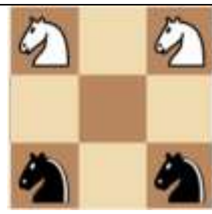
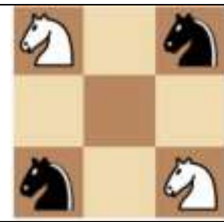
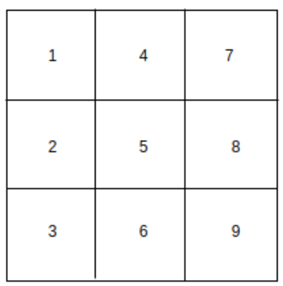
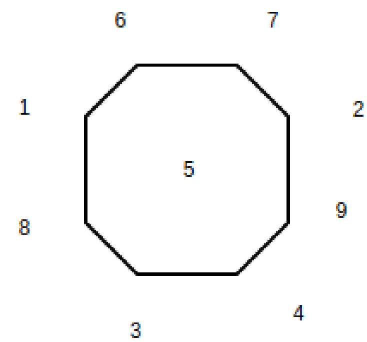
Applications of Graph Theory: (Graph Coloring)

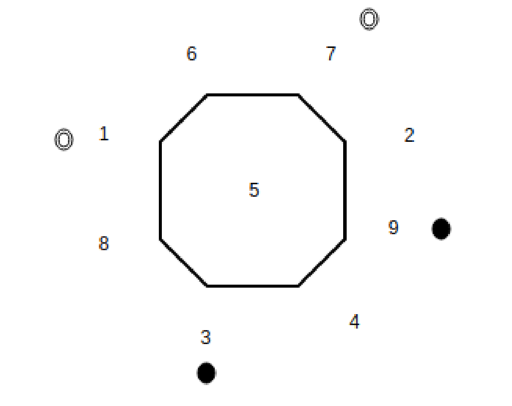
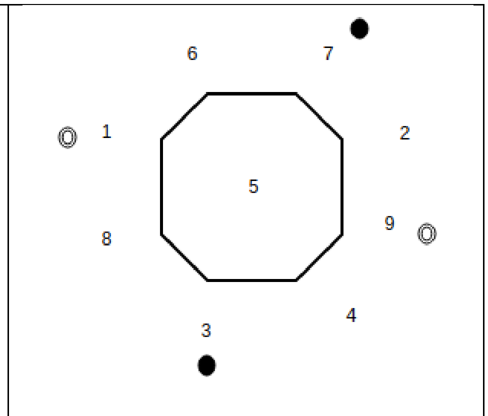
1) The figure below shows an arrangement of knights on a 3*3 grid.
Is it possible to reach to the expected final state as shown below using valid knight's moves?

<table>
<tr><td></td><td></td></tr>
<tr><td>Initial Configuration</td><td>Expected Final Configuration</td></tr>
</table>

Solution – NO
Let each of the 9 vertices be represented by a number as shown below.

<table>
<tr><td>
</td><td>
</td></tr>
<tr><td>Vertices Grid</td><td>Grid Graph (Connected as per Knight's moves)</td></tr>
</table>

Now we consider each square of the grid as a vertex in our graph. There exists a edge between two vertices in our graph if a valid knight's move is possible between the corresponding squares in the graph. For example – If we consider square 1. The reachable squares with valid knight's moves are 6 and 8. We can say that vertex 1 is connected to vertices 6 and 8 in our graph.
Clearly vertex 5 can't be reached from any of the squares. Hence none of the edges connect to vertex 5.

	
Initial Graph	Expected Final Graph
hollow circle → a white knight filled circle → a black knight	

Note that in order to achieve the final state there needs to exist a path where two knights (a black knight and a white knight cross-over). We can only move the knights in a clockwise or counter-clockwise manner on the graph (If two vertices are connected on the graph: it means that a corresponding knight's move exists on the grid). However the order in which knights appear on the graph cannot be changed. There is no possible way for a knight to cross over (Two knights cannot exist on one vertex) the other in order to achieve the final state.

➔no matter what the final arrangement is not possible.

- *Ref: https://www.geeksforgeeks.org/graph-theory-practice-questions/*

2) A PUZZLE WITH MULTICOLORED CUBES
 (Toy Store_*Instant Insanity*)
 Step 1: the physical problem is converted into a problem of graph theory
 Step 2: the graph-theory problem is then solved

Problem: We are given four cubes. The six faces of every cube are variously colored blue, green, red, or white. Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the four sides of this column?
(A trial-and-error method is unsatisfactory,41,472 (= 3 × 24 × 24 × 24) possibilities.)

***Solution:* Step 1:** Draw a graph with four vertices *B*, *G*, *R,* and *W*—one for each color

- For cube 1; represent its three pairs of opposite faces by three edges, drawn between the vertices with appropriate colors.

(e.g. if a blue face in cube 1 has a white face opposite to it, draw an edge between vertices *B* and *W* in the graph.)

- Do the same for the remaining two pairs of faces in cube 1. Put label 1 on all three edges resulting from cube 1. A self-loop with the edge labeled 1 at vertex *R,* for instance, would result if cube 1 had a pair of opposite faces both colored red.
- Repeat the procedure for the other three cubes one by one on the same graph until we have a graph with **four vertices and 12 edges**.

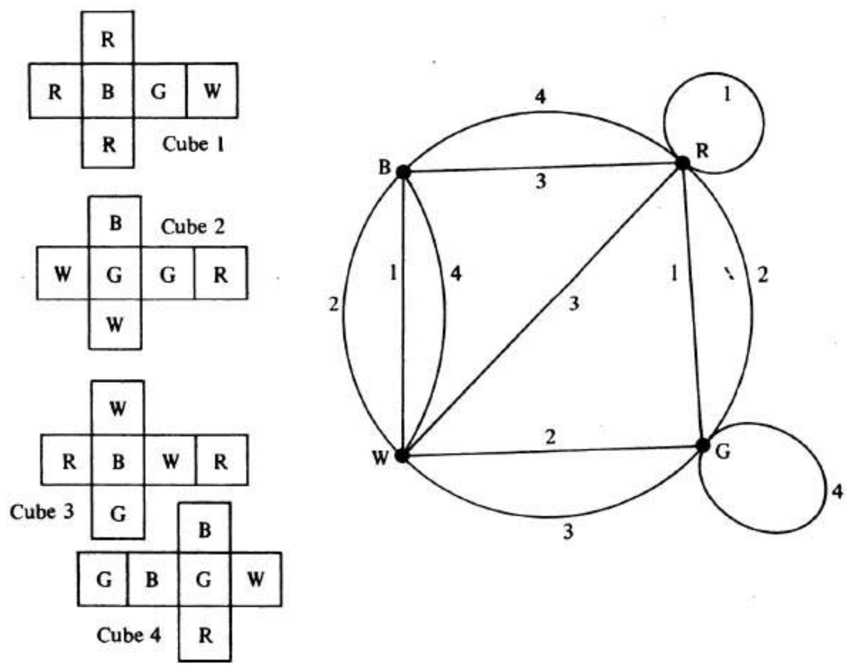


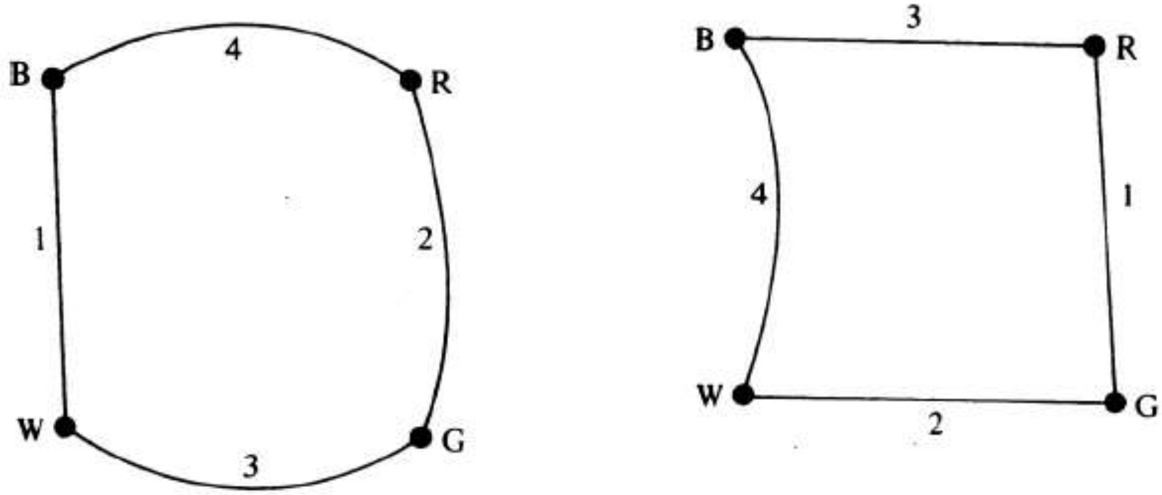
Fig Color_1

Step 2: Consider the graph resulting from this representation. **The degree of each vertex is the total number of faces with the corresponding color**. e.g. we have five blue faces, six green, seven red, and six white (as per one design instance)

Consider two opposite vertical sides of the desired column of four cubes, say facing north and south. **A subgraph (with four edges) will represent these eight faces—** four facing south and four north. Each of the four edges in this subgraph will have a different label—1, 2, 3, and 4. Moreover, no color occurs twice on either the north side or south side of the column **if and only if every vertex in this subgraph is of degree two**. Exactly the same argument applies to the other two sides, east and west, of the column.

Thus the four cubes can be arranged (to form a column such that no color appears more than once on any side) **iff there exist two edge-disjoint subgraphs, each with four edges, each of the edges labeled differently, and such that each vertex is of degree two**. For the set of cubes shown in Fig Color_1

This condition is satisfied, and the two subgraphs (Two edge-disjoint subgraphs of the graph in Fig Color_1) are shown in Fig Color_2.



(a) North-South Subgraph (b) East-West Subgraph

Fig Color_2