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FORMATION OF STRUCTURES IN THE UNIVERSE

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Contents

Abstract

In this project equations which govern the formation of structures and the Friedmann equations were derived .We studied the growth of large-scale structure in an expanding universe through gravitational instability acting on small initial perturbations. Both Newtonian theory and Relativistic theory we used to explain how perturbations grow leading to formation of structures. We also managed to explain how matter perturbations grow when the universe is matter dominated(inside and outside the Horizon) and radiation dominated(inside and outside the Horizon).The universe was assumed to be flat and $\Omega_m = 1$.Due to the fact that pressure forces are present during radiation epoc the growth rate of perturbations was found to be different from the growth rate of perturbations during matter dominated universe.

1 INTRODUCTION

1.1 Cosmology

Cosmology is one of the most interesting and fascinating part of physical science because it forms a nice balance between observations and theory. Cosmology is a combination of both quantum field theory and particle physics with general relativity. General relativity is one of the theory which determine the dynamics of the universe. Cosmology allow us to understand the evolution and the origin of the universe.Cosmological principle states that the universe is homogeneous on large scale and inhomogeneous on small scale.The universe is considered to be homogeneous and isotropic but the study of structures can only be done when the universe is inhomogeneous and anisotropic.[Brandenberger, 1995].

1.2 Inflation

Everyone can observe this inhomogeneities of the our solar system since stars are not randomly distributed in space.This stars are bound to into galaxies . Inflation theory was developed in the early 1980s by Alan Guth.According to the inflation theory,the early universe underwent highly accelerated expansion of space.One of the benefits of inflation is that it provides us with a theory of inhomogeneities in the early universe. These inhomogeneities arise from quantum fluctuations at early times.Quantum fluctuations generate the primordial perturbations which are reliable candidates for origin of structure formation and CMB(T=27K).

In this project we will solve equations and set up initial conditions so that we can calculate the inhomogeneities and anisotropies in the universe.From the fact that matter perturbations depend on some details from radiation perturbations we conclude that matter perturbations are coupled to all other perturbations. The early universe was dense before the formation of stars,planets and galaxies filled with a white fog of hydrogen plasma.The temperature of radiation and hydrogen plasma decreased as the universe expands. When the universe has cooled enough,electrons and protons combined to form atoms(recombination Epoc).The event where this atoms decouple from photons is referred to as photon decoupling.Photon decoupling occurred around 380000 years after the big bang,at redshift z=1090 when the universe temperature is around 3000K[Rubakov, 2005].

Instability is one of the theory which explains the way in which structures developed in the universe.Gravity and the pressure played a roles during the development of structures in the universe.Structures were formed when regions with more matter pull more matter from their surrounding regions by exerting greater gravitational force on them i.e gravitational force acts to increase the over-density but the pressure acts to decrease it .The universe was dominated by Radiation at early times(i.e the density of radiation was high that the density of matter after the big bang).Different cosmological conditions affect the growth of this small perturbations[Dodelson, 2003].We neglect the anisotropic part of their energy-momentum tensor and exclude neutrinos in the radiation.We know that matter have zero pressure ($p = 0$) and gravitational interactions.We will ignore the effect of curvature (K) and cosmological constant[Weinberg, 2002].

1.3 Gauges and a Horizon

In cosmology different have different gauges which are good for different purposes.If we are working on one gauge we can switch to another using calculations.We gonna work on two gauges which are Newtonian and Co-moving gauges.The word 'Co-moving' indicates the motion of matter.In Co-moving the velocity of matter fluid is prevented from vanishing i.e we always assume that matter is always in motion[Peter and Uzan, 2013]. Newtonian theory only focus on scales larger than the Hubble radius(Horizon) $\frac{1}{H}$ but relativistic theory focus on both larger and smaller than the Hubble radius(Horizon) $\frac{1}{H}$ .Large scales are defined to be scales that enter the horizon during matter dominated epoc and small scales are defined to be scales that enter the horizon during radiation epoc[Kurki-Suonio, 2005] .

2 Formation of structures

The universe was dominated by Radiation at early times(i.e the density of radiation was high that the density of matter after the big bang.As the universe expands,the density of radiation drops very fast thus allowing the growth of perturbations . Tiny perturbations were found to be present at early times(i.e after the big bang). Tiny perturbations are the result of microscopic quantum fluctuations in the early universe.Microscopic quantum fluctuations expanded to macroscopic scales as a result of inflation effects. density perturbations grew through different cosmological effects like gravity and pressure and developed into different structures(i.e stars ,galaxies,etc) we see today.Due to radiation domination the gravitational potential fluctuations remain constant but the density fluctuations larger than the cosmic horizon grow proportional to the scale factor(a).

3 Friedmann-Robertson-Walker Universe

The Universe on large scale looks the same around each point,and in each direction i.e it is homogeneous and isotropic on large scale than 100 Mpc but on small scale we observe something different[Aspinal, 2016].consider the Friedmann equation below

$$H^2 = \frac{8\pi G}{3}(\rho) - \frac{K}{a^2} \tag{1}$$

where $H = \frac{a^{\cdot}}{a}$,a(t) is the scale factor which represents the expansion of the universe,$\wedge$ is a cosmological constant and Ω is a density parameter for either matter,radiation or dark energy. The point of transitions between radiation and matter epoc is called matter-radiation equality. The domination of both radiation and matter occurred when the universe is 50000 years.At equality $\rho_m = \rho_r$ .Conservation equations for both radiation and matter dominated universe are $\frac{\rho_r^{'}}{\rho_r} = -4H$ and $\frac{\rho_m^{'}}{\rho_m} = -3H$. Consider a flat Universe with no cosmological constant i.e (K=0),the density of today is the critical density ($\rho = \rho_r$))given by

$$H_0 = \frac{8\pi G\rho_c}{3} \tag{2}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.9h^2 x 10^{-29} gcm^{-3} \tag{3}$$

H_o is said to be the Hubble constant at the present time.Friedmann equal becomes:

$$H^2 = H_o^2\left(\frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} + \frac{\rho_\wedge}{\rho_c} - \frac{K}{a^2}\right) \tag{4}$$

$\Omega = 1$ for a flat universe ,$\Omega < 0$ for an open universe $(K < 0)$ and $\Omega > 0$ for closed universe($K > 0$).In this project we on focus on a flat universe because we try to explain how structures form at early times.This allow to conclude that $\Omega_m = 1$ for a flat universe. The equation of state ($w = \frac{p}{\rho}$)combine both pressure and the density of the particles in the universe.From equation of state and Friedmann equation w=0 ,$a \propto t^{\frac{3}{2}}$ and $\rho_m \propto a^{-4}$ when the universe is dominated by matter,w=$\frac{1}{3}$,$a \propto t^{\frac{1}{2}}$ and $\rho_r \propto a^{-4}$ when universe is dominated by radiation and w=-1 when the universe is dominated by dark energy,dark matter and baryons are referred to as matter.Matter is assumed to be pressureless due to smaller kinetic energy. The present day universe consist of $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$.Dark energy is one which is dominant when it comes to energy distribution and it is responsible for the expansion of the universe.Matter come second when it comes to energy distribution[Oreta, 2016]. The relation which is obeyed by density parameter:

$$1 = \Omega_m + \Omega_r + \Omega_\lambda + \Omega_K \tag{5}$$

4 Linear perturbation(Newtonian theory)

4.1 Jean's Instability

The Poisson equation relate both gravitational potential and the density contract($\delta = \frac{\delta\rho_m}{\rho_m}$).

$$-k^2\phi = 4\pi G a^2 \rho_m \delta \tag{6}$$

Where ρ_m is the matter density.The equation of state (w) is equal to the speed sound(c_s) in a flat and homogeneous universe. The equation of evolution of perturbations is given as

$$\delta'' + 2H\delta + (\frac{k^2 c_s^2}{a^2} - 4\pi G\rho)\delta = 0 \tag{7}$$

Where $(')$=$\frac{d}{dt}$.The above equation is known as Jean's equation of the density contract δ.The first two terms on the right represent time of the density contract δ.The Growth of perturbations decreases as the universe expands.The last term on the right has two competitive source which pressure and gravity[Flender and Schwarz, 2012]. If the term in the brackets is positive ,δ have solutions that oscillate. This means that the pressure forces are stronger than gravity i.e no growth of perturbations.If the term in the brackets is negative ,δ have solutions that do not oscillate.This non oscillating solutions lead to increase or decrease in the density excess.This actually means that gravity dominated collapses the density perturbation when the pressure forces dominate it.The increase of δ result in a bound structure. The physical Jean's wavenumber is defined by $k_J = \frac{\sqrt{4\pi G\rho}}{c_s^2}$ and the Jean's length is defined by $\lambda_J = \frac{2\pi}{k_J}$[Jaffe, 2012].

4.2 Matter and Radiation Dominated Universe

When the universe is dominated by radiation perturbations do not grow inside the horizon(sub-horizon).$D \propto \delta$ therefore the equation of evolution becomes $D'' + 2HD' = 4\pi G\rho_m D = \frac{3H^2 D}{2}$ with a two solutions(one often grow with time the other one decays with time)in matter dominated universe where the sound speed(c_s) term is neglected.The general solution is $D = Bt^{-1} + Ft^{\frac{2}{3}}$. The growing mode $D_+ \propto a \propto \eta^2 \propto t^{\frac{2}{3}}$ and the decaying mode $D_- \propto a^{\frac{-3}{2}} \propto t^{-1}$. We can see that the LHS(equation of evolution of perturbations) has the Jean's length:if $\frac{k}{a}$ does not exceed λ_J , the solution of the equation of evolution of perturbations is oscillatory.Pressure forces can only oppose the gravitational forces effectively for wavelength below the Jean's length (λ_J).In radiation era,$c_s = \frac{c}{\sqrt{3}}$ and so the Jean's length is assumed to be close to the Huble size(Horizon).When matter and radiation are at equality ,the sound speed start to cease or drop.

In radiation dominated universe, the equation of evolution of perturbation becomes $D'' + 2HD' = 4H^2 D$ with a solution $D = Ct + Et^{-1}$. The growing mode $D_+ \propto t \propto a^2$ and the decaying mode $D_- \propto t^{-1} \propto H$.From the results we conclude that $\delta \propto a$ inside and outside the horizon and $\delta \propto a^2$ outside the horizon only.The density contract depend on k.If we consider the behavior of matter in an open curvature dominated universe ,the pressure is negligible(i.e w=0) and $a \propto \eta^2 \propto t$.The equation of evolution of perturbation becomes $\delta'' + \frac{2\delta'}{t} = 0$ .The solution is $D = Q + \frac{Q}{t}$.This allow us to conclude that density perturbations do not grow on all scale.This results also occurs when the universe id dominated by a cosmological constant(w=-1)[Jaffe, 2012].Continuity equation:

$$\delta'_m + ikv_m = 0 \tag{8}$$

Poisson equation:

$$k^2\phi = 4\pi G a^2 \rho \delta_m \tag{9}$$

Euler equation:

$$v'_m + Hv_m = -ik\phi \tag{10}$$

We know that $H = \frac{a'}{a}$,$\eta^2 \propto a$ when the universe is dominated by matter and $\eta \propto a$ when the universe is dominated by radiation.Thus we get that

$$\delta'_m + ikv_m = 0 \tag{11}$$

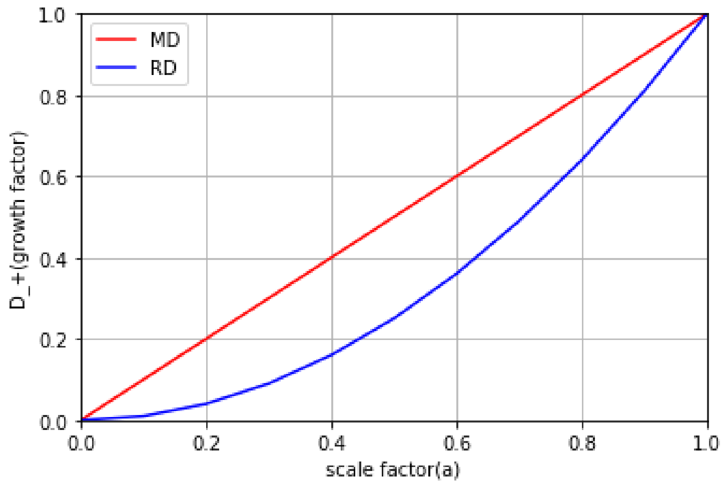

Figure 1: A plot of growth factor when the universe is dominated by radiation and when the universe is dominated by matter.D$\propto$ a inside the horizon and outside the horizon when the universe is dominated by the matter.D$\propto$ a^2 outside the horizon when the universe is dominated by radiation.

$$k^2\phi = \frac{6}{\eta}\delta_m \tag{12}$$

$$v'_m + \frac{2}{\eta}v_m = -ik\phi \tag{13}$$

where $H^2 = \frac{8\pi Ga^2}{3}$ which can be written as $4\pi Ga^2 = \frac{6}{\eta^2}$.From Poisson equation we get that

$$\delta_m = -\frac{\eta^2 k^2 \phi}{6} \tag{14}$$

$$v'_m + Hv_m = -ik\phi \tag{15}$$

The above can be written as

$$\frac{(av_m)'}{a} = -ik\phi \tag{16}$$

taking the integral we get

$$v_m = \frac{-ik\eta\phi}{3} \tag{17}$$

Inserting δ''_m and v'_m in the continuity equation we find the equation for ϕ:

$$\phi' + \frac{\eta\phi''}{6} = 0 \tag{18}$$

Taking the integral we get solution:$\delta_m = q + \frac{t}{\eta^5}$.The equation has the growing mode and the decaying mode .The decaying solution is $\phi \propto \eta^5$ and the growing solution is a constant.We conclude that the velocity perturbation grow as half the scale factor[Flender and Schwarz, 2012] when the universe is dominate.

5 Relativistic theory

5.1 Matter dominated universe

Consider equations of evolution for the density contrasts ,the gravitational potential and the velocity perturbations of matter, and the Poisson equation.

$$\delta_m^N = - \triangle v_m^N + 3\phi \tag{19}$$

$$v_m^{N'} + Hv_m^N = -\phi \tag{20}$$

$$\triangle\phi = 4\pi G\rho a^2 \delta_m^C \tag{21}$$

$$\phi' + H\phi = -4\pi G a^2 \rho_m v_m^N \tag{22}$$

$$\phi'' + 3H\phi' + \left(2H' + H^2\right)\phi = 0 \tag{23}$$

η is conformal time which is defined by the defferential ralation $d\eta = \frac{dt}{a(t)}$. and the conformal Hubble rate is $H = \frac{a'}{a}$ where primes denote a derivative with respect to conformal time. The equation of state of a perfect fluid is characterized by a dimensionless number w, equal to the ratio of its pressure p to its energy density ρ

$$w = \frac{p}{\rho} \tag{24}$$

Since $p = 0$ for matter dominated universe $w = 0$. Assuming that a flat universe ($K = 0$) is dominated by matter with equation of state $w = 0$. The scale factor (a) of the universe dominated by matter is proportional to η^ν where $\nu = 2$ and $c_s^2 = w$ and the pressure is negligible i.e that the anisotropic stress is negligible ,we require that the $\phi = \Psi$. Therefore the linear evolution equation for ϕ is given by

$$\phi'' + 3H\phi' + \left(2H' + H^2\right)\phi = 0 \tag{25}$$

If $a(\eta) \propto \eta^2$ then $a(\eta)' \propto 2\eta$. Whereas $H(\eta) = \frac{2}{\eta}$ and $H(\eta)' = -\frac{2}{\eta^2}$. η is conformal time, and the conformal Hubble rate is $H = \frac{a'}{a}$ where primes denote a derivative with respect to conformal time.

$$\phi'' + \frac{6}{\eta}\phi' + \left(\frac{-4}{\eta^2} + \frac{4}{\eta^2}\right)\phi = 0 \tag{26}$$

$$\phi'' + \frac{6}{\eta}\phi' = 0 \tag{27}$$

The general solution of the above equation is given below as

$$\phi(\eta) = A + \frac{B}{\eta^5} \tag{28}$$

where A and B are constants in time.$\phi(\eta) = A$ as η goes to zero.Therefore we conclude that the gravitational potential frozen (constant) in all scales. Poisons equation relate both gravitational potential and the comoving density.

$$\triangle\phi = 4\pi G\rho a^2 \delta_m^C \tag{29}$$

$$-k^2\phi = 4\pi G\rho_o a^{-1}\delta_m^C \tag{30}$$

We know that $\delta_m^C = \delta_m^N + 3Hv_m$,$-3H = \frac{\rho'}{\rho}$ and $H = \frac{2}{\eta}$.Therefore $\delta_m^N - \frac{6}{\eta}v_m^N \propto \phi a$. We can know deduce that during the matter era, the co-moving density contract evolve as scale factor(a) i.e $\delta_m^C \propto a$ .Using eq 2 we can find the velocity perturbations in matter dominated Universe.

$$v_m^{N'} + Hv_m^N = -\phi \tag{31}$$

$$v_m^N(\eta) \propto \frac{1}{a} \tag{32}$$

That is

$$v_m^N(\eta) = \frac{f}{a} \tag{33}$$

$$v_m^{N'} = -\frac{a'}{f}a^2 + \frac{f\prime}{a} \tag{34}$$

We know that $H = \frac{a\prime}{a}$ therefore eq 2 becomes

$$\frac{f\prime}{a} = -\phi \tag{35}$$

We know that $a \propto \eta^2$

$$\frac{f\prime}{\eta^2} = -\phi \tag{36}$$

Intergration the above equation we get

$$f = -\frac{-\eta^3}{3}\phi - M \tag{37}$$

$$v_m^N(\eta) = \frac{f}{a} = -\frac{\eta}{3}\phi - \frac{M}{\eta^2} \tag{38}$$

If we ignore the decaying mode ,the velocity perturbation (v_m)is proportional to the conformal time (η) i.e $v_m^N = \frac{\phi\eta}{3} \propto \eta \propto a^{\frac{1}{2}}$. The velocity perturbation grows proportional to the square root of the scale factor(i.e $v_m^N \propto \sqrt{a}$). Using $\delta_m^C = \delta_m^N + 3Hv_m^N$ and $v_m^N = \frac{\phi\eta}{3}$ eq(14) becomes

$$-k^2\phi = 4\pi Ga^2\rho_m(\delta_m^N + 2\phi) \tag{39}$$

$$-k^2\phi = \frac{3H^2(\delta_m^N + 2\phi)}{2} \tag{40}$$

$$\delta_m^N = -2\phi - \frac{2k^2\phi}{3H^2} \tag{41}$$

The above equation shows that density perturbation is frozen(constant) at large scale but grows at small scale. For super-horizon,k<<H,the density contract stays constant i.e $\delta_m^N = -2\phi$ and for sub-horizon scales,k>>H the density contract is in the form $\delta_m^N = -\frac{k^2\eta^2\phi}{6} \propto \eta^2 \propto a$. The density perturbations begin to grow when they enter the horizon(i.e $\delta_m = \textit{constant}$ before they enter the horizon) and grow proportional to the scale factor ($\delta_m \propto a$). This is just the conformal-time version of the Newtonian equation $\delta_m'' + 3H\delta_m' - 4\pi G\rho_m a^2\delta_m = 0$ where $' = \frac{d}{dt}$.

5.2 Radiation dominated universe

The gravitational potential can only be affected by only radiation density fluctuations when the universe is dominated by radiation.In this Era the radiation pressure causes the gravitational potentials to decay as modes enter the horizon.Pressure prevent any growth in the radiation perturbation.The growth of matter perturbations is logarithmic in this Era[Boehmer and Caldera-Cabral, 2010]. Consider the Poisson equation:

$$-k^2\phi = 4\pi G a^2 \delta_r^C \tag{42}$$

We know that $a \propto \eta$,$c_s = \frac{1}{3}$ and $H = \frac{a'}{a}$ therefore $H = \frac{1}{\eta}$.

$$\phi'' + 3H\phi' + \left(2H' + H^2\right)\phi + \frac{1}{3}k^2\phi = 0 \tag{43}$$

$$\phi'' + \frac{4\phi}{\eta} + \frac{k^2\phi}{3} = 0 \tag{44}$$

Let $\phi = \frac{u}{\eta}$ therefore $\frac{du}{d\eta} = \eta\phi' + \phi$ and $\frac{du}{d\eta} = 2\phi\prime + \phi\prime\prime$.We can write the above equation as

$$u'' + \frac{2u'}{\eta} + \left(\frac{k^2}{\eta} - \frac{2}{\eta^2}\right)u = 0 \tag{45}$$

The bessens funtions j_v and $n(x)$ are given as

$$j_1(x) = \frac{sin(x) - xcos(x)}{x^2} \tag{46}$$

$$n_1(x) = \frac{-cos(x) - xsin(x)}{x^2} \tag{47}$$

where l=1 and $x = \frac{k\eta}{3^{0.5}}$. The general equation for spherical functions is

$$\frac{d^2 j_l}{dx^2} + \frac{2}{x}\frac{dj_l}{dx} + \left(1 + \frac{l^2 + l}{x^2}\right) j_l = 0 \tag{48}$$

The solutions are

$$u(\eta) = Cj_1(x) + Dn_1(x) \tag{49}$$

As $x \to 0$ we get that $sin(x) \to x - \frac{x^3}{6}$ and $cos(x) \to 1 - \frac{x^2}{2}$ therefore $j_1 \sim \frac{x}{3}$ and $n_1 \sim \frac{1}{x^2} \sim \infty$. Due to this initials conditions the last term vanishes therefore the solution becomes

$$\phi(\eta) = \frac{u(\eta)}{\eta} = 3\phi_p \frac{sin(x) - xcos(x)}{x^3} \tag{50}$$

where x=$\frac{k\eta}{\sqrt{3}}$.The above equation tells us that the gravitational potential decays as soon as the mode enter the horizon and start to oscillate after decaying.As the mode enters the Horizon ,the gravitational potential decays because of the radiation pressure.After decaying ,the potential oscillate as shown in figure 2.Figure 2 shows the evolution of gravitational potential for diffrent values of wavenumber($k = \frac{2\pi}{\lambda}$).$a = 2x10^{-4}$. For super-horizon scales ($k << H$),the gravitational potential becomes

$$\phi(\eta) = \phi_p = contant \tag{51}$$

For sub-horizon scales ($k >> H$), the gravitational potential becomes

$$\phi(\eta) = -\frac{9\phi_p cos(x)}{(x\sqrt{3})^2} \tag{52}$$

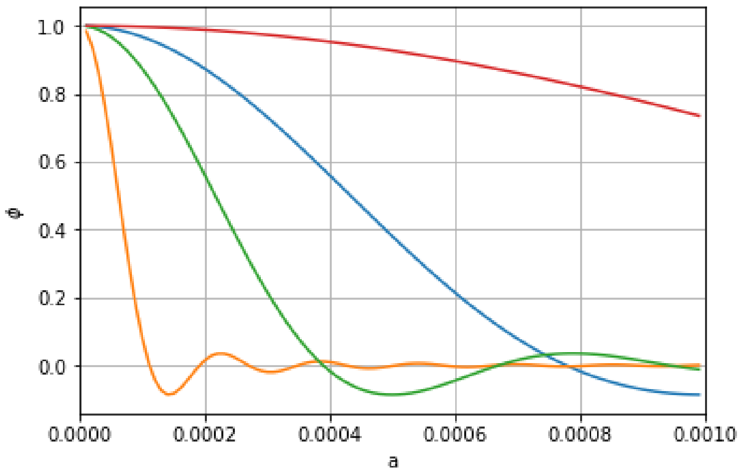

Figure 2: The evolution of gravitational potential as it decays and oscillates with an amplitude decreasing as η^{-2} in the universe dominated by radiation.

where $x = \frac{k\eta}{\sqrt{3}}$.The gravitational potential oscillates inside the horizon with an amplitude which decreases as η^{-2}. The velocity perturbations for sub-horizon scales($k >> H$) can be obtained by using one of the Einstein equations below

$$\phi'\eta + \phi = \frac{3(1+w)v_r}{k\eta} \tag{53}$$

We know that $w = \frac{1}{3}$ so

$$\phi'\eta + \phi = \frac{3(1+\frac{1}{3})v_r}{k\eta} \tag{54}$$

$$v = \frac{(k\eta^2\phi' + k\eta\phi)}{2} = \frac{9\phi_p c_s sin(x)}{2} \tag{55}$$

Using Einstein equations($\phi'\eta + \phi = \frac{3(1+\frac{1}{3})v}{k\eta}$) and $\eta\phi + \phi + \frac{(k\eta)^2\phi}{3} = -\frac{\delta_r^N}{2}$ we get that

$$\delta_r^N = -\frac{4v_m}{k\eta} + \frac{2(k\eta)^2\phi}{3} \tag{56}$$

The first term is the decaying mode which is proportional to η^{-1} and the second term is the growing mode which is proportional to η^2.As η goes to 0 the decaying mode vanishes,thus we have

$$\delta_r^N = \frac{-3(k\eta)^2\phi}{2} = 6\phi_p cos(x) \tag{57}$$

since pressure p=$\frac{\rho}{3}$ then w=$\frac{1}{3}$ and $\frac{\rho_r'}{\rho} = -4H$. The equation which relate both gravitational potential and the Co-moving density contract can be written as $-k^2\phi = 4\pi G\rho_o a^{-2}\delta_r^C$. We know that $\delta_r^C = \delta_r^N + \frac{\rho'}{\rho}v_r^N$,$-3H = \frac{\rho'}{\rho}$ and $H = \frac{1}{\eta}$.Therefore $\delta_r^N - \frac{3}{\eta}v_r^N \propto \phi a^2$ or $\delta_m^C \propto a^2$. We can know deduce that during the radiation era, the super-Hubble Co-moving density perturbations grow proportional to a^2 (i.e $\delta_r^C \propto a^2$) and Newtonian density perturbation is constant outside the horizon.Figure 3 shows the evolution of the Co-moving density contract. Using the below equation we can determine matter density perturbations.The equation relate the both gravitational potential(ϕ) and the density contract(δ_m^N).

$$\delta_m^{N''} + H\delta_m^N = S \tag{58}$$

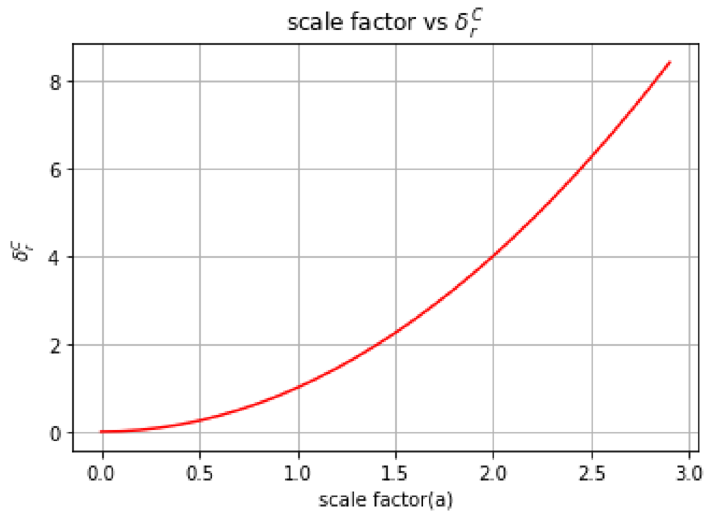

Figure 3: A plot of the Co-moving density contract vs scale factor(a).The plot shows that $\delta_m^C \propto a^2$ when the universe is radiation dominated.

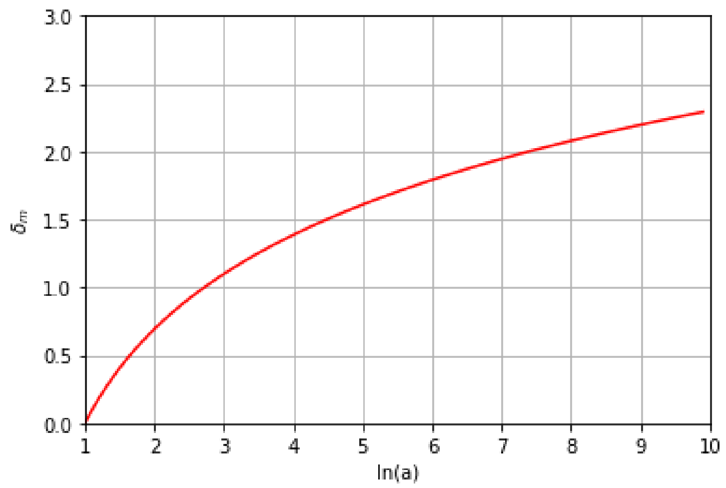

Figure 4: Evolution of the matter density contrast(Both δ_m^N) inside the Horizon when the universe is dominated by radiation.

where $S = 3\phi'' + 3H\phi - k^2\phi$ and $H = \frac{1}{\eta}$.The homogeneous(S=0) solution can be found by intergration which can be written as

$$\delta_m^N = A + ln(\eta) \tag{59}$$

Now we can conclude that in radiation dominated era $\delta_m^N = constant$ and $\delta_m^N = ln(a) = ln(\eta)$.Figure 4 shows the evolution of density contract inside the horizon during radiation era.

The total density for a mixture of both matter and radiation ($\rho = \rho_m + \rho_r = a^{-3} + a^{-4}$) where $p_m = 0$ and $p_r = \frac{1}{3}\rho_r$. Introducing the variable y which is the ratio of the scale factor and the scale factor at equality i.e

$$y = \frac{a}{a_{eq}} = \frac{\rho_m}{\rho_r} \tag{60}$$

$$\frac{\rho_r}{\rho} = \frac{\rho_r}{\rho_m + \rho} = \frac{1}{1+y} \tag{61}$$

$$\frac{\rho_m}{\rho} = \frac{\rho_m}{\rho_m + \rho_r} = \frac{y}{1+y} \tag{62}$$

The equation of state is simply

$$w = \frac{1}{3}\frac{1}{1+y} \tag{63}$$

$$c_2^s = \frac{4}{3(4+3y} \tag{64}$$

We know that $\rho = \rho_m + \rho_r$ and $p = p_m + p_r$ therefore the friedmann equation is given as

$$H^2 = \left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G\rho a^2}{3} = \frac{8\pi G(1+y)\rho_{ro}a_0^4 a^{-2}}{3} \tag{65}$$

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G(1+y)\rho_{ro}a_0^4}{3} \tag{66}$$

$$\frac{dy}{d\eta} = \frac{2a_0^2}{a_{eq}}\left(\frac{2\pi G(1+y)}{3}\right)^{\frac{1}{2}} \tag{67}$$

thus the general solution(Intergration) for the above equation is

$$y = B^2\eta^2 + B\eta \tag{68}$$

where $B = \frac{2a_0^2}{a_{eql}}\left(\frac{2\pi G(1+y)}{3}\right)^{\frac{1}{2}}$.
$y = y_{eq} = 1$ when matter density and radiation density are equal i.e at equality therefore

$$1 = B^2\eta_{eq}^2 + B\eta_{eq} \tag{69}$$

$$b\eta_{eq} = 2^{\frac{1}{2}} + 1 \tag{70}$$

We can right the solution as

$$y = \left(\frac{\eta}{\eta_2}\right)^2 + 2\left(\frac{\eta}{\eta_2}\right) \tag{71}$$

where $\eta_2 = \frac{1}{C} = \frac{\eta_{eql}}{2^{\frac{1}{2}}-1}$ The Hubble parameter is given as $H = \frac{a'}{a} = \frac{y'}{y} = \frac{\eta_2+\eta}{\frac{\eta}{2}+\eta\eta_2}$ At equality the Hubble parameter is simply

$$H_{eq} = \frac{4-8^{\frac{1}{2}}}{\eta_{eq}} = \frac{8^{\frac{1}{2}}}{\eta_2} \tag{72}$$

The universe is dominated by radiation at early times ($\eta << \eta_2$),so the solution is simply

$$y = 2\left(\frac{\eta}{\eta_2}\right) \rightarrow a \propto \eta \rightarrow H = \frac{1}{\eta} \tag{73}$$

When the universe is dominated by matter (i.e at present times($\eta >> \eta_2$))the solution is simply

$$y = \left(\frac{\eta}{\eta_2}\right)^2 \rightarrow a \propto \eta^2 \rightarrow H = \frac{2}{\eta} \tag{74}$$

Using the eq(54) to get η

$$y = \frac{\eta^2 + 2\eta_2\eta}{\eta_2^2} \rightarrow \eta_2^2 y + \eta_2^2 = \eta^2 + 2\eta_2\eta + \eta_2^2 \rightarrow \eta = \left((y+1)^{\frac{1}{2}} - 1\right)\eta_2 \rightarrow \eta = \frac{\left((y+1)^{\frac{1}{2}} - 1\right)\eta_{eq}}{2^{\frac{1}{2}} - 1} \tag{75}$$

The Hubble parameter is simply

$$H = \frac{y\prime}{y} = \frac{2(1+y)^{\frac{1}{2}}}{\eta_2 y} = \frac{H_{eq}(1+y)^{\frac{1}{2}}}{y2^{\frac{1}{2}}} \tag{76}$$

and

$$H^2 = \frac{H_{eq}^2(1+y)}{2y^2} \tag{77}$$

where $\eta_2 = (2^{\frac{1}{2}} + 1)\eta_{eql}$.
Meszaro's equation:
The Meszaros equation gives more details on how cold dark matter density perturbations at equality. The Friedmann equations are

$$H^2 = \frac{8\pi Ga^2\rho}{3} = \frac{8\pi Ga^2(\rho_m + \rho_r)}{3} = \frac{8\pi Ga^2\rho}{3} = \frac{8\pi Ga^2\rho_r(1+y)}{3} \tag{78}$$

$$\left(\frac{a''}{a}\right) = -\frac{4\pi Ga^2(\rho_m + 2\rho_r)}{3} = -\frac{4\pi Ga^2\rho_r(y+2)}{3} \tag{79}$$

Where $('= \frac{d}{dt})$,$\rho_m = \rho_{m0}a^{-3}$,$\rho_r = \rho_{r0}a^{-4}$,$y = \frac{a}{a_{eql}} = \frac{\rho_m}{\rho_r}$, and $\rho = \rho_m + \rho_r$. We introduce the equation of evolution of matter perturbations which is

$$\delta_m'' + H\delta_m' - 4\pi Ga^2\rho_m = 0 \tag{80}$$

We know that $\frac{d}{dt} = \left(\frac{a'}{a}\right) y^2 \frac{d^2}{dy^2}$,there the equation of evolution of matter perturbations can be

$$\frac{d^2\delta_m}{dy^2} + \frac{(2+3y)}{(2y(y+1)}\frac{d\delta_m}{dy} - \frac{3\delta_m}{2y(1+y)} = 0 \tag{81}$$

This equation has a solution as $\delta_m = 2 + 3y$.Density perturbations remain constant ($\delta_m = constant$) for $a < a_{eq}$(Radiation dominated universe) but for $a > a_{eq}$(matter dominated universe) the density perturbations grows as y(i.e $\delta_m \propto y$).

5.3 Transfer function

The transfer function T(k) gives the wavenumber $k = \frac{2\pi}{\lambda}$ dependence of the growth of perturbations in the matter density from early times when the universe was dominated by both matter and radiation until now when the universe is dominated by both dark energy and matter. The rate at which perturbations grow differs due to different cosmological models. When The universe is dominated by radiation and matter the density perturbations grows as scale factor squared(a^2) and scale factor(a). Any primordial perturbations can be affected by different processes :gravity ,pressure and dissipative processes.A perturbation with large wavelength λ will experience the same growth rate($\delta \propto a$) both outside and inside the horizon when the universe is dominated by matter.The transfer function T(k=$\frac{2\pi}{\lambda}$) remain constant when this occurs. Amplitude of perturbation with small wavelength will remain constant once it has entered the horizon during the radiation era.This waves with small wavelengths will experience growth rate($\delta \propto a^2$) outside the horozon but they will stop growing once it enters the horizon[Jaffe, 2012].$\lambda_{eq} = \frac{2\pi}{k_{eq}}$ is the horizon size at equality.

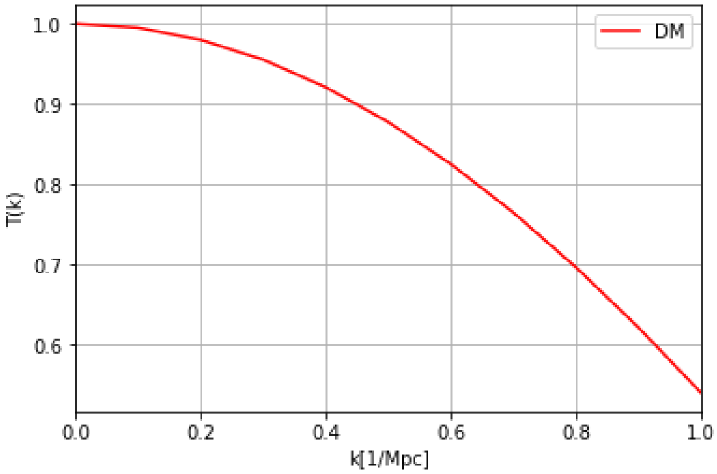

Figure 5: The plot of the transfer function for matter.The transfer function remain of order of unity(T~ 1) for $k < k_{eq}$ and T$\sim (1k)^2$ for $k > k_{eq}$.

$$\lambda_{eq} = 10(h^2)^{-1}Mpc \tag{82}$$

Radiation pressure might prevent the growth of perturbations,leading to oscillation in T(k).We now conclude that

$$T(k) \sim 1.....k << k_{eq} \tag{83}$$

$$T(k) \sim \frac{1}{k^2}......k >> k_{eq} \tag{84}$$

6 Conclusion and summary

Matter and radiation interact gravitationally .From the equations derived above the behavior of the gravitational potential ,density contract and the velocity depend on the mode k. Small perturbations were present at early times when the universe was dominated by radiation.These perturbations grew through self gravity ,pressure and other effects and developed into structures(galaxies,stars,etc) we see today. The gravitational potential also constant when the equation of state(w) changes.Newtonian perturbation theory allow us to conclude that the density contract(δ_m) is proportional to the scale factor(a) ,the gravitational potential perturbation is constant in tim e during the matter-dominated era on all scales when the universe is dominated by matter .

From the solutions obtained above the co-moving density contract(δ_m^C) grow as scale factor(a),while the density contract(δ_m^N) is constant outside the horizon in matter era.Both δ_m^N and δ_m^C grow as scale factor(a) inside the horizon in matter era.The density contractδ_m^N) grow logarithmically(i.e $\delta_m^N \propto ln(a)$)inside the horizon when the universe is dominated by radiation.The co-moving density contract(δ_m^C) grow as a^2. The gravitational potential decrease by the factor of 0.90 in the transition from radiation era to matter era[Baumann, 2014].As we get close to matter era the radiation pressure forces ceases to allow the matter perturbations to grow proportional to the scale factor.δ_m^N and δ_m^C coincide inside the horizon i.e subhorizon fluctuations during radiation era oscillates with a constant amplitude.

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