Category Theory - Lecture 5 (Notes)

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Exercise 0.1. *Let $\mathcal{C}$ be a category.*
Let $f, g : A \longrightarrow B$ and let $u : A' \longrightarrow A$ be an isomorphism. Then,

$$f = g \Leftrightarrow f \circ u = g \circ u$$

Proof. ($\Rightarrow$): It is clear.
($\Leftarrow$): If $u : A' \longrightarrow A$ is an isomorphism, then there exists $u^{-1} : A \longrightarrow A'$ such that

$$\begin{aligned} fu = gu &\Rightarrow fuu^{-1} = guu^{-1} \\ &\Rightarrow f = g \end{aligned}$$

□

1 Natural Transformations

Definition 1.1. *Let $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ be functors.*
*A **natural transformation** $\phi : F \Rightarrow G$ is a family of maps:*

$$\{\phi_A : FA \longrightarrow GA \mid A \in Ob(\mathbf{C})\}$$

such that for all $f : A \longrightarrow B$ in $\mathbf{C}$, we have

$$\begin{tikzcd}
\mathbf{C} & & \mathbf{D} & \\
A \arrow[d, mapsto, "f"] & FA \arrow[rr, "\phi_A"] \arrow[d, "f"] & & GA \arrow[d, "Gf"] \\
B & FB \arrow[rr, "\phi_B"] & & GB
\end{tikzcd}$$

Example 1.1. *Fix $n \in \mathbb{N}$*

- $\mathbf{C} = \underline{CRing} =$ *category of commutative rings*
- $\mathbf{D} =$ *category of monoids*
- $F : \underline{CRing} \longrightarrow \underline{Mon}$ *is the functor*

$$F = M_n : \underline{CRing} \longrightarrow \underline{Mon}$$

Here $M_n(R)$ is $n \times n$ matrix with entries in R. So $\underline{Mon}$ is the category of monoids with $n \times n$ matrices as objects.

$$\begin{tikzcd} R \arrow[r, mapsto, "F"] \arrow[d, "f"] & M_n(R) \arrow[d, "M_n(f)"] \\ S \arrow[r, "F"] & M_n(S) \end{tikzcd}$$

- $G : \underline{CRing} \longrightarrow \underline{Mon}$ *is the functor*

$$G = U : \underline{CRing} \longrightarrow \underline{Mon}$$

Here we have

$$\begin{tikzcd} (R, +, *, 0, 1) \arrow[r, mapsto, "U"] \arrow[d, "f"] & (R, +, 1) \arrow[d, "U(f)"] \\ (S, +, *, 0, 1) \arrow[r, "U"] & (S, *, 1) \end{tikzcd}$$

So U simply returns the underlying monoid of a given ring.

Now we can define a natural transformation

$$det : M_n \Longrightarrow U$$

For $R = (R, +, *, 0, 1) \in \underline{CRing}$, we need

$$det_R : FR \longrightarrow GR$$
$$det_R : M_n(R) \longrightarrow U(R)$$
$$det_R : M_n(R) \longrightarrow (R, *, 1)$$

Here det_R is a monoid homomorphism

This is monoid homomorphism because

$$\begin{cases} det_R(M * N) = det_R(M) * det_R(N) \text{ (it preserves multiplication)} \\ \\ det_R(I_n) = 1_R \text{ (it preserves identity)} \end{cases}$$

*So now need to check that this is a natural transformation. (i.e we need to check its **naturality**).*

$$\begin{tikzcd} \underline{\textbf{Cring}} & \underline{\textbf{Mon}} & \\ R \arrow[d, mapsto, "f"] & M_n(R) \arrow[r, "det_R"] \arrow[d, "M_n(f)"] & R \arrow[d, "f"] \\ S & M_n(S) \arrow[r, "det_S"] & S \end{tikzcd}$$

$$\begin{tikzcd}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[r, mapsto, "det_R"] \arrow[d, "M_n(f)"] & ad-bc \arrow[d, "f"] \\
\begin{pmatrix} fa & fb \\ fc & fd \end{pmatrix} \arrow[r, "U"] & f(ad-bc)
\end{tikzcd}$$

1.1 Notation

1.1.1 Natural transformation diagram

For $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ and $\phi : F \Rightarrow G$ as natural transformation, we write

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbf{D}
\end{tikzcd}$$

1.1.2 Composition diagram

Given $F, G, H : \mathbf{C} \longrightarrow \mathbf{D}$ and $\phi, \psi : F \Rightarrow G$ as natural transformations, we write

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left=70, "F"] \arrow[r, "G" description] \arrow[r, bend right=70, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\psi"] & \mathbf{D}
\end{tikzcd}$$

We define

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\psi\phi"] & \mathbf{D}
\end{tikzcd}$$

Also

$$FA \xrightarrow{(\psi\phi)_A} HA \ =_{def} \ FA \xrightarrow{\phi_A} GA \xrightarrow{\psi_A} HA$$

So we can define composition

$$\begin{tikzcd}
\mathbf{C} & & \mathbf{D} & \\
A \arrow[d, mapsto, "f"] & FA \arrow[r, "\phi_A"] \arrow[d, "f"] & GA \arrow[r, "h_A"] \arrow[d, "Gf"] & HA \arrow[d, "Hf"] \\
B & FB \arrow[r, "\phi_B"] & GB \arrow[r, "h_B"] & HB
\end{tikzcd}$$

Naturality of ϕ Naturality of ψ

1.2 Identity natural transformation

For $F : \mathbf{C} \longrightarrow \mathbf{D}$, we can define

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F"'{name=L}] & \mathbf{D} \arrow[from=U, to=L, Rightarrow, "1_F"] \end{tikzcd}$$

by $FA \xrightarrow{1_{FA}} FA$ for all $A \in \mathbf{C}$.

Naturality of this is

$$\begin{tikzcd} \mathbf{C} & & \mathbf{D} & \\ A \arrow[d, mapsto, "f"] & FA \arrow[rr, "1_{FA}"] \arrow[d, "Ff"] & & FA \arrow[d, "Ff"] \\ B & FB \arrow[rr, "1_{FA}"] & & FB \end{tikzcd}$$

1.3 Functor category

Definition 1.2. *(Functor category)*

For categories $\mathbf{C}, \mathbf{D}$, where $\mathbf{C}$ is a small category, we define the ***functor category*** $[\mathbf{C}, \mathbf{D}]$ *(sometimes denoted as $\mathbf{D}^{\mathbf{C}}$) as having*

- ***objects***: *functors $F : \mathbf{C} \longrightarrow \mathbf{D}$, $G : \mathbf{C} \longrightarrow \mathbf{D}$*
- ***maps***: *natural transformations $\phi : F \Rightarrow G$*

Exercise 1.1. *Check the associativity and unit axioms for functor category.*

1.3.1 Isomorphisms in $\mathbf{D}^{\mathbf{C}}$ or $[\mathbf{C}, \mathbf{D}]$

Fix $\mathbf{C}, \mathbf{D}$.

By definition, an isomorphism in $[\mathbf{C}, \mathbf{D}]$ is $\phi : F \Rightarrow G$ with an inverse

$$\phi^{-1} : G \Rightarrow F$$

This means that

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\phi"] \arrow[dr, Rightarrow, bend right, "1_F"'] & G \arrow[d, Rightarrow, "\phi^{-1}"] & G \arrow[r, Rightarrow, "\phi^{-1}"] \arrow[dr, Rightarrow, bend right, "1_G"'] & F \arrow[d, Rightarrow, "\phi"] \\
 & F & & G
\end{tikzcd}$$

This holds if and only if, for all $A \in \mathbf{C}$, we have

$$\begin{tikzcd}
FA \arrow[r, "\phi_A"] \arrow[dr, bend right, "1_{FA}"'] & GA \arrow[d, "\phi_A^{-1}"] & GA \arrow[r, "\phi_A^{-1}"] \arrow[dr, bend right, "1_{GA}"'] & FA \arrow[d, "\phi_A"] \\
 & FA & & GA
\end{tikzcd}$$

This implies $\forall A \in \mathbf{C}$, $\phi_A : FA \longrightarrow GA$ is an isomorphism.

Proposition 1.1. *Let $\phi : F \Rightarrow G$ be a natural transformation.*

Then ϕ is an isomorphism if and only if for all $A \in \mathbf{C}$

$$\phi_A : FA \longrightarrow GA$$

is an isomorphism.

Note 1.1. *We need ϕ to be natural in order to piece together all isomorphisms $FA \longrightarrow GA$ for all $A \in \mathbf{C}$*