CS-430 - Intro. to Algorithms.
H.W - 1.

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Problem - 1

$$T(n) = \begin{cases} 2 & , \text{ if } n = 2 \\ 2T(n/2) + n & , \text{ if } n = 2^k \text{ for } k > 1 \end{cases}$$

is $T(n) = n \lg n$.

1) Base Case: for $n = 2$

$$T(2) = 2 \log_2 2$$
$$= \underline{\underline{2}}$$

2) Induction Hypothesis: It is true for $n-1$

$n = 2^k$ (n is power of 2)

$\therefore (n-1) = 2^{k-1}$

$$T(n-1) = (n-1)(\log_2(n-1))$$
$$= (n-1) \log_2(n-1)$$

$$\Rightarrow T(2^{k-1}) = 2^{k-1} \log_2(2^{k-1})$$
$$= (k-1) \cdot 2^{k-1} \cdot \log_2 2$$

P.T.O

$$(k-1)\cdot 2^{k-1}\cdot 1$$

$$= \underline{\underline{2^{(k-1)}(k-1)}}$$

3) To prove it is valid for n

$$T(n) = 2T(n/2) + n$$
$$T(n) = 2T(2^k/2^1) + 2^k$$
$$T(n) = 2T(2^{k-1}) + 2^k$$
$$T(n) = 2^1\cdot 2^{k-1}\cdot(k-1) + 2^k$$
$$T(n) = 2^k\cdot(k-1) + 2^k\cdot 1$$
$$T(n) = 2^k\cdot(k-1+1)$$

$$T(n) = 2^k\cdot k = 2^k\cdot \log_2 2^k$$

$$= \underline{\underline{n\cdot \lg n}}$$

Hence

$$\underline{\underline{T(n) = n\cdot \lg n}}$$

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Problem 3)

a) $T(n) = 4T(n/3) + n\lg n$

Master Theorem:-

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$, $a \geq 1$, $b > 1$

now,

$a = 4$ $b = 3$ $f(n) = n\lg n$

$a \cdot f(n/b) = 4 \cdot \frac{n}{3} \lg \frac{n}{3}$

$= \frac{4}{3} \cdot n\lg n/3$

So now, $c = \frac{4}{3}$

$\therefore$ $c > 1$

By Case 2 of Master Theorem,

$T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_3 4})$.

b) $T(n) = 3T(n/3) + n/\lg n$

since $n/\lg n$

Soln $\longrightarrow$ $T(n) = 3T(n/3) + n/\lg n$

$$T(n) = 3\cdot\left(3T(n/9) + \frac{n/3}{\lg n/3}\right) + \frac{n}{\lg n}$$

$$= 9T(n/9) + n/\lg n/3 + \frac{n}{\lg n}$$

$$\Rightarrow 3^i T(n/3^i) + \sum_{j=1}^{i-1} \frac{n}{\lg\left(\frac{n}{3^{j-1}}\right)}$$

So now,

for

$$i = \log_3 n$$

$$T\left(\frac{n}{3i}\right) = \theta.(n)$$

P.T.O

b) By substitution,
We guess

$T(n) = O(n \lg n)$

$\therefore T(n) = \leq cn \lg n$

$T(n) = 3T(n/3) + n/\lg(n)$
$\leq cn(\lg(n)) - cn \lg(3)$
$+ n/\lg(n)$

$= cn \lg(n) + n\left(\frac{1}{\lg(n)} - c\lg(3)\right)$

$\leq cn \lg(n)$

$\therefore T(n) \geq cn^{1-\epsilon}$ for $\forall\ \epsilon > 0$

$\therefore 3^{\epsilon} + n^{\epsilon} / (c \lg(n)) \geq 1$

$\therefore T(n) =$ Theta of n.

$T(n) = \theta(n)$
- soft function.

c) $T(n) = 4T(n/2) + n^2\sqrt{n}$

$T(n) = 4T(n/2) + n^2\, n^{1/2}$

$T(n) = 4T(n/2) + n^2 \cdot n^{1/2}$

$T(n) = 4T(n/2) + n^{5/2}$

for recurrence relation,

$$T(n) = c \qquad n < c_1$$
$$= aT(n/b) + \Theta(n^i), \quad n \geq c_1$$

Has sol^n :-

1) If $a > b^i$ then $T(n) = \Theta(n^{\log_b a})$

2) ~~Inf~~ $a = b^i$ then $T(n) = \Theta(n^i \log_b n)$

3) If $a < b^i$ $T(n) = \Theta(n^i)$

$a = 4$, $b^i = 2^{5/2}$, $f(n) = \cancel{i} = n^{5/2}$

$f(n) = n^{5/2}$

$\therefore i = 5/2$

$\therefore T(n) = \Theta(n^i)$
$= \Theta(n^{5/2})$

$T(n) = \Theta(n^{5/2})$

d) $T(n) = 3T(n/3 - 2) + n/2$

Sol^n → In the above function, it is safe to ignore the subtraction as, for very large values of n, the subtraction of -2 doesn't matter the asymptotics.

$\therefore T(n) = 3T(n/3) + n/2$

Here,

$a = 3, \quad b = 3, \quad f(n) = n/2$

$a \cdot f(n/b) = \frac{3 \cdot n/3}{2}$

$\therefore f = n/2$

$\Rightarrow c = 1$

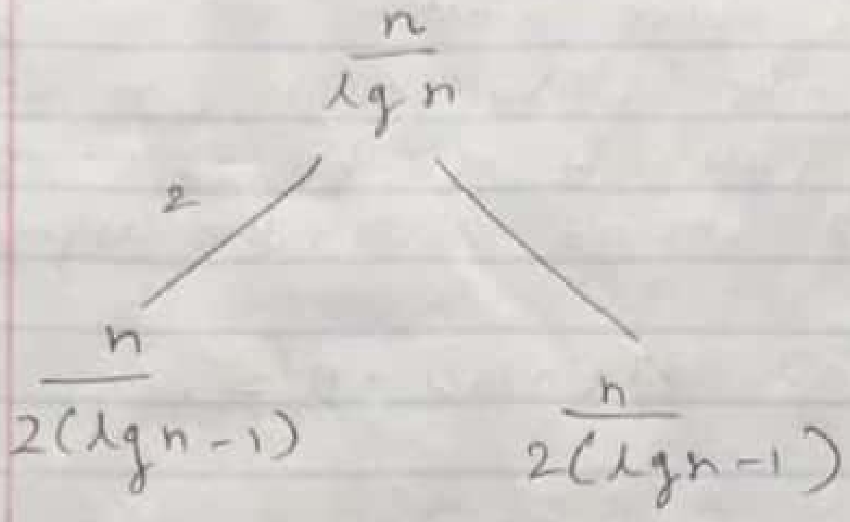
$\therefore T(n) = \theta(f(n) \cdot \log_b n)$

$= \theta(n/2 \cdot \log_3 n)$

$= \theta(n \lg n)$

$\therefore T(n) = \theta(n \lg n)$

e) $T(n) = 2T(n/2) + n/\lg n$.



∴ Master Theorem doesn't work in this example.

$$af(n/b) = \frac{n}{\lg n - 1} \neq f(n)$$

$$T(n) = 2 \cdot \left(2T(n/4) + \frac{n/2}{\lg n/2}\right) + \frac{n}{\lg n}$$

$$T(n) = 4T(n/4) + \frac{n}{\lg n - 1} + \frac{n}{\lg n}$$

$$\Rightarrow 2^i \, T\left(\frac{n}{2^i}\right) + \sum_{j=0}^{i-1} \frac{n}{\lg n - i}$$

e) continued

$$\Rightarrow n \cdot \Theta(1) + n \sum_{j=0}^{\lg n - 1} \frac{1}{\lg n - j}$$

$$\Rightarrow n \cdot \Theta(1) + n \sum_{j=0}^{\lg n - 1} \frac{1}{\lg n - j}$$

$$\Rightarrow \Theta(n) + \sum_{j=1}^{\lg n} \frac{n}{j} \longrightarrow H(g(n))$$

$$\Rightarrow \Theta(n) + n \cdot H_{\lg n}$$

harmonic number

$H_n \approx \log n + \gamma$

$$\Rightarrow \Theta(n \lg \lg n)$$

$\therefore$

$$T(n) = \Theta(n \lg \lg n)$$

Problem 2 -

Rank the following by order of growth.

→ Solution:-

Ranking:-

$O(1)$

$\vee$ → (efficiency is better)

$O(a \lg n)$

$\vee$

$O(an^b)$

$\vee$

$O(ab^n)$

Sol^n :- The final order for the functions in increasing growth is :-

$f_{23} < f_{12} < f_1 < f_{24} < f_{13} < f_{22}$

$< f_{17} < f_3 < f_9 < f_{21} = f_{19} < f_7$

$< f_{18} = f_{10} < f_4 < f_8 < f_2 = f_{20}$

$< f_{14} < f_{15} < f_6 < f_5 < f_{16}$

→ So, in terms of functions the ranking looks like

$1 < n^{1/\lg n} < \lg(\lg^* n) < \lg^*(\lg n)$

$< \ln \ln n < \sqrt{\lg n} < \ln n < n^{\log_6 5}$

$< \left(\frac{3}{4}\right)^n < n = 2^{\lg n} < \left(\frac{4}{3}\right)^n$

$< n \lg n = \lg(n!) < n^2 < n^2 + n$

$< n^{\lg \lg n} = (\lg n)^{\lg n} < 2^n < n \cdot 2^n$

$< 2^{2n} < n! < n^n < 2^{2^n}$.

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Q.2) contd →

working for the problem:-

1) 1 is the least as obvious.

2) $n^{1/\lg n}$ tends to 1 but less than it for higher value of n and is with a decreasing function.

3) $\lg(\lg^* n)$ is a series which not ends but goes on decreasing, its a decreasing graph.

4) $\lg^*(\lg n)$ is greater than $\lg(\lg^* n)$ as it ends unlike $\lg(\lg^* n)$.

5) $\ln(\ln(n))$ is definitely greater than both. and less than $\sqrt{\lg n}$.

6) $n^{\log_6 5}$ is greater than $\ln n$ and less than $\left(\frac{3}{4}\right)^n$

as $\ln n < (0.75)^n$.

→

7) $2^{\lg n} = 2^{\lg_2 n} = n$

$\therefore n = 2^{\lg n}$.

8) $(4/3)^n$ is greater than n hence also greater than $2^{\lg n}$.

9) $\lg(n!) = n$ times $\lg n$

$\therefore n \lg n = \lg(n!)$ which is greater than $\left(\frac{4}{3}\right)^n$.

10) $n^2 + n$ is obviously greater than n^2

11) $(\lg n)^{\lg n}$

$n^{\lg \lg n}$

$\therefore (\lg n)^{\lg n} = n^{\lg \lg n}$.

12) And its evident that

$2^n < n \cdot 2^n < 2^{2n} < n! < n^n < 2^{2^n}$.

$\therefore$ Finally the order looks like.

2^{2^n}

n^n

$n!$

2^{2n}

$n \cdot 2^n$

2^n

$n^{\lg \lg n}$ $\quad$ $(\lg n)^{\lg n}$

$n^2 + n$

n^2

$n \lg n$ $\quad$ $\lg(n!)$

$\left(\frac{4}{3}\right)^n$

$n = 2^{\lg n}$

$\left(\frac{3}{4}\right)^n$

$\cancel{n \log 6^6}$ $\quad$ $n^{\log_6 5}$

$\ln(n)$

$\sqrt{\lg(n)}$

$\ln(\ln(n))$

$\lg^*(\lg n)$

$\lg(\lg^* n)$

1 $\quad$ $n^{1/\lg n}$