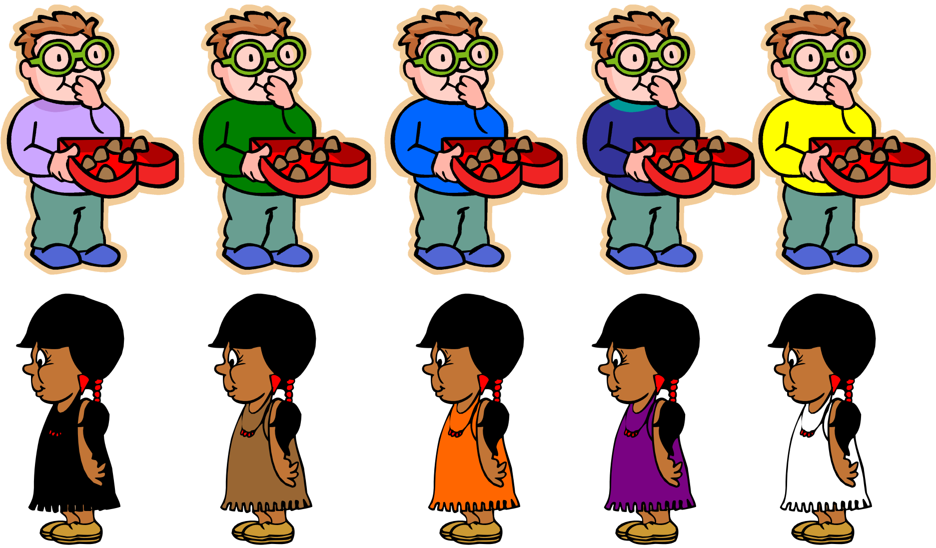
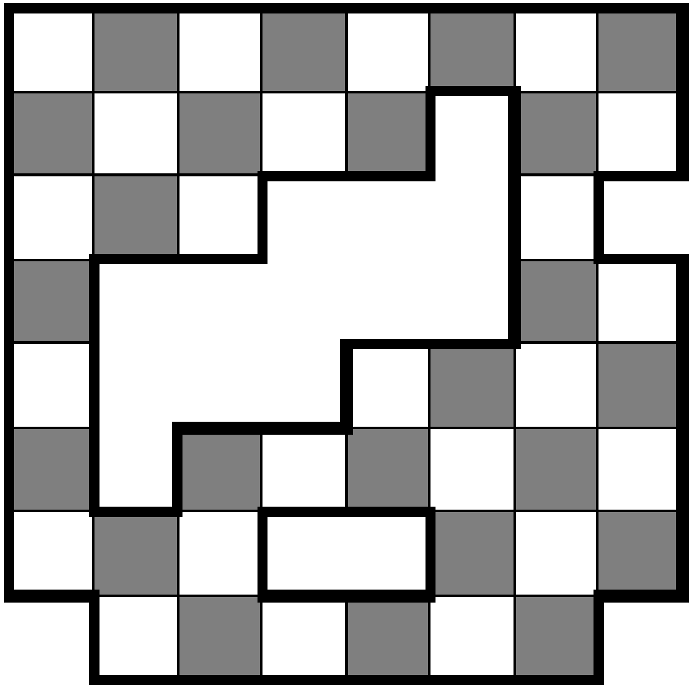
Matching



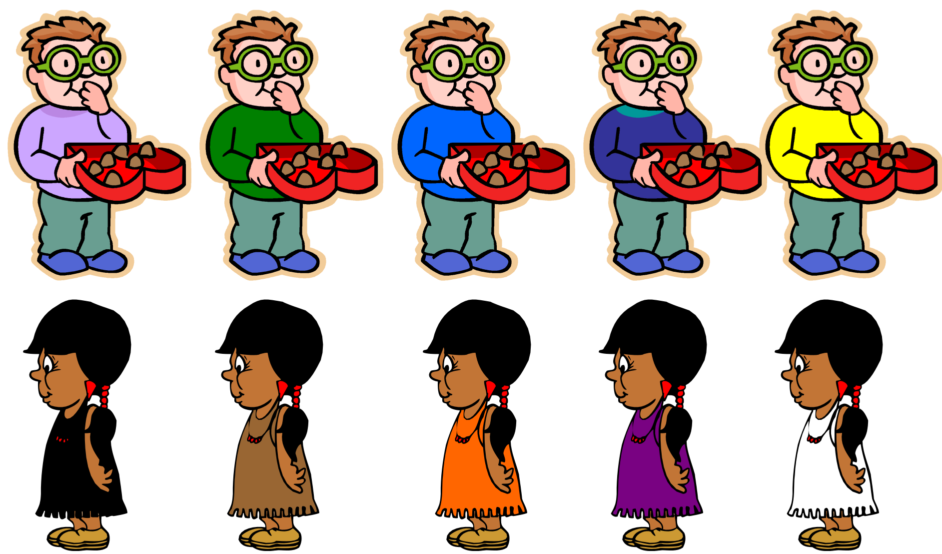


Matching

1 2 3 4 5

Boys

Girls



A B C D E

Today's goal: to "match" the boys and the girls in a "good" way.

Matching

Today's goal: to "match" the boys and the girls in a "good" way.

What is a matching?

- Each boy is married to at most one girl.
- Each girl is married to at most one boy.

What is a good matching?

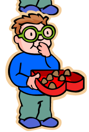
Depending on the information we have.

- ***A* stable matching:** They have no incentive to break up…
- ***A* maximum matching:** To maximize the number of pairs married…

Stable Matching

The Stable Marriage Problem:

- There are n boys and n girls.
- For each boy, there is a preference list of the girls.
- For each girl, there is a preference list of the boys.

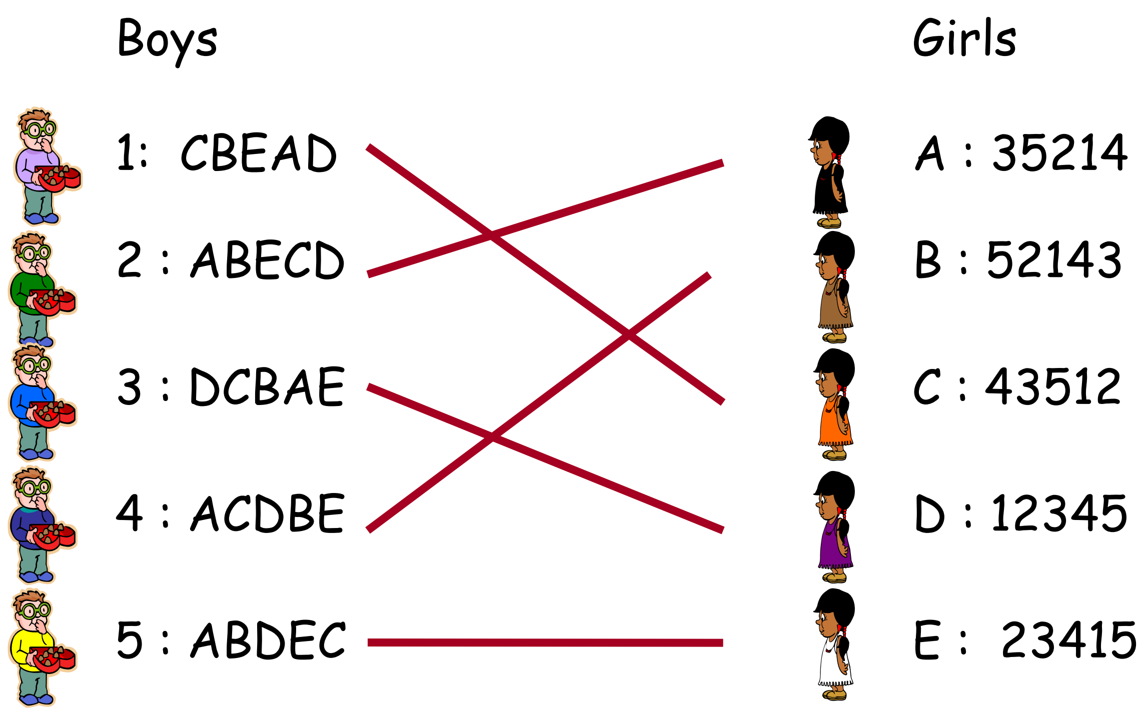
Boys		Girls	
	1: CBEAD		A : 35214
	2 : ABECD		B : 52143
	3 : DCBAE		C : 43512
	4 : ACDBE		D : 12345
	5 : ABDEC		E : 23415

Stable Matching

What is a ***stable*** matching?

Consider the following matching.

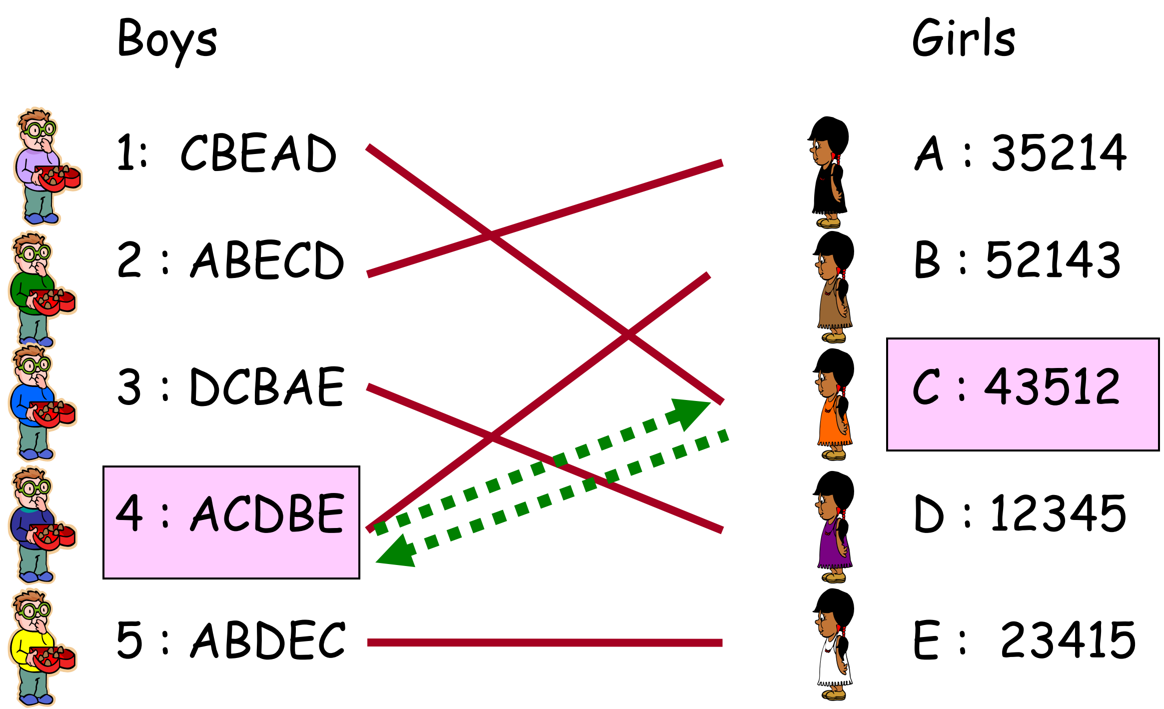
It is **unstable**, why?



Stable Matching

- Boy 4 prefers girl C more than girl B (his current partner).
- Girl C prefers boy 4 more than boy 1 (her current partner).

So they have the incentive to leave their current partners, and switch to each other, we call such a pair an **unstable pair**.



Stable Matching

What is a ***stable*** matching?

A stable matching is a matching with no unstable pair, and every one is married.

Can you find a stable matching in this case?

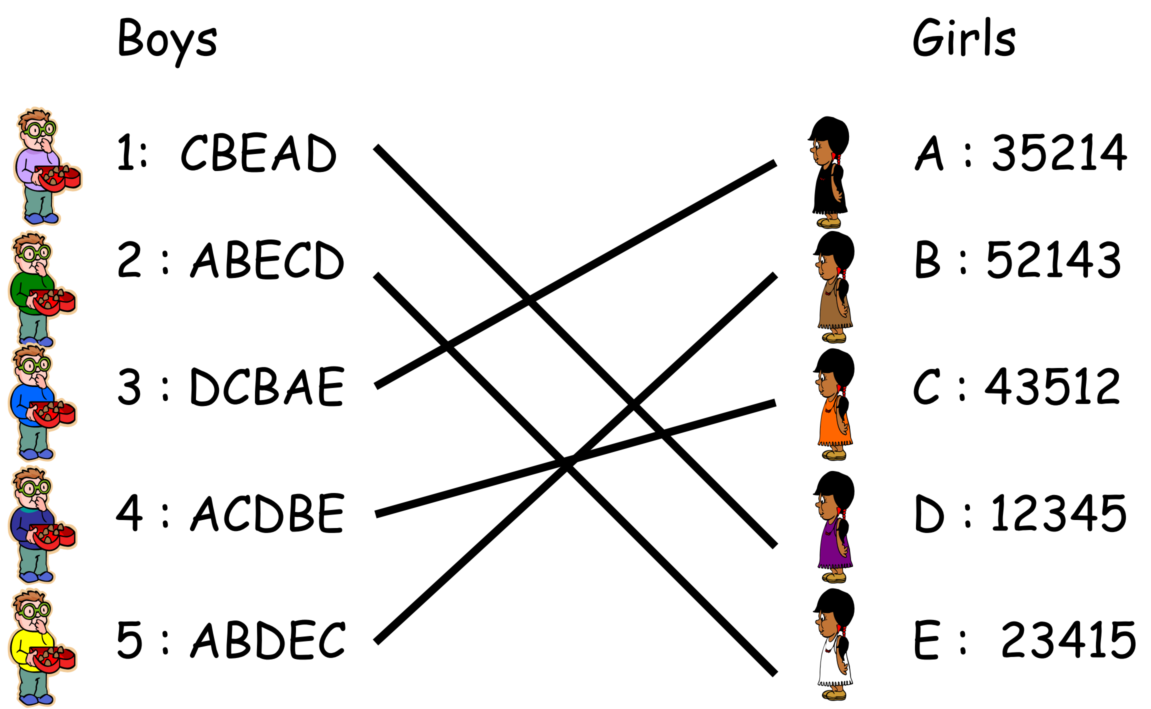
Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

Stable Matching

Does a stable matching always exists?

Not clear…

Can you find a stable matching in this case?



Stable Roommate

The Stable Roommate Problem:

- There are 2n people.
- There are n rooms, each can accommodate 2 people.
- Each person has a preference list of 2n-1 people.
- Find a stable matching (match everyone and no unstable pair).

Does a stable matching always exist?

Not clear…

When is it difficult to find a stable matching?

Idea: **triangle relationship**!

Stable Roommate

Idea: **triangle relationship**!

	1	2	3
a	b	c	d
b	c	a	d
c	a	b	d
d	a	b	c

- a prefers b more than c
- b prefers c more than a
- c prefers a more than b
- no one likes d

So let's say a is matched to b, and c is matched to d.

Then b prefers c more than a, and c prefers b more than d.

No stable matching exists!

Stable Matching

Can you now construct an example where there is no stable marriage? Nope…

Gale,Shapley [1962]:

There is always a stable matching in the stable marriage problem.

This is more than a solution to a puzzle:

- College Admissions (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.
- Matching Dancing Partners.

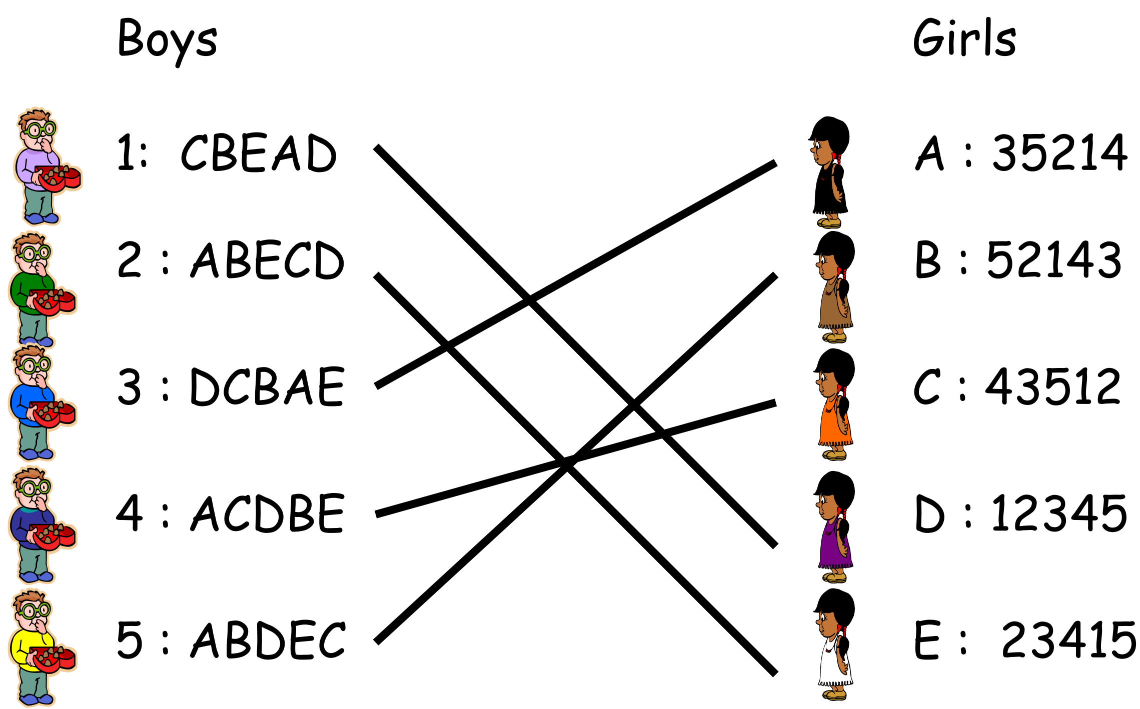
The proof is based on a marriage procedure…

Stable Matching

Why stable marriage is easier than stable roommate?

Intuition: It is enough if we only satisfy one side!

This intuition leads us to a very natural approach.



The Marrying Procedure

Morning: boy propose to their favourite girl

Billy Bob

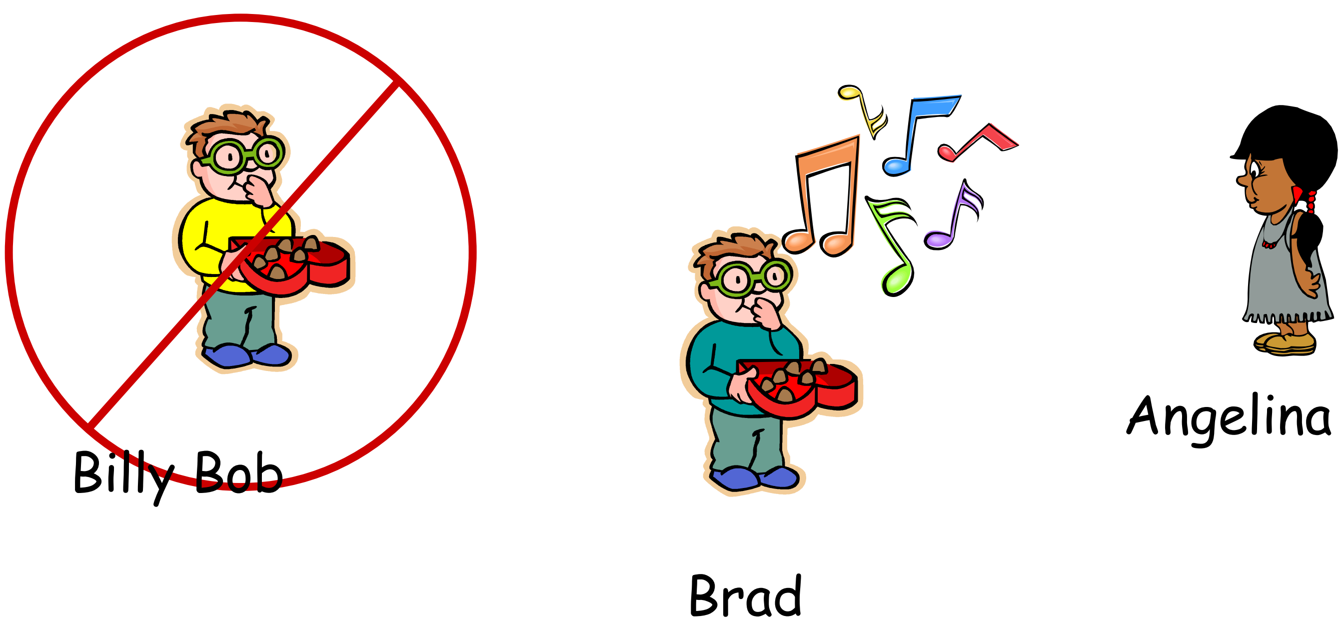
Brad

Angelina

The Marrying Procedure

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite



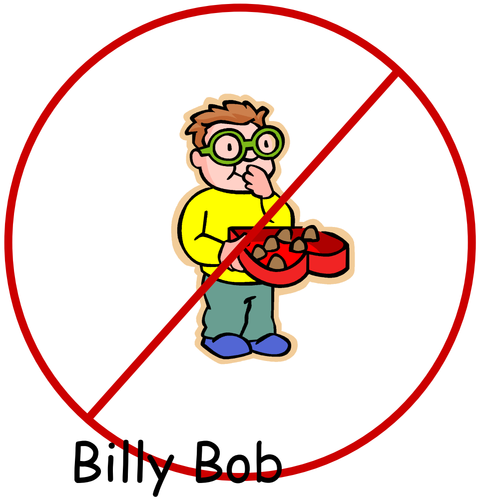
The Marrying Procedure

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite

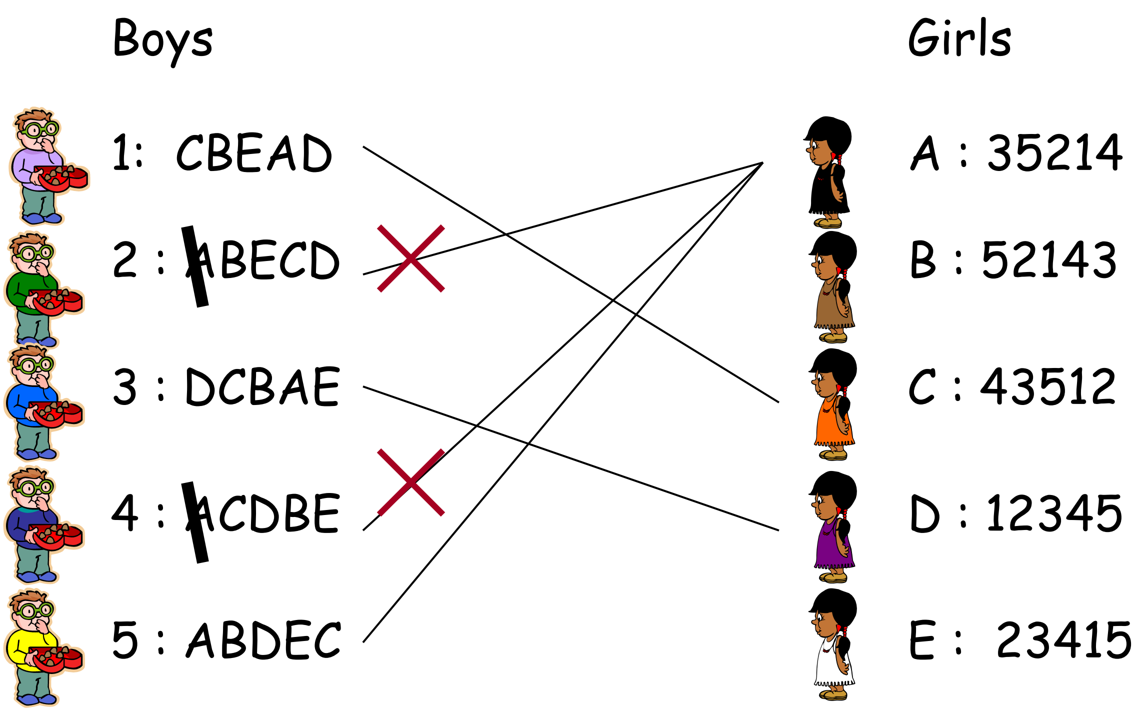
Evening: rejected boy writes off girl

This procedure is then repeated until all boys propose to a different girl



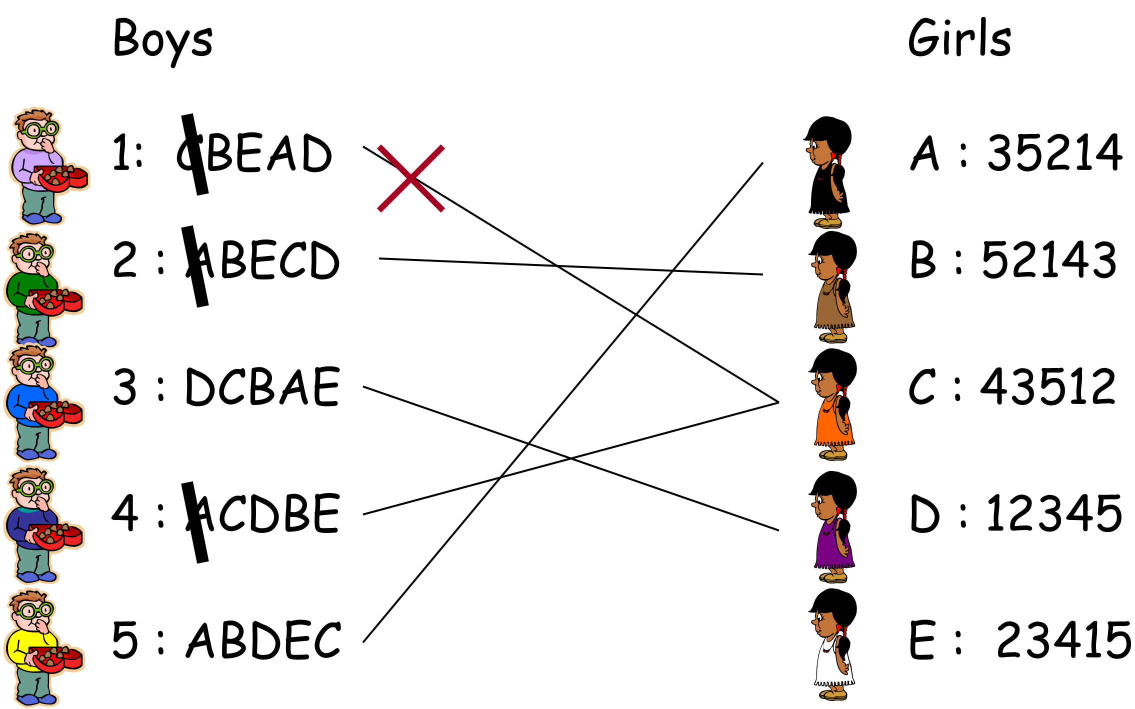
Day 1

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite
Evening: rejected boy writes off girl



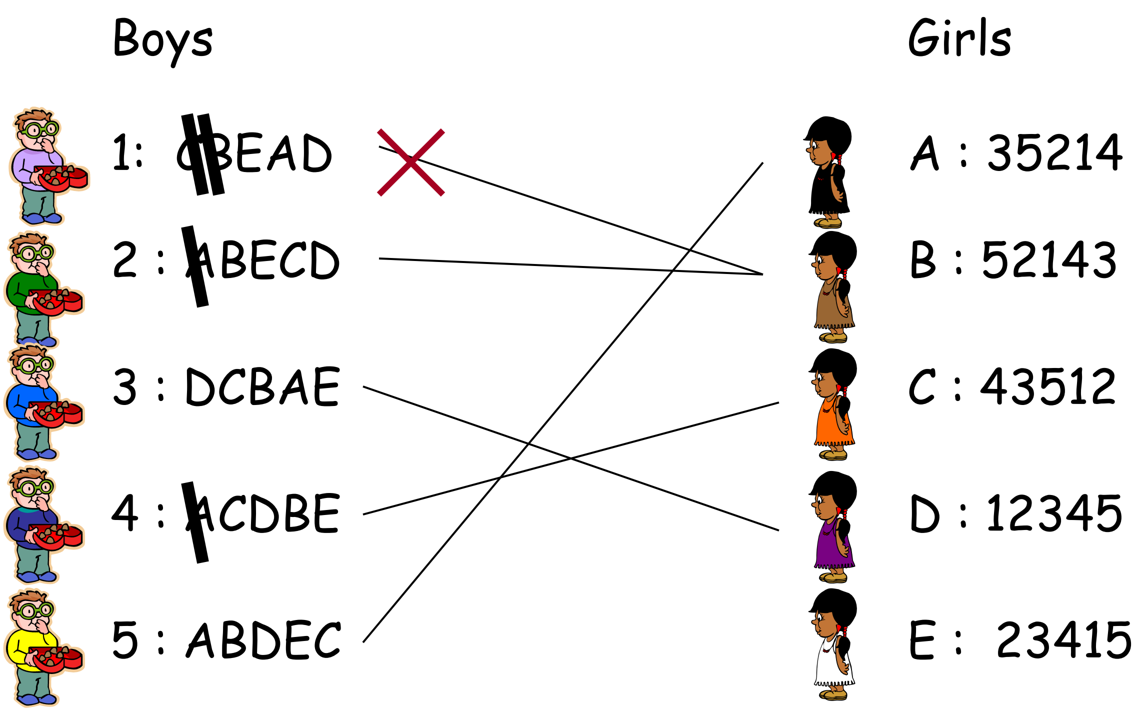
Day 2

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite
Evening: rejected boy writes off girl



Day 3

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite
Evening: rejected boy writes off girl



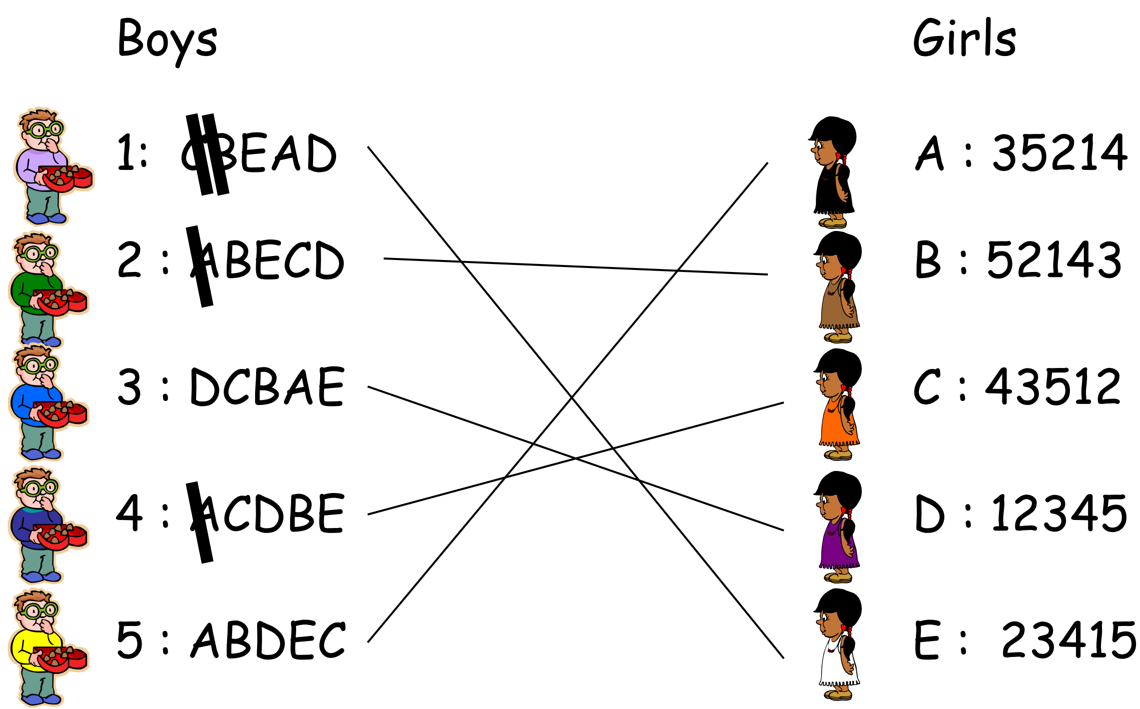
Day 4

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite

Evening: rejected boy writes off girl

OKAY, marriage day!



Proof of Gale-Shapley Theorem

Gale,Shapley [1962]:

This procedure always find a stable matching in the stable marriage problem.

What do we need to check?

1. The procedure will terminate.
2. Everyone is married.
3. No unstable pairs.

Step 1 of the Proof

Claim 1. The procedure will terminate in at most n^2 days.

1. If every girl is matched to exactly one boy,
   then the procedure will terminate.

2. Otherwise, since there are n boys and n girls,
   there must be a girl receiving more than one proposal.

3. She will reject at least one boy in this case,
   and those boys will write off that girl from their lists,
   and propose to their next favourite girl.

4. Since there are n boys and each list has at most n girls,
   the procedure will last for at most n^2 days.

Step 2 of the Proof

Claim 2. Every one is married when the procedure stops.

Proof: by contradiction.

1. If *B* is not married, his list is empty.
2. That is, *B* was rejected by all girls.
3. A girl only rejects a boy if she already has a more preferable partner.
4. Once a girl has a partner, she will be married at the end.
5. That is, all girls are married (to one boy) at the end, but *B* is not married.
6. This implies there are more boys than girls, a contradiction.

Step 3 of the Proof

Claim 3. There is no unstable pair.

Fact. If a girl G rejects a boy B,
then G will be married to a boy (she likes) better than B.

Consider any pair (B,G).

Case 1. If G is on B's list, then B is married to be the best one on his list.
So B has no incentive to leave.

Case 2. If G is not on B's list, then G is married to a boy she likes better.
So G has no incentive to leave.

Proof of Gale-Shapley Theorem

Gale,Shapley [1962]:

There is always a stable matching in the stable marriage problem.

Claim 1. The procedure will terminate in at most n^2 days.

Claim 2. Every one is married when the procedure stops.

Claim 3. There is no unstable pair.

So the theorem follows.

More Questions (Optional)

Intuition: It is enough if we only satisfy one side!

Is this marrying procedure better for boys or for girls??

- All boys get the best partner simultaneously!
- All girls get the worst partner simultaneously!

Why?

That is, among all possible stable matching,
boys get the best possible partners simultaneously.

Can a boy do better by lying? NO!

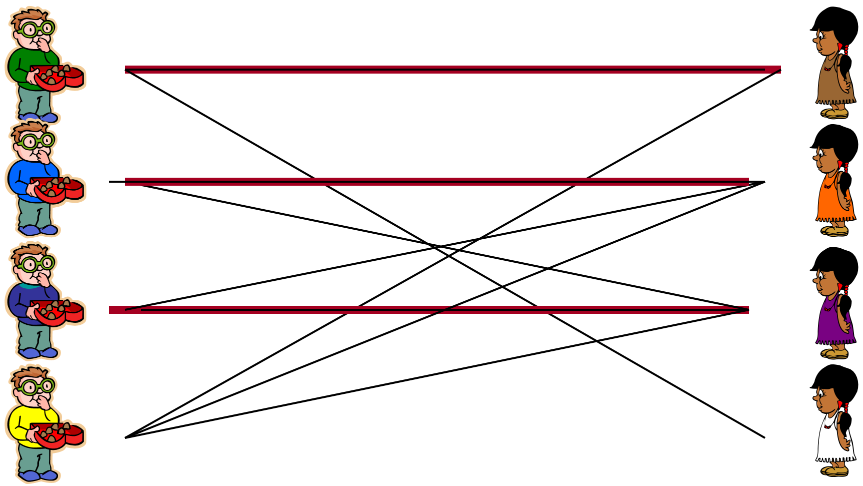
Can a girl do better by lying? YES!

Bipartite Matching

The Bipartite Marriage Problem:

- There are n boys and n girls.
- For each pair, it is either compatible or not.

Goal: to find the maximum number of compatible pairs.



Bipartite Matching

The Bipartite Marriage Problem:

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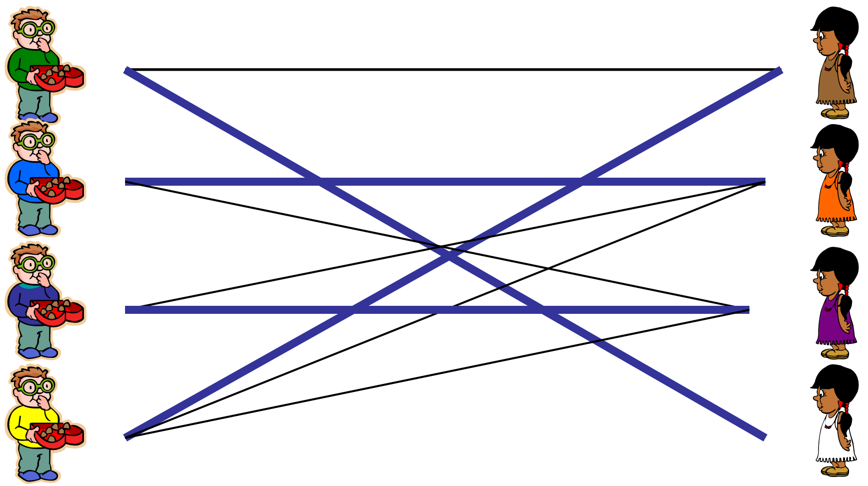
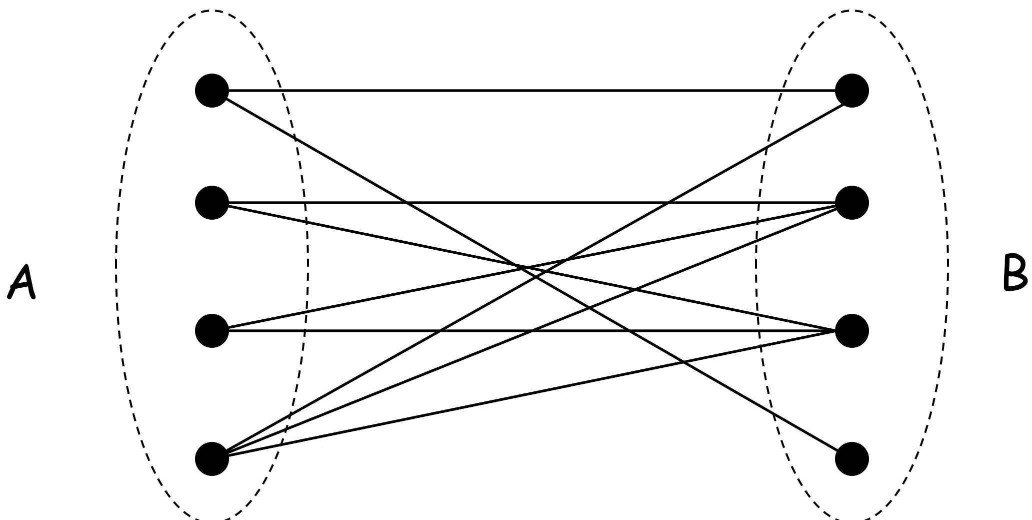


Graph Problem

A graph is **bipartite** if its vertex set can be partitioned into two subsets A and B so that each edge has one endpoint in A and the other endpoint in B.

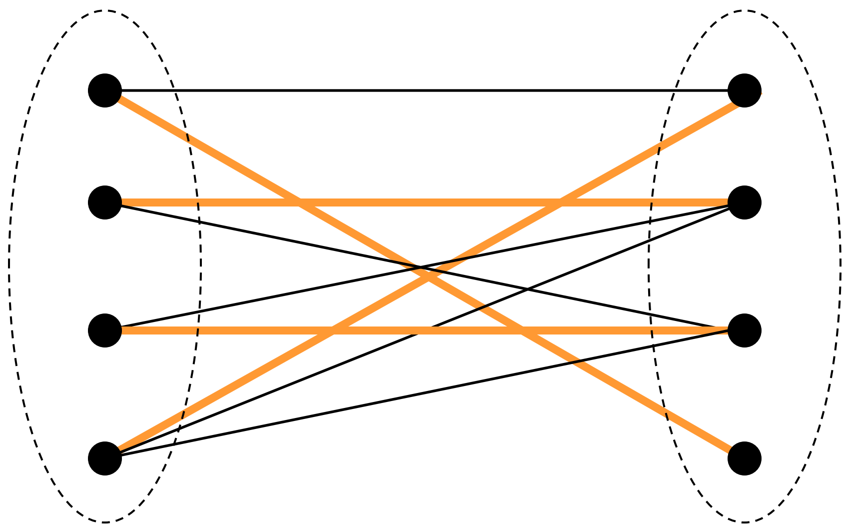


A **matching** is a subset of edges so that every vertex has degree at most one.

Maximum Matching

The bipartite matching problem:

Find a matching with the maximum number of edges.



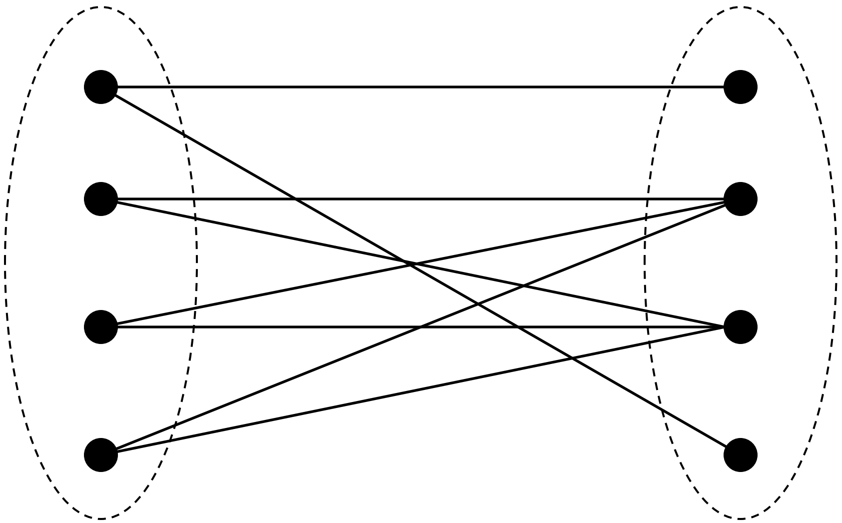
A perfect matching is a matching in which every vertex is matched.

The perfect matching problem: Is there a perfect matching?

Perfect Matching

Does a perfect matching always exist?

Of course not.



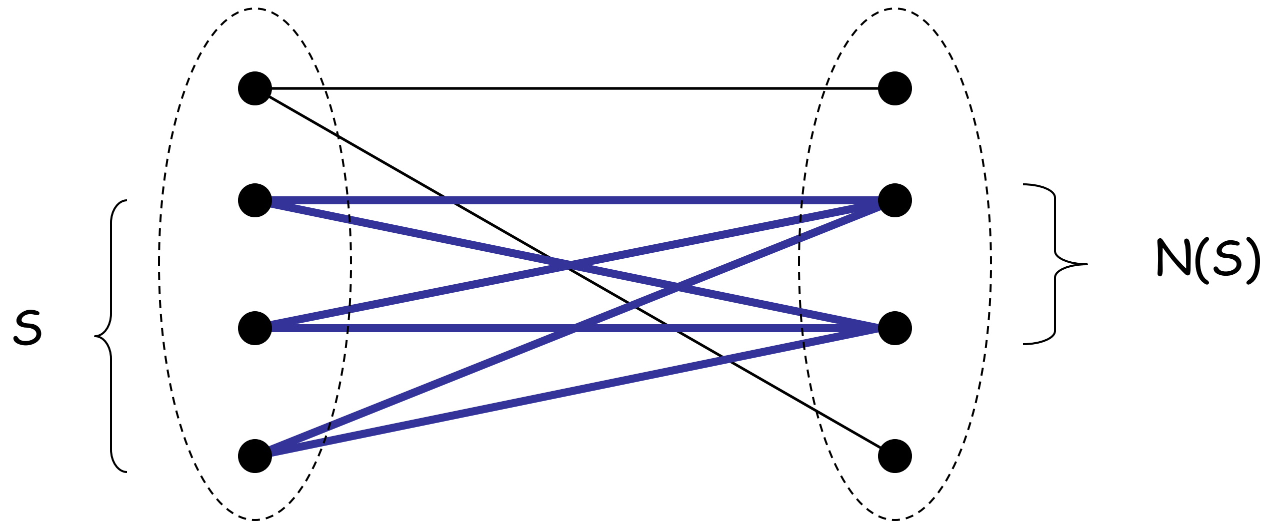
Suppose you work for the King, and your job is to find a perfect matching between 200 men and 200 women. If there is a perfect matching, then you can show it to the King. But suppose there is no perfect matching, how can you convince the King this fact?

Perfect Matching

Does a perfect matching always exist?

Of course not.

If there are more vertices on one side, then of course it is impossible.



Let N(S) be the neighbours of vertices in S.

If $|N(S)| < |S|$, then it is impossible to have a perfect matching.

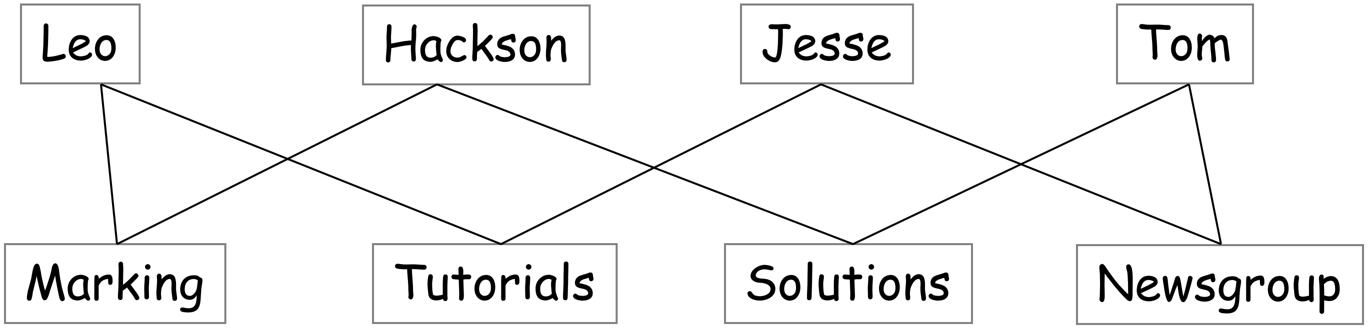
A Necessary and Sufficient Condition

Is it the only situation when a bipartite graph does not have a perfect matching?

Hall's Theorem: A bipartite graph G=(V,W;E) has a perfect matching if and only if |N(S)| >= |S| for every subset S of V and W.

This is a deep theorem.
It tells you exactly when a bipartite graph does not have a perfect matching.
(Now you can convince the king.)

Application of Bipartite Matching



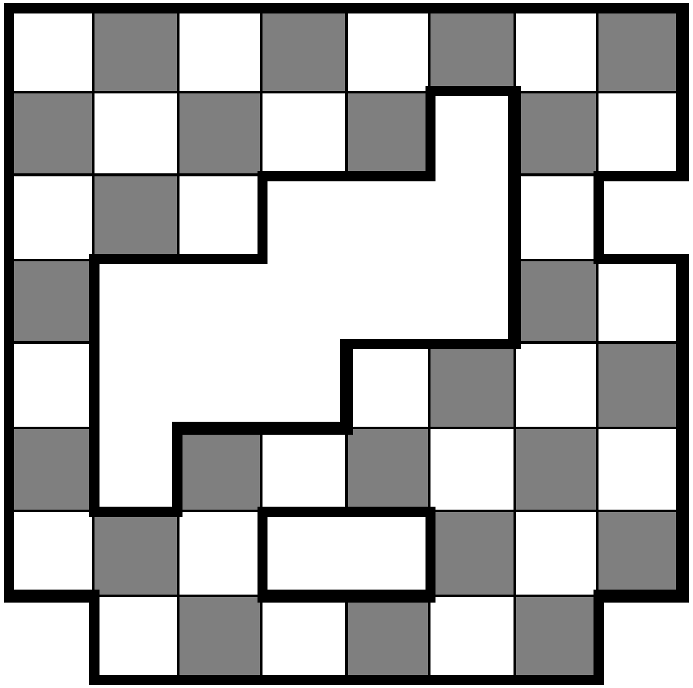
Job Assignment Problem:

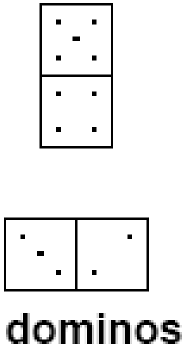
Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of?

In fact, there is an efficient procedure to find such as assignment!

Application of Bipartite Matching





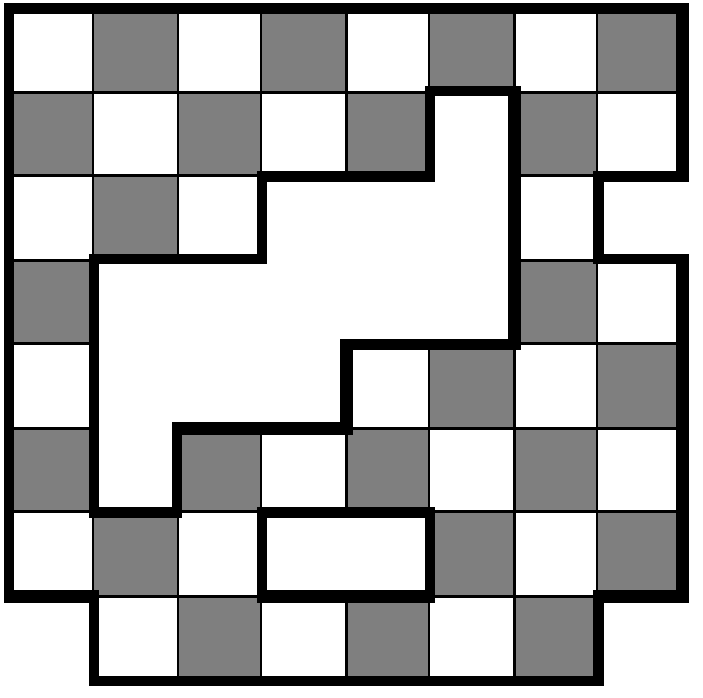
Add a vertex for each square in the board.

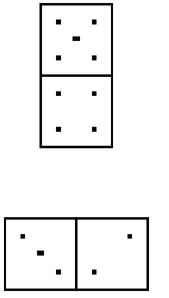
Add an edge for two squares if they are adjacent.

This is a bipartite graph with the black and white squares form the two sides.

A perfect matching in this graph corresponds to a perfect placement of dominos.

Application of Bipartite Matching





dominos

With Hall's theorem, now you can determine exactly when a partial chessboard can be filled with dominos.

Application of Bipartite Matching

Latin Square: a nxn square, the goal is to fill the square with numbers from 1 to n so that:

- Each row contains every number from 1 to n.
- Each column contains every number from 1 to n.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

Application of Bipartite Matching

Suppose you are given a partial Latin Square when some rows are already filled in.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

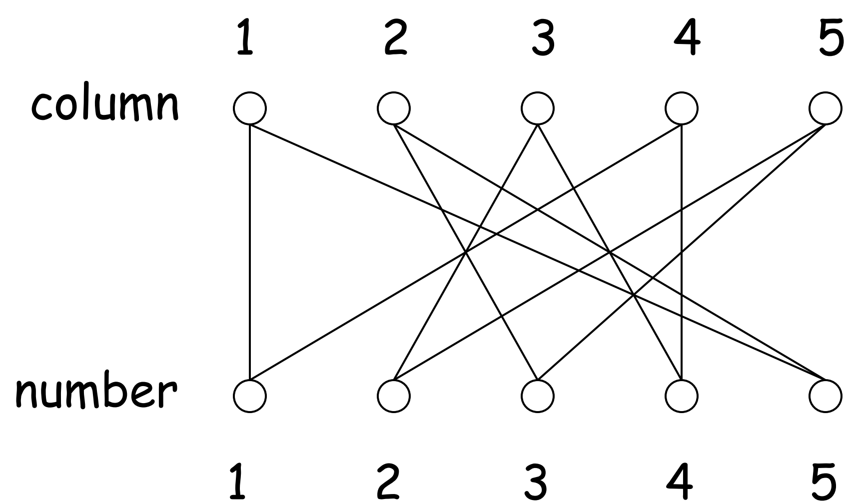
Can you always extend it to a Latin Square?

With Hall's theorem, you can prove that the answer is yes.

Application of Bipartite Matching

Given a partial Latin square, we construct a bipartite graph to fill in the next row.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4



We want to “match” the numbers to the columns.

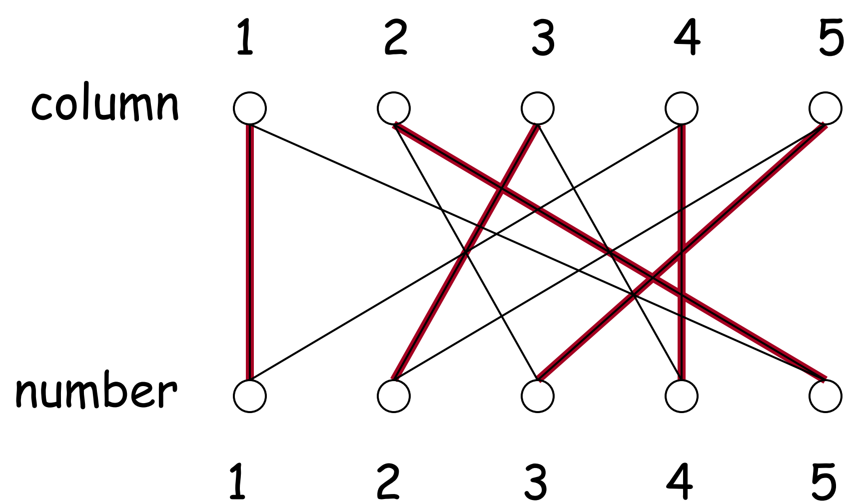
Add one vertex for each column, and one vertex for each number.

Add an edge between column i and color j if color j can be put in column i.

Application of Bipartite Matching

Given a partial Latin square, we construct a bipartite graph to fill in the next row.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4
1	5	2	4	3



A perfect matching corresponds to a valid assignment of the next row.

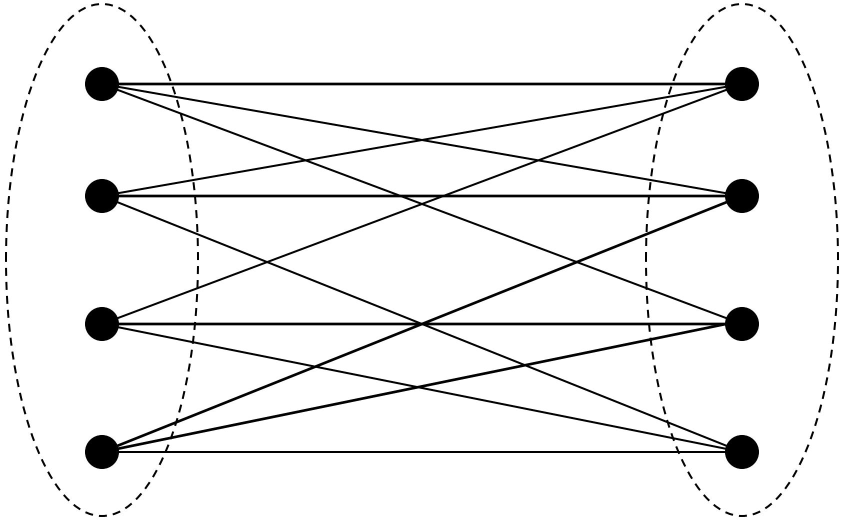
If we can always complete the next row, then by induction we are done.

The key is to prove that the bipartite graph always has a perfect matching.

Using Hall's Theorem

Hall's Theorem: A bipartite graph G=(V,W;E) has a perfect matching if and only if |N(S)| >= |S| for every subset S of V and W.

A graph is **k-regular** if every vertex is of degree k.

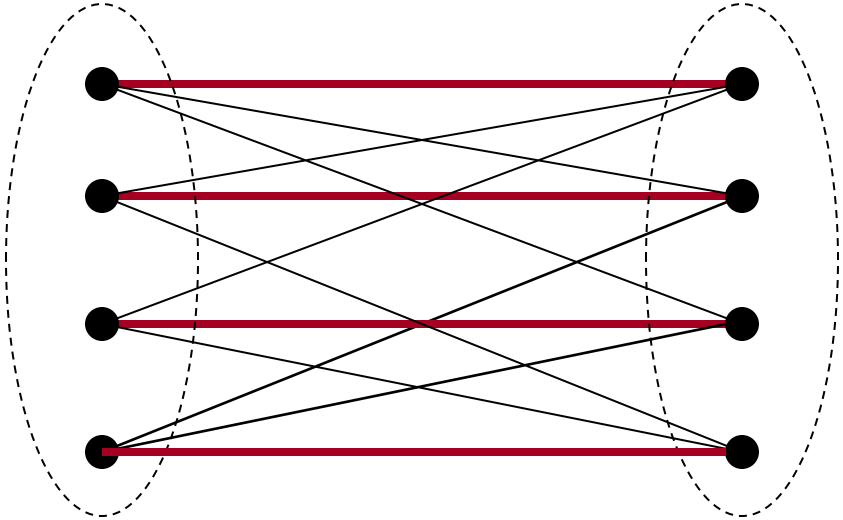


A 3-regular bipartite graph

Using Hall's Theorem

Hall's Theorem: A bipartite graph G=(V,W;E) has a perfect matching if and only if |N(S)| >= |S| for every subset S of V and W.

Claim: Every k-regular bipartite graph has a perfect matching.



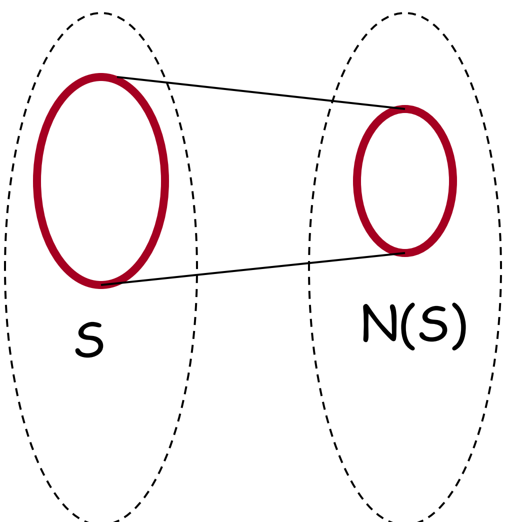
A 3-regular bipartite graph

Using Hall's Theorem

Hall's Theorem: A bipartite graph G=(V,W;E) has a perfect matching if and only if |N(S)| >= |S| for every subset S of V and W.

Claim: Every k-regular bipartite graph has a perfect matching.

To prove this claim using Hall's theorem,
we need to verify |N(S)| >= |S| for every subset S.



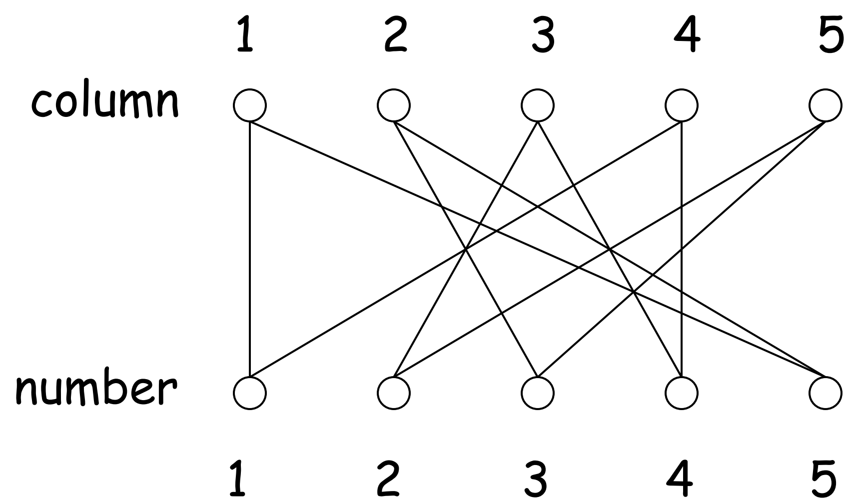
Proof by contradiction:

1. Suppose there is a subset S with |S| > |N(S)|.
2. All the edges from S go to N(S).
3. There are total k|S| edges from S to N(S).
4. There are at most k|N(S)| edges from N(S) to S.
5. A contradiction.

Completing Latin Square

Claim: Every k-regular bipartite graph has a perfect matching.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4



The bipartite graphs coming from Latin square are always regular because:

Suppose there are k unfilled rows.

Then each column already has n-k numbers, and so connected to k numbers.

Each number appeared in n-k columns above, and so connected to k columns.

So, the bipartite graph is k-regular, and thus always has a perfect matching.

Proof of Hall's Theorem

Hall's Theorem: If |N(S)| >= |S| for every subset S of V, then there is a perfect matching.

Case 1: Every subset S has |N(S)| > |S|. (Easy case)

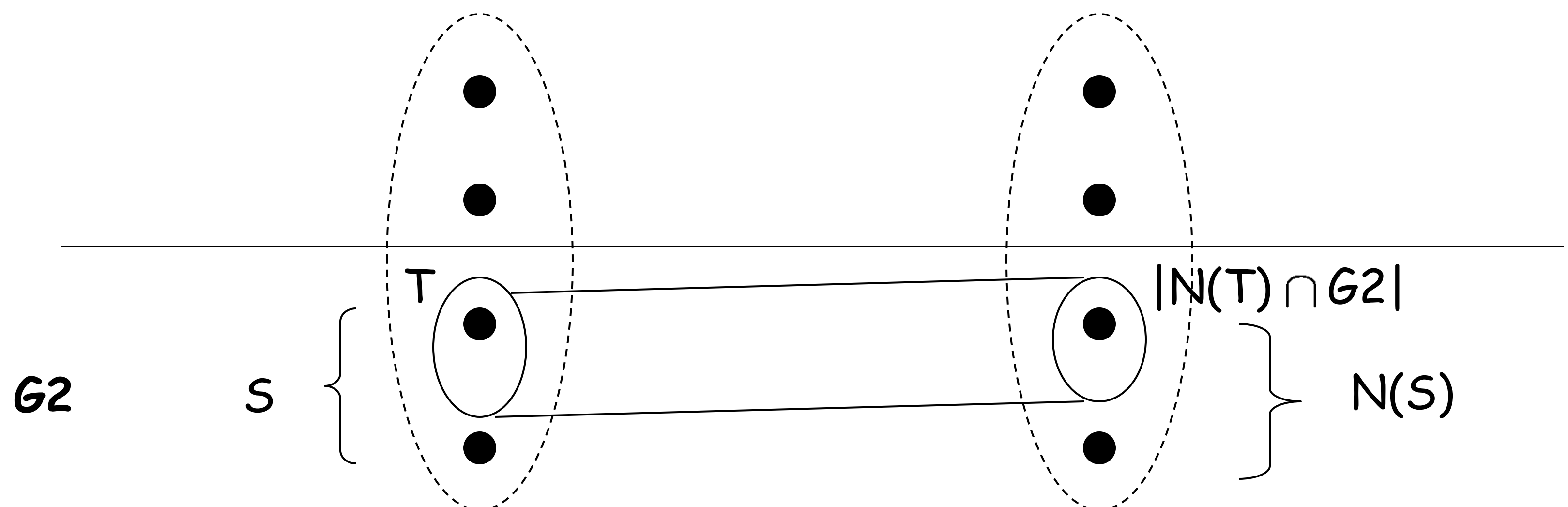
1. Just delete an edge.
2. By deleting an edge, |N(S)| can decrease by at most 1.
3. Since |N(S)| > |S| before,
4. we still have |N(S)| >= |S| after deleting an edge.
5. Since the graph is smaller (one fewer edge), by induction,
6. there is a perfect matching in this smaller graph,
7. hence there is a perfect matching in the original graph.

Proof of Hall's Theorem

Why there is a perfect matching in G2?

Hall's Theorem: If |N(S)| >= |S| for every subset S of V, then there is a perfect matching.

To apply Hall's, we want to show for any subset T of S, $|N(T) \cap G2| >= |T|$.



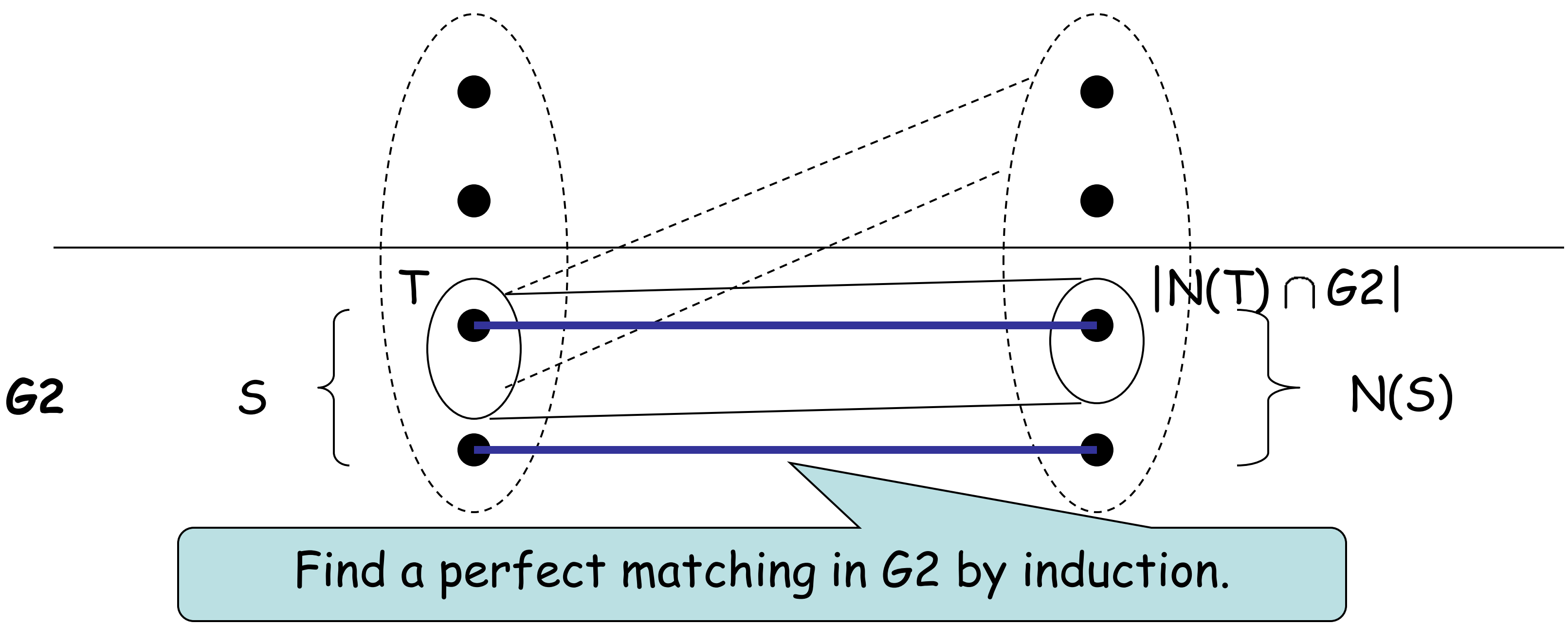
Proof of Hall's Theorem

Why there is a perfect matching in G2?

For any subset $T \subseteq S$, $N(T) \cap G2 = N(T)$.

By assumption, $|N(T) \cap G2| = |N(T)| >= |T|$.

Therefore, by induction, there is a perfect matching in G2.



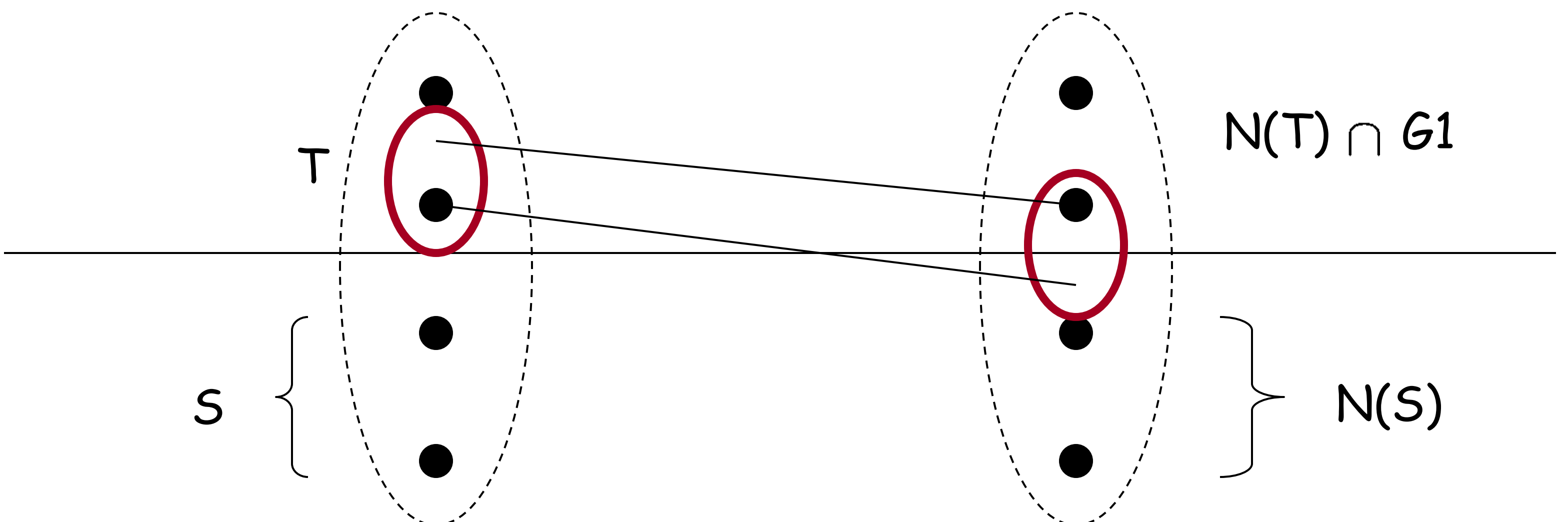
Proof of Hall's Theorem

Why there is a perfect matching in G1?

For any subset T, we want to show $|N(T) \cap G1| >= |T|$ to apply induction.

1. Consider T, by assumption, $|N(T)| >= |T|$
2. Can we conclude that $|N(T) \cap G1| >= |T|$?
3. No, because N(T) may intersect N(S)!

Now what?

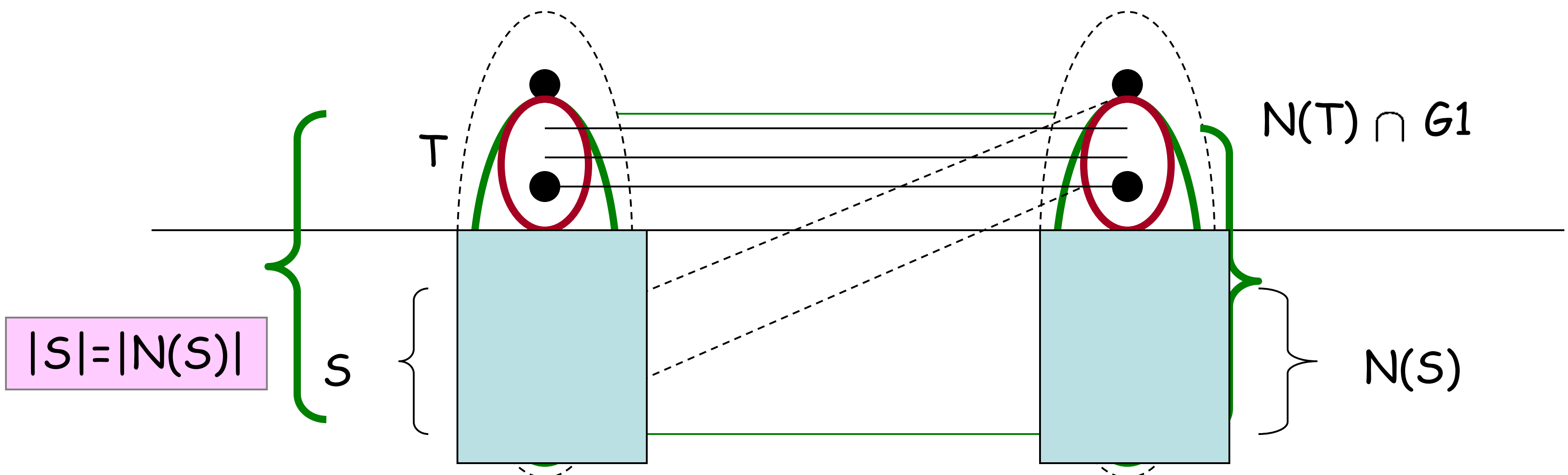


Proof of Hall's Theorem

Why there is a perfect matching in G1?

For any subset T, we want to show $|N(T) \cap G1| >= |T|$ to apply induction.

1. Consider $S \cup T$, by assumption, $|N(S \cup T)| >= |S \cup T|$ (the green areas).
2. Since $|S|=|N(S)|$, $|N(S \cup T) - N(S)| >= |S \cup T - S|$ (the red areas).
3. So $|N(T) \cap G1| = |N(S \cup T) - N(S)| >= |S \cup T - S| = |T|$, as required.



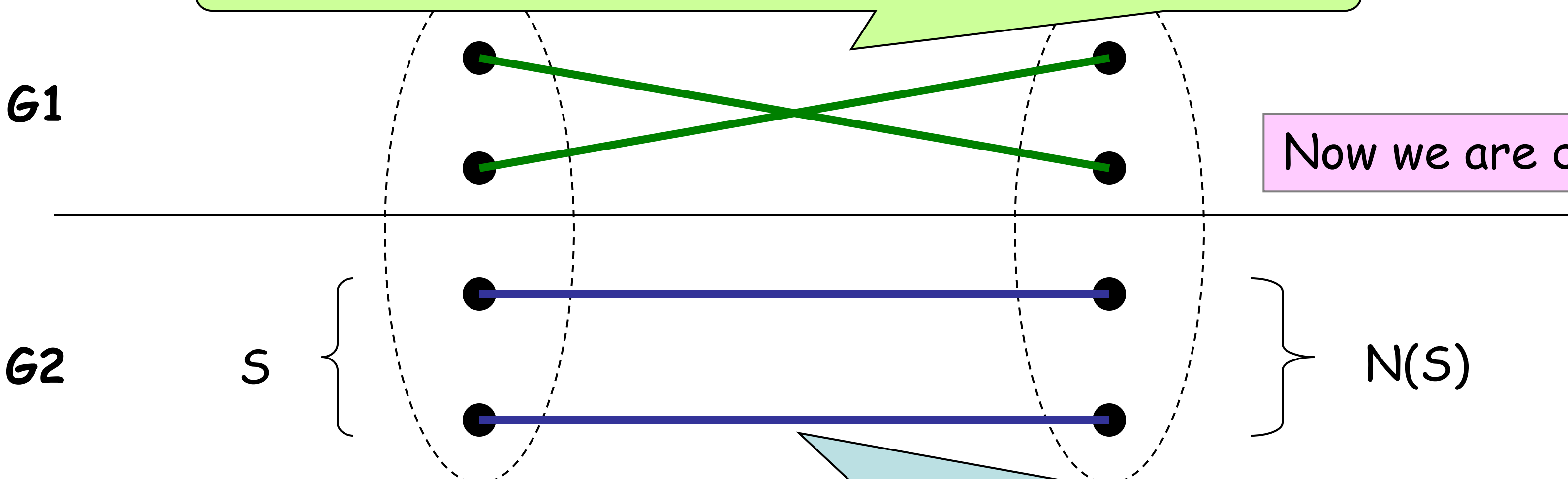
Proof of Hall's Theorem

Hall's Theorem: If |N(S)| >= |S| for every subset S of V, then there is a perfect matching.

Case 2: Suppose there is a subset S with |N(S)| = |S|.

Divide the graph into two smaller graphs G1 and G2 (so we can apply induction)

Find a perfect matching in G1 by induction.



Now we are done.

Find a perfect matching in G2 by induction.

Bipartite Matching and Hall's Theorem

Hall's theorem is a fundamental theorem in graph theory.

In this course, it is important to learn

1. how to use bipartite matching to solve problems, and
2. how to apply Hall's theorem.

The proof of Hall's theorem is optional.