Category Theory - Lecture 6 (Notes)

Vorashil Farzaliyev

September 2024

From last time...

Proposition 0.1. *Let $\phi : F \Rightarrow G$ be a natural transformation.*

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & \mathbf{D} \arrow[Rightarrow, from=U, to=L, "\phi"] \end{tikzcd}$$

Assume that for all $A \in \mathbf{C}$, $\phi_A : FA \longrightarrow GA$ is an isomorphism with inverse $\phi_A^{-1} : GA \longrightarrow FA$. Then $\phi^{-1} : G \Rightarrow F$ is defined by

$$(\phi^{-1})_A = (\phi_A)^{-1} \textit{ for all } A \in \mathbf{C}$$

Proof. We show that the naturality, as a family of maps, is given by

$$\begin{tikzcd} \mathbf{C} & & \mathbf{D} & \\ A \arrow[d, mapsto, "f"] & GA \arrow[rr, "\phi_A^{-1}"] \arrow[d, "Gf"] & & FA \arrow[d, "Ff"] \\ B & GB \arrow[rr, "\phi_B^{-1}"] & & FB \end{tikzcd}$$

But does this commute?

We have

$$\begin{tikzcd} GA \arrow[r, "\phi_A^{-1}"] \arrow[d, "Gf"] \arrow[rr, bend left, "1_{GA}"] \arrow[dr, phantom, "(1)"] & FA \arrow[r, "\phi_A"] \arrow[d, "Ff"] & GA \arrow[d, "Gf"] \\ GB \arrow[r, "\phi_B^{-1}"] \arrow[rr, bend right, "1_{GB}"'] & FB \arrow[r, "\phi_B"] & GB \end{tikzcd}$$

We show that (1) commutes by post-composing it with ϕ_B, ϕ_A, 1_{GA}, 1_{GB} and showing the following equality holds.

$$\phi_A^{-1}Ff = Gf\phi_B^{-1}$$

$\square$

1 Equivalence of categories

1.1 Motivation

We often have

$$\begin{tikzcd} \mathbf{C} \arrow[rr, bend left, "F"] & & \mathbf{D} \arrow[ll, bend left, "G"] \end{tikzcd}$$

but asking

$$GFA = A \text{ for all } A \in \mathbf{C}$$
$$FGX = X \text{ for all } X \in \mathbf{D}$$

is too restrictive. Because we are interested in isomorphisms, not the equality of objects

Definition 1.1. *Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor.*

We say F is an <u>*equivalence*</u> *if there exists a functor $G : \mathbf{D} \longrightarrow \mathbf{C}$ and following natural transformations*

$$\eta : 1_{\mathbf{C}} \Rightarrow GF$$
$$\xi : FG \Rightarrow 1_{\mathbf{D}}$$

Note 1.1. *The components of η and ξ are isomorphisms.*

$$\underbrace{A \xrightarrow{\eta_A} GFA}_{\substack{\textit{in } \mathbf{C} \\ \forall A \in \mathbf{C}}} \qquad \underbrace{FGX \xrightarrow{\xi_X} X}_{\substack{\textit{in } \mathbf{D} \\ \forall X \in \mathbf{D}}}$$

Theorem 1.1. *Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor.*

Then F is an equivalence if and only if F is essentially surjective and fully faithful.

Proof. ($\Rightarrow$): Exercise.

($\Leftarrow$): Assume F is essentially surjective and fully faithful.

$$\begin{aligned} \text{Assume } F \text{ is essentially surjective} &\Rightarrow (\forall X \in \mathbf{D})(\exists A \in \mathbf{C})(\exists \text{ isomorphism } FA \longrightarrow X) \\ &\quad\ \downarrow \text{(axiom of choice)} \\ &\Rightarrow \text{we have a function } G : Ob(\mathbf{D}) \longrightarrow Ob(\mathbf{C}) \\ &\qquad \text{a family of maps } (\xi_X : FGX \longrightarrow X | X \in \mathbf{D}) \\ &\qquad \text{such that } \xi_X : FGX \longrightarrow X \text{ is an isomorphism } \forall X \in \mathbf{D} \end{aligned}$$

Task 1: Extend G to a functor G and ξ to natural transformation.

For $f : X \longrightarrow Y$ in $\mathbf{D}$, we need to define $G(f) : GX \longrightarrow GY$ in $\mathbf{C}$. We use the fact that F is fully faithful

$$\mathbf{C}(GX, GY) \xrightarrow{F} \mathbf{D}(FGX, FGY) \text{ is bijection.}$$

This means for every $FGX \xrightarrow{v} FGY$ there is a unique $GX \xrightarrow{u} GY$ s.t

$$FGX \xrightarrow{Fu} FGY = FGX \xrightarrow{v} FGY$$

So to show that $G(gf) = G(g)G(f))$, we show $G(g)G(f)$ has the same property, i.e

$$F(G(g)G(f)) : FGX \longrightarrow FGZ = FGX \xrightarrow{\xi_X} X \xrightarrow{gf} Z \xrightarrow{\xi_Z^{-1}} FGZ$$

We know

$$\begin{tikzcd}
FGX \arrow[r, "\xi_X"] \arrow[d, "FG(gf)"] & X \arrow[d, "gf"] \\
FGZ \arrow[r, "\xi_Z"] & Z
\end{tikzcd}$$

Consider

$$FGX \xrightarrow{\xi_X} X \xrightarrow{f} Y \xrightarrow{\xi_Y^{-1}} FGY$$

So there is a unique map, which we write $Gf : GX \longrightarrow GY$ such that

$$FGf = FGX \longrightarrow FGY = FGX \xrightarrow{\xi_X} X \xrightarrow{f} Y \xrightarrow{\xi_Y^{-1}} FGY \quad (*)$$

Here

$$(*) \Leftrightarrow \quad \begin{tikzcd}
FGX \arrow[r, "\xi_X"] \arrow[d, "FG(gf)"] & X \arrow[d, "gf"] \\
FGZ \arrow[r, "\xi_Z"] & Z
\end{tikzcd} \quad \text{commutes}$$

To check G is a functor, let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be in $\mathbf{D}$ and show

$$G(gf) : GX \longrightarrow GZ = G(g)G(f) : GX \longrightarrow GZ$$

By definition, $G(gf)$ is the unique map

$$u : GX \longrightarrow GZ$$

such that

$$F(u) : FGX \longrightarrow FGZ = FGX \xrightarrow{\xi_X} X \xrightarrow{f} Z \xrightarrow{\xi_Z^{-1}} FGZ$$

We need to show following diagram commutes

$$\begin{tikzcd}
FGX \arrow[r, "\xi_X"] \arrow[d, "F(GgGf)"'] & X \arrow[d, "gf"] \\
FGZ \arrow[r, "\xi_Z"] & Z
\end{tikzcd}$$

But this follows from

$$\begin{tikzcd}
FGX \arrow[r, "\xi_X"] \arrow[d, "FGf"'] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \\
FGY \arrow[r, "\xi_Y"] \arrow[d, "FGg"'] \arrow[dr, phantom, "(2)"] & Y \arrow[d, "g"] \\
FGZ \arrow[r, "\xi_Z"] & Z
\end{tikzcd}$$

Since both $(1), (2)$ commute.

In the same way we show

$$G(1_X) = 1_{GX}$$

$\square$