

# Category Theory - Lecture 5 (Notes)

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**Exercise 0.1.** *Let  $\mathcal{C}$  be a category.*

*Let  $f, g : A \longrightarrow B$  and let  $u : A' \longrightarrow A$  be an isomorphism. Then,*

$$f = g \Leftrightarrow f \circ u = g \circ u$$

*Proof.*  $(\Rightarrow)$ : It is clear.

$(\Leftarrow)$ : If  $u : A' \longrightarrow A$  is an isomorphism, then there exists  $u^{-1} : A \longrightarrow A'$  such that

$$\begin{aligned} fu = gu &\Rightarrow f u u^{-1} = g u u^{-1} \\ &\Rightarrow f = g \end{aligned}$$

□

## 1 Natural Transformations

**Definition 1.1.** *Let  $F, G : \mathbf{C} \longrightarrow \mathbf{D}$  be functors.*

*A **natural transformation**  $\phi : F \Rightarrow G$  is a family of maps:*

$$\{\phi_A : FA \longrightarrow GA \mid A \in \text{Ob}(\mathbf{C})\}$$

*such that for all  $f : A \longrightarrow B$  in  $\mathbf{C}$ , we have*

$$\begin{array}{ccccc} \mathbf{C} & & \mathbf{D} & & \\ A & FA & \xrightarrow{\phi_A} & GA & \\ \downarrow f & \downarrow f & & \downarrow Gf & \\ B & FB & \xrightarrow{\phi_B} & GB & \end{array}$$

**Example 1.1.** Fix  $n \in \mathbb{N}$

- $\mathbf{C} = \underline{CRing} = \text{category of commutative rings}$
- $\mathbf{D} = \text{category of monoids}$
- $F : \underline{CRing} \longrightarrow \underline{Mon}$  is the functor

$$F = M_n : \underline{CRing} \longrightarrow \underline{Mon}$$

Here  $M_n(R)$  is  $n \times n$  matrix with entries in  $R$ . So  $\underline{Mon}$  is the category of monoids with  $n \times n$  matrices as objects.

$$\begin{array}{ccc} R & \xrightarrow{F} & M_n(R) \\ \downarrow f & & \downarrow M_n(f) \\ S & \xrightarrow{F} & M_n(S) \end{array}$$

- $G : \underline{CRing} \longrightarrow \underline{Mon}$  is the functor

$$G = U : \underline{CRing} \longrightarrow \underline{Mon}$$

Here we have

$$\begin{array}{ccc} (R, +, *, 0, 1) & \xrightarrow{U} & (R, +, 1) \\ \downarrow f & & \downarrow U(f) \\ (S, +, *, 0, 1) & \xrightarrow{U} & (S, *, 1) \end{array}$$

So  $U$  simply returns the underlying monoid of a given ring.

Now we can define a natural transformation

$$\det : M_n \Longrightarrow U$$

For  $R = (R, +, *, 0, 1) \in \underline{CRing}$ , we need

$$\begin{aligned} \det_R : FR &\longrightarrow GR \\ \det_R : M_n(R) &\longrightarrow U(R) \\ \det_R : M_n(R) &\longrightarrow (R, *, 1) \end{aligned}$$

Here  $\det_R$  is a monoid homomorphism

This is monoid homomorphism because

$$\left\{ \begin{array}{l} \det_R(M * N) = \det_R(M) * \det_R(N) \text{ (it preserves multiplication)} \\ \det_R(I_n) = 1_R \text{ (it preserves identity)} \end{array} \right.$$

So now need to check that this is a natural transformation. (i.e we need to check its **naturality**).

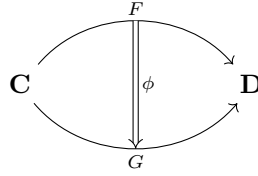
$$\begin{array}{ccccc} \underline{Cring} & & \underline{Mon} & & \\ R & M_n(R) & \xrightarrow{\det_R} & R & \\ \downarrow f & \downarrow M_n(f) & & \downarrow f & \\ S & M_n(S) & \xrightarrow{\det_S} & S & \end{array}$$

$$\begin{array}{ccc}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \xrightarrow{\det_R} & ad - bc \\
\downarrow M_n(f) & & \downarrow f \\
\begin{pmatrix} fa & fb \\ fc & fd \end{pmatrix} & \xrightarrow{U} & f(ad - bc)
\end{array}$$

## 1.1 Notation

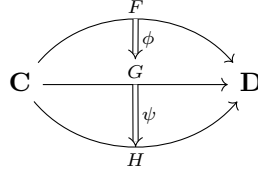
### 1.1.1 Natural transformation diagram

For  $F, G : \mathbf{C} \longrightarrow \mathbf{D}$  and  $\phi : F \Rightarrow G$  as natural transformation, we write

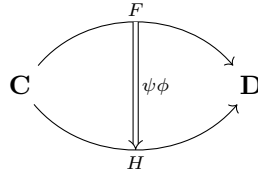


### 1.1.2 Composition diagram

Given  $F, G, H : \mathbf{C} \longrightarrow \mathbf{D}$  and  $\phi, \psi : F \Rightarrow G$  as natural transformations, we write



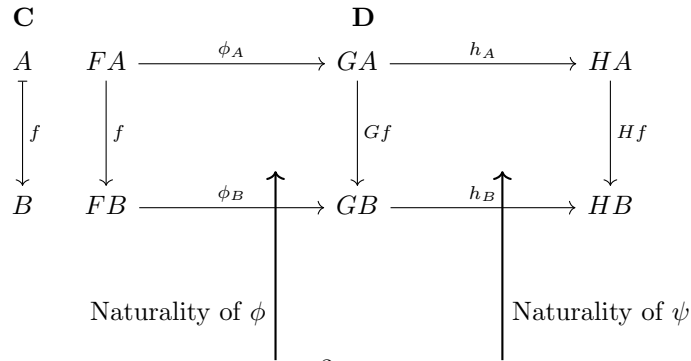
We define



Also

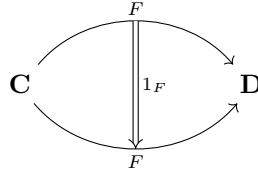
$$FA \xrightarrow{(\psi\phi)_A} HA \stackrel{=_{def}}{=} FA \xrightarrow{\phi_A} GA \xrightarrow{\psi_A} HA$$

So we can define composition



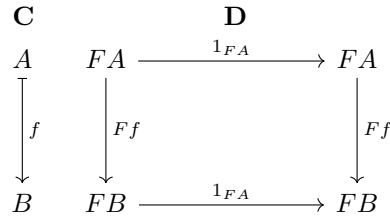
## 1.2 Identity natural transformation

For  $F : \mathbf{C} \longrightarrow \mathbf{D}$ , we can define



by  $FA \xrightarrow{1_{FA}} FA$  for all  $A \in \mathbf{C}$ .

Naturality of this is



## 1.3 Functor category

**Definition 1.2.** (*Functor category*)

For categories  $\mathbf{C}, \mathbf{D}$ , where  $\mathbf{C}$  is a small category, we define the **functor category**  $[\mathbf{C}, \mathbf{D}]$  (sometimes denoted as  $\mathbf{D}^{\mathbf{C}}$ ) as having

- **objects:** functors  $F : \mathbf{C} \longrightarrow \mathbf{D}$ ,  $G : \mathbf{C} \longrightarrow \mathbf{D}$
- **maps:** natural transformations  $\phi : F \Rightarrow G$

**Exercise 1.1.** Check the associativity and unit axioms for functor category.

### 1.3.1 Isomorphisms in $\mathbf{D}^{\mathbf{C}}$ or $[\mathbf{C}, \mathbf{D}]$

Fix  $\mathbf{C}, \mathbf{D}$ .

By definition, an isomorphism in  $[\mathbf{C}, \mathbf{D}]$  is  $\phi : F \Rightarrow G$  with an inverse

$$\phi^{-1} : G \Rightarrow F$$

This means that

$$\begin{array}{ccc} F & \xRightarrow{\phi} & G \\ \searrow 1_F & & \downarrow \phi^{-1} \\ & & F \end{array} \quad \begin{array}{ccc} G & \xRightarrow{\phi^{-1}} & F \\ \searrow 1_G & & \downarrow \phi \\ & & G \end{array}$$

This holds if and only if, for all  $A \in \mathbf{C}$ , we have

$$\begin{array}{ccc} FA & \xrightarrow{\phi_A} & GA \\ \searrow 1_{FA} & & \downarrow \phi_A^{-1} \\ & & FA \end{array} \quad \begin{array}{ccc} GA & \xrightarrow{\phi_A^{-1}} & FA \\ \searrow 1_{GA} & & \downarrow \phi_A \\ & & GA \end{array}$$

This implies  $\forall A \in \mathbf{C}$ ,  $\phi_A : FA \longrightarrow GA$  is an isomorphism.

**Proposition 1.1.** *Let  $\phi : F \Rightarrow G$  be a natural transformation.*

*Then  $\phi$  is an isomorphism if and only if for all  $A \in \mathbf{C}$*

$$\phi_A : FA \longrightarrow GA$$

*is an isomorphism.*

**Note 1.1.** *We need  $\phi$  to be natural in order to piece together all isomorphisms  $FA \longrightarrow GA$  for all  $A \in \mathbf{C}$*