General Relativity

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Lecture 1: Minkowski spacetime

20 paź 2020 First we will formulate a geometry which is equivalent to basic relativistic symmetries known from the special relativity. It was actually the first step towards GR to understand special relativity as symmetries of a given geometry.

Definition 1 (Minkowski spacetime)**.** 4-dimensional affine space M with associated scalar product g (metric tensor) of signature $(-+++)$.

Definition 2 (Affine space)**.** We define,

- Vector space at each point
- Global parallelism (allows to identify vectors at different points)
- $m_0, m_1 \in M$ defines a vector at m_0 : $m_0 + \underbrace{m_1 - m_0}_{v'} = m_1$

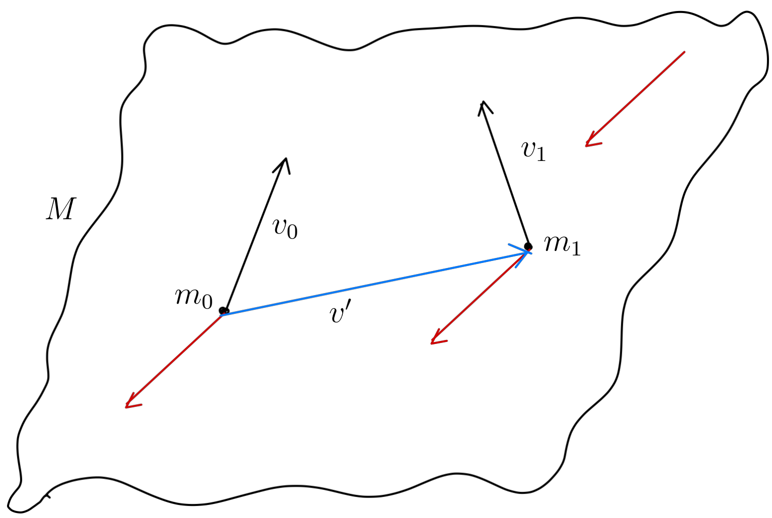

Figure 1: Affine space

In practice you can consider M as a 4D vector space upon a choice of an origin $m_0 \in M$. Let V be 4-dim vector space. We make $V \equiv M$, where M – affine space understood as 4D vector space up to translations.

Definition 3 (Metric tensor)**.**

$$g(u, v) \in \mathbb{R}$$
$$g(u, v) = g(w, u)$$
$$g(\alpha v + \beta u, w) = \alpha g(v, w) + \beta g(u, w)$$

We can see that $\forall g$, that is symmetric and bilinear, $\exists\{e_0, e_1, e_2, e_3\}$ such that:

$$g(e_0, e_0) = \pm 1$$
$$\vdots$$
$$g(e_3, e_3) = \pm 1$$

whenever we take product of two different basis vectors,

$$g\big(e_i, e_j\big) \overset{i\neq j}{=} 0$$

In other words,

$$g\big(e_i, e_j\big) = \pm\delta^i_j$$

Here we have an orthogonal basis. The number of $+$ and $-$ is invariant for a given g. In Minkowski spacetime it is $(-+++)$.
This is the complete definition of Minkowski spacetime.

This scalar product is independent on choice of point $m \in M$ (because the global parallelism holds). In other words, the parallel transport preserves g.

Timelike, spacelike, null vectors

Scalar product distinguishes between different types of vectors that may emerge on M: timelike, spacelike and null.

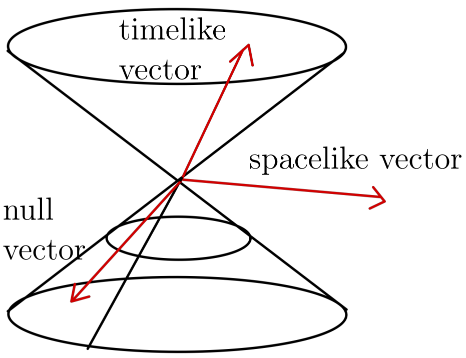

Figure 2: Classification of vectors on M

Definition 4.

$$g(v,v) = \begin{cases} < 0, & \text{timelike} \\ = 0, & \text{null} \\ > 0, & \text{spacelike} \end{cases}$$

Null vectors form a cone. Timelike vectors lay inside these cones, spacelike ones outside.

What is their physical interpretation?

Consider a curve $p\colon [\tau_0, \tau_1] \to M$. We can distinguish its tangent vectors by taking derivatives with respect to parameter:

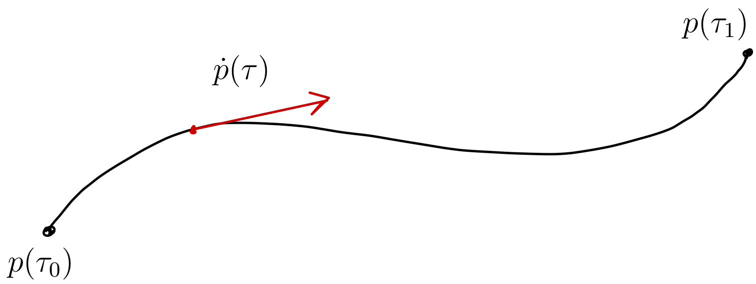

Figure 3: Tangent vector

If $\forall \tau$, all tangent vectors are of the same type, we define:

$$\begin{aligned} g(\dot{p},\dot{p}) &< 0, && \text{timelike curve} \\ g(\dot{p},\dot{p}) &= 0, && \text{null curve} \\ g(\dot{p},\dot{p}) &> 0, && \text{spacelike curve} \end{aligned}$$

A timelike curve is the worldline of a particle at speed $< c$.
A null curve represents particles with speed $= v$.

Proper time, proper distance

Definition 5 (Proper time/distance)**.** Let $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in M$ be timelike. Then, the proper time (as I started my clock and travelled along this curve) will be,

$$T \stackrel{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

For spacelike curves we will have a proper distance:

$$D \stackrel{\text{def}}{=} \int_{\tau_0}^{\tau_1} \sqrt{g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

T and D are reparametrisation invariant.

$$p'(\tau') = p\Big(\tau(\tau')\Big)$$

$$\int_{\tau_0'}^{\tau_1'} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau'}, \frac{\mathrm{d}p}{\mathrm{d}\tau'}\right)}\, \mathrm{d}\tau' = \int_{\tau_0}^{\tau_1} \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\, \mathrm{d}\tau$$

Remark 1. For curves $p(\tau) = \tau e_0$ we can see that $T = (\tau_1 - \tau_0)$. If we assume that $\tau_0 = 0$ and that we calculate proper time at point $p(\tau)$, then $T = \tau$. Vertical lines are those of the proper time.

Inverse triangular inequality Consider straight timelike lines such that they form a triangle.

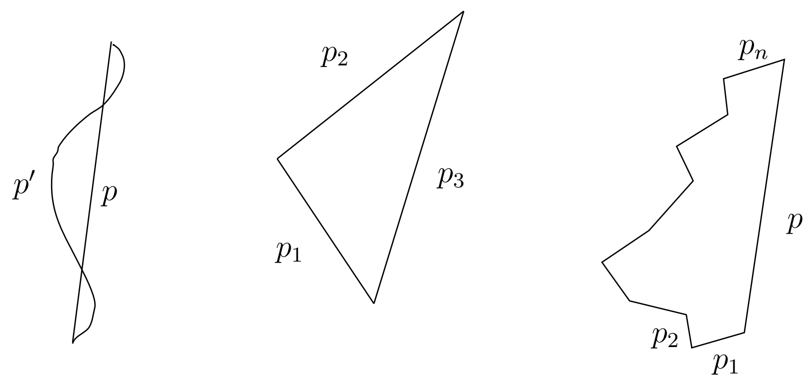

Figure 4: Inverse triangular inequality

Theorem 1.

$$\begin{aligned} T_1 + T_2 &< T_3 \\ T_1 + \cdots + T_n &< T \\ T' &\leq T \end{aligned}$$

Remark 2. The conclusion is that the timelike straight lines (points moving at constant speed $< c$) maximize time.

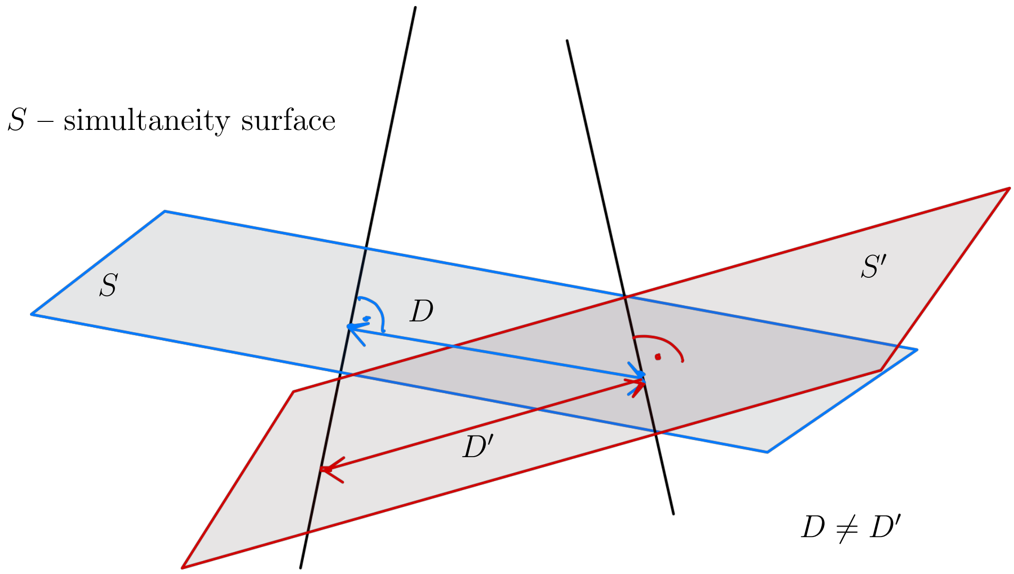

Figure 5: Simultaneity of spacetime events.

Remark 3 (Physical interpretation for spacelike curves)**.** D – distance between two events that we consider to happen simultaneously. Proper distance D measured along that S surface is distance between two events happening simultaneously.

Time and distances dilatation

Time dilatation Consider an observer p' moving at a constant speed with respect to observer p. We take $\tau \in [0, \tau_1]$. $p \in M$ comes with its notion of time T and any relatively moving observer p' comes with its T'. What is the relation between the times that they measure?

Let's calculate the events in spacetime at τ time.

$$\begin{aligned} p(\tau) &= \tau e_0 \\ p'(\tau) &= \tau(e_0 + \beta e_1) \end{aligned}$$

Now we calculate proper times,

$$\begin{aligned}
T^2 &= -g(\tau_1 e_0, \tau_1 e_0) = \tau_1^2 \\
T'^2 &= -\tau_1^2 g(e_0 + \beta e_1, e_0 + \beta e_1) \\
&= -\tau_1^2\left(-1+\beta^2\right) = \tau_1^2\left(1-\beta^2\right) \\
T' &= \sqrt{1-\beta^2} T
\end{aligned}$$

Distances dilatation We have one observer and 2 other parallel observers (moving at the same speeds). Describe this from the point of view of the simultaneity surfaces.

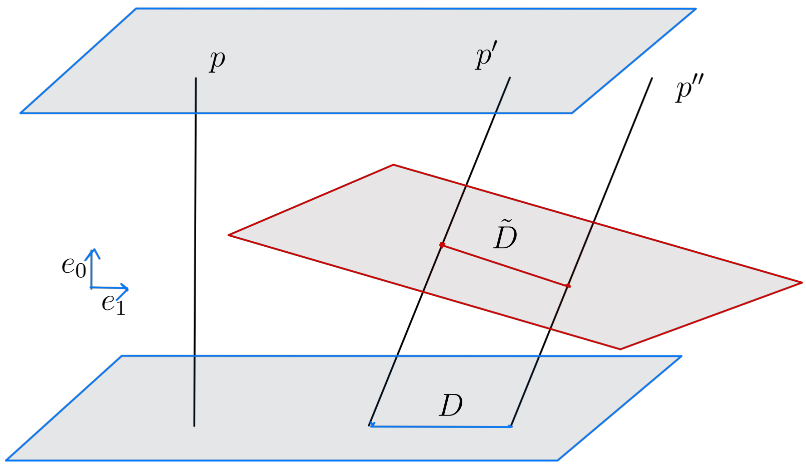

Figure 6: Distances dilatation.

Definition 6 (4-velocity)**.** Consider a timelike curve $p\colon [\tau_0, \tau_1] \to M$. We can always choose such parametrization such that the norm of tangent vectors is $= 1$. Suppose:

$$g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) \neq -1$$

Introduce new parameter $\mathrm{d}\tau' = -\sqrt{g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}\,\mathrm{d}\tau$. τ' is just a proper time. If we take $p'(\tau') \stackrel{\mathrm{def}}{=} p \circ \tau'(\tau)$ then, the 4-velocity is defined as:

$$u = \frac{\mathrm{d}p'}{\mathrm{d}\tau'} = \frac{\mathrm{d}p}{\mathrm{d}\tau} \Big/ \sqrt{-g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right)}$$

It is easy to check, that

$$g(u,u) = -1$$

Then, we can define the acceleration as:

$$a \overset{\text{def}}{=} \frac{\mathrm{d}u}{\mathrm{d}\tau}, \quad \text{where} \quad g\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}, \frac{\mathrm{d}p}{\mathrm{d}\tau}\right) = -1\,(?)$$

What is a timelike circle?

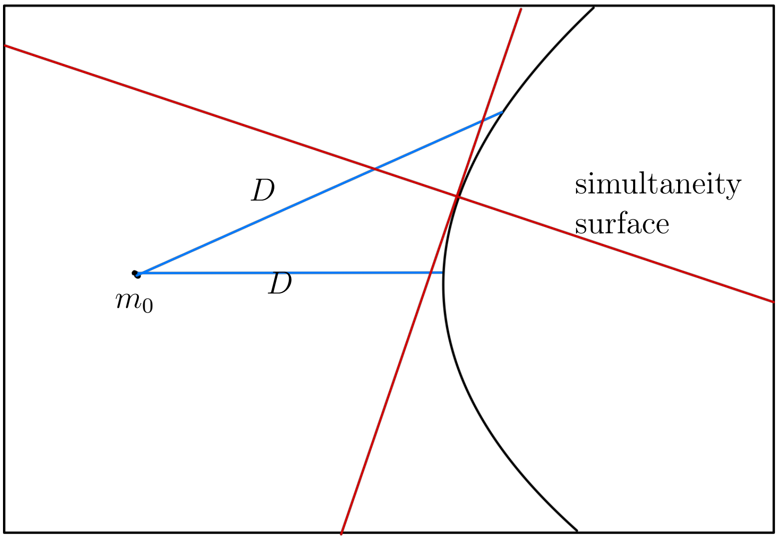

Figure 7: circle

The curve is a hyperbola. There is an isometry between any two tangent observers. The red one concludes there is some acceleration $a = \|\vec{a}\|$. Another one concludes (from symmetry) there is the same constant acceleration.

$$\begin{aligned}
p(\tau) &= D\left[\sinh\left(\frac{\tau}{D}\right)e_0 + \cosh\left(\frac{\tau}{D}\right)e_1\right] \\
g\big(p(\tau), p(\tau)\big) &= D^2\left[-\sinh^2\left(\frac{\tau}{D}\right) + \cosh^2\left(\frac{\tau}{D}\right)\right] = D^2 \\
a &= \frac{1}{D} = \frac{c^2}{D}
\end{aligned}$$

Remark 4. If we consider a timelike curve set in a two-plane by points which have same distance to the fixed points it corresponds to constant acceleration curve.

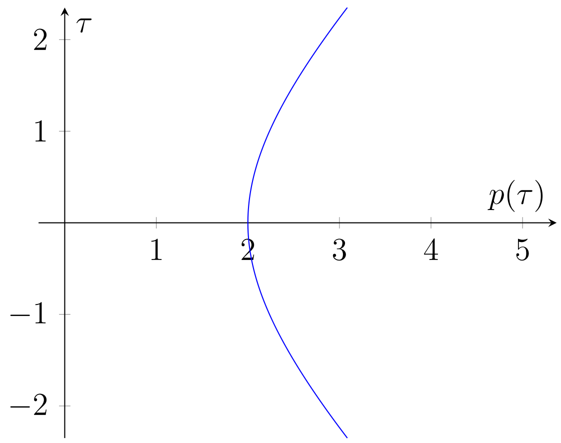

Figure 8: Example of constant acceleration curve with $D = 2$.