

Category Theory - Lecture 2 (Notes)

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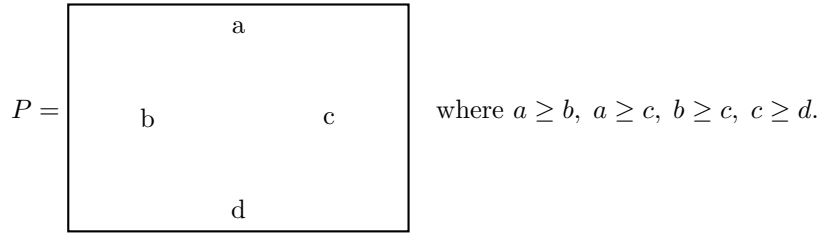
September 2024

1 Examples of categories - Posets

Let (P, \leq) be a poset.

- $a \leq a$ for all $a \in P$.
- $a \leq b$ and $b \leq c$ implies $a \leq c$.
- $a \leq b$ and $b \leq a$ implies $a = b$.

For example,



Definition 1.1 (Category of posets). Let \underline{P} be a category.

- $Ob(\underline{P}) = P$
- $\forall a, b \in P$

$$\underline{P}(a, b) = \begin{cases} \{*\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$$

with the following axioms satisfied:

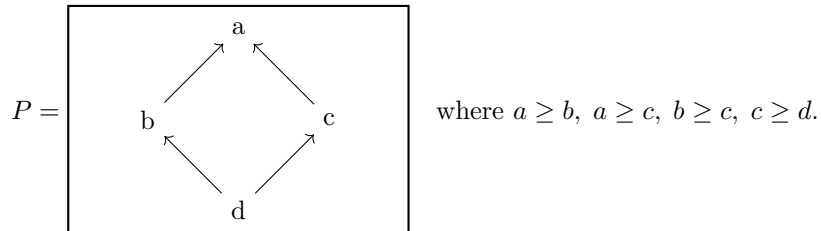
- **Associativity:**

$$\underline{P}(b, c) \times \underline{P}(a, b) \rightarrow \underline{P}(a, c)$$

- **Identity:**

$$I_a \in \underline{P}(a, a) \text{ by reflexivity}$$

We can see how multiplication is associative and identity is reflexive.



Note 1.1 (Size Issue). • By Russel's paradox, there is no set of all sets. So we need to phrase the definition of category carefully.

- For us, a category has a class of objects, and for any two objects, a class of maps between them

Definition 1.2 (Locally Small category). We say that category \mathcal{C} is a **locally small** if for any two objects $A, B \in \text{Ob}(\mathcal{C})$, the class of maps $\mathcal{C}(A, B)$ is a set.

Definition 1.3 (Small category). We say that category \mathcal{C} is a **small** if it's locally small and the class of objects $\text{Ob}(\mathcal{C})$ is a set.

Example 1.1. Let's look at some examples of categories.

- Locally small but not small: Set, Grp, Top
- Small: $\Sigma(G)$, P

2 Isomorphisms

Note 2.1. All isomorphisms in Maths is special case of isomorphisms in Category Theory.

Definition 2.1 (Isomorphism). Let \mathcal{C} be a category.

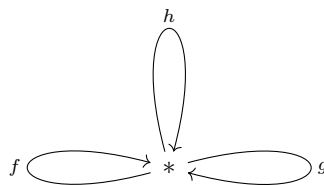
We say that $f : A \rightarrow B$ is an **isomorphism** in \mathcal{C} if there exists a map $g : B \rightarrow A$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow 1_A & \downarrow g \\ & & A \\ & 1_A = gf & \end{array} \quad \begin{array}{ccc} B & \xrightarrow{g} & A \\ & \searrow 1_B & \downarrow f \\ & & B \\ & 1_B = fg & \end{array}$$

We call g the **inverse** of f .

2.1 Examples

- In Set theory: isomorphisms are bijections.
- In Group theory: isomorphisms are group homomorphisms.
- In Topology: isomorphisms are continuous functions.
- In $\Sigma(G)$:



- In P:

$$\begin{array}{ccc} & \leq & \\ A & \xrightarrow{\quad} & B \\ & \leq & \end{array} \Leftrightarrow a = b$$

Identities are always isomorphisms.

2.2 Uniqueness of inverses

Proposition 2.1. *Let $f : A \rightarrow B$ be a map in a category \mathcal{C} . If the inverse of f exists, then it is unique.*

Proof. Let $g_1, g_2 : B \rightarrow A$ be inverses of f .

We claim that $g_1 = g_2$.

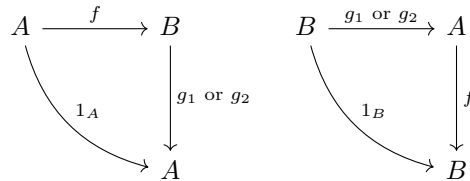
- $g_1 :$

$$g_1 f = 1_A \quad \text{and} \quad f g_1 = 1_B$$

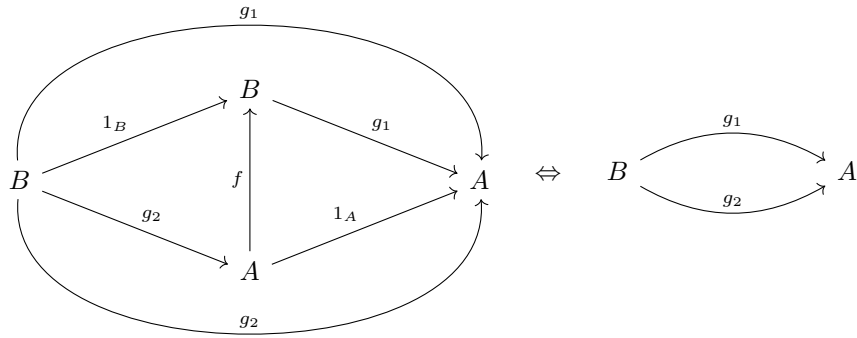
- $g_2 :$

$$g_2 f = 1_A \quad \text{and} \quad f g_2 = 1_B$$

We can write



That is



Using this diagram, we can write

$$\begin{aligned} g_1 &= g_1 1_B \quad \text{by axiom of identity} \\ &= g_1 f g_2 \quad \text{since } f g_2 = 1_B \\ &= 1_A g_2 \quad \text{since } g_1 f = 1_A \\ &= g_2 1_B \quad \text{by axiom of identity} \\ &= g_2 \end{aligned}$$

□

Note 2.2. *When $f : A \rightarrow B$ has an inverse, we can write $f^{-1} : B \rightarrow A$ for the inverse.*

2.3 Terminal and Initial objects

Definition 2.2 (Terminal object). *Let \mathcal{C} be a category.*

*An object T of \mathcal{C} is a **terminal object** if for any object $A \in \text{Ob}(\mathcal{C})$, there exists a unique map $A \rightarrow T$.*

Definition 2.3 (Initial object). *Let \mathcal{C} be a category.*

*An object I of \mathcal{C} is **initial** if for any object $A \in \text{Ob}(\mathcal{C})$, there exists a unique map $f : I \rightarrow A$.*

2.3.1 Examples

- In \underline{Set} , the terminal object is $\{*\}$.
- In \underline{Grp} , the terminal object is $\{*\}$.
- In \underline{P} , the following proposition is true.

Proposition 2.2. *Let \mathcal{C} be a category.*

If T and T' are terminal objects of \mathcal{C} , then T and T' are isomorphic.

Proof. Let $f : T \rightarrow T'$ be the unique map from T to T' .

$$\begin{array}{ccc} T & \xrightarrow{\quad} & T' \\ & \searrow 1_T & \downarrow \\ & & T \end{array} \qquad \begin{array}{ccc} T' & \xrightarrow{\quad} & T \\ & \searrow 1_{T'} & \downarrow \\ & & T' \end{array}$$

□