{=} \frac{\mathrm{d}p_\lambda}{\mathrm{d}\lambda}, \ \dot p \overset{\text{def}}{=} \frac{\mathrm{d}p_\lambda}{\mathrm{d}\tau}$$

And it forms a holonomic base:

$$[X, \dot p] = 0 = \nabla_X \dot p - \nabla_{\dot p} X$$

Suppose that $\nabla_{\dot p}\dot p = 0$ (ie. p is a geodesic). Then, we have the geodesic deviation equation, describing the relative acceleration:

$$\nabla_{\dot p}\nabla_{\dot p} X^i = R^i{}_{jkl}\dot p^j \dot p^k X^l$$

Einstein-Hilbert action

This would consider to consider just gravitational field, without any matter.

$$\begin{aligned} S_{\mathrm{EH}}(g) &= \frac{1}{16\pi G}\int_M R \operatorname{vol} \\ R &= g^{ij}R_{ij} \\ \operatorname{vol} &= \sqrt{\left|\det g_{ij}\right|} e^1 \wedge \cdots \wedge e^n \\ \delta_g S_{\mathrm{SH}} = 0 &\iff R_{ij} - \frac{1}{2}R g_{ij} = G_{ij} = 0 \end{aligned}$$

This action gives precisely the tensor whose covariant divergence is zero.

Palatini actioon. We explain it in $\dim M = 4$. Let $g = g_{ij}e^i \otimes e^j$, $g_{ij} = \text{const.}$, $\left|\det g_{ij}\right| = 1$ and consider $(e^0, \ldots, e^3)$