Category Theory - Lecture 1 (Notes)

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September 2024

1 Introduction

Let's start looking into some examples.

1.1 One element set

Let $1 = \{*\}$ be a one-element set. Then, for every set A there exists a unique function:

$$A \to 1$$

$$a \mapsto *$$

1.2 Ring of integers

Let's remind ourselves of the definition of ring.

1.2.1 Some useful definitions

Definition 1 (Ring) *A ring is a set R with two binary operations $+$ and $\times$ and an element 0 such that:*

- *$(R, +)$ is an abelian group*
- *$(R, \times)$ is a monoid*
- *Multiplication is distributive over addition*

Definition 2 (Monoid) *A monoid is a set M with a binary operation $\times$ and an identity element 1 such that:*

- *$\times$ is associative*
- *$1 \times a = a \times 1 = a$ for all $a \in M$*

Definition 3 (Ring homomorphism) *Let R, S be rings. A ring homomorphism is a function $\phi : R \to S$ such that:*

1. *$\phi(0_R) = 0_S$*
2. *$\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$*
3. *$\phi(a \times b) = \phi(a) \times \phi(b)$ for all $a, b \in R$*

*We can also say that ϕ is a **group homomorphism** because of 1 and 2.*

1.2.2 Z as a ring

So let $\mathbb{Z}$ be the ring of integers. For every ring R there exists a unique ring homomorphism:

$$\mathbb{Z} \to R$$

2 What is a category?

Definition 4 (Category) *A category $\mathcal{C}$ consists of:*

- *a collection of objects $Ob(\mathcal{C})$*
- *for each pair of objects $A, B \in Ob(\mathcal{C})$, a collection of morphisms $\mathcal{C}(A, B)$ from A to B*
- *for all $A, B, C \in Ob(\mathcal{C})$, a function $\circ$*

$$\mathcal{C}(B,C) \times \mathcal{C}(A,B) \xrightarrow{\circ} \mathcal{C}(A,C)$$
$$(g, f) \mapsto g \circ f$$

- *for each object $A \in Ob(\mathcal{C})$, an identity morphism $1_A \in \mathcal{C}(A, A)$*

subject to the following axioms:

- ***Associativity**: for all $A, B, C, D \in Ob(\mathcal{C})$, and all $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(C, D)$,*

$$\underbrace{h \circ \underbrace{(g \circ f)}_{\in \mathcal{C}(A,C)}}_{\in \mathcal{C}(A,D)} = \underbrace{\underbrace{(h \circ g)}_{\in \mathcal{C}(B,D)} \circ f}_{\in \mathcal{C}(A,D)}$$

- ***Unit**: for all $A, B \in Ob(\mathcal{C})$ and for all $f \in \mathcal{C}(A, B)$,*

$$\underbrace{f \circ 1_A = f}_{\mathcal{C}(A,B) \times \mathcal{C}(A,A) \xrightarrow{\circ} \mathcal{C}(A,B)}$$
$$\underbrace{1_B \circ f = f}_{\mathcal{C}(B,B) \times \mathcal{C}(A,B) \xrightarrow{\circ} \mathcal{C}(A,B)}$$

Note 1 (Terminology / Notation) *Fix a category $\mathcal{C}$.*

- *We call all elements of $Ob(\mathcal{C})$ **objects** of $\mathcal{C}$.*
- *for all $A, B \in Ob(\mathcal{C})$, we call $\mathcal{C}(A, B)$ the **maps** (or morphisms, or arrows) from A to B.*
- *for all $f : A \to B$, $g : B \to C$, we call $g \circ f$ the **composition** of f and g.*
- *for all $A \in Ob(\mathcal{C})$, we call 1_A the **identity** on A.*
- *we often write $A \in \mathcal{C}$ to mean $A \in Ob(\mathcal{C})$.*

2.1 Some examples of categories

2.1.1 Set

- $\mathcal{C} = \mathrm{Set}$
- $\mathrm{Ob}(\mathcal{C}) =$ sets A, $B \ldots$
- $\mathcal{C}(A, B) = \mathbf{Set}(A, B) =$ functions from A to B
- Composition

$$\underbrace{A \xrightarrow{f} B \xrightarrow{g} C}_{g \circ f}$$
$$a \mapsto (g \circ f)(a) = g(f(a))$$

- Identity: $1_A : A \to A$ is the identity function with $a \mapsto a$.

2.1.2 Grp

- $\mathcal{C} = \text{Grp}$
- $\text{Ob}(\mathcal{C})$ = groups G, $H \ldots$
- $\mathcal{C}(G,H) = \mathbf{Grp}(G,H)$ = group homomorphisms from G to H
- Composition: composition of group homomorphisms

$$\underbrace{A \xrightarrow{f} B \xrightarrow{g} C}_{g \circ f}$$
$$a \mapsto (g \circ f)(a) = g(f(a))$$

- Identity: $1_G : G \to G$, $g \mapsto g$

2.1.3 Top

- $\mathcal{C} = \text{Top}$ (topological spaces)
- $\text{Ob}(\mathcal{C})$ = topological spaces X, $Y \ldots$
- $\mathcal{C}(X,Y) = \mathbf{Top}(X,Y)$ = continuous functions from X to Y
- Composition: **??????**
- Identity: $1_X : X \to X$, $g \mapsto g$

2.2 Commutative diagrams

Fix a category $\mathcal{C}$.

1. $$A_1 \xrightarrow{f1} A_2 \xrightarrow{f2} A_3 \to \cdots \to A_n \xrightarrow{f_n} A_{n+1}$$

   There is usually a unique map
   $$A_1 \xrightarrow{f_n \circ \cdots \circ f_2 \circ f_1} A_{n+1}$$

2. Pictures lik this are

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"] & & B \arrow[d, "g"] \\ D \arrow[r, "j"] & E \arrow[r, "k"] & C \end{tikzcd}$$

are commutative if $g \circ f = k \circ j \circ h$.

2.3 Other examples of unusual categories

One thing to note is that we can have a category where objects are not sets and morphisms are not functions.

Let $(G, \cdot, 1)$ be a group. Define a category $\Sigma(G)$ as follows:

- $\text{Ob}(\Sigma(G)) = \{*\}$
- $\Sigma(G)(*,*) = G$ (elements of G are maps in $\Sigma(G)$)
- Composition:
$$\Sigma(G)(*,*) \times \Sigma(G)(*,*) \to G$$

- Identity: $1_* : G \to G$, $g \mapsto g$, where $1_* \in \Sigma(G)(*,*)$.
- $1 \in G$