Category Theory - Lecture 2 (Notes)

Vorashil Farzaliyev

September 2024

1 Examples of categories - Posets

Let $(P, \leq)$ be a poset.

- $a \leq a$ for all $a \in P$.
- $a \leq b$ and $b \leq c$ implies $a \leq c$.
- $a \leq b$ and $b \leq a$ implies $a = b$.

For example,

$$P = \boxed{\begin{tikzcd} & \text{a} & \\ \text{b} & & \text{c} \\ & \text{d} & \end{tikzcd}} \quad \text{where } a \geq b,\ a \geq c,\ b \geq c,\ c \geq d.$$

Definition 1.1 (Category of posets)**.** *Let $\underline{P}$ be a category.*

- $Ob(\underline{P}) = P$
- $\forall a, b \in P$

$$\underline{P}(a,b) = \begin{cases} \{*\} & \textit{if } a \leq b, \\ \emptyset & \textit{otherwise.} \end{cases}$$

with the following axioms satisfied:

- ***Associativity****:*

$$\underline{P}(b,c) \times \underline{P}(a,b) \to \underline{P}(a,c)$$

- ***Identity****:*

$$I_a \in \underline{P}(a,a) \quad \textit{by reflexivity}$$

We can see how multiplication is associative and identity is reflexive.

$$P = \boxed{\begin{tikzcd} & \text{a} & \\ \text{b} \arrow[ur] & & \text{c} \arrow[ul] \\ & \text{d} \arrow[ul] \arrow[ur] & \end{tikzcd}} \quad \text{where } a \geq b,\ a \geq c,\ b \geq c,\ c \geq d.$$

Note 1.1 (Size Issue)**.**

- *By Russel's paradox, there is no set of all sets. So we need to phrase the definition of category catefully.*
- *For us, a category has a class of objects, and for any two objects, a class of maps between them*

Definition 1.2 (Locally Small category)**.** *We say that category $\mathcal{C}$ is a **locally small** if for any two objects $A, B \in Ob(\mathcal{C})$, the class of maps $\mathcal{C}(A, B)$ is a set.*

Definition 1.3 (Small category)**.** *We say that category $\mathcal{C}$ is a **small** if it's locally small and the class of objects $Ob(\mathcal{C})$ is a set.*

Example 1.1. *Let's look at some examples of categories.*

- *Locally small but not small:* $\underline{Set}$, $\underline{Grp}$, $\underline{Top}$
- *Small:* $\Sigma(G), \underline{P}$

2 Isomorphisms

Note 2.1. *All isomorphisms in Maths is special case of isomorphisms in Category Theory.*

Definition 2.1 (Isomorphism)**.** *Let $\mathcal{C}$ be a category.*

*We say that $f : A \to B$ is an **isomorphism** in $\mathcal{C}$ if there exists a map $g : B \to A$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, bend right, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[dr, bend right, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

$$1_A = gf \qquad\qquad 1_B = fg$$

*We call g the **inverse** of f.*

2.1 Examples

- *In $\underline{\text{Set theory}}$: isomorphisms are bijections.*
- *In $\underline{\text{Group theory}}$: isomorphisms are group homomorphisms.*
- *In $\underline{\text{Topology}}$: isomorphisms are continuous functions.*
- *In $\Sigma(G)$:*

$$\begin{tikzcd} \ast \arrow[loop above, "h"] \arrow[loop left, "f"] \arrow[loop right, "g"] \end{tikzcd}$$

- *In $\underline{P}$:*

$$\begin{tikzcd} A \arrow[r, bend left, "\leq"] & B \arrow[l, bend left, "\leq"] \end{tikzcd} \Leftrightarrow a = b$$

Identities are always isomorphisms.

2.2 Uniquness of inverses

Proposition 2.1. *Let $f : A \to B$ be a map in a category $\mathcal{C}$. If the inverse of f exists, then it is unique.*

Proof. Let $g_1, g_2 : B \to A$ be inverses of f.

We claim that $g_1 = g_2$.

- g_1 :
$$g_1 f = 1_A \quad \text{and} \quad f g_1 = 1_B$$
- g_2 :
$$g_2 f = 1_A \quad \text{and} \quad f g_2 = 1_B$$

We can write

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "1_A"'] & B \arrow[d, "g_1 \text{ or } g_2"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "g_1 \text{ or } g_2"] \arrow[dr, bend right, "1_B"'] & A \arrow[d, "f"] \\
& B
\end{tikzcd}$$

That is

$$\begin{tikzcd}
& B \arrow[dr, "g_1"] & \\
B \arrow[ur, "1_B"] \arrow[dr, "g_2"'] \arrow[rr, bend left=80, "g_1"] \arrow[rr, bend right=80, "g_2"'] & & A \\
& A \arrow[uu, "f"] \arrow[ur, "1_A"'] &
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & A
\end{tikzcd}$$

Using this diagram, we can write

$$\begin{aligned}
g_1 &= g_1 \ 1_B \quad \text{by axiom of identity} \\
&= g_1 \ f \ g_2 \quad \text{since } f \ g_2 = 1_B \\
&= 1_A \ g_2 \quad \text{since } g_1 \ f = 1_A \\
&= g_2 \ 1_B \quad \text{by axiom of identity} \\
&= g_2
\end{aligned}$$

□

Note 2.2. *When $f : A \to B$ has an inverse, we can write $f^{-1} : B \to A$ for the inverse.*

2.3 Terminal and Initial objects

Definition 2.2 (Terminal object)**.** *Let $\mathcal{C}$ be a category.*

*An object T of $\mathcal{C}$ is a **terminal object** if for any object $A \in Ob(\mathcal{C})$, there exists a unique map $A \to T$.*

Definition 2.3 (Initial object)**.** *Let $\mathcal{C}$ be a category.*

*An object I of $\mathcal{C}$ is **initial** if for any object $A \in Ob(\mathcal{C})$, there exists a unique map $f : I \to A$.*

2.3.1 Examples

- *In $\underline{\mathit{Set}}$, the terminal object is* $\{*\}$.
- *In $\underline{\mathit{Grp}}$, the terminal object is* $\{*\}$.
- *In $\underline{\mathit{P}}$, the following proposition is true.*

Proposition 2.2. *Let $\mathcal{C}$ be a category.*

If T and T' are terminal objects of $\mathcal{C}$, then T and T' are isomorphic.

Proof. Let $f : T \to T'$ be the unique map from T to T'.

$$
\begin{tikzcd}
T \arrow[r] \arrow[dr, bend right, "1_T"] & T' \arrow[d] \\
 & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T' \arrow[r] \arrow[dr, bend right, "1_{T'}"] & T \arrow[d] \\
 & T'
\end{tikzcd}
$$

□