Theoretical Base of the PUCK-Model with Application to Foreign Exchange Markets

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Theoretical Base of the PUCK-Model with application to Foreign Exchange Markets

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Abstract

We analyze statistical properties of a random walker in a randomly changing potential function called the PUCK model both theoretically and numerically. In this model the center of the potential function moves with the moving average of the random walker's trace, and the potential function is given by a quadratic function with its curvature slowly changing around zero. By tuning several parameters the basic statistical properties fit nicely with those of real financial market prices, such as power law price change distribution, very short decay of autocorrelation of price changes, long tails in autocorrelation of the square of price changes and abnormal diffusion in short time scale.

1 Introduction

The first random walk model was introduced by Bachelier in 1900 as a model of market price fluctuations [1]. Independently Einstein proposed a random walk model for the motion of colloid particles [2]. In the case of Einstein's theory the driving force of random walker is random collision of molecules in thermal equilibrium, while in the case of Bachelier's theory the driving force is mass psychology of fickle traders who are predicting the future price individually. Mathematical formulation of Bachelier's theory was developed into highly sophisticated manner after his death in the field of financial technology and it is now widely accepted in the real world finance.

Earnest scientific verification of random walk assumptions for market price fluctuations started in the middle of 1990s with the advent of the new field, Econophysics, until that time high frequency tick-by-tick market data was not accessible for scientists. Roughly speaking the amount of tick-by-tick data is about 10 thousand times denser than traditional daily market data, and intensive analysis of high quality financial market data from physisists' view has clarified that market price fluctuations are not simple random walks [3]. Empirically stylized facts which clearly deviate from a naive random walk model can be summarized by the following 4 characteristics:

1. The distribution of price change in a unit time has nearly symmetric long tails approximated roughly by power laws. This property was firstly pointed out by Mandelbrot [4]. A typical value of the power law exponent is 3, however, it seems not to be a universal constant but depends on market conditions [5].

2. The autocorrelation of price change decays quickly to zero often accompanied by a negative correlation for very short time [3].

3. The magnitude of price changes called the volatility, defined by the square of price changes, is known to have a long correlation often approximated by a power law [3].

4. For large time scale the diffusion property of market price generally follows the normal diffusion in which the variance is proportional to time, however, for short time scale abnormal diffusion is observed, i.e., the variance is approximated by a fractional power of time. The estimated exponent of the power is not universal, but it seems to depend on the market condition and it is closely related to the Hurst exponent of market price fluctuations [6].

There are many variants of random walk models of market prices, however, it is not easy to reproduce all of these characteristics. For example, the Nobel prize laurelled ARCH model [7] roughly satisfies characteristics 1, 2and 3, however, it misses the abnormal diffusion characteristics 4.

The present authors already proposed a new type of random walk model in which random walker moves in a quadratic potential function whose center is given by the moving average of the random walker's trace [8]. This model is named as PUCK model from the abbreviation of Potentials of Unbalanced Complex Kinetics. As the potential function moves according to the motion of the random walker the statistical properties are very different from the case of a fixed potential function, i.e., the Ulenbeck-Ornstein process. We show that the characteristic 4 becomes accessible by the effect of such moving potential function. Also considering the case that the coefficient of the potential function is changing randomly, the price change dynamics is approximated by a random multiplicative process and the distribution of price changes follows a power law satisfying the characteristic 1. Long correlation of volatility is also satisfied by taking into account the effect that the coefficient of the potential function fluctuates with a long correlation.

In the next section we introduce the PUCK model and analyze its basic properties with a fixed potential coefficient in section 2.1. In section 2.2 we consider the case that the coefficient fluctuates randomly and show that the above 4 characteristics are roughly satisfied in some parameter ranges. In section 3 we analyze real market data of Yen-Dollar exchange rates and confirm the above basic characteristics comparing with the PUCK model's results. The final section is devoted for discussion.

2 The PUCK model

We consider the following form of random walk in a moving potential function $U_M(x;t)$:

$$x(t+1) - x(t) = -\frac{d}{dx} U_M(x;t)\,|_{x=x(t)-x_M(t)} + f(t) \quad , \tag{1}$$

$$U_M(x;t) \equiv \frac{b(t)}{M-1} \frac{x^2}{2} \quad , \tag{2}$$

$$x_M(t) \equiv \frac{1}{M} \sum_{k=0}^{M-1} x(t-k) \quad , \tag{3}$$

where $f(t)$ is a random external noise, $b(t)$ is the coefficient of the quadratic potential, M is the size of moving average to define the center of potential function, $x_M(t)$. This model has been used as a new type of time series data analysis characterizing time-dependent stability of markets. Here, we analyze its basic properties both numerically and theoretically.

2.1 The case of constant b

Let us first consider the simplest case of a constant $b(t) = b$. The case of b=0 corresponds to the simplest random walk. For b>0 the random walker is attracted to the moving average of its own traces, so that the diffusion becomes slower than the case of b =0. On the other hand when b<0 the random walker is pushed away from the moving average of its traces and the walker diffuses faster than normal random walk. This property can be confirmed in Fig.1a in which traces for different values of b are plotted for the same random number seed. For larger value of M the behaviors of $x(t)$ are smoother as shown in Fig.1b. There is a sharp transition in diffusion property at $b = -2$. For $0 > b > -2$ the diffusion is faster than the normal case of b =0, however, its long time behavior follows the normal diffusion law, that is, the variance is proportional to the time as shown later. When $b \le -2$ the repulsive force from the center of the potential function is larger than the effect of additive random force, $f(t)$, and the motion of $x(t)$ is approximated by an exponential growth as shown in Fig.1c. These cases are considered to be related to crashes or bubbles in markets. In such a case the direction of growth, either going up or down, is determined by the initial value condition or by the external noise, $f(t)$, as long as the potential function is symmetric. On the other hand for positive large number of b the potential force is so strong that the motion becomes a diverging oscillation as shown in Fig.1d.

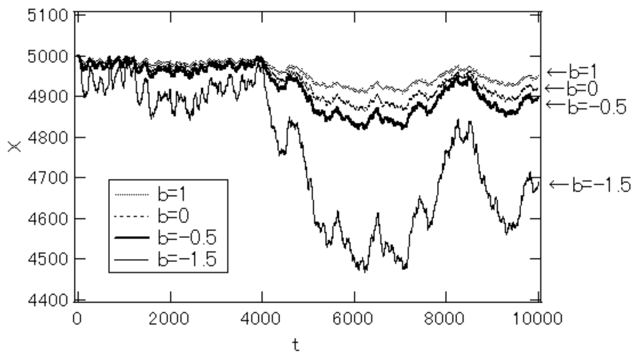

Fig.1a. Examples of $x(t)$ for b=-1.5, -0.5, 0, 1.0 with M=10 in Eq.(3). $f(t)$ is a Gaussian random number with the mean value 0 and the standard deviation unity.

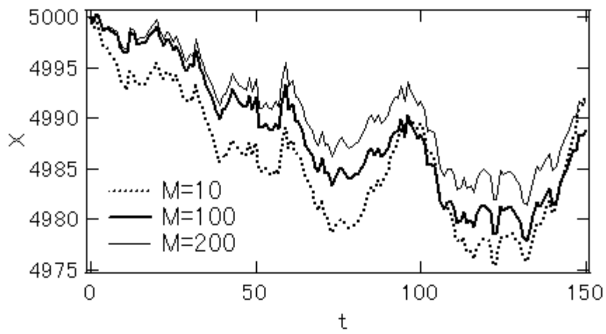

Fig.1 b Examples of $x(t)$ in the case of b=-1.0 for M=10, 100 and 200.

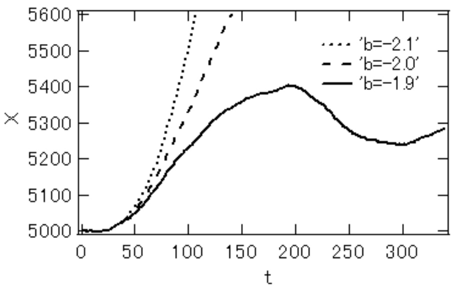

Fig.1c An example of b-dependence around b=-1.9. For b=-2.0 and -2.1 the value of x(t) diverges. Here, the curves look smooth as the scale of the vertical axis is about 20 times larger.

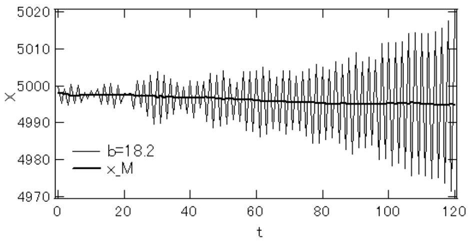

Fig.1d An example of oscillating divergence observed in the case of *b*=18.2 with *M*=10.

The threshold value, $b=-2$, of this exponential divergence can be analyzed theoretically in the following way by representing the basic equation Eq.(1)-(3) in terms of the velocity defined by the price difference $v(t) \equiv x(t) - x(t-1)$;

$$v(t+1) = -\frac{b}{2}\sum_{k=1}^{M-1}\omega_k v(t-k+1) + f(t) \quad , \qquad (4)$$

where the weight function ω_k is given by

$$\omega_k \equiv \frac{2(M-k)}{M(M-1)} \quad , \qquad \sum_{k=1}^{M-1}\omega_k = 1 \quad . \qquad (5)$$

As Eq.(4) can be viewed as an Auto-Regressive process, the condition for realization of statistically steady state can be determined by the condition that all solutions of z of the following equation is within a radius 1 in the complex plain:

$$z^{M-1} = -\frac{b}{2}\sum_{k=1}^{M-1} \omega_k z^{M-k-1} \tag{6}$$

From this analysis it is shown that the stochastic process governed by Eq.(4) is non-stationary when $b \le -2$. At the boundary case of $b = -2$ with $M = 2$ it is confirmed that the velocity satisfies the basic random walk instead of $x(t)$,

$$v(t+1) = v(t) + f(t) \quad . \tag{7}$$

Therefore, in this case that the velocity is a non-stationary variable and the diffusion of $x(t)$ is much faster than that of normal diffusion.

Next we observe the basic statistical properties of PUCK-model with a constant b when the external noise follows a white Gaussian noise. It is easy to show that the first moment, $< v(t) >$, is always zero from Eq.(4). For the second order moments such as the variance, $< v(t)^2 >_c$, we have the following Yule-Walker equation:

$$< v(t+T)v(t) >= \sum_{k=1}^{M-1} (-\frac{b}{2})\omega_k < v(t+T-k)v(t) > + F\delta_T \quad , \tag{8}$$

where F is the variance of the white noise, $< f(t)f(t') >= F\delta_{t-t'}$, and δ_T is the Kronecker delta which is 1 when T=0 and is 0 otherwise. For given b and M the variance, $< v(t)^2 >$, is obtained for T=0 in Eq.(8). In the special case of M=2, the solution is given as follows.

$$< v(t)^2 >= \frac{F}{1-(\frac{b}{2})^2} \quad . \tag{9}$$

This representation is valid for $-2 < b < 2$. In this special case it is easy to prove that the distribution of $v(t)$ follows a Gaussian with the variance given by Eq.(9). Fig.2a shows a typical example of numerical result of distribution of price difference $\Delta x(T;t) \equiv x(t+T) - x(t)$ for a general case of $b$and M, in each case the distribution is well approximated by a Gaussian distribution.

The autocorrelation of $v(t)$, $C_v(T) \equiv< v(t+T)v(t) >_c / < v(t)^2 >_c$, is obtained directly from Eq.(8). In the case of M=2 the solution is given as,

$$C_v(T) \equiv (-\frac{b}{2})^T \quad . \qquad (10)$$

For $0 > b > -2$ this is an exponential damping, and it is an exponential damping with oscillation for $2 > b > 0$. In Fig.2b the autocorrelations for some combinations of *b* and *M* are plotted. As confirmed from this figure, the autocorrelation is always positive and decay exponentially for any negative *b*-value. On the other hand for a positive b-value we can find an oscillatory behavior in general.

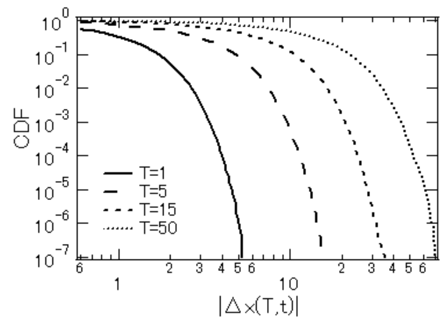

Fig.2a Log-log plot of the cumulative distribution of price difference. M=10、b=－1.5 with T=1, 5, 15, 50.

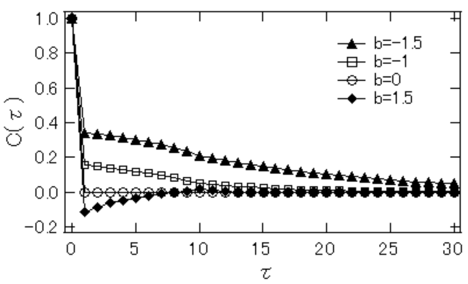

Fig.2b The autocorrelation function for $v(t)$ with M=10, b=-1.5, -0.5, 0.0, 1.5.

The volatility time series is defined by $\{v(t)^2\}$ and its autocorrelation is also analyzed theoretically using Eq.(8). Here, we show analytical result for *M*=2. In this case we have the following equation by taking a square of Eq.(8), and multiplying $v(t-T+1)^2$, then taking average over $\{f(t)\}$:

$$< v(t+1)^2 v(t+1-T)^2 >= \frac{b^2}{4} < v(t)^2 v(t+1-T)^2 > \\ + < f(t)^2 >< v(t+1-T)^2 > \quad (11)$$

From this equation we have the following solution for the volatility autocorrelation:

$$C_{v^2}(T) \equiv < v^2(t+T) v^2(t) >_c / \{< v(t)^2 >\}^2 = (\frac{b}{2})^{2T} \quad . \quad (12)$$

Examples of general cases are numerically estimated as shown in Fig.2c. As known from this result the autocorrelation of volatility always decays exponentially like the case of theoretical solution of *M*=2, and no long-correlation can be observed as far as the value of *b* is a constant.

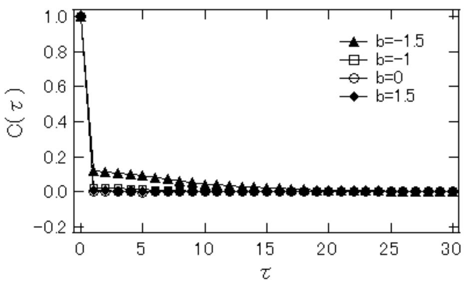

Fig.2 c The autocorrelation of $v(t)^2$ with M=10, b=−1.5, −0.5, 0.0, 1.5.

The diffusion property can be characterized by observing the time evolution of variance $\sigma^2(t) \equiv < \{x(t) - x(0)\}^2 >$ as numerically obtained in Fig.2d. For b>0 the diffusion is slower than the case of b=0 for short time scale, and for large time scale the slope becomes 1 which is equivalent to the normal diffusion. In the case of *M*=2 we can obtain an exact solution also for this quantity. By solving Eq.(4) with *M*=2, we have the following exact representation:

$$v(t) = (-\frac{b}{2})^t v(0) + \sum_{s=1}^{t} (-\frac{b}{2})^{s-1} f(t-s) \quad (13)$$

$$x(t) - x(0) = \frac{1 - (-\frac{b}{2})^t}{1 - (-\frac{b}{2})} v(0) + \sum_{s=0}^{t-1} \frac{1 - (-\frac{b}{2})^{t-s-1}}{1 - (-\frac{b}{2})} f(s) \quad (14)$$

Then, taking average over the square of Eq.(14), we have the solution for $t \geq 1$:

$$< \{x(t) - x(0)\}^2 >= \frac{(1-(-\frac{b}{2})^t)^2}{(1+\frac{b}{2})^2} < \{v(0)\}^2 >$$

$$+ \frac{F}{(1+\frac{b}{2})^2}\{t + b\frac{1-(-\frac{b}{2})^{t-1}}{1+\frac{b}{2}} + \frac{b^2}{4}\frac{1-(\frac{b^2}{4})^{t-1}}{1-\frac{b^2}{4}}\}$$

(15)

We have the diffusion constant for large *t* as follows [9]:

$$D_x = \frac{4F}{(2+b)^2} \quad . \tag{16}$$

Abnormal diffusion at small t can be approximated by assuming the following fractional power law,

$$< \{x(t) - x(0)\}^2 > \propto t^{\alpha} \quad , \tag{17}$$

where α can be determined approximately from small *t*. As an extreme case we can evaluate α from *t* =0,1 and 2:

$$\alpha = \frac{\log\{< x(2)^2 > / < x(1)^2 >\}}{\log 2} = 1 + \frac{\log(1 - \frac{b}{2} + \frac{b^2}{8})}{\log 2} \tag{18}$$

For *b* close to 0 we have the abnormal diffusion exponent as

$$\alpha \approx 1 - 0.72 \cdot b \quad . \tag{19}$$

This approximation holds for the range of -0.5<b<0.5 as shown in Fig.2e. For larger value of *M* the behavior of estimated α deviates from Eq.(18), however, qualitative behaviors are the same, i.e., slower abnormal diffusion for b>0 and faster abnormal diffusion for b<0.

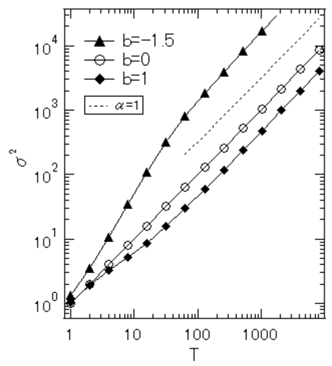

Fig.2 d Log-log plot of the variance of price diffusion for *M*=10, b=−1.5, −0.5, +1

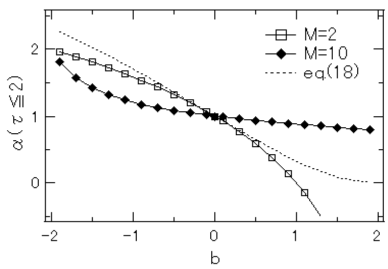

Fig.2e Numerically estimated value of α for small t. The dotted line shows the theoretical value given by Eq.(18)

2.2The case of random $b(t)$

Next we consider the case that the potential coefficient value changes randomly with time. As empirically estimated value of $b(t)$ is known to be fluctuating around 0 [10], we assume that $b(t)$ follows a random walk in a fixed potential function, i.e., an Ornstein-Ulenbeck process as follows:

$$b(t+1) - b(t) = -c_0 b(t) + g(t) \tag{21}$$

where c_0 is a positive constant in the range of [0,1] and $g(t)$ is a normal Gaussian noise with zero mean and the variance *G*. By this effect the value of $b(t)$ fluctuates spontaneously and the statistics of $x(t)$ changes accordingly. Examples of the set of time evolutions of $b(t)$, $x(t)$ and the volatility $|v(t)|$ are shown in Fig.3a-c for typical values of *G* and c_0. The parameters *G* and c_0 plays the central role for the statistical properties of $x(t)$ and $v(t)$.

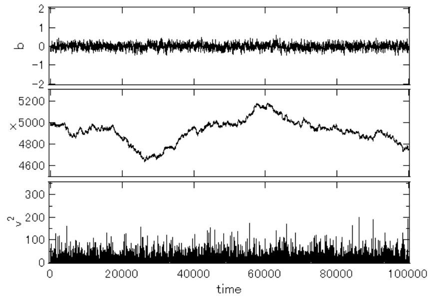

Fig.3a. Simulation results based on Eq.(21) with parameters $c_0 = 0.02$ and G=0.000784. Top; Value of b(t), Middle; Market prices, Bottom; Volatility.

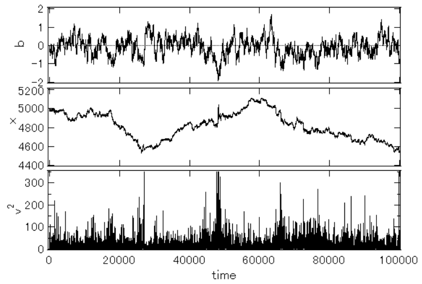

Fig.3b. Simulation results based on Eq.(21) with parameters $c_0 = 0.0015$ and G=0.000784. Top; Value of b(t), Middle; Market prices, Bottom; Volatility.

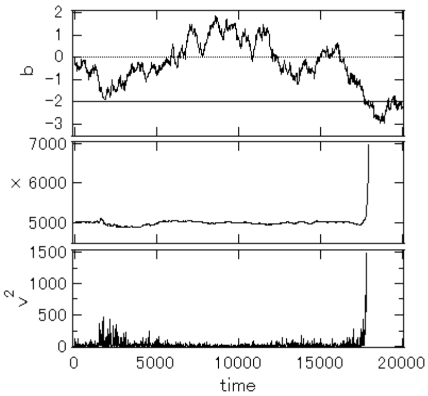

Fig.3c. Simulation results based on Eq.(21) with parameters $c_0 = 0.0001$ and G=0.000784. Top; Value of b(t), Bottom; Market prices. The plot of volatility is skipped as the price changes in the encircled period in which $b(t)$<-2 are too large compared with the other periods.

In the case that the value of c_0 is relatively large compared with the variance of the noise term G as typically shown in Fig.3a, fluctuation of the value of $b(t)$ is concentrated around 0, then the behavior of $x(t)$ looks similar to the simple normal random walk (Fig.3a). In this case the statistical properties of prices are confirmed to be nearly equivalent to the case of normal random walk, i.e., a normal distribution of the price difference, quick decay of the autocorrelations for both the market price and the volatility, and no abnormal diffusion of prices.

As shown in Fig.3b when the value of c_0 is intermediate, it is confirmed that the volatility clustering caused by the fluctuation of $b(t)$ can be observed. When the value of c_0 is close to 0 like the case of Fig.3c the fluctuation amplitude of $b(t)$ becomes very large, and there is a finite possibility that the value of $b(t)$ falls into the parameter range of non-stationary condition $b(t) \le -2$. In such a case the behavior of $x(t)$ switches between random walk phase and exponential growth phase in a random manner and the whole process becomes quite unstable and non-stationary.

The steady distribution of $b(t)$ is solved analytically just like the case of $v(t)$ with M=2 and it is given by the following normal distribution when the random noise $g(t)$ is an independent Gaussian distribution with variance G:

$$p(b(t)) = \frac{1}{\sqrt{2\pi}\sigma_b(c_0)} e^{-\frac{b(t)^2}{2\sigma_b(c_0)^2}} \quad , \quad \sigma_b(c_0)^2 = \frac{G}{2c_0 - c_0{}^2} \quad . \qquad (22)$$

As known from this solution the probability of occurrence of $b(t) \le -2$ always takes a finite value, therefore, theoretically there is a finite possibility that the market price moves nearly monotonically for a finite period. However, in the following discussion we consider the case that c_0 is not so small compared with G that the probability of realization of such non-stationary behavior is negligibly small.

In Fig.3d a typical distributions of $v(t)$ is plotted for different values of c_0. Contrary to the case of constant b it is confirmed that the distributions are well-approximated by power laws in any case. Such power law like behaviors can be understood by considering the case of M=2. As mentioned above the distribution of $v(t)$ in the case of fixed $b(t) = b$ is given by a normal distribution with the variance given by $F/(1-b^2/4)$. Assuming that the change of $b(t)$ is slow enough and we can evaluate the distribution of $v(t)$ by superposition of such normal distributions with the weight of the distribution of $b(t)$:

$$p(v) \approx \int_{-2}^{2} \frac{\sqrt{1-b^2/4}}{\sqrt{2\pi F}} e^{-\frac{1-b^2/4}{2F}v^2} \frac{\sqrt{2c_0 - c_0{}^2}}{\sqrt{2\pi G}} e^{-\frac{2c_0 - c_0{}^2}{2G}b^2} db \qquad (23)$$

By introducing a new variable $B = 1 - b^2 / 4$, we have the following form,

$$p(v) \approx \frac{\sqrt{2c_0 - c_0{}^2}}{2\pi\sqrt{FG}} e^{-2\frac{2c_0 - c_0{}^2}{G}} \int_0^1 \frac{\sqrt{B}}{\sqrt{1-B}} e^{-(\frac{v^2}{2F} - \frac{2c_0 - c_0{}^2}{2G})B} dB \quad . \qquad (24)$$

Evaluating the integral with respect to *B*, the asymptotic behavior for large $|\ v\ |$ is estimated as

$$p(v) \approx \frac{\sqrt{2c_0 - c_0{}^2}}{2\pi\sqrt{FG}} e^{-2\frac{2c_0 - c_0{}^2}{G}} \{0.88(\frac{v^2}{2F})^{-\frac{3}{2}} + 0.66(\frac{v^2}{2F})^{-\frac{5}{2}} + \cdots\} . \qquad (25)$$

We have symmetric power law tails in the distribution of $v(t)$. In this case the cumulative distribution is approximated by the following power law form with β =2.

$$P(\geq |\ v\ |) \propto |\ v\ |^{-\beta} \quad . \qquad (26)$$

Another limit case is solved theoretically when the change of $v(t)$ is very fast in the case of *M*=2. Applying the formula of random multiplicative noise [11] for Eq.(4), the power law exponent β of the steady state cumulative distribution of $v(t)$ is given by solving the following equation for β:

$$1 = <|\ \frac{b}{2}\ |^{\beta}> = \frac{1}{\sqrt{2^{\beta}\pi}} \Gamma(\frac{\beta+1}{2})(\frac{G}{2c_0 - c_0{}^2})^{\beta} \quad , \qquad (27)$$

where $\Gamma(\frac{\beta+1}{2})$ is the gamma function. In view of this limit the power law exponent is given by the parameters *G* and c_0 characterizing the distribution of $b(t)$. Empirical values of β are known to lie mainly in the range $2 < \beta < 4$, which is realizable by Eq.(27) by tuning the value of *G* and c_0.

The autocorrelations of $v(t)$ in the case of random $b(t)$ are plotted in Fig.3e. It is confirmed that the autocorrelations decay always rapidly to zero. This vanish of autocorrelation is not trivial as the autocorrelation for a fixed $b(t)$ is not zero as mentioned above. Accumulation of various values of $b(t)$ with 0 mean causes this phenomenon.

The volatility autocorrelation is plotted in Fig.3f. We can find that the tail part becomes longer for smaller value of c_0. In the case that non-stationary price motion is included, the autocorrelation tends to converge to a non-zero value for large time difference.

The diffusion properties are analyzed in Fig.3g. We can find abnormal diffusion for small time scales, however, for large time scales the normal diffusion property is retained in any case.

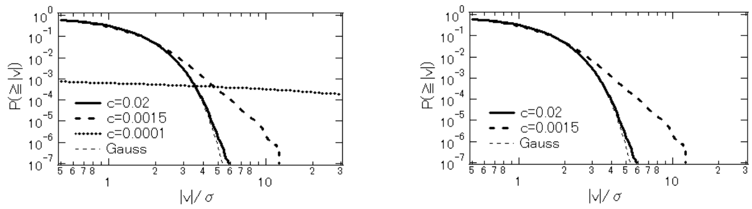

Fig.3dLog-log plot of the cumulative distribution of price difference. The parameters are c_0 =0.02, 0.0015, 0.0001 with *G*=0.000784.

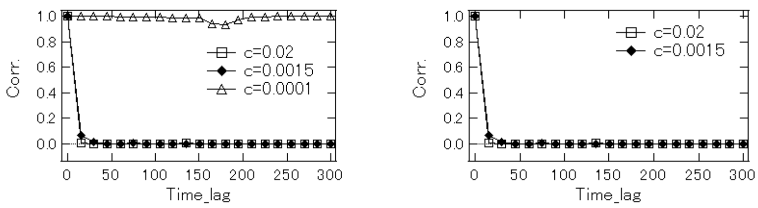

Fig.3e The autocorrelation function for v(t). The parameters are c_0 =0.02, 0.0015, 0.0001 with*G*=0.000784.

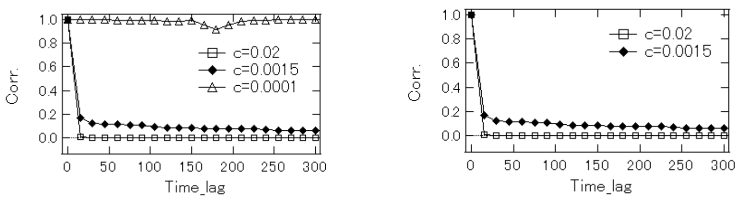

Fig.3f The autocorrelation of $v(t)^2$. The parameters are c_0 =0.02, 0.0015, 0.0001 with *G*=0.000784.

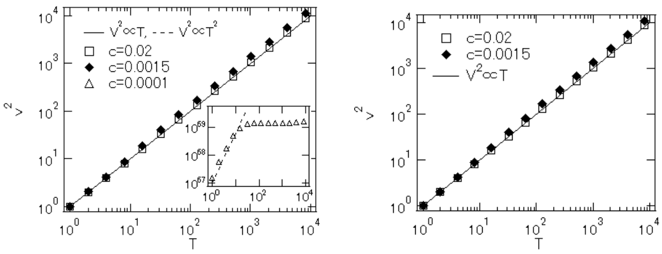

Fig.3g Log-log plot of the variance of price diffusion. The parameters are c_0 =0.02, 0.0015, 0.0001 with *G*=0.000784.

3 Statistical properties of a real financial market

In this section we compare the model's properties with the real market data of US-dollar Japanese-yen exchange rates. This is the price of 1 dollar paid by Yen in the electronic broker system provided by ICAP. In this broker system major international banks are dealing continuously 24 hours excepts weekends with the minimum unit of a deal 1 million dollar. The observation period is 2006 with time stamp in second. The number of total data points is about 3 million ticks. In the data there are 4 kinds of prices, deal-ask, deal-bid, best-ask and best-bid. Here, we apply deal prices which are actually transacted prices taken on the bid-side and ask-side, namely, offered orders to buy and sell.

For given time series of market prices, the PUCK analysis is done in the following procedures [8]: Firstly, we calculate the autocorrelation of price changes from the raw time series, and we apply the noise separation process based on Yule-Walker method to derive the optimal moving average [12]. After this procedure the smoothed market price time series $\{x(t)\}$ is used to define the super-moving average $x_M(t)$ by Eq.(3) with a fixed value of moving average size *M*. Then, we plot $x(t+1)-x(t)$ vs. $\{x(t)-x_M(t)\}/(M-1)$ for *N* data points using $\{x(t-N), x(t-N+1), \cdots, x(t+1)\}$, where the data number *N* is typically 500. The slope of the best-fit line for these scattered points gives the value of $-b(t)$, the curvature of the potential force.

By numerical tests using the artificial random data produced by the PUCK model with a constant value of $b(t)$, the magnitude of estimation error of $b(t)$ is estimated to be less than 0.3 and the occurrence probability of $b(t) < -1$ or $b(t) > 1$ are less than 0.1 % in the statistical test [10].

Fig.4a shows an example of Dollar-Yen exchange rate time series, and Fig.4b gives an estimated time series of *b*(*t*) for the time series of Fig.4a. As known from these figures, it is found that the value of $b(t)$ is always fluctuating in various scales in the market.

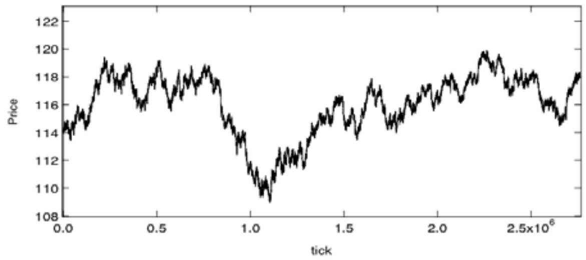

Fig.4a An example of Dollar-Yen exchange rate time series in 2006

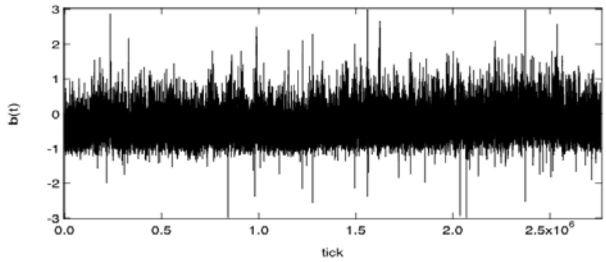

Fig.4b The estimated values of $b(t)$ for Fig.4a.

Here, we apply Eq.(21) as a numerical model of time evolution of $b(t)$. For the time sequence shown in Fig.4b, we can approximate the dynamics by Eq.(21). Using such empirically estimated potential function of $\phi(b)$, the time evolution of $b(t)$ can be simulated by applying a Gaussian white noise for g(t) [13]. Fig.5a is a simulated variant for real data version Fig.4b. Fig.5b gives a simulated Dollar-Yen rate fluctuation corresponding to the real data of Fig.4a.

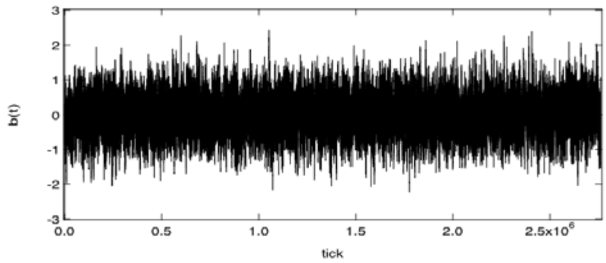

Fig.5a An example of simulated time series of $b(t)$.

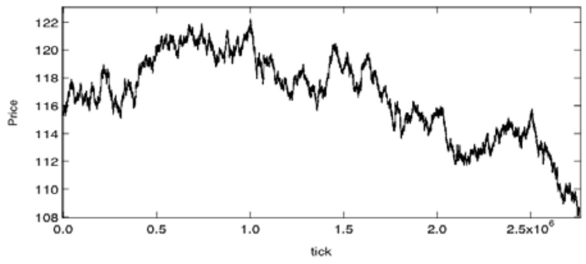

Fig.5b An example of time series of market price assuming the fluctuation of potential coefficient $b(t)$ shown in Fig.5a.

In Figs.6a to 6d the basic statistics of this empirical model are compared with the real data [13]: The distribution of price difference, the autocorrelation of price change, the autocorrelation of volatility and the diffusion properties. It is confirmed that basic properties are roughly satisfied in all cases.

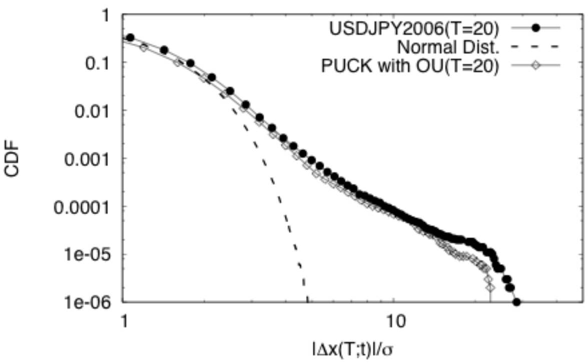

Fig.6a Log-log plot of the cumulative distribution of Dollar-Yen rate changes, $|\Delta x(20;t)|/\sigma$, in 2006.The line with black circle shows the real data and the line with diamond shows the model.

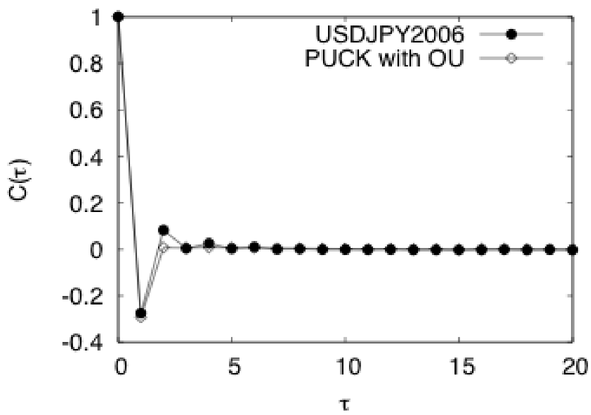

Fig.6b The autocorrelation function for Dollar-Yen rate changes.

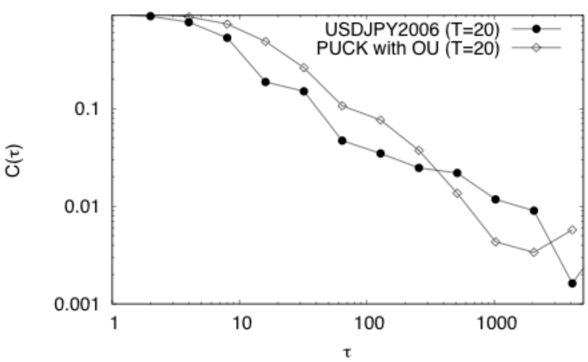

Fig.6c The autocorrelation of volatility, $\Delta x(20;t)^2$, of Dollar-Yen exchange rates. The line with black circle shows the real data and the line with diamond shows the model.

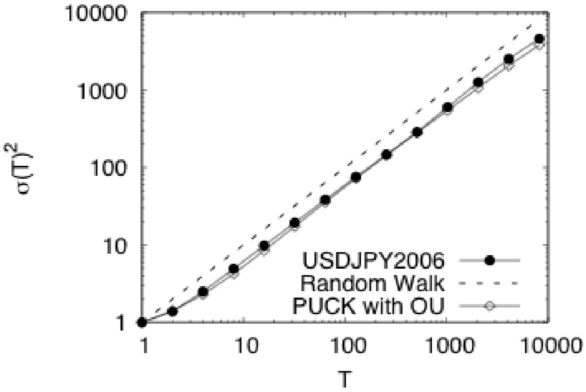

Fig.6dLog-log plot of the variance of exchange rate diffusion. The line with black circle shows the real data, the line with diamond shows the model and the dotted line shows the case of normal diffusion.

As demonstrated in these figures our model can reproduce most of the basic market properties. However, it should be noted that the basic statistics of markets are not universal in quantitative sense. For example, the slopes of the distribution or the functional forms for different observation periods may be slightly different. In the case of material systems non-stationary properties of this type rarely appear in general and we can expect permanent properties, however, for modeling human systems like the financial markets it is important to prepare a model with a wide variety and flexibility to cope with such non-stationary phenomena. In this sense the PUCK model and its generalized variants are flexible enough for description of markets as it can represent normal random walk, slower diffusion, faster diffusion and even an exponential growth by tuning the parameter, b(t), and its statistics.

Fig.7a gives the special example of Yen-Dollar rate in which the market condition changed suddenly by an external news, the 911 terrorism 2001, and Fig.7b shows the corresponding value of $b(t)$. Before the terrorism Yen-Dollar market was calm and the value of $b(t)$ was larger than 1, the attractive stable state. A little before 9 am in New York time, the first airplane hit the World Trade Center. The market price did not respond to this event clearly, however, the market became a neutral state, namely, the estimated value of $b(t)$ became close to 0. Right after the second airplane crashed the World Trade Center, the Dollar rate sharply dropped and accordingly the $b(t)$ value went into a negative region as clearly seen from Fig.7b. The market kept a highly turbulent state for several hours.

As known from this example the response of $b(t)$ value is rather quick, about a few minutes delay. This quick response is non-trivial as the value of $b(t)$ is calculated using several hundreds data points which is about a few hours in real time.

The reason for this sharp response is that when the market price changed suddenly the super moving average shifts certain amount and this shift causes change of the plot of $x(t+1)-x(t)$ vs. $\{x(t)-x_M(t)\}/(M-1)$. This example clearly shows that the market is reflecting the external news sensitively. Generally speaking any market model neglecting the effect of real time external news has only limited ability of description of real market.

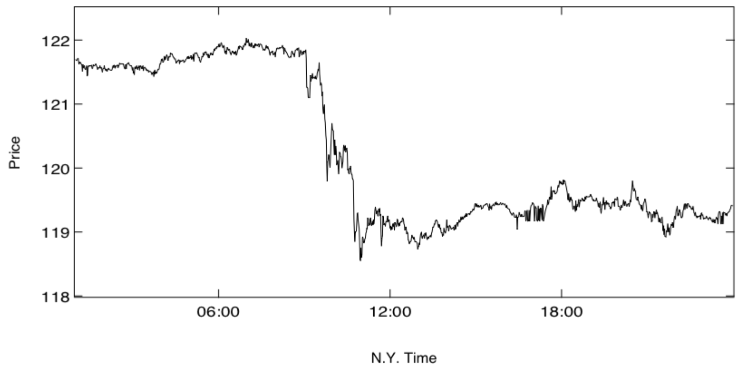

Fig.7a Dollar-Yen exchange rates on September 11th 2001.

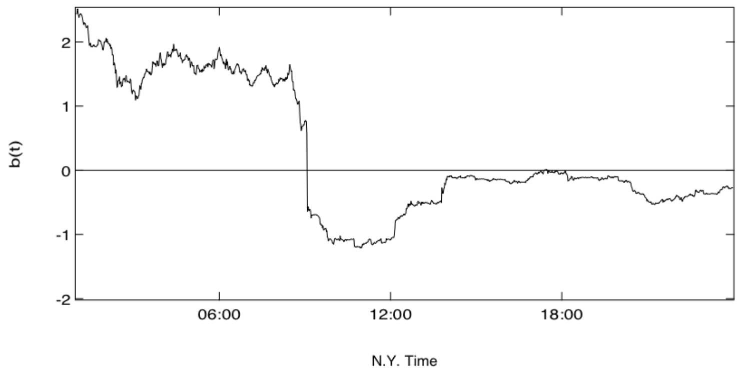

Fig.7b Estimated value of $b(t)$ for Fig.7a.

4 Summary and Discussions

In this paper we theoretically analyzed the detail statistical properties of the new type of random walk model with a moving potential force, the PUCK model. As the center of the potential force is given by the moving average, the model with a constant potential curvature shows peculiar characteristics. When the curvature $b(t)$ is positive the random walk is slower than the case of simple random walk and when $-2 < b(t) < 0$ the random walk shows abnormally fast diffusion in short time scale, however, the large scale diffusion properties are the same as the case of simple random walk. When $b(t) \le -2$ the system no longer shows a random walk behavior, rather it follows a dynamical exponential movement. Considering the generalized case with random autoregressive motion of $b(t)$ the basic statistical properties of this stochastic system becomes similar to that of real market price fluctuations as demonstrated in this paper.

From empirical point of view this PUCK model reproduces the major 4 statistical properties of market price fluctuations. Power law of price difference is explained by the effect of temporal change of $b(t)$. The quick decay of autocorrelation of price changes is realized by the statistical property that the average of the curvature is nearly zero, $< b(t) > \approx 0$. The long time autocorrelation of volatility is caused by the long time correlation in the changes of $b(t)$. The abnormal diffusion in small time scale is a direct reflection that the value of $b(t)$ takes non-zero values. The normal diffusion property in large time scale is due to the effect of that most of the values of $b(t)$ satisfies the stationary condition, $|b(t)| < 2$.

From a theoretical view point the appearance of market's potential force has already been analyzed by our group using the theoretical dealer model [14]. It is shown that when dealers are all trend-followers who predict the near future price by a linear trend of latest market price changes, then, there appear a negative curvature in the potential function, $b(t) < 0$, in the market price fluctuations. On the contrary when the dealers are contrarians who predict that the market trend will turn in the near future, then, there appears a positive potential force in the market. Namely, the moving potential reflects the averaged response of the dealers to the market price changes. The value of $b(t)$ also depends both on the number of dealers and on other characteristics, so that the market potential function slowly changes in the real market spontaneously and sometimes very quickly by news like the case of 911.

The PUCK model can be generalized in various ways. One direction of generalization is taking into account higher order terms in the potential function, $U_M(x;t)$ in Eq.(2). In the case that this potential function includes odd order terms of x such as x^3, it implies that the potential force works either up or down in uneven way causing a directional motion of market price [15]. It is already shown that in the special case that an asymmetric potential function changes its sign randomly at every time step, the PUCK model is reduced to the ARCH model in financial technology [10].

Another direction of generalization is the continuum limit and macroscopic limit of the PUCK model [16]. In this paper we consider discrete tick-by-tick time as a standard of time, however, we can consider a continuous time version of the model. In the continuum limit it is shown that the PUCK model is described by a Langevin type stochastic equation with time dependent mass and viscosity. In the case of stable potential function the corresponding mass is negative and the viscosity is positive. In the special case $b(t)=0$ the corresponding mass is 0 and the price follows an ordinary diffusion equation. In the case $-2<b(t)<0$ both the mass and viscosity are positive, a situation similar to colloid particles in water. For $b(t)<-2$ the mass is positive and the viscosity takes a negative value, hence any small fluctuation is magnified indefinitely.

Macroscopic limit can be considered by applying renormalization to the PUCK model [16]. It is shown that the PUCK model becomes a macroscopic inflation equation by a renormalization limit. Applying such renormalization technique to the PUCK model we may expect to bridge the microscopic market phenomena and macroscopic social behaviors.

The potential forces in the market are expected to be strongly related to the distribution of demand and supply in the market called the order book. We hope to apply our model to market data with full order book information.

Interaction with other market prices is also an interesting open problem. It is well accepted that each market superficially look changing independently, however, in the case of foreign exchange market any 3 combination of currencies are interacting through so called the triangular arbitrage. Triangular arbitrage is the chance of getting more money by simply circulating currencies, for example, buy US dollar with Yen, buy Euro with US dollar, and buy Yen with Euro [17]. If each combination of currency exchange is done independently there occurs situation that such circulation of money causes an increase of dealer's asset. Then, those arbitrage transactions will change the market prices so that the arbitrage opportunity vanishes quickly. Also, many market prices are intuitively interacting with many others, however, mathematical description of such interaction is yet to be done.

Finally, as known from the mathematical formulation the applicability of the present model is not limited to market data only, but also to any time sequential data for finding hidden potential dynamics by analyzing the motion of moving averages in various scales. We hope that this method will contribute to analyze complicated dynamics from given data in wide field of science.

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References

[1] L. Bachelier, *Théorie de la Spéculation*, Doctoral disseration. Annales Scientifiques de l'Ecole Normale Superieure (1900). Translation: The RandomCharacter of Stock MarketPrices (Editor P.H.Cootner, MIT Press, Cambridge), 21-86 (1964).
[2] A. Einstein, Analen der Physik 17, 549-560 (1905).
[3] T. N. Mantegna, and H. E. Stanley, An Introduction to Econophysics: Correlation and Complexity in Finance,(Cambridge University Press, Cambridge, 2000).
[4] B.B.Mandelbrot, J. of Business 36, 394-419 (1963).
[5] P. Gopikrishnan, M. Meyer, L.A.N. Amaral, and H. E. Stanley, Euro. Phys. J. B: Rapid Comm.**3**, 139-140 (1998).
[6] N. Vandewalle and M. Ausloos, Int. J. Mod. Phys. C 9, 711-719 (1998).
[7] R. Engle, Econometrica 50, 987-1008 (1982).
[8] M.Takayasu, T. Mizuno, T. Ohnishi and H. Takayasu, Proceedings of "Practical Fruits of Econophysics", (Editor H. Takayasu, Springer, Tokyo), 29-32 (2005).
[9] M. Takayasu, T. Mizuno and H. Takayasu, Physica A370, 96-97 (2006).
[10] M. Takayasu, T. Mizuno and H. Takayasu, Physica A383, 115-119 (2007).
[11] H. Takayasu, A.-H.Sato and M. Takayasu, Phys. Rev. Lett., 79, 966-969(1997).
[12] T. Ohnishi, T. Mizuno, K. Aihara, M. Takayasu, and H. Takayasu, Physica A 344, 207-210 (2004).
[13] M. Takayasu, K. Watanabe and H.Takayasu, submitted to Phys. Rev. Lett.
[14] K. Yamada, H. Takayasu, T. Ito and M. Takayasu, Phys. Rev. E79, 051120-(1-12)(2009).
[15] K. Watanabe, H. Takayasu and M. Takayasu, submitted to Phys. Rev. E.
[16] M. Takayasu and H. Takayasu, Prog. Theor. Phys. Supplement 179, 1-7(2009).
[17] Y. Aiba, N. Hatano, H. Takayasu, K. Marumo and T. Shimizu, Physica A324, 253-257 (2003).