Category Theory - Lecture 3 (Notes)

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1 Functors

Definition 1.1 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories.*

*A **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of:*

- *A function*

$$\begin{gathered} F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D}) \\ A \longmapsto F(A) \end{gathered}$$

- $\forall A, B \in Ob(\mathcal{C})$

$$\begin{gathered} F : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B)) \\ f \longmapsto F(f) \end{gathered}$$

with the following "functoriality" axioms satisfied:

- ***Associativity**: for all $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in $\mathcal{C}$,*

$$F(g \underbrace{\circ}_{\text{composition in } \mathcal{C}} f) = F(g) \underbrace{\circ}_{\text{composition in } \mathcal{D}} F(f)$$

- ***Identity** for all $A \in Ob(\mathcal{C})$,*

$$F(1_A) = 1_{F(A)}$$

Observation 1.1. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.*

- *The action of F on any string of composable maps is well defined. i.e*

$$\text{for all } \underbrace{A_1 \xrightarrow{f1} A_2 \xrightarrow{f2} A_3 \longrightarrow \ldots \longrightarrow A_n \xrightarrow{f_n} A_{n+1}}_{f_n \ldots f_2 \ldots f_1 : A_1 \longrightarrow A_{n+1}} \text{ in } \mathcal{C}$$

we have a unique map

$$\underbrace{FA_1 \xrightarrow{F(f_1)} FA_2 \xrightarrow{F(f_2)} F(A_3) \longrightarrow \ldots \longrightarrow FA_n \xrightarrow{F(f_n)} FA_{n+1}}_{F(f_n \ldots f_2 \ldots f_1) : FA_1 \longrightarrow FA_{n+1}} \text{ in } \mathcal{D}$$

- *Since the composition of maps is well defined, we can say that F preserves the commutative diagrams.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "g"] & & FA \arrow[r, "Ff"] \arrow[d, "Fh"] & FB \arrow[d, "Fg"] \\
C \arrow[r, "k"] & D & \Rightarrow & FC \arrow[r, "Fk"] & FD
\end{tikzcd}$$

commutes in $\mathcal{C}$ $\qquad$ *commutes in $\mathcal{D}$*

$gf = kh$ $\qquad$ $FgFh = FkFh$

Remark 1.1. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.*

If $f : A \longrightarrow B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \longrightarrow F(B)$ is an isomorphism in $\mathcal{D}$.

Proof. If $f : A \longrightarrow B$ is an isomorphism in $\mathcal{C}$, then we have $f^{-1} : B \longrightarrow A$ in $\mathcal{C}$ s.t

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "1_A"'] & B \arrow[d, "f^{-1}"] & B \arrow[r, "f^{-1}"] \arrow[dr, bend right, "1_B"'] & A \arrow[d, "f"] \\
& A & & B
\end{tikzcd}$$

We want to show that $F(f)$ is an isomorphism in $\mathcal{D}$. In other words, we want to show that

$$(Ff)^{-1} : F(B) \longrightarrow F(A)$$

is given by $F(f^{-1}) : FB \longrightarrow FA$. To check this, we need to show that following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[dr, bend right, "1_{FA}"'] & FB \arrow[d, "F(f^{-1})"] & FB \arrow[r, "F(f^{-1})"] \arrow[dr, bend right, "1_{FB}"'] & FA \arrow[d, "Ff"] \\
& FA & & FB
\end{tikzcd}$$

This is equivalent to showing that

$$\begin{aligned}
F(f^{-1})F(f) &= F(f^{-1}f) \quad \text{(according to functoriality axioms)} \\
&= F(1_A) \\
&= 1_{FA}
\end{aligned}$$

$$\begin{aligned}
F(f)F(f^{-1}) &= F(ff^{-1}) \quad \text{(according to functoriality axioms)} \\
&= F(1_B) \\
&= 1_{FB}
\end{aligned}$$

$\square$

Example 1.1. *(Forgetful functor) Functor $U : \mathcal{C} \longrightarrow \mathcal{D}$ that forgets the group structure.*

1. $U : \underline{Grp} \longrightarrow \underline{Set}$ forgets the group structure on the objects and maps the group operation to the set operation.

$$\begin{aligned} U : \underline{Grp} &\longrightarrow \underline{Set} \\ (G, *) &\longmapsto G \end{aligned}$$

2. $U : \underline{Grp} \longrightarrow \underline{Mon}$ forgets the group structure on the objects and maps the group operation to the monoid operation.

$$\begin{tikzcd} (G,*,1) \arrow[r, "U"] \arrow[d, "f"] & G = U(G,*,1) \arrow[d, "f=U(f)"] \\ (H,*,1) \arrow[r, "U"] & H = U(H,*,1) \end{tikzcd}$$

3. $U : \underline{Ring} \longrightarrow \underline{Grp}$

$$\begin{tikzcd} (R,+,0,*,1) \arrow[r, "U"] \arrow[d, "f"] & (R,+,0) \arrow[d, "f=U(f)"] \\ (S,+,0,*,1) \arrow[r, "U"] & (S,+,0) \end{tikzcd}$$

4. $U : \underline{Ab\ Grp} \longrightarrow \underline{Grp}$ forgets the abelian group property.

Example 1.2. *(Free Functors)*

Let's start reminding ourselves of the definition of free group on S

- ***Elements:*** *words like $x^2y^3z^{-2}$, where $x, y, z \in S$ subject to group axioms.*
- ***Operation:*** *concatenation of words, subject to group axioms.*
- ***Identity:*** *empty word.*

So we can define a free functor $F : \underline{Set} \longrightarrow \underline{Grp}$ as follows:

$$\begin{aligned} F : \underline{Set} &\longrightarrow \underline{Grp} \\ S &\longmapsto F(S) \end{aligned}$$

Here $F(S)$ is the free group on S.

$$\begin{tikzcd}
x, y, z \arrow[d, mapsto, "f"] & \in & S \arrow[d, "f"] & F(S) \arrow[d, "F(f)"] & x^2y^3z^{-2} \arrow[d, mapsto] \\
f(x), f(y), f(z) & \in & T & F(T) & f(x)^2 f(y)^3 f(z)^{-2}
\end{tikzcd}$$

Example 1.3. *(Topological Space with a base point)*

Let X be a topological space with a base point $x \in X$.

We can define a functor $\Pi_1 : \underline{Top_*} \longrightarrow \underline{Grp}$ that maps a topological space with a base point (i.e Top_*) to its fundamental group:

$$Top_X : \underline{Grp} \xrightarrow{\Pi_1} \underline{Set}$$
$$(X, O(x), x) \longmapsto \Pi_1(X)$$

Here X is the fundamental group of X.

Exercise 1.1. *1. Let G, H be groups. Define following functor:*

$$F : \Sigma(G) \longrightarrow \Sigma(H)$$

2. Let P, Q be posets. Define following functor:

$$F : \underline{P} \longrightarrow \underline{Q}$$

Proposition 1.1. *Small categories and functors form a category $\underline{Cat}$.*

Proof. Formally,

- **objects**: $Ob(\underline{Cat})$ = small categories
- **maps**: for small categories $\mathcal{C}, \mathcal{D} \in Ob(\underline{Cat})$,

$$\underline{Cat}(\mathcal{C}, \mathcal{D}) = \{F | F : \mathcal{C} \longrightarrow \mathcal{D}\}$$

- **composition**:

$$\underline{Cat}(\mathcal{D}, \mathcal{E}) \times \underline{Cat}(\mathcal{C}, \mathcal{D}) \xrightarrow{\circ} \underline{Cat}(\mathcal{C}, \mathcal{E})$$

$$(G \times F) \longmapsto GF$$

composition of functors G and F

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & G(F(A)) \arrow[d, "G(F(f))"] \\
B \arrow[r] & G(F(B))
\end{tikzcd}$$

To conclude the proof we need to show

Exercise 1.2. *Show that the following axioms are satisfied to conclude the proof:*

- ***associativity***: *let* F, G, H *be functors. Show* $H(GF) = GH(F)$.
- ***identity***: *let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor. Show* $F = F1_{\mathcal{C}} = 1_{\mathcal{D}}F$.

$\square$