# Category Theory - Lecture 4 (Notes)

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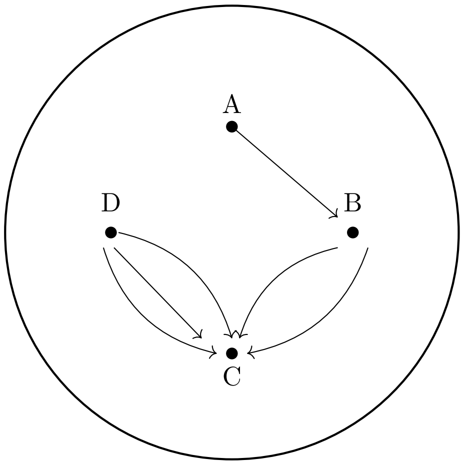
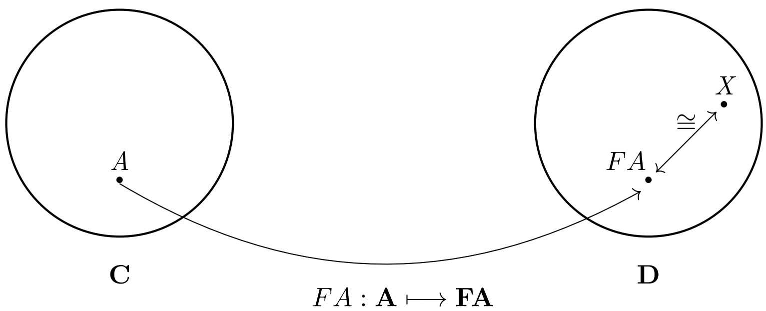
## 1 Surjective functors

Write $A \cong B$ to mean that $A$ and $B$ are isomorphic,
i.e., $f : A \longrightarrow B$ is an isomorphism.




**Definition 1.1.** *(Surjective functor) Let* $F$*:* $\mathbf{C} \longrightarrow \mathbf{D}$ *be a functor.*

*We say that* $F$ *is* ***essentially surjective*** *if for all* $X \in \mathbf{D}$*, there exists a* $A$ *such that* $F(A) \cong X$*.*




$FA : \mathbf{A} \longmapsto \mathbf{FA}$

**Example 1.1.** *Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor.*

- *For* $\mathbf{C}$ =

$$\mathbf{C} = \text{ category of } \begin{cases} [n] = \{1, 2, \dots, n\} \text{ for } n \in \mathbb{N}\} \\ \\ [n] \xrightarrow{f} [m] \end{cases}$$

- *For* $\mathbf{D}$ = <u>*FinSet*</u>

$$\underline{FinSet} = \text{ category of } \begin{cases} \text{finite sets} \\ \\ \text{functions} \end{cases}$$

$$\begin{tikzcd} {[n]} \arrow[r, "F"] \arrow[d, "f"] & \{1, 2, \dots, n\} \arrow[d, "Ff=f"] \\ {[m]} \arrow[r, "F"] & \{1, 2, \dots, m\} \end{tikzcd}$$

*For example,* $\{1, 3\} \in \underline{FinSet}$ *is not in the (essential) image of* $F$. *However, we have* $[2] \in \mathbf{C}$ *and*

$$F([2]) \cong \{1, 3\} \in \underline{FinSet}$$

# 2 Full and Faithful Functors

**Definition 2.1.** *(Faithful functor)*

*Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor. We say that $F$ is* ***faithful*** *if $\forall A, B \in \mathbf{C}$ the function*

$$\mathbf{C}(A, B) \longrightarrow \mathbf{D}(F(A), F(B))$$

*is injective.*

**Definition 2.2.** *(Full functor)*

*Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor. We say that $F$ is* ***full*** *if $\forall A, B \in \mathbf{C}$ the function*

$$\mathbf{C}(A, B) \longrightarrow \mathbf{D}(F(A), F(B))$$

*is surjective.*

**Note 2.1.** *(What does it mean for a functor to be full?)*

*For all $A, B \in \mathbf{C}$, we have*

$$\mathbf{C}(A, B) \longrightarrow \mathbf{D}(F(A), F(B)) \text{ is surjective}$$

*means $\forall u : FA \longrightarrow FB$ in $\mathbf{D}$ there exists $f : A \longrightarrow B$ in $\mathbf{C}$ such that*

$$Ff = u$$

**Note 2.2.** *(What does it mean for a functor to be faithful?)*

*For all $A, B \in \mathbf{C}$, we have*

$$\mathbf{C}(A, B) \longrightarrow \mathbf{D}(F(A), F(B)) \text{ is injective}$$

*means $\forall f_1, f_2 : A \longrightarrow B$ in $\mathbf{C}$*

$$F(f_1) = F(f_2) \Rightarrow f_1 = f_2$$

## 2.1 Examples

**Example 2.1.** *($F : \underline{Grp} \longrightarrow \underline{Set}$)*

$$\underline{Grp} \longrightarrow \underline{Set}$$

$$\begin{tikzcd}
(G,\circ) \arrow[r, "F"] \arrow[d, "f"] & G = F(G,\circ) \arrow[d, "Ff=f"] \\
(H,*) \arrow[r, "F"] & H = F(H,*)
\end{tikzcd}$$

*So we have*

$$\underline{Grp}((G,\circ),(H,*)) \longmapsto \underline{Set}(G,H)$$
$$f \longmapsto f$$

*not full, but faithful.*

- ***not full**: Because a map in $\underline{Grp}$ is a group homomorphism, but not every map in $\underline{Set}$ corresponds to a group homomorphism. So we can fund $u : G \longrightarrow H$ in $\underline{Set}$ such that there is no $f : (G,\circ) \longrightarrow (H,*)$ in $\underline{Grp}$ such that $Ff = u$.*
- ***faithful**: since it injectively maps every map $f : G \longrightarrow H$ in $\underline{Grp}$ to a map $Ff : GF \longrightarrow FH$ in $\underline{Set}$.*

**Example 2.2.** *($F : \underline{Ab} \longrightarrow \underline{Grp}$)*

$$\underline{Ab} \longrightarrow \underline{Grp}$$

$$\begin{tikzcd}
(G,\circ) \arrow[r, "F"] \arrow[d, "f"] & (G,\circ) = F(G,\circ) \arrow[d, "Ff=f"] \\
(H,*) \arrow[r, "F"] & (H,*) = F(H,*)
\end{tikzcd}$$

*So we have*

$$\underline{Grp}((G,\circ),(H,*)) \longmapsto \underline{Set}(G,H)$$
$$f \longmapsto f$$

*full and faithful.*

- ***full**: every group homomorphism between $F(G,\circ)$ and $F(H,*)$ comes from an abelian group homomorphism between $(G,\circ)$ and $(H,*)$.*
- ***faithful**: $F$ is faithful because it injectively maps abelian group homomorphisms to group homomorphisms*

**Exercise 2.1.** *Find a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ that is full but not faithful.*

- *think small categories*
- $\underline{P} \longrightarrow \underline{Q}$

# 3 Subcategories

**Definition 3.1.** *Let* $\mathbf{C}$ *be a category.*

*A* ***subcategory*** $\mathbf{D}$ *of* $\mathbf{C}$ *consists of*

- *a subclass*
$$Ob(\mathbf{D}) \subseteq Ob(\mathbf{C})$$
- $\forall A, B \in Ob(\mathbf{D}) \subseteq Ob(\mathbf{C})$
$$\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B)$$

*which is closed under composition and identities:*

- $\forall A, B, C \in Ob(\mathbf{D})$
$$f \in \mathbf{D}(A, B) \text{ and } g \in \mathbf{D}(B, C) \Rightarrow g \circ f \in \mathbf{D}(A, C)$$
- $\forall A \in Ob(\mathbf{D})$
$$1_A \in \mathbf{D}(A, A)$$

**Definition 3.2.** *Let* $\mathbf{C}$ *be a category and* $\mathbf{D}$ *be a subcategory of* $\mathbf{C}$.

*We say that* $\mathbf{D}$ *is* ***full*** *if*
$$\mathbf{D}(A, B) = \mathbf{C}(A, B)$$
*for all* $A, B \in Ob(\mathbf{D})$.

**Example 3.1.** *(Subcategories of* $\underline{Set}$*)*

- $\underline{Finset} \subseteq \underline{Set}$ *is full subcategory of* $\underline{Set}$ *with objects as finite sets*
- $\underline{Finset}_{i,j} \subseteq \underline{Set}$*, the subcategory of* $\underline{Set}$ *with objects finite sets and bijections between them*

**Exercise 3.1.** *Let* $G$ *be a group. Describe explicitly in terms of* $G$*, what is the subcategory of* $\Sigma(G)$.

**Exercise 3.2.** *Let* $\underline{P}$ *be a poset. Describe explicitly in terms of* $\underline{P}$*, what is the subcategory of* $\underline{P}$.