

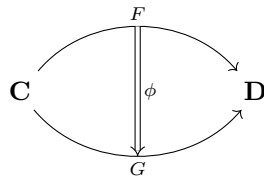
# Category Theory - Lecture 6 (Notes)

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From last time...

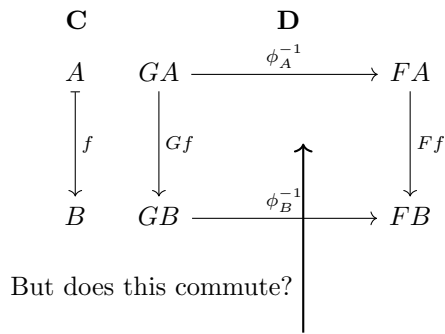
**Proposition 0.1.** *Let  $\phi : F \Rightarrow G$  be a natural transformation.*



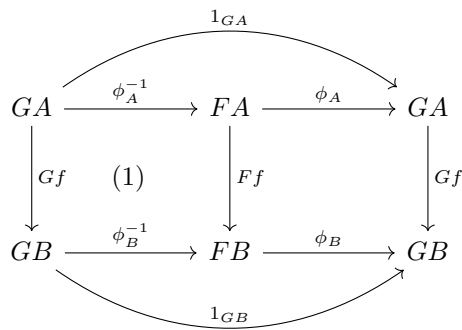
Assume that for all  $A \in \mathbf{C}$ ,  $\phi_A : FA \longrightarrow GA$  is an isomorphism with inverse  $\phi_A^{-1} : GA \longrightarrow FA$ .  
Then  $\phi^{-1} : G \Rightarrow F$  is defined by

$$(\phi^{-1})_A = (\phi_A)^{-1} \text{ for all } A \in \mathbf{C}$$

*Proof.* We show that the naturality, as a family of maps, is given by



We have



We show that (1) commutes by post-composing it with  $\phi_B$ ,  $\phi_A$ ,  $1_{GA}$ ,  $1_{GB}$  and showing the following equality holds.

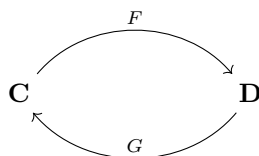
$$\phi_A^{-1} F f = G f \phi_B^{-1}$$

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## 1 Equivalence of categories

## 1.1 Motivation

We often have



but asking

$$\begin{aligned} GFA &= A \text{ for all } A \in \mathbf{C} \\ FGX &= X \text{ for all } X \in \mathbf{D} \end{aligned}$$

is too restrictive. Because we are interested in isomorphisms, not the equality of objects

**Definition 1.1.** *Let  $F : \mathbf{C} \longrightarrow \mathbf{D}$  be a functor.*

We say  $F$  is an equivalence if there exists a functor  $G : \mathbf{D} \longrightarrow \mathbf{C}$  and following natural transformations

$$\begin{array}{l} \eta : 1_{\mathbf{C}} \Rightarrow GF \\ \xi : FG \Rightarrow 1_{\mathbf{D}} \end{array}$$

**Note 1.1.** *The components of  $\eta$  and  $\xi$  are isomorphisms.*

$$\underbrace{A \xrightarrow{\eta_A} GFA}_{\substack{\text{in } \mathbf{C} \\ \forall A \in \mathbf{C}}} \qquad \underbrace{FGX \xrightarrow{\xi_X} X}_{\substack{\text{in } \mathbf{D} \\ \forall X \in \mathbf{D}}}$$

**Theorem 1.1.** *Let  $F : \mathbf{C} \longrightarrow \mathbf{D}$  be a functor.*

*Then  $F$  is an equivalence if and only if  $F$  is essentially surjective and fully faithful.*

*Proof.* ( $\Rightarrow$ ): Exercise.

( $\Leftarrow$ ): Assume  $F$  is essentially surjective and fully faithful.

$$\begin{aligned} \text{Assume } F \text{ is essentially surjective} &\Rightarrow (\forall X \in \mathbf{D})(\exists A \in \mathbf{C})(\exists \text{ isomorphism } FA \longrightarrow X) \\ &\quad \downarrow (\text{axiom of choice}) \\ &\Rightarrow \text{we have a function } G : Ob(\mathbf{D}) \longrightarrow Ob(\mathbf{C}) \\ &\quad \text{a family of maps } (\xi_X : FGX \longrightarrow X | X \in \mathbf{D}) \\ &\quad \text{such that } \xi_X : FGX \longrightarrow X \text{ is an isomorphism } \forall X \in \mathbf{D} \end{aligned}$$

**Task 1:** Extend  $G$  to a functor  $G$  and  $\xi$  to natural transformation.

For  $f : X \longrightarrow Y$  in  $\mathbf{D}$ , we need to define  $G(f) : GX \longrightarrow GY$  in  $\mathbf{C}$ . We use the fact that  $F$  is fully faithful

$$\mathbf{C}(GX, GY) \xrightarrow{F} \mathbf{D}(FGX, FGY) \text{ is bijection.}$$

This means for every  $FGX \xrightarrow{v} FGY$  there is a unique  $GX \xrightarrow{u} GY$  s.t

$$FGX \xrightarrow{Fu} FGY = FGX \xrightarrow{v} FGY$$

So to show that  $G(gf) = G(g)G(f)$ , we show  $G(g)G(f)$  has the same property, i.e

$$F(G(g)G(f)) : FGX \longrightarrow FGZ = FGX \xrightarrow{\xi_x} X \xrightarrow{gf} Z \xrightarrow{\xi_z^{-1}} FGZ$$

We know

$$\begin{array}{ccc} FGX & \xrightarrow{\xi_x} & X \\ \downarrow FG(gf) & & \downarrow gf \\ FGZ & \xrightarrow{\xi_z} & Z \end{array}$$

Consider

$$FGX \xrightarrow{\xi_x} X \xrightarrow{f} Y \xrightarrow{\xi_Y^{-1}} FGY$$

So there is a unique map, which we write  $Gf : GX \longrightarrow GY$  such that

$$FGf = FGX \longrightarrow FGY = FGX \xrightarrow{\xi_x} X \xrightarrow{f} Y \xrightarrow{\xi_Y^{-1}} FGY \quad (*)$$

Here

$$(*) \Leftrightarrow \begin{array}{ccc} FGX & \xrightarrow{\xi_x} & X \\ \downarrow FG(gf) & & \downarrow gf \\ FGZ & \xrightarrow{\xi_z} & Z \end{array} \text{ commutes}$$

To check  $G$  is a functor, let  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow Z$  be in  $\mathbf{D}$  and show

$$G(gf) : GX \longrightarrow GZ = G(g)G(f) : GX \longrightarrow GZ$$

By definition,  $G(gf)$  is the unique map

$$u : GX \longrightarrow GZ$$

such that

$$F(u) : FGX \longrightarrow FGZ = FGX \xrightarrow{\xi_x} X \xrightarrow{f} Z \xrightarrow{\xi_Z^{-1}} FGZ$$

We need to show following diagram commutes

$$\begin{array}{ccc}
FGX & \xrightarrow{\xi_X} & X \\
\downarrow F(GgGf) & & \downarrow gf \\
FGZ & \xrightarrow{\xi_Z} & Z
\end{array}$$

But this follows from

$$\begin{array}{ccc}
FGX & \xrightarrow{\xi_X} & X \\
\downarrow FGf & (1) & \downarrow f \\
FGY & \xrightarrow{\xi_Y} & Y \\
\downarrow FGg & (2) & \downarrow g \\
FGZ & \xrightarrow{\xi_Z} & Z
\end{array}$$

Since both (1), (2) commute.

In the same way we show

$$G(1_X) = 1_{GX}$$

□