Category Theory - Lecture 3 (Notes)

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1 Functors

Definition 1.1 (Functor). Let C and D be categories.

A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of:

• A function

$$F: Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$$
$$A \longmapsto F(A)$$

• $\forall A, B \in Ob(\mathcal{C})$

$$F: \mathcal{C}(A,B) \longrightarrow \mathcal{D}(F(A),F(B))$$

 $f \longmapsto F(f)$

with the following "functoriality" axioms satisfied:

• Associativity: for all $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in C,

$$F(g \mathrel{\circ} f) = F(g) \mathrel{\circ} F(f)$$

$$composition \ in \ \mathcal{D}$$

• *Identity* for all $A \in Ob(\mathcal{C})$,

$$F(1_A) = 1_{F(A)}$$

Observation 1.1. Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

ullet The action of F on any string of composable maps is well defined. i.e

for all
$$\underbrace{A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{} \dots \xrightarrow{} A_n \xrightarrow{f_n} A_{n+1}}_{f_n \xrightarrow{f_2} f_1 A_1 \xrightarrow{} A_{n+1}}$$
 in C

we have a unique map

$$\underbrace{FA_1 \xrightarrow{F(f_1)} FA_2 \xrightarrow{F(f_2)} F(A_3) \longrightarrow \ldots \longrightarrow FA_n \xrightarrow{F(f_n)} FA_{n+1}}_{F(f_n \dots f_2 \dots f_1): FA_1 \longrightarrow FA_{n+1}} \text{ in } \mathcal{D}$$

• Since the composition of maps is well defined, we can say that F preserves the commutative diagrams.

$$A \xrightarrow{f} B \qquad FA \xrightarrow{Ff} FB$$

$$\downarrow h \qquad \downarrow g \qquad \Rightarrow \qquad \downarrow_{Fh} \qquad \downarrow_{Fg}$$

$$C \xrightarrow{k} D \qquad FC \xrightarrow{Fk} FD$$

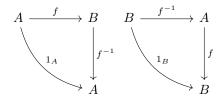
$$commutes in C \qquad commutes in D$$

$$gf = kh \qquad FgFh = FkFh$$

Remark 1.1. Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

If $f: A \longrightarrow B$ is an isomorphism in C, then $F(f): F(A) \longrightarrow F(B)$ is an isomorphism in D.

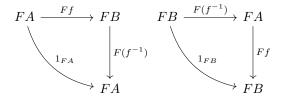
Proof. If $f: A \longrightarrow B$ is an isomorphism in \mathcal{C} , then we have $f^{-1}: B \longrightarrow A$ in \mathcal{C} s.t



We want to show that F(f) is an isomorphism in \mathcal{D} . In other words, we want to show that

$$(Ff)^{-1}: F(B) \longrightarrow F(A)$$

is given by $F(f^{-1}): FB \longrightarrow FA$. To check this, we need to show that following diagram commutes:



This is equivalent to showing that

$$F(f^{-1})F(f) = F(f^{-1}f)$$
 (according to functoriality axioms)
= $F(1_A)$
= 1_{FA}

$$F(f)F(f^{-1}) = F(ff^{-1})$$
 (according to functoriality axioms)
= $F(1_B)$
= 1_{FB}

Example 1.1. (Forgetful functor) Functor $U: \mathcal{C} \longrightarrow \mathcal{D}$ that forgets the group structure.

1. $U: \underline{Grp} \longrightarrow \underline{Set}$ forgets the group structure on the objects and maps the group operation to the set operation.

$$U: \underline{Grp} \longrightarrow \underline{Set}$$

$$(G, *) \longmapsto G$$

2. $U: \underline{Grp} \longrightarrow \underline{Mon}$ forgets the group structure on the objects and maps the group operation to the monoid operation.

$$(G,*,1) \xrightarrow{U} G = U(G,*,1)$$

$$\downarrow^f \qquad \qquad \downarrow^{f=U(f)}$$

$$(H,*,1) \xrightarrow{U} H = U(H,*,1)$$

3. $U : \underline{Ring} \longrightarrow \underline{Grp}$

$$(R, +, 0, *, 1) \xrightarrow{U} (R, +, 0)$$

$$\downarrow^{f} \qquad \qquad \downarrow^{f=U(f)}$$

$$(S, +, 0, *, 1) \xrightarrow{U} (S, +, 0)$$

4. $U: \underline{Ab\ Grp} \longrightarrow \underline{Grp}$ forgets the abelian group property.

Example 1.2. (Free Functors)

Let's start reminding ourselves of the definition of free group on S

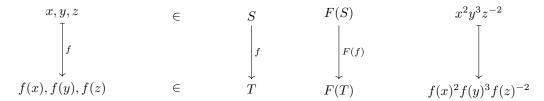
- **Elements:** words like $x^2y^3z^{-2}$, where $x, y, z \in S$ subject to group axioms.
- Operation: concatenation of words, subject to group axioms.
- Identity: empty word.

So we can define a free functor $F: \underline{Set} \longrightarrow Grp$ as follows:

$$F: \underbrace{Set} \longrightarrow \underbrace{Grp} \\ S \longmapsto F(S)$$

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Here F(S) is the free group on S.



Example 1.3. (Topological Space with a base point)

Let X be a topological space with a base point $x \in X$.

We can define a functor $\Pi_1: \underline{Top_*} \longrightarrow \underline{Grp}$ that maps a topological space with a base point (i.e Top_*) to its fundamental group:

$$Top_X: \underline{Grp} \xrightarrow{\Pi_1} \underline{Set}$$

$$(X, O(x), x) \longmapsto \Pi_1(X)$$

Here X is the fundamental group of X.

Exercise 1.1. 1. Let G, H be groups. Define following functor:

$$F: \Sigma(G) \longrightarrow \Sigma(H)$$

2. Let P,Q be posets. Define following functor:

$$F:\underline{P}\ \longrightarrow\ Q$$

Proposition 1.1. Small categories and functors form a category <u>Cat</u>.

Proof. Formally,

- **objects**: $Ob(\underline{Cat}) = \text{small categories}$
- maps: for small categories $C, D \in Ob(\underline{Cat})$,

$$\underline{Cat}(\mathcal{C}, \mathcal{D}) = \{ F | F : \mathcal{C} \longrightarrow \mathcal{D} \}$$

• composition:

$$\underline{Cat}(\mathcal{D},\mathcal{E})\times\underline{Cat}(\mathcal{C},\mathcal{D})\overset{\circ}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-}\underline{Cat}(\mathcal{C},\mathcal{E})$$

$$(G\times F)\longmapsto GF$$
 composition of functors G and F

$$A \longrightarrow G(F(A))$$

$$\downarrow^f \qquad \qquad \downarrow^{G(F(f))}$$

$$B \longrightarrow G(F(B))$$

To conclude the proof we need to show

Exercise 1.2. Show that the following axioms are satisfied to conclude the proof:

- associativity: let F, G, H be functors. Show H(GF) = GH(F).
- identity: let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Show $F = F1_{\mathcal{C}} = 1_{\mathcal{D}}F$.